1.3. Vector Equations

3 Representations of the Same Problem

1.1-1.2. System of Equations

$$\left\{\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 3x_3 & = & 9 \\ -4x_1 & + & 2x_2 & & & = & -20 \\ 2x_1 & + & x_2 & - & 2x_3 & = & 0 \end{array}\right\}$$

1.3. Vector Equation

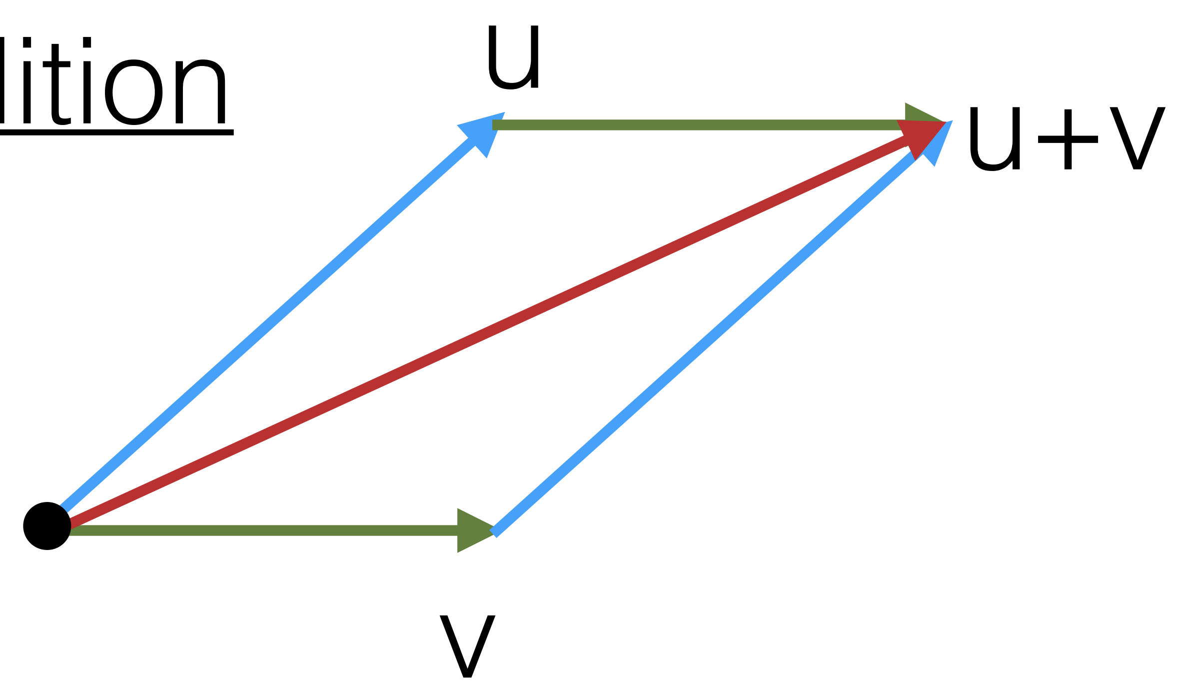
$$x_1\begin{bmatrix}1\\-4\\2\end{bmatrix}+x_2\begin{bmatrix}2\\2\\1\end{bmatrix}+x_3\begin{bmatrix}3\\0\\-2\end{bmatrix}=\begin{bmatrix}9\\-20\\0\end{bmatrix}.$$

1.4. Matrix Equation

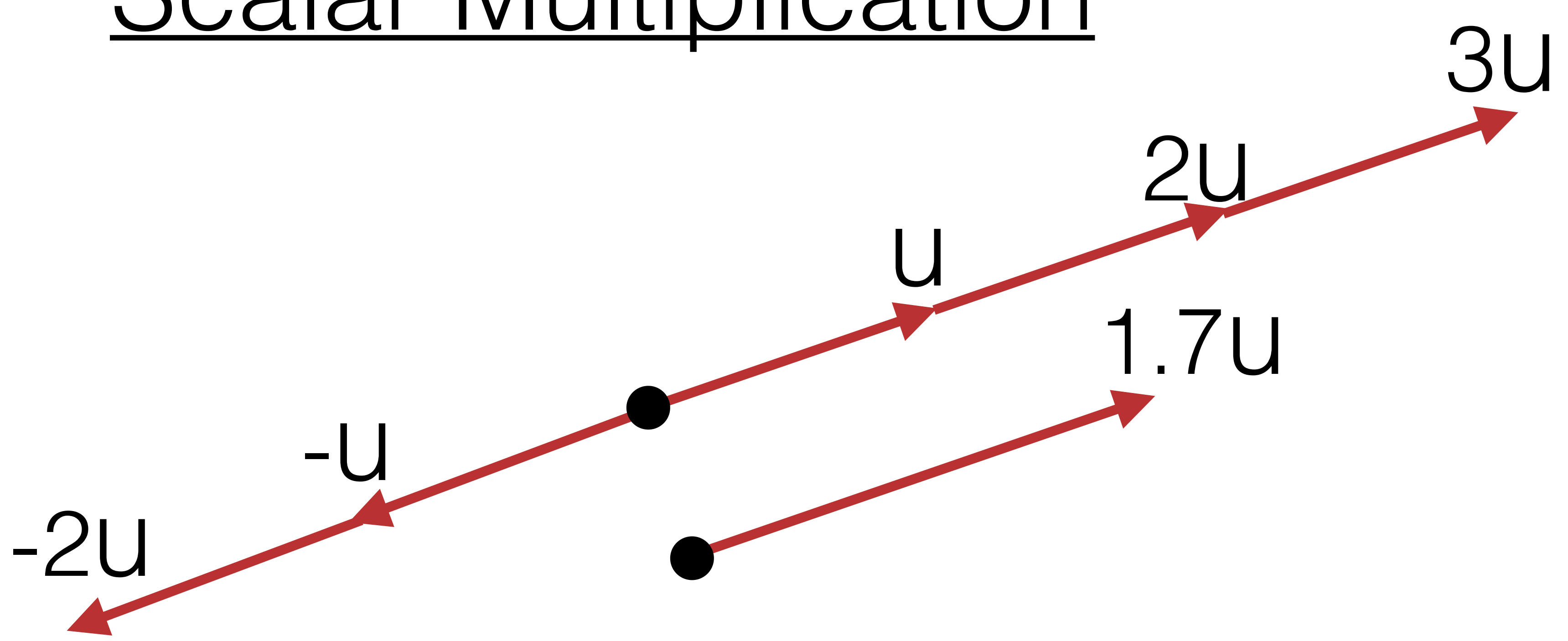
$$\begin{bmatrix}1 & 2 & 3\\-4 & 2 & 0\\2 & 1 & -2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}9\\-20\\0\end{bmatrix}$$

