4.4. Coordinates

Basis

Definition

A set of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ is a **basis** for a subspace $S \subseteq \mathbb{R}^n$ when

1. $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ are linearly independent
2. $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ span S

Bases

Which of these are bases of $\mathbb{R}^3$?

(a) $\mathcal{S} = \left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}$ Yes! Standard basis

Put in matrix and row reduce

(b) $\mathcal{B}_1 = \left\{ \begin{bmatrix}1\\-2\\3\end{bmatrix}, \begin{bmatrix}2\\-3\\1\end{bmatrix}, \begin{bmatrix}1\\1\\-2\end{bmatrix} \right\}$

$$\begin{bmatrix}1 & 2 & 1\\-2 & -3 & 1\\3 & 1 & -2\end{bmatrix} \longrightarrow \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

Yes! Must get identity matrix

(c) $\mathcal{B}_2 = \left\{ \begin{bmatrix}1\\2\\-1\end{bmatrix}, \begin{bmatrix}2\\-1\\2\end{bmatrix}, \begin{bmatrix}-1\\3\\-3\end{bmatrix} \right\}$

linearly dependent

$$\begin{bmatrix}1 & 2 & -1\\2 & -1 & 3\\-1 & 2 & -3\end{bmatrix} \longrightarrow \begin{bmatrix}1 & 0 & 1\\0 & 1 & -1\\0 & 0 & 0\end{bmatrix}$$

No!

do not span

(d) $\mathcal{B}_3 = \left\{ \begin{bmatrix}2\\3\\2\end{bmatrix}, \begin{bmatrix}3\\-3\\-1\end{bmatrix}, \begin{bmatrix}-2\\-1\\3\end{bmatrix}, \begin{bmatrix}-3\\3\\-1\end{bmatrix} \right\}$

$$\begin{bmatrix}2 & 3 & -2 & -3\\3 & -3 & -1 & 3\\2 & -1 & 3 & -1\end{bmatrix} \longrightarrow \begin{bmatrix}1 & 0 & 0 & -\frac{18}{59}\\0 & 1 & 0 & -\frac{67}{59}\\0 & 0 & 1 & -\frac{30}{59}\end{bmatrix}$$

Too many vectors

(e) $\mathcal{B}_4 = \left\{ \begin{bmatrix}1\\-1\\2\end{bmatrix}, \begin{bmatrix}4\\2\\-1\end{bmatrix} \right\}$

$$\begin{bmatrix}1 & 4\\-1 & 2\\2 & -1\end{bmatrix} \longrightarrow \begin{bmatrix}1 & 0\\0 & 1\\0 & 0\end{bmatrix}$$

Too few vectors

Bases of $\mathbb{R}^n$

Theorem A. Any basis of $\mathbb{R}^n$ has n vectors

B. More than n vectors must be linearly dependent

C. Fewer than n vectors cannot span.

Proof. Put the vectors in a matrix and row reduce.

$$\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_p \\ | & | & & | \end{bmatrix} \longrightarrow$$

cannot have free variables or it will be dependent

cant have a row of zeros or it wont span

must have a pivot in every row and column ⇒ square

has to have fv's

more than n (wide) → dependent

Fewer than n (tall) → cant span

$$\begin{matrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{matrix}$$

← must have at least one row of 0's

Coordinates

Below are three bases of $\mathbb{R}^3$

Express the vector v in each of these bases. $\mathbf{v} = \begin{bmatrix} 2 \\ -3 \\ 11 \end{bmatrix}$

