5.5. Complex Eigenvalues

The Rotation-Dilation Matrix

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \quad c(\lambda) = \begin{vmatrix} a-\lambda & -b \\ b & a-\lambda \end{vmatrix} = (a-\lambda)^2 + b^2$$

$$= a^2 - 2a\lambda + \lambda^2 + b^2$$

$$= \lambda^2 - 2a\lambda + (a^2+b^2)$$

Quadratic Formula

$$\lambda = \frac{2a \pm \sqrt{(2a)^2 - 4(a^2+b^2)}}{2} = \frac{2a \pm \sqrt{4a^2 - 4a^2 - 4b^2}}{2}$$

$$= \frac{2a \pm \sqrt{-4b^2}}{2} = \frac{2a \pm 2bi}{2} = \boxed{a \pm bi}$$

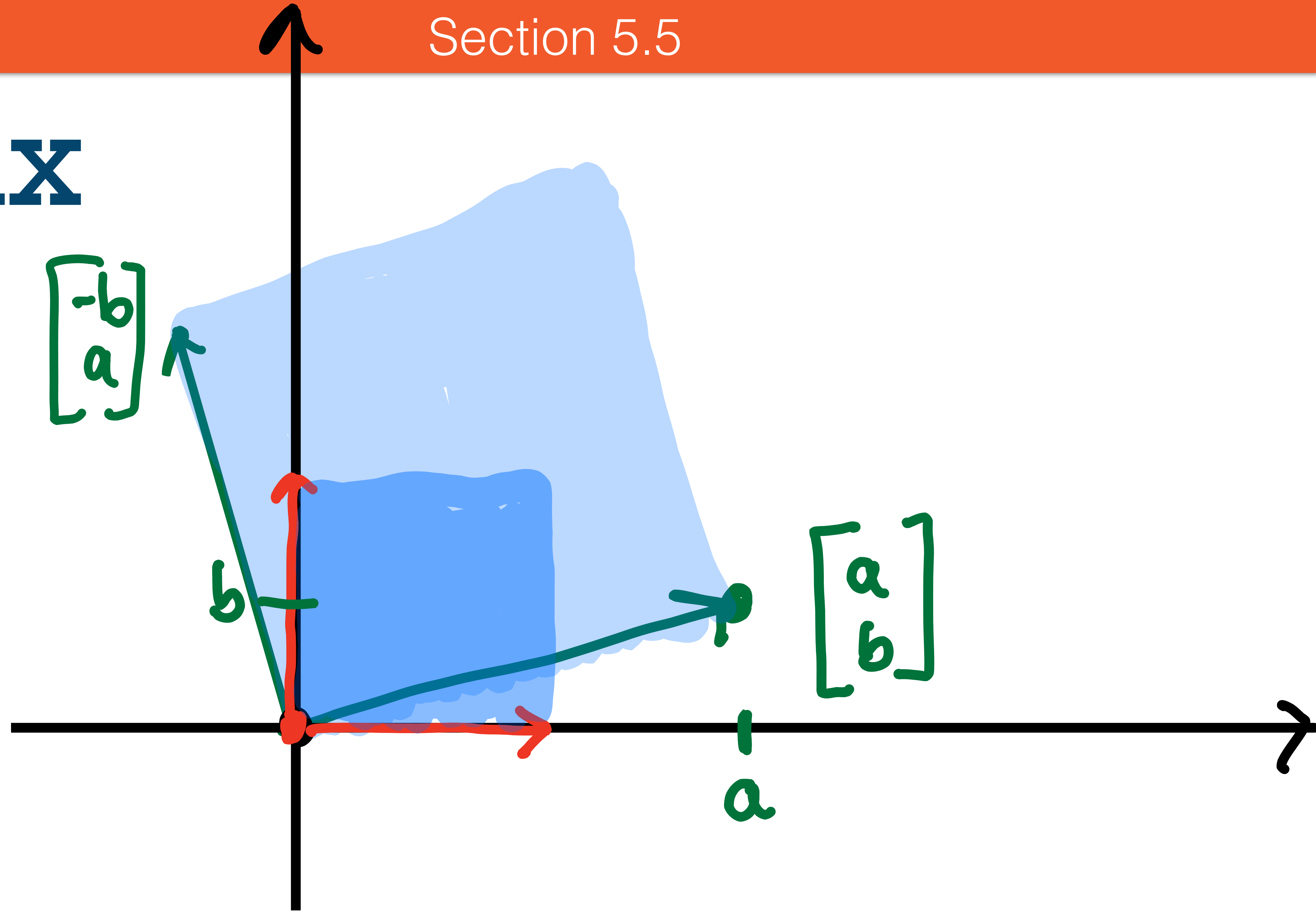
The Rotation-Dilation Matrix

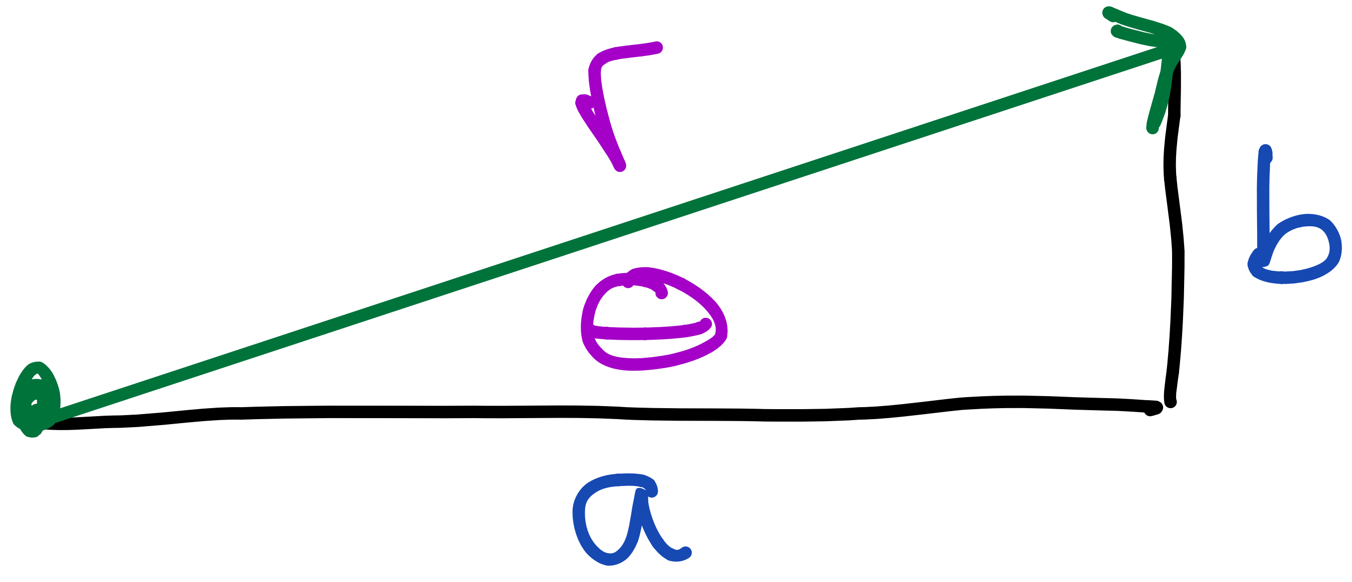
$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} r\cos(\theta) & -r\sin(\theta) \\ r\sin(\theta) & r\cos(\theta) \end{bmatrix}$$

$\lambda_1 = a + bi$

$\lambda_2 = a - bi$

$$= r\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$





$$\boxed{\begin{aligned} \theta &= \tan^{-1}\left(\frac{b}{a}\right) = \text{Arg}(\lambda) \\ r &= \sqrt{a^2 + b^2} = |\lambda| \end{aligned}}$$