Geometry Vectors

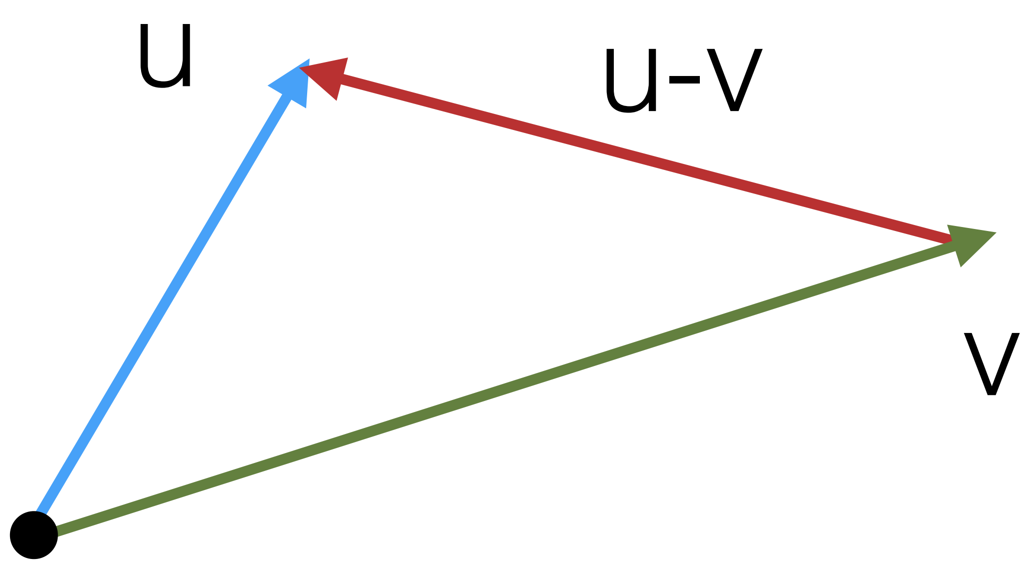
Addition



Scalar Multiplication



Subtraction



v + (u-v) = u

Multiplication

No good notion of vector multiplication

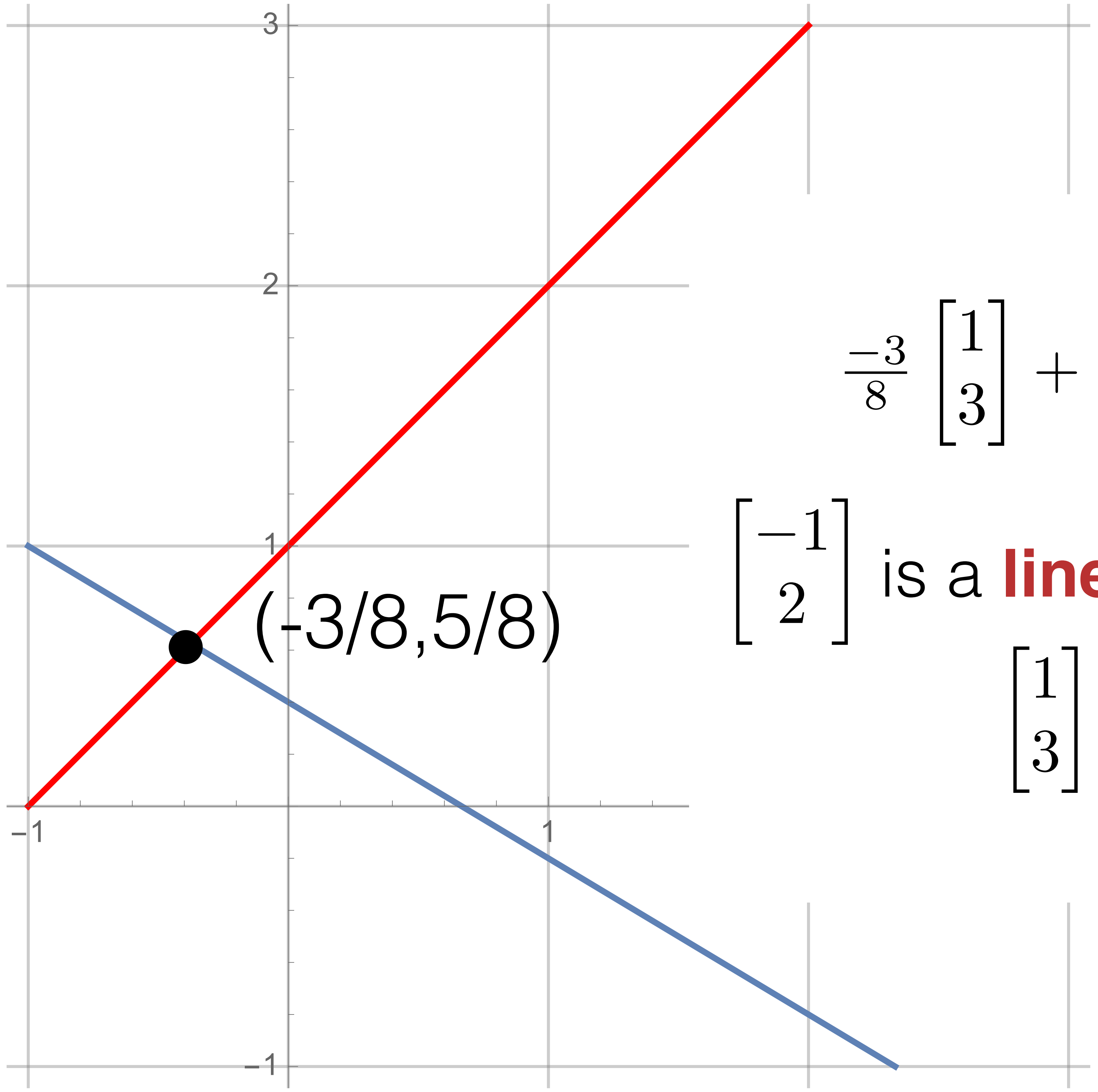
Later in the course we will see a scalar product

v ▪ w = constant (not a vector)