Coordinates of v in the 3 bases

$$\mathcal{S} = \left\{ \underset{\mathbf{e}_1}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}, \underset{\mathbf{e}_2}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}, \underset{\mathbf{e}_3}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} \right\} \qquad \mathbf{v} = 2\mathbf{e}_1 - 3\mathbf{e}_2 + 11\mathbf{e}_3 \qquad \mathbf{v} = \begin{bmatrix} 2 \\ -3 \\ 11 \end{bmatrix}_{\mathcal{S}}$$

$$\mathbf{v} = 5\mathbf{v}_1 - 2\mathbf{v}_2 + \mathbf{v}_3$$

$$\mathcal{B}_1 = \left\{ \underset{\mathbf{v}_1}{\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}}, \underset{\mathbf{v}_2}{\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}}, \underset{\mathbf{v}_3}{\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}} \right\} \qquad \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ -2 & -3 & 1 & -3 \\ 3 & 1 & -2 & 11 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right] \qquad \mathbf{v} = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}_{\mathcal{B}_1}$$

$$\mathbf{v} = 0\mathbf{u}_1 + \mathbf{u}_2 + 0\mathbf{u}_3 = \mathbf{u}_2$$

$$\mathcal{B}_2 = \left\{ \underset{\mathbf{u}_1}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}, \underset{\mathbf{u}_2}{\begin{bmatrix} 2 \\ -3 \\ 11 \end{bmatrix}}, \underset{\mathbf{u}_3}{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}} \right\} \qquad \left[\begin{array}{ccc|c} 1 & 2 & 0 & 2 \\ 1 & -3 & 1 & -3 \\ 0 & 11 & 1 & 11 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right] \qquad \mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}_{\mathcal{B}_2}$$

Changing Coordinates (Sec 4.7)

The matrix with basis vectors down the column, is the **change of basis matrix**.

$$\mathcal{B}_1 = \left\{ \begin{array}{c}\mathbf{v}_1\\ \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}\end{array}, \begin{array}{c}\mathbf{v}_2\\ \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}\end{array}, \begin{array}{c}\mathbf{v}_3\\ \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}\end{array} \right\} \qquad B = \begin{bmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 3 & 1 & -2 \end{bmatrix}$$

It converts from B coordinates to standard coordinates:

$$\begin{array}{ccc}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3\end{array}$$
$$\begin{bmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}_{\mathcal{B}_1} = 5\mathbf{v}_1 - 2\mathbf{v}_2 + \mathbf{v}_3 = \begin{bmatrix} 2 \\ -3 \\ 11 \end{bmatrix}_{\mathcal{S}}$$

Its ***inverse*** converts from standard coordinates to B coordinates:

$$B^{-1} = \frac{1}{10} \begin{bmatrix} 5 & 5 & 5 \\ -1 & -5 & -3 \\ 7 & 5 & 1 \end{bmatrix} \qquad \frac{1}{10} \begin{bmatrix} 5 & 5 & 5 \\ -1 & -5 & -3 \\ 7 & 5 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ 11 \end{bmatrix}_{\mathcal{S}} = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}_{\mathcal{B}_1}$$

computed using software

$$B = \begin{array}{c c} & \begin{array}{ccc}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3\end{array} \\ \begin{array}{c}\mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\end{array} & \begin{bmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 3 & 1 & -2 \end{bmatrix}\end{array}$$

$$B^{-1} = \begin{array}{c c} & \begin{array}{ccc}\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3\end{array} \\ \begin{array}{c}\mathbf{v}_1\\ \mathbf{v}_2\\ \mathbf{v}_3\end{array} & \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{10} & -\frac{1}{2} & -\frac{3}{10} \\ \frac{7}{10} & \frac{1}{2} & \frac{1}{10} \end{bmatrix}\end{array}$$

Coordinates in the Plane

$$\mathcal{S} = \left\{ \mathbf{e}_1 = \begin{bmatrix}1\\0\end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix}0\\1\end{bmatrix} \right\}$$

$$\mathcal{B} = \left\{ \mathbf{v}_1 = \begin{bmatrix}1\\3\end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix}-1\\5\end{bmatrix} \right\}$$

