4.1. Subspaces of $\mathbb{R}^n$

$\mathbf{V} = \mathbb{R}^n$ is a Vector Space

As we have seen:

$$\mathbb{R}^n = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \;\middle|\; a_1, a_2, \ldots, a_n \in \mathbb{R} \right\}$$

is the set of all n-dimensional vectors

addition: $\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{bmatrix}$

scalar multiplication: $c \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} c\,a_1 \\ c\,a_2 \\ \vdots \\ c\,a_n \end{bmatrix}$

Algebraic Properties

For all vectors **u** and **v** in **V**

1. **u** + **v** is in **V**
2. c**u** is in **V** for each scalar c in $\mathbb{R}$
3. There is a zero vector **0** in **V** so **u** + **0** = **u**
4. **u** + **v** = **v** + **u**
5. (**u**+**v**) + **w** = **u** + (**v** + **w**)
6. There is a vector -**u** so that **u** - **u** = **0**
7. c(**u**+**v**) = c**u** + c**v**
8. (c+d)**u** = c**u** + d**u** for scalars c and d.
9. (cd)**u** = c(d**u**)
10. 1**u** = **u**
11. 0**u** = **0**

This is the definition of a vector space

Subspaces

A subset **S** of $\mathbb{R}^n$ is a **subspace** if it also satisfies these rules

Good news: 4-11 will be automatically true for any subset. True for all vectors in $\mathbb{R}^n$.

Algebraic Properties

For all vectors **u** and **v** in **V**:

1. **u** + **v** is in **V**
2. *c* **u** is in **V** for each scalar *c* in $\mathbb{R}$
3. There is a zero vector **0** in **V** so **u** + **0** = **u**

Def: A **subspace** is a subset S of $\mathbb{R}^n$ that satisfies the following properties:

For all vectors **u** and **v** in **S**:

1. **u** + **v** is in **S** ← *closed* under addition
2. *c* **u** is in **S** for each scalar *c* in $\mathbb{R}$ ← *closed* under scalar multiplication

3. **0** is in **S** ← Follows from 2 with c = 0
4. *c* **u** + *d* **v** is in **S** ← Follows from 1 and 2

closed under scalar multiplication

Subspaces

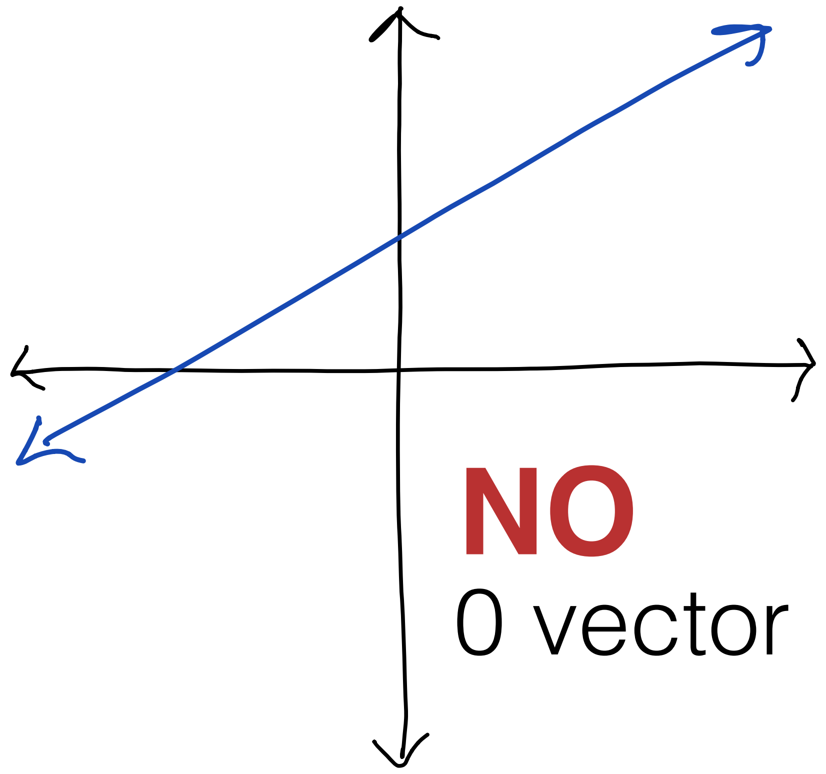
Which of these *subsets* of $\mathbb{R}^2$ are *subspaces* of $\mathbb{R}^2$?

Def: A **subspace** is a subset S of $\mathbb{R}^n$ that satisfies the following properties:

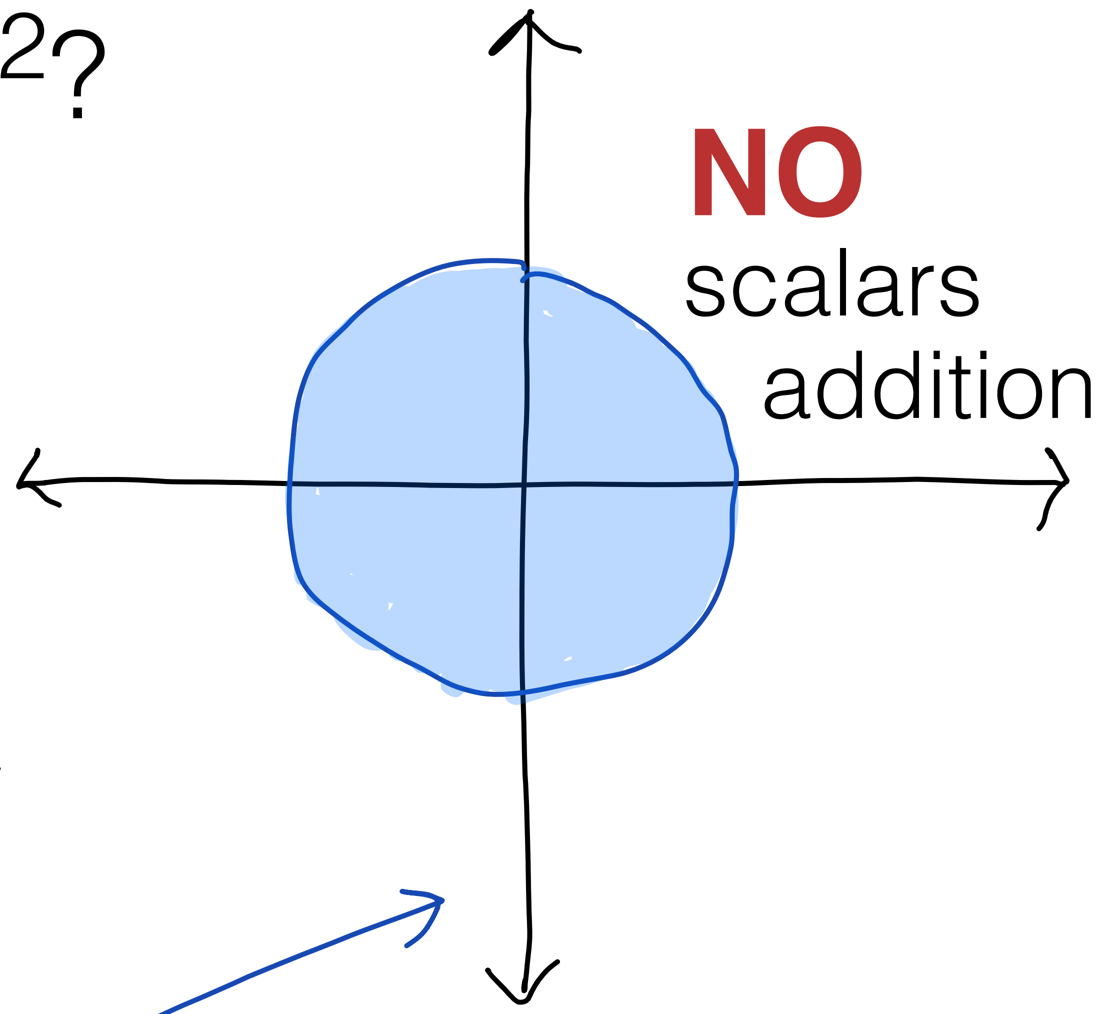
For all vectors **u** and **v** in **S**:

1. **u** + **v** is in **S**
2. c **u** is in **S** for each scalar c in $\mathbb{R}$

