3.1-3.3. Determinants

Determinant of a 2x2 Matrix

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

What does it tell us?

$$\boxed{\text{A is invertible if and only if } \det(A) \neq 0}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Determinant of a 3x3 Matrix?

$$\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix}^{-1} = \frac{1}{\text{det}(A)} \begin{bmatrix} a_{2,2}a_{3,3} - a_{2,3}a_{3,2} & a_{1,3}a_{3,2} - a_{1,2}a_{3,3} & a_{1,2}a_{2,3} - a_{1,3}a_{2,2} \\ a_{2,3}a_{3,1} - a_{2,1}a_{3,3} & a_{1,1}a_{3,3} - a_{1,3}a_{3,1} & a_{1,3}a_{2,1} - a_{1,1}a_{2,3} \\ a_{2,1}a_{3,2} - a_{2,2}a_{3,1} & a_{1,2}a_{3,1} - a_{1,1}a_{3,2} & a_{1,1}a_{2,2} - a_{1,2}a_{2,1} \end{bmatrix}$$

$$\boxed{\text{det}(A) = a_{1,1}a_{2,2}a_{3,3} + a_{1,2}a_{2,3}a_{3,1} + a_{1,3}a_{2,1}a_{3,2} - a_{1,1}a_{2,3}a_{3,2} - a_{1,2}a_{2,1}a_{3,3} - a_{1,3}a_{2,2}a_{3,1}}$$

It is easier to invert the matrix by row reducing than it is to use the inverse formula

In fact, it is as expensive (computationally) to invert the matrix by row reducing as it is to compute the determinant.

Determinant of a 3x3 Matrix

123 231 312

$$\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix} \quad \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix} \quad \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix}$$

$$\det\left(\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix}\right) = a_{1,1}a_{2,2}a_{3,3} + a_{1,2}a_{2,3}a_{3,1} + a_{1,3}a_{2,1}a_{3,2}$$

132 213 321

$$\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix} \quad \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix} \quad \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix}$$

$$-a_{1,1}a_{2,3}a_{3,2} - a_{1,2}a_{2,1}a_{3,3} - a_{1,3}a_{2,2}a_{3,1}$$

$$\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix} \begin{matrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{matrix}$$

only works for 2x2 and 3x3

Determinant of a 4x4 Matrix

$$\det\left(\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{bmatrix}\right) =$$

$$\begin{aligned}
&a_{1,4}\,a_{2,3}\,a_{3,2}\,a_{4,1} - a_{1,3}\,a_{2,4}\,a_{3,2}\,a_{4,1} - a_{1,4}\,a_{2,2}\,a_{3,3}\,a_{4,1} + a_{1,2}\,a_{2,4}\,a_{3,3}\,a_{4,1} + \\
&a_{1,3}\,a_{2,2}\,a_{3,4}\,a_{4,1} - a_{1,2}\,a_{2,3}\,a_{3,4}\,a_{4,1} - a_{1,4}\,a_{2,3}\,a_{3,1}\,a_{4,2} + a_{1,3}\,a_{2,4}\,a_{3,1}\,a_{4,2} + \\
&a_{1,4}\,a_{2,1}\,a_{3,3}\,a_{4,2} - a_{1,1}\,a_{2,4}\,a_{3,3}\,a_{4,2} - a_{1,3}\,a_{2,1}\,a_{3,4}\,a_{4,2} + a_{1,1}\,a_{2,3}\,a_{3,4}\,a_{4,2} + \\
&a_{1,4}\,a_{2,2}\,a_{3,1}\,a_{4,3} - a_{1,2}\,a_{2,4}\,a_{3,1}\,a_{4,3} - a_{1,4}\,a_{2,1}\,a_{3,2}\,a_{4,3} + a_{1,1}\,a_{2,4}\,a_{3,2}\,a_{4,3} + \\
&a_{1,2}\,a_{2,1}\,a_{3,4}\,a_{4,3} - a_{1,1}\,a_{2,2}\,a_{3,4}\,a_{4,3} - a_{1,3}\,a_{2,2}\,a_{3,1}\,a_{4,4} + a_{1,2}\,a_{2,3}\,a_{3,1}\,a_{4,4} + \\
&a_{1,3}\,a_{2,1}\,a_{3,2}\,a_{4,4} - a_{1,1}\,a_{2,3}\,a_{3,2}\,a_{4,4} - a_{1,2}\,a_{2,1}\,a_{3,3}\,a_{4,4} + a_{1,1}\,a_{2,2}\,a_{3,3}\,a_{4,4}
\end{aligned}$$

