4.5. Rank and Row Space

Rank

Def. The **dimension** of a vector space V is the number of vectors in a basis of V.

Def. The **rank** of a matrix A is the dimension of Col(A) = # pivots(A) = # pivots(A^T)

$$A = \begin{bmatrix} \boxed{\begin{matrix}1\\1\\2\\4\end{matrix}} & \boxed{\begin{matrix}1\\-1\\1\\-1\end{matrix}} & \begin{matrix}-1\\-3\\-3\\-9\end{matrix} & \begin{matrix}2\\4\\5\\13\end{matrix} & \boxed{\begin{matrix}1\\0\\0\\1\end{matrix}} \end{bmatrix} \longrightarrow \begin{bmatrix} \boxed{\begin{matrix}1\\0\\0\\0\end{matrix}} & \boxed{\begin{matrix}0\\1\\0\\0\end{matrix}} & \begin{matrix}-2\\1\\0\\0\end{matrix} & \begin{matrix}3\\-1\\0\\0\end{matrix} & \boxed{\begin{matrix}0\\0\\1\\0\end{matrix}} \end{bmatrix}$$

rank(A) = 3

$$A^T = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 1 & -1 & 1 & -1 \\ -1 & -3 & -3 & -9 \\ 2 & 4 & 5 & 13 \\ 1 & 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & \frac{7}{3} \\ 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank

Def. The **dimension** of a vector space V is the number of vectors in a basis of V.

Def. The **rank** of a matrix A is the dimension of Col(A) = # pivots(A) = # pivots(A^T)

$$A=\begin{pmatrix} 0 & -1 & 8 & 8 & -18 & -1 & -9 & 8 & -4 & 0 & 19 & -4 \\ 7 & -5 & 7 & 1 & -11 & 1 & 4 & -5 & 7 & 8 & 0 & -5 \\ -6 & 4 & -2 & 5 & -1 & -1 & 5 & -5 & -2 & 1 & -25 & 4 \\ 8 & 6 & 2 & -2 & 20 & 7 & -2 & 6 & 2 & 2 & 2 & 0 \\ -3 & -5 & -9 & 3 & -7 & -2 & -4 & -3 & -1 & -3 & 11 & -5 \\ 8 & -4 & -7 & 7 & 0 & 8 & -4 & -9 & -1 & -1 & 12 & -5 \\ 3 & 0 & 2 & 6 & -5 & 8 & 8 & 3 & -5 & -8 & 6 & -6 \\ 1 & -5 & 5 & 8 & -22 & -6 & -2 & 7 & 5 & -4 & 24 & 9 \\ -4 & 4 & 8 & -4 & 0 & 8 & -9 & -3 & -1 & 3 & -9 & 5 \\ 1 & 4 & -3 & -4 & 16 & -2 & -9 & -7 & -3 & -2 & -3 & -1 \\ -1 & -6 & -7 & 9 & -15 & -5 & 7 & 1 & 3 & 4 & 1 & 6 \\ -3 & 8 & -1 & 1 & 13 & 7 & -5 & 8 & 9 & -5 & -1 & -4 \\ -3 & 4 & 2 & -6 & 9 & 0 & 3 & 9 & 6 & 2 & -7 & -6 \\ -6 & -2 & -2 & -9 & 1 & 1 & -2 & 6 & 1 & -1 & 7 & -2 \\ -5 & 3 & -1 & 5 & -3 & -8 & 8 & 8 & -7 & 0 & -11 & 5 \\ 8 & -7 & -9 & -8 & 11 & -1 & 2 & -3 & -2 & -9 & 26 & -5 \\ 3 & 2 & -5 & -1 & 13 & -9 & 9 & -6 & 7 & 9 & -25 & 8 \\ -4 & 5 & 3 & 8 & -5 & 8 & -9 & 0 & 1 & 1 & -6 & 1 \\ 1 & -1 & 4 & 9 & -14 & -8 & 7 & -1 & -1 & -9 & 4 & 2 \\ -6 & -5 & 5 & 0 & -21 & -9 & -7 & -8 & 8 & -4 & 7 & 9 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

