1. On Friday, we diagonalized three matrices. What are the eigenvalues and eigenvectors of these matrices?

$$A = \begin{bmatrix} -10 & 6 \\ -18 & 11 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}$$

$$B = \begin{bmatrix} -2 & -1 & 1 \\ -3 & -2 & 3 \\ -3 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

$\lambda = 2, 2, -1$

$$C = \begin{bmatrix} -4 & 9 & -3 \\ -6 & 11 & -3 \\ -12 & 18 & -4 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 0 & 1 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1/3 & 0 & -1/6 \\ 2/3 & -1 & 1/6 \\ -2/3 & 1 & 1/3 \end{bmatrix}$$

alg mult of $\lambda=2$ is 2
$\dim(E_2) = 2$

2. Below is a matrix $A$ and its eigenvalues and eigenvectors. Note that its columns sum to 1, so that it is a probability (stochastic) matrix. Such matrices always have a largest eigenvalue equal to 1 (we will be able to prove this later). Didn't get to this one. (practice for E3)

(a) Diagonalize $A$:

$$A = \begin{bmatrix} 0.6 & 0.3 & 0.25 \\ 0.2 & 0.4 & 0.25 \\ 0.2 & 0.3 & 0.50 \end{bmatrix}$$

$$\lambda_1 = 1.0 \quad \lambda_2 = 0.34 \quad \lambda_3 = 0.16$$

$$\begin{bmatrix} -.692 \\ -.462 \\ -.554 \end{bmatrix} \quad \begin{bmatrix} -0.79 \\ 0.21 \\ 0.58 \end{bmatrix} \quad \begin{bmatrix} 0.21 \\ -0.79 \\ 0.58 \end{bmatrix}$$

(b) What does and eigenvalue of $\lambda = 1$ tell us?

(c) Rescale the eigenvector of eigenvalue $\lambda = 1$ so that it sums to 1.

(d) Use the diagonalization $A = PDP^{-1}$ to compute $A^n$:

(e) Compute $\lim_{n\to\infty} A^n$:

3. When is a matrix diagonalizable?

(a) Eigenvectors corresponding to different eigenvalues are linearly independent, so if A has distinct eigenvalues, it has an eigenbasis.

4x4

$$A = \begin{bmatrix} -3 & 4 & 3 & -1 \\ -2 & 3 & 2 & 0 \\ -5 & 4 & 5 & -1 \\ -5 & 4 & 5 & -1 \end{bmatrix} = \underbrace{\begin{bmatrix} | & | & | & | \\ v_1 & v_2 & v_3 & v_4 \\ | & | & | & | \end{bmatrix}}_{P} \begin{bmatrix} 3 & & & \\ & 2 & & \\ & & -1 & \\ & & & 0 \end{bmatrix} \begin{bmatrix} P^{-1} \end{bmatrix}$$

Distinct

$\lambda_1 = 3$ $\quad \lambda_2 = 2$ $\quad \lambda_3 = -1$ $\quad \lambda_4 = 0$ ← A is not invertible

$$\underset{v_1}{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}} \quad \underset{v_2}{\begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \end{bmatrix}} \quad \underset{v_3}{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}} \quad \underset{v_4}{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}}$$

Is this matrix invertible? Does it have any steady-state vectors?

No. $\lambda = 1$ not an eigenvalue

(b) When it has repeated eigenvalues it can be diagonalizable. See (a) above. The geometric multiplicity equals the algebraic multiplicity for each eigenvalue.

c in problem 1 has eigenvalue with multiplicity > 1 an is diagonalizable

(c) When it has repeated eigenvalues it might not be diagonalizable. Such matrices are $\boxed{\text{“defective.”}}$ The matrix $A$ below has characteristic polynomial $\boxed{f_A(\lambda) = (\lambda - 3)(\lambda - 2)^2}$ and eigenvalues $\lambda = 3, 2, 2$.

3, 2, 2

algebraic multiplicity = 2

$$A = \begin{bmatrix} 3 & 2 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$A - 3I = \begin{bmatrix} 0 & 2 & 1 \\ -1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -\frac{3}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

$E_3 = \text{span}\left( \begin{bmatrix} 3/2 \\ -1/2 \\ 1 \end{bmatrix} \right)$

$$\boxed{A - 2I = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}}$$

$E_2 = \text{span}\left( \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right)$

geometric mult of $\lambda = 2$ is $\dim(E_2) = 1$