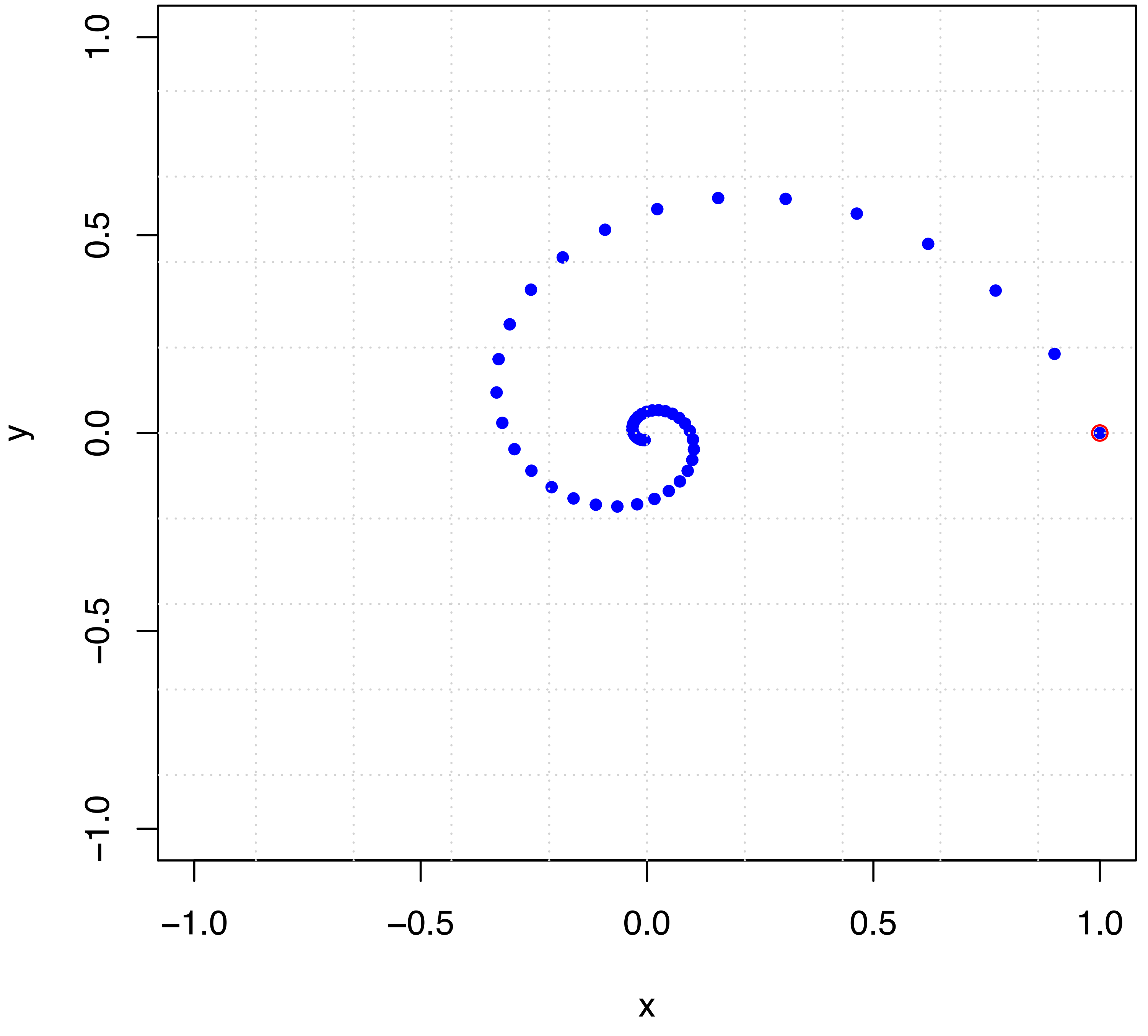
$$\boxed{\begin{aligned} a &= r\cos(\theta) \\ b &= r\sin(\theta) \end{aligned}}$$

Examples

$A = \begin{bmatrix} .9 & -.2 \\ .2 & .9 \end{bmatrix}$ $|\lambda| = \sqrt{.9^2 + .2^2} = .92$