Express $\vec{\mathbf{w}} = \begin{bmatrix}11\\9\end{bmatrix}$ in each basis

$$\vec{\mathbf{w}} = \begin{bmatrix}11\\9\end{bmatrix} = 11\begin{bmatrix}1\\0\end{bmatrix} + 9\begin{bmatrix}0\\1\end{bmatrix} = 11\mathbf{e}_1 + 9\mathbf{e_2} = \begin{bmatrix}11\\9\end{bmatrix}_{\mathcal{S}}$$

$$\left[\begin{array}{cc|c} 1 & -1 & 11 \\ 3 & 5 & 9 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 8 \\ 0 & 1 & -3 \end{array}\right]$$

$$\vec{\mathbf{w}} = \begin{bmatrix}11\\9\end{bmatrix} = 8\begin{bmatrix}1\\3\end{bmatrix} - 3\begin{bmatrix}-1\\5\end{bmatrix} = 8\mathbf{v}_1 - 3\mathbf{v}_2 = \begin{bmatrix}8\\-3\end{bmatrix}_{\mathcal{B}}$$

Using Change-of-Basis Matrices

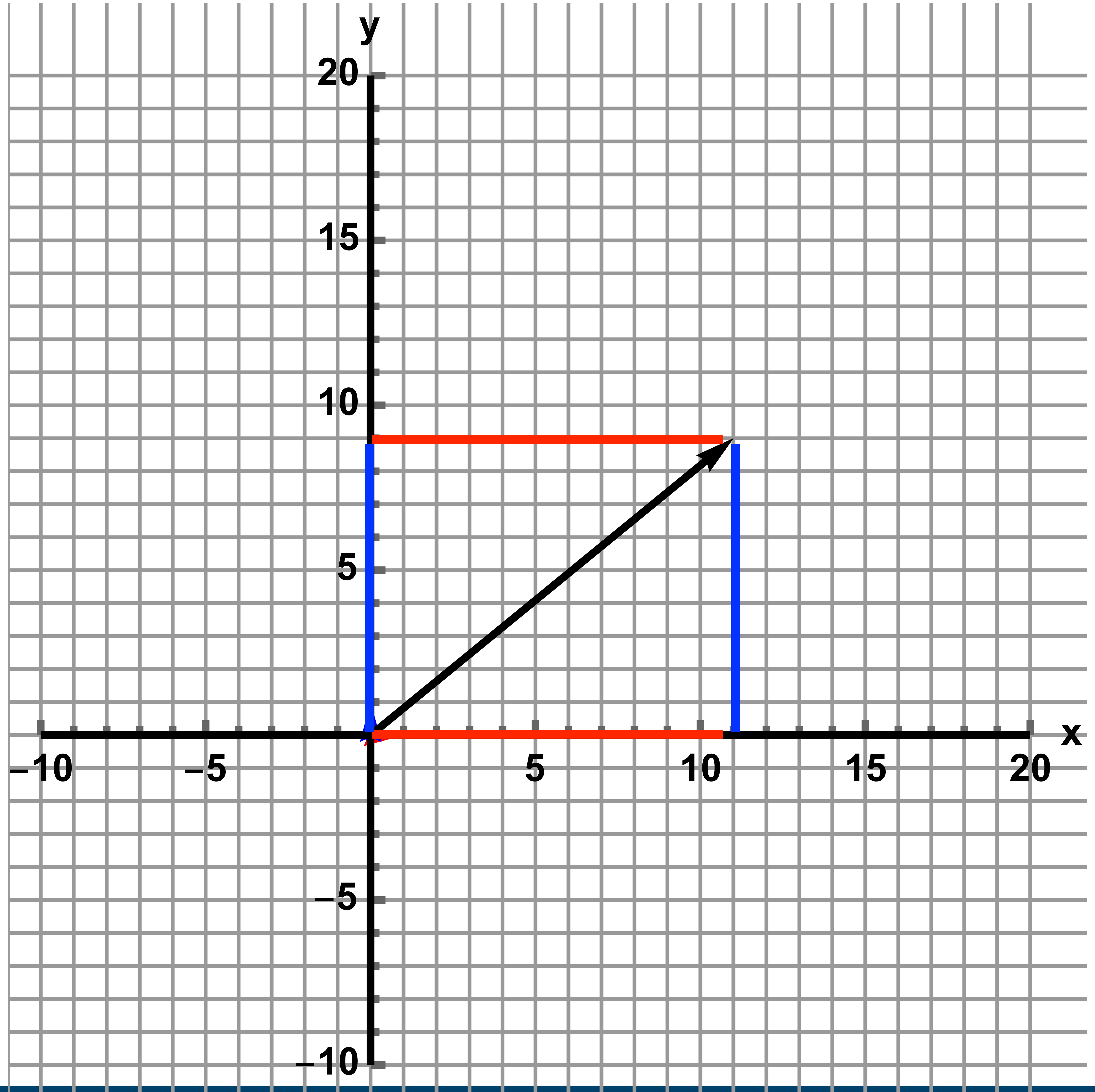
$$B = \underbrace{\begin{bmatrix}1 & -1\\3 & 5\end{bmatrix}}_{\mathcal{B}\to\mathcal{S}}$$

