Review of Coordinates.

1. Here are two bases of $\mathbb{R}^3$:

$$\mathcal{B} = \left\{ \vec{v}_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}, \vec{v}_2 = \begin{bmatrix}0\\1\\1\end{bmatrix}, \vec{v}_3 = \begin{bmatrix}1\\1\\1\end{bmatrix} \right\}, \qquad \mathcal{S} = \left\{ \mathbf{e}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}, \mathbf{e}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix} \right\},$$

Fill in the blanks.

$$\begin{bmatrix}5\\-3\\4\end{bmatrix}_{\mathcal{B}} = \begin{bmatrix}9\\6\\1\end{bmatrix}_{\mathcal{S}} \qquad 5\begin{bmatrix}1\\1\\0\end{bmatrix} - 3\begin{bmatrix}0\\1\\1\end{bmatrix} + 4\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}9\\6\\1\end{bmatrix}$$

$$\begin{bmatrix}6\\4\\3\end{bmatrix}_{\mathcal{S}} = \begin{bmatrix}1\\-2\\5\end{bmatrix}_{\mathcal{B}}$$

(i) $$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 6 \\ 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & 3 \end{array}\right] \xrightarrow[R]{\text{used}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

B ← change of basis matrix

(ii) or find B^{-1} and $B^{-1}\begin{bmatrix}6\\4\\3\end{bmatrix} = \begin{bmatrix}1\\-2\\5\end{bmatrix}$

$$\mathbf{e}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}_{\mathcal{S}} = \begin{bmatrix}1\\1\\-1\end{bmatrix}_{\mathcal{B}}$$

by definition

By inspection: $\begin{bmatrix}0\\1\\0\end{bmatrix} = \underline{1}\begin{bmatrix}1\\1\\0\end{bmatrix} + \underline{1}\begin{bmatrix}0\\1\\1\end{bmatrix} + \underline{(-1)}\begin{bmatrix}1\\1\\1\end{bmatrix}$

or augment and row reduce

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & & & 1 \\ & 1 & & 1 \\ & & 1 & -1 \end{array}\right]$$

$$\vec{v}_2 = \begin{bmatrix}0\\1\\1\end{bmatrix}_{\mathcal{S}} = \begin{bmatrix}0\\1\\0\end{bmatrix}_{\mathcal{B}}$$

$v_2 = 0v_1 + 1\cdot v_2 + 0\cdot v_3$

what we are given

Dimension. The *dimension* of a vector space V (including subspaces, which are vector spaces), denoted $\dim(V)$ is the number of vectors in a basis of V.

Key points:

- a vector space has infinitely many bases
- they all have the same # of vectors ← That's the dimension

Examples:

1. $\dim(\mathbb{R}) =$ 1
2. $\dim(\mathbb{R}^2) =$ 2
3. $\dim(\mathbb{R}^3) =$ 3
4. $\dim(\mathbb{R}^n) =$ n
5. $\dim(\{\mathbf{0}\}) =$ 0

6. If L is a line through the origin in $\mathbb{R}^3$ and P is a plane through the origin in $\mathbb{R}^3$ then

   $\dim(L) =$ 1 $\qquad\qquad \dim(P) =$ 2

7. Find the dimension of $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5, \vec{v}_6, \vec{v}_7\} \subseteq \mathbb{R}^5$.
   To do so, we make $S = \mathsf{Col}(\mathsf{A})$ and row reduce A.

$$\mathsf{A} = \begin{bmatrix} | & | & | & | & | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 & \vec{v}_5 & \vec{v}_6 & \vec{v}_7 \\ | & | & | & | & | & | & | \end{bmatrix} \longrightarrow \begin{array}{c} \begin{array}{ccccccc} & & r & & s & & t \end{array} \\ \begin{bmatrix} 1 & 0 & 2 & 0 & 1 & 0 & -3 \\ 0 & 1 & -3 & 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$\dim(S) = \dim(C(A)) = \text{rank}(A) = 4 = \#$ pivots

8. Also find a basis of the null space of the matrix above:

$$\begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 0 \\ -2 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

$\dim(\text{Nul}(A)) = \text{nullity}(A) = 3$

9. The dimension of the column space $\mathsf{Col}(\mathsf{A})$ is called the *rank* of the matrix A and the dimension of the null space $\mathsf{Nul}(\mathsf{A})$ is called the *nullity* of the matrix A. Find the rank and the nullity of the following matrix. Give bases of $\mathsf{Col}(\mathsf{A})$ and $\mathsf{Nul}(\mathsf{A})$.

rank=3

$$\begin{bmatrix} 1 & 1 & 3 & 1 & 2 \\ -2 & 1 & -3 & -1 & -9 \\ -1 & 1 & -1 & 2 & -3 \\ 4 & 1 & 9 & 3 & 13 \\ -2 & 3 & -1 & 2 & -10 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 0 & 3 \\ 0 & 1 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

nullity=2

rank=3

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 2 \\ -1 & 1 & 3 \\ 2 & 1 & 1 \\ 4 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

nullity=0

For a general matrix A:

