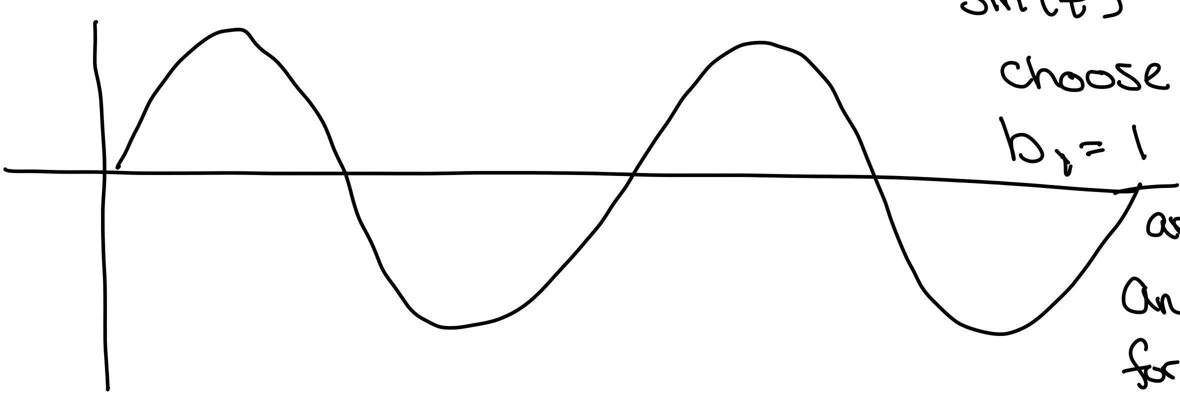
Partial Differential Equations (PDEs)
are differential equations with two or more independent variables

$$\frac{\partial u}{\partial x}, \quad \frac{\partial u}{\partial t} \qquad u(x,t)$$

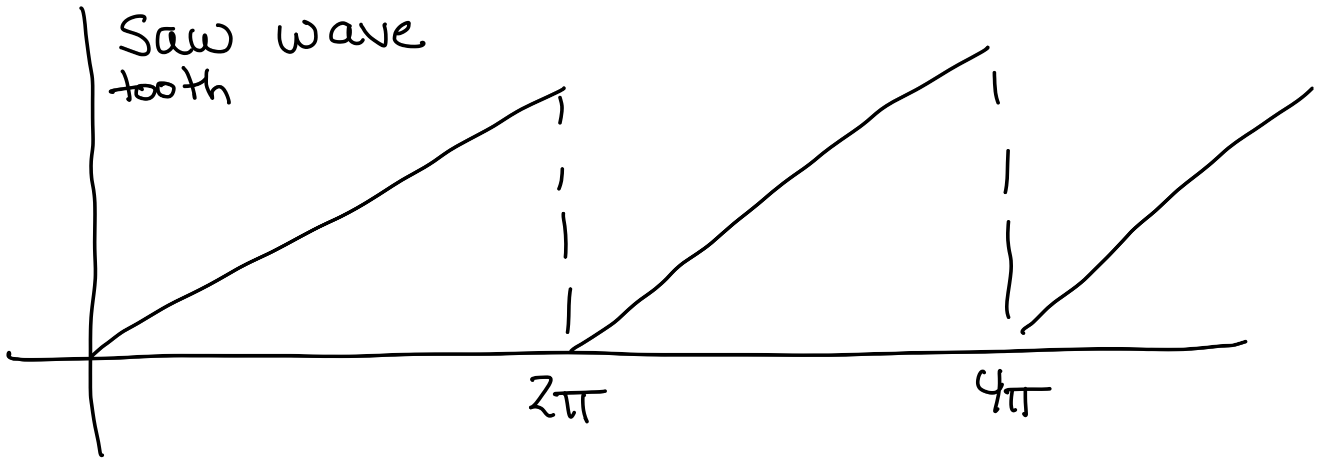
Section 6.1: A Crash Course on Fourier Series

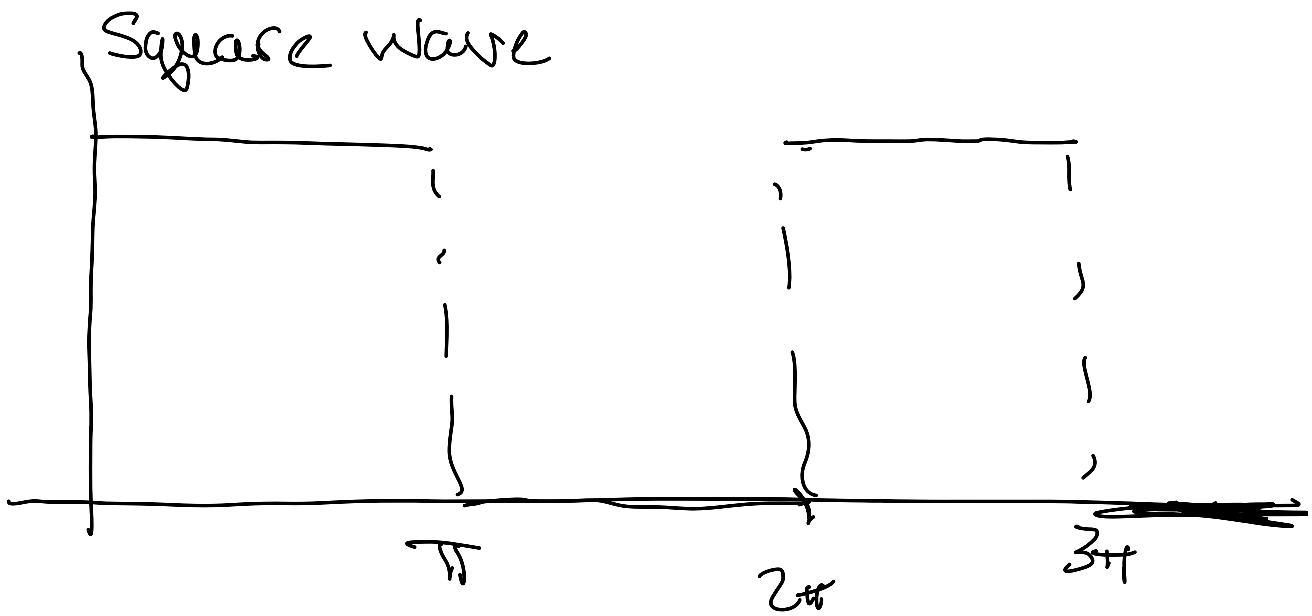
Fourier conjecture every 2π-periodic function $f(t)$ can be represented by an infinite trigonometric series of the form

