Review of Null and Column space:

1. If A is a 6×9 matrix. Then $\mathsf{Nul}(\mathsf{A}) \subseteq \mathbb{R}^n$ for what n? And Then $\mathsf{Col}(\mathsf{A}) \subseteq \mathbb{R}^m$ for what m?

$$\mathsf{A} = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

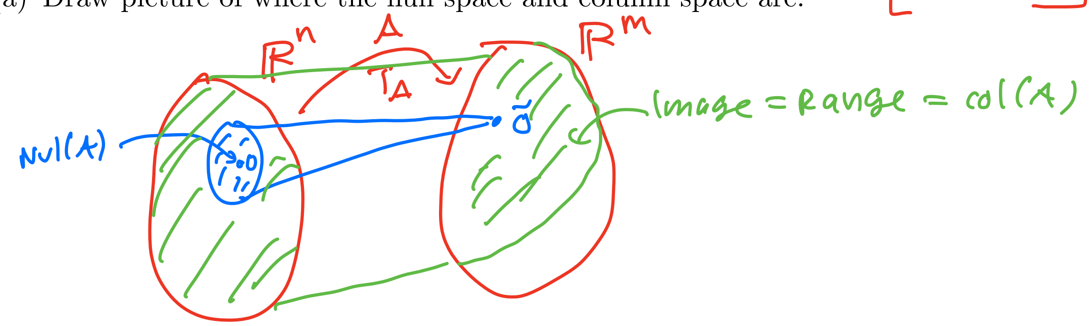
$\mathrm{Nul}(A) \subseteq \mathbb{R}^9$

$\mathrm{Col}(A) \subseteq \mathbb{R}^6$

2. If $T_\mathsf{A} : \mathbb{R}^n \to \mathbb{R}^m$ is the linear transformation with matrix A, then

$$\begin{array}{cc} & n \\ m & \begin{bmatrix} \quad A \quad \end{bmatrix} \end{array}$$

(a) Draw picture of where the null space and column space are:



(b) If T is one-to-one, then what, if anything, can you say about $\mathsf{Nul}(A)$ and $\mathsf{Col}(A)$?

i. $\mathsf{Nul}(A)$

ii. $\mathsf{Col}(A)$

(c) If T is onto, then what, if anything, can you say about $\mathsf{Nul}(A)$ and $\mathsf{Col}(A)$?

i. $\mathsf{Nul}(A)$

ii. $\mathsf{Col}(A)$

(d) If $n = m$ so that A is square, then what, if anything, can we say about the relationship between $\mathsf{Nul}(A)$ and $\mathsf{Col}(A)$.

i. If $\mathsf{Nul}(A) = \{0\}$ then ...

ii. If $\mathsf{Col}(A) = \mathbb{R}^n$ then ...

Basis: The idea of a basis is one of the most important in mathematics:

Def: A **basis** of a vector space (or a subspace, which itself is a vector space) V is a set of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ satisfying two properties:

1. The vectors *span* V. This means that for every vector $\vec{v} \in V$ there are weights $c_1, c_2, \ldots, c_n$ such that $\vec{v}$ can be written as a linear combination
$$\boxed{\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n.}$$
2. The vectors are *linearly independent*. This means that the only linear combination of these vectors to reach the $\mathbf{0}$ vector,
$$\boxed{\mathbf{0} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n}$$
is with $c_1 = c_2 = \cdots = c_n = 0$. An alternative way to say this is that none of these vectors can be written as a linear combination of the others, and so there are no redundancies.

An equivalent way to say this, in a single statement, is that for every vector $\vec{v} \in V$ there is a *unique* set of weights $c_1, c_2, \ldots, c_n$ such that $\vec{v}$ can be written as a linear combination
$$\boxed{\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n.}$$
only one set of scalars

Key point: they *span* the space as *efficiently* as possible (i.e., with no redundancy).

