Linear Transformations

The matrix-vector product that sends $T(x) = Ax$ is a linear transformation. A *linear transformation* T is a function from $\mathbb{R}^n$ to $\mathbb{R}^m$ that satisfies the following two properties, for all vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$ and all scalars $c \in \mathbb{R}$.

L1. $T(\mathbf{0}) = \mathbf{0}$.

L2. $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$

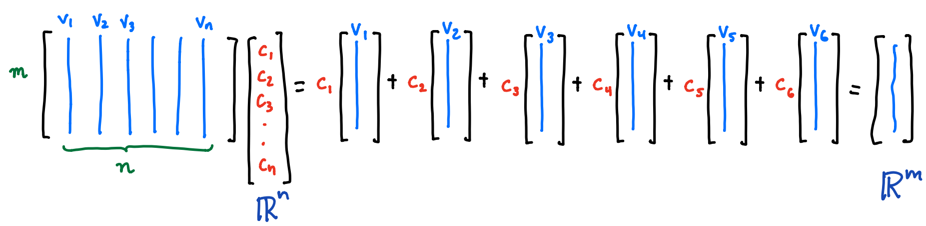
L3. $T(c\vec{u}) = cT(\vec{u})$

L1 is really a special case of L3

L4. $T(c\vec{u} + d\vec{v}) = cT(\vec{u}) + dT(\vec{v})$

L2 and L3 combined

Notice that if A is an $m \times n$ matrix, then A transforms n-dimensional vectors into m-dimensional vectors.



Example

1. Only one of these functions is linear. Which one is it? For the one that is not linear, show that it is not by demonstrating that one of the rules above fails for some *specific vectors*. For the one that is linear, show that it is linear by showing that the rules above work for *arbitrary* vectors $\vec{u}$ and $\vec{v}$.

$$T_1\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2^2 \\ x_1 - 3x_2 \end{bmatrix} \qquad T_2\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 5x_1 - x_2 \\ x_1 + x_2 \\ 2x_1 + 2x_2 \end{bmatrix}$$

T_1 is not linear with a specific counter example

$$T\left(3\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right) = T\left(\begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}\right) = \begin{bmatrix} -33 \\ -15 \end{bmatrix}$$

$$3\,T\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right) = 3\begin{bmatrix} -3 \\ -5 \end{bmatrix} = \begin{bmatrix} -9 \\ -15 \end{bmatrix}$$

show the rules work for arbitrary vectors.

(2) $$T\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} u_1+v_1 \\ u_2+v_2 \end{bmatrix}\right) = \begin{bmatrix} 5(u_1+v_1)-(u_2+v_2) \\ (u_1+v_1)+(u_2+v_2) \\ 2(u_1+v_1)+2(u_2+v_2) \end{bmatrix} = \begin{bmatrix} 5u_1 - u_2 + 5v_1 - v_2 \\ u_1+u_2+v_1+v_2 \\ 2u_1+2u_2+2v_1+2v_2 \end{bmatrix}$$

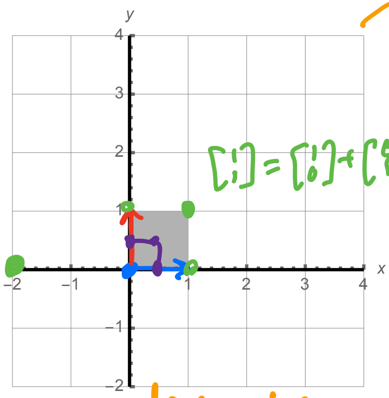
$$T\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) + T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \begin{bmatrix} 5u_1 - u_2 \\ u_1+u_2 \\ 2u_1+2u_2 \end{bmatrix} + \begin{bmatrix} 5v_1 - v_2 \\ v_1+v_2 \\ 2v_1+2v_2 \end{bmatrix} = \begin{bmatrix} \text{same} \end{bmatrix}$$

(3) $$T\left(c\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix}\right) = \begin{bmatrix} 5cu_1 - cu_2 \\ cu_1 + cu_2 \\ 2cu_1 + 2cu_2 \end{bmatrix}$$

