**Geometry in $\mathbb{R}^n$**

- The *dot product* (aka, *inner product*) of two vectors $\vec{v} = (v_1, v_2, \ldots, v_n)^T$ and $\vec{w} = (w_1, w_2, \ldots, w_n)^T$ in $\mathbb{R}^n$ is the scalar computed by

$$\vec{v} \cdot \vec{w} \;=\; \vec{v}^T \vec{w} = \quad \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} \quad = \quad v_1 w_2 + v_2 w_2 + \cdots v_n w_n \quad \in \mathbb{R} \qquad \text{(a scalar).}$$

- The *length* or *magnitude* of a vector $\vec{v} = (v_1, v_2, \ldots, v_n)^T \in \mathbb{R}^n$ is given by the following formula (which is the $n$-dimensional version of the Pythagorean theorem).

$$||\vec{v}_n|| = \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

- The *distance* between two vectors $\vec{v} = (v_1, v_2, \ldots, v_n)^T$ and $\vec{w} = (w_1, w_2, \ldots, w_n)^T$ in $\mathbb{R}^n$ is the length of the vector $\vec{v} - \vec{w}$ between them

$$d(\vec{v}, \vec{w}) = ||\vec{v} - \vec{w}|| = \sqrt{(\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w})} = \sqrt{(v_1 - w_1)^2 + (v_2 - w_2)^2 + \cdots (v_n - w_n)^2}.$$

- The *angle* $\theta$ between two vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^n$ is computed with the dot product (this formula comes from the law of cosines):

$$\cos(\theta) = \frac{\vec{v} \cdot \vec{w}}{||\vec{v}||||\vec{w}||} \qquad \Rightarrow \qquad \theta = \arccos\left(\frac{\vec{v} \cdot \vec{w}}{||\vec{v}||||\vec{w}||}\right)$$

- The vectors $\vec{v}, \vec{w} \in \mathbb{R}^n$ are *orthogonal* if their dot product is 0 (and thus $\cos(\pi/2) = 0$ so $\arccos(0) = \pi/2$).

- 1. $\vec{v} \cdot \vec{w} = 0 \quad \Rightarrow$
  2. $\vec{v} \cdot \vec{w} > 0 \quad \Rightarrow$
  3. $\vec{v} \cdot \vec{w} < 0 \quad \Rightarrow$

**Examples**

1. Here are two vectors in $\mathbb{R}^5$. Find the distance between them, the cosine of the angle between them, and the angle between them:

$$\vec{v} = \begin{bmatrix} 4 \\ 2 \\ 0 \\ 1 \\ 2 \end{bmatrix} \qquad \vec{w} = \begin{bmatrix} 3 \\ 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}$$

2. Find a nonzero vector orthogonal to $\vec{w}$.

3. *Unitize* $\vec{v}$. That is, find a unit vector $\vec{u}$ in the same direction as $\vec{v}$.

- If $W \subseteq \mathbb{R}^n$ is a subspace, then the *orthogonal complement* of $W$ in $\mathbb{R}^n$ is

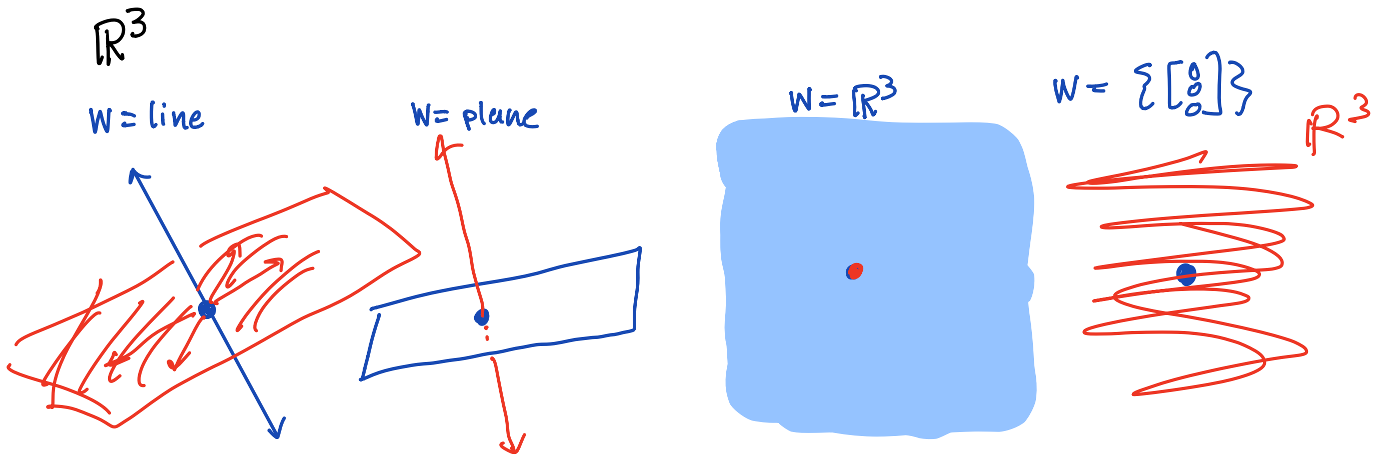
  $$W^{\perp} = \{\vec{v} \in \mathbb{R}^n \mid \vec{v} \cdot \vec{w} = 0 \text{ for every } \vec{w} \in W\}.$$

