Daily vocabulary: vectors, linear combination, weights, span.

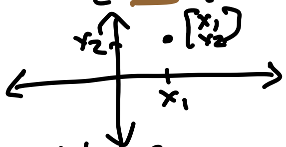
$\boxed{\mathbb{R}^n}$: n-dimensional space

$\mathbb{R}$ = real number line

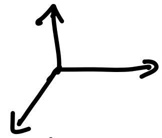
… −1 0 1 2 …

"such that"

$\mathbb{R}^2 = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \;\middle|\; x_1, x_2 \in \mathbb{R} \right\}$ "is an element of" $\quad \mathbb{R}^3 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \;\middle|\; x_1, x_2, x_3 \in \mathbb{R} \right\} \quad \mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \;\middle|\; x_1, x_2, \dots, x_n \in \mathbb{R} \right\}$



plane


3d space

Class Discussion: Vector Equations

1. From CP-1.3: Consider the following linear system.

(A) $\left\{ \begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & x_3 & = & 10 \\ 4x_1 & + & 5x_2 & + & x_3 & = & 34 \\ 7x_1 & + & 8x_2 & - & x_3 & = & 60 \end{array} \right.$

augment $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 10 \\ 4 & 5 & 1 & 34 \\ 7 & 8 & -1 & 60 \end{array}\right] \longrightarrow$ row reduce $\begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right] \end{array}$

(a) ~~(a)~~ Write down the "general solution" to this system. Is it the empty set, a point, a line, a plane, something else?

0 1 ∞

(b) Write the system of equations as a vector equation of the form: $x_1\vec{\mathbf{u}} + x_2\vec{\mathbf{v}} + x_3\vec{\mathbf{w}} = \vec{\mathbf{b}}$

(c) Write the solution(s) as a linear combination of vectors:

(d) The **span** of $\{\vec{\mathbf{u}}, \vec{\mathbf{v}}, \vec{\mathbf{w}}\}$ is the set of all vectors that can be written as a linear combination of $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, w. What is the span of these vectors?

$\boxed{\begin{array}{l} x_1 = 5 \\ x_2 = 3 \\ x_3 = -1 \end{array}}$

$$\underbrace{\underset{5}{x_1}\underset{u}{\begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}} + \underset{3}{x_2}\underset{v}{\begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}} + \underset{-1}{x_3}\underset{w}{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}}}_{\text{linear combination}} = \underset{b}{\begin{bmatrix} 10 \\ 34 \\ 60 \end{bmatrix}}$$

b is a lin comb of u, v, w in exactly one way.

b is in the span of u, v, w

span(u, v, w) = set of all possible linear combs of u, v, w

$= \{ x_1\vec{u} + x_2\vec{v} + x_3\vec{w} \mid x_1, x_2, x_3 \in \mathbb{R} \}$

2. Repeat with B and C:

(B) $\left\{\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 3x_3 & = & 0 \\ 4x_1 & + & 5x_2 & + & 6x_3 & = & 3 \\ 7x_1 & + & 8x_2 & + & 9x_3 & = & 0 \end{array}\right\}$ $\quad \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 3 \\ 7 & 8 & 9 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$ no solution

(C) $\left\{\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 3x_3 & = & 3 \\ 4x_1 & + & 5x_2 & + & 6x_3 & = & 9 \\ 7x_1 & + & 8x_2 & + & 9x_3 & = & 15 \end{array}\right\}$ $\quad \left[\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 4 & 5 & 6 & 9 \\ 7 & 8 & 9 & 15 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$ free ← consistent

$$x_1 \underbrace{\begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}}_{u} + x_2 \underbrace{\begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}}_{v} + x_3 \underbrace{\begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}}_{w} = \underbrace{\begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}}_{b}$$

b is not in the span of u, v, w

$$\underset{\substack{1 \\ (1+1d)}}{x_1} \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + \underset{(1-2id)}{x_2} \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} + \underset{\substack{0 \\ id}}{x_3} \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \\ 15 \end{bmatrix}$$

$$\left\{\begin{array}{l} x_1 \quad - x_3 = 1 \\ \quad x_2 + 2x_3 = 1 \end{array}\right\}$$

gen solution

$x_1 = 1 + x_3$

$x_2 = 1 - 2x_3$

$x_3 = \text{free}$

3. In the plot below, identify the vectors $\vec{\mathbf{u}}, \vec{\mathbf{v}}, \vec{\mathbf{w}}$ and use the picture to write $\vec{\mathbf{w}}$ as a linear combination of $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$. Write down the matrix that we would row reduce to solve the problem.