Vector Equations

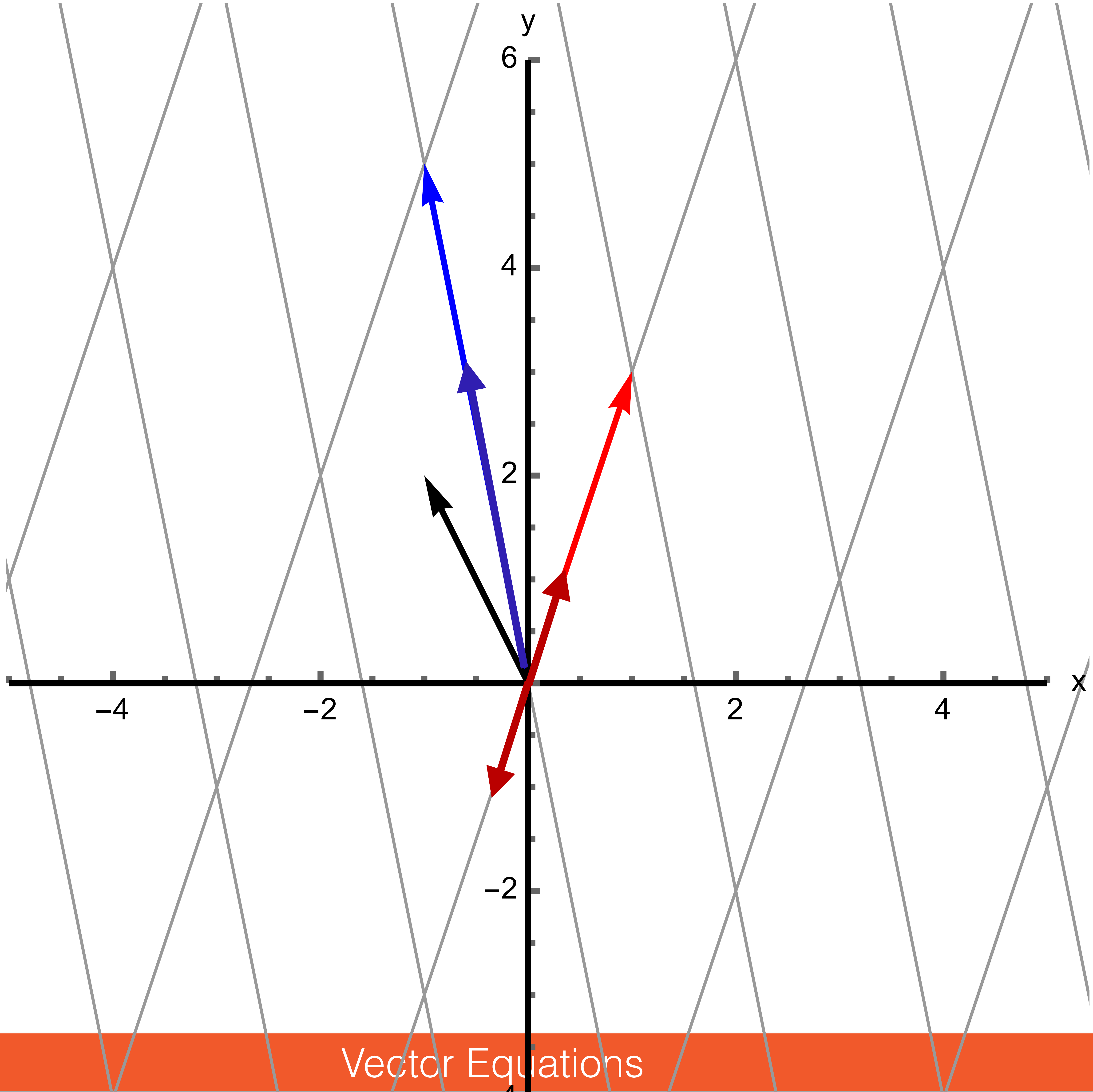
$$\left\{\begin{array}{rcrcr} x & - & y & = & -1 \\ 3x & + & 5y & = & 2 \end{array}\right\}$$

$$\begin{bmatrix} x \\ 3x \end{bmatrix} + \begin{bmatrix} -y \\ 5y \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \longrightarrow x\begin{bmatrix} 1 \\ 3 \end{bmatrix} + y\begin{bmatrix} -1 \\ 5 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$



$$\tfrac{-3}{8}\begin{bmatrix} 1 \\ 3 \end{bmatrix} + \tfrac{5}{8}\begin{bmatrix} -1 \\ 5 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ is a **linear combination** of $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 5 \end{bmatrix}$



Another Example

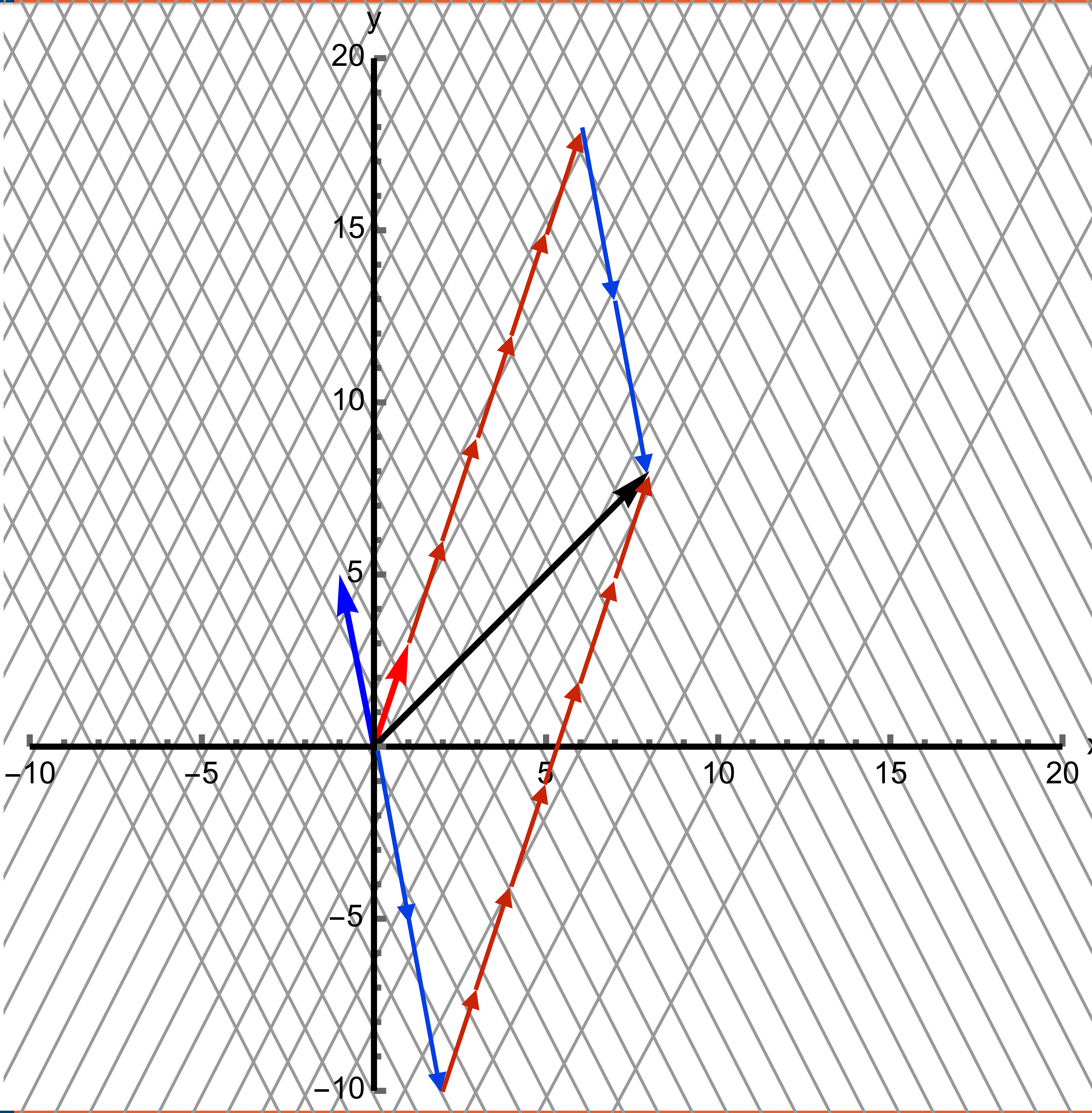
Convert this to a vector equation and solve. Describe you're answer using "linear combination"

$$\left\{\begin{array}{rcrcr} x & - & y & = & 8 \\ 3x & + & 5y & = & 8 \end{array}\right\}$$

Vector Equation: $x\begin{bmatrix}1\\3\end{bmatrix} + y\begin{bmatrix}-1\\5\end{bmatrix} = \begin{bmatrix}8\\8\end{bmatrix}$

Solution (unique): $6\begin{bmatrix}1\\3\end{bmatrix} - 2\begin{bmatrix}-1\\5\end{bmatrix} = \begin{bmatrix}8\\8\end{bmatrix}$

The vector $\begin{bmatrix}8\\8\end{bmatrix}$ is a linear combination of the vectors $\begin{bmatrix}1\\3\end{bmatrix}$ and $\begin{bmatrix}-1\\5\end{bmatrix}$ (uniquely)



Linear Combinations

Definition

Given vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \in \mathbb{R}^n$ and constants $c_1, c_2, \ldots, c_k$, the vector

$$\vec{y} = c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_k\vec{v}_k$$

is a **linear combination** of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ with weights $c_1, c_2, \ldots, c_k$.

