1.2. Row Reduction and Row Echelon Form

$$\left\{\begin{array}{ccccccc} 2x_1 & + & 4x_2 & + & 6x_3 & = & 0 \\ 4x_1 & + & 5x_2 & + & 6x_3 & = & 3 \\ 7x_1 & + & 8x_2 & + & 9x_3 & = & 6 \end{array}\right\} \longrightarrow \left[\begin{array}{ccc|c} 2 & 4 & 6 & 0 \\ 4 & 5 & 6 & 3 \\ 7 & 8 & 9 & 6 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Elementary Row Operations

(E1) Replacement: Add a multiple of one row to another

$$\left[\begin{array}{ccc|c} 1 & -3 & 1 & 4 \\ 2 & -8 & 8 & -2 \\ -6 & 3 & -15 & 9 \end{array}\right] \longrightarrow \left[\begin{array}{lll|l} 1 & -3 & 1 & 4 \\ 2+(-2)1 & -8+(-2)-3 & 8+(-2)1 & -2+(-2)4 \\ -6 & 3 & -15 & 9 \end{array}\right] = \left[\begin{array}{rrr|r} 1 & -3 & 1 & 4 \\ 0 & -6 & 6 & -10 \\ -6 & 3 & -15 & 9 \end{array}\right]$$

(E2) Exchange two rows

$$\left[\begin{array}{rrr|r} 1 & -3 & 1 & 4 \\ 2 & -8 & 8 & -2 \\ -6 & 3 & -15 & 9 \end{array}\right] \longrightarrow \left[\begin{array}{rrr|r} 2 & -8 & 8 & -2 \\ 1 & -3 & 1 & 4 \\ -6 & 3 & -15 & 9 \end{array}\right]$$

Key Point: these operations do not change the set of solutions

(E3) Scale: Multiply a row by a nonzero constant

$$\left[\begin{array}{rrr|r} 1 & -3 & 1 & 4 \\ 2 & -8 & 8 & -2 \\ -6 & 3 & -15 & 9 \end{array}\right] \longrightarrow \left[\begin{array}{rrr|r} 1 & -3 & 1 & 4 \\ \frac{1}{2}2 & \frac{1}{2}(-8) & \frac{1}{2}8 & \frac{1}{2}(-2) \\ -6 & 3 & -15 & 9 \end{array}\right] = \left[\begin{array}{rrr|r} 1 & -3 & 1 & 4 \\ 1 & -4 & 4 & -1 \\ -6 & 3 & -15 & 9 \end{array}\right]$$

Row Echelon Form

Row Echelon Form: a staircase of zeros that descends from left to right. All steps must be height 1.

Leading Entry: first nonzero entry in a row

Reduced Row Echelon Form:
- Leading entry must be 1
- Leading entry is the only nonzero entry in the column

EXAMPLE 1 The following matrices are in echelon form. The leading entries (■) may have any nonzero value; the starred entries ($*$) may have any value (including zero).

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \end{bmatrix}$$

The following matrices are in reduced echelon form because the leading entries are 1's, and there are 0's below *and above* each leading 1.

$$\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

■

The Gaussian Elimination Algorithm

1. Begin with the leftmost nonzero column. This is the **Pivot Column**
   - The pivot position is the top row
   - Exchange rows if necessary to move the desired nonzero entry to the top

2. Use **Replacement** to create all zeros below the pivot.

3. Move down one row and right one column and repeat until you reach the bottom of the matrix. The matrix is now in **Row Echelon Form**.

