Section 6.2: PDEs and the Heat Equation

Partial differential equation - a differential equation where derivatives are w.r.t. more than one independent variable.

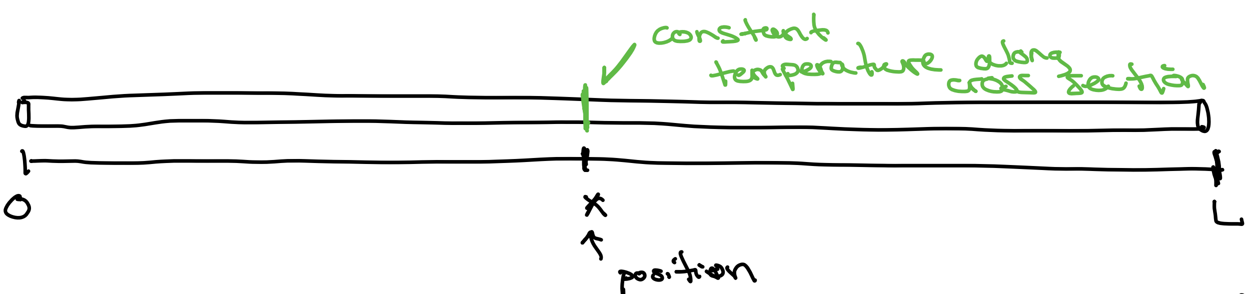
$u(x,t)$

$u_x(x,t)$ $\quad \frac{\partial u}{\partial x}$
(partial derivatives in x variable)

$u_t(x,t)$ $\quad \frac{\partial u}{\partial t}$
(partial derivatives in t variable

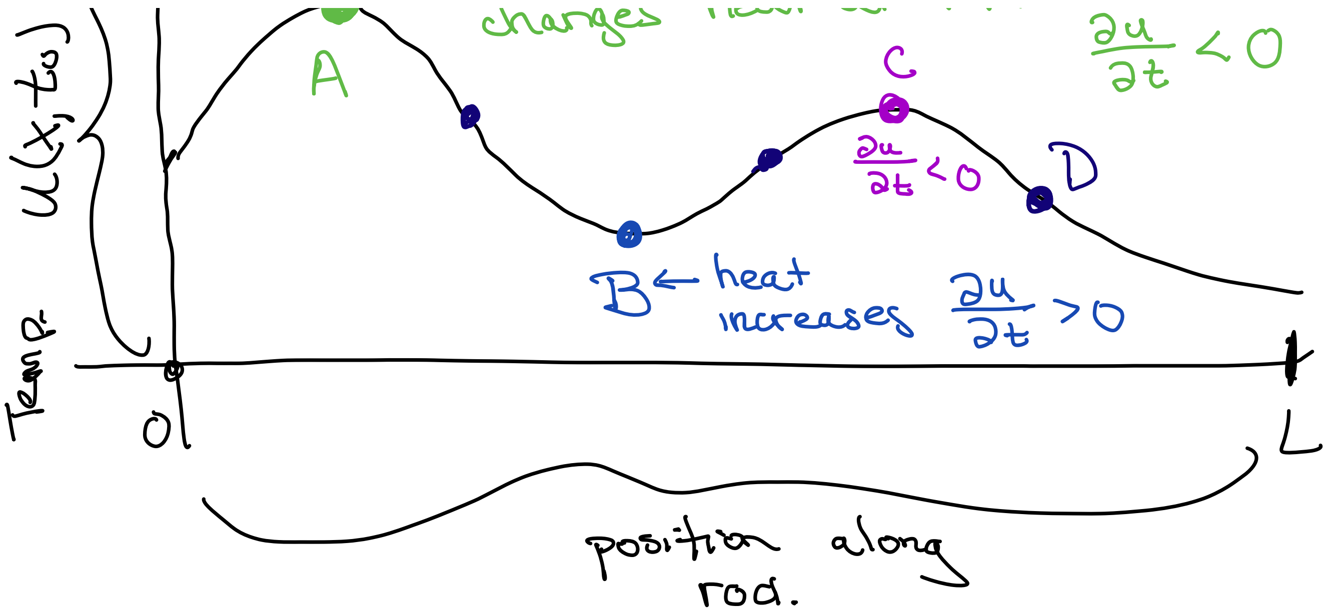
$u(x,y,t)$
$u(x_1,x_2,x_3,\dots,x_n,t)$

To build the build the one dimensional heat equation.



Time slice of $u(x,t)$ at time $t=t_0$ temperature at time t and position x

as time ... heat at A will decrease



Heat Equation

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$$

inflection points along $u(x, t_0)$ for fixed t_0.

↑ thermal diffusivity constant

$$u_t = \alpha u_{xx}$$

$$\dot{u} = \alpha \nabla^2 u$$

Nabla

$$u_t = \alpha \Delta u$$

Laplace

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

$$\nabla^2 := \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

$$= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

initial data.

Initial Condition - specify a condition for $t=0$.

tells us how heat is distributed along the rod at time 0.

$$u(x,0)=f(x)$$

Boundary Conditions

Dirichlet Boundary Conditions - explicitly prescribe values of u on the boundary (end points) of rod.

$$u(0,t)=c_1 \qquad u(L,t)=c_2$$

(For example, keeping left and right tips dipped in water baths of temp. c_1 and c_2.)