Eg. $$6\begin{bmatrix}1\\3\end{bmatrix} - 2\begin{bmatrix}-1\\5\end{bmatrix} = \begin{bmatrix}8\\8\end{bmatrix}$$

So $\begin{bmatrix}8\\8\end{bmatrix}$ is a linear combination of $\begin{bmatrix}1\\3\end{bmatrix}$ and $\begin{bmatrix}-1\\5\end{bmatrix}$ with weights 6 and -2.

Eg. $$2\underset{\mathbf{u}}{\begin{bmatrix}1\\1\\1\end{bmatrix}} - 3\underset{\mathbf{v}}{\begin{bmatrix}-1\\2\\1\end{bmatrix}} + 4\underset{\mathbf{w}}{\begin{bmatrix}3\\2\\1\end{bmatrix}} = \underset{\mathbf{b}}{\begin{bmatrix}17\\4\\3\end{bmatrix}}$$

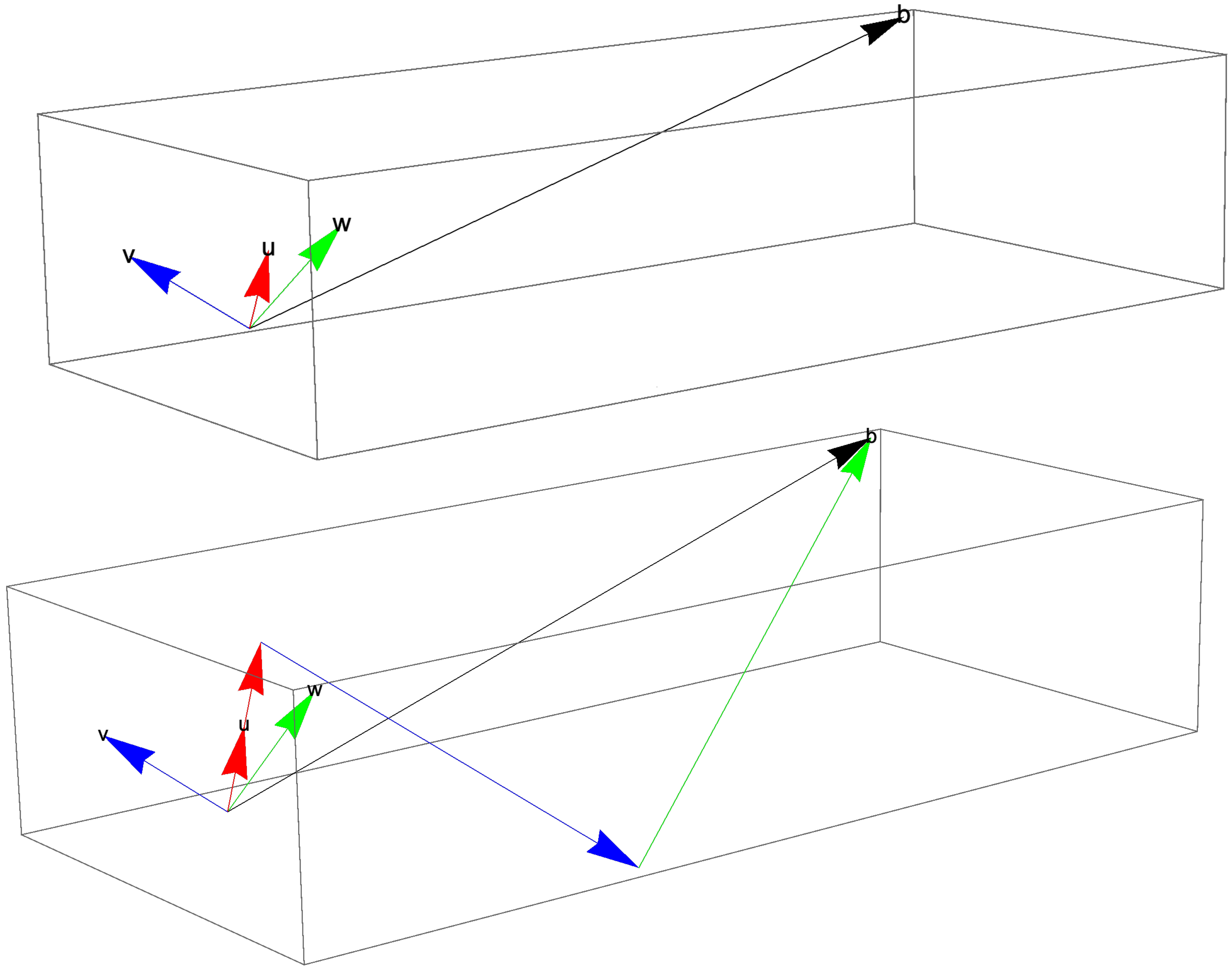
So **b** is a linear combination of **u**, **v**, and **w** with weights 2, -3, 4

Linear Combinations

Eg. $$\underset{}{2\begin{bmatrix}1\\1\\1\end{bmatrix}}-3\begin{bmatrix}-1\\2\\1\end{bmatrix}+4\begin{bmatrix}3\\2\\1\end{bmatrix}=\begin{bmatrix}17\\4\\3\end{bmatrix}$$

