# Section 4.3: Operational Properties of the Laplace Transform.

**First Translation Theorem:** If $\mathcal{L}\{f(t)\} = F(s)$ and $a$ is any real number, then

$$\mathcal{L}\{e^{at} f(t)\} = F(\underbrace{s-a}_{\text{shift in } s \text{ by } a \text{ units}})$$

<u>Proof:</u>

$$\mathcal{L}\{e^{at} f(t)\} = \int_0^\infty e^{-st} \cdot e^{at} f(t)\, dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t)\, dt$$

$$\stackrel{*}{=} F(s-a) \qquad s-a>0$$

We can use

$$\mathcal{L}\{e^{at} f(t)\} = \mathcal{L}\{f(t)\}\underbrace{{}_{s\to s-a}}_{s \text{ gets replaced by } s-a.}$$

Example: Evaluate $\mathcal{L}\{e^{6t}(t^2+\sin(4t))\}$.

$$\mathcal{L}\{e^{\overset{a=6}{6t}}(t^2+\sin(4t))\} = \mathcal{L}\{e^{6t}t^2\} + \mathcal{L}\{e^{6t}\sin(4t)\}.$$

$$= \mathcal{L}\{t^2\}\Big|_{s\to s-6} + \mathcal{L}\{\sin(4t)\}\Big|_{s\to s-6}$$

$$= \frac{2}{s^3}\Big|_{s\to s-6} + \frac{4}{s^2+16}\Big|_{s\to s-6}$$

$$= \frac{2}{(s-6)^3} + \frac{4}{(s-6)^2+16}$$

$$\mathcal{L}^{-1}\{F(s-a)\} = \mathcal{L}^{-1}\{F(s)\,|_{s\to s-a}\}$$
$$= e^{at}f(t) \quad \text{where } \mathcal{L}\{f(t)\} = F(s)$$

Example: $\mathcal{L}^{-1}\left\{\dfrac{2s+4}{s^2+6s+11}\right\}$.

complete square

$$\frac{2s+4}{s^2+6s+11} = \frac{2s+4}{s^2+6s+9+2} = \frac{2s+4}{\underbrace{(s+3)}_{s-(-3)}{}^2+2}$$

$$= \frac{2(s+3)-2}{(s+3)^2+2} = \frac{2(s+3)}{(s+3)^2+2} - \frac{2}{(s+3)^2+2}.$$

$$\mathcal{L}^{-1}\left\{\frac{2s+4}{s^2+6s+11}\right\} = 2\mathcal{L}^{-1}\left\{\frac{s+3}{(s+3)^2+2}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{(s+3)^2+2}\right\}$$

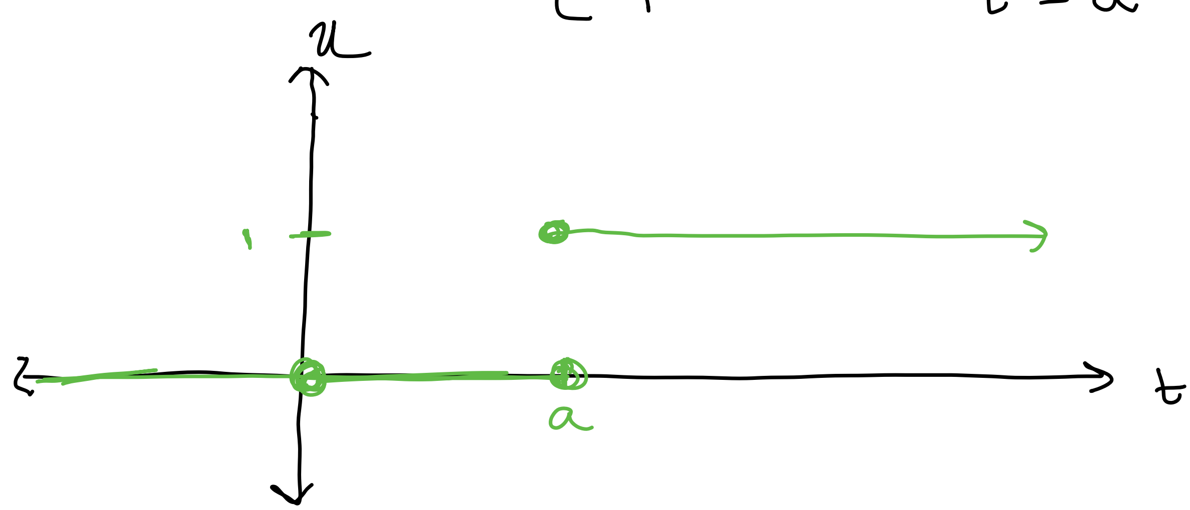
$$\mathcal{L}^{-1}\{\;\} = 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+2}\Big|_{s\to s+3}\right\} - \frac{2}{\sqrt{2}}\mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{s^2+2}\Big|_{s\to s+3}\right\}$$

$$= 2e^{-3t}\cos(\sqrt{2}t) - \frac{2}{\sqrt{2}}e^{-3t}\sin(\sqrt{2}t)$$

Translation in $t$-axis.