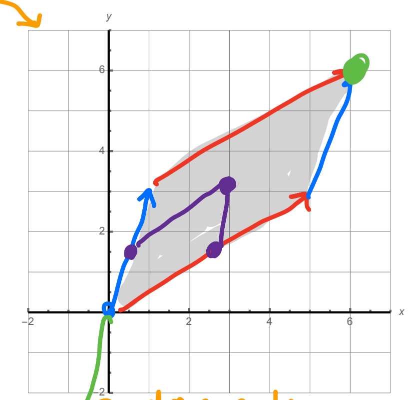
$$cT\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = c\begin{bmatrix} 5u_1 - u_2 \\ u_1+u_2 \\ 2u_1+2u_2 \end{bmatrix} = \begin{bmatrix} 5cu_1 - cu_2 \\ cu_1 + cu_2 \\ 2cu_1 + 2cu_2 \end{bmatrix}$$

2. If T is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ and $\boxed{T(\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]) = \left[\begin{smallmatrix}1\\3\end{smallmatrix}\right]}$ and $\boxed{T(\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]) = \left[\begin{smallmatrix}5\\3\end{smallmatrix}\right]}$ then draw where T sends the unit square. Hint: $\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right] = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] + \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$.

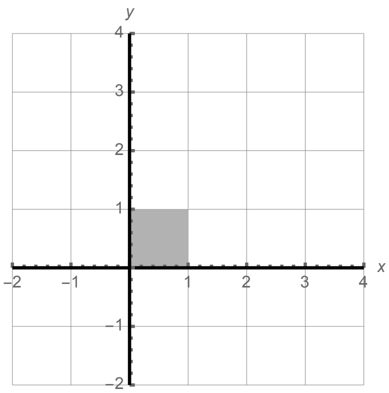
$$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = T\left(\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}\right) = \underbrace{T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)}_{\text{blue}} + \underbrace{T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)}_{\text{red}}$$

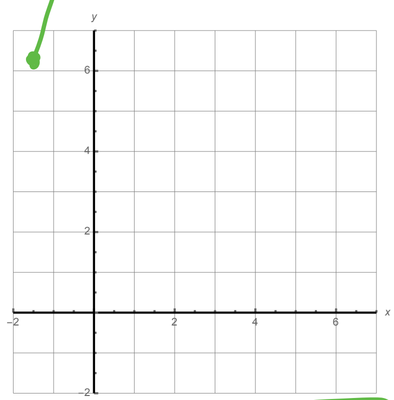


T



Repeat for $T(\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]) = \left[\begin{smallmatrix}2\\1\end{smallmatrix}\right]$ and $T(\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]) = \left[\begin{smallmatrix}4\\2\end{smallmatrix}\right]$





3. If T is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$ and $\boxed{T(\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]) = \begin{bmatrix}3\\2\\1\end{bmatrix}}$ and $\boxed{T(\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]) = \begin{bmatrix}1\\0\\1\end{bmatrix}}$ then how much do we know about T? For example, can we compute any of these?

Tells us everything!!

$$T\left(\begin{bmatrix}0\\0\end{bmatrix}\right) = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = T\left(\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}3\\2\\1\end{bmatrix} + \begin{bmatrix}1\\0\\1\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\99\end{bmatrix}\right) = T\left(99\begin{bmatrix}0\\1\end{bmatrix}\right) = 99\,T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = 99\begin{bmatrix}1\\0\\1\end{bmatrix}$$

$$T\left(\begin{bmatrix}7\\-11\end{bmatrix}\right) = T\left(\underset{x}{7}\begin{bmatrix}1\\0\end{bmatrix} - \underset{y}{11}\begin{bmatrix}0\\1\end{bmatrix}\right) = \underset{x}{7}\begin{bmatrix}3\\2\\1\end{bmatrix} - \underset{y}{11}\begin{bmatrix}1\\0\\1\end{bmatrix}$$

$$T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) =$$

4. Work together on PS2 problems 3.3 and 3.4 together in your groups.