Section 3.2: Exponential Substitution.

Motivation: $\underline{y' = 3y}$

$y' - 3y = 0$

$(D-3)y = 0$

$\boxed{y(x) = Ae^{3x}}$

$\int \frac{1}{3y}\,dy = \int 1\,dx$

$\int \frac{1}{y}\,dy = 3\int 1\,dx$

$\ln|y| = 3x + C$

$y = e^{3x+C}$

$= Ae^{3x}$

Solve higher order linear differential equations of the form

$$Ly = 0$$ homogeneous.

L is a polynomial in D and the coefficients are constants

Example: $y^{(3)} - 3y'' + 6y' + 18y = 0$

$$(D^3 - 3D^2 + 6D + 18)y = 0$$

Restrict to setting where coefficients are constants

Exponential Substitution:

Assume that the solution to an equation is exponential $y = e^{\lambda x}$. Then we substitute assume solution into equation and solve for λ

Example: Consider $y'' - 2y' - 3y = 0$.

Assume $y = e^{\lambda x}$ is a solution.

$$y' = \lambda e^{\lambda x}$$

$$y'' = \lambda^2 e^{\lambda x}$$

$$\lambda^2 e^{\lambda x} - 2\lambda e^{\lambda x} - 3e^{\lambda x} = 0$$

$$e^{\lambda x}(\lambda^2 - 2\lambda - 3) = 0$$

$$\Rightarrow \lambda^2 - 2\lambda - 3 = 0$$

$$(\lambda - 3)(\lambda + 1) = 0$$

$$\lambda = 3 \qquad \lambda = -1$$

So $y=e^{3x}$ and $y=e^{-x}$ are solutions to the equation.

$$y''-2y'-3y=0$$

$$\underbrace{(D^2-2D-3)y=0}$$

$$(D-3)(D+1)=0$$

$$\underbrace{y=e^{3x}},\ \underbrace{y=e^{-x}}$$

Roots $D=3$ $D=-1$

We've found that $\{e^{3x}, e^{-x}\}$ is a solution set for $y''-2y'-3y=0$

$$L(y)=0$$

Since $L=D^2-2D-3$ is linear,

$$y=c_1e^{3x}+c_2e^{-x}$$

is also solution for any $c_1, c_2 \in \mathbb{R}$.

$$L(c_1e^{3x}+c_2e^{-x})=c_1L(e^{3x})+c_2L(e^{-x})$$

$$=c_1\cdot 0+c_2\cdot 0$$

$$=0$$

Nonsolution

$$\underbrace{L(c_1 e^{3x} + c_2 e^{-x} + 5)}$$

$$= c_1 L(e^{3x}) + c_2 L(e^{-x}) + L(5)$$

$$= c_1 \cdot 0 + c_2 \cdot 0 + 0 - 2(0) - 3(5)$$

$$= -15$$

$$y'' - 2y' - 3y = -15$$

From linear algebra we get that the solution for $y'' - 2y' - 3y = 0$ is

$$\text{span}\{e^{3x}, e^{-x}\}$$

Theorem: Let L be a linear differential operator of order n. (Polynomial in D of degree n with constant coefficients). Then the solution to the equation $Ly = 0$ form an n-dimensional vector space. In particular if the polynomial in D has

has n real distinct roots $\lambda_1, \lambda_2, \dots, \lambda_n$ then a basis for the solution space

$$\{e^{\lambda_1 x}, e^{\lambda_2 x}, \dots, e^{\lambda_n x}\}$$

The collection of solutions of $Ly = 0$ is the nullspace of the polynomial in D

$$[p(D)]y = \underset{\text{nullspace}}{0}$$

1. $y'' + 2y' - 8y = 0$

$(D+4)(D-2)$

$y = c_1 e^{-4x} + c_2 e^{2x}$

2. $y^{(3)} + y'' - 14y' - 24y = 0$

$$\frac{\pm \text{ some factor of } 24}{\pm \text{ some factor of } 1}$$

integer

some integer.

