**Def.** If A is an $n \times n$ matrix, then a *nonzero* vector $\vec{v} \in \mathbb{R}^n$ is an *eigenvector* of A if

$$A\vec{v} = \lambda\vec{v}, \qquad \text{for some } \lambda \in \mathbb{R}^n.$$

The scalar $\lambda$ is the *eigenvalue* corresponding to $\vec{v}$.

**Computations**:

1. If you have an eigenvector $\vec{v}$ for a matrix A, how do you find the eigenvalue?

$$\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -4 & 6 \\ 2 & -10 & 12 \end{bmatrix}$$

2. If you have an eigenvalue $\lambda$ for a matrix A, how do you find the eigenvectors?
   e.g., $\lambda = 3$ is another eigenvalue, find the eigenspace:

$$\begin{bmatrix} 1 & -2 & 3 \\ 2 & -4 & 6 \\ 2 & -10 & 12 \end{bmatrix}$$

3. If you don't have the eigenvalues or eigenvectors.

$$\begin{bmatrix} 1 & -2 & 3 \\ 2 & -4 & 6 \\ 2 & -10 & 12 \end{bmatrix}$$

4. Find the other eigenvectors:

$$\begin{bmatrix} 1-6 & -2 & 3 \\ 2 & -4-6 & 6 \\ 2 & -10 & 12-6 \end{bmatrix} = \begin{bmatrix} -5 & -2 & 3 \\ 2 & -10 & 6 \\ 2 & -10 & 6 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 0 & -1/3 \\ 0 & 1 & -2/3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 3 \\ 2 & -4 & 6 \\ 2 & -10 & 12 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

5. What does it mean if $\lambda = 0$ is an eigenvalue?

**Examples**: Today's checkpoint: Find the eigenvalues of the matrices below.

1. $\mathsf{A} = \begin{bmatrix} -7 & -10 \\ 5 & 8 \end{bmatrix}$

$$\det(A - \lambda I_2) = \begin{vmatrix} -7-\lambda & -10 \\ 5 & 8-\lambda \end{vmatrix} = (-7-\lambda)(8-\lambda) + 50$$
$$= -56 - \lambda + \lambda^2 + 50$$
$$= \lambda^2 - \lambda - 6$$
$$= (\lambda - 3)(\lambda + 2) \qquad \boxed{\lambda = 3, -2}$$

2. $\mathsf{B} = \begin{bmatrix} 4 & 0 & 0 \\ 3 & -7 & -10 \\ 17 & 5 & 8 \end{bmatrix}$

$$\det(B - \lambda I_3) = \begin{vmatrix} 4-\lambda & 0 & 0 \\ 3 & -7-\lambda & -10 \\ 17 & 5 & 8-\lambda \end{vmatrix} = (4-\lambda)\begin{vmatrix} -7-\lambda & -10 \\ 5 & 8-\lambda \end{vmatrix}$$
$$= (4-\lambda)(\lambda-3)(\lambda+2) \qquad = -x^3 + 5x^2 + 2x - 24$$
$$\boxed{\lambda = 4, 3, -2}$$

$$= \left|\begin{array}{ccc|cc} 4-\lambda & 0 & 0 & 4-\lambda & 0 \\ 3 & -7-\lambda & -10 & 3 & -7-\lambda \\ 17 & 5 & 8-\lambda & 17 & 5 \end{array}\right| = (4-\lambda)(-7-\lambda)(8-\lambda) - (4-\lambda)\,5\cdot 10$$
$$= (4-\lambda)\Big[\underbrace{(-7-\lambda)(8-\lambda) - 5\cdot 10}_{\text{same as above}}\Big]$$
$$= (4-\lambda)(\lambda-3)(\lambda+2)$$

3. Here are a few more characteristic polynomials.

(a) Another $2 \times 2$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \qquad f_A(\lambda) = \begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 1 = 1 - 2\lambda + \lambda^2 - 1 = \lambda^2 - \lambda - 1$$