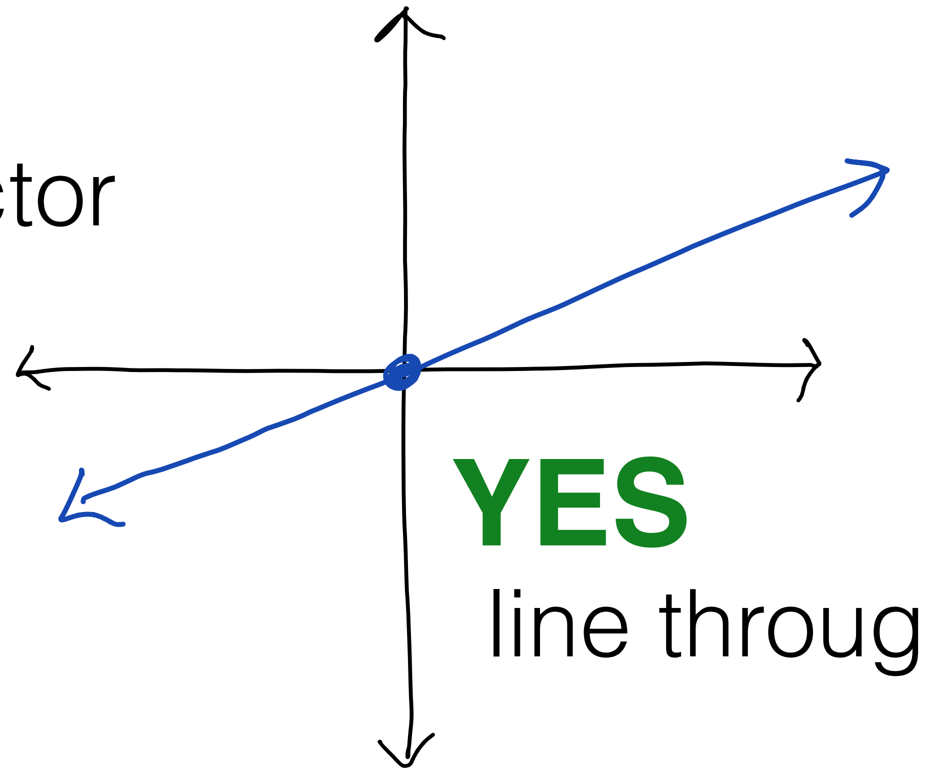
3. **0** is in **S**



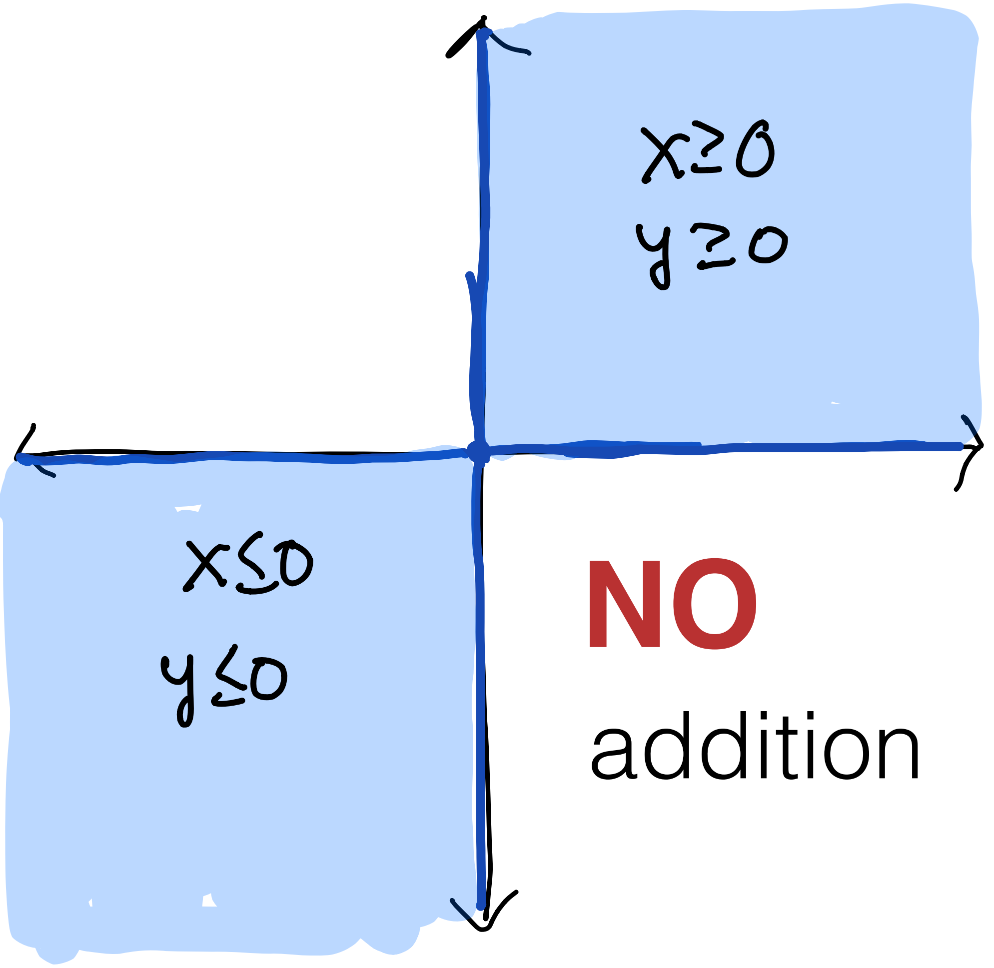
NO
0 vector