24 summands

Recursive Definition Using Cofactors

$$\boxed{\det(A) = a_{1,1}a_{2,2}a_{3,3} + a_{1,2}a_{2,3}a_{3,1} + a_{1,3}a_{2,1}a_{3,2} - a_{1,1}a_{2,3}a_{3,2} - a_{1,2}a_{2,1}a_{3,3} - a_{1,3}a_{2,2}a_{3,1}}$$

$$= a_{1,1}(a_{2,2}a_{3,3} - a_{2,3}a_{3,2}) + a_{1,2}(a_{2,3}a_{3,1} - a_{2,1}a_{3,3}) + a_{1,3}(a_{2,1}a_{3,2}c - a_{2,2}a_{3,1})$$

$$= + \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix} - \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix} + \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix}$$

$$\begin{bmatrix} + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \end{bmatrix}$$

Can perform the expansion along any row or column using the +/- from this table

Example

$$\det(A) = +\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix} - \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix} + \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix}$$

Calculate the determinant of $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 1 \\ 2 & 0 & 4 \end{bmatrix}$

$$\det(A) = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 4 & 1 \\ 2 & 0 & 4 \end{vmatrix} = +1 \cdot \begin{vmatrix} 4 & 1 \\ 0 & 4 \end{vmatrix} - 2 \cdot \begin{vmatrix} 3 & 1 \\ 2 & 4 \end{vmatrix} + 3 \cdot \begin{vmatrix} 3 & 4 \\ 2 & 0 \end{vmatrix}$$

$$= (16 - 0) - 2(12 - 2) + 3(-8) = 16 - 20 - 24 = -28$$

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 3 & 4 & 1 \\ 2 & 0 & 4 \end{vmatrix} = -2\begin{vmatrix} 3 & 1 \\ 2 & 4 \end{vmatrix} + 4\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} - 0\begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} = -2(10) + 4(-2) - 0(-8) = -28$$

Example 2: Triangular Matrices

Calculate the determinants of the following matrices.

1. $A = \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix}$ $\det(A) =$ 6

2. $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 5 & 0 \\ 4 & 0 & 3 \end{bmatrix}$ $\det(A) =$ 30

3. $A = \begin{bmatrix} 4 & 7 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 3 \end{bmatrix}$. $\det(A) =$ 0

Make a conjecture about the value of the determinant of a triangular matrix.

The determinant of a triangular matrix is the product of its diagonal entries.

Interacts Nicely With Row Operations

1. **Swap** two rows: $\boxed{\text{Multiply determinant by } \mathbf{-1}}$
2. **Scale** a row by **c**: $\boxed{\text{Scale determinant by } \mathbf{c}}$
3. **Replace** a row by itself plus a multiply of another row: $\boxed{\text{Determinant does not change!}}$

$$\det\left(\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix}\right) = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix} \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix} \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix}$$

$$a_{1,1}a_{2,2}a_{3,3} + a_{1,2}a_{2,3}a_{3,1} + a_{1,3}a_{2,1}a_{3,2}$$

$$\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix} \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix} \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix}$$

$$-a_{1,1}a_{2,3}a_{3,2} - a_{1,2}a_{2,1}a_{3,3} - a_{1,3}a_{2,2}a_{3,1}$$

Can perform row operations: until it is upper triangular

Interacts Nicely With Row Operations

Here is why in the 3×3 case the replacement row operation does not change the determinant:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ \mathbf{t}a_1 + b_1 & \mathbf{t}a_2 + b_2 & \mathbf{t}a_3 + b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} =$$

$$- a_3 (\mathbf{t}a_2 + b_2) c_1 + a_2 (\mathbf{t}a_3 + b_3) c_1 + a_3 (\mathbf{t}a_1 + b_1) c_2$$
$$- a_1 (\mathbf{t}a_3 + b_3) c_2 - a_2 (\mathbf{t}a_1 + b_1) c_3 + a_1 (\mathbf{t}a_2 + b_2) c_3$$

$$= -a_3b_2c_1 + a_2b_3c_1 + a_3b_1c_2$$
$$- a_1b_3c_2 - a_2b_1c_3 + a_1b_2c_3$$
$$- a_3\mathbf{t}a_2c_1 + a_2\mathbf{t}a_3c_1 + a_3\mathbf{t}a_1c_2$$
$$- a_1\mathbf{t}a_3c_2 - a_2\mathbf{t}a_1c_3 + a_1\mathbf{t}a_2c_3$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \mathbf{t} \begin{vmatrix} a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Use Row Operations to Compute the Following Determinant

