Section 1.2 (Continued)

Two fundamental question

1. Existence: Does a solution to the problem exist?
2. Uniqueness: If a solution exists, is it unique?

Theorem (Existence of a Unique Solution): Let $R = [a,b] \times [c,d]$ be a rectangular region that contains a point (x_0, y_0) in it's interior. If $f(x,y)$ and $\frac{\partial f}{\partial y}$ are continuous on R, then there exist a unique function $y(x)$ defined on an Interval $I \subset [a,b]$ that is a solution to

$$\text{Solve } \frac{dy}{dx} = f(x,y)$$
$$\text{Subject to } y(x_0) = y_0$$

Previous example:

Solve: $\frac{dy}{dx} = y$ $\quad f(x,y) = y \quad \frac{\partial f}{\partial y} = 1$

Subject to $y(0) = 5$

E+U Theorem applies!

$y = 5e^{x}$ is the only solution

A non-example:

Solve: $\frac{dy}{dx} = \boxed{2xy^{1/2}}$

Subject to: $y(0) = 0$

$f(x,y) = 2xy^{1/2}$

$\frac{\partial f}{\partial y} = \frac{x}{y^{1/2}}$

not defined

Verify

- $y \equiv 0$ is a solution
- $y = \frac{1}{4}x^4$ is a solution

$(y' = x^3, \quad 2x\left(\frac{1}{4}x^4\right)^{1/2} = x^3$

Example:

Solve: $\frac{dy}{dx} = 2xy^{1/2}$

Subject to: $y(2) = 4$

Verify that $y = \frac{1}{4}x^4$ is a solution

$y(2) = \frac{1}{4}(2)^4 = 4$ ✓

Take $R = [1,3] \times [1,10]$

$(2,4) \in R$.

$f(x,y)$ $2xy^{1/2}$ is continuous on R

$\frac{\partial f}{\partial y} = \frac{x}{y^{1/2}}$ is continuous on R.

So $y = \frac{1}{4}x^4$ is the unique solution

domain : $(1,3) = I$