# 1.1. Systems of Linear Equations

## Linear Systems of Equations

(a) $$\left\{\begin{array}{rcrcrcr} x & - & 3y & + & z & = & 4 \\ 2x & - & 8y & + & 8z & = & -2 \\ -6x & + & 3y & - & 15z & = & 9 \end{array}\right\}$$

3 equations and 3 unknowns

(b) $$\left\{\begin{array}{rcrcrcrcr} 4x & + & y & & & + & 5w & = & 1 \\ 2x & - & 2y & + & 8z & - & 2w & = & 2 \\ 3x & + & 11y & + & 3z & + & 5w & = & 3 \end{array}\right\}$$

3 equations and 4 unknowns

(c) $$\left\{\begin{array}{rcrcrcr} 4x & + & y & + & z & = & 1 \\ 2x & - & 2y & + & 8z & = & 2 \\ 3x & + & 11y & + & 3z & = & 3 \\ & & y & + & 10z & = & 0 \\ x & + & y & + & z & = & 1 \end{array}\right\}$$

5 equations and 3 unknowns

## Nonlinear Systems of Equations

$$\left\{\begin{array}{rcrcrcr} 4xy & + & y^2 & + & z & = & 1 \\ 2\frac{1}{x} & - & 2y & + & 8\sqrt{z} & = & 2 \end{array}\right\}$$

$$\left\{\begin{array}{rcr} \cos(x^2) + 5\sin(2y) & = & 1 \\ y + Ae^{5x^2+2x+3y} & = & 0 \end{array}\right\}$$

What about?

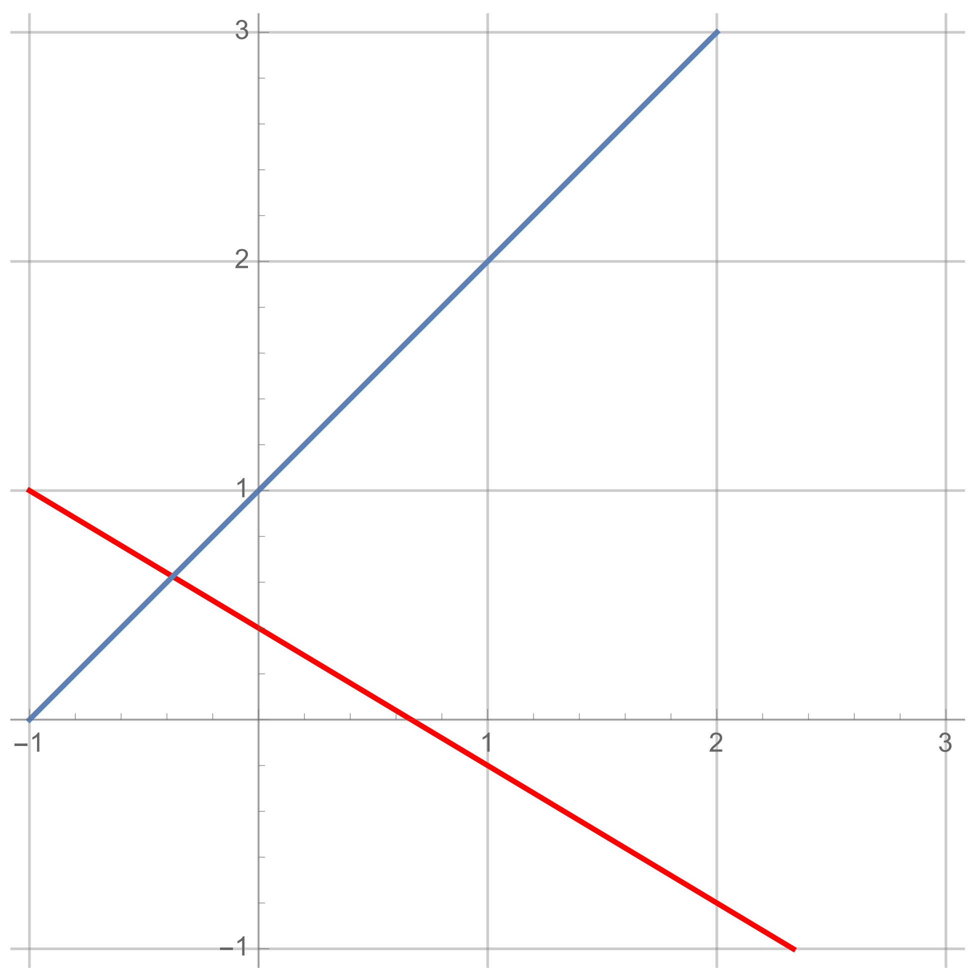
$$\left\{\begin{array}{r} 1 + 5x - 2y = 10 \\ 4x - 10y + 3x = 11 \end{array}\right\}$$ Yes!

rearrange $\rightarrow$ $$\left\{\begin{array}{rcr} 5x - 2y & = & 9 \\ 7x - 10y & = & 11 \end{array}\right\}$$