A is 12 x 20

rank(A) = 10

A^T is 20 x 12

$$A^T=\begin{pmatrix} 0 & 7 & -6 & 8 & -3 & 8 & 3 & 1 & -4 & 1 & -1 & -3 & -3 & -6 & -5 & 8 & 3 & -4 & 1 & -6 \\ -1 & -5 & 4 & 6 & -5 & -4 & 0 & -5 & 4 & 4 & -6 & 8 & 4 & -2 & 3 & -7 & 2 & 5 & -1 & -5 \\ 8 & 7 & -2 & 2 & -9 & -7 & 2 & 5 & 8 & -3 & -7 & -1 & 2 & -2 & -1 & -9 & -5 & 3 & 4 & 5 \\ 8 & 1 & 5 & -2 & 3 & 7 & 6 & 8 & -4 & -4 & 9 & 1 & -6 & -9 & 5 & -8 & -1 & 8 & 9 & 0 \\ -18 & -11 & -1 & 20 & -7 & 0 & -5 & -22 & 0 & 16 & -15 & 13 & 9 & 1 & -3 & 11 & 13 & -5 & -14 & -21 \\ -1 & 1 & -1 & 7 & -2 & 8 & 8 & -6 & 8 & -2 & -5 & 7 & 0 & 1 & -8 & -1 & -9 & 8 & -8 & -9 \\ -9 & 4 & 5 & -2 & -4 & -4 & 8 & -2 & -9 & -9 & 7 & -5 & 3 & -2 & 8 & 2 & 9 & -9 & 7 & -7 \\ 8 & -5 & -5 & 6 & -3 & -9 & 3 & 7 & -3 & -7 & 1 & 8 & 9 & 6 & 8 & -3 & -6 & 0 & -1 & -8 \\ -4 & 7 & -2 & 2 & -1 & -1 & -5 & 5 & -1 & -3 & 3 & 9 & 6 & 1 & -7 & -2 & 7 & 1 & -1 & 8 \\ 0 & 8 & 1 & 2 & -3 & -1 & -8 & -4 & 3 & -2 & 4 & -5 & 2 & -1 & 0 & -9 & 9 & 1 & -9 & -4 \\ 19 & 0 & -25 & 2 & 11 & 12 & 6 & 24 & -9 & -3 & 1 & -1 & -7 & 7 & -11 & 26 & -25 & -6 & 4 & 7 \\ -4 & -5 & 4 & 0 & -5 & -5 & -6 & 9 & 5 & -1 & 6 & -4 & -6 & -2 & 5 & -5 & 8 & 1 & 2 & 9 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.6 & 0.7 & 0.2 & -0.4 & 1.2 & -0.8 & -0.2 & -0.1 & -0.6 & -0.3 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0.9 & 0 & -0.3 & 0.7 & -0.7 & 0.5 & 0.4 & -0.8 & -0.7 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.2 & -1.5 & -0.6 & 0.9 & -1.3 & 0.2 & 0.8 & 0.4 & 0.5 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -0.6 & -2.1 & -0.1 & 2.4 & -3. & 0.8 & 1.8 & -0.6 & 2.1 & 0.3 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -0.3 & 0.7 & 0.3 & -1.3 & 1.3 & 0 & -1.1 & 0.7 & -0.9 & -0.8 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1.7 & 1.5 & 0.2 & -1.9 & 2.2 & -1.5 & -1 & 0.6 & -2.4 & -1.3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0.3 & 0.2 & 0 & 0.4 & 0.3 & -0.2 & -0.6 & -0.3 & 0.3 & -0.4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -0.2 & -1.2 & -1 & 0 & -0.3 & -0.1 & 0.3 & -0.4 & 0.6 & -0.1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0.9 & 0.2 & 0.3 & 0.8 & 0.4 & -0.8 & 0.5 & -0.3 & -0.4 & 0.4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -0.8 & -0.6 & -0.3 & -0.7 & -2.2 & 0.9 & 0.9 & 0.5 & 0.5 & 0.4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

rank(A^T) = 10

Row Space of A

Row(A) = span of the rows of A (subspace of $\mathbb{R}^5$)

$$A = \begin{bmatrix} 1 & 1 & -1 & 2 & 1 \\ 1 & -1 & -3 & 4 & 0 \\ 2 & 1 & -3 & 5 & 0 \\ 4 & -1 & -9 & 13 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -2 & 3 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank(A) = 3.