$$\begin{bmatrix}1 & -1\\3 & 5\end{bmatrix}\begin{bmatrix}8\\-3\end{bmatrix}_{\mathcal{B}} = 8\begin{bmatrix}1\\3\end{bmatrix} - 3\begin{bmatrix}-1\\5\end{bmatrix} = \begin{bmatrix}11\\9\end{bmatrix}_{\mathcal{S}}$$

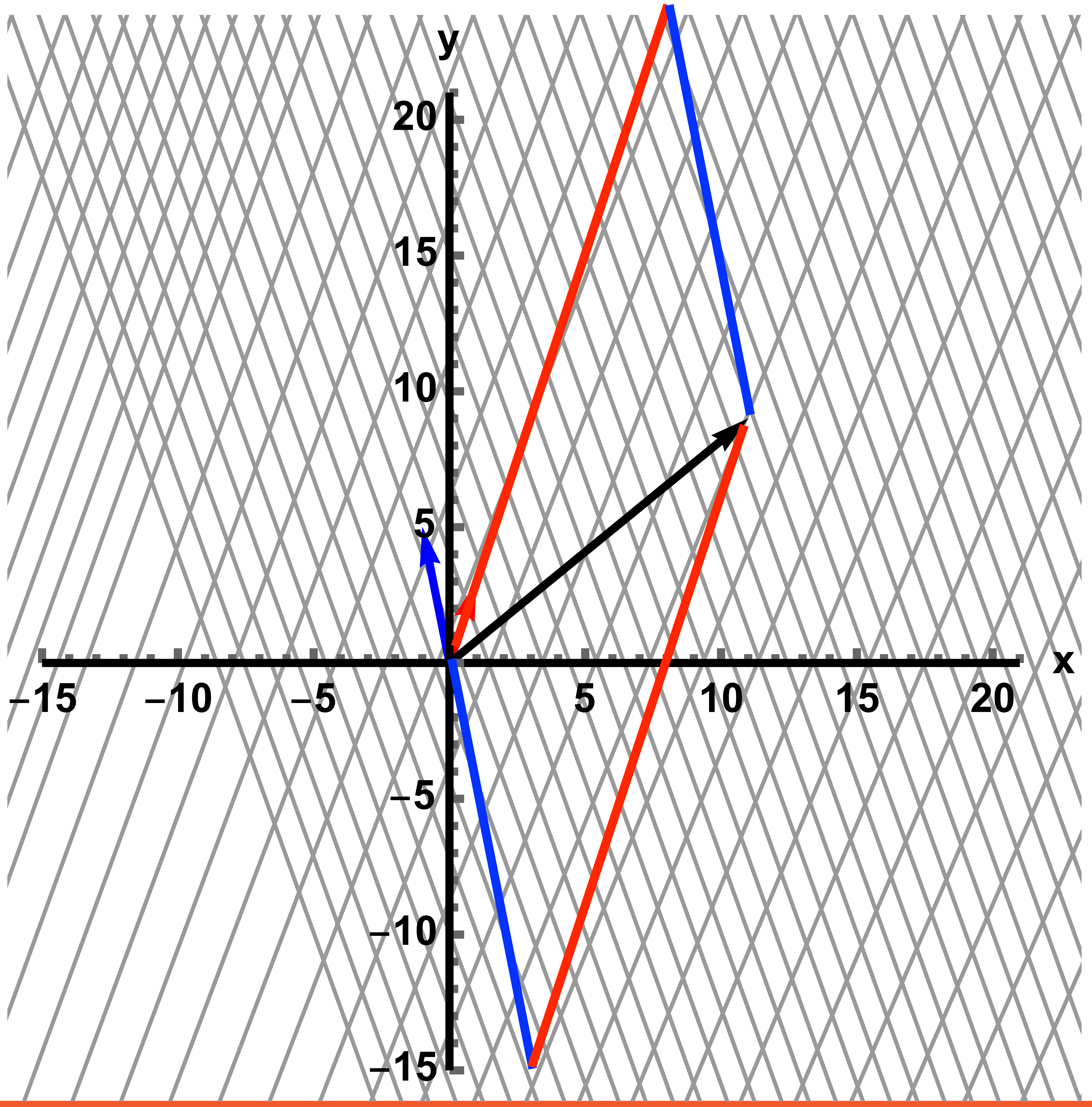
$$B^{-1} = \frac{1}{8}\underbrace{\begin{bmatrix}5 & 1\\-3 & 1\end{bmatrix}}_{\mathcal{S}\to\mathcal{B}}$$

