Daily vocabulary: linearly dependent and linearly independent.

Warm Up

1. Here is an $\mathsf{A}x = b$ problem:

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ -2 & 1 & 1 & -2 & 1 \\ -1 & -1 & 1 & 2 & 2 \\ 3 & -1 & 1 & 10 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ 8 \\ 20 \end{bmatrix}$$

along with a helpful row reduction:

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 1 & 1 & 9 \\ -2 & 1 & 1 & -2 & 1 & 5 \\ -1 & -1 & 1 & 2 & 2 & 8 \\ 3 & -1 & 1 & 10 & 2 & 20 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 2 & 0 & 3 \\ 0 & 1 & 0 & -1 & 0 & 2 \\ 0 & 0 & 1 & 3 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

(a) Give the solution to $\mathsf{A}x = b$ in parametric form

(b) Describe the geometry of the solution (e.g. "a plane in $\mathbb{R}^4$").

(c) What does the fact that there is not a row of 0s in $\mathsf{rref(A)}$ mean about A?

Linear Dependence

Def. A set of vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_k$ is *linearly dependent* if there exist a set of scalars $c_1, c_2, \ldots, c_k$, that are not all zero, such that

$$c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + \cdots + c_k\vec{\mathbf{v}}_k = \vec{\mathbf{0}}$$

Otherwise, the vectors are *linearly independent.*

1. Each of these have a fairly easy-to-see dependence relation. Find one in each case.

(a) $\_\_\_ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \_\_\_ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \_\_\_ \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$

(b) $\_\_\_ \begin{bmatrix} 1 \\ 2 \\ -2 \\ 1 \end{bmatrix} + \_\_\_ \begin{bmatrix} -3 \\ -6 \\ 6 \\ 3 \end{bmatrix} + \_\_\_ \begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$

(c) $\_\_\_ \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} + \_\_\_ \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} + \_\_\_ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \_\_\_ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$

2. An equivalent definition of linear independence is the following: A set of vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_k$ is linearly dependent if and only if you can write at least one of them as a linear combination of the others. Show that this is true in the above examples.

(a)

(b)

(c)

3. Remember that the matrix vector product gives a linear combination of the columns of a matrix A.

$$\begin{bmatrix} | & | & & | \\ \vec{\mathbf{v}}_1 & \vec{\mathbf{v}}_2 & \cdots & \vec{\mathbf{v}}_k \\ | & | & & | \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix} = c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + \cdots + c_k\vec{\mathbf{v}}_k = \vec{\mathbf{0}}.$$

This means that:

$$\text{nonzero solution to } \mathsf{A}x = \vec{\mathbf{0}} \quad \Longleftrightarrow \quad \text{dependence relation among columns of } \mathsf{A}.$$

4. Use #3 to find a dependence relation among the following vectors:

$$\_\_\_ \begin{bmatrix} 2 \\ 1 \\ 3 \\ 5 \end{bmatrix} + \_\_\_ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} + \_\_\_ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + \_\_\_ \begin{bmatrix} -4 \\ 5 \\ -7 \\ 9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

As usual, the following row reduction is helpful.

$$\begin{bmatrix} 2 & 1 & 0 & -4 \\ 1 & 2 & 1 & 5 \\ 3 & 1 & 0 & -7 \\ 5 & 1 & 1 & -9 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

5. Decide if the columns of theses matrices are linearly independent?. Do the columns span $\mathbb{R}^4$? If they are dependent, give a dependence relation.

(a) linearly dependent or linearly independent; span $\mathbb{R}^4$? yes no; <u>dependence relation</u>

$$\begin{bmatrix} 1 & 1 & 1 & 4 \\ 2 & -1 & 1 & 3 \\ 2 & 1 & 1 & 7 \\ 1 & -1 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) linearly dependent or linearly independent; span $\mathbb{R}^4$? yes no; <u>dependence relation</u>

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 1 \\ 2 & -1 & 1 & 5 & 0 & 7 \\ 2 & 1 & 1 & 1 & 1 & 2 \\ 1 & -1 & 0 & 3 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -2 & 0 & -2 \\ 0 & 0 & 1 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

(c) linearly dependent or linearly independent; span $\mathbb{R}^4$? yes no; <u>dependence relation</u>

$$\begin{bmatrix} 4 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 1 \\ -3 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$