The column space of A is a 3-dimensional subspace of $\mathbb{R}^4$ with basis:

$$\mathcal{B}_{\text{col}} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

The row space of A is a 3-dimensional subspace of $\mathbb{R}^5$ with basis:

$$\mathcal{B}_{\text{row}} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Elementary row operations

- swap two rows
- rescale a row
- (row 2) $\longleftrightarrow$ (row 1) + c (row 2)

These operations are linear combinations of the rows, so you stay in the row space when you do row operations

The nonzero rows of RREF(A) span Row(A)

Row operations:

- change the column space
- preserve relations among the columns
- do not change the row space

Col(A) = Row(A^T)

Row(A) = span of the rows of A = Col(A^T).

$$A = \begin{bmatrix} 1 & 1 & -1 & 2 & 1 \\ 1 & -1 & -3 & 4 & 0 \\ 2 & 1 & -3 & 5 & 0 \\ 4 & -1 & -9 & 13 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -2 & 3 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 1 & -1 & 1 & -1 \\ -1 & -3 & -3 & -9 \\ 2 & 4 & 5 & 13 \\ 1 & 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & \frac{7}{3} \\ 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Two bases of Col(A)

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$ consists of a subset of the columns of A

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 7/3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1/3 \end{bmatrix} \right\}$$ Has nice 0-1 property

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Is $\begin{bmatrix} 1 \\ 3 \\ 6 \\ 10 \end{bmatrix}$ in Col(A)?

$$\begin{bmatrix} 1 \\ 3 \\ 6 \\ 10 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 7/3 \end{bmatrix} + 6 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \\ 0 \\ 7 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}$$ Yes!

In the other basis: augment and row reduce.

Rank(A) = Rank(A^T)

Rank(A) = number of pivots in A

= dim(Col(A)).

= dim(Row(A^T))

= number of pivots in A^T

= rank(A^T)

$$A = \begin{bmatrix} 1 & 1 & -1 & 2 & 1 \\ 1 & -1 & -3 & 4 & 0 \\ 2 & 1 & -3 & 5 & 0 \\ 4 & -1 & -9 & 13 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -2 & 3 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad A^T = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 1 & -1 & 1 & -1 \\ -1 & -3 & -3 & -9 \\ 2 & 4 & 5 & 13 \\ 1 & 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & \frac{7}{3} \\ 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Two Bases of A Span of a Set of Vectors

Find a basis of the following subspace of $\mathbb{R}^4$

$$S = \text{span}\left\{ \begin{bmatrix} 4 \\ 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ -7 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ -5 \\ 7 \end{bmatrix} \right\}$$

$$\boxed{\text{rank}(\mathbf{A}) = \dim(S) = \text{rank}(\mathbf{A}^T)}$$

$S = \text{Col}(\mathbf{A})$

$$\begin{bmatrix} \boxed{\begin{matrix} 4 \\ 1 \\ -2 \\ 0 \end{matrix}} & \boxed{\begin{matrix} 1 \\ 1 \\ 0 \\ 0 \end{matrix}} & \begin{matrix} 2 \\ -1 \\ -2 \\ 0 \end{matrix} & \boxed{\begin{matrix} 1 \\ 2 \\ 3 \\ -7 \end{matrix}} & \begin{matrix} 4 \\ 0 \\ -5 \\ 7 \end{matrix} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{B}_1 = \text{span}\left\{ \begin{bmatrix} 4 \\ 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ -7 \end{bmatrix} \right\}$$

Is chosen from the original vectors

$S = \text{Row}(\mathbf{A}^T)$

$$\begin{bmatrix} 4 & 1 & -2 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & -1 & -2 & 0 \\ 1 & 2 & 3 & -7 \\ 4 & 0 & -5 & 7 \end{bmatrix} \to \begin{bmatrix} \boxed{1 \quad 0 \quad 0 \quad -2} \\ \boxed{0 \quad 1 \quad 0 \quad 2} \\ \boxed{0 \quad 0 \quad 1 \quad -3} \\ 0 \quad 0 \quad 0 \quad 0 \\ 0 \quad 0 \quad 0 \quad 0 \end{bmatrix}$$

Has standard basis feature with 0-1s

$$\mathcal{B}_2 = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -3 \end{bmatrix} \right\}$$

