Daily vocabulary: parametric form of solution to $\mathsf{A}x = \vec{\mathbf{b}}$. Comparing the solutions to $\mathsf{A}x = \vec{\mathbf{0}}$, $\mathsf{A}x = \vec{\mathbf{b}}$, $\mathsf{A}x = \vec{\mathbf{c}}$.

Warm Up

1. Fill in the missing entry

$$\begin{bmatrix} 2 & -2 & -3 & 2 & -3 \\ 2 & -3 & 2 & 1 & -3 \\ -1 & 2 & 2 & -1 & 1 \\ -3 & -2 & -2 & 1 & -3 \\ 3 & 2 & -1 & 3 & 1 \\ -2 & 0 & 2 & -3 & -3 \\ -1 & -3 & 2 & -3 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ -1 \\ \\ 0 \\ 3 \\ -5 \\ -6 \end{bmatrix}$$

2. "Compute" the product:

$$\begin{bmatrix} | & | & | & | \\ v_1 & v_2 & v_3 & v_4 \\ | & | & | & | \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 5 \\ 2 \end{bmatrix} =$$

3. If $A\vec{\mathbf{v}} = \vec{\mathbf{b}}$ and $A\vec{\mathbf{u}} = \vec{\mathbf{d}}$ then what are the following matrix-vector products?

$A(2\vec{\mathbf{v}}) =$ $\qquad$ $A(\vec{\mathbf{v}}+\vec{\mathbf{w}}) =$ $\qquad$ $A(5\vec{\mathbf{v}}+3\vec{\mathbf{w}}) =$

4. A system equations of the form $Ax = \vec{\mathbf{0}}$ is a *homogeneous* system of equation (when the right-hand side is 0). Otherwise, it is called a nonhomogeneous system of equations. Here are two homogeneous systems of equations and their row reductions. *Describe the solution to each of them.*

(a) $\begin{bmatrix} 1 & 2 & 1 \\ 4 & 5 & 1 \\ 7 & 8 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 4 & 5 & 1 & 0 \\ 7 & 8 & -1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & -6 & -8 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 3 & 4 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

(b) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -3 & -6 & 0 \\ 0 & -6 & -12 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

(c) Discuss: What can you say, in general, about homogeneous systems of equations? Eg., how many solutions can they have? What can you say about the corresponding vector equation?

Parametric Solutions

5. Here is a matrix problem $\mathsf{A}x = \vec{\mathbf{b}}$. Write out the corresponding system of equations and the corresponding vector problem.

$$\begin{bmatrix} 1 & 1 & -1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 12 \\ 2 \\ 15 \\ 3 \end{bmatrix}$$

(a) All three problems are solved by "augmenting and row reducing". Use the row reduction to write down the parametric solution to $\mathsf{A}x = \vec{\mathbf{b}}$.

$$\left[\begin{array}{ccccc|c} 1 & 1 & -1 & 1 & 1 & 12 \\ 1 & 0 & 1 & 1 & 0 & 2 \\ 1 & 1 & -1 & 0 & 2 & 15 \\ 0 & 0 & 0 & -1 & 1 & 3 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 1 & 5 \\ 0 & 1 & -2 & 0 & 1 & 10 \\ 0 & 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(b) Draw a picture of the geometry of this solution: It is a subset of $\mathbb{R}^5$. Is it the empty set, a point, a line, a plane, a circle, a 3-dimensional space, a sphere, or something else?

(c) Write down the parametric solution to $\mathsf{A}x = \vec{\mathbf{0}}$ (the corresponding homogenous equations).

(d) Discuss the relation between your answers to part (a) and part (c).

6. True-False. Continued with the same problem.

$$\mathsf{A} = \begin{bmatrix} 1 & 1 & -1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}, \quad \vec{\mathbf{b}} = \begin{bmatrix} 12 \\ 2 \\ 15 \\ 3 \end{bmatrix}, \qquad \left[\begin{array}{ccccc|c} 1 & 1 & -1 & 1 & 1 & 12 \\ 1 & 0 & 1 & 1 & 0 & 2 \\ 1 & 1 & -1 & 0 & 2 & 15 \\ 0 & 0 & 0 & -1 & 1 & 3 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 1 & 5 \\ 0 & 1 & -2 & 0 & 1 & 10 \\ 0 & 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Here are some true and false questions to discuss. See if you can also agree on *why* they are T or F.

(a) T F $\mathsf{A}x = \vec{\mathbf{d}}$ has a solution for all right-hand sides $\vec{\mathbf{d}} \in \mathbb{R}^4$

(b) T F If $\mathsf{A}x = \vec{\mathbf{d}}$ has a solution, then it has infinitely many solutions.

(c) T F If $\vec{\mathbf{p}}$ and $\vec{\mathbf{q}}$ are both solutions to $\mathsf{A}x = \vec{\mathbf{b}}$, then $\vec{\mathbf{p}} - \vec{\mathbf{q}}$ is a solution to $\mathsf{A}x = \vec{\mathbf{0}}$.

(d) T F If $\vec{\mathbf{p}}$ and $\vec{\mathbf{q}}$ are both solutions to $\mathsf{A}x = \vec{\mathbf{b}}$, then $\frac{1}{2}\vec{\mathbf{p}} + \frac{1}{2}\vec{\mathbf{q}}$ is a solution to $\mathsf{A}x = \vec{\mathbf{b}}$.

7. Consider the matrix equation $\mathsf{A}x = \vec{\mathbf{b}}$:

$$A = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 0 & 3 & -1 \\ -1 & 1 & -5 & 1 \\ 1 & 2 & -1 & 1 \end{bmatrix}, \qquad \text{and} \qquad \vec{\mathbf{b}} = \begin{bmatrix} 2 \\ 4 \\ -6 \\ 4 \end{bmatrix},$$

Create an R markdown file R to do the row-reduction. Here are R commands for defining A and $\vec{\mathbf{b}}$ and the augmented matrix:

```
A = cbind(c(1,1,-1,1),c(1,0,1,2),c(1,3,-5,-1),c(-1,-1,1,1))
b = c(2,4,-6,4)
Ab = cbind(A,b)
A
Ab
```

Remember that you need to include pracma to get rref:

```
require('pracma')
```

(a) Row reduce the matrix using rref. Write out the solution to $\mathsf{A}x = \vec{\mathbf{b}}$ in parametric form.

(b) Find *two* specific, and different, solutions to $Ax = b$. Call them x_1 and x_2. Name them x1 and x2 in R. Show that they are solutions to $\mathsf{A}x = \vec{\mathbf{b}}$ by multiplying by A. The matrix-vector product in R is funny. You use the following syntax to do it.

```
A %*% x1
```

(c) Show, with a matrix-vector product, that $x_h = x_1 - x_2$ is a solution to the homogeneous equation $\mathsf{A}x = \vec{\mathbf{0}}$.

(d) Are any of these vectors solutions to $\mathsf{A}x = \vec{\mathbf{b}}$? Confirm with a matrix vector product:

i. $x_1 + x_2$

ii. $2x_1$

iii. $\frac{1}{2}x_1 + \frac{1}{2}x_2$

iv. $x_1 + 2021x_h$