# Solutions to Linear Systems of Equations

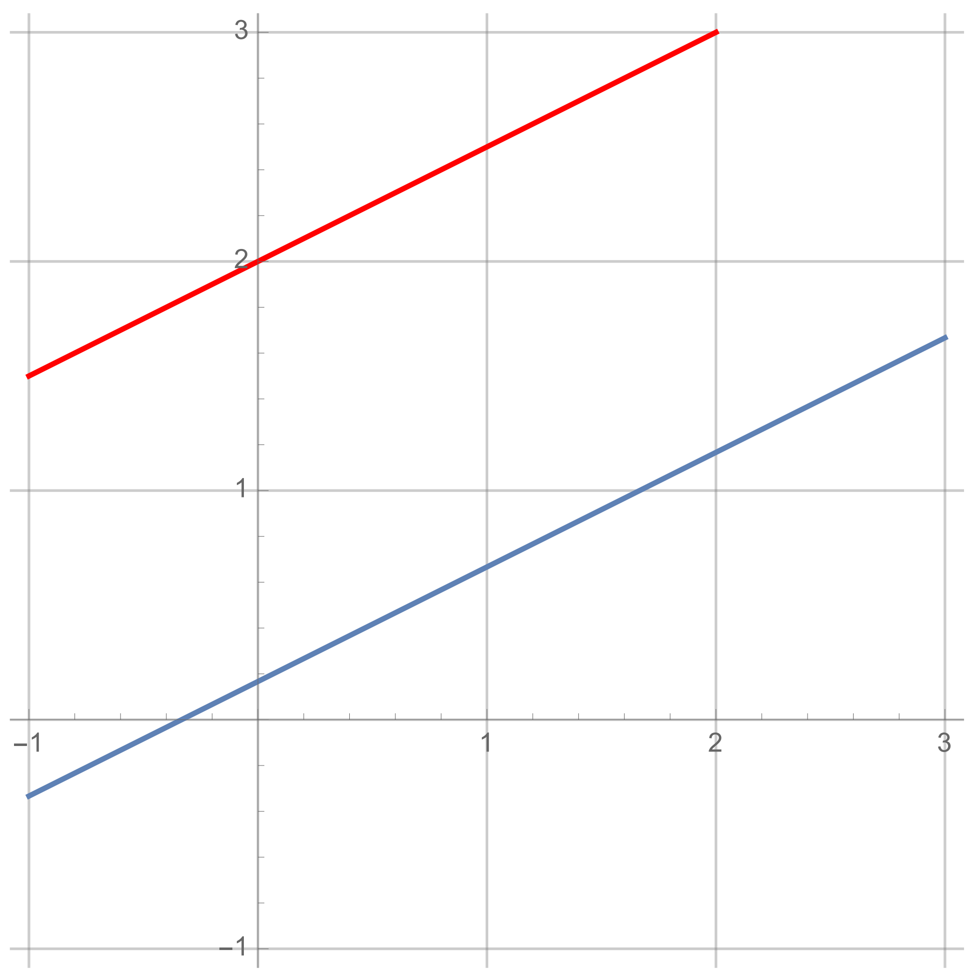
Q1. How many solutions do each of these systems of 2x2 linear equations have?
How do you find them?

