6.5. Least-Squares Projections

Back to where it all began

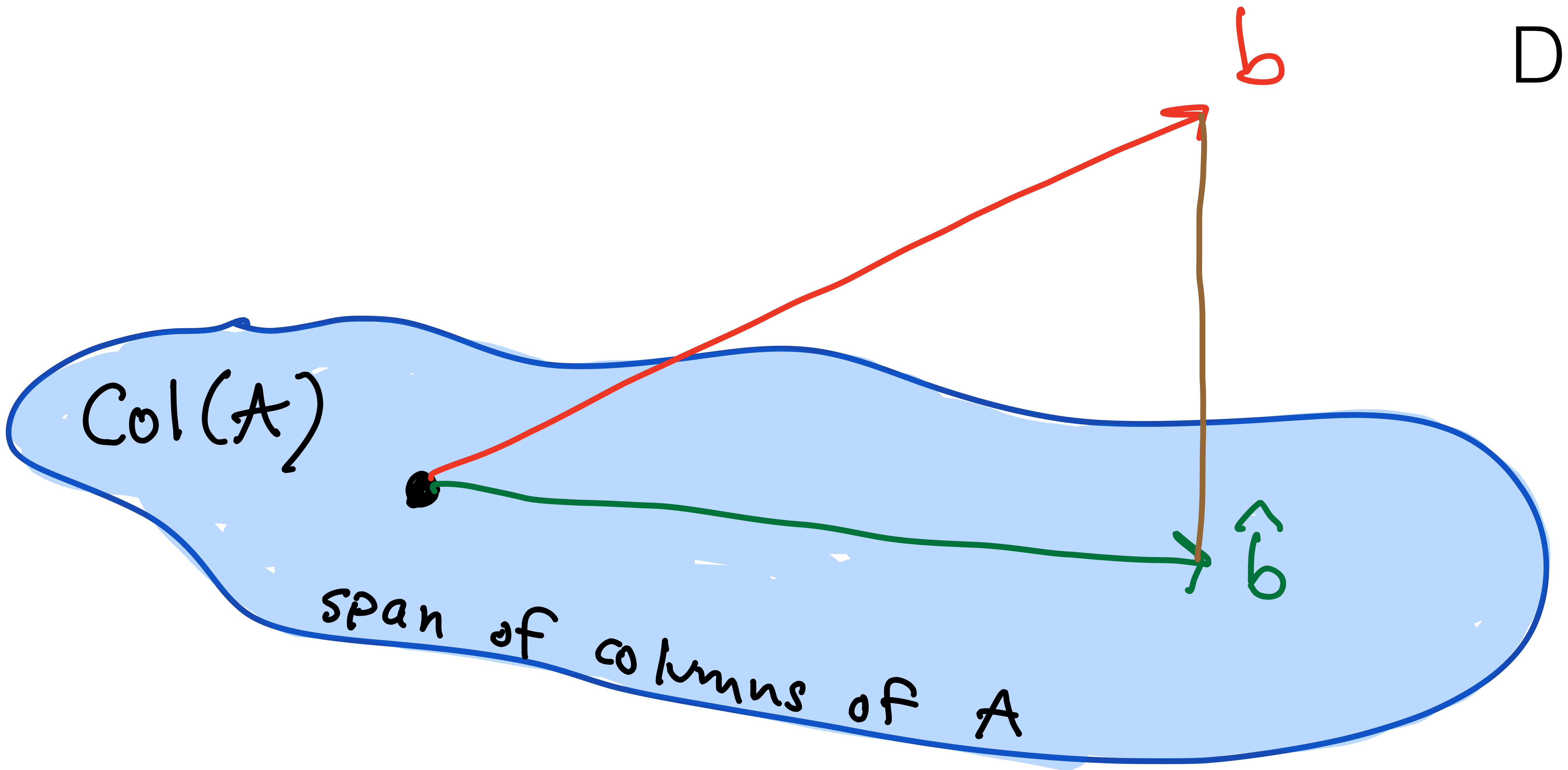
Solve: $\mathbf{A}\mathbf{x} = \mathbf{b}$

$$\underbrace{\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_k \\ | & | & & | \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix} = \begin{bmatrix} | \\ b \\ | \end{bmatrix}$$

But suppose that $\mathbf{A}\mathbf{x} = \mathbf{b}$ is inconsistent.

