5.3. Diagonalization and Eigenbases

Example

$$A = \begin{bmatrix} -2 & -1 & 1 \\ -3 & -2 & 3 \\ -3 & -1 & 2 \end{bmatrix}$$

has characteristic polynomial: $f_A(\lambda) = (\lambda+2)(\lambda+1)(\lambda-1)$

and thus eigenvalues: $\lambda_1 = -2, \lambda_2 = -1, \lambda_3 = 1$

corresponding eigenvectors: $\mathcal{B} = \left\{ \underset{\lambda=-2}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}, \underset{\lambda=-1}{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}, \underset{\lambda=1}{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}} \right\}$

$$A B = A \underbrace{\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}}_{B} = \begin{bmatrix} Av_1 & Av_2 & Av_3 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 v_1 & \lambda_2 v_2 & \lambda_3 v_3 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}}_{D}$$

These are linearly independent, and

$$B = \begin{array}{c} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \\ \begin{matrix} v_1 & v_2 & v_3 \end{matrix} \end{array}$$

is invertible

$$\boxed{\begin{aligned} AB &= BD \\ \Rightarrow \quad B^{-1}AB &= D \\ A &= BDB^{-1} \end{aligned}}$$

Diagonalization

$$A = \begin{bmatrix} -2 & -1 & 1 \\ -3 & -2 & 3 \\ -3 & -1 & 2 \end{bmatrix}$$

$\boxed{\lambda_1 = 2}$

$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\boxed{\lambda_2 = -1}$

$v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$\boxed{\lambda_3 = 1}$

$v_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$$\underbrace{\begin{bmatrix} -2 & -1 & 1 \\ -3 & -2 & 3 \\ -3 & -1 & 2 \end{bmatrix}}_{A} = \underbrace{\overset{\text{eigenvectors}}{\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}}}_{B} \underbrace{\overset{\text{eigenvalues}}{\begin{bmatrix} 2 & & \\ & -1 & \\ & & 1 \end{bmatrix}}}_{D} \underbrace{\begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix}}_{B^{-1}}$$

$$\underbrace{\overset{\text{eigenvalues}}{\begin{bmatrix} 2 & & \\ & -1 & \\ & & 1 \end{bmatrix}}}_{D} = \underbrace{\begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix}}_{B^{-1}} \underbrace{\begin{bmatrix} -2 & -1 & 1 \\ -3 & -2 & 3 \\ -3 & -1 & 2 \end{bmatrix}}_{A} \underbrace{\overset{\text{eigenvectors}}{\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}}}_{B}$$

Diagonalization

If an n x n matrix $\mathbf{A}$ has n linearly independent eigenvectors, then there is a basis of $\mathbb{R}^n$ consisting of eigenvectors of $\mathbf{A}$. We will call this an **eigenbasis**.

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\overbrace{\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}}^{\text{eigenvectors}}}_{P} \overbrace{\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}}^{\text{eigenvalues}} \begin{bmatrix} P^{-1} \end{bmatrix}$$

Diagonalization: More Examples

(a) $$\begin{bmatrix} -2 & 2 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 6 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} \frac{1}{9} & \frac{2}{9} \\ -\frac{4}{9} & \frac{1}{9} \end{bmatrix}$$

(b) $$\begin{bmatrix} -7 & 6 & -3 \\ -3 & 2 & -3 \\ 3 & -6 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 2 & -1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix}$$

(c) $$\begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{4}{3} & -\frac{1}{2} & -\frac{1}{3} \\ \frac{4}{3} & -\frac{1}{2} & -\frac{2}{3} \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{12} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{3}{11} & \frac{3}{11} & \frac{3}{11} \\ -\frac{36}{11} & \frac{30}{11} & \frac{8}{11} \\ 3 & -3 & 0 \end{bmatrix}$$

Diagonalization is changing coordinates

$$\underset{[A]_S}{\begin{array}{cc} & \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{array}{c} e_1\\ e_2\\ e_3 \end{array} & \begin{bmatrix} -7 & 6 & -3 \\ -3 & 2 & -3 \\ 3 & -6 & -1 \end{bmatrix} \end{array}} = \underbrace{\begin{array}{cc} & \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{array}{c} e_1\\ e_2\\ e_3 \end{array} & \begin{bmatrix} -1 & 2 & -1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix} \end{array}}_{B \quad \mathcal{B}\to S} \underset{[A]_{\mathcal{B}}}{\begin{array}{cc} & \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{array}{c} v_1\\ v_2\\ v_3 \end{array} & \begin{bmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 2 \end{bmatrix} \end{array}} \underbrace{\begin{array}{cc} & \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{array}{c} v_1\\ v_2\\ v_3 \end{array} & \begin{bmatrix} -\frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix} \end{array}}_{B^{-1} \quad S\to\mathcal{B}}$$