**(A)** $\left\{ \begin{array}{rcrcr} x & - & y & = & -1 \\ 3x & + & 5y & = & 2 \end{array} \right\}$



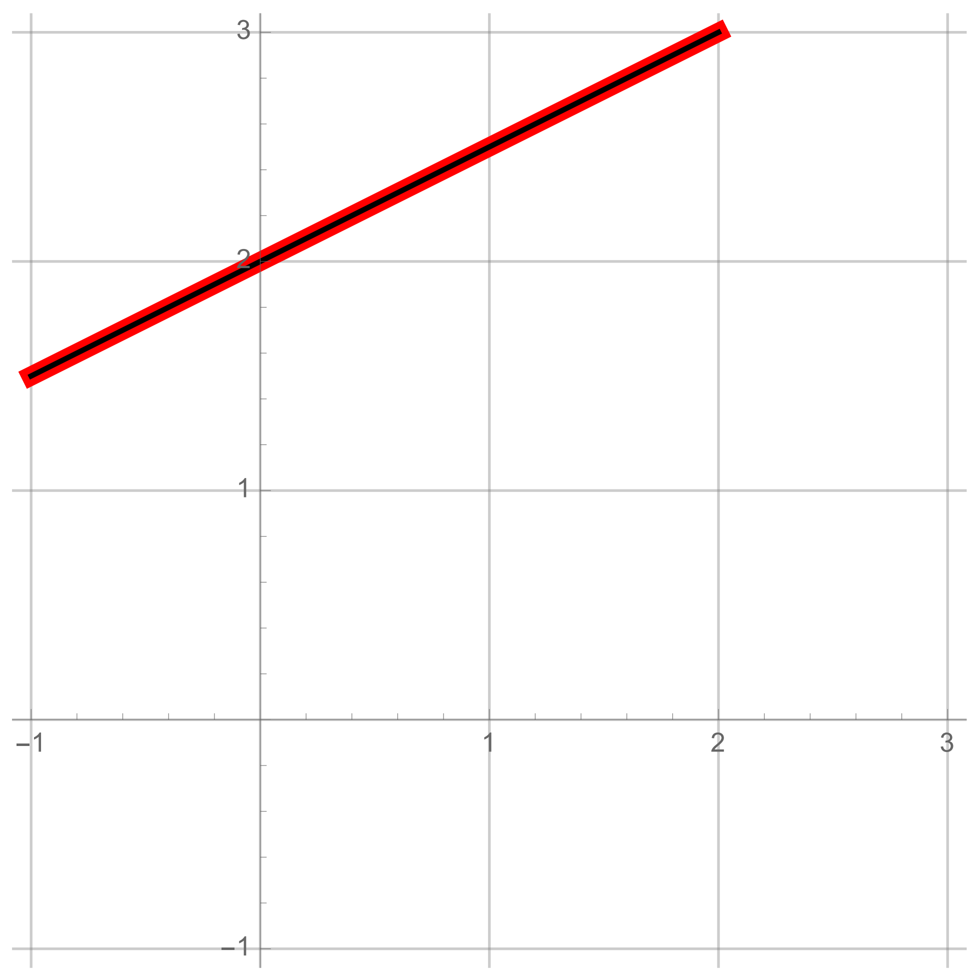
1. Exactly One Solution

**(B)** $\left\{ \begin{array}{rcrcr} -x & + & 2y & = & 4 \\ -3x & + & 6y & = & 1 \end{array} \right\}$



2. No Solutions

**(C)** $\left\{ \begin{array}{rcrcr} x & - & 2y & = & -4 \\ -3x & + & 6y & = & 12 \end{array} \right\}$



3. Infinitely Many Solutions

# Solving Linear Systems of Equations

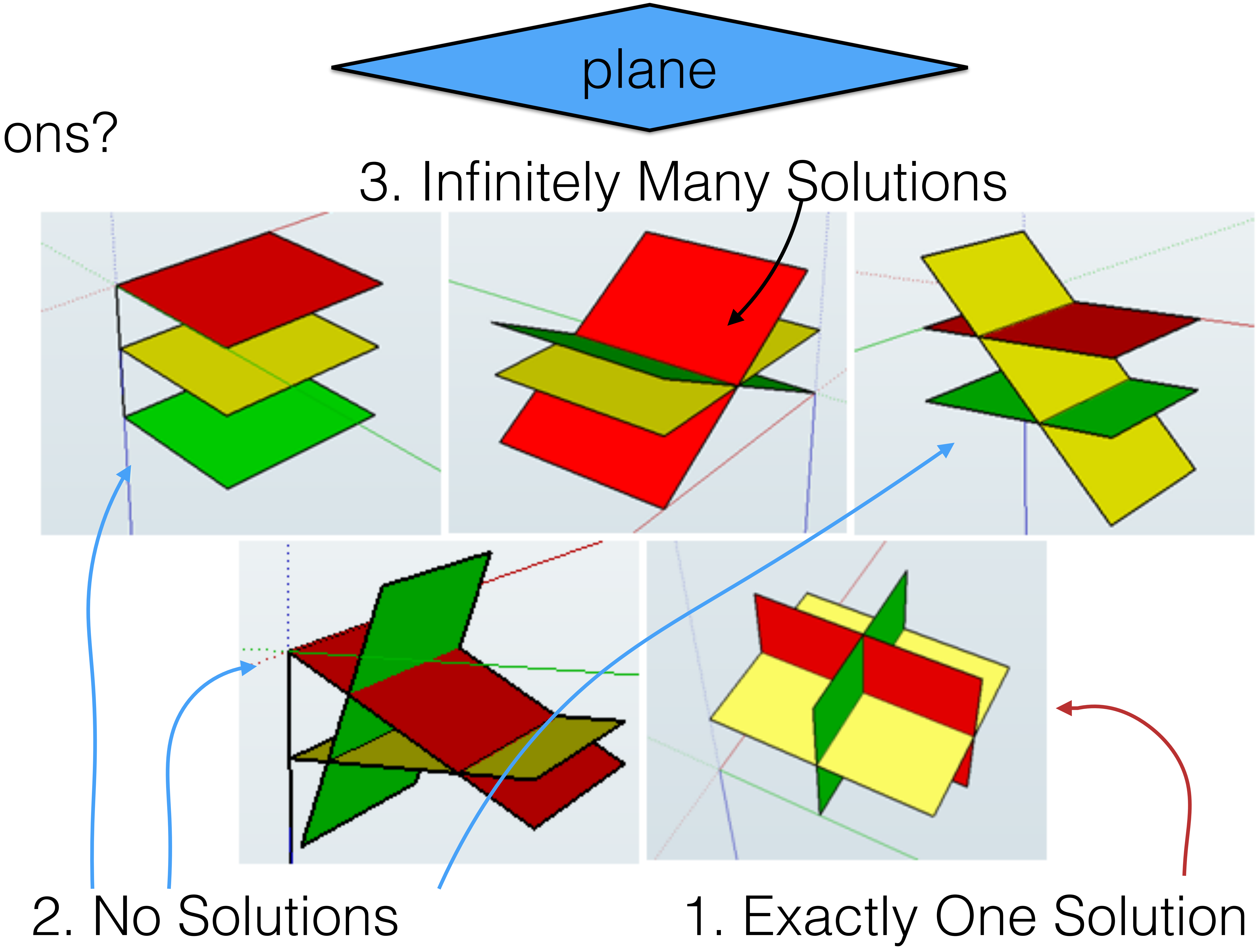
Q2. How about this 3x3 system of equations?