u **v** **w** **b**

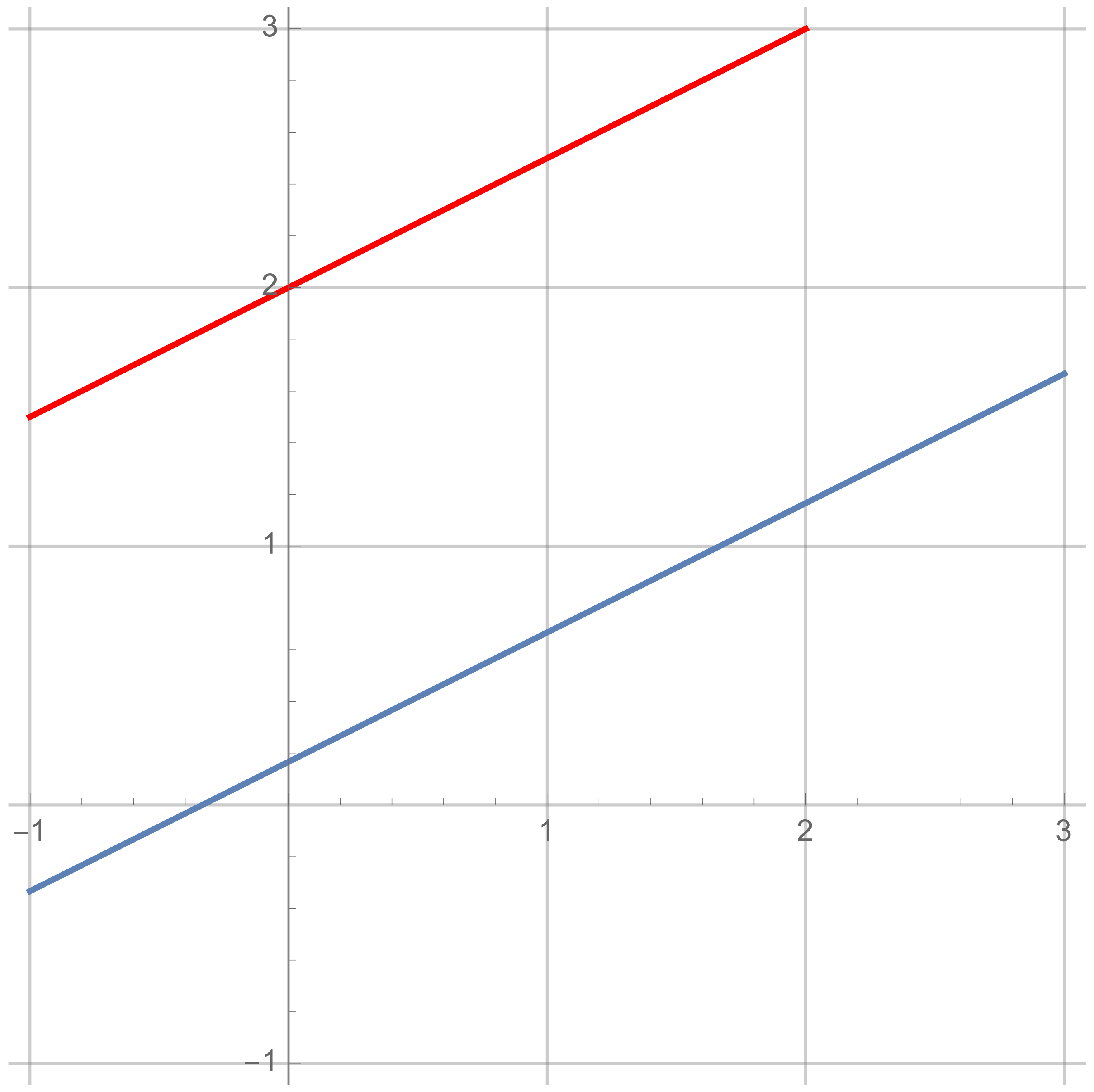
So **b** is a linear combination of **u**, **v**, and **w** with weights 2, -3, 4

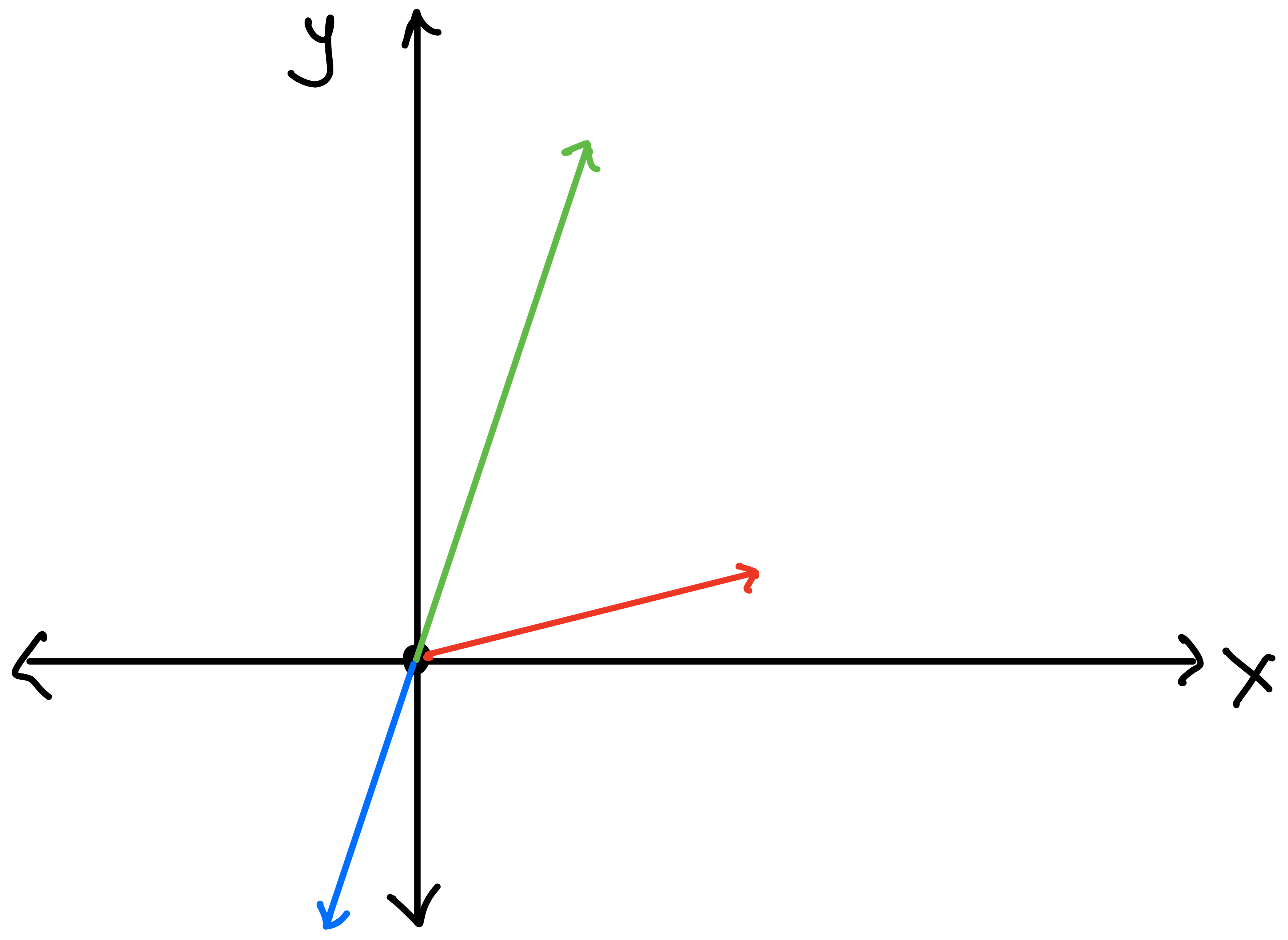


Linear Combinations: Can't Get There From Here

(B) $\left\{ \begin{array}{rcrcr} -x & + & 2y & = & 4 \\ -3x & + & 6y & = & 1 \end{array} \right\}$

$$x \begin{bmatrix} -1 \\ -3 \end{bmatrix} + y \begin{bmatrix} 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

