1.9. The Matrix of a Linear Transformation

Linear Transformations

A **transformation** is a function $T : \mathbb{R}^n \to \mathbb{R}^m$

that assigns to each $\mathbf{x} \in \mathbb{R}^n$ a vector $T(\mathbf{x}) \in \mathbb{R}^m$

A transformation is **linear** if for all vectors **u** and **v** and all scalars *c*

(1) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$

(2) $T(c\,\mathbf{u}) = c\,T(\mathbf{u})$

(3) $T(\mathbf{0}_n) = \mathbf{0}_m$

(4) $T(c\,\mathbf{u} + d\,\mathbf{v}) = c\,T(\mathbf{u}) + d\,T(\mathbf{v})$

Matrix transformations are linear

A (**u** + **v**) = **A u** + **A v**

A (*c* **u**) = *c* **A u**

TODAY: Linear Transformations are matrix transformations

for a given linear transformation we will find matrix that performs the transformation

The Key Idea in an Example

Suppose that T is a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ and we know that

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}3\\5\end{bmatrix} \qquad T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\1\end{bmatrix} \qquad T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}-1\\1\end{bmatrix}$$

Then what if anything can we say about

$$T\left(\begin{bmatrix}1\\2\\3\end{bmatrix}\right) = T\left(1\underbrace{\begin{bmatrix}1\\0\\0\end{bmatrix}}_{e_1} + 2\underbrace{\begin{bmatrix}0\\1\\0\end{bmatrix}}_{e_2} + 3\underbrace{\begin{bmatrix}0\\0\\1\end{bmatrix}}_{e_3}\right) = 1T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) + 2T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) + 3T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)$$

$$= 1\begin{bmatrix}3\\5\end{bmatrix} + 2\begin{bmatrix}1\\1\end{bmatrix} + 3\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}2\\10\end{bmatrix}$$

T(**u** + **v**) = T(**u**) + T(**v**)

T(c **u**) = c T(**u**)

T(**0**) = **0**

T(c**u**+d**v**) = cT(**u**)+dT(**u**)

Knowing what T does to the standard basis tells us everything about T

$$\mathbf{e}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}, \quad \mathbf{e}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$$ are the standard basis vectors in $\mathbb{R}^3$

The Key Idea in an Example

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}3\\5\end{bmatrix}\qquad T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)=\begin{bmatrix}1\\1\end{bmatrix}\qquad T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}-1\\1\end{bmatrix}$$

$$A=\begin{array}{c} \begin{array}{ccc} T(e_1) & T(e_2) & T(e_3) \\ \downarrow & \downarrow & \downarrow \end{array} \\ \begin{bmatrix} 3 & 1 & -1 \\ 5 & 1 & 1 \end{bmatrix} \end{array}$$

Suppose that we want to find a matrix **A** that does this same transformation

$$\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}a_{11}\\a_{21}\end{bmatrix}$$

$$\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}a_{12}\\a_{22}\end{bmatrix}$$

$$\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}a_{13}\\a_{23}\end{bmatrix}$$

Check:

$$A\begin{bmatrix}1\\2\\3\end{bmatrix}=\begin{bmatrix}3 & 1 & -1\\ 5 & 1 & 1\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix}=1\begin{bmatrix}3\\5\end{bmatrix}+2\begin{bmatrix}1\\1\end{bmatrix}+3\begin{bmatrix}-1\\1\end{bmatrix}=\begin{bmatrix}2\\10\end{bmatrix}$$

The Matrix of a Linear Transformation

Matrix Transformations are linear transformations

$\mathbf{A}(\mathbf{u}+\mathbf{v}) = \mathbf{A}\mathbf{u} + \mathbf{A}\mathbf{v}$

$\mathbf{A}(c\,\mathbf{u}) = c\,\mathbf{A}$

$\longleftrightarrow$

Linear transformations can be performed as matrix transformations

$$T(\mathbf{x}) = \mathsf{A}\mathbf{x}$$

Theorem

If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation then there is a unique matrix $m \times n$ matrix A such that $T(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^n$.