The Rank-Nullity Theorem

rank(A) = dim(Col(A)) = # pivots in A

nullity(A) = dim(Nul(A)) = # free variables in A

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 3 & 1 & 2 \\ -2 & 1 & -3 & -1 & -9 \\ -1 & 1 & -1 & 2 & -3 \\ 4 & 1 & 9 & 3 & 13 \\ -2 & 3 & -1 & 2 & -10 \end{bmatrix} \longrightarrow \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & s & x_4 & t \end{array} \\ \begin{bmatrix} 1 & 0 & 2 & 0 & 3 \\ 0 & 1 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = s \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 2 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

rank($\mathbf{A}$) = 3

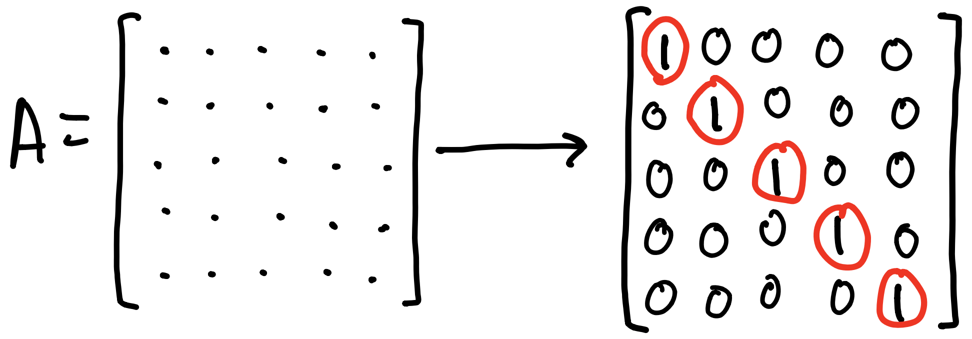
nullity($\mathbf{A}$) = 2

Theorem If A is an m x n matrix, then rank(A) + nullity(A) = n.

Proof. Each of the n columns is either a pivot column (thus, contributing to the rank) or corresponds to a free variable (thus, contributing to the nullity.). ∎

Invertible Matrix Theorem Revisited

If A is an *n*x*n* matrix, then the following statements are equivalent



- A is invertible
- $RREF(A) = I_n$
- A has a pivot in every row
- A has a pivot in every column
- $T(x) = Ax$ is one-to-one
- $T(x) = Ax$ is onto
- The columns of A span $\mathbb{R}^n$
- The columns of A are linearly independent
- $Ax = b$ has exactly one solution for all $b \in \mathbb{R}^n$
- $Ax = 0$ has only the 0 solution
- The columns of A are a basis of $\mathbb{R}^n$
- $Col(A) = \mathbb{R}^n$
- $\dim(Col(A)) = \mathbb{R}^n$
- $rank(A) = n$
- $Nul(A) = \{0\}$
- $nullity(A) = 0$

You Try!

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 8 & 0 & 0 \\ 1 & 1 & 2 & 0 & 2 \\ 0 & 2 & -6 & 1 & 3 \\ 0 & 2 & -6 & 2 & 4 \\ 2 & 0 & 10 & 0 & 2 \\ 2 & 0 & 10 & -2 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 5 & 0 & 1 \\ 0 & 1 & -3 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{A}^T = \begin{bmatrix} 1 & 1 & 0 & 0 & 2 & 2 \\ -1 & 1 & 2 & 2 & 0 & 0 \\ 8 & 2 & -6 & -6 & 10 & 10 \\ 0 & 0 & 1 & 2 & 0 & -2 \\ 0 & 2 & 3 & 4 & 2 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & -1 & 1 & 3 \\ 0 & 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Fill in the blanks:

a) rank($\mathbf{A}$) =

b) nullity($\mathbf{A}$) =

c) rank($\mathbf{A}^T$) =

d) nullity($\mathbf{A}^T$) =

i) Col($\mathbf{A}$) is a ____ dimensional subspace of ______

ii) Nul($\mathbf{A}$) is a ____ dimensional subspace of ______

iii) Row($\mathbf{A}$) is a ____ dimensional subspace of ______

iv) Null($\mathbf{A}^T$) is a ____ dimensional subspace of ______