

# 1.5-b. Algebraic Properties of Solutions to $Ax = b$

# Properties of Matrix-Vector Product

$$A(\mathbf{u} + \mathbf{w}) = A\mathbf{u} + A\mathbf{w}$$

$$\begin{aligned}
 & \left[ \begin{array}{ccc|c} 1 & 1 & 1 & u_1 \\ v_1 & v_2 & v_3 & u_2 \\ 1 & 1 & 1 & u_3 \\ \hline & & & w_1 \\ & & & w_2 \\ & & & w_3 \end{array} \right] \left( \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \right) = \left[ \begin{array}{ccc|c} 1 & 1 & 1 & u_1 + w_1 \\ v_1 & v_2 & v_3 & u_2 + w_2 \\ 1 & 1 & 1 & u_3 + w_3 \\ \hline & & & \end{array} \right] \left( \begin{bmatrix} u_1 + w_1 \\ u_2 + w_2 \\ u_3 + w_3 \end{bmatrix} \right) \\
 & = (u_1 + w_1) \begin{bmatrix} 1 \\ v_1 \\ 1 \end{bmatrix} + (u_2 + w_2) \begin{bmatrix} 1 \\ v_2 \\ 1 \end{bmatrix} + (u_3 + w_3) \begin{bmatrix} 1 \\ v_3 \\ 1 \end{bmatrix} \\
 & = \underbrace{u_1 \begin{bmatrix} 1 \\ v_1 \\ 1 \end{bmatrix} + u_2 \begin{bmatrix} 1 \\ v_2 \\ 1 \end{bmatrix} + u_3 \begin{bmatrix} 1 \\ v_3 \\ 1 \end{bmatrix}}_{+} + \underbrace{w_1 \begin{bmatrix} 1 \\ v_1 \\ 1 \end{bmatrix} + w_2 \begin{bmatrix} 1 \\ v_2 \\ 1 \end{bmatrix} + w_3 \begin{bmatrix} 1 \\ v_3 \\ 1 \end{bmatrix}}_{+} \\
 & = \left[ \begin{array}{ccc|c} 1 & 1 & 1 & u_1 \\ v_1 & v_2 & v_3 & u_2 \\ 1 & 1 & 1 & u_3 \\ \hline & & & w_1 \\ & & & w_2 \\ & & & w_3 \end{array} \right] \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \left[ \begin{array}{ccc|c} 1 & 1 & 1 & w_1 \\ v_1 & v_2 & v_3 & w_2 \\ 1 & 1 & 1 & w_3 \\ \hline & & & \end{array} \right] \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}
 \end{aligned}$$

# Properties of Matrix-Vector Product

$$A(c \mathbf{u}) = c A\mathbf{u}$$

$$\begin{aligned}
 & \left[ \begin{array}{ccc|c} 1 & 1 & 1 & u_1 \\ v_1 & v_2 & v_3 & u_2 \\ 1 & 1 & 1 & u_3 \end{array} \right] \left( c \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \right) = \left[ \begin{array}{ccc|c} 1 & 1 & 1 & cu_1 \\ v_1 & v_2 & v_3 & cu_2 \\ 1 & 1 & 1 & cu_3 \end{array} \right] \begin{bmatrix} cu_1 \\ cu_2 \\ cu_3 \end{bmatrix} \\
 & = c u_1 \begin{bmatrix} 1 \\ v_1 \\ 1 \end{bmatrix} + c u_2 \begin{bmatrix} 1 \\ v_2 \\ 1 \end{bmatrix} + c u_3 \begin{bmatrix} 1 \\ v_3 \\ 1 \end{bmatrix} \\
 & = c \left( u_1 \begin{bmatrix} 1 \\ v_1 \\ 1 \end{bmatrix} + u_2 \begin{bmatrix} 1 \\ v_2 \\ 1 \end{bmatrix} + u_3 \begin{bmatrix} 1 \\ v_3 \\ 1 \end{bmatrix} \right) \\
 & = c \left[ \begin{array}{ccc|c} 1 & 1 & 1 & u_1 \\ v_1 & v_2 & v_3 & u_2 \\ 1 & 1 & 1 & u_3 \end{array} \right] \left( \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \right)
 \end{aligned}$$