To make the matrix of T:

$$A = \begin{bmatrix} | & | & & | \\ T(e_1) & T(e_2) & \cdots & T(e_n) \\ | & | & & | \end{bmatrix}$$

Examples

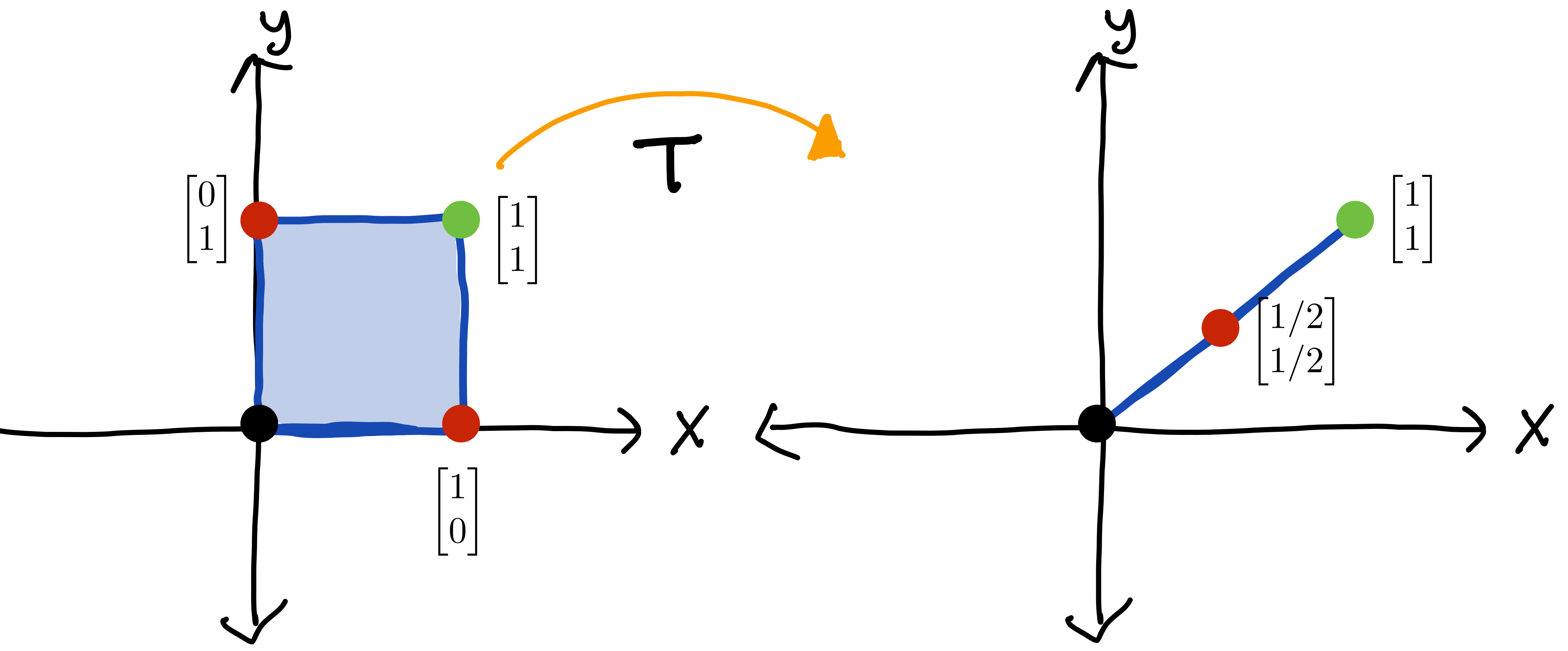
1. Make the matrix of $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 + 3x_4 \\ 5x_1 - x_3 + x_4 \\ x_1 + x_2 + x_3 + x_4 \end{bmatrix}$