$\mathsf{rank}(\mathsf{A}) =$ # pivots in A

$\mathsf{nullity}(\mathsf{A}) =$ # free vars in A

Rank-Nullity Theorem: If A is an $m \times n$ matrix then $\mathsf{rank}(A) + \mathsf{nullity}(A) =$ n (# of columns)

Invertible Matrices: An $n \times n$ matrix A is invertible if and only if $\boxed{\mathsf{rank}(\mathsf{A}) = n}$ if and only if $\boxed{\mathsf{nullity}(\mathsf{A}) = 0}$

Discussion

1. (a) Find the rank and nullity of the following matrices

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 1 & 1 \\ -1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 & 1 & -2 & 3 & 0 \\ -2 & 1 & -1 & -2 & -2 & -5 \\ -1 & 1 & 2 & -9 & 2 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 2 & 1 & 1 \\ 0 & 1 & 0 & -1 & 1 & -2 \\ 0 & 0 & 1 & -3 & 1 & 1 \end{bmatrix}$$

$r=4,\ n=0$ full row rank + full column rank

Full row rank

$r=3,\ n=3$

$$C = \begin{bmatrix} 1 & 3 & 5 & -2 & 4 \\ 1 & 2 & 4 & -1 & 3 \\ 2 & 1 & 5 & 1 & 3 \\ 2 & 0 & 4 & 2 & 2 \\ -1 & 1 & -1 & -2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 1 & 1 \\ 0 & 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 1 & 1 \\ -2 & 1 & -1 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \\ -2 & 3 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Full column rank

$r=2 \quad n=3$

$r=3,\ n=0$

(b) An $m \times n$ matrix has *full row rank* if $\mathsf{rank}(A) = m$ and it has *full column rank* if $\mathsf{rank}(A) = n$. Do any of the matrices above have full row rank or full column rank?

(c) What do you know if a matrix has full row rank?

① the vectors in the columns of A span $\mathbb{R}^m$

② T_A is onto

③ $Ax=b$ is consistent for all b

(d) What do you know if a matrix has full column rank?

① the vectors in the columns of A are linearly independent

② T_A is one-to-one

③ $Ax=b$ has a unique solution if it is consistent

2. Here is a basis $\mathcal{B}$ of $\mathbb{R}^4$ and two vectors $\vec{\mathrm{u}}$ and $\vec{\mathrm{w}}$ in $\mathbb{R}^4$.

$$\mathcal{B} = \left\{ \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix} \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_4 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\} \qquad \vec{w} = \begin{bmatrix} 2 \\ -1 \\ 3 \\ 2 \end{bmatrix}_{\mathcal{B}}, \qquad \vec{u} = \begin{bmatrix} 3 \\ 1 \\ 1 \\ 2 \end{bmatrix}_{\mathcal{S}}.$$

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A = cbind(c(1,2,-1,0),c(1,1,1,1),c(1,1,0,0),c(0,1,1,1))
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(a) The vector $\vec{w}$ is given in $\mathcal{B}$. Express $\vec{w}$ in standard coordinates.

(b) The vector $\vec{u}$ is given in standard $\mathcal{S}$. Express $\vec{u}$ in $\mathcal{B}$ coordinates.

$$B = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 1 & 1 \\ -1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$ change of basis matrix

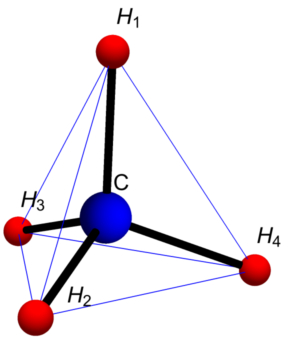
ⓐ multiply $B \begin{bmatrix} 2 \\ -1 \\ 3 \\ 2 \end{bmatrix} = \cdots$

ⓑ $\boxed{i}$ augment and row reduce $\left[B \,\middle|\, \begin{matrix} 3 \\ 1 \\ 1 \\ 2 \end{matrix} \right] \longrightarrow \cdots$

or $\boxed{ii}$ find B^{-1} and $B^{-1} \begin{bmatrix} 3 \\ 1 \\ 1 \\ 2 \end{bmatrix} = \cdots$

3. The following problem is on the next problem set.

In practice, we change bases because problems are computationally easier in another coordinate system or because we learn something by looking at a problem from a different point of view. The following example illustrates this with ideas that arises both in chemistry and computer graphics. Below is the tetrahedral molecule methane, CH_4 along with the coordinates of its atoms



$$\mathsf{C} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \mathsf{H}_1 = \begin{bmatrix} 0 \\ 0 \\ \frac{3}{2\sqrt{6}} \end{bmatrix}, \mathsf{H}_2 = \begin{bmatrix} -\frac{1}{2\sqrt{3}} \\ -\frac{1}{2} \\ -\frac{1}{2\sqrt{6}} \end{bmatrix}, \mathsf{H}_3 = \begin{bmatrix} -\frac{1}{2\sqrt{3}} \\ \frac{1}{2} \\ -\frac{1}{2\sqrt{6}} \end{bmatrix}, \mathsf{H}_4 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ 0 \\ -\frac{1}{2\sqrt{6}} \end{bmatrix}$$

since $H_1 = 1 \cdot H_1 + 0H_2 + 0H_3$

(a) Find a dependence relation among the vectors $\mathsf{H}_1, \mathsf{H}_2, \mathsf{H}_3, \mathsf{H}_4$. Hint: add them together (you can do it "by hand" by just looking at the sum).