That is $\mathbf{b} \notin \text{Col}(\mathbf{A})$

i.e., you can't get to $\mathbf{b}$ using a linear combination of the columns of $\mathbf{A}$.



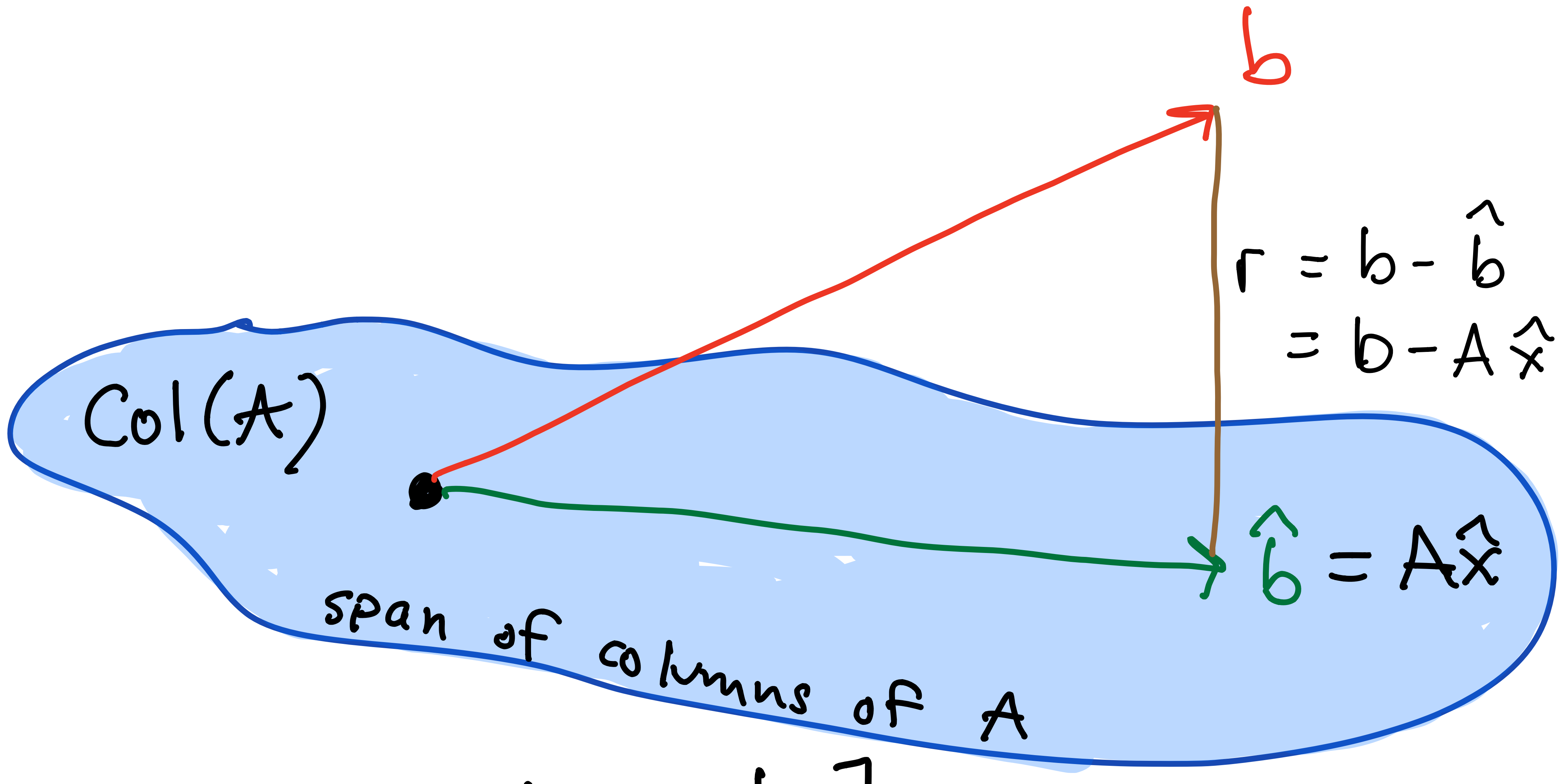
Don't give up!