$$\frac{1}{8}\begin{bmatrix}5 & 1\\-3 & 1\end{bmatrix}\begin{bmatrix}11\\9\end{bmatrix}_{\mathcal{S}} = \frac{1}{8}\begin{bmatrix}64\\-24\end{bmatrix} = \begin{bmatrix}8\\-3\end{bmatrix}_{\mathcal{B}}$$

$$\mathcal{S} = \left\{ \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \qquad \vec{\mathbf{w}} = \begin{bmatrix} 11 \\ 9 \end{bmatrix}_{\mathcal{S}}$$



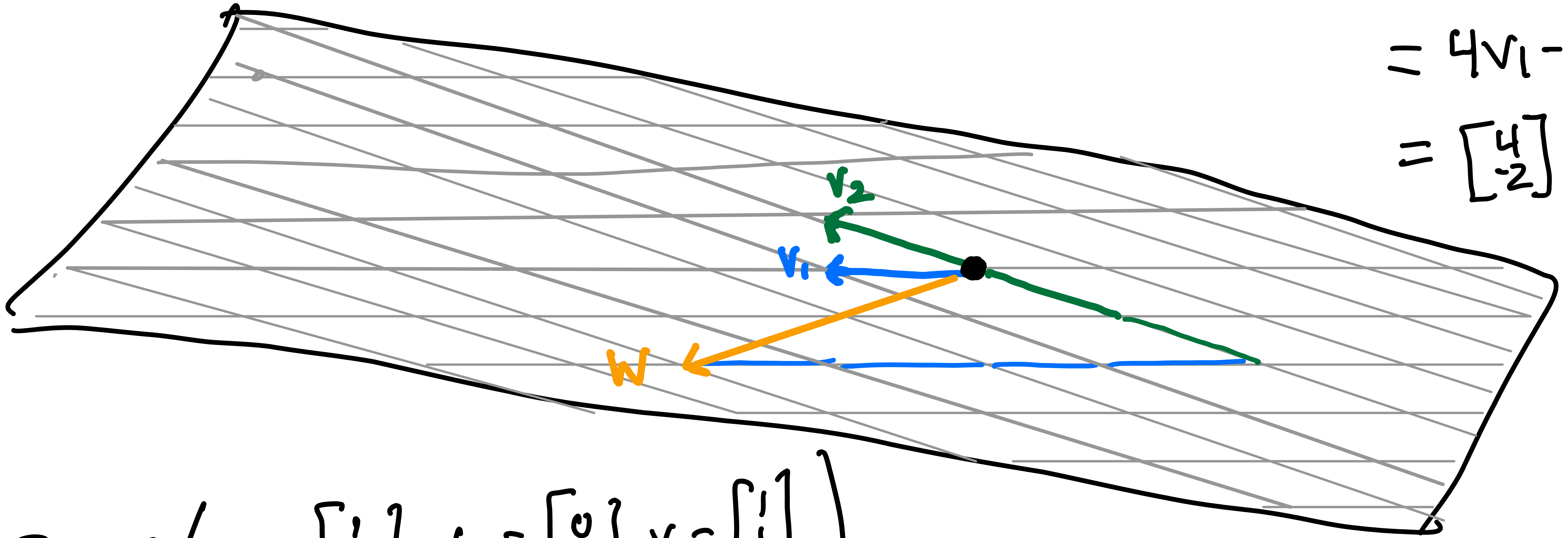
$$\mathcal{B} = \left\{ \mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1 \\ 5 \end{bmatrix} \right\} \qquad \vec{\mathbf{w}} = \begin{bmatrix} 8 \\ -3 \end{bmatrix}_{\mathcal{B}}$$



