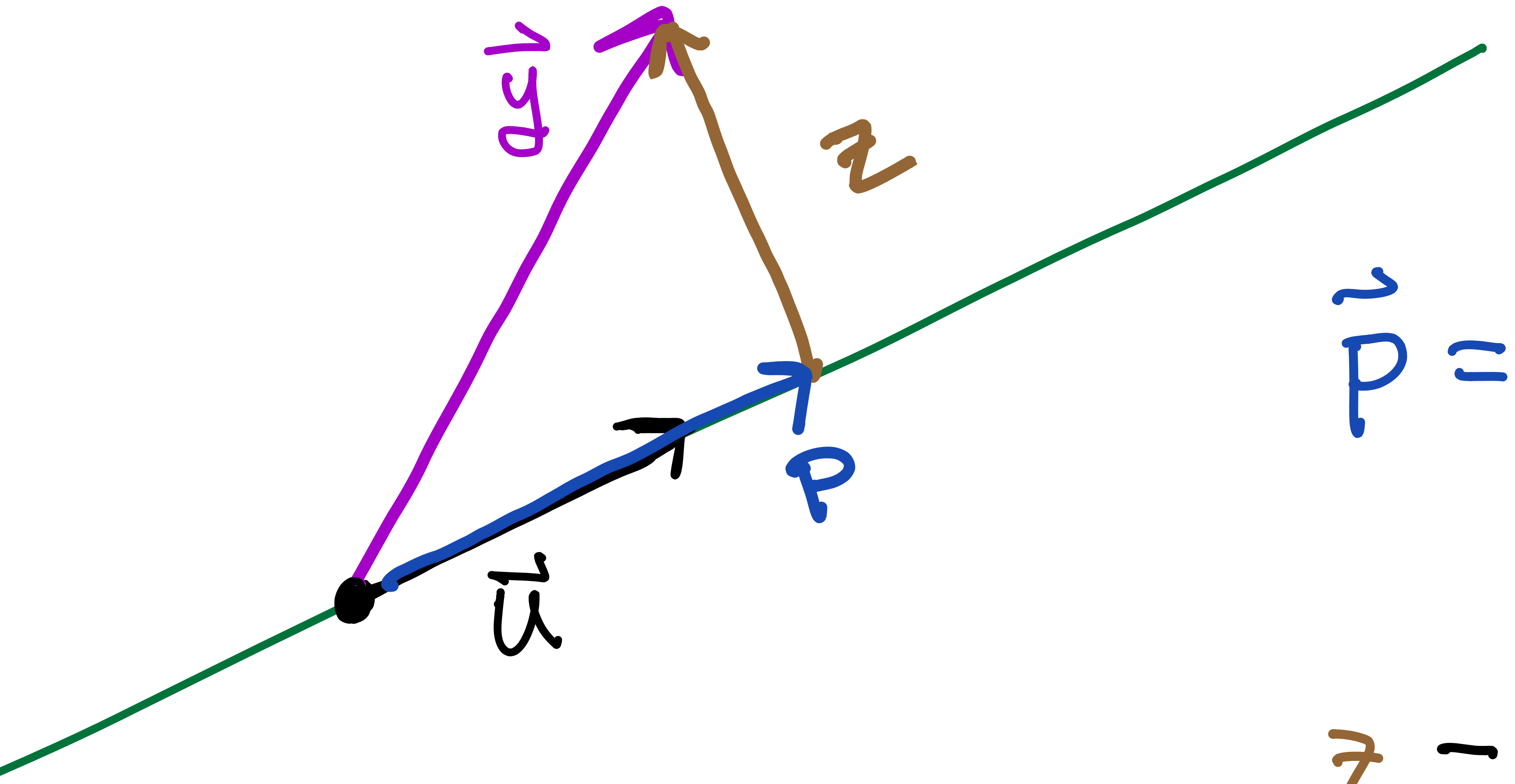
# 6.3. Orthogonal Projections

# Projection Onto a Line




$L = \text{span}(\vec{u})$

$\vec{p} = \alpha \vec{u}$ projection of $\vec{y}$ onto $L = \text{span}(\vec{u})$

