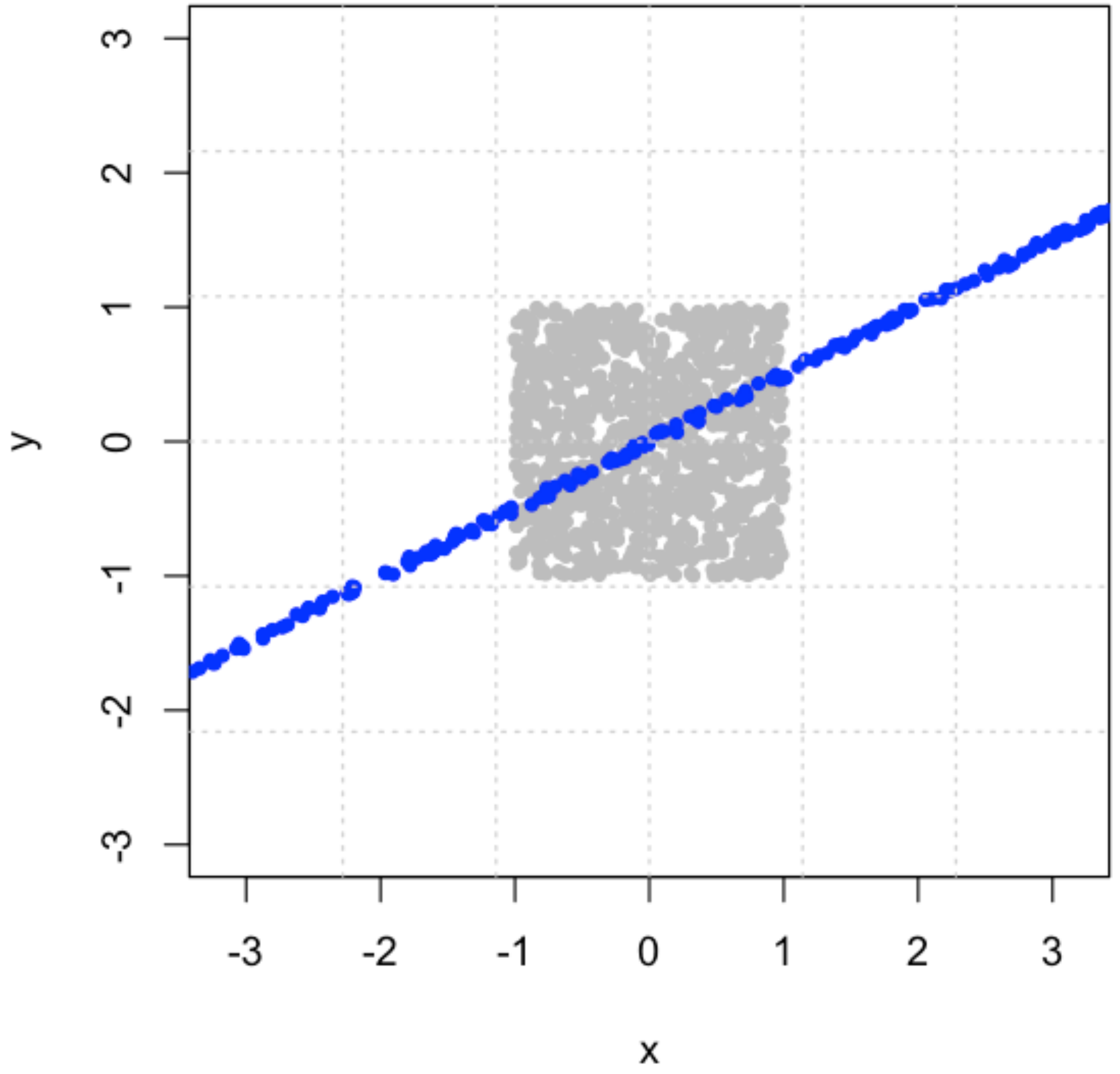
5.1. Eigenvectors

$$A = \frac{1}{30}\begin{bmatrix} 31 & 4 \\ 2 & 29 \end{bmatrix}$$ corresponds to a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$

Let's watch what it does to points on the unit square.