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix} \qquad T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \qquad T\left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 5 & 0 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

2. Make the matrix of this transformation and check that it sends (1,1) to (1,1).



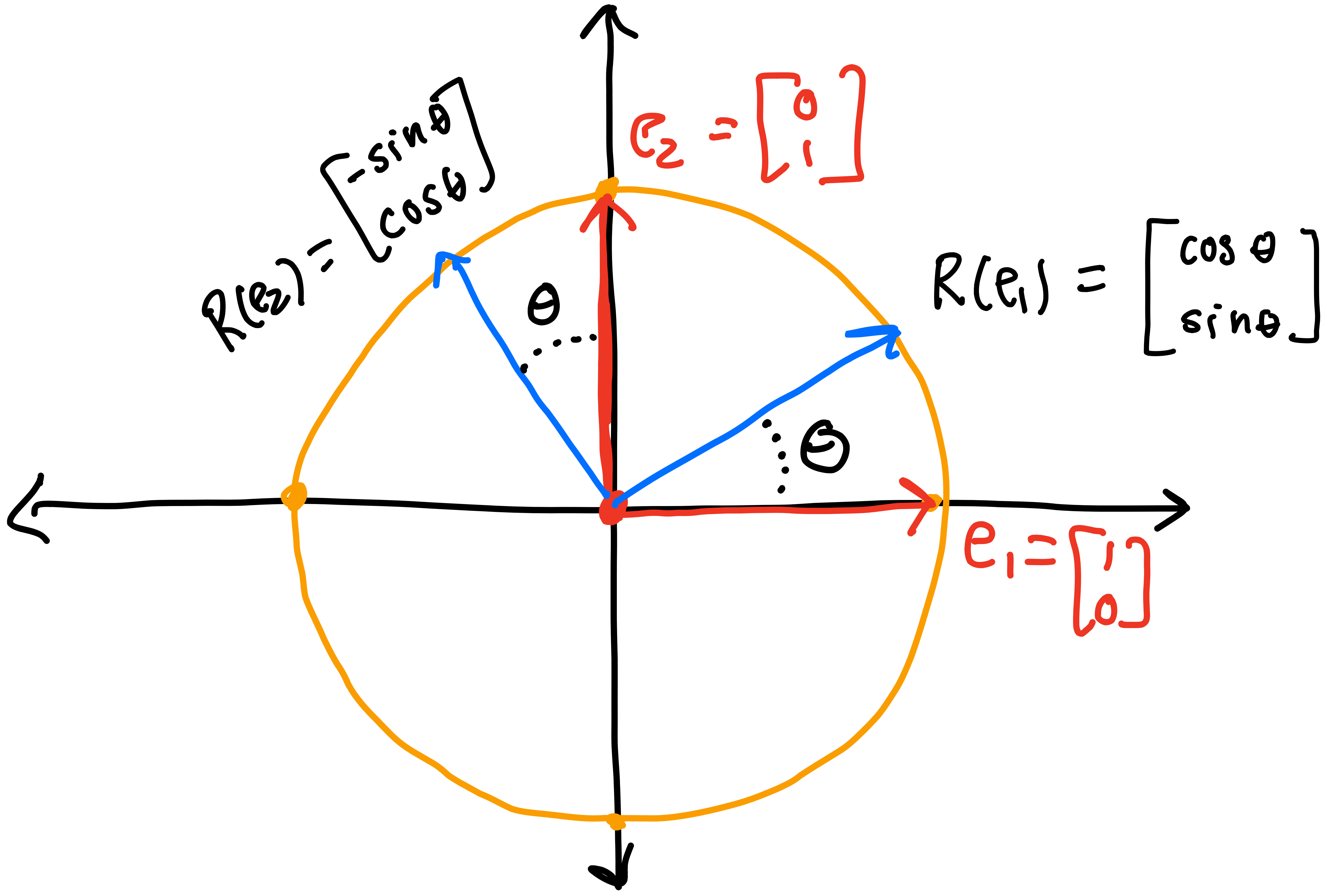
$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} \qquad T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 + 1/2 \\ 1/2 + 1/2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