$$\left\{ \begin{array}{ccccccc} x & + & 2y & + & 3z & = & 6 \\ 4x & + & 5y & + & 6z & = & 3 \\ x & + & 2y & + & 2z & = & 4 \end{array} \right\}$$

How many possible solutions could a situation like this have?

How do we solve this particular system?

What is the geometry?



# Matrix Short Hand

**System of Equations** | **Augmented Coefficient Matrix**

$$\left\{\begin{array}{rcrcrcr} x & - & 3y & + & z & = & 4 \\ 2x & - & 8y & + & 8z & = & -2 \\ -6x & + & 3y & - & 15z & = & 9 \end{array}\right\} \longleftrightarrow \left[\begin{array}{rrr|r} 1 & -3 & 1 & 4 \\ 2 & -8 & 8 & -2 \\ -6 & 3 & -15 & 9 \end{array}\right]$$

$$\left\{\begin{array}{rcrcrcrcr} 4x & + & y & & & + & 5w & = & 1 \\ 2x & - & 2y & + & 8z & - & 2w & = & 2 \\ 3x & + & 11y & + & 3z & + & 5w & = & 3 \end{array}\right\} \longleftrightarrow \left[\begin{array}{cccc|c} 4 & 1 & 0 & 5 & 1 \\ 2 & -2 & 8 & -2 & 2 \\ 3 & 11 & 3 & 5 & 3 \end{array}\right]$$

$$\left\{\begin{array}{rcrcrcr} 4x & + & y & + & z & = & 1 \\ 2x & - & 2y & + & 8z & = & 2 \\ 3x & + & 11y & + & 3z & = & 3 \\ & & y & + & 10z & = & 0 \\ x & + & y & + & z & = & 1 \end{array}\right\} \longleftrightarrow \left[\begin{array}{ccc|c} 4 & 1 & 1 & 1 \\ 2 & -2 & 8 & 2 \\ 3 & 11 & 3 & 3 \\ 0 & 1 & 10 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right]$$

$$\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \underbrace{\begin{bmatrix} 1 & -3 & 1 \\ 2 & -8 & 8 \\ -6 & 3 & -15 \end{bmatrix}}_{\text{Coefficient Matrix}} \underbrace{\begin{bmatrix} 4 \\ -2 \\ 9 \end{bmatrix}}_{\text{RHS}} \end{array}$$

# Elementary Row Operations

**(E1) Replacement**: Add a multiple of one row to another

$$\left[\begin{array}{rrr|r} 1 & -3 & 1 & 4 \\ 2 & -8 & 8 & -2 \\ -6 & 3 & -15 & 9 \end{array}\right] \longrightarrow \left[\begin{array}{lll|l} 1 & -3 & 1 & 4 \\ 2+(-2)1 & -8+(-2)-3 & 8+(-2)1 & -2+(-2)4 \\ -6 & 3 & -15 & 9 \end{array}\right] = \left[\begin{array}{rrr|r} 1 & -3 & 1 & 4 \\ 0 & -6 & 6 & -10 \\ -6 & 3 & -15 & 9 \end{array}\right]$$

**(E2) Exchange** two rows

$$\left[\begin{array}{rrr|r} 1 & -3 & 1 & 4 \\ 2 & -8 & 8 & -2 \\ -6 & 3 & -15 & 9 \end{array}\right] \longrightarrow \left[\begin{array}{rrr|r} 2 & -8 & 8 & -2 \\ 1 & -3 & 1 & 4 \\ -6 & 3 & -15 & 9 \end{array}\right]$$

**Key Point**: these operations do not change the set of solutions

**(E3) Scale**: Multiply a row by a nonzero constant

$$\left[\begin{array}{rrr|r} 1 & -3 & 1 & 4 \\ 2 & -8 & 8 & -2 \\ -6 & 3 & -15 & 9 \end{array}\right] \longrightarrow \left[\begin{array}{rrr|r} 1 & -3 & 1 & 4 \\ \frac{1}{2}2 & \frac{1}{2}(-8) & \frac{1}{2}8 & \frac{1}{2}(-2) \\ -6 & 3 & -15 & 9 \end{array}\right] = \left[\begin{array}{rrr|r} 1 & -3 & 1 & 4 \\ 1 & -4 & 4 & -1 \\ -6 & 3 & -15 & 9 \end{array}\right]$$

