Thursday, Sept 8

$\boxed{1}$ Welcome!

$\boxed{2}$ Small Work

$\boxed{3}$ Sage Stuff

↳ install guide pending

↳ Sage Math Cell

$\boxed{4}$ HW 1 up tonight

$\boxed{5}$ Questions?

$\boxed{6}$ Geometry of LPs

$\boxed{7}$ Outro

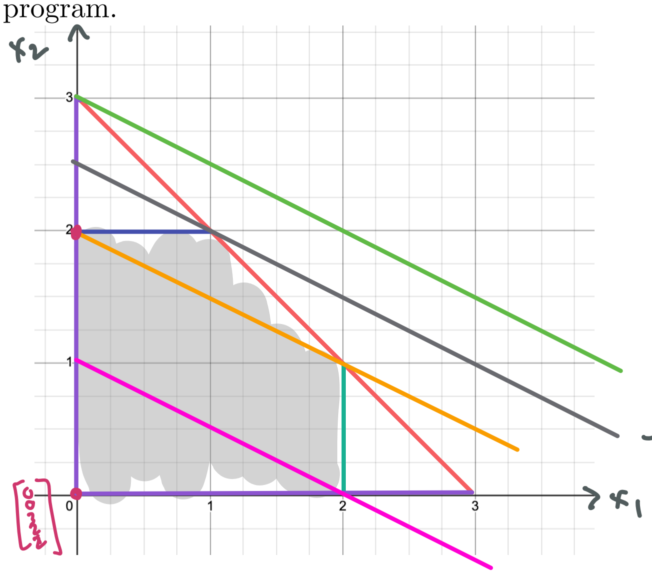
↳ small work:

Welcome back to another day of LPs! At this point, we know how to formulate them, and talked through how a computer can be used to get a solution. However, we've got no idea how that solution actually comes about. To dig into that, we've gotta get into some of the geometry of these spaces. Let's start with our favorite geometry, $\mathbb{R}^2$.

Example: Consider the linear program below:

$$\begin{cases} \max & \frac{1}{2}x_1 + x_2 \quad =1 \quad =2 \quad =3 \\ & x_1 + x_2 \le 3 \\ \text{s.t.} & x_1 \le 2 \\ & x_2 \le 2 \\ & x_i \ge 0 \end{cases}$$

Graph the constraints and a few contours of the objective. Use this to solve the linear program.



this contour maximizes the LP, looks like 2.5

Definition. *(feasible region, feasible solution) For a linear program, a solution is feasible if it satisfies all constraints. The set of all feasible solutions is called the feasible region.*

Take a second to shade the feasible region of the LP above. It's a polygon. Moreover, the linear program's optimal solution occurred at one of the corners. *It turns out, this will always be the case**, even as we move to higher dimensions. For this move, we'll need some vocabulary.

Definition. *(hyperplane, halfspace) Let a be a nonzero vector in $\mathbb{R}^n$ and let $b \in \mathbb{R}$. Then*

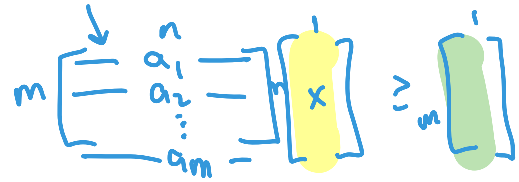
- *the set $\{x \in \mathbb{R}^n : a^T x = b\}$ is a hyperplane.*
- *the set $\{x \in \mathbb{R}^n : a^T x \ge b\}$ is a halfspace.* can be $\le$

Pause: does this agree with our three dimensional understanding of planes?

$3x_1 + 2x_2 + 1x_3 = 4 \rightarrow$ 2 degrees of freedom, so 2D object

$$\begin{bmatrix}3 & 2 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = 4$$

each row defines a half space.



Definition. *(polyhedron) A polyhedron is a set of the form*

$$\{x \in \mathbb{R}^n : Ax \geq b\}$$

for some $m \times n$ matrix A and $b \in \mathbb{R}^m$.

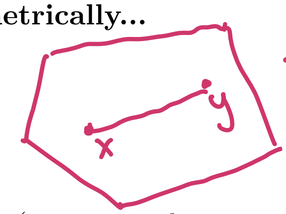
So this means...

* polyhedron is an intersection of m half spaces
* the feasible region of an LP is a polyhedron.

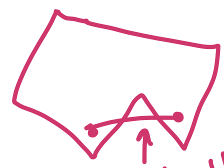
Notice: a polyhedron may not be ___bounded___. If it is, we'll call it a ___polytope___.