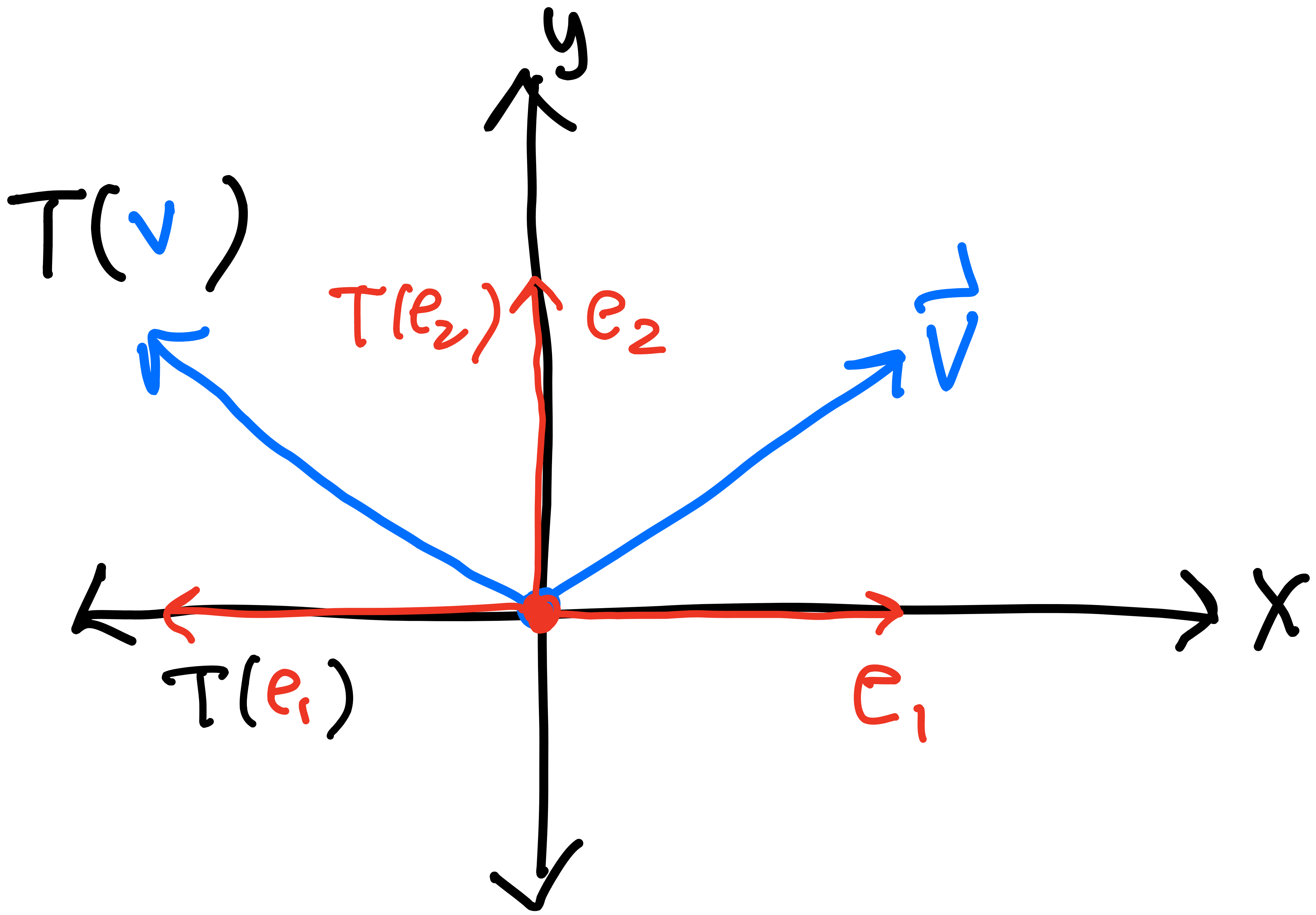
Example

$R : \mathbb{R}^2 \to \mathbb{R}^2$ is rotation by θ radians counter clockwise



$$\begin{array}{cc} T(e_1) & T(e_2) \\ \downarrow & \downarrow \end{array}$$
$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

