Subspaces: We are interested in subsets $S \subseteq \mathbb{R}^n$ that are also vector spaces. These are called *subspaces*. To be a subspace, S must satisfy two properties:

0. $\mathbf{0} \in S$
1. If $\vec{u}, \vec{v} \in S$ then $\vec{u} + \vec{v} \in S$.
2. If $\vec{u} \in S$, then $c\vec{u} \in S$.

1. The span of a set of vectors is a subspace. Describe the span of the vectors below

$$S = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \right\}$$

done on board
in class

2. The solutions to $Ax = \mathbf{0}$ is a subspace called the *null space*. Describe the the null space of the matrix A below

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 5 & -1 & 4 \end{bmatrix}$$

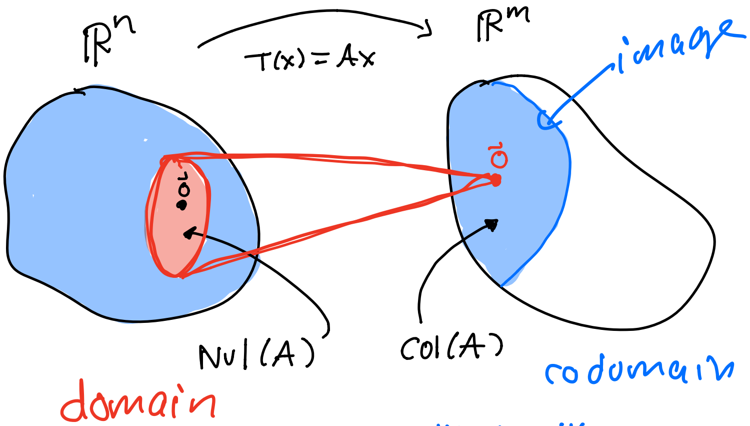
$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 5 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & -1 \\ 0 & -6 & -6 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

done on board
in class

Null space and Column space: If A is an $m \times n$ matrix then its *null space* and *column space* are

$$\boxed{\mathsf{Nul(A)} = \{x \in \mathbb{R}^n \mid Ax = \mathbf{0}\} \subseteq \mathbb{R}^n}$$

$$\boxed{\begin{aligned} \mathsf{Col(A)} &= \{\vec{\mathsf{b}} \in \mathbb{R}^m \mid Ax = \vec{\mathsf{b}} \text{ is consistent } \} \\ &= \{\mathsf{A}x \mid x \in \mathbb{R}^n\} \\ &= \text{span of the columns of A} \\ &= \text{the range or the image of the transformation} \end{aligned}}$$



Remember

$$A\mathbf{x} = \begin{bmatrix} | & | & | & & | \\ v_1 & v_2 & v_3 & \cdots & v_n \\ | & | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 v_1 + x_2 v_2 + \cdots + x_n v_n = b$$

1. Consider the matrix

$$\mathsf{A} = \begin{array}{ccc} w_1 & w_2 & w_3 \\ \end{array} \begin{bmatrix} 1 & 2 & 1 \\ -2 & -7 & 1 \\ 3 & 8 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & 3 \\ 0 & 2 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{array}{c} \begin{array}{ccc} \downarrow & \downarrow & s \end{array} \\ \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

(a) Describe the null space Nul(A).

$$\mathrm{Nul}(A) = \left\{ s\begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} \,\middle|\, s \in \mathbb{R} \right\} \subseteq \mathbb{R}^3$$

$$= \mathrm{span}\left(\begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}\right)$$

$$3w_1 - w_2 = w_3$$

(b) Describe the column space Col(A).

$$\mathrm{Col}(A) = \mathrm{span}(w_1, w_2, w_3)$$

$$= \mathrm{span}(w_1, w_2)$$

Discussion

1. Sometimes we can turn other subspaces into column spaces and null spaces.

(a) The set below is a subspace of $\mathbb{R}^3$. Show that this set is the null space of a matrix A (and find A).

$$S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \middle| \underbrace{3x_1 = 2x_2, x_1 + x_2 + x_3 = 0}_{\text{set of linear equations}} \right\}.$$

$$\left\{ \begin{array}{l} 3x_1 - 2x_2 = 0 \\ x_1 + x_2 + x_3 = 0 \end{array} \right\}$$

$$\begin{bmatrix} 3 & -2 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$= \text{Nul}\left(\begin{bmatrix} 3 & -2 & 0 \\ 1 & 1 & 1 \end{bmatrix} \right)$$