  – The orthogonal complement $W^{\perp}$ is a subspace

  – It is enough to check that $\vec{v}$ is orthogonal to a basis of $W$ (i.e., you don't have to check every vector in $W$; if you are orthogonal to the basis then you are orthogonal to $W$).

**Examples**

3. Draw the orthogonal complement to the subspaces of $\mathbb{R}^3$ below.




4. **Key Idea**: The orthogonal complement of the row space of $A$ is

$$\begin{bmatrix} — & w_1 & — \\ — & w_2 & — \\ — & w_3 & — \\ & \vdots & \\ — & w_k & — \end{bmatrix} \begin{bmatrix} | \\ v \\ | \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{array}{l} \leftarrow w_1 \cdot v \\ \leftarrow w_2 \cdot v \\ \leftarrow w_3 \cdot v \\ \\ \leftarrow w_k \cdot v \end{array}$$

$$\mathrm{Row}(A)^{\perp} = \mathrm{Nul}(A)$$

5. Find the orthogonal complement of $\mathbf{W} = \text{span}\left\{ \vec{w}_1 = \begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix}, \vec{w}_2 = \begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}, \vec{w}_3 = \begin{bmatrix}1\\2\\2\\2\\1\end{bmatrix}, \vec{w}_4 = \begin{bmatrix}3\\5\\6\\7\\7\end{bmatrix}, \vec{w}_5 = \begin{bmatrix}0\\2\\1\\0\\-4\end{bmatrix} \right\} \subseteq \mathbb{R}^5$

When in doubt, row reduce:

$$A = \begin{array}{c} \begin{array}{ccccc} w_1 & w_2 & w_3 & w_4 & w_5 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 3 & 0 \\ 2 & 1 & 2 & 5 & 2 \\ 3 & 1 & 2 & 6 & 1 \\ 4 & 1 & 2 & 7 & 0 \\ 5 & 1 & 1 & 7 & -4 \end{bmatrix} \end{array} \to \begin{bmatrix} 1 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 1 \\ 3 & 5 & 6 & 7 & 7 \\ 0 & 2 & 1 & 0 & -4 \end{bmatrix} \to \begin{array}{c} \begin{array}{ccccc} & & & s & t \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & -4 \\ 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

(a) What is $\dim(\mathbf{W})$? 3 What is $\dim(\mathbf{W}^\perp)$? 2 What is $\dim(\mathbf{W}) + \dim(\mathbf{W}^\perp)$? 5 $n$

(b) Give a basis (or two if you can) of $\mathbf{W}$ and give a basis of $\mathbf{W}^\perp$

$$\mathcal{B}_W = \left\{ \begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix}, \begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\2\\2\\1\end{bmatrix} \right\} \qquad \mathcal{B}_{W^\perp} = \left\{ \begin{bmatrix}0\\1\\-2\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\4\\-4\\0\\1\end{bmatrix} \right\}$$

(c) Are these vectors in $\mathbf{W}$, $\mathbf{W}^\perp$, or neither?

$$\vec{a} = \begin{bmatrix}6\\10\\11\\12\\10\end{bmatrix}, \qquad \vec{b} = \begin{bmatrix}-1\\5\\-6\\1\\1\end{bmatrix}, \qquad \vec{c} = \begin{bmatrix}5\\15\\5\\13\\11\end{bmatrix}.$$