$\pm\ 24, 12, 6, 4, 3, 2, 1$

$\lambda = -2 \rightarrow \lambda + 2$

$\lambda^3 + \lambda^2 - 14\lambda - 24$

$\lambda^2 - \lambda - 12$

$$\lambda+2 \,\overline{\big)\, \lambda^3+\lambda^2-14\lambda-24}$$

$$-(\lambda^3+2\lambda^2)$$

$$-\lambda^2-14\lambda-24$$

$$-(-\lambda^2-2\lambda)$$

$$-12\lambda-24$$

$$-(-12\lambda-24)$$

$$0$$

$$(\lambda+2)(\lambda^2-\lambda-12)$$

$\lambda = -2$
$\lambda = 4$
$\lambda = -3$

$$y = c_1e^{-2x} + c_2e^{4x} + c_3e^{-3x}$$

Nondistinct Roots

Example: $y'' + 2y' + y = 0$

$$(D^2+2D+1)(y) = 0$$

$\lambda^2+2\lambda+1$ has roots $\lambda=-1$
$\lambda=-1$

$$\{e^{-x}, xe^{-x}\}$$

these are linearly independent.

Check: $y = e^{-x}$ is a solution.
$y = xe^{-x}$ is a solution

Lemma: The differential operator $(D-\lambda)^n$ has an n-dimensional nullspace
$\{e^{\lambda x}, xe^{\lambda x}, x^2 e^{\lambda x}, \ldots, x^{n-1} e^{\lambda x}\}$

It follows that for $L = (D-\lambda)^n$, $L(y) = 0$ has the general solution

$$(*) \quad y(x) = c_1 e^{\lambda x} + c_2 x e^{\lambda x} + \ldots + c_n x^{n-1} e^{\lambda x}.$$

worksheet problems

$$\boxed{\begin{array}{l} y'' + 4y' + 4 = 0 \\ y^{(3)} + 7y'' + 8y' - 16y = 0 \end{array}}$$

(1) $y = c_1 e^{-2x} + c_2 x e^{-2x}$

(2) $y = c_1 e^{x} + c_2 e^{-4x} + c_3 x e^{-4x}$

Purely Imaginary Roots

Example: $y'' + y = 0$

$$(D^2 + 1)y = 0$$

$\lambda^2 + 1 = 0$ $\quad \pm i = \lambda$ are solutions

$$i^2+1=-1+1=0$$

$$(-i)^2+1=(-1)^2(i)^2+1=1\cdot(-1)+1=0.$$

$$\lambda^2+1=(\lambda-i)(\lambda+i)$$

Solution: $y(x)=c_1e^{ix}+c_2e^{-ix}$

$$y'(x)=ic_1e^{ix}-ic_2e^{-ix}$$

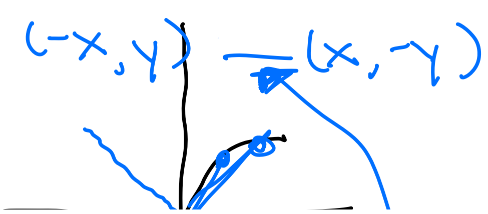
$$y''(x)=i^2c_1e^{ix}+i^2c_2e^{-ix}$$
$$=-c_1e^{ix}-c_2e^{-ix}$$

$$y''+y=(-c_1e^{ix}-c_2e^{-ix})+(c_1e^{ix}+c_2e^{-ix})$$
$$=0 \quad \checkmark$$

Check Taylor Series Expansions

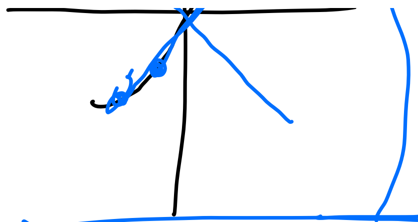
Euler's Formula: $e^{ix}=\cos(x)+i\sin(x)$

