Def. A *linear transformation* T is a function from $\mathbb{R}^n$ to $\mathbb{R}^m$ that satisfies the following properties, L1 and L2, for all vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$ and all scalars $c \in \mathbb{R}$. Properties L3 and L4 are consequences of L1 and L2, so they must be true for a linear transformation.

L1. $L(\mathbf{0}) = \mathbf{0}$

L2. $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$

L3. $T(c\vec{u}) = cT(\vec{u})$

L4. $L(c\vec{u} + d\vec{v}) = cL(\vec{u}) + dL(\vec{v})$

A matrix transformation is linear: $T(x) = \mathsf{A}x$. <u>And</u> every linear transformation has a matrix!

Finding the Matrix of a Transformation

1. Frst observe the following about the matrix vector product

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \end{bmatrix}}_{\substack{3\times 5 \\ \mathbb{R}^5 \to \mathbb{R}^3}} \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{e_2} = \begin{bmatrix} 2 \\ 7 \\ 12 \end{bmatrix}$$

$T(e_2) = V_2 \leftarrow$ 2<u>nd</u> coulumn of A

$T(e_i) =$ ith column of matrix

2. Suppose that T is a linear transformation with the properties below. Find the matrix A of T.

$$T\Bigg(\underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{e_1}\Bigg) = \underbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}}_{v_1}, \quad T\Bigg(\underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{e_2}\Bigg) = \underbrace{\begin{bmatrix} e \\ \pi \\ 0 \\ -1 \end{bmatrix}}_{v_2}, \quad \text{and} \quad T\Bigg(\underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{e_3}\Bigg) = \underbrace{\begin{bmatrix} 2 \\ 0 \\ 2 \\ 1 \end{bmatrix}}_{v_3}$$

$$A = \begin{array}{c} \begin{array}{ccc} T(e_1) & T(e_2) & T(e_3) \end{array} \\ \begin{bmatrix} 1 & e & 2 \\ 2 & \pi & 0 \\ 3 & 0 & 2 \\ 4 & -1 & 1 \end{bmatrix} \end{array} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$\boxed{A = \begin{bmatrix} | & | & & | \\ T(e_1) & T(e_2) & \cdots & T(e_n) \\ | & | & & | \end{bmatrix}}$$

3. Find the matrix of the following linear transformation and show that it performs the same function by applying it to $\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right]$.

$$T_1\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 \\ x_1 + 2x_2 \\ 2x_1 - x_2 \end{bmatrix}$$

$$T\Bigg(\underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{e_1}\Bigg) = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \qquad T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$$

$$A = \begin{array}{c} \begin{array}{cc} T(e_1) & T(e_2) \end{array} \\ \begin{bmatrix} 1 & -1 \\ 1 & 2 \\ 2 & -1 \end{bmatrix} \end{array} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ x_1 + 2x \\ 2x_1 - x_2 \end{bmatrix}$$

You Try

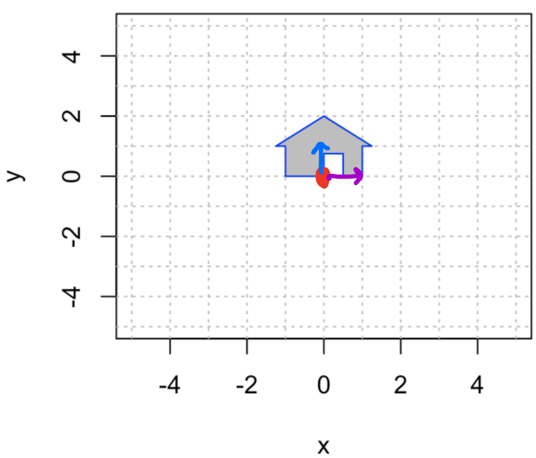
1. Find the matrix for the following linear transformations:

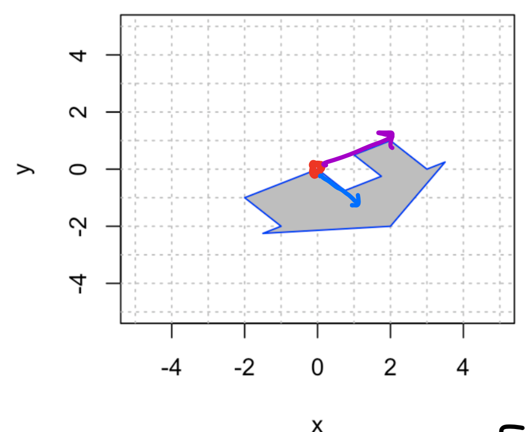
   (a) $T_1\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = 3\begin{bmatrix} x_2 \\ x_3 \\ x_1 \end{bmatrix} - 2\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$T_1\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}-2\\0\\3\end{bmatrix} \qquad T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}3\\-2\\0\end{bmatrix} \qquad T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\3\\-2\end{bmatrix}$$

$$A = \begin{bmatrix} -2 & 3 & 0 \\ 0 & -2 & 3 \\ 3 & 0 & -2 \end{bmatrix}$$

   (b) $T : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation that sends my house on the left to the house on the right.





