Vector Spaces: The set of n-dimensional vectors $\mathbb{R}^n = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \middle| a_1, a_2, \dots, a_n \in \mathbb{R} \right\}$ forms a vector space.

A set V is a *vector space* if it satisfies the following rules for all vectors $\vec{u}, \vec{v}, w \in V$ and all scalars $c, d \in \mathbb{R}$.

1. $\vec{u} + \vec{v} \in V$,
2. $c\vec{u} \in V$,
3. $\vec{v} + \mathbf{0} = \mathbf{0} + \vec{v} = \vec{v}$,
4. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$,
5. $(\vec{u} + \vec{v}) + w = \vec{u} + (\vec{v} + w)$,
6. $\vec{u} + (-\vec{u}) = \mathbf{0}$,
7. $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$,
8. $(c + d)\vec{u} = c\vec{u} + d\vec{u}$,
9. $(cd)\vec{u} = c(d\vec{u})$,
10. $1\vec{u} = \vec{u}$,
11. $0\vec{u} = \mathbf{0}$.

Subspaces: We are interested in subsets $S \subseteq \mathbb{R}^n$ that are also vector spaces. These are called *subspaces*. To be a subspace, S must satisfy three properties:

0. $\mathbf{0} \in S$ (contains the zero vector)
1. If $\vec{u}, \vec{v} \in S$ then $\vec{u} + \vec{v} \in S$ (closed under addition)
2. If $\vec{u} \in S$ and $c \in \mathbb{R}$, then $c\vec{u} \in S$ (closed under scalar multiplication).

From these two properties, we can also derive: (3) $c\vec{u} + d\vec{v} \in S$ (closed under linear combinations).

Three Important Examples

A. The set of solutions to $Ax = \mathbf{0}$ forms a subspace: $S = \{ \vec{x} \mid A\vec{x} = \mathbf{0} \}$

0. $A\vec{0} = \vec{0}$, so $\boxed{\vec{0} \in S}$

1. if $x_1, x_2 \in S$ then $Ax_1 = 0$ and $Ax_2 = 0$
show $x_1 + x_2 \in S$: $A(x_1 + x_2) = Ax_1 + Ax_2 = 0 + 0 = 0 \Rightarrow \boxed{x_1 + x_2 \in S}$

2. if $x_1 \in S$, then $Ax_1 = 0$
show $c_1 x \in S$: $A(cx_1) = cAx_1 = c\vec{0} = \vec{0}. \Rightarrow \boxed{cx_1 \in S}$

B. The set of solutions to $\boxed{Ax = \vec{b}}$ (with $\vec{b} \neq \mathbf{0}$) does *not* form a subspace: $S = \{ \vec{x} \mid A\vec{x} = \vec{b} \}$

$\cancel{0}$ $A\vec{0} = \vec{0} \neq b$

$\cancel{1}$ $A(x_1 + x_2) = Ax_1 + Ax_2 = b + b = 2b \neq b$

$\cancel{2}$ $A(cx_1) = cAx_1 = cb \neq b$

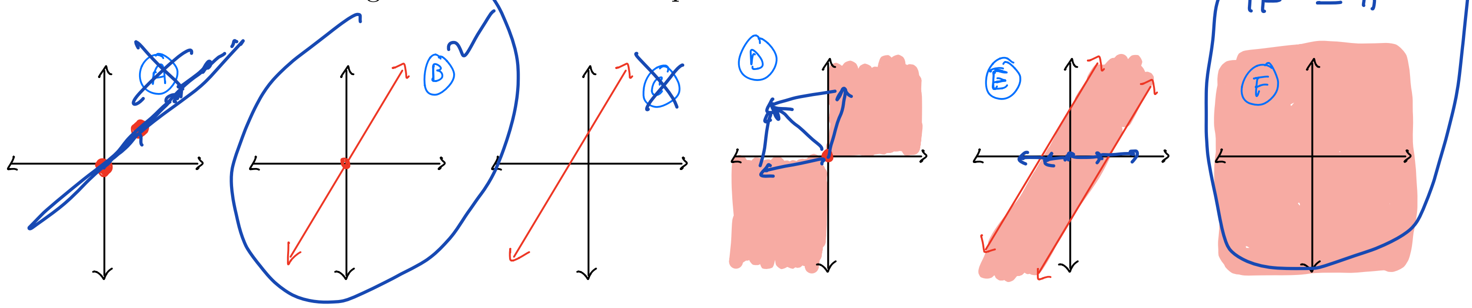
C. The span of a set of vectors forms a subspace: $S = \text{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\} = \boxed{\{x_1\vec{v}_1 + x_2\vec{v}_2 + \dots + x_n\vec{v}_n \mid x_i \in \mathbb{R}\}}$

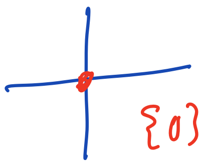
0. $0 = 0v_1 + 0v_2 + \dots + 0v_n$

1. $$\begin{array}{r} x_1 v_1 + x_2 v_2 + \dots + x_n v_n \\ + \; y_1 v_1 + y_2 v_2 + \dots + y_n v_n \\ \hline (x_1 + y_1) v_1 + (x_2 + y_2) v_2 + \dots + (x_n + y_n) v_n \end{array}$$