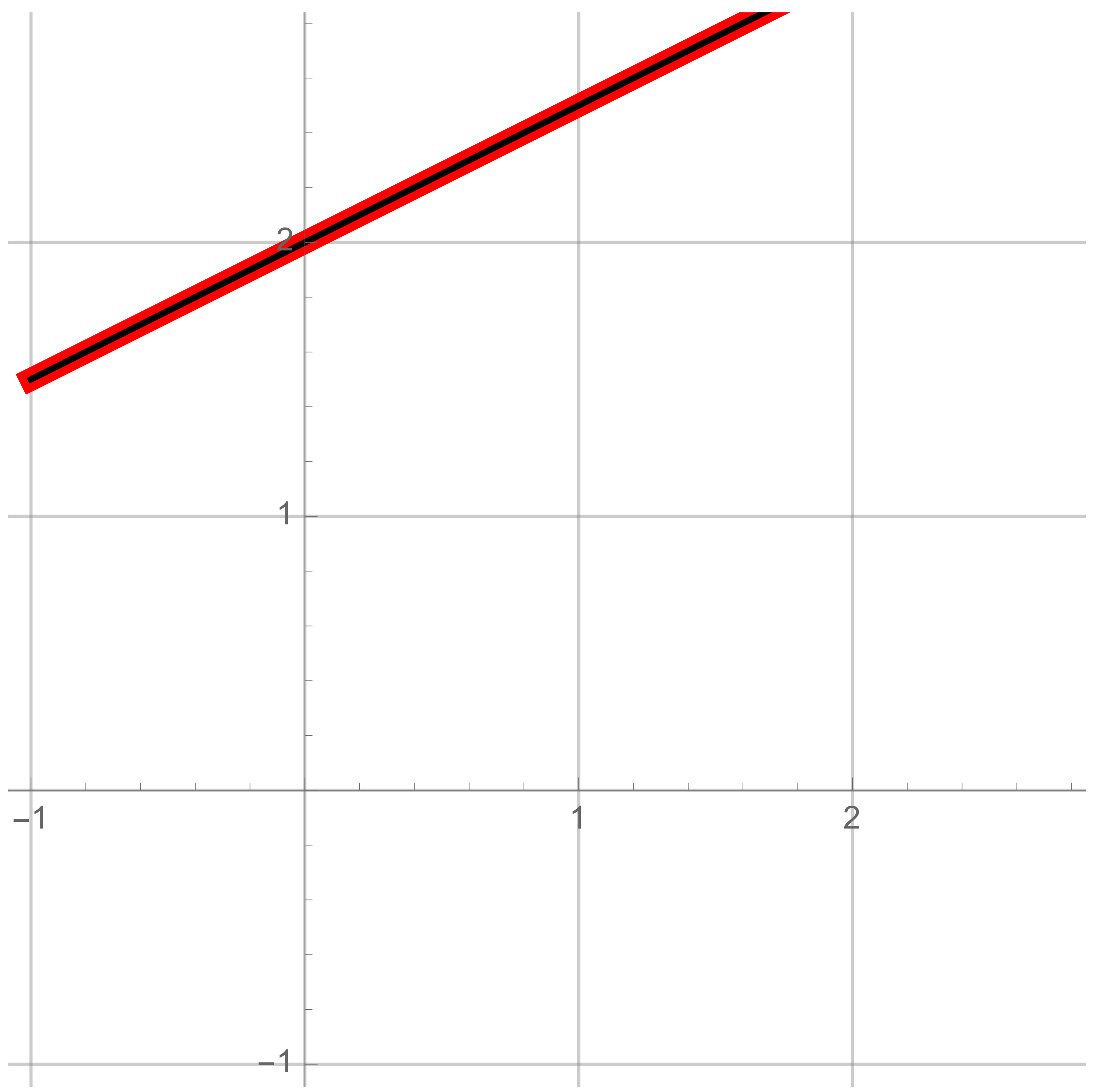


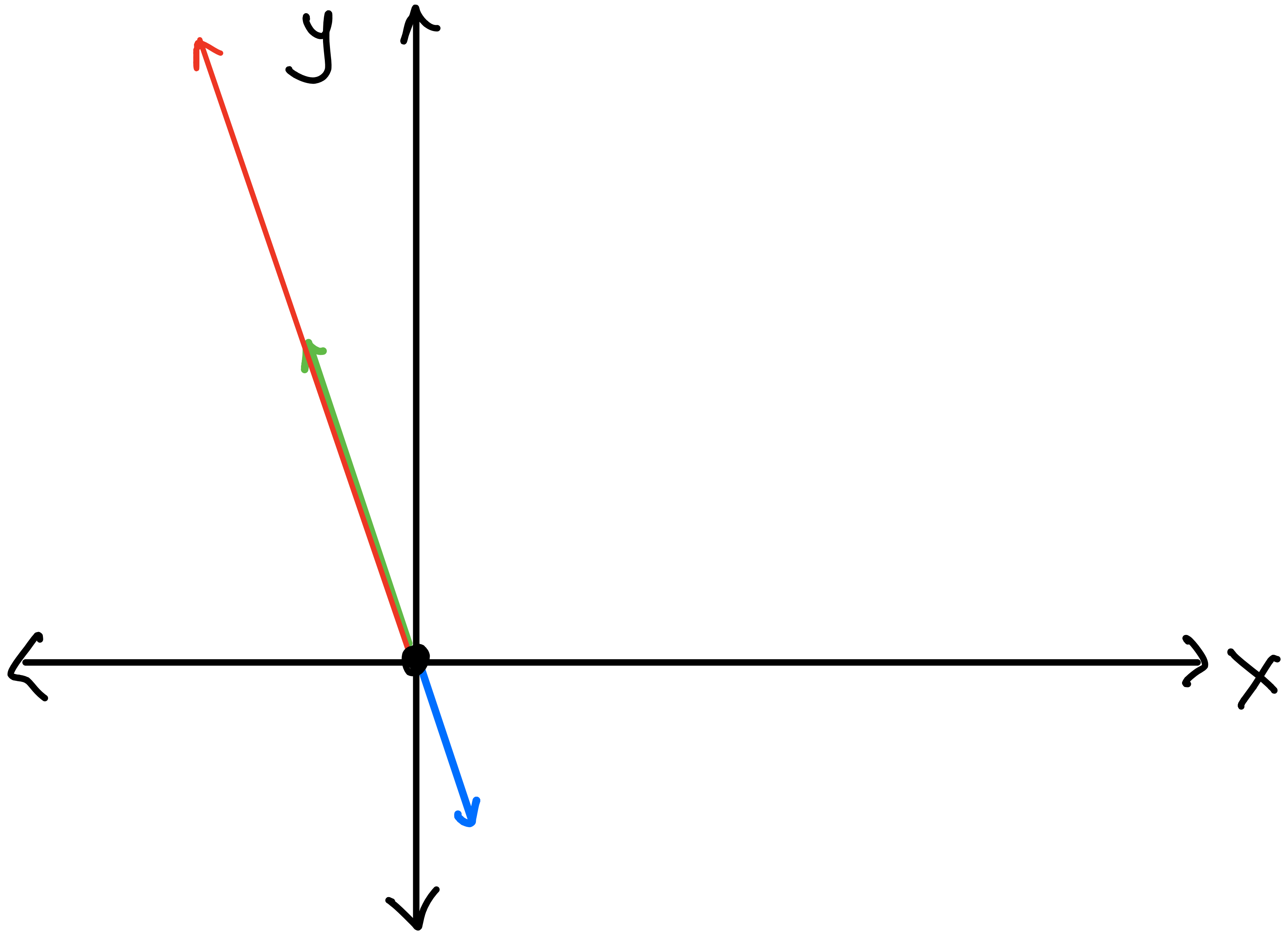


Linear Combinations: Can Get There in Multiple Ways

(C) $\left\{ \begin{array}{rcrcr} x & - & 2y & = & -4 \\ -3x & + & 6y & = & 12 \end{array} \right\}$

$$x\begin{bmatrix}1\\-3\end{bmatrix} + y\begin{bmatrix}-2\\6\end{bmatrix} = \begin{bmatrix}-4\\12\end{bmatrix}$$





Span

Definition

The **span** of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \in \mathbb{R}^n$, denoted $\operatorname{span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k)$, is the set of all vectors of the form

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_k\vec{v}_k$$

where $c_1, c_2, \ldots, c_k \in \mathbb{R}$ are scalars.

