1.5-b. Algebraic Properties of Solutions to Ax = b

Properties of Matrix-Vector Product

$$A(\mathbf{u} + \mathbf{w}) = A\mathbf{u} + A\mathbf{w}$$

$$\begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix}\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}\right) = \begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix}\left(\begin{bmatrix} u_1 + w_1 \\ u_2 + w_2 \\ u_3 + w_3 \end{bmatrix}\right)$$

$$= (u_1 + w_1)\begin{bmatrix} | \\ v_1 \\ | \end{bmatrix} + (u_2 + w_2)\begin{bmatrix} | \\ v_2 \\ | \end{bmatrix} + (u_3 + w_3)\begin{bmatrix} | \\ v_3 \\ | \end{bmatrix}$$

$$= \underbrace{u_1\begin{bmatrix} | \\ v_1 \\ | \end{bmatrix} + u_2\begin{bmatrix} | \\ v_2 \\ | \end{bmatrix} + u_3\begin{bmatrix} | \\ v_3 \\ | \end{bmatrix}} + \underbrace{w_1\begin{bmatrix} | \\ v_1 \\ | \end{bmatrix} + w_2\begin{bmatrix} | \\ v_2 \\ | \end{bmatrix} + w_3\begin{bmatrix} | \\ v_3 \\ | \end{bmatrix}}$$

$$= \begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

Properties of Matrix-Vector Product

$$A\,(c\,\mathbf{u}) = c\,A\mathbf{u}$$

$$\begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix}\left(c\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) = \begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix}\begin{bmatrix} c\,u_1 \\ c\,u_2 \\ c\,u_3 \end{bmatrix}$$

$$= c\,u_1\begin{bmatrix} | \\ v_1 \\ | \end{bmatrix} + c\,u_2\begin{bmatrix} | \\ v_2 \\ | \end{bmatrix} + c\,u_3\begin{bmatrix} | \\ v_3 \\ | \end{bmatrix}$$

$$= c\left(u_1\begin{bmatrix} | \\ v_1 \\ | \end{bmatrix} + u_2\begin{bmatrix} | \\ v_2 \\ | \end{bmatrix} + u_3\begin{bmatrix} | \\ v_3 \\ | \end{bmatrix}\right)$$

$$= c\begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix}\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right)$$

Properties of Solutions to $\mathbf{A}\mathbf{x} = \mathbf{b}$

1. $\mathbf{A}\,\mathbf{0} = \mathbf{0}$ so $\mathbf{A}\,x = \mathbf{0}$ always has a solution: the zero vector

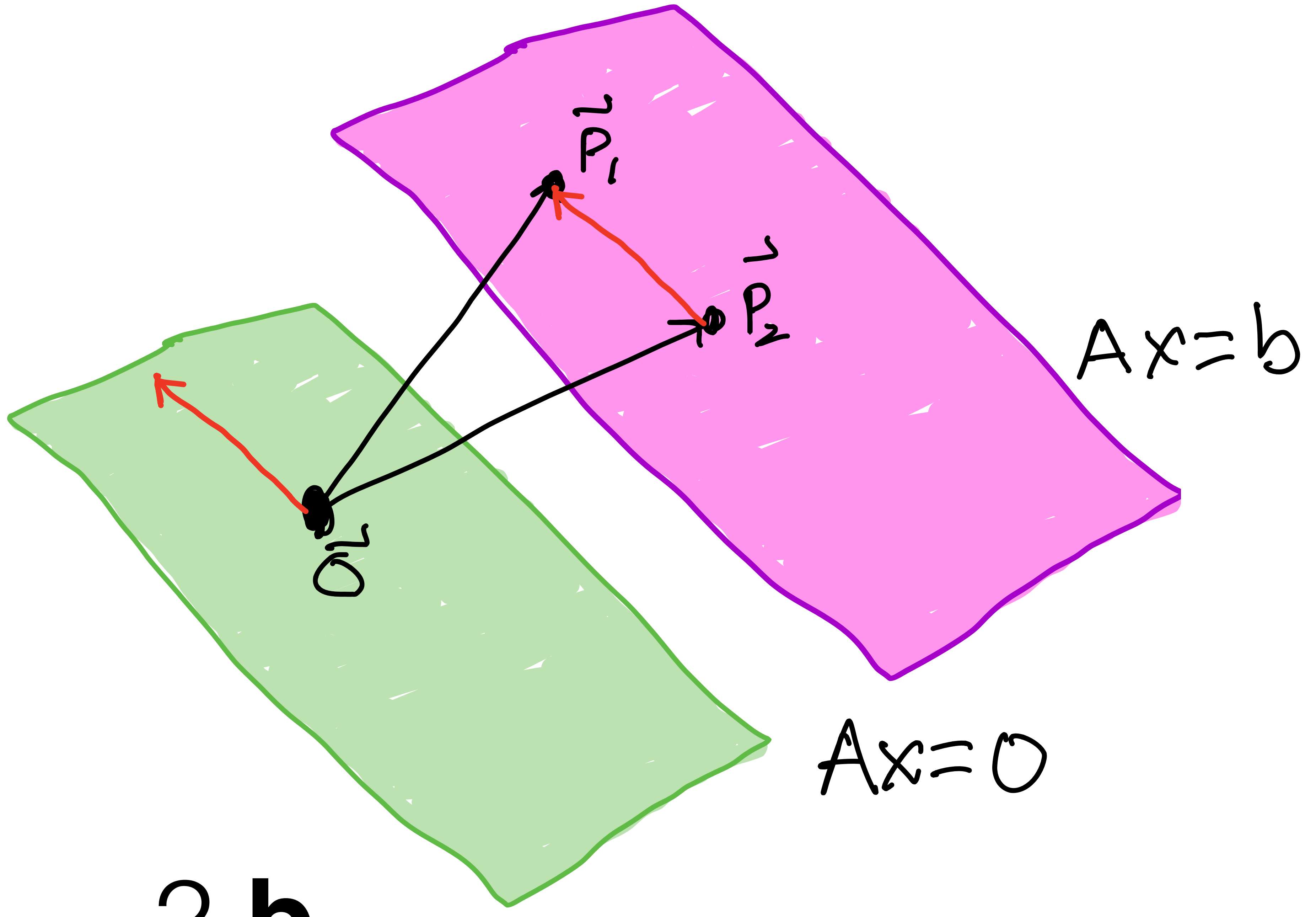
2. Suppose that $\mathbf{p}_1$ and $\mathbf{p}_2$ are solutions to $\mathbf{A}\,x = \mathbf{b}$

   i.e., $\mathbf{A}\,\mathbf{p}_1 = \mathbf{b}$ and $\mathbf{A}\,\mathbf{p}_2 = \mathbf{b}$

   Then: $\mathbf{A}\,(\,\mathbf{p}_1 - \mathbf{p}_2\,) = \mathbf{A}\,\mathbf{p}_1 - \mathbf{A}\,\mathbf{p}_2$

   $= \mathbf{b} - \mathbf{b}$

   $= \mathbf{0}$



3. $\mathbf{A}\,(\,\mathbf{p}_1 + \mathbf{p}_2\,) = \mathbf{A}\,\mathbf{p}_1 + \mathbf{A}\,\mathbf{p}_2 = \mathbf{b} + \mathbf{b} = 2\,\mathbf{b}$