4. If necessary **Scale** the pivot to be 1 (this can be done earlier).
   Working from *right to left* use **Elimination** to create all zeros above the pivot.
   The matrix is now in **Reduced Row Echelon Form**

EXAMPLE: from day 1

$\boxed{\text{TEMPERATURE GRID}}$

$$\left\{\begin{array}{rrrrl} 4w & -x & -y & & = 185 \\ -w & +4x & & -z & = 156 \\ -w & & +4y & -z & = 104 \\ & -x & -y & +4z & = 129 \end{array}\right\}$$

Linear System
4 linear equations
in 4 unknowns

$$\left[\begin{array}{cccc|c} 4 & -1 & -1 & 0 & 185 \\ -1 & 4 & 0 & -1 & 156 \\ -1 & 0 & 4 & -1 & 104 \\ 0 & -1 & -1 & 4 & 129 \end{array}\right]$$

COEFFICIENT MATRIX

RHS

AUGMENTED MATRIX

EXAMPLE: from day 1

$$\left[\begin{array}{cccc|c} 4 & -1 & -1 & 0 & 185 \\ -1 & 4 & 0 & -1 & 156 \\ -1 & 0 & 4 & -1 & 104 \\ 0 & -1 & -1 & 4 & 129 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} -1 & 4 & 0 & -1 & 156 \\ 4 & -1 & -1 & 0 & 185 \\ -1 & 0 & 4 & -1 & 104 \\ 0 & -1 & -1 & 4 & 129 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & -4 & 0 & 1 & -156 \\ 4 & -1 & -1 & 0 & 185 \\ -1 & 0 & 4 & -1 & 104 \\ 0 & -1 & -1 & 4 & 129 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & -4 & 0 & 1 & -156 \\ 0 & 15 & -1 & -4 & 809 \\ 0 & -4 & 4 & 0 & -52 \\ 0 & -1 & -1 & 4 & 129 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & -4 & 0 & 1 & -156 \\ 0 & -1 & -1 & 4 & 129 \\ 0 & -4 & 4 & 0 & -52 \\ 0 & 15 & -1 & -4 & 809 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & -4 & 0 & 1 & -156 \\ 0 & 1 & 1 & -4 & -129 \\ 0 & -4 & 4 & 0 & -52 \\ 0 & 15 & -1 & -4 & 809 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & -4 & 0 & 1 & -156 \\ 0 & 1 & 1 & -4 & -129 \\ 0 & 0 & 8 & -16 & -568 \\ 0 & 0 & -16 & 56 & 2744 \end{array}\right]$$

EXAMPLE: from day 1

$$\left[\begin{array}{cccc|c} 1 & -4 & 0 & 1 & -156 \\ 0 & 1 & 1 & -4 & -129 \\ 0 & 0 & 8 & -16 & -568 \\ 0 & 0 & -16 & 56 & 2744 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & -4 & 0 & 1 & -156 \\ 0 & 1 & 1 & -4 & -129 \\ 0 & 0 & 1 & -2 & -71 \\ 0 & 0 & -16 & 56 & 2744 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & -4 & 0 & 1 & -156 \\ 0 & 1 & 1 & -4 & -129 \\ 0 & 0 & 1 & -2 & -71 \\ 0 & 0 & 0 & 24 & 1608 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} 1 & -4 & 0 & 1 & -156 \\ 0 & 1 & 1 & -4 & -129 \\ 0 & 0 & 1 & -2 & -71 \\ 0 & 0 & 0 & 1 & 67 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & -4 & 0 & 0 & -223 \\ 0 & 1 & 1 & 0 & 139 \\ 0 & 0 & 1 & 0 & 63 \\ 0 & 0 & 0 & 1 & 67 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & -4 & 0 & 0 & -223 \\ 0 & 1 & 0 & 0 & 76 \\ 0 & 0 & 1 & 0 & 63 \\ 0 & 0 & 0 & 1 & 67 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 81 \\ 0 & 1 & 0 & 0 & 76 \\ 0 & 0 & 1 & 0 & 63 \\ 0 & 0 & 0 & 1 & 67 \end{array}\right]$$

http://www.wolframalpha.com/

Computational Complexity

$\boldsymbol{O}(n^3)$: an nxn matrix takes approximately n^3 operations to row reduce

$(2n)^3 = 8n^3$