1.5. Describing Solutions to Ax = b

$$\left\{\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 3x_3 & = & 9 \\ -4x_1 & + & 2x_2 & & & = & -20 \\ 2x_1 & + & x_2 & - & 2x_3 & = & 0 \end{array}\right\}$$

$$x_1 \begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 9 \\ -20 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ -4 & 2 & 0 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ -20 \\ 0 \end{bmatrix}$$

The General Solution to Ax = b in Parametric Form

$$[\,A \mid b\,] \to \begin{array}{c} \text{pivot} \qquad\qquad \overset{\text{free}}{\downarrow s} \qquad\qquad \downarrow t \\ \begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & \\ \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & -1 & 5 \\ 0 & 1 & -3 & 0 & 2 & 7 \\ 0 & 0 & 0 & 1 & 3 & 11 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

GENERAL SOLUTION

$$\begin{cases} x_1 = 5 - 2x_3 + x_5 & = 5 - 2s + t \\ x_2 = 7 + 3x_3 - 2x_5 & = 7 + 3s - 2t \\ x_3 = \text{free } s & = s \\ x_4 = 11 - 3x_5 & = 11 - 3t \\ x_5 = \text{free } t & = t \end{cases}$$

for all $s, t \in \mathbb{R}$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 5 - 2s + t \\ 7 + 3s - 2t \\ s \\ 11 - 3t \\ t \end{bmatrix} = \overset{\vec{p}}{\begin{bmatrix} 5 \\ 7 \\ 0 \\ 11 \\ 0 \end{bmatrix}} + s \overset{\vec{u}}{\begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}} + t \overset{\vec{v}}{\begin{bmatrix} 1 \\ -2 \\ 0 \\ -3 \\ 1 \end{bmatrix}}$$

PARAMETRIC FORM of General solution

The General Solution to Ax = 0 in Parametric Form

"Homogeneous" equations

Solution to Ax = 0
"homogeneous" equations

Pivot Free $\downarrow s$ $\downarrow t$

$$
\left[\begin{array}{c|c} A & \begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \end{matrix} \end{array}\right]
\rightarrow
\begin{array}{cccccc}
x_1 & x_2 & x_3 & x_4 & x_5 & \\
\end{array}
\left[\begin{array}{ccccc|c}
1 & 0 & 2 & 0 & -1 & 0 \\
0 & 1 & -3 & 0 & 2 & 0 \\
0 & 0 & 0 & 1 & 3 & \vdots \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
$$

GENERAL SOLUTION

$$
\begin{cases}
x_1 = -2x_3 + x_5 = -2s + t \\
x_2 = +3x_3 - 2x_5 = +3s - 2t \\
x_3 = \text{free } s = s \\
x_4 = -3x_5 = -3t \\
x_5 = \text{free } t = t
\end{cases}
$$

$$
\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2s + t \\ +3s - 2t \\ s \\ -3t \\ t \end{bmatrix} = \boxed{s \overset{\vec{u}}{\begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}} + t \overset{\vec{v}}{\begin{bmatrix} 1 \\ -2 \\ 0 \\ -3 \\ 1 \end{bmatrix}}}
$$

The General Solution to Ax = b in Parametric Form

pivot free

$$[\,A \mid b\,] \rightarrow \begin{array}{ccccc|c} x_1 & x_2 & x_3 \downarrow s & x_4 & x_5 \downarrow t & \\ \left[\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & -1 & 5 \\ 0 & 1 & -3 & 0 & 2 & 7 \\ 0 & 0 & 0 & 1 & 3 & 11 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

Particular solution to Ax = b

Solution to Ax = 0
"homogeneous" equations

GENERAL SOLUTION

$$\begin{cases} x_1 = 5 - 2x_3 + x_5 & = 5 - 2s + t \\ x_2 = 7 + 3x_3 - 2x_5 & = 7 + 3s - 2t \\ x_3 = \text{free } s & = s \\ x_4 = 11 - 3x_5 & = 11 - 3t \\ x_5 = \text{free } t & = t \end{cases}$$

for all $s, t \in \mathbb{R}$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 5 - 2s + t \\ 7 + 3s - 2t \\ s \\ 11 - 3s - 3t \\ t \end{bmatrix} = \underbrace{\begin{bmatrix} 5 \\ 7 \\ 0 \\ 11 \\ 0 \end{bmatrix}}_{\vec{p}} + \boxed{s \underbrace{\begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\vec{u}} + t \underbrace{\begin{bmatrix} 1 \\ -2 \\ 0 \\ -3 \\ 1 \end{bmatrix}}_{\vec{v}}}$$

PARAMETRIC FORM of General solution

The Geometry of the Solution

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 5 - 2s + t \\ 7 + 3s - 2t \\ s \\ 11 - 3s - 3t \\ t \end{bmatrix} = \overset{\vec{p}}{\begin{bmatrix} 5 \\ 7 \\ 0 \\ 11 \\ 0 \end{bmatrix}} \boxed{+ s \overset{\vec{u}}{\begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}} + t \overset{\vec{v}}{\begin{bmatrix} 1 \\ -2 \\ 0 \\ -3 \\ 1 \end{bmatrix}}}$$

Particular solution to Ax = b

Solution to Ax = 0
“homogeneous” equations