Examples

1. These are all bases of $\boxed{\mathbb{R}^3}$.

(a) $\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}$

(b) $\left\{ \begin{bmatrix}1000\\0\\0\end{bmatrix}, \begin{bmatrix}0\\-19.34\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0.01024\end{bmatrix} \right\}$

(c) $\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix} \right\}$

(d) $\left\{ \begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}0\\1\\-1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix} \right\}$

(e) $\left\{ \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}.8\\.25\\.5\end{bmatrix}, \begin{bmatrix}1\\.9\\.1\end{bmatrix} \right\}$

(f) $\left\{ \begin{bmatrix}-0.99\\-0.38\\0.75\end{bmatrix}, \begin{bmatrix}0.77\\-0.76\\-0.43\end{bmatrix}, \begin{bmatrix}-0.64\\-0.26\\-0.51\end{bmatrix} \right\}$

In each of these examples, if you put the vectors in a matrix and row reduce, you get:

$$\begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$$

2. What do bases in $\mathbb{R}^3$ look like?

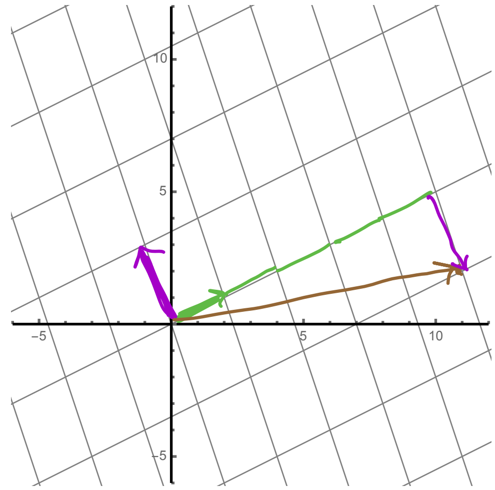
3. From Day 3 of this course: Consider the basis $\mathcal{B} = \{\vec{u}, \vec{v}\}$ below. It gives us a new coordinate system. Describe $\vec{b}$ both in terms of the standard basis and in terms of the basis $\mathcal{B}$.

$$\mathcal{B} = \left\{ \vec{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \vec{v} = \begin{bmatrix} -1 \\ 3 \end{bmatrix} \right\}$$

$$\vec{b} = \begin{bmatrix} 11 \\ 2 \end{bmatrix} = \boxed{5u - v}\ \mathcal{B}$$

$$= \boxed{11\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \end{bmatrix}}\ S$$

standard



$W \subseteq \mathbb{R}^5$

4. Find a basis of the subspace of $\mathbb{R}^5$ defined by $\mathbf{W} = \text{span}$
$$\left\{ \begin{array}{cccccc} w_1 & w_2 & w_3 & w_4 & w_5 & w_6 \\ \begin{bmatrix} 1 \\ 5 \\ 2 \\ -1 \\ 3 \end{bmatrix}, & \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, & \begin{bmatrix} 0 \\ 4 \\ 3 \\ 0 \\ 2 \end{bmatrix}, & \begin{bmatrix} 3 \\ 2 \\ 0 \\ 2 \\ 3 \end{bmatrix}, & \begin{bmatrix} 3 \\ 7 \\ 0 \\ -3 \\ 5 \end{bmatrix}, & \begin{bmatrix} 5 \\ 8 \\ 1 \\ 0 \\ 7 \end{bmatrix} \end{array} \right\}$$

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A = cbind(c(1,5,2,-1,3),c(1,1,-1,-1,1),c(0,4,3,0,2),c(3,2,0,2,3),c(3,7,0,-3,5),c(5,8,1,0,7))
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$W = \text{span}(w_1, w_2, \cancel{w_3}, w_4, \cancel{w_5}, \cancel{w_6})$

$= \text{span}(w_1, w_2, w_4)$

$\mathcal{B} = \{w_1, w_2, w_4\}$ is a basis

(Fibonacci)

5. (**Cool trick!** Use on PS5.3b – bake the condition into the definition) Find basis of the following subspace of $\mathbb{R}^4$:

$$\begin{bmatrix} 1 \\ -2 \\ 3 \\ -2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} 5 \\ -11 \\ 8 \\ -2 \end{bmatrix}$$

$$Z = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \middle| \, x_1 + x_2 + x_3 + x_4 = 0 \right\} \subseteq \mathbb{R}^4$$