$$\underset{[A]_{\mathcal{B}}}{\begin{array}{cc} & \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{array}{c} v_1\\ v_2\\ v_3 \end{array} & \begin{bmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 2 \end{bmatrix} \end{array}} = \underbrace{\begin{array}{cc} & \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{array}{c} v_1\\ v_2\\ v_3 \end{array} & \begin{bmatrix} -\frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix} \end{array}}_{B^{-1} \quad S\to\mathcal{B}} \underset{[A]_S}{\begin{array}{cc} & \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{array}{c} e_1\\ e_2\\ e_3 \end{array} & \begin{bmatrix} -7 & 6 & -3 \\ -3 & 2 & -3 \\ 3 & -6 & -1 \end{bmatrix} \end{array}} \underbrace{\begin{array}{cc} & \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{array}{c} e_1\\ e_2\\ e_3 \end{array} & \begin{bmatrix} -1 & 2 & -1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix} \end{array}}_{B \quad \mathcal{B}\to S}$$

When is a matrix diagonalizable?

$\boxed{\text{A is diagonalizable}} \Longleftrightarrow \boxed{\text{A has an eigenbasis}}$

THEOREM 2
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If $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ of an $n \times n$ matrix A, then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is linearly independent.

$\boxed{\text{Means: A is diagonalizable if it has } \textit{\textbf{distinct}} \text{ eigenvalues}}$

Proof For the sake of contradiction, suppose $v_1, v_2, \ldots, v_r$ are dependent.

Let p be smallest index such that $v_{p+1} = c_1 v_1 + c_2 v_2 + \cdots + c_p v_p$

multiply by A $\quad A v_{p+1} = c_1 A v_1 + c_2 A v_2 + \cdots + c_p A v_p$

$$\lambda_{p+1} v_{p+1} = c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 + \cdots + c_p \lambda_p v_p$$

multiply by λ_{p+1}

$$\lambda_{p+1} v_{p+1} = c_1 \lambda_{p+1} v_1 + c_2 \lambda_{p+1} v_2 + \cdots + c_p \lambda_{p+1} v_p$$

not 0 since eigenvalues distinct

$$0 = c_1 \underbrace{(\lambda_1 - \lambda_{p+1})} v_1 + c_2 \underbrace{(\lambda_2 - \lambda_{p+1})} v_2 + \cdots + c_p \underbrace{(\lambda_p - \lambda_{p+1})} v_p$$

smaller dependence relation : contradiction

Example: distinct eigenvalues

$$\mathbf{A} = \begin{bmatrix} -78 & -19 & -82 & 22 \\ 104 & 25 & 108 & -28 \\ 82 & 19 & 86 & -22 \\ 111 & 28 & 115 & -31 \end{bmatrix} = \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} -15 & -7 & -1 & -3 \\ 20 & 9 & 1 & 4 \\ 16 & 7 & 1 & 3 \\ 21 & 10 & 1 & 4 \end{bmatrix} \end{array} \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & -1 & -1 & 1 \\ -3 & -2 & 1 & -1 \\ -2 & 3 & -3 & -2 \end{bmatrix}$$

4 distinct eigenvalues: $\lambda_1 = 4, \lambda_2 = -3, \lambda_3 = 1, \lambda_4 = 0$

Thus, A is **diagonalizable**.

Is A **invertible**? No! 0 is an eigenvalue

$$A\vec{v}_4 = 0\vec{v}_4 = \vec{0}$$

$$\mathrm{Nul}(A) = \mathrm{span}(\vec{v}_4)$$

A is diagonalizable, but not invertible!

Two Examples: Repeated Eigenvalues

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & -3 \\ 4 & 0 & 6 \\ 2 & -1 & 5 \end{bmatrix}$$

algebraic multiplicity = 2

$$c(\lambda) = -(\lambda-2)^2(\lambda-1)$$

$$\lambda_1 = 2, \lambda_2 = 2, \lambda_3 = 1$$

$$\begin{bmatrix} -2 & 1 & -3 \\ 4 & -2 & 6 \\ 2 & -1 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -\frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Geometric multiplicity $\dim(E_2) = 2$

$$\begin{bmatrix} -1 & 1 & -3 \\ 4 & -1 & 6 \\ 2 & -1 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & -3 \\ 4 & 0 & 6 \\ 2 & -1 & 5 \end{bmatrix} = \begin{bmatrix} -3 & 1 & -1 \\ 0 & 2 & 2 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} & 2 \\ 2 & -\frac{1}{2} & 3 \\ -2 & 1 & -3 \end{bmatrix}$$

diagonalizable and invertible

$$\mathbf{B} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix}$$

algebraic multiplicity = 2

$$c(\lambda) = -(\lambda-2)(\lambda-1)^2$$

$$\lambda_1 = 2, \lambda_2 = 1, \lambda_2 = 1$$

$$\begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 2 & -5 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & -5 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Geometric multiplicity $\dim(E_1) = 1$

invertible but not diagonalizable

Diagonalizable if algebraic multiplicity equals geometric multiplicity for all eigenvalues