$= \text{proj}_{\vec{u}} \vec{y} = \text{proj}_L \vec{y}$

$z = \vec{y} - \vec{p}$ is orthogonal to $\vec{u}$

$= \vec{y} - \alpha \vec{u}$

$(\vec{y} - \alpha \vec{u}) \cdot \vec{u} = 0$

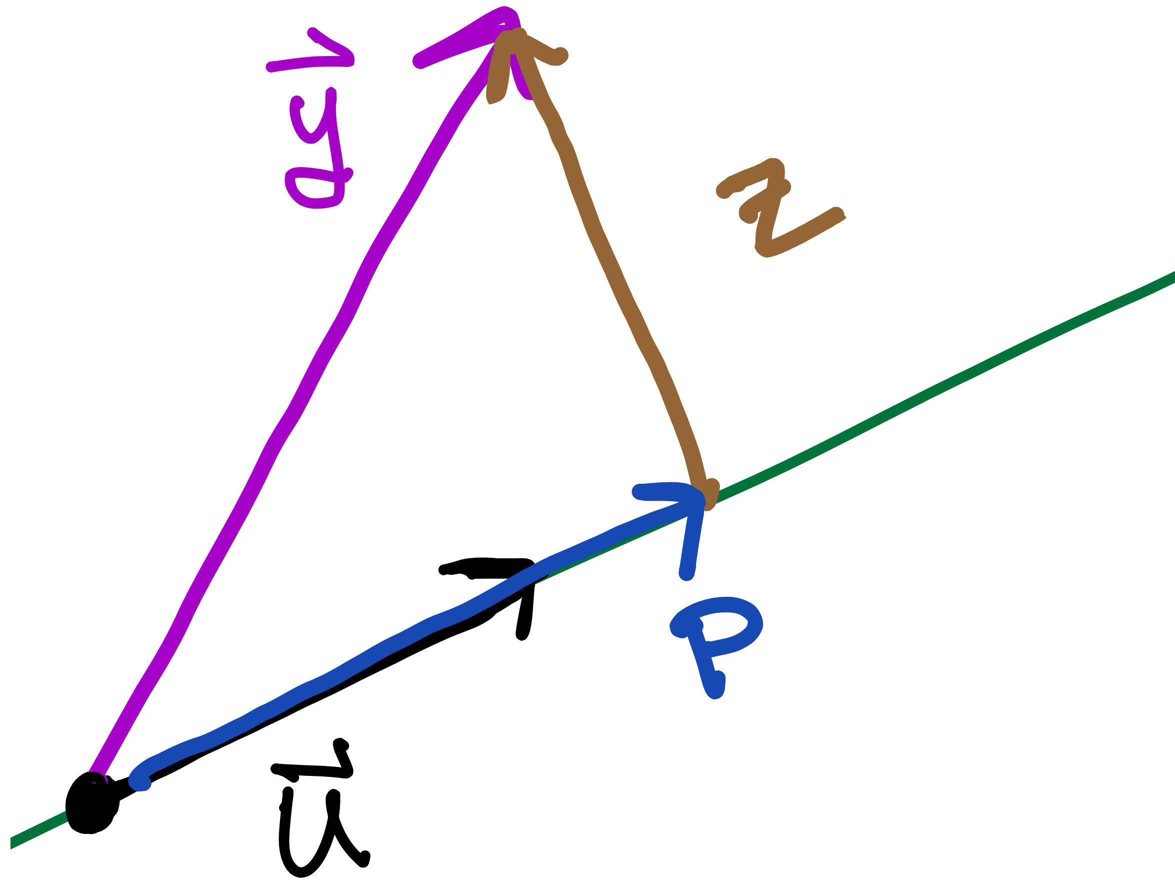
$\vec{y} \cdot \vec{u} - \alpha (\vec{u} \cdot \vec{u}) = 0$

$\vec{y} \cdot \vec{u} = \alpha (\vec{u} \cdot \vec{u})$

$\Rightarrow \boxed{\alpha = \dfrac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}}$

# Example

Find the projection of $\vec{y} = \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}$ onto $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$




and find the residual vector.

$$\boxed{\begin{aligned} \vec{p} &= \mathrm{proj}_{\vec{u}}\vec{y} = \left(\frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\right)\vec{u} \\ \vec{z} &= \vec{y} - \vec{p} \end{aligned}}$$

$$\frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}} = \frac{2-3-2}{1+1+1} = \frac{-3}{3} = -1$$