$x_4 = -x_1 - x_2 - x_3$

$$Z = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ -x_1 - x_2 - x_3 \end{bmatrix} \middle| \, x_1, x_2, x_3 \right\}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ -x_1 - x_2 - x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ 0 \\ -x_1 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2 \\ 0 \\ -x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ x_3 \\ -x_3 \end{bmatrix}$$

$$= x_1 \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}}_{v_1} + x_2 \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}}_{v_2} + x_3 \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}}_{v_3}$$

v_1, v_2, v_3 span Z

independent 0's and 1's property

6. $A = \begin{array}{c} \begin{array}{ccccc} w_1 & w_2 & w_3 & w_4 & w_5 \end{array} \\ \begin{bmatrix} 1 & 1 & 3 & 1 & 2 \\ -2 & 1 & -3 & -1 & -9 \\ -1 & 1 & -1 & 2 & -3 \\ 4 & 1 & 9 & 3 & 13 \\ -2 & 3 & -1 & 2 & -10 \end{bmatrix} \end{array} \to \begin{bmatrix} 1 & 0 & 2 & 0 & 3 \\ 0 & 1 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

$$\boxed{\text{Nul}(A) = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = s \underbrace{\begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{v_1} + t \underbrace{\begin{bmatrix} -3 \\ 2 \\ 0 \\ -1 \\ 1 \end{bmatrix}}_{v_2} \,\middle|\, s, t \in \mathbb{R} \right\}}$$

Null space

0's and 1's property

parametric solution vectors are always a basis for Nul(A)

(a) Find a basis of the **null space** of this matrix.

$$\boxed{\mathcal{B}_{nul} = \{v_1, v_2\}}$$

(b) Find a basis of the **column space** of this matrix.

$\text{Col}(A) = \text{span}(w_1, w_2, \cancel{w_3}, w_4, \cancel{w_5})$

$$\boxed{\mathcal{B}_{col} = \{w_1, w_2, w_4\}}$$

pivot columns of A

7. Find bases for the column and null spaces of the following matrices.

(a) $A = \begin{bmatrix} 4 & 2 & -8 \\ -2 & -1 & 4 \\ 6 & 3 & -12 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1/2 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(b) $B = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -2 & 0 \\ -3 & 1 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$

(c) $C = \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 1 & 1 \end{bmatrix} \end{array} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\mathrm{Nul}(C) = \{0\}$ no basis

$\mathrm{Col}(A) = \mathrm{span}\underbrace{(v_1, v_2, v_3)}_{\text{basis}} = \mathbb{R}^3$

8 4

8. How many vectors are there in a basis for Col(A) and Nul(A) for the following matrix:

$$\begin{bmatrix} 1 & -2 & 6 & 5 & 7 & 3 & -2 & 5 & -26 & 0 & 10 & -20 \\ 0 & 3 & 1 & 5 & -18 & -3 & 3 & 1 & -5 & 6 & -5 & -8 \\ 1 & -2 & 4 & -5 & 25 & -3 & 0 & 1 & -20 & -2 & -2 & -10 \\ -5 & -2 & -2 & -1 & -19 & 4 & 0 & 1 & -1 & 2 & 5 & -9 \\ 6 & -3 & -6 & -6 & 45 & -3 & -1 & 4 & 8 & -1 & 2 & 19 \\ -3 & -5 & 3 & -1 & 5 & 5 & 5 & -2 & -18 & -2 & -2 & -12 \\ 6 & 3 & 6 & 2 & 23 & 1 & 2 & 1 & -9 & 3 & 0 & -18 \\ -2 & 5 & -5 & 1 & -32 & -3 & 4 & 3 & 10 & 0 & -4 & 14 \\ 3 & -3 & 5 & 1 & 27 & 3 & -4 & 0 & -11 & -1 & 7 & -12 \\ -6 & -2 & -5 & 4 & -37 & -4 & 3 & 4 & -10 & 4 & -3 & 5 \end{bmatrix} \longrightarrow \begin{array}{c} \begin{array}{cccccccccccc} & & & & s & & & & t & & r & q \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -3 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & -3 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

9. Use the trick from #4 to find a basis for the subspace $S = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \;\middle|\; b = 2a,\ d = 2c \right\} \subseteq \mathbb{R}^4$