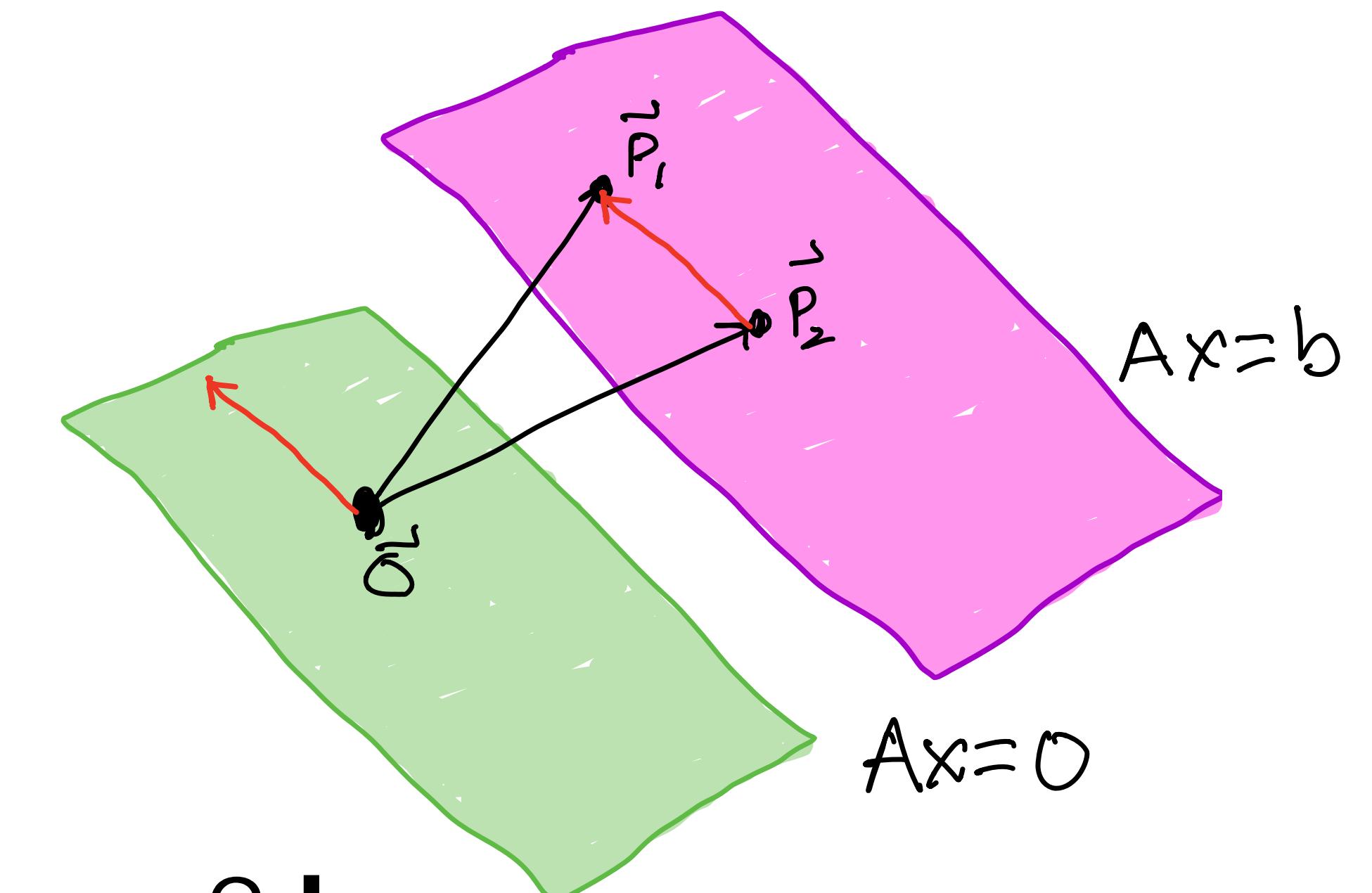
# Properties of Solutions to $\mathbf{A}x = \mathbf{b}$

1.  $\mathbf{A} \mathbf{0} = \mathbf{0}$  so  $\mathbf{A} x = \mathbf{0}$  always has a solution: the zero vector

2. Suppose that  $\mathbf{p}_1$  and  $\mathbf{p}_2$  are solutions to  $\mathbf{A} x = \mathbf{b}$

i.e.,  $\mathbf{A} \mathbf{p}_1 = \mathbf{b}$  and  $\mathbf{A} \mathbf{p}_2 = \mathbf{b}$

$$\begin{aligned}\text{Then: } \mathbf{A} (\mathbf{p}_1 - \mathbf{p}_2) &= \mathbf{A} \mathbf{p}_1 - \mathbf{A} \mathbf{p}_2 \\ &= \mathbf{b} - \mathbf{b} \\ &= 0\end{aligned}$$



3.  $\mathbf{A} (\mathbf{p}_1 + \mathbf{p}_2) = \mathbf{A} \mathbf{p}_1 + \mathbf{A} \mathbf{p}_2 = \mathbf{b} + \mathbf{b} = 2 \mathbf{b}$