# Elementary Row Operations: Why do these work?

**(E1) Replacement**: Add a multiple of one row to another

**(E2) Exchange** two rows

**(E3) Scale**: Multiply a row by a nonzero constant

**Key Point**: these operations do not change the set of solutions

$$\begin{cases} 5x + 3y - 6z = 10 \\ 9x - 2y + 4z = 5 \end{cases} \Leftrightarrow \begin{cases} 5x + 3y - 6z - 10 = 0 \\ 9x - 2y + 4z - 5 = 0 \end{cases}$$

$$\begin{cases} \text{Equation } 1 = 0 \\ \text{Equation } 2 = 0 \end{cases} \Leftrightarrow \begin{cases} \text{Equation } 1 = 0 \\ c\,\text{Equation } 1 + \text{Equation } 2 = 0 \end{cases}$$

# Eg. Solve the following 3 x 3 system of equations

$$\left\{\begin{array}{ccccccc} x & + & 2y & + & 3z & = & 6 \\ 4x & + & 5y & + & 6z & = & 3 \\ x & + & 2y & + & 2z & = & 4 \end{array}\right\}$$

At this point can we see how many solutions there will be?

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 4 & 5 & 6 & 3 \\ 1 & 2 & 2 & 4 \end{array}\right] \longrightarrow \begin{array}{c} row_2 - 4\,row_1 \rightarrow \\ row_3 - row_1 \rightarrow \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -3 & -6 & -21 \\ 0 & 0 & -1 & -2 \end{array}\right]$$

Exactly one solution:
x = - 6, y = 3, z = 2

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 7 \\ 0 & 0 & 1 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right] \longrightarrow \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right] \end{array}$$

how do we check the answer?

# Comparing 3 Examples

Exactly one solution

$$\left\{\begin{array}{ccccccc} x & + & 2y & + & 3z & = & 6 \\ 4x & + & 5y & + & 6z & = & 3 \\ x & + & 2y & + & 2z & = & 4 \end{array}\right\} \quad \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 4 & 5 & 6 & 3 \\ 1 & 2 & 2 & 4 \end{array}\right] \longrightarrow \cdots \longrightarrow \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right] \end{array}$$

x = -6, y = 3, z = 2

$$\left\{\begin{array}{ccccccc} x_1 & + & 2x_2 & + & 3x_3 & = & 0 \\ 4x_1 & + & 5x_2 & + & 6x_3 & = & 3 \\ 7x_1 & + & 8x_2 & + & 9x_3 & = & 0 \end{array}\right\} \quad \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 3 \\ 7 & 8 & 9 & 0 \end{array}\right] \longrightarrow \cdots \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

No solutions 0=1 !!

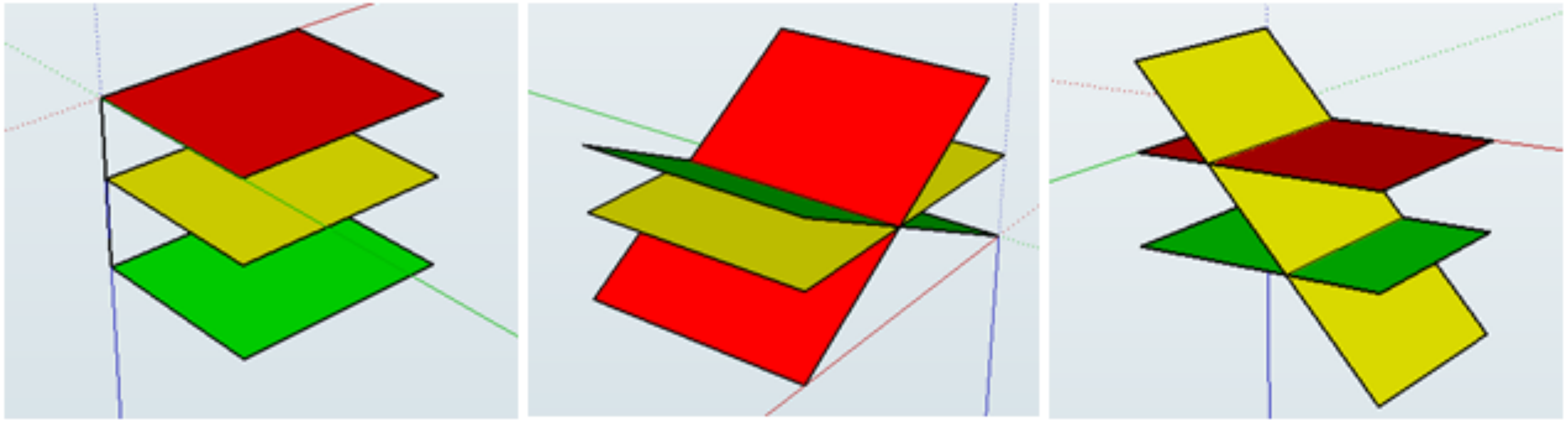
$$\left\{\begin{array}{ccccccc} 2x_1 & + & 4x_2 & + & 6x_3 & = & 0 \\ 4x_1 & + & 5x_2 & + & 6x_3 & = & 3 \\ 7x_1 & + & 8x_2 & + & 9x_3 & = & 6 \end{array}\right\} \quad \left[\begin{array}{ccc|c} 2 & 4 & 6 & 0 \\ 4 & 5 & 6 & 3 \\ 7 & 8 & 9 & 6 \end{array}\right] \longrightarrow \cdots \longrightarrow \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