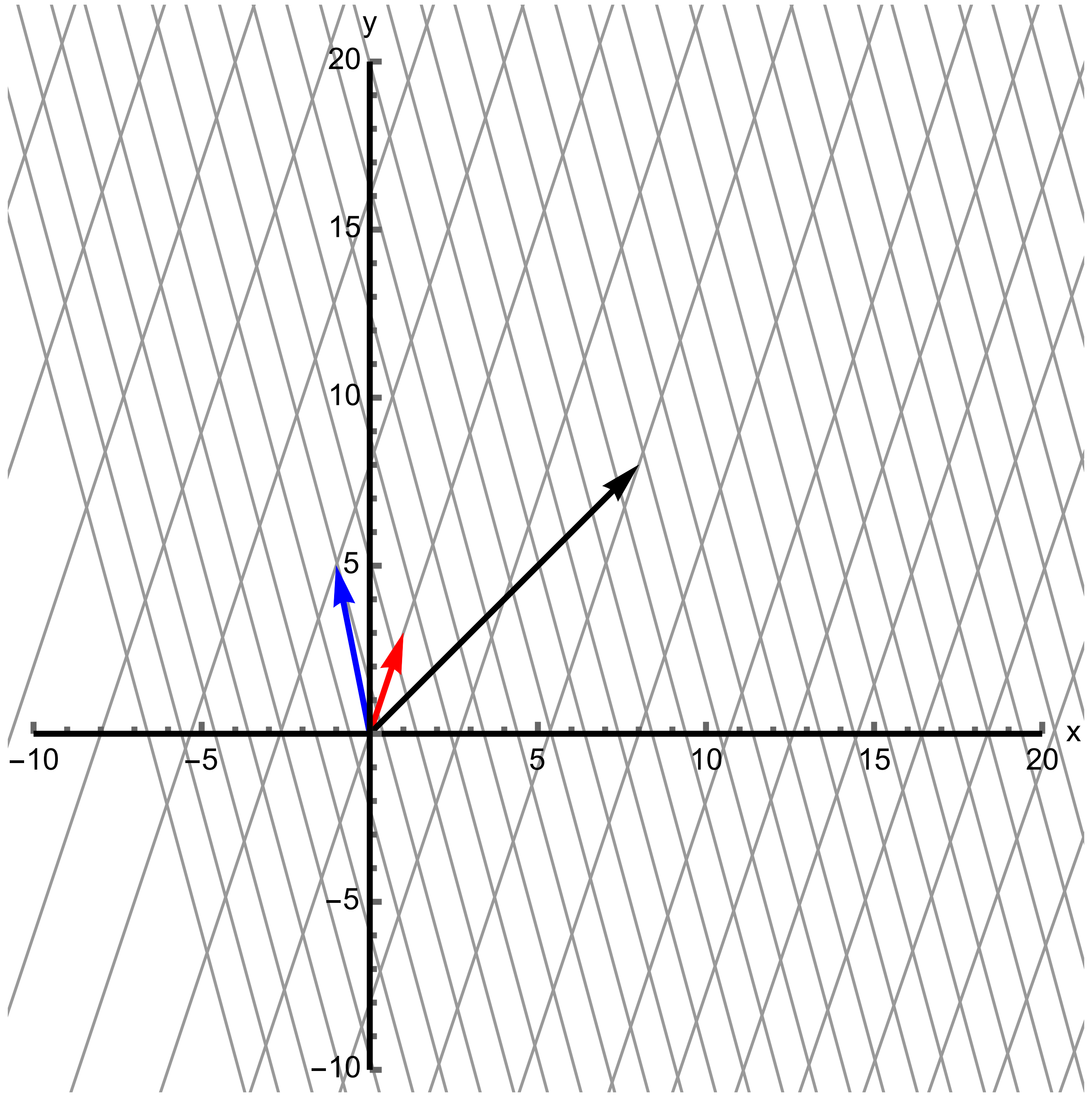
In other words, $\operatorname{span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k)$ is the set of **all linear combinations** of the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$.

Definition

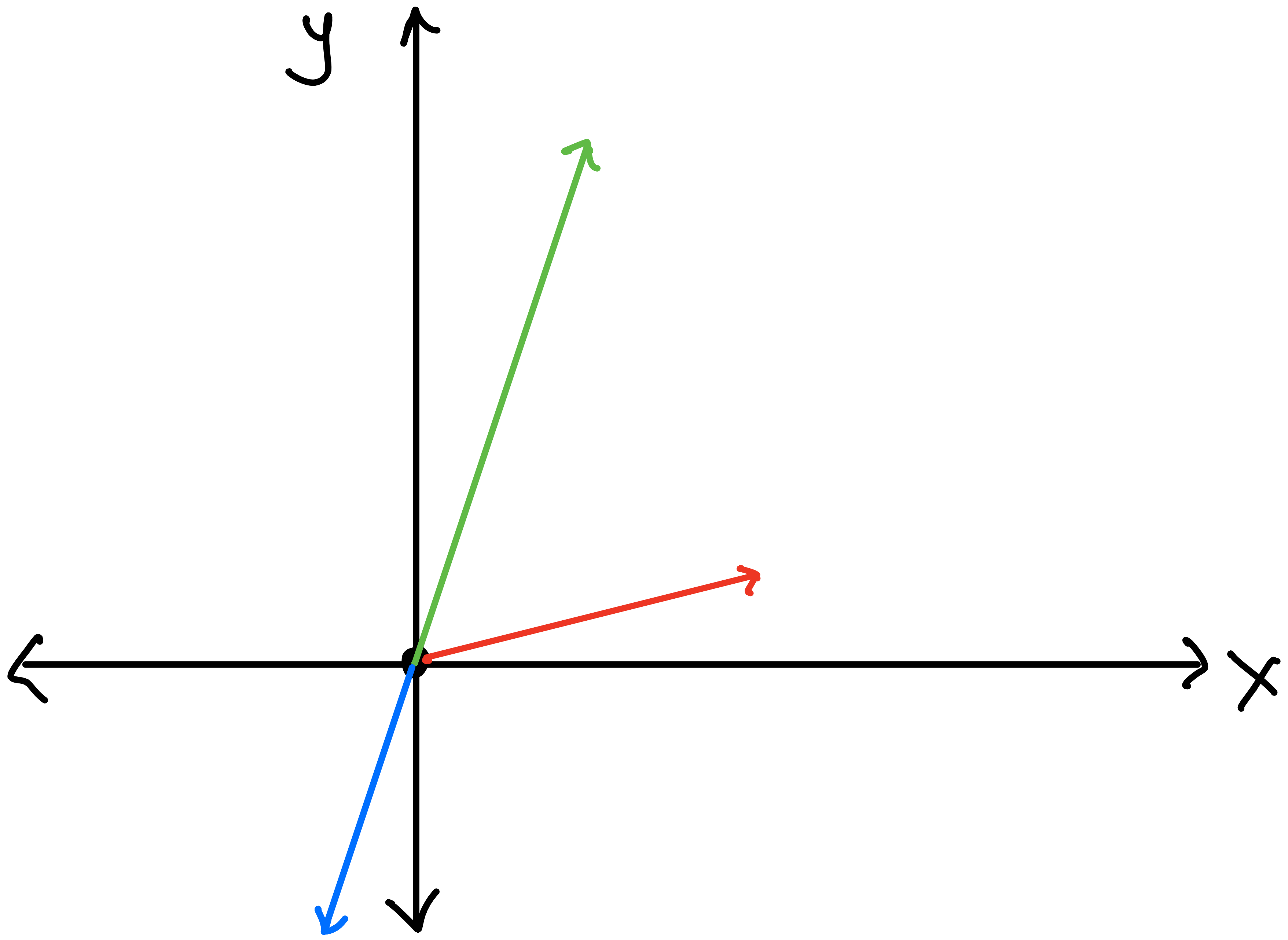
The **span** of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \in \mathbb{R}^n$, denoted $\operatorname{span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k)$, is the set of all vectors of the form