$\theta = \tan^{-1}\left(\frac{.2}{.9}\right) = .21 \text{ rad}$
$= 12.5^\circ$

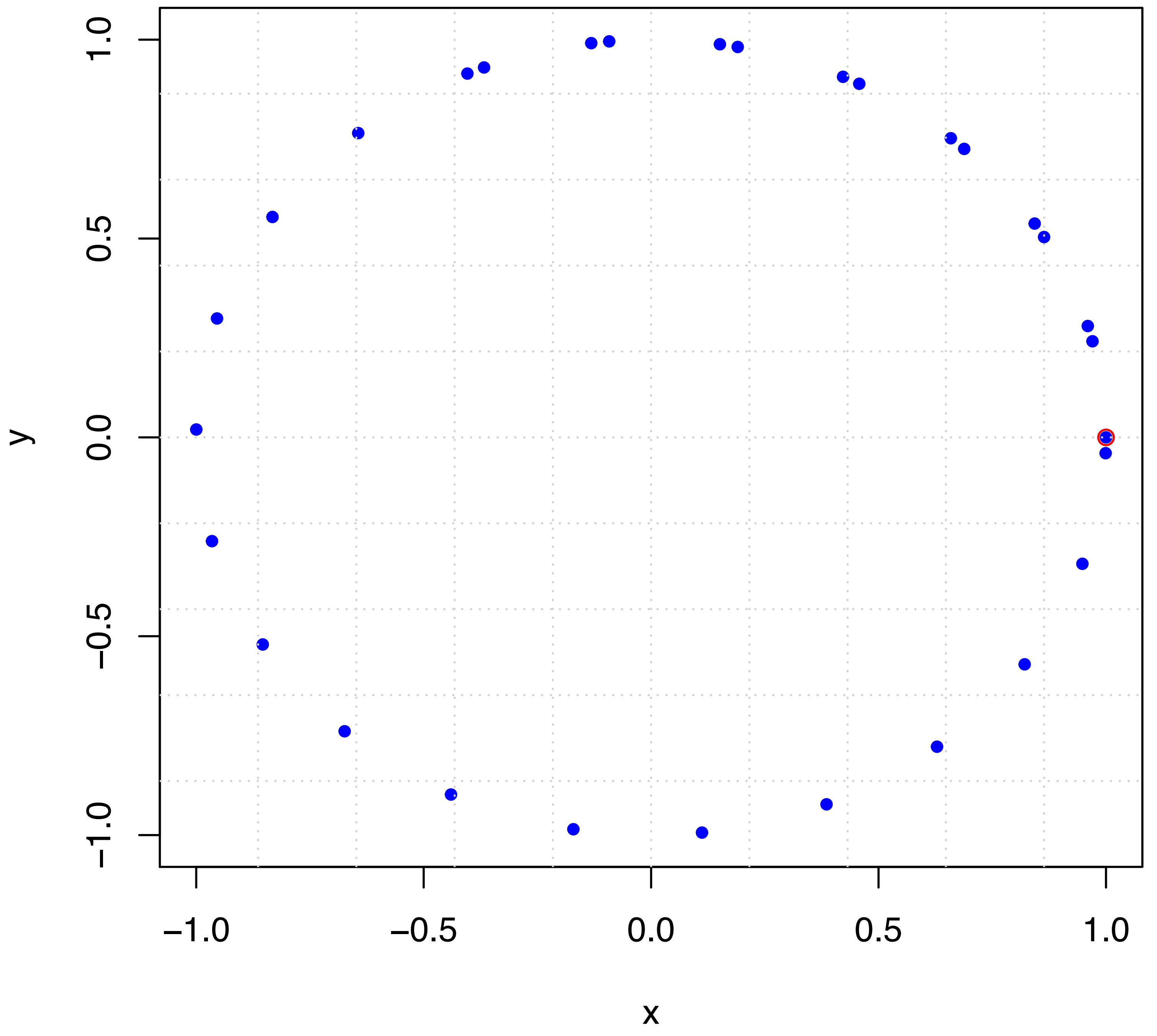
$\lambda = .9 \pm .2i$



$B = \begin{bmatrix} .96 & -.28 \\ .28 & .96 \end{bmatrix}$ $|\lambda| = \sqrt{.96^2 + .28^2} = 1.00$