- Project $\mathbf{b}$ onto Col($\mathbf{A}$)
- Solve $\mathbf{A}\mathbf{x} = \hat{\mathbf{b}}$ instead

This is the "least squares" solution to Ax = b

Least Squares

- Want to solve $\mathbf{Ax} = \mathbf{b}$
- But $\mathbf{b} \notin \mathrm{Col}(\mathbf{A})$
- Project $\mathbf{b}$ onto Col($\mathbf{A}$) to get
- Solve $\mathbf{Ax} = \hat{\mathbf{b}}$ instead



$$A = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_k \\ | & | & & | \end{bmatrix}$$

Important calculation:

If $A\hat{x} = \hat{b}$ is the solution

Then $A\hat{x}$ is the projection of b onto Col(A)

$\Rightarrow$ $b - A\hat{x}$ is orthogonal to Col(A)

$\Rightarrow$ $A^T(b - A\hat{x}) = 0$

$\Rightarrow$ $A^Tb - A^TA\hat{x} = 0$

$\Rightarrow$ $A^TA\hat{x} = A^Tb$

The normal equations for $Ax = b$

$\Rightarrow$ $\hat{x}$ is a solution to $\boxed{A^TAx = A^Tb}$

solve these equations instead!

Example

- **Solve Ax =b**

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ 1 \\ 11 \\ -4 \end{bmatrix}$$