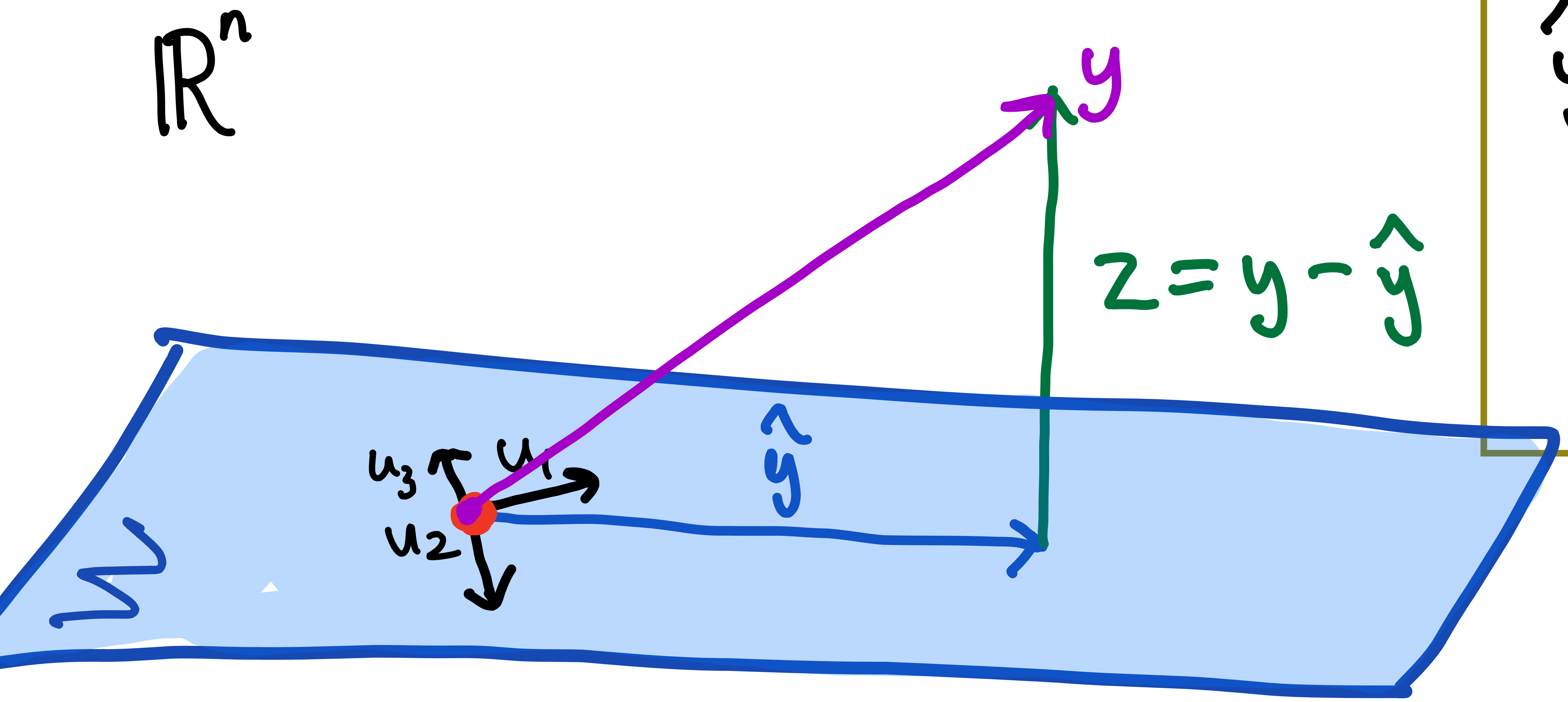
$$\vec{p} = -\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{z} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

Check:

$$\vec{p} + \vec{z} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix} = \vec{y}$$

$$\vec{u} \cdot z = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} = 1\cdot 3 + 1\cdot(-2) + (-1)\cdot 1 = 0$$