$\theta = \tan^{-1}\left(\frac{.28}{.96}\right) = .28 \text{ rad}$
$= 16.3^\circ$

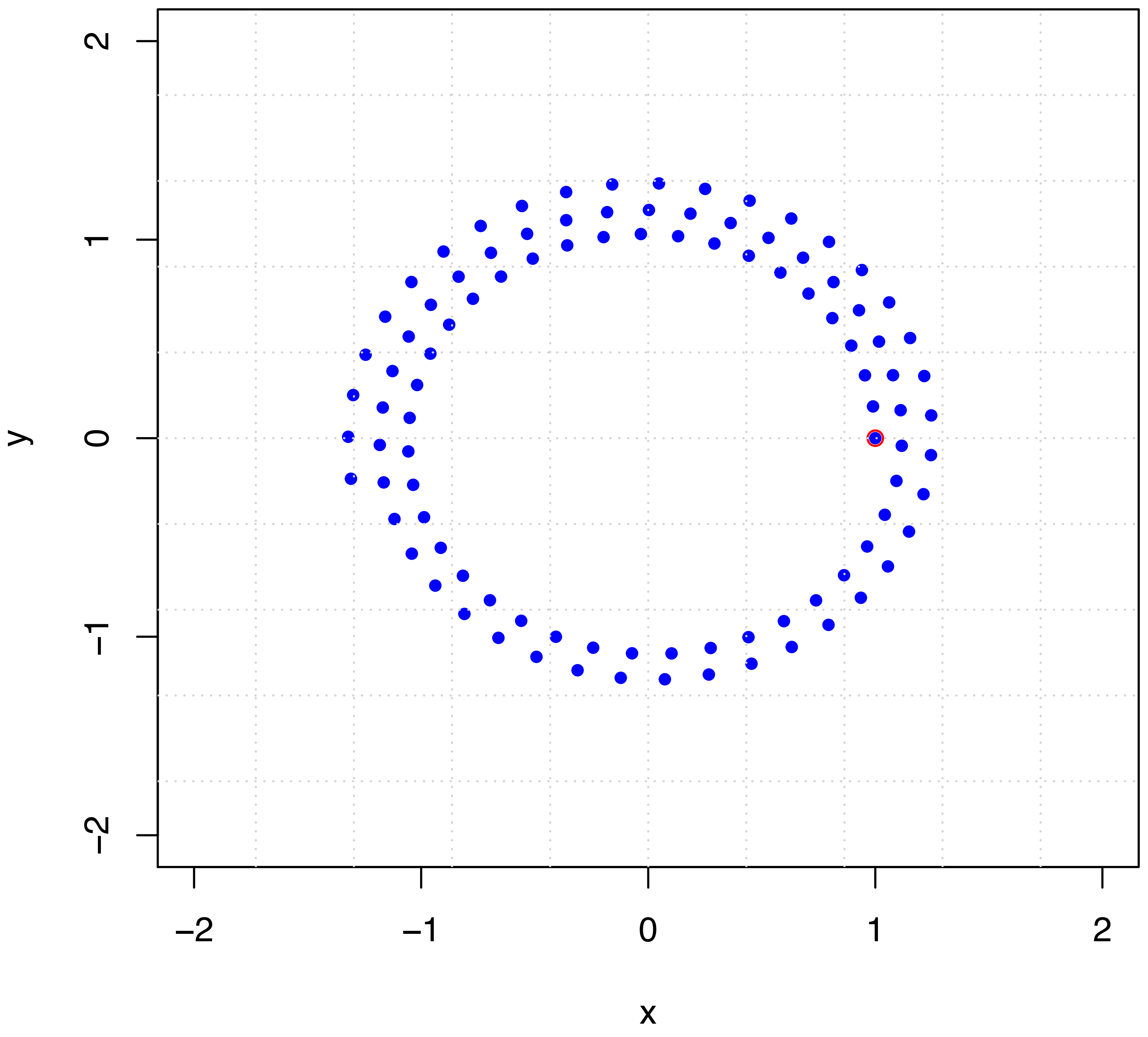
$\lambda = .96 \pm .28i$



$C = \begin{bmatrix} .99 & -.16 \\ .16 & .99 \end{bmatrix}$ $|\lambda| = \sqrt{.99^2 + .16^2} = 1.002$