$$c_1e^{ix}+\underbrace{c_2e^{-ix}}_{c_2e^{i(-x)}}=c_1\cos(x)+c_1i\sin(x)$$



$$+c_2\cos(-x)+c_2i\sin(-x)$$

$$=c_1\cos(x)+c_1i\sin(x)$$

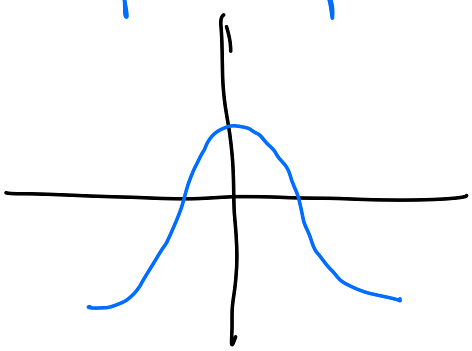


$+c_2\cos(x) - c_2 i\sin(x)$

$= (c_1+c_2)\cos(x) + (c_1-c_2)i\sin(x)$

Odd function:
$f(-x) = -f(x)$

Even function
$f(-x) = f(x)$



$c_1 + c_2$ $\qquad$ $(c_1 - c_2)i$

We need $c_1 + c_2$ and $(c_1 - c_2)i$ to be real valued.

So c_1 and c_2 are complex conjugates:

Example $c_1 = 2-4i$ $\quad$ $c_2 = 2+4i$

$c_1 = 3+5i$ $\quad$ $c_2 = 3-5i$

$c_1 + c_2 = 2-2i+2+2i$
$= 4$ ✓

$\boxed{c_1 = 2-2i \quad c_2 = 2+2i}$

$(c_1 - c_2)i = (2-2i-2-2i)i$ $\quad$ $c_1 = -i$ $\quad$ $c_2 = i$

$= (-4i)i = 4$

$c_1 = a+bi$ $\quad$ $c_2 = a-bi$

Now set $C_1 = c_1 + c_2$ $\qquad C_2 = (c_1 - c_2)i$

Real Numbers

$$y(x) = C_1 \cos(x) + C_2 \sin(x)$$

Given an equation of the form
$$y'' + \lambda^2 y = 0 \qquad \lambda \in \mathbb{R}$$
a general solution is

$$y(x) = C_1 \cos(\lambda x) + C_2 \sin(\lambda x)$$
where $C_1, C_2 \in \mathbb{R}$.

If an equation is given by
$(D^2 + \lambda^2)^2 y = 0$, then

$$y(x) = C_1 \cos(\lambda x) + C_2 \sin(\lambda x) + C_3 x \cos(\lambda x) + C_4 x \sin(\lambda x)$$

$(D^2 + 1)^2 y = 0$ $\qquad$ Roots of $(\lambda^2 + 1)^2 = 0$ are $\lambda = \pm i$ (of mult. 2)

$$y(x) = C_1 \cos(x) + C_2 \sin(x)$$

$+ C_3 x\cos(x) + C_4 x\sin(x)$

Fully Complex Roots.

Example: $y'' - 4y' + 8y = 0.$

$\lambda^2 - 4\lambda + 8 = 0$ quadratic formula

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$a\lambda^2 + b\lambda + c = 0$$

$\lambda_1 = 2 + 2i \qquad \lambda_2 = 2 - 2i$

$\sqrt{16} \cdot \sqrt{-1}$

$$\frac{4 \pm \sqrt{-16}}{2} = \frac{4 \pm 4i}{2} = 2 \pm 2i$$

$$y(x) = c_1 e^{(2+2i)x} + c_2 e^{(2-2i)x}$$

$$= e^{2x}\left(c_1 e^{2ix} + c_2 e^{-2ix}\right)$$

$$= e^{2x}\left(C_1 \cos(2x) + C_2 \sin(2x)\right)$$

Given an equation

$$ay'' + by' + cy = 0$$

with associated roots $A \pm Bi$, the general solution

$$y(x) = C_1 e^{Ax} \cos(Bx) + C_2 e^{Ax} \sin(Bx)$$