Definition. *(convex) A set $S \subseteq \mathbb{R}^n$ is convex if, for any two points $x, y \in S$ and any $\lambda \in [0, 1]$, $\lambda x + (1 - \lambda)y$ is in S.*

or, geometrically...



the line segment b/t any 2 points is contained in the set.



bad!

Definition. *(convex combination, convex hull) Let $\{x^i\}_{i=1}^k$ be a set of vectors, each in $\mathbb{R}^n$, and let $\{\lambda_i\}_{i=1}^k$ be a set of nonnegative scalars such that $\sum_{i=1}^k \lambda_i = 1$.*

- *The vector $\sum_{i=1}^k \lambda_i x^i$ is a convex combination of $\{x^i\}$.* (like a lin. comb.)
- *The convex hull of $\{x^i\}_{i=1}^k$ is the set of all convex combinations of $\{x^i\}_{i=1}^k$.* (like span).

Claim. *Polyhedra are convex.*

Step 1. Halfspaces are convex.

Let $a^T x \geq b$ be a halfspace, called H. Let $x, y \in H$.
Then $a^T x \geq b$ and $a^T y \geq b$. for any $\lambda \in [0,1]$,
$a^T(\lambda x + (1-\lambda)y) = \lambda a^T x + (1-\lambda) a^T y \geq \lambda b + (1-\lambda) b = b$

Step 2. The finite intersection of convex sets is convex.

Let $\{C_i\}$ be a finite set of convex sets. Let $x, y \in \cap C_i$
Then $x, y \in C_i \ \forall i$. Then $\lambda x + (1-\lambda) y \in C_i \ \forall i$
Thus $\lambda x + (1-\lambda) y \in \cap C_i$, so $\cap C_i$ is convex.

A quick aside: The proof we did for finite intersections actually works for infinitely many convex sets. What might an intersection of infinitely many halfspaces look like? sphere

> **Two more big ideas about convexity and convex hulls:**
>
> **Claim.** *A convex combination of a finite number of elements in a convex set is in the set.*
>
> *Proof:* probably on homework 1, but have you considered induction?
>
> **Claim.** *The convex hull of a finite number of vectors is convex*
>
> *Proof:* probably on homework 1, chase some definitions around.

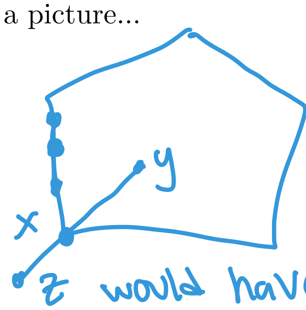
So now we have an idea of what feasible regions look like in higher dimensions. Now we just need to sort out what we mean by *corners*. We'll call them vertices or extreme points.

Definition. *(extreme point) A vector x is an extreme point of a polyhedron P if we cannot find two vectors y, z and $\lambda \in [0,1]$ such that $x = \lambda y + (1-\lambda)z$.* $y, z \neq x$

In other words...

we cannot write it as a conv. comb. of 2 other things in P.

In a picture...



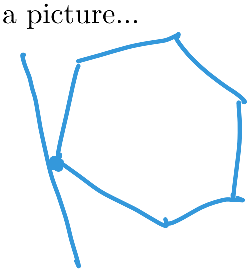
would have to be outside!

Definition. *(vertex) A vector x vertex of a polyhedron P if there exists some c such that $c^T x < c^T y$ for all other $y \in P$, $y \neq x$.*

In other words...

there is a tangent hyperplane @ x.

In a picture...



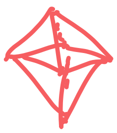
We'll come back to how these relate to each other in a minute (spoiler, they're gonna be the same), but we pause here for a nice observation.

> **Claim.** *Every polytope is the convex hull of its vertices.*
>
> halfspace formulation $\longleftrightarrow$ convex hull of points

So we now have a reasonable geometric understanding of what the vertices/extreme points are in our polyhedra. However, it's not as nice to work with from an algebraic/algorithmic standpoint. We'll fix this by considering *active constraints*.

Definition. *A constraint $a^T x \leq b$ (or $\geq b$) is* active *(or tight or binding) at x^* if $a^T x^* = b$.*

Big Idea: To look for vertices/extreme points of the polytope, first require any equality constraints to be active. Then, look for places where any n *linearly independent* constraints are active. This determines a unique vector.



$$A = \begin{bmatrix} - a_1^T - \\ - a_2^T - \\ \vdots \\ - a_m^T - \end{bmatrix} \leq \begin{bmatrix} b \end{bmatrix}$$

- What do we mean by linearly independent constraints?

  the vectors a_i that define the active constraints are lin. ind.