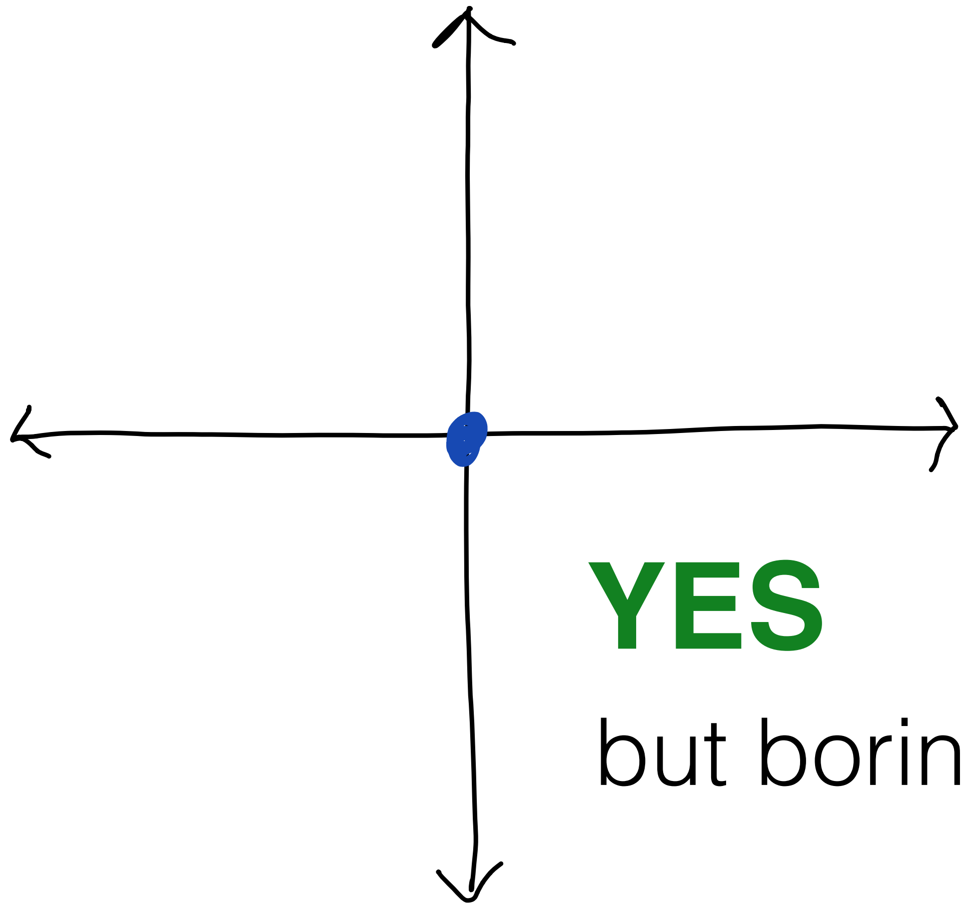
NO
scalars
addition



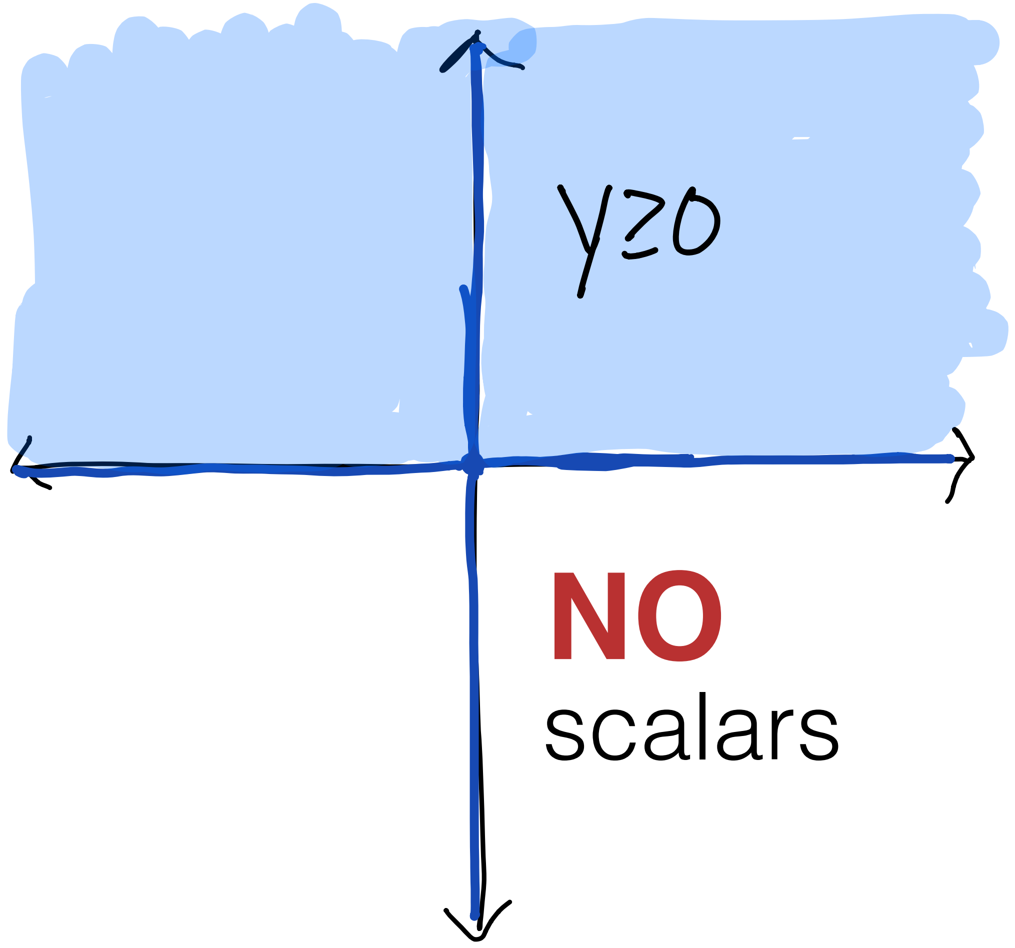
YES
line through the origin!