III. Row Reduction

$$\begin{vmatrix} 2 & 1 & -3 \\ -1 & 2 & 1 \\ 2 & 3 & 2 \end{vmatrix} \overset{\text{row swap}}{=} -\begin{vmatrix} -1 & 2 & 1 \\ 2 & 1 & -3 \\ 2 & 3 & 2 \end{vmatrix} \overset{\text{factor out } -1}{=} \begin{vmatrix} 1 & -2 & -1 \\ 2 & 1 & -3 \\ 2 & 3 & 2 \end{vmatrix} \overset{\text{pivot}}{=} \begin{vmatrix} 1 & -2 & -1 \\ 0 & 5 & -1 \\ 0 & 7 & 4 \end{vmatrix} \overset{\text{factor out } 5}{=} 5\begin{vmatrix} 1 & 2 & 1 \\ 0 & 1 & -1/5 \\ 0 & 7 & 4 \end{vmatrix} \overset{\text{pivot}}{=} 5\begin{vmatrix} 1 & 2 & 1 \\ 0 & 1 & -1/5 \\ 0 & 0 & \frac{27}{5} \end{vmatrix} = 5\cdot\underbrace{1\cdot 1\cdot \tfrac{27}{5}}_{\substack{\text{det of } \Delta \text{ matrix} \\ \text{is product} \\ \text{on diagonal}}} = \boxed{27}$$

II. COFACTOR EXPANSION

EXPAND ALONG THE TOP ROW: + − +

$$\begin{vmatrix} 2 & 1 & -3 \\ -1 & 2 & 1 \\ 2 & 3 & 2 \end{vmatrix} \qquad \begin{vmatrix} 2 & 1 & -3 \\ -1 & 2 & 1 \\ 2 & 3 & 2 \end{vmatrix} \qquad \begin{vmatrix} 2 & 1 & -3 \\ -1 & 2 & 1 \\ 2 & 3 & 2 \end{vmatrix}$$

$$= 2\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} - 1\begin{vmatrix} -1 & 1 \\ 2 & 2 \end{vmatrix} + (-3)\begin{vmatrix} -1 & 2 \\ 2 & 3 \end{vmatrix}$$

$$= 2\cdot(1) - 1(-4) + (-3)(-7)$$

$$= 2 + 4 + 21 = \boxed{27}$$

I. PERMUTATION METHOD

$$\begin{array}{ccc|cc} 2 & 1 & -3 & 2 & 1 \\ -1 & 2 & 1 & -1 & 2 \\ 2 & 3 & 2 & 2 & 3 \\ - & - & - & + & + \quad + \end{array}$$