- Why is the vector unique?

  n equations,
  n unknowns.

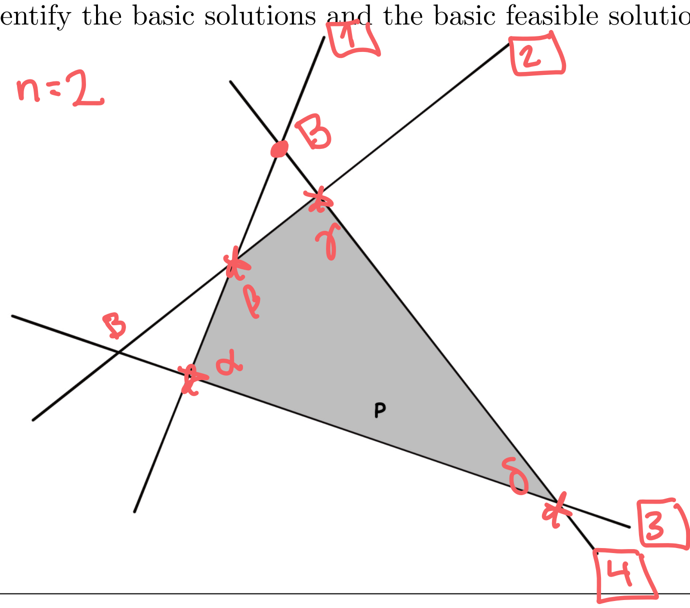
Definition. *(basic solution, basic feasible solution) Consider a polytope P defined by linear equality and inequality constraints, and let $x^* \in \mathbb{R}^n$.*

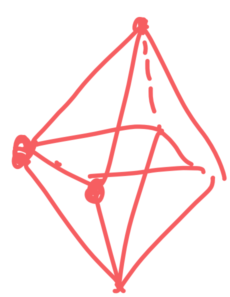
B • *x^* is a basic solution if all equality constraints are met and there are n linearly independent active constraints at x^*.*

✗ • *x^* is a basic feasible solution if x^* satisfies all the constraints of the polytope.*

Example: In the image below, a polytope P is defined by 4 inequality constraints. Identify the basic solutions and the basic feasible solutions.

$n=2$





Claim. *Basic feasible solutions, extreme points, and vertices are equivalent.*

Pf: let $P = \{Ax \geq b\}$

($v \to ep$) s/p x^* is a vertex. There exists c st $c^T x^* < c^T y$ $\forall y \in P$, $y \neq x$. Let $y, z \in P$, $y, z \neq x^*$. Note, $c^T y > c^T x^*$ and $c^T z > c^T x^*$

Let $\lambda \in [0,1]$. Then $c^T(\lambda y + (1-\lambda) z) = \lambda c^T y + (1-\lambda) c^T z > \lambda c^T x^* + (1-\lambda) c^T x^*$

So $c^T x^* < c^T(\lambda y + (1-\lambda) z)$ so $x^* \neq \lambda y + (1-\lambda) z$. So x^* is an ep by def.

($ep \to bfs$)

s/p x^* is not a bfs. Let I be the set of tight constraints: $I = \{i : a_i^T x^* = b_i\}$

There are fewer than n of these that are lin. ind. This means we can find a vector d st. $a_i^T d = 0$ $\forall i$.

Let $y = x^* + \varepsilon d$, $z = x^* - \varepsilon d$ (ε can be picked small enough)

$\hookrightarrow a_i^T z = a_i^T x^* - \varepsilon \underbrace{a_i^T d}_{0}$

we can show $y, z \in P$.

$\varepsilon |a_i d| < a_i^T x - b_i$

$i \notin I$

then $x^* = \frac{1}{2} y + \frac{1}{2} z$

($bfs \to$ vertex) s/p x^* is a bfs. Then let $I = \{i \mid a_i^T x^* = b_i\}$

Ok, so? So this was a ton of work. We now know how to define the corners three different ways and took a whole journey to get there. Why do we care?

$c = \sum_{i \in I} a_i$

* need to show

$\boxed{1}$ $c^T y \geq c^T x$ $\forall y \in P$

$\boxed{2}$ equality only happens @ x.