Review Eigenvectors and Eigenvalues

A number λ is said to be <u>eigenvalue</u> of a matrix A if there exists a nonzero vector $\vec{k}$ that satisfies

$$\underbrace{A\vec{k} = \lambda\vec{k}}$$

$\vec{k}$ is the <u>eigenvector</u> corresponding to eigenvalue λ.

$$(A-\lambda I)\vec{k} = \vec{0}$$

I identity $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

To find eigenvalues:

$$\det(A-\lambda I) = 0$$

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

Example: Find eigenvalues and eigenvectors for

$$A = \begin{bmatrix} 1 & -4 \\ -2 & 3 \end{bmatrix}$$

$$\det\left(\begin{bmatrix} 1 & -4 \\ -2 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} 1-\lambda & -4 \\ -2 & 3-\lambda \end{bmatrix}\right)$$

$$= (1-\lambda)(3-\lambda) - (-4)(-2)$$

$$= \lambda^2 - 4\lambda - 5 = 0$$

characteristic polynomial

$$\lambda^2 - 4\lambda - 5 = 0$$
$$(\lambda - 5)(\lambda + 1) = 0$$

$$\underbrace{\lambda = 5 \quad \lambda = -1}_{\text{eigenvalues}}$$

For $\lambda = -1$:

$$\underbrace{(A - \lambda I)\vec{k} = \vec{0} \Rightarrow \left(\begin{bmatrix} 1 & -4 \\ -2 & 3 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\right)\vec{k} = \vec{0}}$$

$$\begin{bmatrix} 2 & -4 \\ -2 & 4 \end{bmatrix} \vec{k} = \vec{0}$$

$$\left[\begin{array}{cc|c} 2 & -4 & 0 \\ -2 & 4 & 0 \end{array}\right] \xrightarrow{R_1 \to \frac{1}{2}R_1} \left[\begin{array}{cc|c} 1 & -2 & 0 \\ -2 & 4 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 + 2R_1} \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

free variable
$k_2 = C$

$$\vec{k} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 2k_2 \\ k_2 \end{bmatrix} = \begin{bmatrix} 2c \\ c \end{bmatrix} = c \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$k_1 - 2k_2 = 0$
$k_1 = 2k_2$

$\vec{k}$ can be any nonzero multiple of $\langle 2, 1 \rangle$.

For $\lambda = 5$, $\vec{k} = \langle -1, 1 \rangle$ is an eigenvector
(any constant multiple of $\langle -1, 1 \rangle$ works)

Exponential substitution for systems of linear equations

Suppose a system of equations is

written in matrix form $\vec{x}' = A\vec{x}$

$$\underbrace{\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix}}_{\vec{x}'} = \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{=A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\vec{x}}$$

Assume a solution is exponential

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} k_1 e^{\lambda t} \\ k_2 e^{\lambda t} \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} e^{\lambda t}.$$

If this is the case, then

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} \lambda k_1 e^{\lambda t} \\ \lambda k_2 e^{\lambda t} \end{bmatrix} = \lambda \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} e^{\lambda t}$$

Then since this should solve the system

$$\underbrace{\lambda \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} e^{\lambda t}}_{\vec{x}'} = \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} e^{\lambda t}}_{\vec{x}}$$

So

$$\lambda \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

$$\Rightarrow \lambda \vec{k} = A\vec{k} \qquad \Rightarrow A\vec{k} = \lambda \vec{k}$$

λ is an eigenvalue of A, and $\vec{k}$ is corresponding eigenvector!

Distinct Real Eigenvalues:

Theorem: Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be n distinct real eigenvalues of a coefficient matrix A for the system

$$\vec{x}\,' = A\vec{x}.$$

Let $\vec{k_1}, \vec{k_2}, \ldots, \vec{k_n}$ be the corresponding eigenvectors. Then a solution to the system is

$$\vec{x} = c_1\vec{k_1}e^{\lambda_1 t} + c_2\vec{k_2}e^{\lambda_2 t} + \ldots + c_n\vec{k_n}e^{\lambda_n t}.$$

Example:

$$\frac{dx}{dt} = x - 4y \qquad \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$
$$\frac{dy}{dt} = -2x + 3y$$

Eigenvalues $\lambda_1 = -1 \qquad \lambda_2 = 5$

$$\vec{k_1} = \begin{bmatrix} 2 \\ \end{bmatrix} \qquad \vec{k_2} = \begin{bmatrix} -1 \\ \end{bmatrix}$$

$[\,1\,] \qquad [\,1\,]$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{5t}.$$

$$x(t) = c_1 2e^{-t} - c_2 e^{5t}$$
$$y(t) = c_1 e^{-t} + c_2 e^{5t}.$$