(b) Rental car problem

$$M = \begin{bmatrix} 0.85 & 0.30 & 0.35 \\ 0.09 & 0.60 & 0.05 \\ 0.06 & 0.10 & 0.60 \end{bmatrix}, \qquad f_M(\lambda) = -\lambda^3 + 2.05\lambda^2 - 1.327\lambda + 0.277 = (x-1)(x^2 - 1.05x + 0.277)$$

(c) A 4 x 4 example:

$$B = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 5 & 8 & -3 & -5 \\ -2 & 0 & 2 & 2 \\ 5 & 6 & -3 & -3 \end{bmatrix}, \qquad f_B(\lambda) = \lambda^4 - 7\lambda^3 + 12\lambda^2 + 4\lambda - 16 = (\lambda-4)(\lambda-2)^2(\lambda+1).$$

(d) How many eigenvalues can an $n \times n$ matrix have?

4. Find bases for the eigenspaces of these matrices

(a) $\mathsf{A} = \begin{bmatrix} -10 & 6 \\ -18 & 11 \end{bmatrix}$ has characteristic polynomial

$$f_A(\lambda) = \begin{vmatrix} -10-\lambda & 6 \\ -18 & 11-\lambda \end{vmatrix} = (-10-\lambda)(11-\lambda)+108 = -110-\lambda+\lambda^2+108 = \lambda^2-\lambda-2 = (\lambda-2)(\lambda+1).$$

Use the information below to describe the eigenspaces.

i. $\mathsf{A} - 2\mathbf{I}_2 = \begin{bmatrix} -12 & 6 \\ -18 & 9 \end{bmatrix} \to \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} \to \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix}$

ii. $\mathsf{A} + \mathbf{I}_2 = \begin{bmatrix} -9 & 6 \\ -18 & 12 \end{bmatrix} \to \begin{bmatrix} 3 & -2 \\ 3 & -2 \end{bmatrix} \to \begin{bmatrix} 3 & -2 \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2/3 \\ 0 & 0 \end{bmatrix}$

(b) $\mathsf{B} = \begin{bmatrix} -2 & -1 & 1 \\ -3 & -2 & 3 \\ -3 & -1 & 2 \end{bmatrix}$ has characteristic polynomial $p(\lambda) = -\lambda^3 - 2\lambda^2 + \lambda + 2 = -(\lambda+2)(\lambda+1)(\lambda-1)$.

Use the information below to describe the eigenspaces.

i. $\mathsf{B} + 2\mathbf{I}_3 = \begin{bmatrix} 0 & -1 & 1 \\ -3 & 0 & 3 \\ -3 & -1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

ii. $\mathsf{B} + \mathbf{I}_3 = \begin{bmatrix} -1 & -1 & 1 \\ -3 & -1 & 3 \\ -3 & -1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

iii. $\mathsf{B} - \mathbf{I}_3 = \begin{bmatrix} -3 & -1 & 1 \\ -3 & -3 & 3 \\ -3 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

(c) $\mathsf{C} = \begin{bmatrix} -4 & 9 & -3 \\ -6 & 11 & -3 \\ -12 & 18 & -4 \end{bmatrix}$ has characteristic polynomial $p(\lambda) = -\lambda^3 + 3\lambda^2 - 4 = -(\lambda-2)^2(\lambda+1)$.

i. $\mathsf{C} - 2\mathbf{I}_3 = \begin{bmatrix} -6 & 9 & -3 \\ -6 & 9 & -3 \\ -12 & 18 & -6 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{3}{2} & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

ii. $\mathsf{C} + \mathbf{I}_3 = \begin{bmatrix} -3 & 9 & -3 \\ -6 & 12 & -3 \\ -12 & 18 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$

5. Diagonalize the matrices on the previous page

(a) $\mathsf{A} = \begin{bmatrix} -10 & 6 \\ -18 & 11 \end{bmatrix} =$

(b) $\mathsf{B} = \begin{bmatrix} -2 & -1 & 1 \\ -3 & -2 & 3 \\ -3 & -1 & 2 \end{bmatrix} =$

(c) $\mathsf{C} = \begin{bmatrix} -4 & 9 & -3 \\ -6 & 11 & -3 \\ -12 & 18 & -4 \end{bmatrix} =$