(b) We can see visually that $\mathcal{M} = \{\mathsf{H}_1, \mathsf{H}_2, \mathsf{H}_3\}$ is a basis of $\mathbb{R}^3$, which we will call the *tetrahedral basis*. You can see from the plot that these vectors are linearly independent (not all on the same plane). Give the coordinates of each of the vectors $\mathsf{H}_1, \mathsf{H}_2, \mathsf{H}_3, \mathsf{H}_4$ in the $\mathcal{M}$ basis (for H_4 you will need to use part a).

$H_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}_{\mathcal{M}}$

(c) In chemistry and physics, we are interested in symmetry operations. These are linear transformations such that the atom looks the same after the transformation as it did before. For example one such operation is rotation r by 120^o around the H_4 axis. This rotation sends H_1 to H_3, H_3 to H_2, and H_2 to H_1. Give the matrix of r in the $\mathcal{M}$ basis. The columns should be the result of applying the symmetry operation to each of the basis vectors and then expressing the answer in the $\mathcal{M}$ basis.

$H_2 = \begin{bmatrix} \; \\ \; \\ \; \end{bmatrix}_{\mathcal{M}}$

$H_3 = \begin{bmatrix} \; \\ \; \\ \; \end{bmatrix}_{\mathcal{M}}$ $\quad H_4 = \begin{bmatrix} \; \\ \; \\ \; \end{bmatrix}_{\mathcal{M}}$ ← The most interesting one

$$r = \begin{array}{c} \\ \mathsf{H}_1 \\ \mathsf{H}_2 \\ \mathsf{H}_3 \end{array} \begin{array}{c} \begin{array}{ccc} \mathsf{H}_1 & \mathsf{H}_2 & \mathsf{H}_3 \end{array} \\ \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}_{\mathcal{M}} \end{array}$$

- compute $r(H_1)$ in $\mathcal{M}$-coords and put down first column
- same with second and third

(d) Show, by multiplying by hand, that your matrix r sends H_1 to H_3, H_3 to H_2, H_2 to H_1, *and* H_4 to H_4.

(e) Another symmetry operation is a rotation s by 180^o around the axis that passes through the midpoint between H_1 and H_2 and the midpoint between H_3 and H_4. This rotation exchanges H_1 and H_2 and exchanges H_3 and H_4. Find the matrix of s in the $\mathcal{M}$ basis. By hand, apply it to each of the four hydrogen atoms and show that they go to the right place.

(f) Now we will convert our matrix for r to standard coordinates. You will do this in R Studio. The vectors $\mathsf{H}_1, \mathsf{H}_2, \mathsf{H}_3, \mathsf{H}_4$ are given to you under the PS6 link on the handbook. Here is the recipe.

i. Enter the change of basis matrix T that converts from the tetrahedral basis $\mathcal{M}$ to the standard basis $\mathcal{S}$ and compute its inverse that converts from the standard basis back to $\mathcal{M}$. .

ii. Enter the matrix of the rotation r from part c above.

iii. Compute the matrix of $[r]_{\mathcal{S}}$ in the standard basis by computing the matrix product below. Notice that, working from right to left, it first converts from standard coordinates to $\mathcal{M}$ coordinates. Then it does the rotation in $\mathcal{M}$ coordinates. Then it converts the answer back to standard coordinates.

$$[r_4]_{\mathcal{S}} = \underbrace{\begin{array}{c} \\ \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{array} \begin{array}{c} \begin{array}{ccc} \mathsf{H}_1 & \mathsf{H}_2 & \mathsf{H}_3 \end{array} \\ \begin{pmatrix} \text{convert the} \\ \text{answer} \\ \text{back to} \\ \text{standard} \\ \text{coordinates} \end{pmatrix} \end{array}}_{\mathcal{M}\to\mathcal{S}} \begin{array}{c} \\ \mathsf{H}_1 \\ \mathsf{H}_2 \\ \mathsf{H}_3 \end{array} \begin{array}{c} \begin{array}{ccc} \mathsf{H}_1 & \mathsf{H}_2 & \mathsf{H}_3 \end{array} \\ \begin{pmatrix} \text{perform the} \\ \text{rotation in} \\ \mathcal{M}\text{-coordinates} \end{pmatrix} \end{array} \underbrace{\begin{array}{c} \\ \mathsf{H}_1 \\ \mathsf{H}_2 \\ \mathsf{H}_3 \end{array} \begin{array}{c} \begin{array}{ccc} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{array} \\ \begin{pmatrix} \text{convert from} \\ \text{standard} \\ \text{coordinates} \\ \mathcal{M}\text{ coordinates} \end{pmatrix} \end{array}}_{\mathcal{S}\to\mathcal{M}}.$$

← will discuss on Friday!