$$\underbrace{\frac{a_0}{2}}_{\text{some constant term}} + \sum_{n=1}^{\infty} \left(a_n \cos(nt) + b_n \sin(nt)\right)$$



$\sin(t)$
choose $b_1 = 1$
and $a_n = 0$ for all n,
$b_n = 0$ when $n \neq 1$





$$\int_{-\pi}^{\pi} \cos(mt)\cos(nt)\,dt = \begin{cases} \pi & m=n \\ 0 & m\neq n \end{cases}$$

$$\int_{-\pi}^{\pi} \sin(mt)\sin(nt)\,dt = \begin{cases} \pi & m=n \\ 0 & m\neq n \end{cases}$$

$$\int^{\pi} \cos(mt)\sin(nt)\,dt = 0 \quad \text{for all } m,n$$

$$\int_{-\pi}^{\pi} \frac{1}{2}\cos(\underbrace{(m-n)}_{\text{integer}}t) - \cos(\underbrace{(m+n)}_{\text{integer}}t)\,dt$$

Orthogonal vectors: $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$

(dot product $1\cdot0+0\cdot1=0$)

Important property of trig functions:
$\sin(nt)$, $\cos(nt)$, $n=1,2,3,\ldots$
form a mutually orthogonal set.

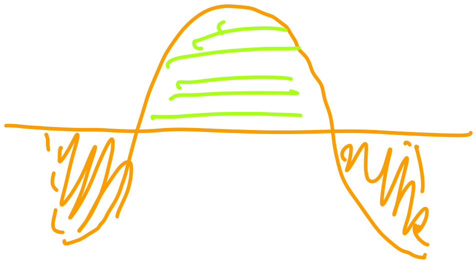
$$u(t)\cdot v(t) = \int_{-\pi}^{\pi} u(t)v(t)\,dt$$

Suppose we know $f(t)$ has a Fourier Series Representation

$$* \quad f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n\cos(nt) + b_n\sin(nt))$$

By orthogonality:

$$\int_{-\pi} f(t)\,dt = \frac{a_0}{2}\int_{-\pi} 1\,dt + \cancel{\sum_{n=1} a_n \int_{-\pi} \cos(nt)\,dt}$$



$$+\cancel{\sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin(nt)\,dt}$$

$$= \pi a_0 .$$

$$\Rightarrow a_0 = \frac{2}{\pi}\int_{-\pi}^{\pi} f(t)\,dt.$$

$$\int_{-\pi}^{\pi} f(t)\cdot\cos(mt)\,dt$$

$$= \frac{a_0}{2}\int_{-\pi}^{\pi}\cos(mt)\,dt + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos(mt)\cos(nt)\,dt$$

$$+ \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \cos(mt)\sin(nt)\,dt$$

$$= 0 + a_m\cdot\pi + 0$$

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(mt)\,dt$$

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$$\int_{-\pi} f(t)\sin(mt)\,dt = b_m \cdot \pi$$

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(mt)\,dt$$

What happen if f is periodic but not 2π-periodic.

Suppose f has period $P = 2L$
(L is half of a period)

Define $g(u) := f\left(\frac{Lu}{\pi}\right)$ for all $u \in \mathbb{R}$.

$$\begin{aligned} g(u+2\pi) &= f\left(\frac{L}{\pi}(u+2\pi)\right) \\ &= f\left(\frac{L}{\pi}\cdot u + \frac{2L\pi}{\pi}\right) \\ &= f\left(\frac{L}{\pi}u + 2L\right) \\ &= f\left(\frac{L}{\pi}u\right) \quad \text{since } f \text{ is } 2L \text{ periodic} \\ &= g(u) \end{aligned}$$

$g(u)$ has a Fourier Series

$$g(u) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nu) + b_n \sin(nu)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} g(u)\,du$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(u)\cos(nu)\,du$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(u)\sin(nu)\,du$$

Set $t = \frac{Lu}{\pi} \Leftrightarrow u = \frac{t\cdot\pi}{L} \Rightarrow f(t) = g(u)$

$$f(t) = g(u) = g\left(\frac{t\pi}{L}\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(n\frac{t\pi}{L}\right) + b_n \sin\left(n\frac{t\pi}{L}\right)$$

Where $a_n = \frac{1}{\pi}\int_{u=-\pi}^{u=\pi} g(u)\cos(nu)\,du$ $\quad u = \frac{\pi t}{L}$, $du = \frac{\pi}{L}dt$

$$= \frac{1}{\cancel{\pi}}\int_{-L}^{L} g\left(\frac{t\pi}{L}\right)\cos\left(\frac{n\pi t}{L}\right)\cdot\frac{\cancel{\pi}}{L}\,dt$$

$$= \frac{1}{L}\int^{L} f(t)\cos\left(\frac{n\pi t}{L}\right)dt$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi t}{L}\right) dt.$$