(b) The set below is a subspace of $\mathbb{R}^3$. Show that this set is the columns space of a matrix A (and find A).

$$S = \left\{ \begin{bmatrix} a + 2b \\ 2a - b \\ a + b \end{bmatrix} \middle| a, b \in \mathbb{R} \right\}.$$

$$= \left\{ a \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + b \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \middle| a, b \in \mathbb{R} \right\}$$

$$= \text{span}\left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right)$$

$$= \text{Col}\left(\begin{bmatrix} 1 & 2 \\ 2 & -1 \\ 1 & 1 \end{bmatrix} \right)$$

2. If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation with $m \times n$ matrix A shown below, then (i) find a set of vectors which span $\mathsf{Col}(A)$ and find a set of vectors which span $\mathsf{Nul}(A)$. (ii) Determine if your vectors are linearly independent or not. (iii) If they are not, can you remove redundancies to get a linearly independent set that still spans?

(a) $$A = \begin{array}{cccccc} w_1 & w_2 & w_3 & w_4 & w_5 & w_6 \\ \end{array}$$
$$A = \begin{bmatrix} 1 & 1 & 1 & -2 & 3 & 0 \\ -2 & 1 & -1 & -2 & -2 & -5 \\ -1 & 1 & 2 & -9 & 2 & -1 \end{bmatrix} \to \begin{array}{cccccc} & & & s & t & u \\ \end{array} \begin{bmatrix} 1 & 0 & 0 & 2 & 1 & 1 \\ 0 & 1 & 0 & -1 & 1 & -2 \\ 0 & 0 & 1 & -3 & 1 & 1 \end{bmatrix}$$

$$\mathrm{Nul}(A) = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = s \underbrace{\begin{bmatrix} -2 \\ 1 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{v_1} + t \underbrace{\begin{bmatrix} -1 \\ -1 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{v_2} + u \underbrace{\begin{bmatrix} -1 \\ 2 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{v_3} \;\middle|\; s,t,u \in \mathbb{R} \right\}$$

$\boxed{\text{is a 3-dimension subspace of } \mathbb{R}^6}$

$= \mathrm{span}(v_1, v_2, v_3)$

$\mathrm{Col}(A) = \mathrm{span}(w_1, w_2, w_3, w_4, w_5, w_6) = \mathrm{span}(w_1, w_2, w_3)$

$\boxed{\text{is all of } \mathbb{R}^3}$

(b) $$A = \begin{array}{c} \begin{array}{ccccc} w_1 & w_2 & w_3 & w_4 & w_5 \end{array} \\ \begin{bmatrix} 1 & 1 & \cancel{3} & 1 & \cancel{2} \\ -2 & 1 & \cancel{-3} & -1 & \cancel{-9} \\ -1 & 1 & \cancel{-1} & 2 & \cancel{-3} \\ 4 & 1 & \cancel{9} & 3 & \cancel{13} \\ -2 & 3 & \cancel{-1} & 2 & \cancel{-10} \end{bmatrix} \end{array} \to \begin{array}{c} \begin{array}{ccccc} & & s & & t \end{array} \\ \begin{bmatrix} 1 & 0 & 2 & 0 & 3 \\ 0 & 1 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$\mathrm{Col}(A) = \mathrm{span}(w_1, w_2, w_3, w_4, w_5) = \mathrm{span}(w_1, w_2, w_4)$

$\boxed{\text{is a 3-dimensional subspace of } \mathbb{R}^5}$

$$\mathrm{Nul}(A) = \left\{ s \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 2 \\ 0 \\ -1 \\ 1 \end{bmatrix} \;\middle|\; s,t \in \mathbb{R} \right\}$$

$\boxed{\text{is a 2-dimensional subspace of } \mathbb{R}^5 \quad \text{a line}}$

3. Work on some problems from PS5. Here are some recommended things to try in no particular order.

5.1 This is like the problems above. You will need to row reduce the matrix in R.

5.2 This is like a problem on Monday's in-class exercises. You can look at Monday's solutions for ideas.

5.3 Just work on (a) for now. We will cover the other ideas on Friday and have time to work on it then.

5.4 Also about column and null space.