$$A\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{30}\begin{bmatrix} 31 & 4 \\ 2 & 29 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 11/5 \\ 11/10 \end{bmatrix}$$

$$= \frac{11}{10}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= 1.1\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Eigenvectors

eigenvalue

$$\mathbf{A}\mathrm{v} = \lambda \mathrm{v}$$

An nxn matrix

An **eigenvector**

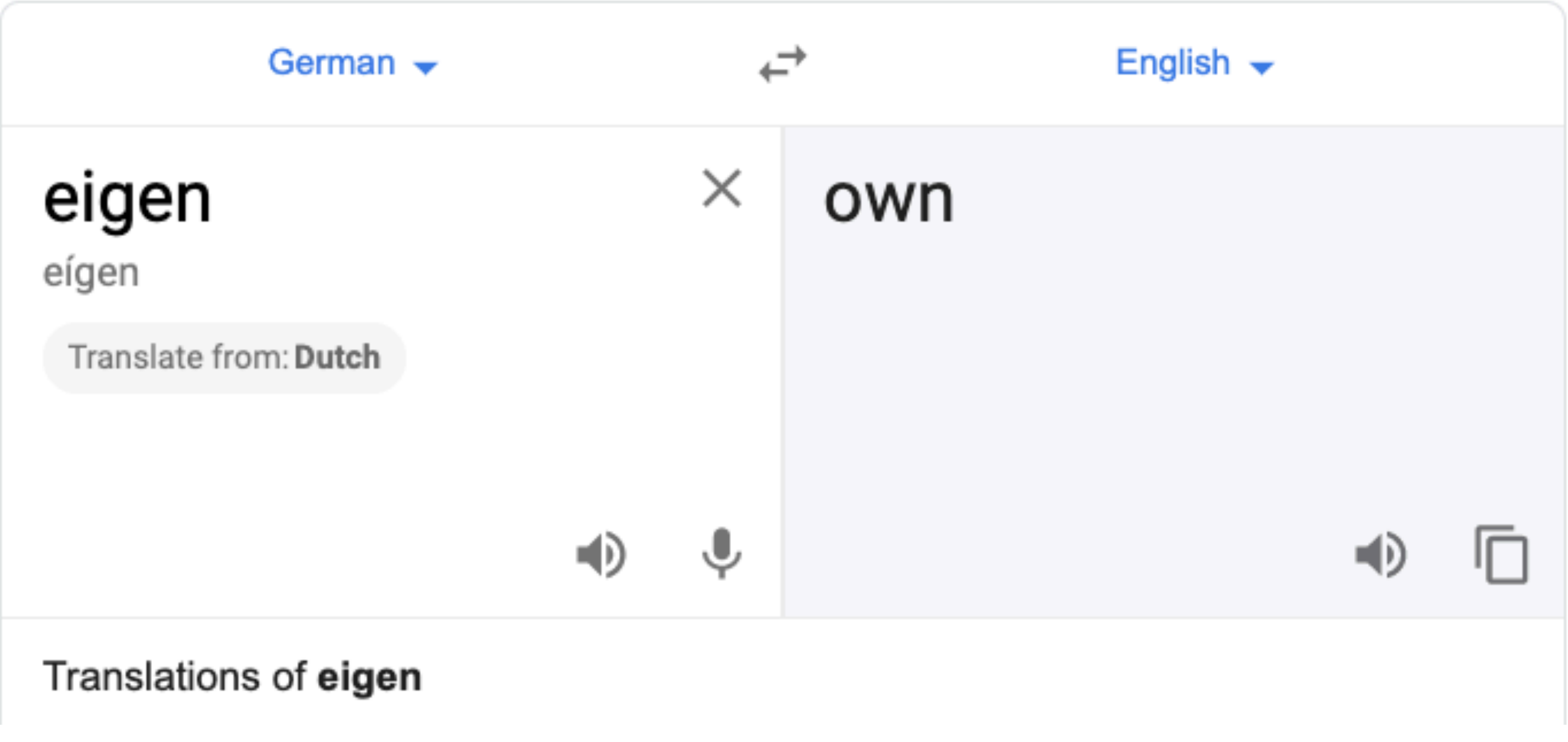
Scalar

$$\lambda \in \mathbb{R}$$

v is rescaled by $\mathbf{A}$

$\boxed{\mathbf{v} = \mathbf{0} \text{ is } \textit{not} \text{ an eigenvector}}$

- Observe that $\mathbf{A}$ does not change the direction of **v**.
- It simply rescales it.
- Note: if λ is negative then it also flips over.



Example

Is $v = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$ an eigenvector for $A = \begin{bmatrix} 3 & -5 & 7 \\ -4 & 7 & 2 \\ -14 & 25 & -6 \end{bmatrix}$ if so find the eigenvalue.

$$\begin{bmatrix} 3 & -5 & 7 \\ -4 & 7 & 2 \\ -14 & 25 & -6 \end{bmatrix}\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ -3 \end{bmatrix} = \underline{\mathbf{-3}} \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$$

YES! v is an eigenvector for A of eigenvalue λ = - 3.

