Section 4.3: Operational Properties of the Laplace Transform.

First Translation Theorem: If $\mathcal{L}\{f(t)\} = F(s)$ and a is any real number, then

$$\mathcal{L}\{e^{at} f(t)\} = F(\underbrace{s-a}_{\text{shift in } s \text{ by } a \text{ units}})$$

<u>Proof:</u>

$$\mathcal{L}\{e^{at} f(t)\} = \int_0^\infty e^{-st} \cdot e^{at} f(t)\, dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t)\, dt$$

$$\doteq F(s-a) \qquad s-a > 0$$

$\blacksquare$

We can use

$$\mathcal{L}\{e^{at} f(t)\} = \mathcal{L}\{f(t)\}\underbrace{{}_{s \to s-a}}_{s \text{ gets replaced by } s-a.}$$

Example: Evaluate $\mathcal{L}\{e^{6t}(t^2+\sin(4t))\}$

$a=6$

$$\mathcal{L}\{e^{6t}(t^2+\sin(4t))\} = \mathcal{L}\{e^{6t}t^2\} + \mathcal{L}\{e^{6t}\sin(4t)\}$$

$$= \mathcal{L}\{t^2\}\Big|_{s\to s-6} + \mathcal{L}\{\sin(4t)\}\Big|_{s\to s-6}$$

$$= \frac{2}{s^3}\Big|_{s\to s-6} + \frac{4}{s^2+16}\Big|_{s\to s-6}$$

$$= \frac{2}{(s-6)^3} + \frac{4}{(s-6)^2+16}$$

$$\mathcal{L}^{-1}\{F(s-a)\} = \mathcal{L}^{-1}\{F(s)\,|_{s\to s-a}\}$$
$$= e^{at}f(t) \quad \text{where } \mathcal{L}\{f(t)\} = F(s)$$

Example: $\mathcal{L}^{-1}\left\{\frac{2s+4}{s^2+6s+11}\right\}$

complete square

$$\frac{2s+4}{s^2+6s+11} = \frac{2s+4}{s^2+6s+9+2} = \frac{2s+4}{(\underbrace{s+3}_{s-(-3)})^2+2}$$

$$= \frac{2(s+3)-2}{(s+3)^2+2} = \frac{2(s+3)}{(s+3)^2+2} - \frac{2}{(s+3)^2+2}.$$

$$\mathcal{L}^{-1}\left\{\frac{2s+4}{s^2+6s+11}\right\} = 2\mathcal{L}^{-1}\left\{\frac{s+3}{(s+3)^2+2}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{(s+3)^2+2}\right\}$$

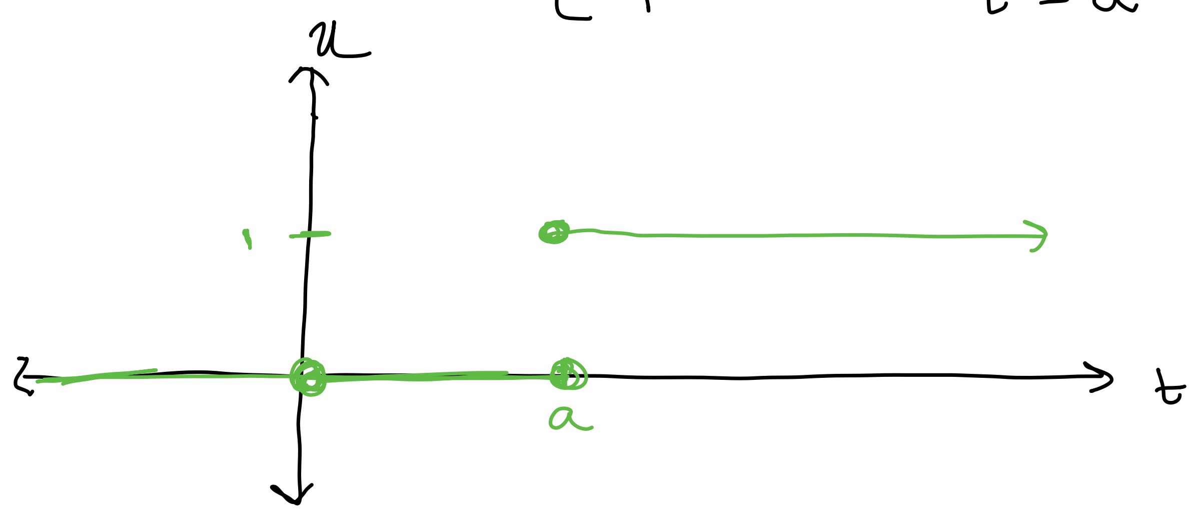
$$\xi = 2\mathcal{L}^{-1}\left\{ \underbrace{\frac{s}{s^2+2}} \Big|_{s\to s+3} \right\} - \frac{2}{\sqrt{2}}\mathcal{L}^{-1}\left\{ \frac{\sqrt{2}}{s^2+2} \Big|_{s\to s+3} \right\}$$

$$= 2e^{-3t}\cos(\sqrt{2}t) - \frac{2}{\sqrt{2}}e^{-3t}\sin(\sqrt{2}t)$$

Translation in t-axis.

The <u>unit step function</u> (or Heaviside Function) $\mathcal{U}(t-a)$ is defined by

$$\mathcal{U}(t-a) = \begin{cases} 0 & 0 \le t < a \\ 1 & t \ge a \end{cases}$$



Let $$f(t) = \begin{cases} g(t) & 0 \le t < a \\ h(t) & t \ge a \end{cases}$$

Then

* $f(t) = g(t) - g(t)\mathcal{U}(t-a) + h(t)\mathcal{U}(t-a)$

When $0 \le t < a$

$$g(t) - \cancel{g(t)} \cdot 0 + \cancel{h(t)} \cdot 0 = g(t) \checkmark$$

$t \ge a$

$$\cancel{g(t)} - \cancel{g(t)} \cdot 1 + h(t) \cdot 1 = h(t) \checkmark$$

$$f(t) = \begin{cases} 0 & 0 \le t < a \\ g(t) & a \le t < b \\ 0 & t \ge b \end{cases}$$

$$\begin{aligned} f(t) &= g(t)\mathcal{U}(t-a) - g(t)\mathcal{U}(t-b) \\ &= g(t)(\mathcal{U}(t-a) - \mathcal{U}(t-b)). \end{aligned}$$

Second Translation Theorem: If

$$F(s) = \mathcal{L}\{f(t)\},$$

and $a > 0$, then

$$\mathcal{L}\{f(\underline{t-a})\mathcal{U}(\underline{t-a})\} = e^{-as}F(s)$$