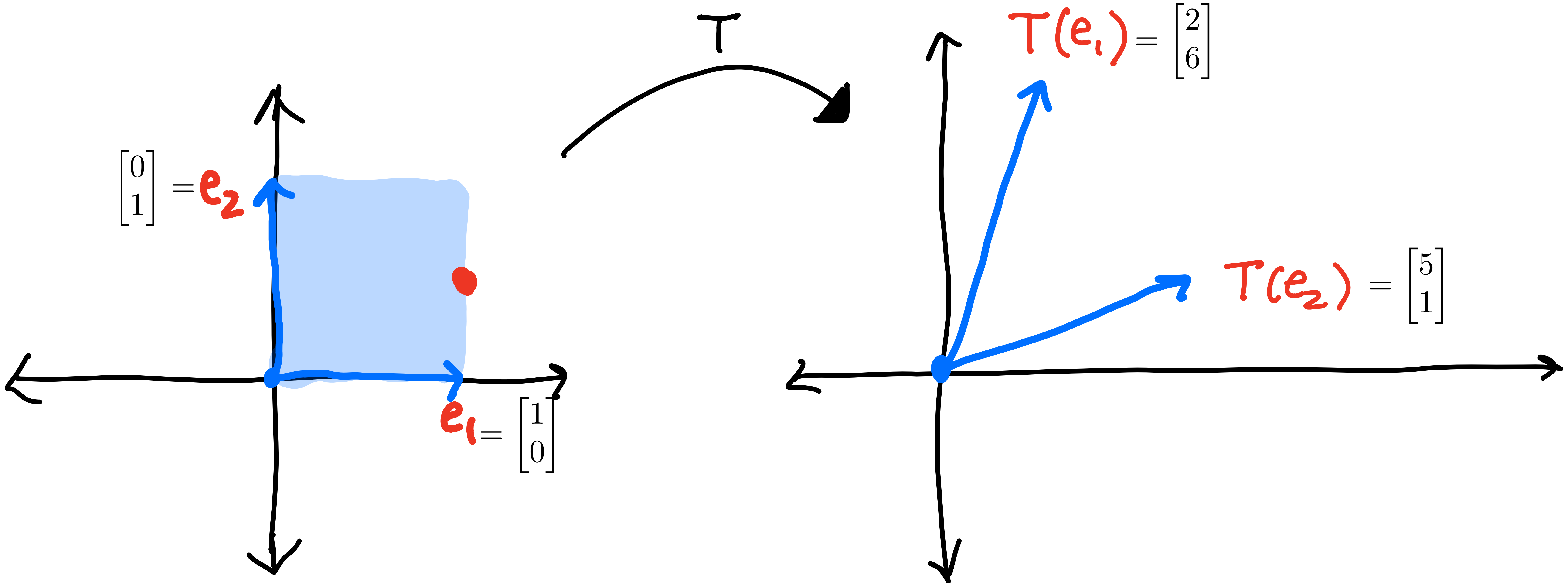
Example $T : \mathbb{R}^2 \to \mathbb{R}^2$ is reflection over the y axis



$$\begin{array}{cc} T(e_1) & T(e_2) \\ \downarrow & \downarrow \end{array}$$
$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

You Try

A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is illustrated below.



(a) Find the matrix of T
(b) The red dot is halfway up the side of the unit square. Where does it get mapped?
(c) Where does (-2,3) get mapped?