But: $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 11 \\ -1 & -1 & -1 & -4 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$ inconsistent $b \notin \text{Col}(A)$

- Form the normal equations:

$$A^TA = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \qquad A^Tb = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 11 \\ -4 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 15 \end{bmatrix}$$

- Solve the normal equations:

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 4 \\ 1 & 2 & 1 & 5 \\ 1 & 1 & 2 & 15 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 9 \end{array}\right]$$

So the least squares solution is $\hat{x} = \begin{bmatrix} -2 \\ -1 \\ 9 \end{bmatrix}$

Example

- We found a least-squares solution to the Ax = b problem:

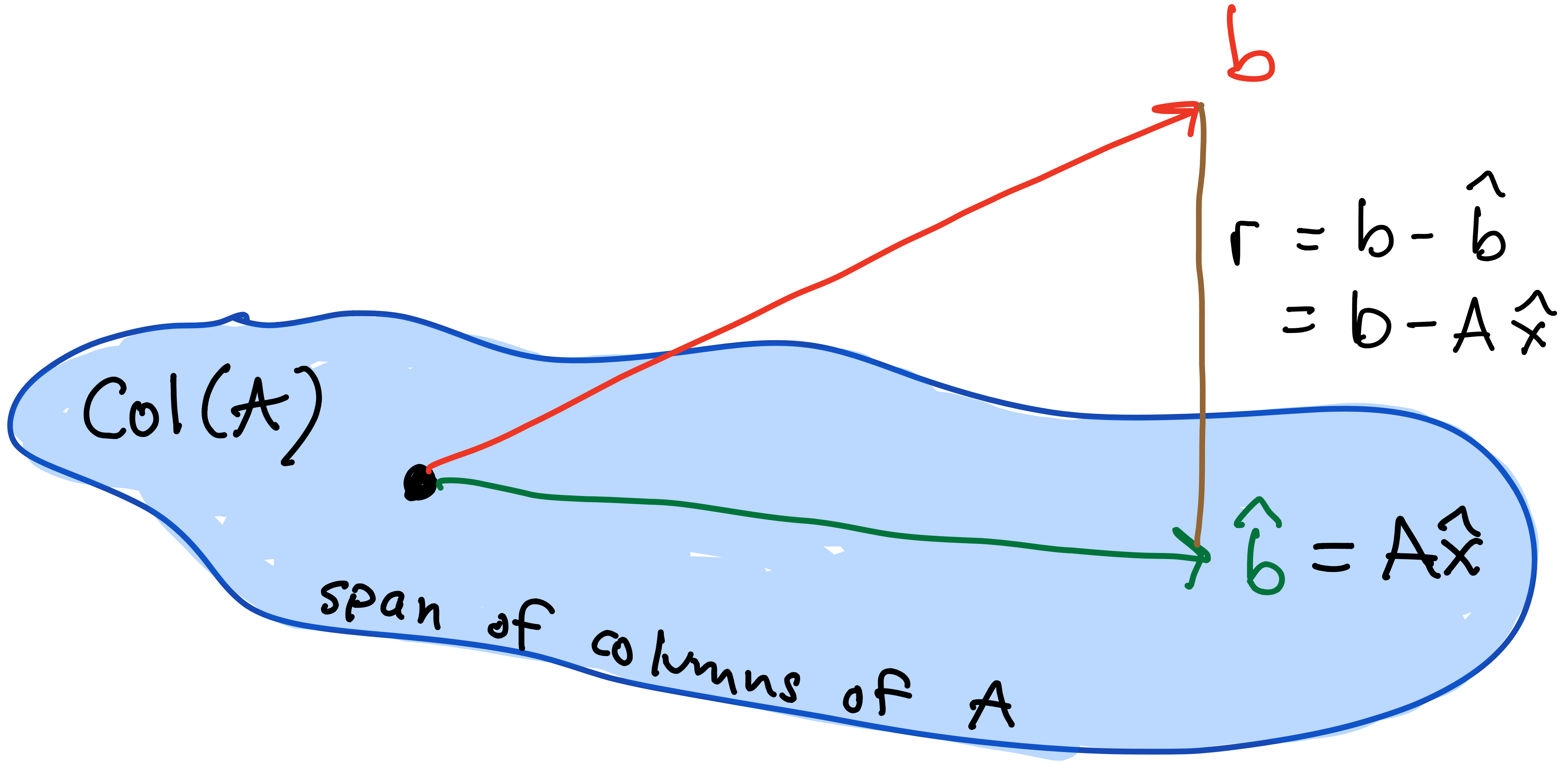
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 11 \\ -4 \end{bmatrix}$$ has least squares solution: $$\hat{x} = \begin{bmatrix} -2 \\ -1 \\ 9 \end{bmatrix}$$