$\theta = \tan^{-1}\left(\frac{.16}{.99}\right) = .16 \text{ rad}$
$= 9.2^\circ$

$\lambda = .99 + .16i$

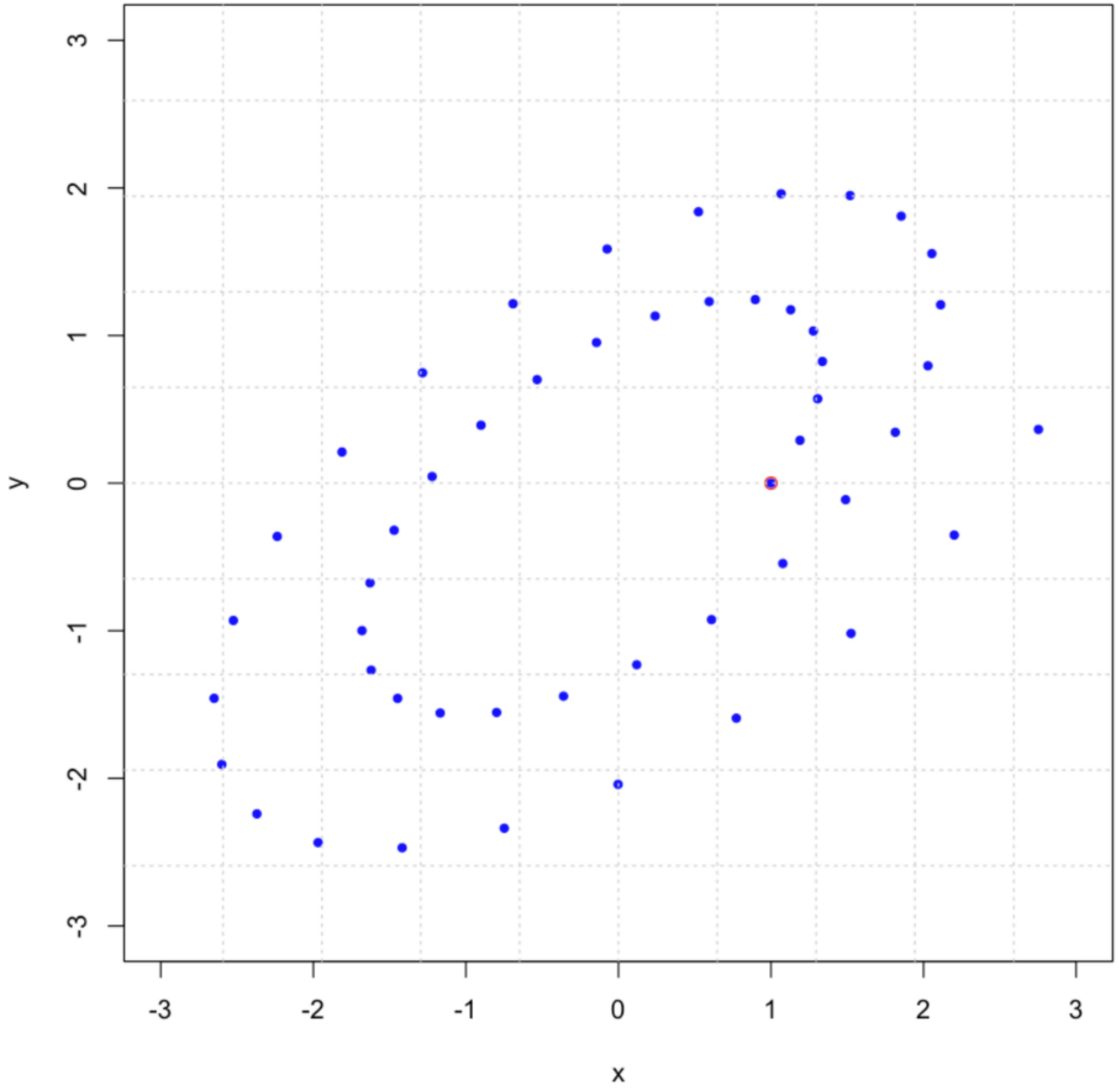


More Generally

$$A = \begin{bmatrix} 1.19 & -0.39 \\ 0.29 & 0.79 \end{bmatrix}$$

$\lambda_1 = .98 + .26\,i$ $\qquad$ $\lambda_1 = .98 - .26\,i$

$v_1 = \begin{bmatrix} 0.75 \\ 0.41 \end{bmatrix} + \begin{bmatrix} 0.00 \\ -.52 \end{bmatrix} i$ $\qquad$ $v_1 = \begin{bmatrix} 0.75 \\ 0.41 \end{bmatrix} - \begin{bmatrix} 0.00 \\ -.52 \end{bmatrix} i$



Diagonalize

Eigenvalues λ_1 λ_2

Eigenvectors v_1 v_2

$$A = \underbrace{\begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} P^{-1} \end{bmatrix}}_{P^{-1}}$$

Rotation-Dilation-alize

$\lambda_1 = a + bi$, $\lambda_2 = a - bi$

$v_1 = u + wi$ $v_2 = u - wi$

$$A = \underbrace{\begin{bmatrix} | & | \\ w & u \\ | & | \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} a & -b \\ b & a \end{bmatrix}}_{R} \underbrace{\begin{bmatrix} P^{-1} \end{bmatrix}}_{P^{-1}}$$

Why does this work?

$$\lambda_1 = a + bi \qquad \lambda_2 = a - bi$$

$$v_1 = u + wi \qquad v_2 = u - wi$$