Example in 3D

$$Z = \text{span}\left(v_1 = \begin{bmatrix}1\\0\\-1\end{bmatrix}, v_2 = \begin{bmatrix}0\\1\\-1\end{bmatrix}\right) \subseteq \mathbb{R}^3$$

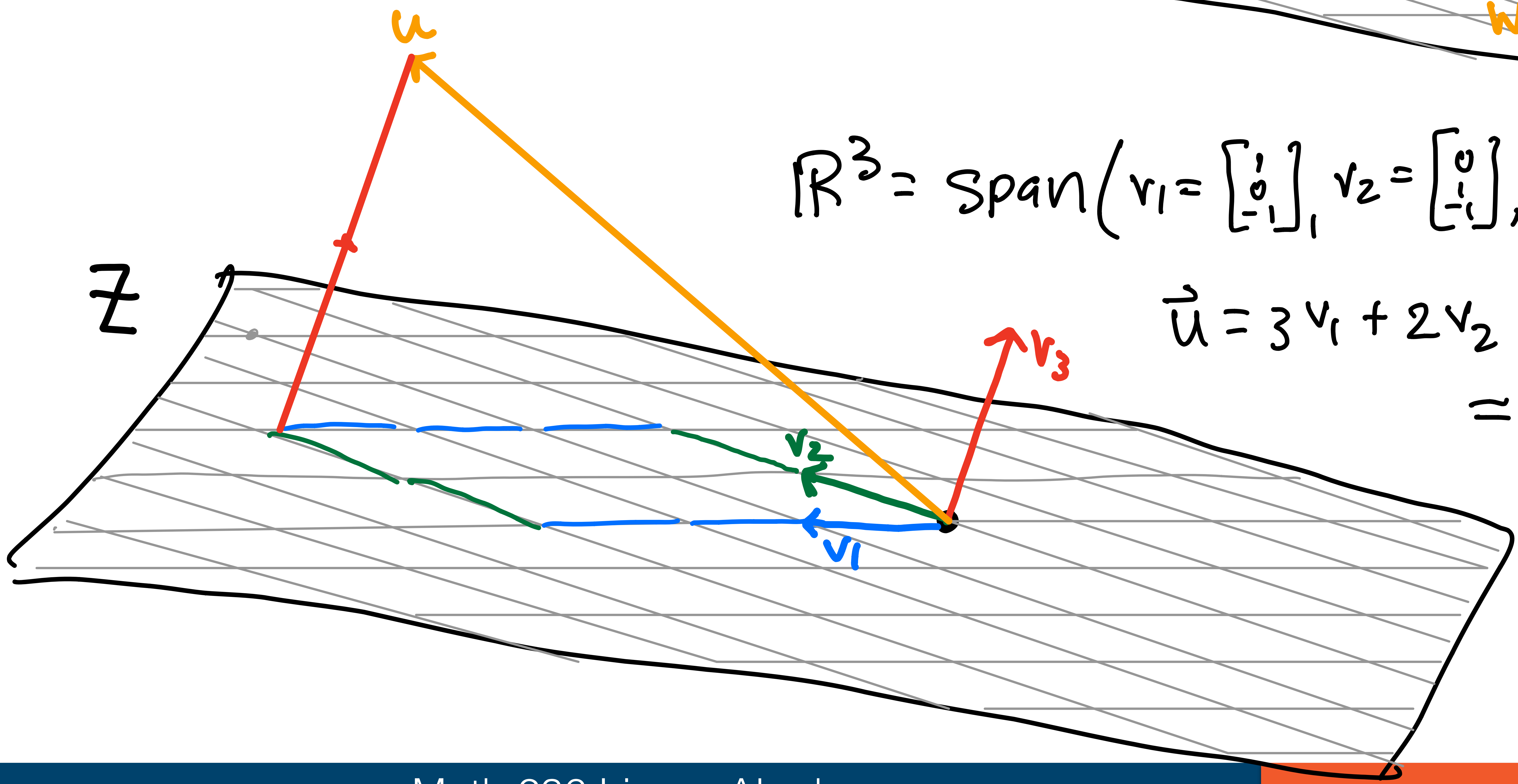
$$w = \begin{bmatrix}4\\-2\\-2\end{bmatrix} = 4v_1 - 2v_2 = \begin{bmatrix}4\\-2\end{bmatrix}_{\mathcal{B}}$$



