Tuesday, Sept 13

1. Welcome!
2. Look out for mask survey
3. Homework due Thurs. night / Friday morning
4. Quiz 1 Week from Thurs.
5. look out for topics list
6. Questions?
7. Feasibility
8. Outro
   - ↳ Small work

HW #2

$-2 \leq x \leq 2$

MSCS student advisory board applications due soon!

Last time we talked a lot about the geometry that underpins linear programs, and we made the claim (though have not proven) that optimal solutions (should they exist) appear at the extreme points of polyhedron. We'll prove that claim today and get back to our original goal of figuring out how to actually solve linear programs. First, the proof:

Claim: Consider the linear program of minimizing $c^T x$ over a polyhedron P. Suppose that P has at least one extreme point and at least an optimal solution. Then there exists an optimal solution at an extreme point.

Pf: let $P = \{Ax \geq b\}$. Let v be the min value of $c^T x$. Then, let $Q = \{Ax \geq b,\ c^T x = v\}$. Q is the polyhedron of optimal sol. Since P has an extreme point, so does Q. Let x^* be an e.p. of Q.

Cl: it's an ep of P. Suppose it's not. Then $x^* = \lambda y + (1-\lambda) z$ for some $y, z \in P$, $y, z \neq x^*$. Note $c^T y \geq v$, $c^T z \geq v$, and

$v = c^T x^* = c^T(\lambda y + (1-\lambda) z) = \lambda c^T y + (1-\lambda) c^T z \geq v$. Thus $c^T y = v$, $c^T z = v$, so $y, z \in Q$. This $\lightning$ x^* being an ep of Q.

Thus x^* is an ep of P, and it is an optimal sol. of the LP.

Ok, again, all well and good, except *we're still not solving linear programs!* Let's get to this by considering the three possible outcomes of a linear program (indeed, this idea is called the fundamental theorem of linear programming):

Theorem (2.12)**.** *Let P be an LP problem. Then exactly one of the following holds:*

- *P is infeasible,*
- *P is unbounded, or*
- *P has an optimal solution.*

To get to this, we'll define a form of LP called the *standard equality form* (SEF).

Definition. *An LP is in standard equality form if it is of the form*

$$\max\{c^T x : Ax = b, x \geq 0\}.$$

Question: what do we notice about this linear program?

- maximization problem
- all constraints are equalities
- variables must be nonneg.

All our feasibility arguments are going to be for SEFs, which means we're gonna need to argue why any LP can be put into SEF. Specifically, we want to develop an equivalent linear program in SEF. We're gonna have to look at three ideas, but let's do it in parallel with an LP that needs each one:

$$\begin{cases} \overset{\text{min}}{\cancel{\max}} & -1x_1 + 2x_2 - 4x_3 \\ \text{s. t.} & \begin{bmatrix} 1 & 5 & 3 \\ 2 & -1 & 2 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{matrix} \geq \\ \leq \\ = \end{matrix} \begin{bmatrix} 5 \\ 4 \\ 2 \end{bmatrix} \\ & x_1, x_2 \geq 0 \end{cases}$$

1. How do we deal with a minimization problem?

multiply c by -1.

$\boxed{\text{ex}}$ $\min\ -x_1 + 2x_2 - 4x_3$

$\downarrow$

$\max\ x_1 - 2x_2 + 4x_3$

2. How do we deal with the unbounded variables?

let an unbounded variable x_i be $x_i = x_i^+ - x_i^-$

$x_i^{\pm} \geq 0$

$\boxed{\text{ex}}$ $\max\ x_1 - 2x_2 + 4(x_3^+ - x_3^-)$

$$\begin{bmatrix} 1 & 5 & 3 & -3 \\ 2 & -1 & 2 & -2 \\ 1 & 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3^+ \\ x_3^- \end{bmatrix} \begin{matrix} \geq \\ \leq \\ = \end{matrix} \begin{bmatrix} 5 \\ 4 \\ 2 \end{bmatrix}$$

3. How do we deal with inequalities?

build in slack

for each ineq. constraint, build in a variable s_i

$$\boxed{} \pm s_i = \boxed{}$$

const. value

$$\begin{array}{cccccc} x_1 & x_2 & x_3^+ & x_3^- & s_1 & s_2 \end{array}$$
$$\begin{bmatrix} 1 & 5 & 3 & -3 & -1 & 0 \\ 2 & -1 & 2 & -2 & 0 & 1 \\ 1 & 2 & -1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3^+ \\ x_3^- \\ s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 2 \end{bmatrix}$$

$$x_1 + 5x_2 + 3x_3^+ - 3x_3^- - s_1 = 5$$

Example: Put the following LP into SEF.