$$= +2\cdot2\cdot2 + 1\cdot1\cdot2 + (-3)(-1)(3)$$

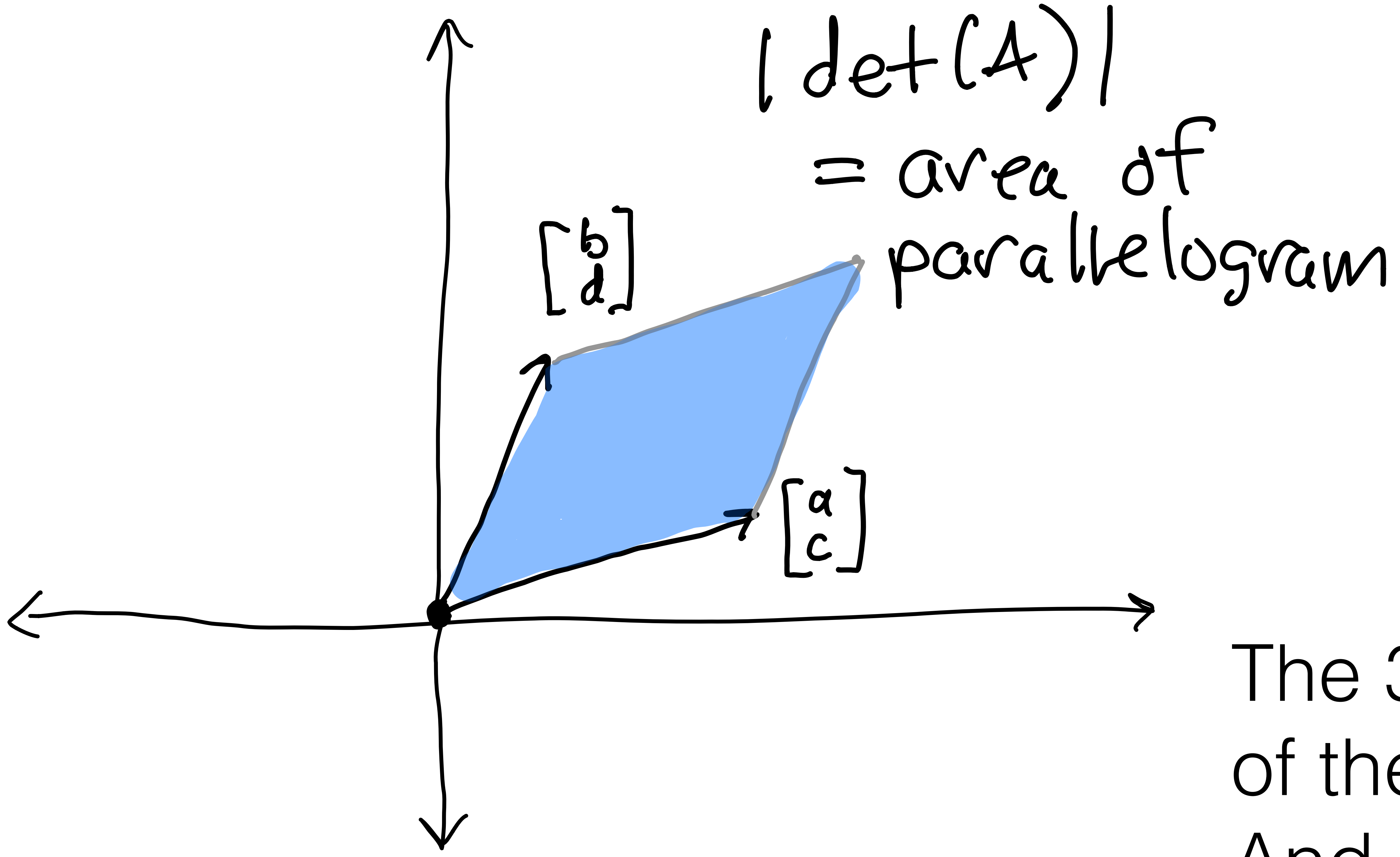
$$- 2\cdot2(-3) - 3\cdot1\cdot2 - 2(-1)(1)$$

