Section 3.3: General Solutions to Nonhomogeneous equations

We already know how to solve

$$L(y) = 0.$$

where L is a polynomial in D with constant coefficients

Goal: Solve

$$L(y) = f(x)$$

where L is a polynomial in D with constant coefficients, and $f(x)$ is in the nullspace of some polynomial in D.

Method of Annihilators

- Other methods: Method of Undetermined Coefficients, Variation of Parameters*

Example: $y'' + 3y' + 2y = e^{3x}$.

$$L := D^2 + 3D + 2$$

$$L(y) = \boxed{e^{3x}}$$

e^{3x} : Can you reverse the ideas from previous section in order to find an operator M such that

$$M(e^{3x}) = 0 \ ?$$

$y = e^{3x}$ $\quad M(y) = 0$

$$\boxed{M := D - 3}$$

$$\boxed{M(e^{3x})} = D(e^{3x}) - 3e^{3x} \quad *$$

$$= 3e^{3x} - 3e^{3x} = 0 \ ✓$$

$$L(y) = e^{3x}$$

$$M(L(y)) = M(e^{3x})$$

$$M(L(y)) = 0 \quad \text{homogeneous}$$

$$(D-3)(D^2+3D+2)(y) = 0$$

polynomial in D

$(*)\ (D-3)(D+1)(D+2)(y)=0$

General Solution to $(*)$

$\Rightarrow y = C_1 e^{-2x} + C_2 e^{-x} + C_3 e^{3x}$.

$L(y) = L(C_1 e^{-2x} + C_2 e^{-x} + C_3 e^{3x})$

Hint: Use the fact that L is Linear and we know solutions to L come from roots λ_1, λ_2 of $\lambda^2+3\lambda+2$

$e^{3x} = L(y) = \cancel{C_1 L(e^{-2x})} + \cancel{C_2 L(e^{-x})} + C_3 L(e^{3x})$

$= C_3 (D^2 + \underline{3D} + 2)(e^{3x})$

$= C_3 (9e^{3x} + 9e^{3x} + 2e^{3x})$

$= 20 c_3 e^{3x}$

$e^{3x} = 20 c_3 e^{3x}$ *

$1 = 20 c_3 \quad \Rightarrow c_3 = \frac{1}{20}$ *

General Solution $L(y) = e^{3x}$ is

$$y = c_1 e^{-2x} + c_2 e^{-x} + \frac{1}{20} e^{3x}.$$