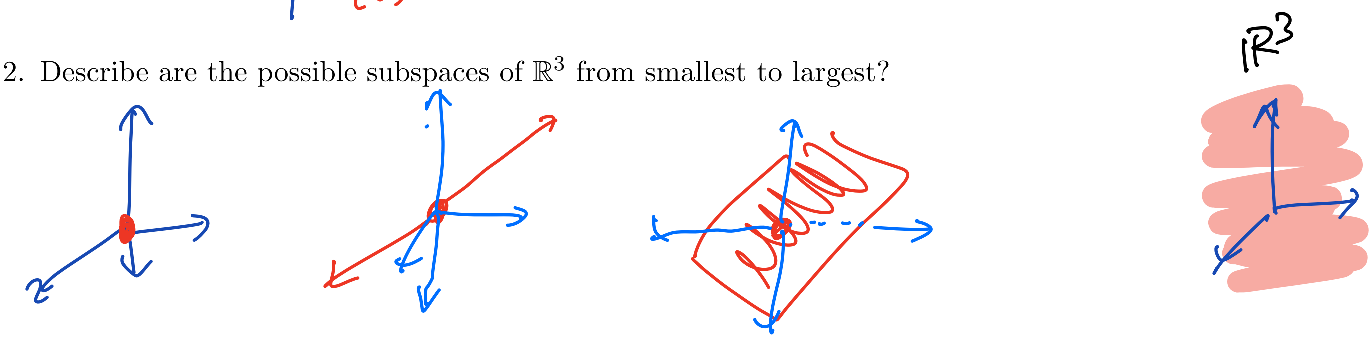
2. $c(x_1 v_1 + x_2 v_2 + \dots + x_n v_n) = (cx_1) v_1 + (cx_2) v_2 + \dots + (cx_n) v_n$

1. Which of the following subsets of $\mathbb{R}^2$ are subspaces?





2. Describe are the possible subspaces of $\mathbb{R}^3$ from smallest to largest?



3. Which of the following subsets of $\mathbb{R}^3$ are subspaces?

$$A = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \;\middle|\; x_1x_2x_3 \geq 0 \right\}, \quad \boxed{B = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \;\middle|\; x_1^2 + x_2^2 + x_3^2 \leq 1 \right\}}, \quad C = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \;\middle|\; \underline{5x_1 - 2x_2 - x_3 = 0} \right\}.$$

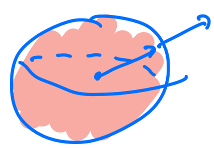
(a) For the ones that are not subspaces, give a specific example of vectors that break rules 1 and 2 above.

(b) For those that are, show that the rules above are satisfied for any vectors $\vec{u}, \vec{v}$ and any scalar c.

$\cancel{A}$ $\quad (-1)\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$

in $\quad$ out

$\cancel{B}$



$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix}$$

$5x_1 - 2x_2 - x_3 = 0 \qquad 5y_1 - 2y_2 - y_3 = 0$

$$5(x_1+y_1) - 2(x_2+y_2) - (x_3+y_3)$$
$$= \underbrace{5x_1 - 2x_2 - x_3} + \underbrace{5y_1 - 2y_2 - y_3}$$
$$= 0 + 0$$
$$= 0$$

$$c\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} cx_1 \\ cx_2 \\ cx_3 \end{bmatrix}$$

$$5(cx_1) - 2(cx_2) - (cx_3) = c(5x_1 - 2x_2 - x_3)$$
$$= c \cdot 0 = 0$$

4. Decide if the following subsets are subspaces. As in the previous examples, if it is not, give a specific example of vectors that break rules 1 and 2 above, and if it is, show that the rules above are satisfied for any vectors $\vec{u}, \vec{v}$ and any scalar c.

(a) $S = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \middle| \, b = 2a, \; c = 2b, \; d = 2c \right\} \subseteq \mathbb{R}^4$

① closed under addition

$$\begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{bmatrix} + \begin{bmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 \\ b_1 + b_2 \\ c_1 + c_2 \\ d_1 + d_2 \end{bmatrix}$$

$b_1 = 2a_1$, $c_1 = 2b_1$, $d_1 = 2c_1$

$b_2 = 2a_2$, $c_2 = 2b_2$, $d_2 = 2c_2$

$2(a_1+a_2) = 2a_1 + 2a_2 = b_1 + b_2$

$2(b_1+b_2) = 2b_1 + 2b_2 = c_1 + c_2$

$2(c_1+c_2) = 2c_1 + 2c_2 = d_1 + d_2$ ✓

② closed under scalar multiplication

$$k \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} ka \\ kb \\ kc \\ kd \end{bmatrix}$$