# Projection Onto a Subspace




$$\hat{y} = \text{proj}_W(y)$$
$$= \left(\frac{y \cdot u_1}{u_1 \cdot u_1}\right) u_1 + \left(\frac{y \cdot u_2}{u_2 \cdot u_2}\right) u_2 + \cdots + \left(\frac{y \cdot u_k}{u_k \cdot u_k}\right) u_k$$

- $z = y - \hat{y} \in W^{\perp}$

W has orthogonal basis $u_1, u_2, \dots, u_k$

- $\hat{y} \in \text{span}(u_1, u_2, \dots, u_k) \in W$
- if $z = y - \hat{y}$ then $z \cdot u_i = 0$ for each $i$

$$z \cdot u_i = (y - \hat{y}) \cdot u_i$$
$$= y \cdot u_i - \hat{y} \cdot u_i$$
$$= y \cdot u_i - \left(\frac{y \cdot u_i}{\cancel{u_i \cdot u_i}}\right)(\cancel{u_i \cdot u_i})$$
$$= y \cdot u_i - y \cdot u_i$$
$$= 0$$

# Orthogonal Projection Onto A Subspace

We can project onto any subspace **W** of **R**$^n$ with an *orthogonal basis*.

## Theorem

Let $W$ be a subspace of $\mathbb{R}^n$ with **orthogonal basis** $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k$. For any $\vec{y} \in \mathbb{R}^n$, we have

$$\vec{y} = \hat{y} + \vec{z}$$

where $\hat{y} \in W$ and $\vec{z} \in W^\perp$. Furthermore,

$$\hat{y} = \text{proj}_W \vec{y} = c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_k\vec{u}_k$$

where

$$c_i = \frac{\vec{y} \bullet \vec{u}_i}{\vec{u}_i \bullet \vec{u}_i} \quad \text{for } 1 \leq i \leq k$$

is called the **projection** of $\vec{y}$ onto $W$ and $\vec{z}$ is called the **residual vector**.

# Example

Let $\vec{u} = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \quad \vec{v} = \begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}, \quad \vec{y} = \begin{bmatrix}1\\3\\1\\7\end{bmatrix}$

**W** = span(***u***,***v***), a subspace of $\mathbb{R}^4$

Find the projection of **y** into span(**u**,**v**) and its corresponding residual vector

Note that **u•v** = 0, so **u** and **v** are orthogonal

$$\frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}} = \frac{1+3+1+7}{1+1+1+1} = \frac{12}{4} = 3$$

$$\frac{\vec{y}\cdot\vec{v}}{\vec{u}\cdot\vec{v}} = \frac{1-3-1+7}{1+1+1+1} = \frac{4}{4} = 1$$

$$\hat{y} = 3\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + \begin{bmatrix}1\\-1\\-1\\1\end{bmatrix} = \begin{bmatrix}4\\2\\2\\4\end{bmatrix}$$

$$\vec{z} = \vec{y} - \hat{y} = \begin{bmatrix}1\\3\\1\\7\end{bmatrix} - \begin{bmatrix}4\\2\\2\\4\end{bmatrix} = \begin{bmatrix}-3\\1\\-1\\3\end{bmatrix}$$

Check:

$$\hat{y} + z = \begin{bmatrix}4\\2\\2\\4\end{bmatrix} + \begin{bmatrix}-3\\1\\-1\\3\end{bmatrix} = \begin{bmatrix}1\\3\\1\\7\end{bmatrix} = y$$

$$\begin{bmatrix}1\\1\\1\\1\end{bmatrix} \cdot \begin{bmatrix}-3\\1\\-1\\3\end{bmatrix} = -3+1-1+3=0 \qquad \begin{bmatrix}1\\-1\\-1\\1\end{bmatrix} \cdot \begin{bmatrix}-3\\1\\-1\\3\end{bmatrix} = -3-1+1+3=0$$

# You Try

Find the orthogonal projection $\hat{\mathsf{w}} = \mathsf{proj}_{\mathbf{S}}(\vec{\mathsf{w}})$ of $\vec{\mathsf{w}}$ onto the subspace $\mathbf{S}$ and find the residual $z = \vec{\mathsf{w}} - \hat{\mathsf{w}}$. Then check that the residual is orthogonal to $\mathbf{S}$

$$\mathbf{S} = \mathsf{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \right\} \subseteq \mathbb{R}^4$$

$$\vec{\mathsf{w}} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$