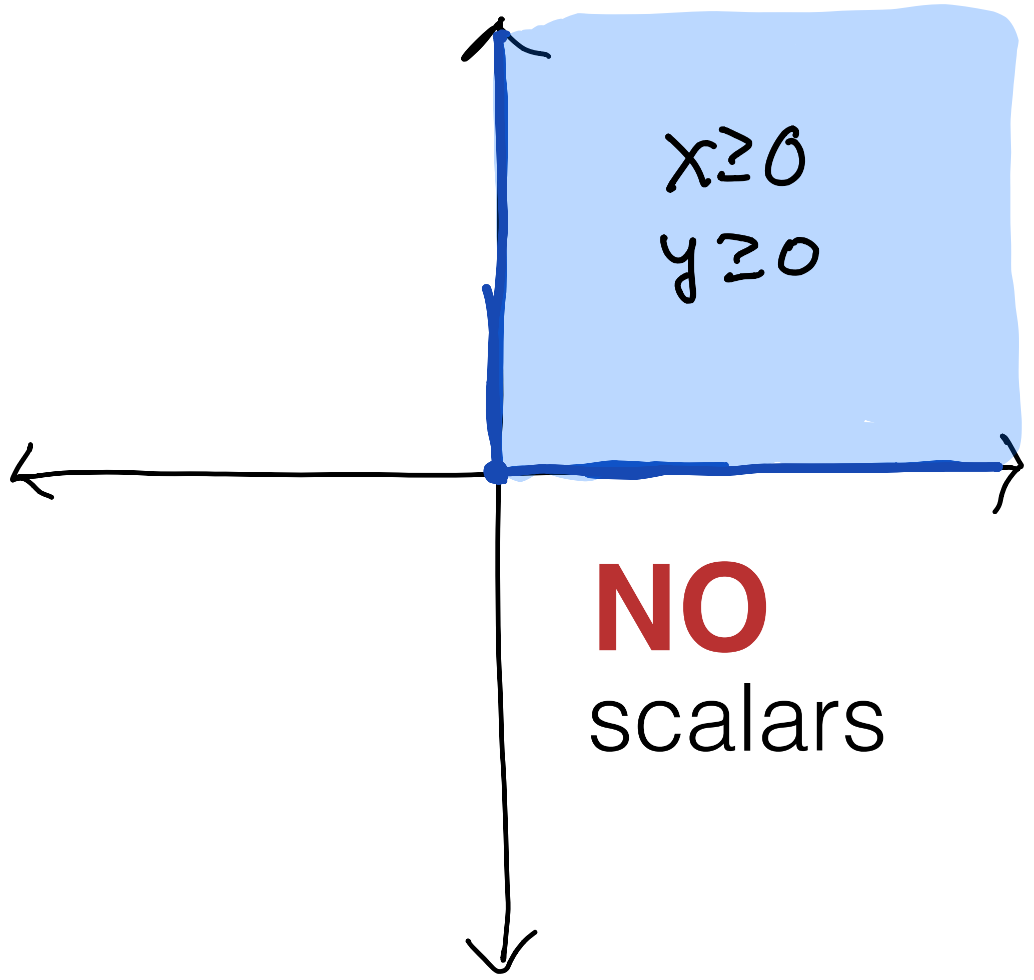
NO
addition



YES
but boring



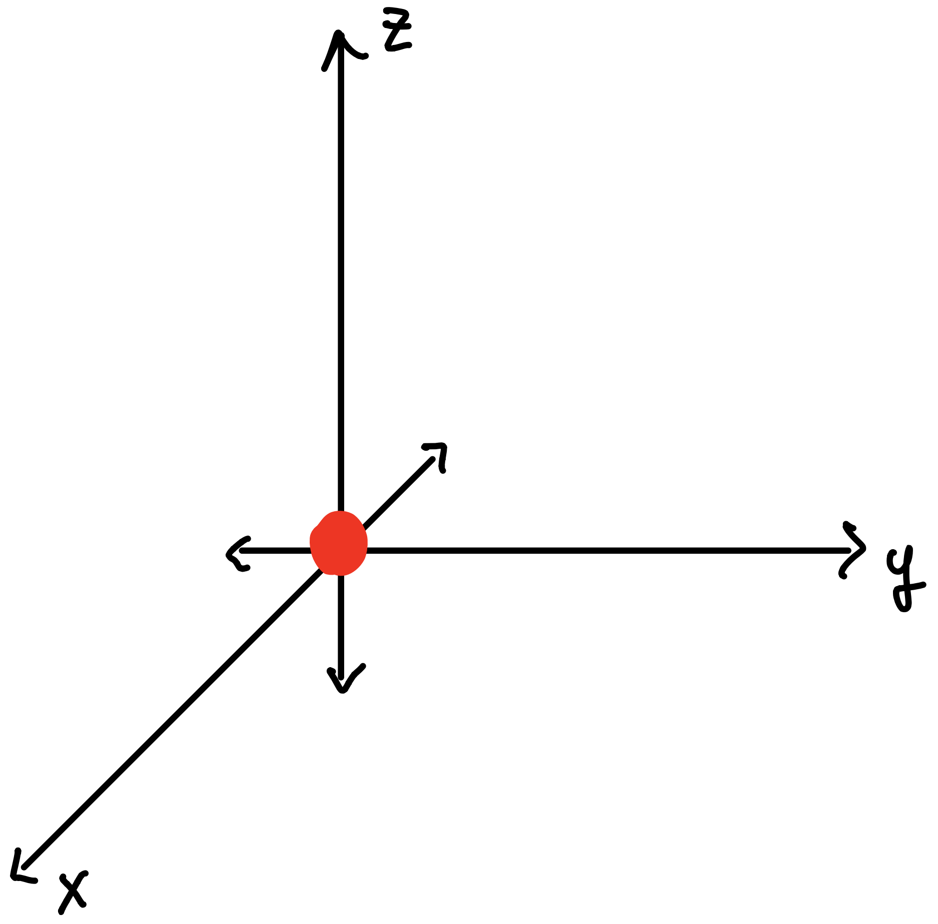
NO
scalars



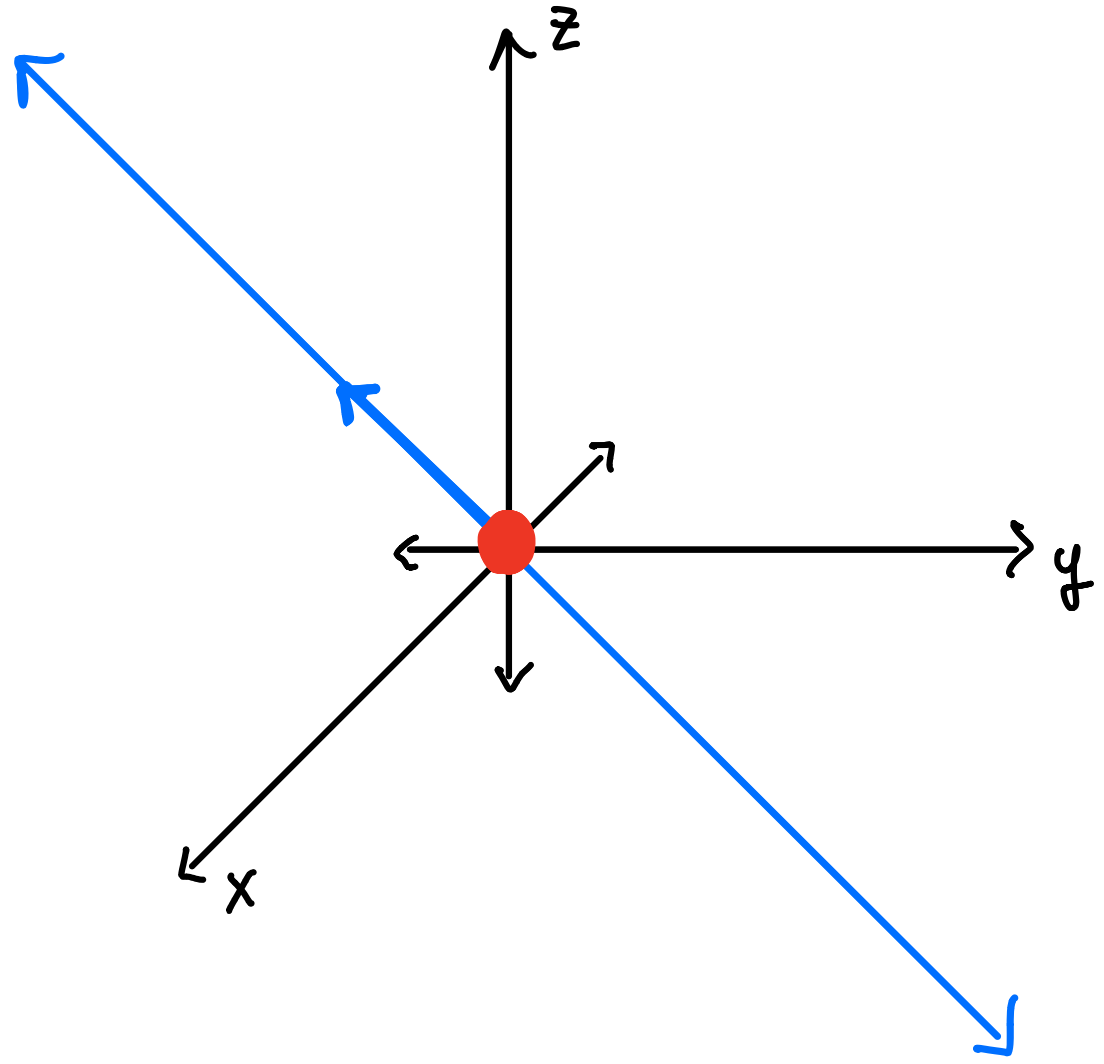
NO
scalars

Subspaces of $\mathbb{R}^3$

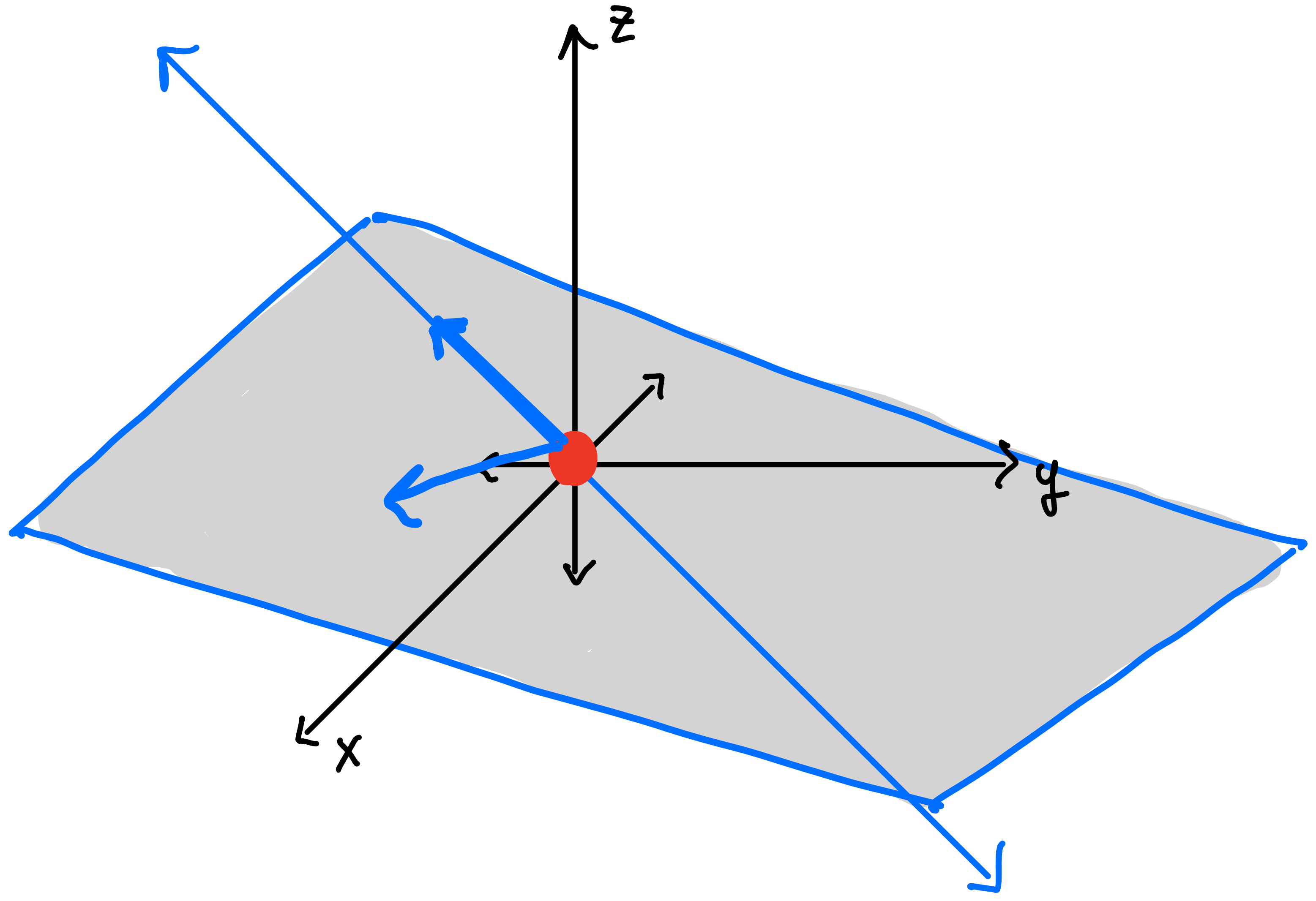
$S = \{\vec{0}\}$



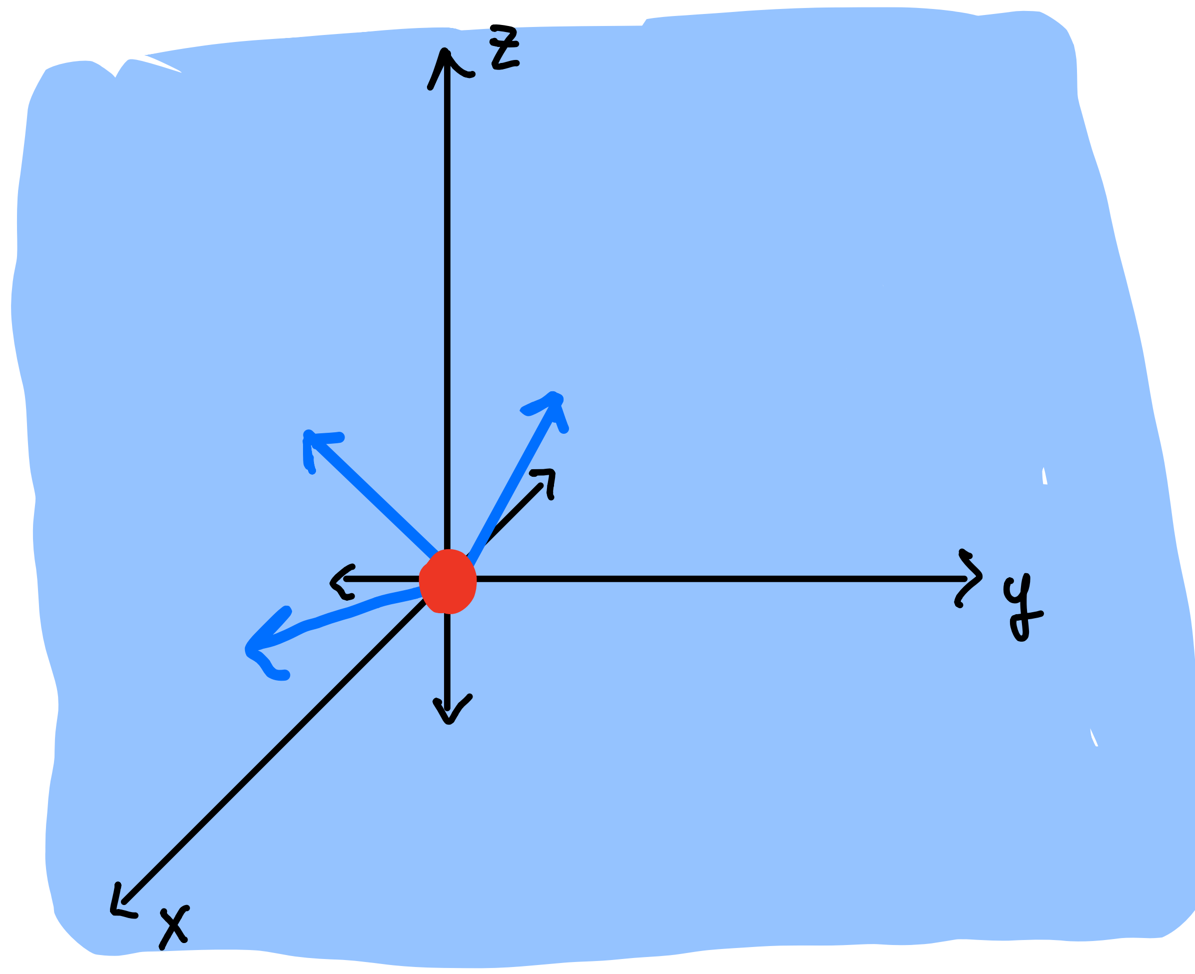
$S =$ line through the origin



$S =$ plane through the origin



$S =$ all of $\mathbb{R}^3$



Subspaces of $\mathbb{R}^n$

Decide whether the subsets below are subspaces.

If it is a subspace, prove that the three subspace properties hold true, if it is not a subspace, show that it is not by demonstrating that one of the rules is broken.

$$A = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \middle| \; a+b+c+d=0 \right\} \subseteq \mathbb{R}^4$$

$$\begin{bmatrix} 3 \\ -2 \\ 4 \\ -5 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ -4 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 2 \\ -8 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 5 \\ -9 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$C = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \middle| \; abcd=0 \right\} \subseteq \mathbb{R}^4$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 1 \\ -17 \end{bmatrix}$$

$$B = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \middle| \; a+b+c+d=1 \right\} \subseteq \mathbb{R}^4$$

The zero vector is not in this subset, so it is not a subspace

$$\begin{bmatrix} 3 \\ -2 \\ 4 \\ -4 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \cancel{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}$$

Subspaces of $\mathbb{R}^n$

Decide whether the subsets below are subspaces.

$$A = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \;\middle|\; a+b+c+d=0 \right\} \subseteq \mathbb{R}^4$$

$\boxed{\text{show that if } u, v \in S \text{ then } u+v \in S.}$ closed under +

choose $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}$ and $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$ in S. Then $a_1+a_2+a_3+a_4=0$, and $b_1+b_2+b_3+b_4=0$.