Neumann Boundary Condition

$$u_x(0,t)=c_1 \quad , \quad u_x(L,t)=c_2$$

(For example, $u_x(0,t)=u_x(L,t)=0$

corresponds to pipe with insolated endpoints)

Robin boundary condition

Example at position L.

$$u(L,t) + u_x(L,t) = 0 \quad \text{for all } t$$

Example: $u_1(x,t) = e^{-t}\sin x$ solves the heat equation $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$ specific $\alpha = 1$

$$\frac{\partial u_1}{\partial t} = -e^{-t}\sin x$$

$$\frac{\partial u_1}{\partial x} = e^{-t}\cos x \qquad \frac{\partial^2 u_1}{\partial x^2} = -e^{-t}\sin x$$

$$-e^{-t}\sin x = -e^{-t}\sin x \quad \checkmark$$

$u_2(x,t) = e^{-4t}\sin(2x)$ solves $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} = -4e^{-4t}\sin(2x)$$

Heat equation is linear!

$$u(x,t) = c_1 u_1(x,t) + c_2 u_2(x,t)$$
$$= c_1 e^{-t}\sin x + c_2 e^{-4t}\sin(2x)$$

also satisfies

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

Add boundary and initial conditions:

↙ Dirichlet Boundary Condition

$$u(0,t) = u(\pi,t) = 0$$
$$u(x,0) = 30\sin x - 4\sin(2x)$$

$$c_1 = 30 \quad , \quad c_2 = -4$$

$$u(x,t) = 30e^{-t}\sin x - 4e^{-4t}\sin(2x)$$

Often we need an infinite series of "building block" functions to solve a boundary value problem.

$$u(x,t) = \sum_{n=1}^{\infty} c_n \underbrace{u_n(x,t)}_{\text{exponential times sine or cosine function}}$$

Separation of Variables for PDEs

Find a solution to

$$u_t = \alpha u_{xx} \quad , \quad \underbrace{u(0,t) = u(L,t) = 0}$$

Main idea: ↙ solution function

$u(x,t)$

can be written as a (infinite) sum of "building block functions" $u_1, u_2, u_3, \ldots$ such that

$$\underbrace{u_i(x,t) = X_i(x)\,T_i(t)}$$

Work with $u(x,t) = X(x)\,T(t)$

Assume u satisfies the heat equation

$$u_t = \alpha u_{xx}$$

$$\underset{\substack{\text{constant}\\ \text{w.r.t. } t}}{X(x)}\,T'(t) = \alpha X''(x)\,\underset{\substack{\text{constant}\\ \text{w.r.t. } x}}{T(t)}$$

$$XT' = \alpha X'' T$$

$$\frac{T'}{\alpha T} = \frac{X''}{X} = -\lambda$$

a constant

$\lambda > 0$

$$\frac{X''}{X} = -\lambda \Rightarrow X'' + \lambda X = 0$$

$$\frac{T'}{\alpha T} = -\lambda \Rightarrow T' + \lambda\alpha T = 0$$

ODEs that we already know how to solve.

$$X'' + \lambda X = 0.$$

Recall Boundary Conditions:

$$0 = u(0,t) = X(0)T(t)$$

assume $T(t) \neq 0$

$$\Rightarrow \underline{X(0) = 0}$$

$$0 = u(L,t) = X(L)T(t)$$

$x^2 + \lambda^2 = 0$

$$\Rightarrow \underline{X(L)} = 0$$

By exponential substitution

(Roots $\pm \lambda i$)

$$X(x) = c_1 \cos(\sqrt{\lambda}\, x) + c_2 \sin(\sqrt{\lambda}\, x)$$

$$0 = X(0) = c_1 \cos(0) + c_2 \sin(0)$$

$c_1 = 0$

$$0 = X(L) = \cancel{c_1 \cos(\sqrt{\lambda} L)} + c_2 \sin(\sqrt{\lambda}\, L)$$

$c_1 = 0$

If $c_2 \neq 0$

$$\sin(\sqrt{\lambda}\, L) = 0 \Rightarrow \underline{\lambda = \frac{n^2 \pi^2}{L^2}} \quad n \in \mathbb{Z}$$

$$\sin(n\pi) = 0$$

$n\pi = \sqrt{\lambda} L$

$$X_n(x) = \sin\left(\sqrt{\frac{n^2\pi^2}{L^2}} \cdot x\right)$$

$$= \sin\left(\frac{n\pi}{L} x\right)$$

$T' + \lambda \alpha T = 0$

$$T_n' + \frac{n^2\pi^2}{L^2} \alpha T_n = 0$$

By exp. substitution

$$T_n(t) = e^{-\left(\frac{n^2\pi^2}{L^2} \cdot \alpha t\right)}$$

$$u_n(x,t) = X_n(x) \cdot T_n(t) = e^{-\left(\frac{n^2\pi^2}{L^2} \alpha t\right)} \cdot \sin\left(\frac{n\pi}{L} x\right)$$

By construction $u_n(x,t)$ satisfies

$$u_t = \alpha u_{xx} \quad \text{and} \quad u(0,t) = u(L,t) = 0$$

Suppose we have an initial condition

$$u(x,0) = f(x)$$ Find Fourier Series for f

Guess that a solution to problem

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\left(\frac{n^2\pi^2}{L^2}\alpha t\right)} \sin\left(\frac{n\pi}{L}x\right)$$

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{L}x\right)$$

Form of Fourier Series for an odd function

Use coefficients for Fourier Series of f as c_n in sum for u!

$$u(x,0) = f(x) = \begin{cases} 25 & 0 < x \le 5 \\ -25 & -5 \le x < 0 \end{cases}$$