$$\vec{\mathbf{u}} = \begin{bmatrix}3\\1\end{bmatrix} \quad \vec{\mathbf{v}} = \begin{bmatrix}-1\\1\end{bmatrix} \quad \vec{\mathbf{w}} = \begin{bmatrix}10\\-2\end{bmatrix}$$

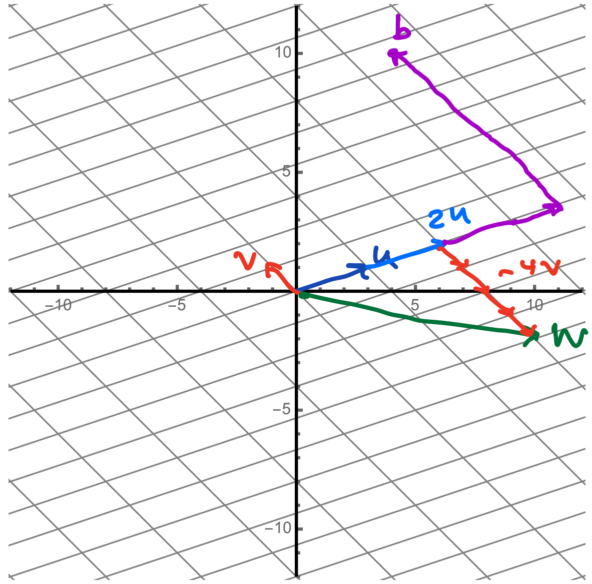
$$\underset{2}{x_1} \begin{bmatrix}3\\1\end{bmatrix} + \underset{-4}{x_2} \begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}10\\-2\end{bmatrix}$$

$$\left[\begin{array}{cc|c}3 & -1 & 10\\1 & 1 & -2\end{array}\right] \to \left[\begin{array}{cc|c}1 & 1 & -2\\3 & -1 & 10\end{array}\right]$$

$$\to \left[\begin{array}{cc|c}1 & 1 & -2\\0 & -4 & 16\end{array}\right]$$

$$\to \left[\begin{array}{cc|c}1 & 1 & -2\\0 & 1 & -4\end{array}\right]$$

$$\to \left[\begin{array}{cc|c}1 & 0 & 2\\0 & 1 & -4\end{array}\right]$$



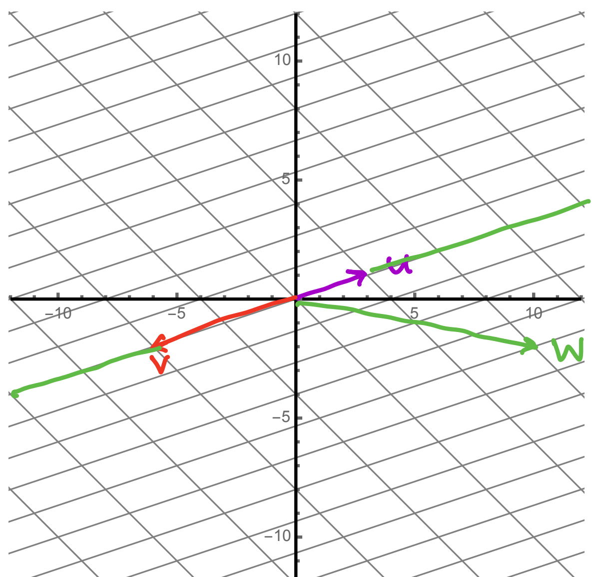
4. Repeat the same question with slightly different vectors.

$$\vec{\mathbf{u}} = \begin{bmatrix}3\\1\end{bmatrix} \quad \vec{\mathbf{v}} = \begin{bmatrix}-6\\-2\end{bmatrix} \quad \vec{\mathbf{w}} = \begin{bmatrix}10\\-2\end{bmatrix}$$

$$\left[\begin{array}{cc|c}3 & -6 & 10\\1 & -2 & -2\end{array}\right] \to \left[\begin{array}{cc|c}1 & -2 & -2\\3 & -6 & 10\end{array}\right]$$

$$\to \left[\begin{array}{cc|c}1 & -2 & -2\\0 & 0 & 16\end{array}\right]$$

inconsistent



Didn't get to: See Friday (Day 4)

5. In each example below there are 3 vectors, $\vec{\mathbf{u}}, \vec{\mathbf{v}}, \vec{\mathbf{w}}$, in $\mathbb{R}^3$. Describe the span of the vectors. A useful row reduction has been done for you in each case.

