Section 2.2: Separation of Variables.

A first order separable equation is a DE of the form

only involves dependent variable

$$\frac{dy}{dx} = \overbrace{f(y)} \cdot \underbrace{g(x)}$$

only involves independent variable

Examples

$$\frac{dy}{dx} = \overbrace{y}^{f}\overbrace{(x+4)}^{g}$$

$$y' = \underbrace{(e^y + 4y)}_{f} \cdot \underbrace{2x}_{g}$$

$$(y^2 + 2y + 3)\frac{dy}{dx} = x^5 + 5$$

$$\hookrightarrow \frac{dy}{dx} = \underbrace{\left(\frac{1}{y^2+2y+3}\right)}_{f}\underbrace{(x^5+5)}_{g}$$

Nonexample: $\frac{dy}{dx} = y + x + 2$

addition not multiplication!

Steps for solving a separable equation

(Separation of Variables technique).

1. Write equation as

$$h(y)\frac{dy}{dx} = g(x)$$

2. Treat "$\frac{dy}{dx}$" like a fraction and multiply both sides by dx to get

$$h(y)\,dy = g(x)\,dx$$

justify later!

then integrate

$$\underbrace{\int h(y)\,dy}_{\text{integral w.r.t. } y} = \underbrace{\int g(x)\,dx}_{\text{integral w.r.t. } x}$$

$$H(y) = G(x) + C$$

3. If told to find an explicit solution, solve for an equation in the form $y =$ (just involves x) and if given initial conditions solve for C.

Example: Solve $y' = (y+4)(x^2+4)$ Subject to $y(0) = 1$.

$$\frac{1}{y+4}\frac{dy}{dx} = x^2+4$$

$$\int \frac{1}{y+4}\,dy = \int x^2+4\,dx$$

$$\ln|y+4| = \frac{1}{3}x^3 + 4x + C$$

one parameter family of solutions to $\frac{dy}{dx} = (y+4)(x^2+4)$ (implicit)

Initial condition $y(0)=1$

$$\ln|1+4| = \frac{1}{3}(0)^3 + 4(0) + C$$

$$\ln|5| = C$$

$\log(x) = \log_e(x) = \ln(x)$

Solution to IVP:

$$\ln|y+4| = \frac{1}{3}x^3 + 4x + \ln 5$$

$$\frac{d}{dx}\left(\ln|y+4|\right) = \frac{1}{y+4}\cdot\frac{dy}{dx}$$

$$\frac{d}{dx}\left(\frac{1}{3}x^3 + 4x + \ln 5\right) = x^2 + 4 + 0$$

Why can we treat $\frac{dy}{dx}$ like a fraction?

Start with generic equation

$$h(y)\frac{dy}{dx} = g(x)$$

y is a dependent variable (y is a function of x: $y = y(x)$)

Rewrite equation with explicit notation including x

$$h(y(x))\frac{dy}{dx} = g(x)$$

$$h(y(x))\,y'(x) = g(x)$$

Integrate with respect to x

$$\int h(y(x))\,y'(x)\,dx = \int g(x)\,dx$$

$u = y(x)$ $\quad du = y'(x)\,dx \quad \frac{du}{dx} = y'(x)$

$$\int h(u)\,\underline{du} = \int g(x)\,\underline{dx}$$

$$\int h(y)\,dy = \int g(x)\,dx$$

already established u-substitution techniques allow us to to integrate with respect to different variables