$$\begin{cases} \min & 2x_1 - x_2 + 4x_3 - x_4 \\ \text{s.t.} & \begin{bmatrix} 1 & 2 & 4 & 7 \\ 2 & 8 & 9 & 0 \\ 1 & 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \begin{matrix} \geq \\ \leq \\ \leq \end{matrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \\ & x_1, x_3 \geq 0 \end{cases}$$

$$\max \quad -2x_1 + x_2^+ - x_2^- - 4x_3 + x_4^+ - x_4^-$$

$$\begin{bmatrix} 1 & 2 & -2 & 4 & 7 & -7 & -1 & 0 & 0 \\ 2 & 8 & -8 & 9 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & 2 & -2 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2^+ \\ x_2^- \\ x_3 \\ x_4^+ \\ x_4^- \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

With these three ideas, we can move any program to an SEF. Now all our discussions about feasibility can be stated and proven for that form, which will make things easier for us. Let's do unfeasibility first:

Theorem. *(2.1) Let P be an LP in standard form:*

$$\max\{c^T x : Ax = b, x \geq 0\}$$

$$y^T A x = y^T b$$

Then P is infeasible if there exists a y such that

- $y^T A \geq 0$
- $y^T b < 0$.

Such a y is called a certificate of infeasibility.

Proof: $y^T A x \geq 0$ for any $x \geq 0$ but $y^T b < 0$.

Example: Find a certificate of infeasibility for an SEF LP where

$$A = \begin{bmatrix} 4 & 10 & -6 & -2 \\ -2 & 2 & -4 & 1 \\ -7 & -2 & 0 & 4 \end{bmatrix} \quad b = \begin{bmatrix} 6 \\ 5 \\ 3 \end{bmatrix}$$

$$y = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \qquad y^T A = [1 \ 4 \ 2 \ 0] \qquad y^T b = -1$$

The proof that every infeasible solution has such a certificate has to wait, but it has to do with Farkas' Lemma, and I'll flag it down when we get there.

Theorem. *(2.2) Let P be an LP in standard form:*

$$\max\{c^T x : Ax = b, x \geq 0\}$$

Then P is unbounded if there exists a feasible solution x and vector d such that

- $Ad = 0$
- $d \geq 0$, *and*
- $c^T d > 0$.

This pair of vectors x, d are a certificate of unboundedness.

Proof: $x_t = x + td$

$Ax_t = A(x+td) = Ax + tAd = b$ * every x_t is feasible

$c^T x_t = c^T(x+td) = c^T x + t\underbrace{c^T d}_{>0} \longrightarrow \infty$ as $t \to \infty$

Example: Suppose P is an SEF LP such that

$$A = \begin{bmatrix} 1 & 1 & -3 & 1 & 2 \\ 0 & 1 & -2 & 2 & -2 \\ -2 & -1 & 4 & 1 & 0 \end{bmatrix} \quad b = \begin{bmatrix} 7 \\ -2 \\ -3 \end{bmatrix} \quad c = \begin{bmatrix} -1 \\ 0 \\ 3 \\ 7 \\ -1 \end{bmatrix}$$

Find the certificate of unboundedness.

$x = [2\ 0\ 0\ 1\ 2]$ $\quad d = [1\ 2\ 1\ 0\ 0]$

As we can see, these certificates can often be hard to find in practice. It would be great if we could find a single algorithm that would either (1) prove unbounded, (2) prove infeasible, or (3) find the optimal solution. It turns out that's exactly what the simplex algorithm will do. Let's preview it with one step of a linear program we've seen before:

$$\begin{cases} \max & \frac{1}{2}x_1 + x_2 \\ & x_1 + x_2 \leq 3 \\ \text{s. t.} & x_1 \leq 2 \\ & x_2 \leq 2 \\ & x_i \geq 0 \end{cases}$$

First, we gotta put it in SEF:

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}$$

Because of the way it is, we can find a feasible solution pretty quickly. We can also calculate its objective value:

$$\begin{bmatrix} 0 \\ 0 \\ 3 \\ 2 \\ 2 \end{bmatrix} \qquad \frac{1}{2}x_1 + x_2 = 0 \qquad \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Note: such a feasible solution is called canonical.

Let's try to make it better, meaning we find a feasible solution w/ better obj. value. It looks like if we increase x_2, we can make our objective value better. We'll increase it to t

$$\begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + s_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + s_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} \rightarrow \underbrace{\begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix}}_{\geq 0} = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix} - t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

which means that $t =$ ____2____. At that point, our feasible solution is

feasible soln: $\begin{bmatrix} 0 \\ 2 \\ 1 \\ 2 \\ 0 \end{bmatrix}$ obj. value: 2

which is interesting when we look at it in the context of the graph! The simplex algorithm is going to improve objective over time by checking the basic feasible solution.

Next time: recall that we originally talked about basic feasible solutions having a linearly independent set of constraints. We'll use this idea to build a starting basic feasible solution and complete the algorithm.