1. On Friday, we diagonalized three matrices. What are the eigenvalues and eigenvectors of these matrices?

$$A = \begin{bmatrix} -10 & 6 \\ -18 & 11 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}$$

$$B = \begin{bmatrix} -2 & -1 & 1 \\ -3 & -2 & 3 \\ -3 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} -4 & 9 & -3 \\ -6 & 11 & -3 \\ -12 & 18 & -4 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 0 & 1 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1/3 & 0 & -1/6 \\ 2/3 & -1 & 1/6 \\ -2/3 & 1 & 1/3 \end{bmatrix}$$

2. Below is a matrix A and its eigenvalues and eigenvectors. Note that its columns sum to 1, so that it is a probability (stochastic) matrix. Such matrices always have a largest eigenvalue equal to 1 (we will be able to prove this later).

(a) Diagonalize A:

$$A = \begin{bmatrix} 0.6 & 0.3 & 0.25 \\ 0.2 & 0.4 & 0.25 \\ 0.2 & 0.3 & 0.50 \end{bmatrix}$$

$$\lambda_1 = 1.0 \quad \lambda_2 = 0.34 \quad \lambda_3 = 0.16$$
$$\begin{bmatrix} -.692 \\ -.462 \\ -.554 \end{bmatrix} \quad \begin{bmatrix} -0.79 \\ 0.21 \\ 0.58 \end{bmatrix} \quad \begin{bmatrix} 0.21 \\ -0.79 \\ 0.58 \end{bmatrix}$$

(b) What does and eigenvalue of $\lambda = 1$ tell us?

(c) Rescale the eigenvector of eigenvalue $\lambda = 1$ so that it sums to 1.

(d) Use the diagonalization $A = PDP^{-1}$ to compute A^n:

(e) Compute $\lim_{n \to \infty} A^n$:

3. When is a matrix diagonalizable?

(a) Eigenvectors corresponding to different eigenvalues are linearly independent, so if A has distinct eigenvalues, it has an eigenbasis.

$$\mathsf{A} = \begin{bmatrix} -3 & 4 & 3 & -1 \\ -2 & 3 & 2 & 0 \\ -5 & 4 & 5 & -1 \\ -5 & 4 & 5 & -1 \end{bmatrix}$$

$$\begin{array}{cccc} \lambda_1 = 3 & \lambda_2 = 2 & \lambda_3 = -1 & \lambda_4 = 0 \\ \begin{bmatrix}1\\1\\1\\1\end{bmatrix} & \begin{bmatrix}2\\2\\1\\1\end{bmatrix} & \begin{bmatrix}1\\0\\1\\1\end{bmatrix} & \begin{bmatrix}1\\0\\1\\0\end{bmatrix} \end{array}$$

Is this matrix invertible? Does it have any steady-state vectors?

(b) When it has repeated eigenvalues it can be diagonalizable. See (c) above. The geometric multiplicity equals the algebraic multiplicity for each eigenvalue.

(c) When it has repeated eigenvalues it might not be diagonalizable. Such matrices are "defective." The matrix A below has characteristic polynomial $f_A(\lambda) = (\lambda - 3)(\lambda - 2)^2$ and eigenvalues $\lambda = 3, 2, 2$.

$$\mathsf{A} = \begin{bmatrix} 3 & 2 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$\mathsf{A} - 3I = \begin{bmatrix} 0 & 2 & 1 \\ -1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -\frac{3}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathsf{A} - 2I = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$