(a) $\vec{\mathbf{u}} = \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}, \vec{\mathbf{v}} = \begin{bmatrix} 3 \\ 12 \\ -9 \end{bmatrix}, \vec{\mathbf{w}} = \begin{bmatrix} -2 \\ -8 \\ 6 \end{bmatrix}$ $\qquad \begin{bmatrix} 1 & 3 & -2 \\ 4 & 12 & -8 \\ -3 & -9 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(b) $\vec{\mathbf{u}} = \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}, \vec{\mathbf{v}} = \begin{bmatrix} 3 \\ 12 \\ -9 \end{bmatrix}, \vec{\mathbf{w}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\qquad \begin{bmatrix} 1 & 3 & 1 \\ 4 & 12 & 1 \\ -3 & -9 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

(c) $\vec{\mathbf{u}} = \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}, \vec{\mathbf{v}} = \begin{bmatrix} 3 \\ 12 \\ -9 \end{bmatrix}, \vec{\mathbf{w}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\qquad \begin{bmatrix} 1 & 3 & 1 \\ 4 & 12 & 1 \\ -3 & -9 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

(d) $\vec{\mathbf{u}} = \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}, \vec{\mathbf{v}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{\mathbf{w}} = \begin{bmatrix} 2 \\ 5 \\ -2 \end{bmatrix}$ $\qquad \begin{bmatrix} 1 & 1 & 2 \\ 4 & 1 & 5 \\ -3 & 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

(e) $\vec{\mathbf{u}} = \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}, \vec{\mathbf{v}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{\mathbf{w}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $\qquad \begin{bmatrix} 1 & 1 & 1 \\ 4 & 1 & 2 \\ -3 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Done in R

R Computations

1. Working together we will use R to solve the following system of equations.

$$\left\{\begin{array}{rcrcrcrcr} x_1 & + & 2x_2 & + & x_3 & + & x_4 & = & 4 \\ x_1 & + & 2x_2 & + & -x_3 & + & -3x_4 & = & 6 \\ & & x_2 & + & x_3 & + & x_4 & = & 0 \\ -x_1 & + & x_2 & + & -x_3 & + & -4x_4 & = & -1 \end{array}\right\}$$

2. Discuss problem PS1.5 (Traffic Flow) at your tables. (a) Agree on the system of equations that needs to be solved and put the system in matrix form. (b) Use R to solve it. Help each other at your table to make sure everyone gets the system solved. Discuss and find a solutions for (c) and (d).

3. Time permitting, work with your table mates to answer this question using R.

   (a) Below are three vectors $\vec{\mathbf{u}}, \vec{\mathbf{v}}, \vec{\mathbf{w}}$ in $\mathbb{R}^4$. Is the vector $\vec{\mathbf{b}}$ in the span of $\vec{\mathbf{u}}, \vec{\mathbf{v}}, \vec{\mathbf{w}}$? If so, what weights are use to get to $\vec{\mathbf{b}}$?

did not get to

$$\vec{\mathbf{u}} = \begin{bmatrix} 5 \\ 3 \\ 1 \\ 2 \end{bmatrix}, \vec{\mathbf{v}} = \begin{bmatrix} 1 \\ 4 \\ -1 \\ 3 \end{bmatrix}, \vec{\mathbf{w}} = \begin{bmatrix} -1 \\ -1 \\ 2 \\ 1 \end{bmatrix}. \qquad \vec{\mathbf{b}} = \begin{bmatrix} 57 \\ -6 \\ 40 \\ 4 \end{bmatrix}.$$

   (b) Find a vector $\vec{\mathbf{b}}$ in $\mathbb{R}^4$ that is not in the span of $\vec{\mathbf{u}}, \vec{\mathbf{v}}, \vec{\mathbf{w}}$.