$b = 2a$, $c = 2b$, $d = 2c$

$2(ka) = k2a = kb$

$2(kb) = k2b = kc$

$2(kc) = k(2c) = kd$

(b) $S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \middle| \, x_3 = \max(x_1, x_2) \right\} \subseteq \mathbb{R}^3$

not a subspace:

eg. $\begin{bmatrix} 1 \\ 5 \\ 5 \end{bmatrix} + \begin{bmatrix} 9 \\ 3 \\ 9 \end{bmatrix} = \begin{bmatrix} 10 \\ 8 \\ 14 \end{bmatrix}$ ← specific counter example

in S in S not in S

(c) Here $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation and T is the set of *fixed points*:

$S = \{\vec{x} \in \mathbb{R}^n \mid T(\vec{x}) = \vec{x}\} \subseteq \mathbb{R}^n$

0. $T(0) = 0 \Rightarrow 0 \in S$

1. if $x_1, x_2 \in S$ then $T(x_1) = x_1$ and $T(x_2) = x_2$
show $x_1 + x_2 \in S$:
$T(x_1 + x_2) = T(x_1) + T(x_2) = x_1 + x_2 \Rightarrow x_1 + x_2 \in S$

2. If $x \in S$, then $T(x) = x$. Show $cx \in S$: $T(cx) = cT(x) = cx$ $\Rightarrow cx \in S$. ∎

(d) Here $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation and S is the image:

$S = \{\vec{y} \in \mathbb{R}^m \mid T(\vec{x}) = \vec{y} \text{for some } \vec{x} \in \mathbb{R}^n\} \subseteq \mathbb{R}^m$

0. $T(\vec{0}) = \vec{0} \Rightarrow \vec{0} \in S$.

1. If $y_1, y_2 \in S$ then $T(x_1) = y_1$ and $T(x_2) = y_2$ for some $x_1, x_2 \in \mathbb{R}^n$
show $y_1 + y_2 \in S$. Indeed, $T(x_1 + x_2) = T(x_1) + T(x_2) = y_1 + y_2$
$\Rightarrow y_1 + y_2 \in S$.

2. If $y \in S$, then $T(x) = y$ for some $x \in \mathbb{R}^n$
show $cy \in S$. Behold, $T(cx) = cT(x) = cy \Rightarrow cy \in S$ ∎

(e) We can encode polynomials $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ with vectors as follows:

$$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \qquad = \qquad \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

For example, if we consider the set $\mathcal{P}_3$ of polynomials of degree 3 or less, here are some examples of their encoding as 4 dimensional vectors in $\mathbb{R}^4$:

$$1+2x+3x^2+4x^3 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \qquad x = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \qquad 2-5x^2-3.14x^3 = \begin{bmatrix} 2 \\ 0 \\ -5 \\ 3.14 \end{bmatrix}, \qquad 1+x-x^2+x^3 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}.$$

Notice that the polynomials in $\mathcal{P}_n$, of degree n or less, correspond to $\mathbb{R}^{n+1}$.

Observe that polynomial addition is the same as the corresponding vector addition:

$$\begin{array}{rcl} p(x) & = & 1+2x+3x^2+4x^3 \\ q(x) & = & 1+\ x-\ x^2+\ x^3 \\ \hline p(x)+q(x) & = & 2+3x+2x^2+5x^3 \end{array} \quad \text{in } \mathcal{P}_3 \qquad \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 2 \\ 5 \end{bmatrix} \quad \text{in } \mathbb{R}^4.$$

and scalar multiplication of polynomials is the same as scalar multiplication of vectors:

$$2p(x) = 2+4x+6x^2+8x^3 \quad \text{in } \mathcal{P}_3 \qquad 2\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 6 \\ 8 \end{bmatrix} \quad \text{in } \mathbb{R}^4.$$

Thus $\mathcal{P}_n$ is really the same as $\mathbb{R}^{n+1}$. The only difference is cosmetic. We say that $\mathcal{P}_n$ and $\mathbb{R}^{n+1}$ are *isomorphic* vector spaces.

Here are two subsets of $\mathcal{P}_4$. Decide if they are subspaces. In each case, if it is not a subspace, give examples using specific polynomials to show that one of the rules is broken, and if it is a subspace, show that the subspace rule holds for any two polynomials $p(x), q(x)$ and any constant $c \in \mathbb{R}$.

i. $U = \{p(x) \in \mathcal{P}_4 \mid p(1) = 0\}$

① If $p(x), q(x) \in U$, then $p(1)=0$ and $q(1)=0$.

Show $p(x)+q(x) \in U$. Indeed $p(1)+q(1) = 0+0=0$

$\Rightarrow p(x)+q(x) \in U$

② Show $cp(x) \in U$. Indeed, $cp(1) = c0 = 0$

$\Rightarrow cp(x) \in U$

ii. $V = \{p(x) \in \mathcal{P}_4 \mid p(0) = 1\}$

If $p(x), q(x) \in V$ then

$p(0)=1$ and $q(0)=1$

not a subspace

But $p(0)+q(0) = 1+1 = 2 \neq 1$

$\Rightarrow p(x)+q(x) \notin V$.