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_k\vec{v}_k$$

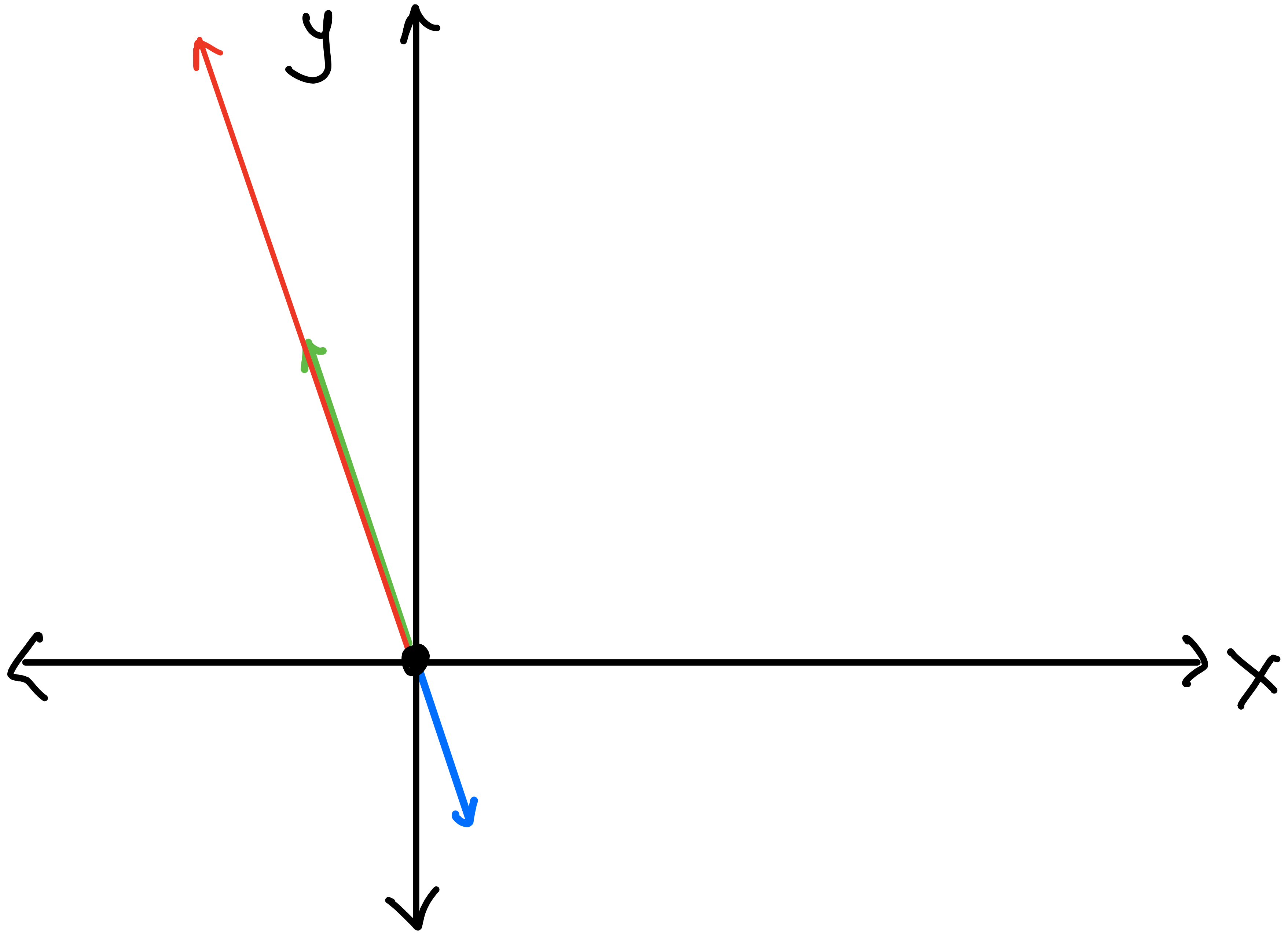
where $c_1, c_2, \ldots, c_k \in \mathbb{R}$ are scalars.



(A)



(B)



(C)

3D examples: Write as vector equations and discuss their solutions using the terms **linear combination** and **span**

(A)
$$\left\{\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & x_3 & = & 10 \\ 4x_1 & + & 5x_2 & + & x_3 & = & 34 \\ 7x_1 & + & 8x_2 & - & x_3 & = & 60 \end{array}\right\} \qquad \left[\begin{array}{rrr|r} 1 & 2 & 1 & 10 \\ 4 & 5 & 1 & 34 \\ 7 & 8 & -1 & 60 \end{array}\right] \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

(B)
$$\left\{\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 3x_3 & = & 0 \\ 4x_1 & + & 5x_2 & + & 6x_3 & = & 3 \\ 7x_1 & + & 8x_2 & + & 9x_3 & = & 0 \end{array}\right\} \qquad \left[\begin{array}{rrr|r} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 3 \\ 7 & 8 & 9 & 0 \end{array}\right] \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

(C)
$$\left\{\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 3x_3 & = & 3 \\ 4x_1 & + & 5x_2 & + & 6x_3 & = & 9 \\ 7x_1 & + & 8x_2 & + & 9x_3 & = & 15 \end{array}\right\} \qquad \left[\begin{array}{rrr|r} 1 & 2 & 3 & 3 \\ 4 & 5 & 6 & 9 \\ 7 & 8 & 9 & 15 \end{array}\right] \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

3D examples: Write as vector equations and discuss their solutions using the terms **linear combination** and **span**

(A) $\left\{\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & x_3 & = & 10 \\ 4x_1 & + & 5x_2 & + & x_3 & = & 34 \\ 7x_1 & + & 8x_2 & - & x_3 & = & 60 \end{array}\right\}$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 10 \\ 4 & 5 & 1 & 34 \\ 7 & 8 & -1 & 60 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Vector equation: $x_1 \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 10 \\ 34 \\ 60 \end{bmatrix}$

b can be written as a linear combination of **u**, **v**, and **w** as **b** = 5 **u** + 3 **v** - **w**

$$\begin{array}{ccccccc} & \mathbf{u} & & \mathbf{v} & & \mathbf{w} & & \mathbf{b} \\ 5 & \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} & +3 & \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} & +(-1) & \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} & = & \begin{bmatrix} 10 \\ 34 \\ 60 \end{bmatrix} \end{array}$$

Solution: $5 \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} + (-1) \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 10 \\ 34 \\ 60 \end{bmatrix}$

b is in the span of **u**, **v**, and **w**