$$= 8 + 2 + 9 + 12 - 6 + 2$$

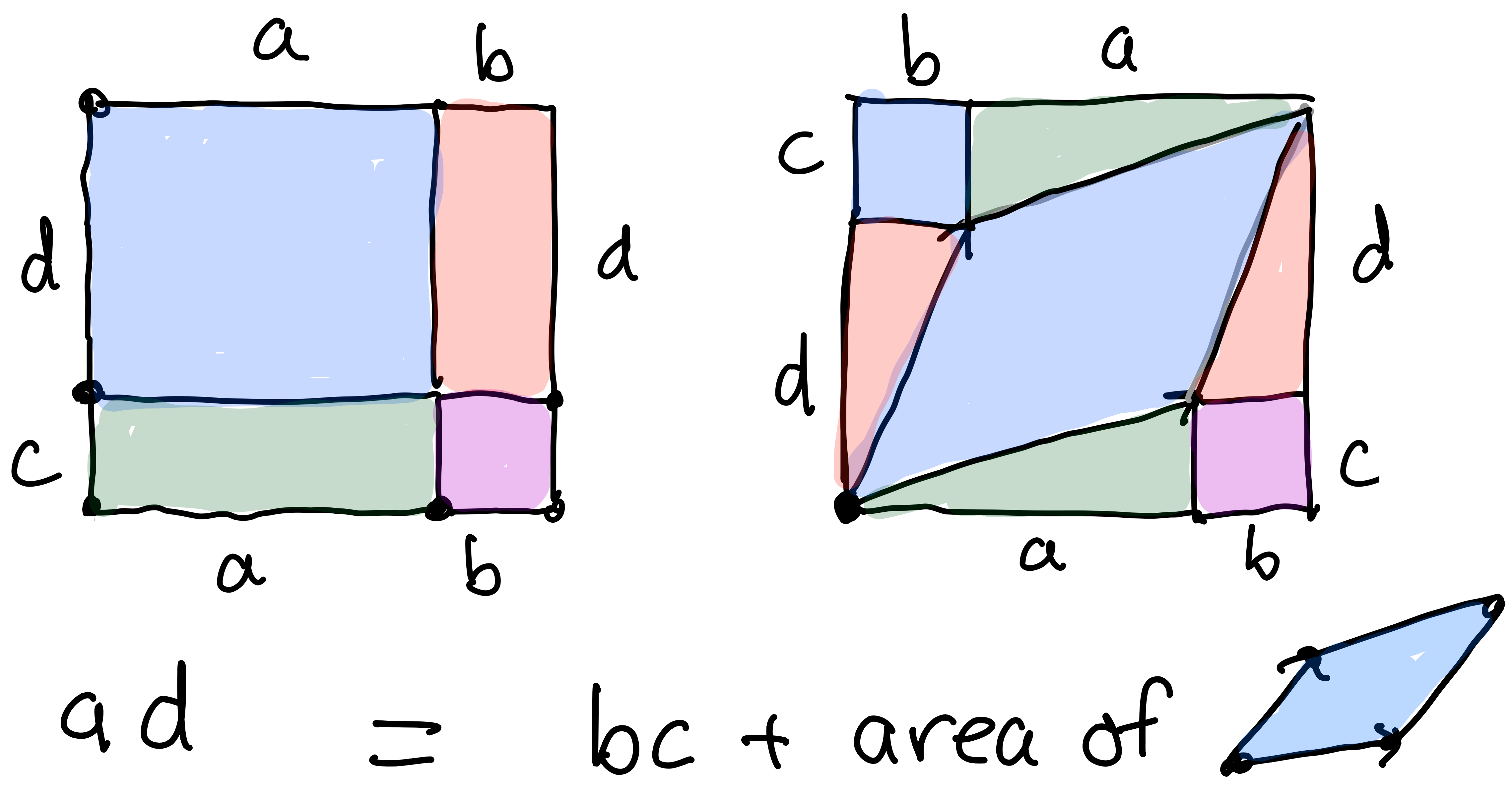
$$= 33 - 6 = \boxed{27}$$

2D Determinant as Area of Parallelogram

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$



Awesome proof by picture



The 3D determinant det(A) is the signed volume of the parallelepiped created by the columns of A. And so on for higher dimensions.

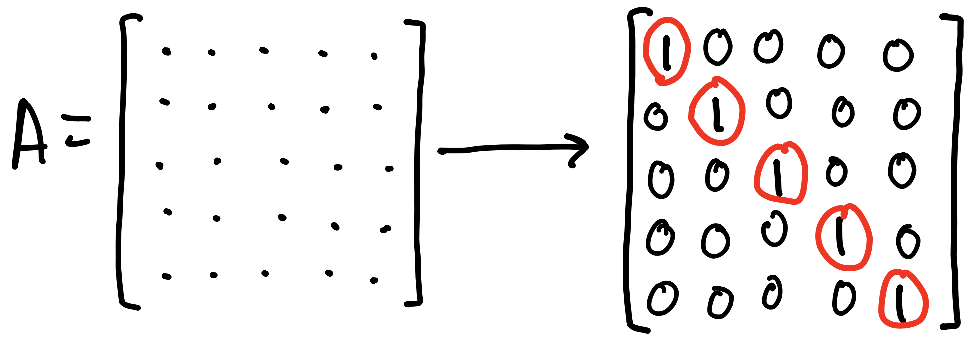
You Try!

Compute the determinant of the following 3x3 matrix in all 3 ways: (i) by row reduction to a diagonal matrix, (ii) by cofactor expansion, and (iii) using forward and backwards diagonals.

$$\det \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 8 \\ 3 & 1 & 2 \end{bmatrix}$$

Invertible Matrix Theorem Revisited

If A is an *nxn* matrix, then the following statements are equivalent



- A is invertible
- $RREF(A) = I_n$
- A has a pivot in every row
- A has a pivot in every column
- $T(x) = Ax$ is one-to-one
- $T(x) = Ax$ is onto
- The columns of A span $\mathbb{R}^n$
- The columns of A are linearly independent
- $Ax = b$ has exactly one solution for all $b \in \mathbb{R}^n$
- $Ax = 0$ has only the 0 solution

- The columns of A are a basis of $\mathbb{R}^n$
- $Col(A) = \mathbb{R}^n$
- $\dim(Col(A)) = \mathbb{R}^n$
- $rank(A) = n$
- $Nul(A) = \{0\}$
- $nullity(A) = 0$

- $\det(A) \neq 0$