$$\mathbb{R}^3 = \text{span}\left(v_1 = \begin{bmatrix}1\\0\\-1\end{bmatrix}, v_2 = \begin{bmatrix}0\\1\\-1\end{bmatrix}, v_3 = \begin{bmatrix}1\\1\\1\end{bmatrix}\right)$$

$$\vec{u} = 3v_1 + 2v_2 + 2v_3 = \begin{bmatrix}3\\2\\2\end{bmatrix}_{\mathcal{U}}$$

Z



Unique Representation Theorem

THEOREM 7

p. 216

The Unique Representation Theorem

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V. Then for each $\mathbf{x}$ in V, there exists a unique set of scalars $c_1, \ldots, c_n$ such that

$$\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n \tag{1}$$

i.e., the B-coordinates are unique

Proof.

$$x = c_1 b_1 + c_2 b_2 + c_3 b_3 + \cdots + c_n b_n$$

$$x = d_1 b_1 + d_2 b_2 + d_3 b_3 + \cdots + d_n b_n$$

subtract

$$0 = (c_1 - d_1) b_1 + (c_2 - d_2) b_2 + \cdots + (c_n - d_n) b_n$$

linear independence $\Rightarrow$

$$\begin{aligned} c_1 - d_1 &= 0 \\ c_2 - d_2 &= 0 \\ &\vdots \\ c_n - d_n &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} c_1 &= d_1 \\ c_2 &= d_2 \\ &\vdots \\ c_n &= d_n \end{aligned}$$

$\blacksquare$

span means I can get to x

linear independence means I can get to x uniquely

Equivalent Definition of Basis

Definition

A set of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ is a **basis** for a subspace $S \subseteq \mathbb{R}^n$ when

1. $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ are linearly independent
2. $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ span S

Definition

A set of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ is a **basis** for a subspace $S \subseteq \mathbb{R}^n$ if for every vector $\vec{v} \in S$ there is a *unique* set of weights $c_1, c_2, \ldots, c_k \in \mathbb{R}$ so that $\vec{v}$ can be written as a linear combination

$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k.$$

You Try

Below are two bases of $\mathbb{R}^3$ from earlier in these slides.

$$\mathcal{S} = \left\{ \overset{\mathbf{e}_1}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}, \overset{\mathbf{e}_2}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}, \overset{\mathbf{e}_3}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} \right\} \qquad \mathcal{B}_1 = \left\{ \overset{\mathbf{v}_1}{\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}}, \overset{\mathbf{v}_2}{\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}}, \overset{\mathbf{v}_3}{\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}} \right\}$$

(a) Express $\vec{\mathbf{w}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{\mathcal{B}_1}$ in the standard basis.

$$\vec{\mathbf{w}} = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}_{\mathcal{S}}$$

(b) Express $\vec{\mathbf{u}} = \begin{bmatrix} 2 \\ 4 \\ -4 \end{bmatrix}_{\mathcal{S}}$ in the $\mathcal{B}_1$-basis.

$$\vec{\mathbf{u}} = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}_{\mathcal{B}_1}$$