Complex Eigenvalues

Complex eigenvalues come in conjugate pairs

$\lambda_1 = a+bi \qquad\qquad \lambda_2 = a-bi = \overline{\lambda_1}$

Theorem: Let A be a coefficient matrix for a system of differential equations and let $\vec{k_1}$ be an eigenvector corresponding to eigenvalue λ_1. Then

complex conjugate vector

$$\vec{x} = c_1 \vec{k_1} e^{\lambda_1 t} + c_2 \overline{\vec{k_1}}\, e^{\overline{\lambda_1} t}$$

$= \lambda_2$

Solve:

$$\left.\begin{aligned} \frac{dx}{dt} &= 4x+5y \\ \frac{dy}{dt} &= -2x+6y \end{aligned}\right\} \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ -2 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\det(A-\lambda I) = \det\left(\begin{bmatrix} 4-\lambda & 5 \\ -2 & 6-\lambda \end{bmatrix}\right)$$

$$= (4-\lambda)(6-\lambda) - (-10)$$

$$= 24 - 10\lambda + \lambda^2 + 10$$

$$= \lambda^2 - 10\lambda + 34$$

$$\lambda = \frac{10 \pm \sqrt{(-10)^2 - 4(1)(34)}}{2} = \frac{10}{2} \pm \frac{\sqrt{-36}}{2}$$

$$= 5 \pm 3i$$

$\lambda_1 = 5 + 3i$

$$\left[\begin{array}{cc|c} a-\lambda & b & 0 \\ c & d-\lambda & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 4-(5+3i) & 5 & 0 \\ -2 & 6-(5+3i) & 0 \end{array}\right]$$

$$= \left[\begin{array}{cc|c} -1-3i & 5 & 0 \\ -2 & 1-3i & 0 \end{array}\right]$$

$(-1-3i)(-1+3i)$

$= (-1)^2 - 3i$

$+3i - (3i)^2$

$= 1 + 9$

$$\xrightarrow{R_1 \to (-1+3i)R_1} \left[\begin{array}{cc|c} 10 & 5(-1+3i) & 0 \\ -2 & 1-3i & 0 \end{array}\right]$$

$$\xrightarrow{R_1 \to \frac{1}{10}R_1} \left[\begin{array}{cc|c} 1 & \frac{1}{2}(-1+3i) & 0 \\ -2 & 1-3i & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 + 2R_1} \left[\begin{array}{cc|c} 1 & \frac{1}{2}(-1+3i) & 0 \end{array}\right.$$

$$\left[\, 0 \quad 0 \;\middle|\; 0 \,\right]$$

k_2 is free

$$k_1 + \left(-\tfrac{1}{2} + \tfrac{3}{2} i\right) k_2 = 0$$

$$k_1 = \left(\tfrac{1}{2} - \tfrac{3}{2} i\right) k_2$$

$$\vec{k} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

any nonzero constant

$$= \begin{bmatrix} \left(\frac{1}{2} - \frac{3}{2} i\right) k_2 \\ k_2 \end{bmatrix} = k_2 \begin{bmatrix} \frac{1}{2} - \frac{3}{2} i \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} \frac{1}{2} - \frac{3}{2} i \\ 1 \end{bmatrix} e^{(5+3i)t} + c_2 \begin{bmatrix} \frac{1}{2} + \frac{3}{2} i \\ 1 \end{bmatrix} e^{(5-3i)t}.$$

$$\vec{k_1} = \begin{bmatrix} k_1 \\ \frac{1}{\left(-\frac{1}{2} + \frac{3}{2} i\right)} k_1 \end{bmatrix}$$

Theorem: Let $\lambda_1 = a + bi$ be a complex eigenvalue of coefficient matrix A. Let

$$\underline{\vec{b_1} = \operatorname{Re}(\vec{k_1})}, \text{ and } \vec{b_2} = \operatorname{Im}(\vec{k_1})$$

where $\vec{k_1}$ is eigenvector corresponding to λ_1. Then

$$\vec{x} = c_1 \left[\vec{b_1} \cos(bt) - \vec{b_2} \sin(bt) \right] e^{at}$$

$$+ c_2 [\vec{b_2} \cos(bt) + \vec{b_1} \sin(bt)] e^{at}$$

is a (real-valued) solution to the system

Example (continued)

$$\vec{k} = \begin{bmatrix} \frac{1}{2} - \frac{3}{2} i \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{3}{2} \\ 0 \end{bmatrix} i$$

$$\vec{b_1} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \qquad \vec{b_2} = \begin{bmatrix} -\frac{3}{2} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \left(\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \cos(3t) - \begin{bmatrix} -\frac{3}{2} \\ 0 \end{bmatrix} \sin(3t) \right) e^{5t}$$

$$+ c_2 \left(\begin{bmatrix} -\frac{3}{2} \\ 0 \end{bmatrix} \cos(3t) + \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \sin(3t) \right) e^{5t}$$

Real-repeated eigenvalue case.

For 2×2 matrices with repeated eigenvalue there is only 1 linearly independent eigenvector

From exponential sub for systems

$$\vec{k}e^{\lambda t} \qquad (\vec{k} \text{ eigenvector corresponding to } \lambda)$$

is solution

Guess (motivated by section 3.2)

$$\vec{k}te^{\lambda t}$$

is a solution. Check if it works:

$$\frac{d}{dt}(\vec{k}te^{\lambda t}) = \begin{bmatrix} \frac{d}{dt}(k_1te^{\lambda t}) \\ \frac{d}{dt}(k_2te^{\lambda t}) \end{bmatrix}$$

$$= \begin{bmatrix} k_1e^{\lambda t} + k_1\lambda te^{\lambda t} \\ k_2e^{\lambda t} + k_2\lambda te^{\lambda t} \end{bmatrix}$$

$$= \vec{k}e^{\lambda t} + \lambda\vec{k}te^{\lambda t}$$

LHS. RHS

$$\vec{k}e^{\lambda t} + \underbrace{\lambda\vec{k}te^{\lambda t}} = \underbrace{A\vec{k}te^{\lambda t}}$$

"$te^{\lambda t}$" terms $\lambda \vec{k}\, t e^{\lambda t} = A \vec{k}\, t e^{\lambda t}$

"$e^{\lambda t}$" terms $\vec{k}\, e^{\lambda t} = \vec{0}\, e^{\lambda t}$.