- Find the projection $\hat{b}$

$$\hat{b} = A\hat{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \\ 9 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 9 \\ -6 \end{bmatrix}$$

- Find the residual:

$$r = b - \hat{b} = \begin{bmatrix} 0 \\ 1 \\ 11 \\ -4 \end{bmatrix} - \begin{bmatrix} -2 \\ -1 \\ 9 \\ -6 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}$$



Important Application

Find the line that best fits the data (1,1), (2,2.4), (3,3.6), (4,4)

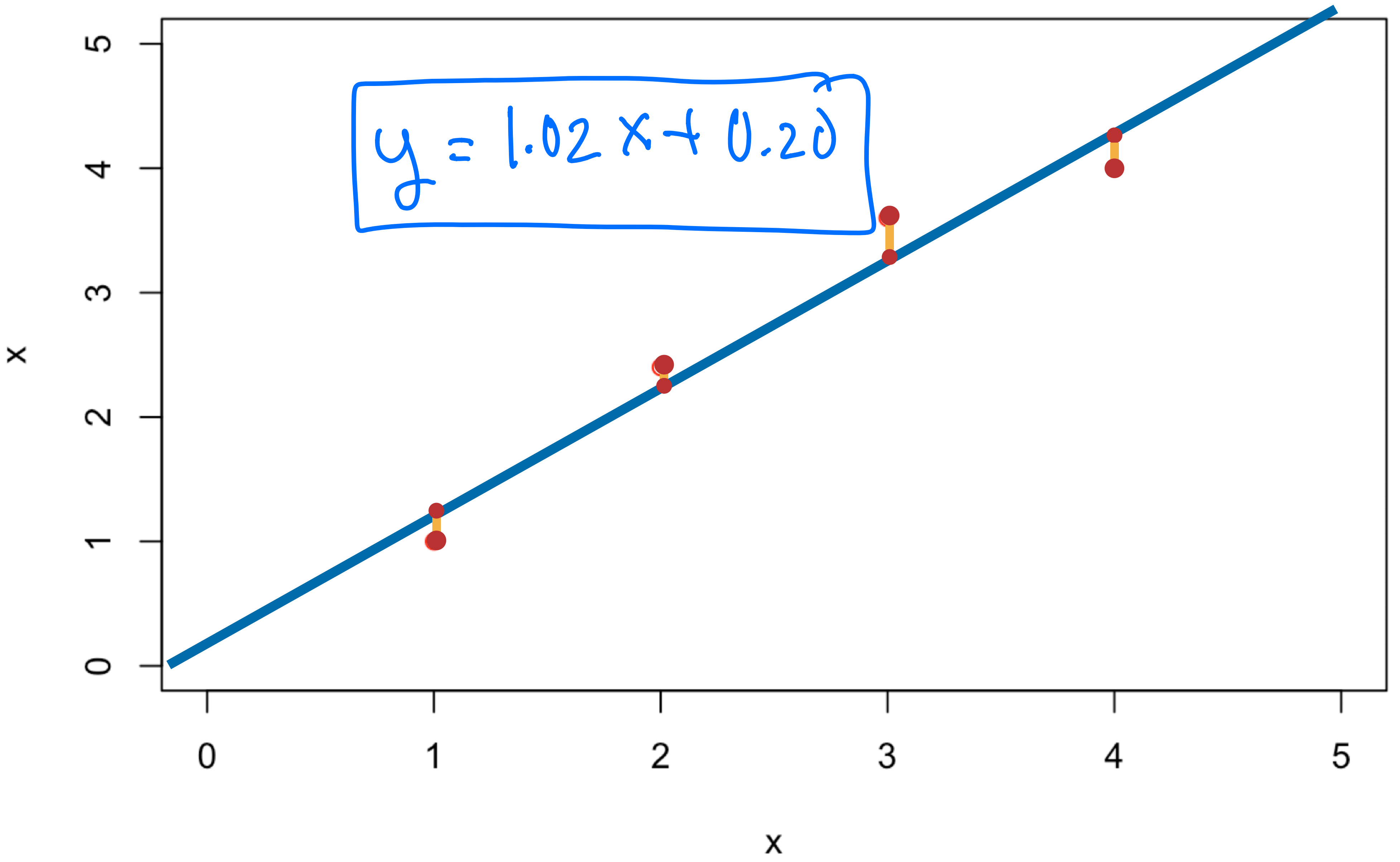
If the line is $y = ax + b$ then

we want

$$\begin{aligned} a\cdot 1 + b &= 1 \\ a\cdot 2 + b &= 2.4 \\ a\cdot 3 + b &= 3.6 \\ a\cdot 4 + b &= 4 \end{aligned}$$

or

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 1 \\ 2.4 \\ 3.6 \\ 4 \end{bmatrix}}_{b}$$



$b \notin \mathrm{Col}(A)$, so we find a least squares solution

$$A^TA = \begin{bmatrix} 30 & 10 \\ 10 & 4 \end{bmatrix}$$

$$A^Tb = \begin{bmatrix} 32.6 \\ 11.4 \end{bmatrix}$$

Normal equations

Solve $\begin{bmatrix} 30 & 10 \\ 10 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 32.6 \\ 11.0 \end{bmatrix}$

augment and now reduce or use software

$$\rightsquigarrow \begin{bmatrix} a \\ b \end{bmatrix} = \underbrace{\begin{bmatrix} 1.02 \\ 0.20 \end{bmatrix}}_{\hat{x}}$$