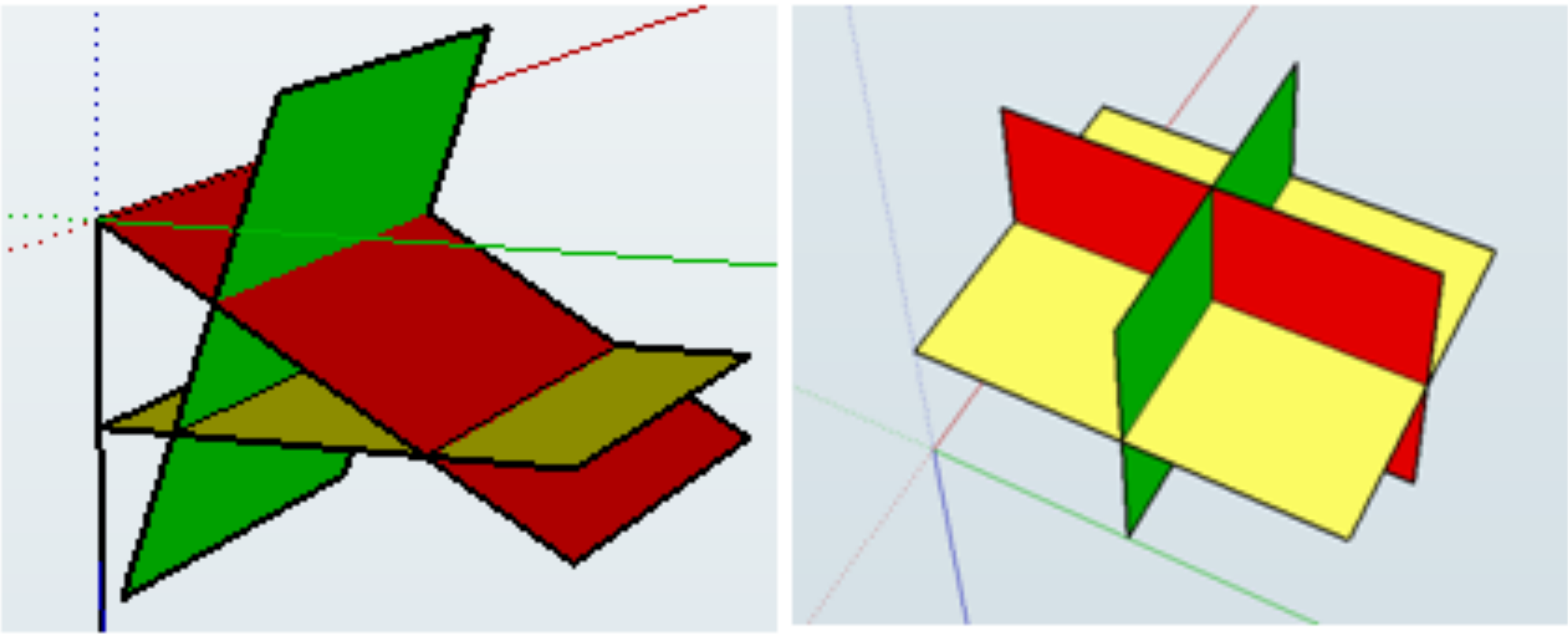
x = 2 + z
y = -1 - 2 z
z free to be anything

x - z = 2
y + 2z = -1

Infinitely many solutions

# Comparing 3 Examples





Exactly one solution

$$\begin{array}{c} \begin{array}{cccc} x & y & z & \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right] \end{array}$$

$\boxed{x = -6,\ y = 3,\ z = 2}$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

No solutions $\boxed{0=1 \text{ !!}}$

$$\begin{array}{c} \begin{array}{cccc} x & y & z & \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

Infinitely many solutions

$x = 2 + z$
$y = -1 - 2z$
$z$ free to be anything

$x - z = 2$
$y + 2z = -1$

# Two Fundamental Questions

1. Is the system **consistent**? Is there at least one solution
2. If there is a solution is it **unique** (that is, is it the only one or are there infinitely many)?

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 4 & 5 & 6 & 3 \\ 1 & 2 & 2 & 4 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

x = -6, y = 3, z = 2

Consistent and unique

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 3 \\ 7 & 8 & 9 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

No solutions 0=1 !!