$$A = \underbrace{\begin{bmatrix} | & | \\ w & u \\ | & | \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} a & -b \\ b & a \end{bmatrix}}_{R} \underbrace{\begin{bmatrix} \; P^{-1} \; \end{bmatrix}}_{P^{-1}}$$

$$A(u + wi) = (a + bi)(u + wi)$$

$$= au + bwi^2 + bui + awi$$

$$Au + Awi = au - bw + (bu + aw)i$$

$$\Rightarrow Au = au - bw$$

$$Aw = bu + bw$$

$$[A]_{\{w,u\}} = \begin{array}{c c} & \begin{array}{cc} w & u \end{array} \\ \begin{array}{c} w \\ u \end{array} & \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \end{array}$$

Example: rotation-dilationalizing

$$A = \begin{bmatrix} 1.19 & -0.39 \\ 0.29 & 0.78 \end{bmatrix} \qquad \lambda_1 = .98 + .26i \qquad \lambda_1 = .98 - .26i$$

$$v_1 = \begin{bmatrix} 0.75 \\ 0.41 \end{bmatrix} + \begin{bmatrix} 0.00 \\ -.52 \end{bmatrix} i \qquad v_1 = \begin{bmatrix} 0.75 \\ 0.41 \end{bmatrix} - \begin{bmatrix} 0.00 \\ -.52 \end{bmatrix} i$$

$$|\lambda_1| = \sqrt{.98^2 + .26^2} = 1.0139$$

$$\theta = \arctan\left(\frac{0.29}{1.19}\right) = .239 \text{ rad} = 13.696^\circ$$

$$A = \begin{bmatrix} 1.19 & -0.39 \\ 0.29 & 0.78 \end{bmatrix} = \begin{bmatrix} 0.00 & 0.75 \\ -.52 & 0.41 \end{bmatrix} \begin{bmatrix} .98 & -.26 \\ .26 & .98 \end{bmatrix} \begin{bmatrix} 1.05 & -1.92 \\ 1.33 & 0.00 \end{bmatrix}$$