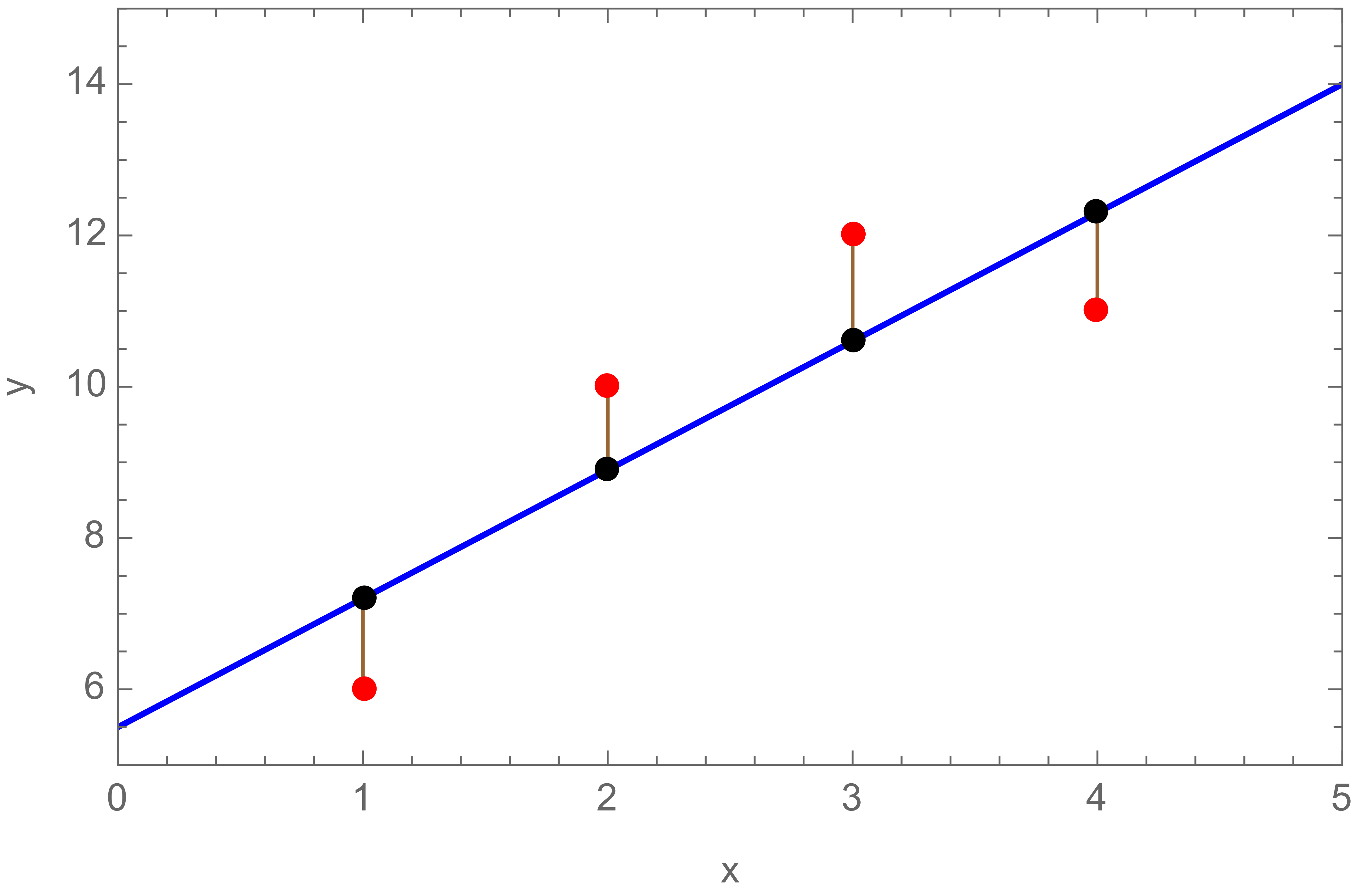
$$\underbrace{\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1.02 \\ 0.20 \end{bmatrix}}_{\hat{x}} = \underbrace{\begin{bmatrix} 1.22 \\ 2.24 \\ 3.26 \\ 4.29 \end{bmatrix}}_{\hat{b}}$$

$$r = \underbrace{\begin{bmatrix} 1 \\ 2.4 \\ 3.6 \\ 4 \end{bmatrix}}_{b} - \underbrace{\begin{bmatrix} 1.22 \\ 2.24 \\ 3.26 \\ 4.29 \end{bmatrix}}_{\hat{b}} = \begin{bmatrix} -.22 \\ .16 \\ .34 \\ -.28 \end{bmatrix}$$

$$\|r\| = \sqrt{(-.22)^2 + .16^2 + .34^2 + (-.28)^2} = 0.517$$

You Try

Fit a line to the points: (1,6), (2,10),(3,12),(4,11)



$$y = ax + b$$
$$6 = a1 + b$$
$$10 = a2 + b$$
$$12 = a3 + b$$
$$11 = a4 + b$$

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ 12 \\ 11 \end{bmatrix}$$