1. Below is a subspace of $\mathbb{R}^5$. Find a basis for S. And find the dimension of S.

$$S = \left\{ \underbrace{\begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix}}_{\mathbb{R}^5} \;\middle|\; \overbrace{\underbrace{c = \frac{1}{2}(a+b)}_{①}, \underbrace{d = a - b}_{②}, \underbrace{e = a}_{3}}^{\text{condition}} \right\} \subseteq \mathbb{R}^5. \qquad 5\begin{bmatrix} 1 \\ 0 \\ 1/2 \\ 1 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \\ 1/2 \\ -1 \\ 0 \end{bmatrix} = \vec{v} = \begin{bmatrix} 5 \\ 3 \\ 4 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}_{\mathcal{B}}$$

The vector $\vec{v}$ is in S. Give its coordinates with respect to your basis.

$$\begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} a \\ b \\ \frac{1}{2}a + \frac{1}{2}b \\ a - b \\ a \end{bmatrix} = a\underbrace{\begin{bmatrix} 1 \\ 0 \\ 1/2 \\ 1 \\ 1 \end{bmatrix}}_{u_1} + b\underbrace{\begin{bmatrix} 0 \\ 1 \\ 1/2 \\ -1 \\ 0 \end{bmatrix}}_{u_2}$$

$S = \text{span}(u_1, u_2)$

$\boxed{\mathcal{B} = \{u_1, u_2\}}$

$\boxed{\dim(S) = 2}$

$\boxed{\text{0's-1's property} \quad \text{std basis prop}}$

$\boxed{\text{alternative}}$

$$\begin{cases} \frac{1}{2}a + \frac{1}{2}b - c = 0 \\ a - b - d = 0 \\ a - e = 0 \end{cases}$$

$$S = \text{Nul}\left(\begin{bmatrix} 1/2 & 1/2 & -1 & 0 & 0 \\ 1 & -1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 \end{bmatrix}\right)$$

- row reduce
- find null space

2. Below a matrix A is row reduced, and so is its transpose. Compute rank(A), nullity(A) and find bases for Col(A), Row(A), Nul(A).

$$A = \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & & \end{array} \\ \begin{bmatrix} 1 & 3 & 2 & 3 & 7 \\ 1 & -2 & 0 & 0 & -5 \\ 0 & 4 & -1 & 5 & 7 \\ -1 & -5 & -1 & -6 & -10 \\ -1 & 1 & 2 & -3 & 5 \end{bmatrix} \end{array} \to \begin{array}{c} \begin{array}{ccccc} & & & s & t \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}, \quad A^T = \begin{bmatrix} 1 & 1 & 0 & -1 & -1 \\ 3 & -2 & 4 & -5 & 1 \\ 2 & 0 & -1 & -1 & 2 \\ 3 & 0 & 5 & -6 & -3 \\ 7 & -5 & 7 & -10 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -\frac{11}{13} & \frac{7}{13} \\ 0 & 1 & 0 & -\frac{2}{13} & -\frac{20}{13} \\ 0 & 0 & 1 & -\frac{9}{13} & -\frac{12}{13} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Col(A) = Row(A^T)

The vectors below are in Col(A), Row(A), Nul(A). There is one vector in each. Which one is in which subspace?

$$\underset{\text{Null}}{\vec{u} = \begin{bmatrix} 8 \\ -1 \\ -5 \\ -3 \\ 2 \end{bmatrix}}, \quad \overset{\text{Col}}{\vec{v} = \begin{bmatrix} 13 \\ 26 \\ 0 \\ -15 \\ -33 \end{bmatrix}}, \quad \vec{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}, \text{ Row}$$

rank(A) = # pivots(A) = 3
= dim(Col(A))
= dim(Row(A))

Basis: $\mathcal{B}_{Col} = \{v_1, v_2, v_3\} = \left\{ \underset{w_1}{\begin{bmatrix} 1 \\ 0 \\ 0 \\ -11/13 \\ 7/13 \end{bmatrix}}, \underset{w_2}{\begin{bmatrix} 0 \\ 1 \\ 0 \\ -2/13 \\ -20/13 \end{bmatrix}}, \underset{w_3}{\begin{bmatrix} 0 \\ 0 \\ 1 \\ -9/13 \\ -12/13 \end{bmatrix}} \right.$

parametriz solution $Ax = 0$

$$\mathcal{B}_{Nul} = \left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

nullity(A) = # free vars
= dim(Nul(A))

$$\mathcal{B}_{Row} = \left\{ \underset{r_1}{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \\ -1 \end{bmatrix}}, \underset{r_2}{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}}, \underset{r_3}{\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}} \right\}$$

$A = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}$

3. If A is 3×3 matrix, are either of the subsets below, $\boxed{\text{subspaces}}$ of $\mathbb{R}^3$? Justify your answers.

$$S_1 = \{\vec{v} \in \mathbb{R}^3 \mid A\vec{v} = -\vec{v}\} \qquad S_2 = \left\{\vec{v} \in \mathbb{R}^3 \mid A\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}.$$

$A\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

all 3 properties fail

subspace

(0) $\vec{0} \in S$

(1) if $\vec{u}, \vec{v} \in S$ then $\vec{u} + \vec{v} \in S$

(2) if $\vec{u} \in S$ then $c\vec{u} \in S$

(1) if $u, v \in S_1$ then $Au = -u$ and $Av = -v$

then $A(u+v) = Au + Av$

$= -u - v$

$= -(u+v)$

(2) $A(cu) = c(Au) = c(-u) = -(cu)$

4. Below is a basis of $\mathbb{R}^3$. Let $\mathcal{S} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ denote the standard basis. Find the missing change of coordinates for one of the examples at the right. For the other, describe *two* ways of finding the missing entries.

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} \right\}. \qquad \begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} -3 \\ 12 \\ 6 \end{bmatrix}_{\mathcal{S}} \qquad \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}_{\mathcal{S}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}_{\mathcal{B}}$$

$B = \begin{bmatrix} 1 & 2 & -1 \\ 1 & -1 & 2 \\ 1 & -1 & -1 \end{bmatrix}$

$\begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix}_{\mathcal{B}} = 5\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 3\begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} + 2\begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ 12 \\ 6 \end{bmatrix}_{S}$

$\underbrace{\begin{bmatrix} 1 & 2 & -1 \\ 1 & -1 & 2 \\ 1 & -1 & -1 \end{bmatrix}}_{B} \begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 12 \\ 6 \end{bmatrix}$

(i) $B^{-1}\begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}_S = \begin{bmatrix} a \\ b \\ c \end{bmatrix}_{\mathcal{B}}$

(ii) $\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 1 & -1 & 2 & 6 \\ 1 & -1 & -1 & 9 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & & & a \\ & 1 & & b \\ & & 1 & c \end{array}\right]$

5. See also: E2 review problems.

6. See also: PS6 problems.