Section 3.1: The Derivative as a linear Operator.

Goal: Solve higher order linear equations

An <u>operator</u> is a function that takes in a function and returns a new function

Use the letter D to denote derivative as an operator.

$$D(x^2 + 3\sin x) = 2x + 3\cos x$$

We can build new operators from D.

$$L := \underbrace{2D^2}_{2\cdot D\cdot D} - 4D + 2.$$

$\frac{d}{dx}\left(\frac{d}{dx}y\right) = \frac{d^2}{dx^2}y$

$2DD(y) = 2D(y') = 2y''$

L takes in a function y and returns the function

$$2y'' - 4y' + 2y$$

$$L(y) = (2D^2 - 4D + 2)(y)$$

$$= 2\underline{D^2(y)} - 4\underline{D(y)} + 2\underline{y}$$

$$L = 2D^2 - 4\cdot D - \frac{2}{y}$$

$L\,)$

An operator L is a <u>linear operator</u> if it satisfies

(1) $L(cu) = cL(u)$ for any constant c and function u.

(2) $L(u+v) = L(u) + L(v)$ for any functions u and v.

(It's enough to show
$L(c\cdot u + v) = cL(u) + L(v)$)

Derivative is <u>linear</u>. Can prove using limit definition of derivative.

Example: Define $L := D^2 + x^2 D + 4x$
We'll show L is linear.

$$L(cu+v) = (D^2 + x^2 D + 4x)(cu+v)$$

$$= D^2(cu+v) + x^2 D(cu+v) + 4x(cu+v)$$

linearity of D

$$= cD^2u + D^2 v + x^2 c D(u) + x^2 D(v) + 4cxu + 4xv$$

$$= c(D^2 u + x^2 Du + 4xu) + D^2 v + x^2 Dv + 4xv$$

$$= c(D^2 + x^2 D + 4x)(u) + (D^2 + x^2 D + 4x)(v)$$

$$= \underline{c L(u) + L(v)}$$

Any operator defined as a polynomial in D is linear.

(Non) example: Define N by

$$N(y) := (y')^2$$

N is not linear since for $y = x^2$.

$$N(5x^2) = (10x)^2 = 100x^2$$

Not equal!

$$5N(x^2) = 5(2x)^2 = 5(4x^2) = 20x^2$$

Try $y = \sin x$

$$N(2\sin x) = (2\cos x)^2 = \underline{4}\cos^2 x$$

$$2N(\sin x) = 2(\cos x)^2 = 2\cos^2 x$$

Linear Differential Equations

A differential equation is <u>linear</u> if it can be written in the form.

$$Ly = f(x)$$

Where
(1) L is a linear operator
(2) f is a function only of x (independent variable.)

If $f(x) = 0$ for all x $(Ly = 0)$ then the equation is <u>homogeneous</u>.
Otherwise, the equation is nonhomogenous (inhomogeneous)

Example: $\frac{dy}{dx} = ky$. is a homogeneous linear DE.

$$\frac{dy}{dx} - ky = 0$$

$$(D - k)y = 0$$

polynomial in D

$L := D - k$

$Ly = 0$