Inconsistent

$$\left[\begin{array}{ccc|c} 2 & 4 & 6 & 0 \\ 4 & 5 & 6 & 3 \\ 7 & 8 & 9 & 6 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

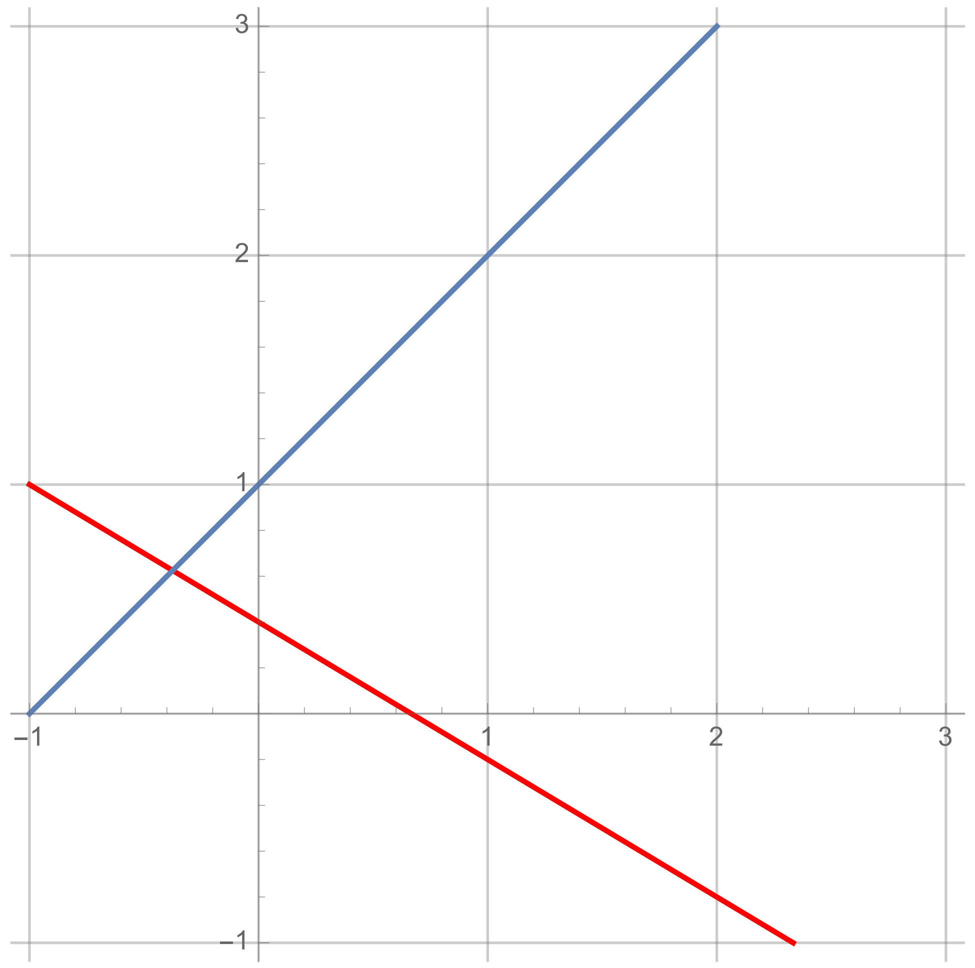
x = 2 + z
y = -1 - 2 z
z free to be anything

Consistent and not unique: infinitely many

# Two fundamental questions about a linear systems of equations

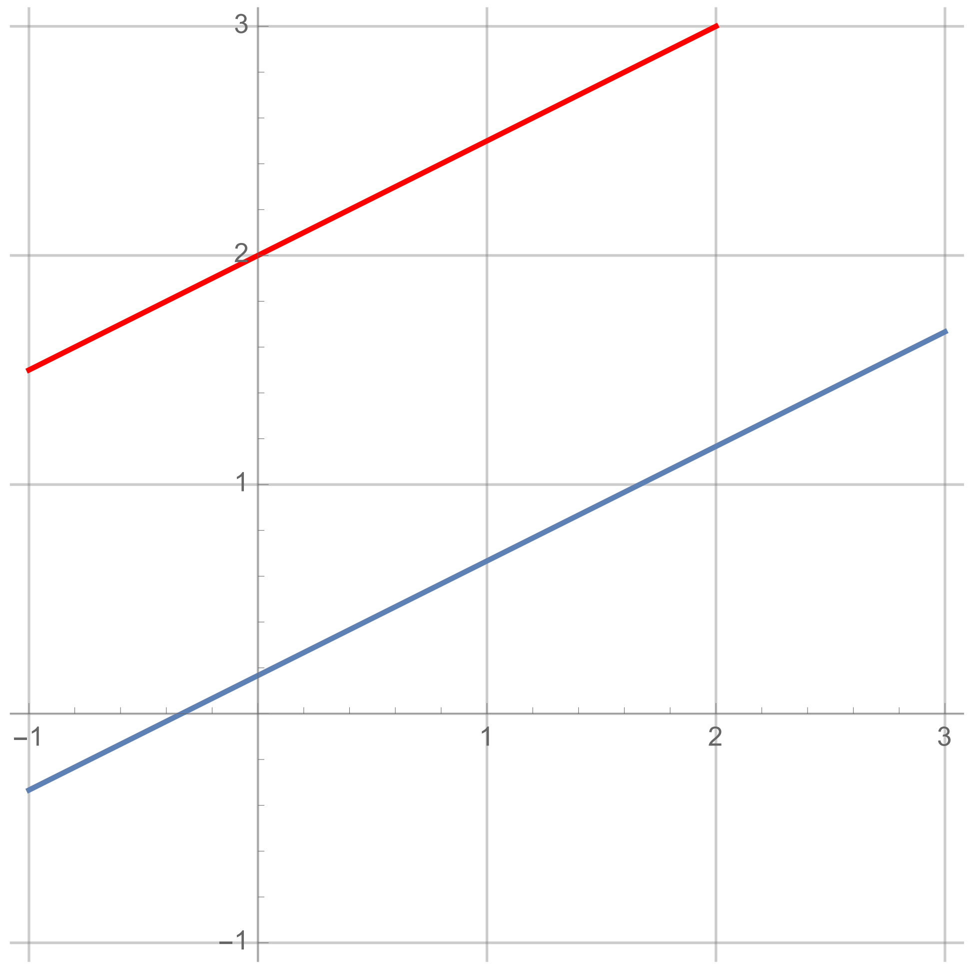
1. Is the system **consistent**? Is there at least one solution
2. If there is a solution is it **unique** (that is, is it the only one or are there infinitely many)?