What about these?

$$\begin{bmatrix} 3 & -5 & 7 \\ -4 & 7 & 2 \\ -14 & 25 & -6 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix}$$

YES! $\lambda=5$

$$\begin{bmatrix} 3 & -5 & 7 \\ -4 & 7 & 2 \\ -14 & 25 & -6 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 12 \\ 30 \end{bmatrix}$$

NO!

$$\begin{bmatrix} 3 & -5 & 7 \\ -4 & 7 & 2 \\ -14 & 25 & -6 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$$

YES! $\lambda=2$

Eigenvalues of Diagonal Matrices

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ b \\ 0 \end{bmatrix} = b\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix} = c\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The eigenvalues are the diagonal entries of the matrix.

And the eigenvectors are the standard basis vectors

Eigenvectors and Null Space

This is important!

$$A\mathbf{v} = \lambda \mathbf{v} \iff A\mathbf{v} = \lambda I \mathbf{v}$$
$$\iff A\mathbf{v} - \lambda I \mathbf{v} = 0$$
$$\iff (A - \lambda I)\mathbf{v} = 0$$
$$\iff \mathbf{v} \in \mathrm{Nul}(A - \lambda I)$$

$\boxed{\text{NOTE}}$

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \Rightarrow A - \lambda I = \begin{bmatrix} a-\lambda & b & c \\ d & e-\lambda & f \\ g & h & i-\lambda \end{bmatrix}$$

subtract λ from the diagonal of A

$$E_\lambda = \{\mathbf{v} \in \mathbb{R}^n \mid A\mathbf{v} = \lambda \mathbf{v}\} = \mathrm{Nul}(A - \lambda I)$$

Eigenspace of A   Null space of $A - \lambda I$

Eigenspace

Definition: If $\lambda \in \mathbb{R}$ is an eigenvalue for the n x n matrix **A** then the ***eigenspace*** associated with λ is the set $\mathbf{E}_\lambda = \{ \mathbf{v} \in \mathbb{R}^n \mid \mathbf{A}\,\mathbf{v} = \lambda\,\mathbf{v} \}$.

This is the set of all eigenvectors with eigenvalue λ plus the **0** vector.

An eigenspace is a nullspace

$$\mathbf{E}_\lambda = \{ \mathbf{v} \in \mathbb{R}^n \mid \mathbf{A}\,\mathbf{v} = \lambda\,\mathbf{v} \} = \mathbf{Nul}\,(\mathbf{A} - \lambda\,\mathbf{I}).$$

So it is a **subspace** of $\mathbb{R}^n$

$$A(cv) = cAv = c(\lambda v) = \lambda(cv)$$

$$A(v+w) = Av + Aw = \lambda v + \lambda w = \lambda(v+w)$$

Example

Given that 6 and -3 are eigenvalues of the matrix below find its eigenvectors.

$$A = \begin{bmatrix} -2 & 2 \\ 4 & 5 \end{bmatrix}$$

Key property

$$\boxed{A\vec{v} = \lambda\vec{v} \iff (A-\lambda I)\vec{v} = \vec{0} \iff \vec{v} \in \mathrm{Nul}(A-\lambda I)}$$

$\lambda = 6$

$$\boxed{\begin{bmatrix} -2-6 & 2 \\ 4 & 5-6 \end{bmatrix} = \begin{bmatrix} -8 & 2 \\ 4 & -1 \end{bmatrix} \to \begin{bmatrix} 4 & -1 \\ -8 & 2 \end{bmatrix} \to \begin{bmatrix} 4 & -1 \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1/4 \\ 0 & 0 \end{bmatrix}}$$

$$E_6 = \left\{ s\begin{bmatrix} 1/4 \\ 1 \end{bmatrix} \middle| s \in \mathbb{R} \right\} = \mathrm{Span}\begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

Eigenspace

$\lambda = -3$

$$\boxed{\begin{bmatrix} -2+3 & 2 \\ 4 & 5+3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}}$$

$$E_{-3} = \left\{ s\begin{bmatrix} -2 \\ 1 \end{bmatrix} \middle| s \in \mathbb{R} \right\} = \mathrm{Span}\begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Eigenspace

You Try!

4 and -3 are eigenvalues of the matrix below. Find the corresponding eigenspaces.

$$A = \begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix}$$