1.7. Linear Independence

Definition

The vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \in \mathbb{R}^n$ are **linearly dependent** if there exist constants $c_1, c_2, \ldots, c_k$ **not all zero** such that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_k\vec{v}_k = \vec{0}.$$

Otherwise, the vectors are **linearly independent**.

Example: $\begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

are **linearly dependent** because

$$\begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Example: $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

are **linearly independent**

$$c_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 + c_3 \\ c_2 + c_3 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

More Examples

(a) Dependent: $\begin{bmatrix} -1 \\ 2 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ -6 \\ 0 \\ -12 \end{bmatrix}$

since $3\begin{bmatrix} -1 \\ 2 \\ 0 \\ 4 \end{bmatrix} + \begin{bmatrix} 3 \\ -6 \\ 0 \\ -12 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

(b) Dependent: $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

since $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} - 2\begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

(c) Independent $c_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

since must have: $c_1 = c_2 = c_3 = 0$

(d) Dependent $0\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(e) Independent: $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

(f) Independent: $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$

Dependence implies Redundancy

A dependence relation:

$3\vec{v}_1 + 5\vec{v}_2 + 0\vec{v}_3 - 2\vec{v}_4 = \vec{0}$

Lets you solve for one of the variables in terms of the others.

$5\vec{v}_2 = -3\vec{v}_1 + 2\vec{v}_4$

$\vec{v}_2 = -\frac{3}{5}\vec{v}_1 + \frac{2}{5}\vec{v}_4$

Note: I could not have solved for v_3 because of the 0 weight

Example:

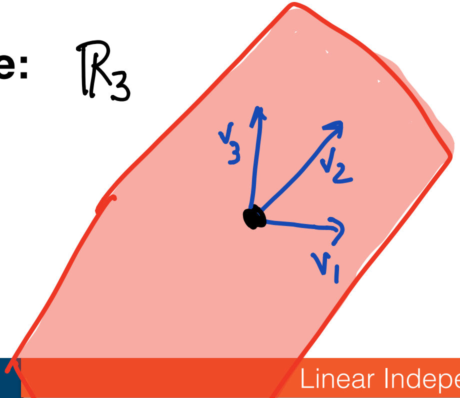
$$-2\begin{bmatrix}1\\1\\0\\1\end{bmatrix} + 3\begin{bmatrix}1\\2\\2\\1\end{bmatrix} - 4\begin{bmatrix}-1\\1\\1\\-1\end{bmatrix} - \begin{bmatrix}5\\0\\2\\5\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \quad \mathbf{v}_4$

$\mathbf{v}_4 = -2\,\mathbf{v}_1 + 3\,\mathbf{v}_2 - 4\,\mathbf{v}_3$

$\textbf{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) = \textbf{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$

Example: $\mathbb{R}^3$



$\vec{v}_3 = \vec{v}_2 - \vec{v}_1$

$\vec{v}_1 - \vec{v}_2 + \vec{v}_3 = 0$

Finding Dependence Relations from A x = 0

Vector form:

$$-2\begin{bmatrix}1\\1\\0\\1\end{bmatrix}+3\begin{bmatrix}1\\2\\2\\1\end{bmatrix}-4\begin{bmatrix}-1\\1\\1\\-1\end{bmatrix}-\begin{bmatrix}5\\0\\2\\5\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$\mathbf{v}_1$ $\mathbf{v}_2$ $\mathbf{v}_3$ $\mathbf{v}_4$

Matrix form:

$$\begin{bmatrix}1&1&-1&5\\1&2&1&0\\0&2&1&2\\1&1&-1&5\end{bmatrix}\begin{bmatrix}-2\\3\\-4\\-1\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

IMPORTANT: Dependence relations come from nonzero solutions to Ax = **0**

Q: Are the following vectors linearly independent?

$$\underset{\mathbf{v}_1}{\begin{bmatrix}1\\1\\-1\\-1\end{bmatrix}},\ \underset{\mathbf{v}_2}{\begin{bmatrix}0\\1\\1\\1\end{bmatrix}},\ \underset{\mathbf{v}_3}{\begin{bmatrix}2\\1\\1\\0\end{bmatrix}},\ \underset{\mathbf{v}_4}{\begin{bmatrix}0\\4\\0\\1\end{bmatrix}},\ \underset{\mathbf{v}_5}{\begin{bmatrix}-1\\0\\2\\2\end{bmatrix}}$$

I. Solve Ax = 0: $\mathbf{A}=\left[\begin{array}{ccccc|c}1&0&2&0&-1&0\\1&1&1&4&0&0\\-1&1&1&0&2&0\\-1&1&0&1&2&0\end{array}\right]\rightarrow\left[\begin{array}{ccccc|c}1&0&0&2&-1&0\\0&1&0&3&1&0\\0&0&1&-1&0&0\\0&0&0&0&0&0\end{array}\right]$

II. Parametric solution: $\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix}=x_4\begin{bmatrix}-2\\-3\\1\\1\\0\end{bmatrix}+x_5\begin{bmatrix}1\\-1\\0\\0\\1\end{bmatrix}$

III. Dependence Relations:

$-2\,\mathbf{v}_1 - 3\,\mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4 = \mathbf{0}$

$\mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_5 = \mathbf{0}$

More Vectors than the Dimension

Are the columns of the following matrix linearly independent?

$$\mathbf{A} = \begin{bmatrix} -1 & 0 & 3 & -6 & 2 & -2 & 2 \\ 2 & 0 & -2 & 8 & 0 & 3 & 3 \\ 1 & 0 & 2 & 1 & 0 & -2 & 1 \\ -1 & 2 & 2 & -1 & -3 & 3 & 1 \\ 3 & -1 & -2 & 9 & -2 & -2 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 3 & 0 & 0 & -23/17 \\ 0 & 1 & 0 & 2 & 0 & 0 & -505/17 \\ 0 & 0 & 1 & -1 & 0 & 0 & 157/17 \\ 0 & 0 & 0 & 0 & 1 & 0 & -93/17 \\ 0 & 0 & 0 & 0 & 0 & 1 & 137/17 \end{bmatrix}$$

THEOREM 8 If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.

$$\begin{array}{cc} & p \\ n & \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix} \end{array}$$

3 vectors in the plane must be linearly dependent

4 vectors in 3-space must be linearly dependent

Etc

Example

The matrices A and B each contain 4-dimensional vectors down their columns.

In each case:

(i) Decide if the vectors in the column are linearly independent or linearly dependent. If they are ~~in~~dependent, give a dependence relation among them.

(ii) Decide if the vectors in the column span $\mathbf{R}^4$

$$\mathbf{A} = \begin{bmatrix} | & | & | & | \\ \vec{\mathbf{v}}_1 & \vec{\mathbf{v}}_2 & \vec{\mathbf{v}}_3 & \vec{\mathbf{v}}_4 \\ | & | & | & | \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} | & | & | & | & | \\ \vec{\mathbf{w}}_1 & \vec{\mathbf{w}}_2 & \vec{\mathbf{w}}_3 & \vec{\mathbf{w}}_4 & \vec{\mathbf{w}}_5 \\ | & | & | & | & | \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$