The <u>unit step function</u> (or Heaviside Function) $\mathcal{U}(t-a)$ is defined by

$$\mathcal{U}(t-a) = \begin{cases} 0 & 0 \le t < a \\ 1 & t \ge a \end{cases}$$




Let
$$f(t) = \begin{cases} g(t) & 0 \le t < a \\ h(t) & t \ge a \end{cases}$$

Then

$$* \quad f(t) = g(t) - g(t)\mathcal{U}(t-a) + h(t)\mathcal{U}(t-a)$$

When $0 \leq t < a$

$$g(t) - \cancel{g(t) \cdot 0} + \cancel{h(t) \cdot 0} = g(t) \checkmark$$

$t \geq a$

$$\cancel{g(t)} - \cancel{g(t) \cdot 1} + h(t) \cdot 1 = h(t) \checkmark$$

$$f(t) = \begin{cases} 0 & 0 \leq t < a \\ g(t) & a \leq t < b \\ 0 & t \geq b \end{cases}$$

$$\begin{aligned} f(t) &= g(t)\mathcal{U}(t-a) - g(t)\mathcal{U}(t-b) \\ &= g(t)\left(\mathcal{U}(t-a) - \mathcal{U}(t-b)\right). \end{aligned}$$

Second Translation Theorem: If

$$F(s) = \mathcal{L}\{f(t)\},$$

and $a > 0$, then

$$\mathcal{L}\{\underline{f(t-a)}\,\underline{\mathcal{U}(t-a)}\} = e^{-as}F(s)$$

Proof: By definition

$$\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = \int^{\infty} e^{-st} f(t-a)\mathcal{U}(t-a)\,dt$$

$$= \int_0^a e^{-st} f(t-a)\underbrace{\mathcal{U}(t-a)}_{\equiv 0}\,dt + \int_a^\infty e^{-st} f(t-a)\underbrace{\mathcal{U}(t-a)}_{\equiv 1}\,dt.$$

$$= \int_a^\infty e^{-st} f(t-a)\,dt$$

By change of variables, $v = t-a$

$$\int_{t=a}^{t=\infty} e^{-st} f(t-a)\,dt = \int_{v=0}^{v=\infty} e^{-s(v+a)} f(v)\,dv$$

$$= e^{-sa} \underbrace{\int_0^\infty e^{-sv} f(v)\,dv}$$

$$= e^{-sa}\,\mathcal{L}\{f\}$$

Corollary: $\mathcal{L}\{\mathcal{U}(t-a)\} = \dfrac{e^{-as}}{s}$

(take $f \equiv 1$)

Example

$$\mathcal{L}\{\underbrace{2\,\mathcal{U}(t-4)}_{\text{-4s} \quad 2} + \underbrace{e^{2(t-2)}\,\mathcal{U}(t-2)}\}$$

shift by 2      shift by 2

$$= 2\frac{e}{s} + \underline{e^{-2s}} \cdot \frac{1}{s-2}$$

$$\mathcal{L}\{e^{5(t-2)}\,\mathcal{U}(t-2)\} = e^{-2s}\,\mathcal{L}\{e^{5t}\}$$
$$= e^{-2s}\,\frac{1}{s-5}$$

$$\mathcal{L}\{e^{5(t+2)}\,\mathcal{U}(t+2)\} = e^{-(-2)s}\,\mathcal{L}\{e^{5t}\}$$

Inverse Laplace

If $f(t) = \mathcal{L}^{-1}\{F(s)\}$ and $a > 0$ then

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\,\mathcal{U}(t-a)$$

Example: $\mathcal{L}^{-1}\left\{\frac{4}{s^2+16}\,e^{-\pi s}\right\}$

$$= \sin(4(t-\pi))\,\mathcal{U}(t-\pi)$$

Alternate Second Translation Theorem:

$$\mathcal{L}\{\underline{g(t)}\,\underline{\mathcal{U}(t-a)}\} = e^{-as}\,\mathcal{L}\{g(t+a)\}$$

shifts don't match

g(t)

Example: $\mathcal{L}\{t^2\,\mathcal{U}(t-2)\}$

$$\mathcal{L}\{t^2\,\mathcal{U}(t-2)\} = e^{-2s}\,\mathcal{L}\{\overset{g(t+2)}{(t+2)^2}\}$$

$$= e^{-2s}\,\mathcal{L}\{\underbrace{t^2+4t+4}\}$$

$$= e^{-2s}\left(\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right).$$

$$\mathcal{L}\{\overset{g(t)}{\cos(2\pi t)}\,\mathcal{U}(\underline{t-1})\} = e^{-s}\,\mathcal{L}\{\cos(2\pi(t+1))\}$$

$$= e^{-s}\,\mathcal{L}\{\cos(2\pi t + 2\pi)\}$$

$$= e^{-s}\,\mathcal{L}\{\cos(2\pi t)\}$$

$$= e^{-s}\,\frac{s}{s^2+(2\pi)^2}.$$