Thursday, Sept 29

1 Welcome!
2 Topics list up, try to have ideas/pairings next week
3 maybe hws due Friday?
4 Questions?
5 Duality day 2

Today's our last day of linear programs! Or, well, learning about them for their own sake. Our subsequent discussions of them will be in context of (mostly) graph problems, and they'll be much more of a tool for solving other problems. We'll close our discussion with a few more duality examples and a discussion of sensitivity.

Example: Consider a two player game where Kevin and Joy choose between 3 different options (kinda like rock paper scissors). Depending on what they both choose, some sort of exchange of chips occurs. We'll show a *payout* matrix from Kevin's perspective.

	J's Choice			
K's Choice		A	B	C
	A	1	-1	-2
	B	-1	1	1
	C	2	-1	-1

We want to try to maximize Kevin's *minimum expected* winnings by employing a *mixed* strategy, meaning he'll chose between the three options randomly, but with different probabilities.

a.) Write a linear program that does this.

decision var. are x_i, probability of picking i.

$$\begin{cases} \max \; x_0 \\ \text{st. } x_1+x_2+x_3 = 1 \\ x_1 - x_2 + 2x_3 - x_0 \geq 0 \\ -x_1 + x_2 - x_3 - x_0 \geq 0 \\ -2x_1 + x_2 - x_3 - x_0 \geq 0 \\ x_i \geq 0 \quad x_0 \text{ free} \end{cases}$$

$\rightarrow c = [1, 0, 0, 0]$

$\rightarrow A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ -1 & 1 & -1 & 2 \\ -1 & -1 & 1 & -1 \\ -1 & -2 & 1 & -1 \end{bmatrix}$

b.) Write a dual program for this and interpret it in the context of the problem.

$$\begin{cases} \min \; y_0 \\ \text{st } -y_1 - y_2 - y_3 = 1 \\ \quad \leq 0 \\ A^T \quad \leq 0 \\ \quad \leq 0 \end{cases}$$

c.) Solve both in SAGE and interpret the solutions.

Let's talk about sensitivity. Sensitivity analysis is a broad term in mathematics given to the study of how perturbations to a system change it. For us, we'll explore how messing with the constraints, objective functions, and variables may change our programs. This benefits us in two ways: first, it allows for us to maybe not *fully* resolve a problem should we need to add something. Second, it may convince us of our correctness for our model. If we can show that the program that we built is fairly *stable*, or invariant under small changes, we won't need to be so careful over what numbers we specifically use.

As we start, suppose we have a linear program $\max\{c^T x : Ax = b, x \geq 0\}$ (though most of the work we'll do here can also be extended to non equality based systems). Further, suppose we've already solved this program, meaning we found a basis B such that

$$c - c_B B^{-1} A \leq 0 \qquad\qquad B^{-1} b \geq 0$$

Wait, what does this mean?

Possible changes in the linear program (see Bertsimas, Tsitsiklis *Introduction to Linear Optimization*, 5.1)

- Adding a new variable x_n
  - Problem:
  - Solution:
- Adding a new inequality constraint $a_{m+1}^T x \leq b_{m+1}$
  - Problem:
  - Solution:
- Adding a new equality constraint $a_{m+1}^T x = b_{m+1}$
  - Problem:
  - Solution:
- Changing b_i to $b_+\delta$
  - Problem:
  - Solution:
- Changing c_j to $c_j + \delta$
  - Problem:
  - Solution:
- Changing a nonbasic column of A to $A_{ij} + \delta$
  - Problem:
  - Solution:
- Changing a basic column of A $A_{ij} + \delta$
  - Problem:
  - Solution:

In preparation for our pivot to graph applications, let's do a quick review of graphs. Note, this gets more difficult as it goes on, and you may not have seen some of these. It'll help me gauge where to pitch our review!

Review: Define the following vocabulary words from graph theory.

- graph (two set definition)
- vertex, vertices, vertex set
- edges, edge set
- degree
- path
- cycle
- tree
- spanning tree
- bipartite
- Eulerian
- Hamiltonian
- coloring
- chromatic number
- weighted graph
- directed graph
- arc, directed edge
- directed cycle
- matching
- vertex cover
- cut
- flow