**(A)** $\left\{ \begin{array}{rcrcr} x & - & y & = & -1 \\ 3x & + & 5y & = & 2 \end{array} \right\}$



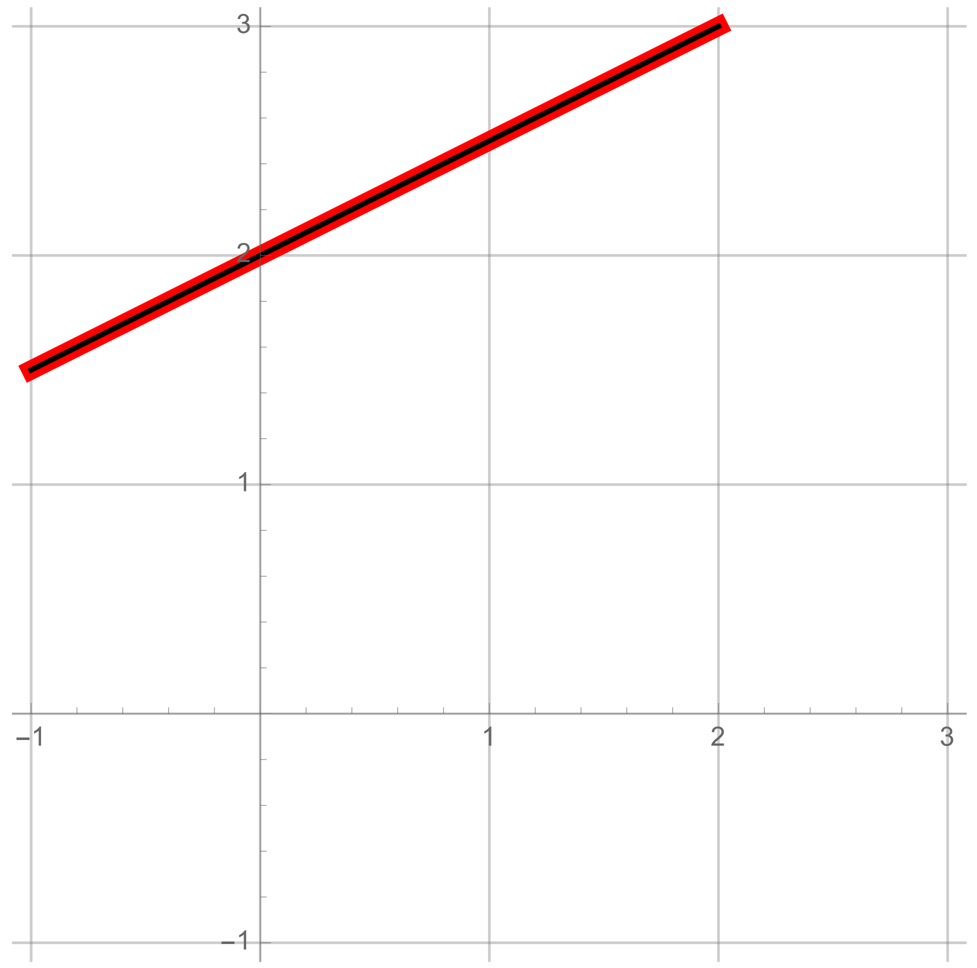

Consistent and unique

**(B)** $\left\{ \begin{array}{rcrcr} -x & + & 2y & = & 4 \\ -3x & + & 6y & = & 1 \end{array} \right\}$




Inconsistent

**(C)** $\left\{ \begin{array}{rcrcr} x & - & 2y & = & -4 \\ -3x & + & 6y & = & 12 \end{array} \right\}$




Consistent and not unique: infinitely many