Then $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} a_1+b_1 \\ a_2+b_2 \\ a_3+b_3 \\ a_4+b_4 \end{bmatrix}$. check that this vector is in S by summing its coordinates

$$(a_1+b_1)+(a_2+b_2)+(a_3+b_3)+(a_4+b_4) = \underbrace{a_1+a_2+a_3+a_4}_{0} \underbrace{+b_1+b_2+b_3+b_4}_{0}$$

$$= 0+0=0$$

$\boxed{\text{show that if } u \in S \text{ then } cu \in S}$ closed under scalar multiplication

Let $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \in S$. Then $a_1+a_2+a_3+a_4=0$

Let $c \in \mathbb{R}$ be a scalar.

Then $c\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} ca_1 \\ ca_2 \\ ca_3 \\ ca_4 \end{bmatrix}$. check that this vector is in S.

$$ca_1+ca_2+ca_3+ca_4 = c(\underbrace{a_1+a_2+a_3+a_4}_{0})$$

$$= c\cdot 0 = 0$$

You try

Either that the three subspace properties hold true or show that it is not a subspace by demonstrating that one of the rules is broken.

$$C = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \;\middle|\; abcd = 0 \right\} \subseteq \mathbb{R}^4$$

Span

The **span** of a set of vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3, \dots, \vec{v}_p \in \mathbb{R}^n$

is the subset of all possible linear combinations of those vectors:

$$S = \operatorname{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \dots, \vec{v}_p\} = \{ c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p \mid c_1, \dots, c_p \in \mathbb{R} \}$$

$\boxed{\text{Example}}$

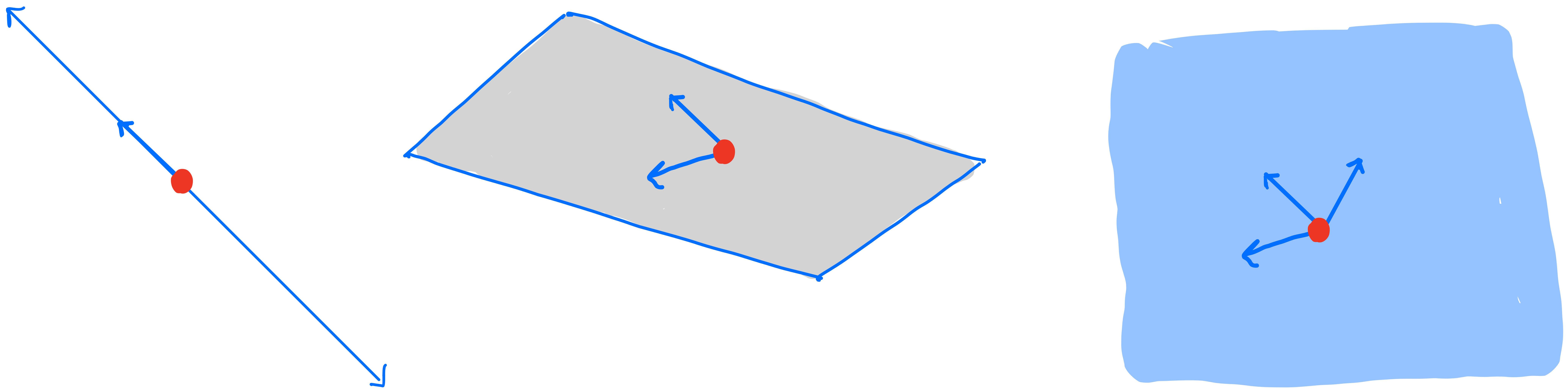
$$\operatorname{Span}\left(\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \right) = \left\{ a\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} + b\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \;\middle|\; a, b \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} a+b \\ -a+2b \\ a+3b \\ -a+4b \end{bmatrix} \;\middle|\; a, b \in \mathbb{R} \right\}$$

Span

The **span** of a set of vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots, \vec{v}_p \in \mathbb{R}^n$

is the subset of all possible linear combinations of those vectors:

$$S = \text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots, \vec{v}_p\} = \{c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p \mid c_1, \ldots, c_p \in \mathbb{R}\}$$



Span is a Subspace

$$S = \text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \dots, \vec{v}_p\} = \{c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p \mid c_1, \dots, c_p \in \mathbb{R}\}$$

Proof. $\boxed{\text{sums}}$ if $u = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p$ and

$v = d_1\vec{v}_1 + d_2\vec{v}_2 + \cdots + d_p\vec{v}_p$

Then $u + v = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p + d_1\vec{v}_1 + d_2\vec{v}_2 + \cdots + d_p\vec{v}_p = (c_1 + d_1)\vec{v}_1 + (c_2 + d_2)\vec{v}_2 + \cdots + (c_p + d_p)\vec{v}_p \in S$

$\boxed{\text{scalar multiples}}$ if $u = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p$ and $c \in \mathbb{R}$

then $cu = (c\,c_1)\vec{v}_1 + (c\,c_2)\vec{v}_2 + \cdots + (c\,c_p)\vec{v}_p$