$$A = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$$

<u>note</u> peak of house is $\begin{bmatrix}2\\0\end{bmatrix} \to \begin{bmatrix}2\\-2\end{bmatrix}$

$\Rightarrow \begin{bmatrix}1\\0\end{bmatrix} \to \begin{bmatrix}1\\-1\end{bmatrix}$

   (c) The matrix that $T : \mathbb{R}^3 \to \mathbb{R}^3$ does the following. First it doubles the first coordinate. Then it adds the first coordinate to the second coordinate. Then it triples the middle coordinate. Then it exchanges the second and third coordinates.

$$\begin{bmatrix}1\\0\\0\end{bmatrix} \xrightarrow{\text{double}} \begin{bmatrix}2\\0\\0\end{bmatrix} \xrightarrow{\text{add}} \begin{bmatrix}2\\2\\0\end{bmatrix} \xrightarrow{\text{triple}} \begin{bmatrix}2\\6\\0\end{bmatrix} \xrightarrow{\text{exchange}} \begin{bmatrix}2\\0\\6\end{bmatrix}$$

$$\begin{bmatrix}0\\1\\0\end{bmatrix} \longrightarrow \begin{bmatrix}0\\1\\0\end{bmatrix} \longrightarrow \begin{bmatrix}0\\1\\0\end{bmatrix} \longrightarrow \begin{bmatrix}0\\3\\0\end{bmatrix} \longrightarrow \begin{bmatrix}0\\0\\3\end{bmatrix}$$

$$\begin{bmatrix}0\\0\\1\end{bmatrix} \longrightarrow \begin{bmatrix}0\\0\\1\end{bmatrix} \longrightarrow \begin{bmatrix}0\\0\\1\end{bmatrix} \longrightarrow \begin{bmatrix}0\\0\\1\end{bmatrix} \longrightarrow \begin{bmatrix}0\\1\\0\end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 6 & 3 & 0 \end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}2x\\z\\6x+3y\end{bmatrix}$$

does that process to <u>any</u> vector: cool!

2. T is the linear transformation given by $T(\vec{x}) = A\vec{x}$ where A is the matrix below. I also give you a useful calculation to the right.

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}\right) \underbrace{\begin{bmatrix} 1 & 0 & 0 & -1 & 1 \\ 1 & 1 & 1 & -2 & 2 \\ 2 & 1 & -2 & 3 & 1 \\ 1 & 1 & 1 & -2 & 1 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & -1 & 1 & 5 \\ 1 & 1 & 1 & -2 & 2 & 14 \\ 2 & 1 & -2 & 3 & 1 & 2 \\ 1 & 1 & 1 & -2 & 1 & 10 \end{array}\right] \longrightarrow \begin{array}{c} \qquad\qquad\quad t \\ \left[\begin{array}{ccccc|c} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right] \end{array}$$

(a) T is a function that sends vectors from $\mathbb{R}^n$ to $\mathbb{R}^m$ where $n =$ 5 and $m =$ 4

(b) Compute these values: $T\left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 3 \\ 5 \\ 2 \end{bmatrix}$ $\quad T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix}$

(c) Find all input values $\vec{x}$ so that $T(\vec{x}) = \begin{bmatrix} 5 \\ 14 \\ 2 \\ 10 \end{bmatrix}$.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \\ 4 \end{bmatrix} + t\begin{bmatrix} 1 \\ -1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

(d) True or False?

T i. (onto) For all $b \in \mathbb{R}^4$ there exists at least one $\vec{x} \in \mathbb{R}^5$ such that $T(\vec{x}) = \vec{b}$.

F ii. (one-to-one) For all $b \in \mathbb{R}^4$ there exists at most one $\vec{x} \in \mathbb{R}^5$ such that $T(\vec{x}) = \vec{b}$.

F iii. There exists a unique $\vec{x} \in \mathbb{R}^5$ such that $T(\vec{x}) = \mathbf{0}$.

F iv. The columns of A are linearly dependent

T v. The columns of A span $\mathbb{R}^4$.

3. Match the matrix with the statement about the corresponding linear transformation.

A (a) The transformation is both one-to-one and onto.

D (b) The transformation is one-to-one but not onto.

B (c) The transformation is onto but not one-to-one.

C (d) The transformation is neither onto nor one-to-one.

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 3 \\ 1 & 2 & 0 & 1 & 2 \\ -1 & 2 & 1 & 0 & 1 \\ -1 & 2 & 0 & -1 & 0 \\ 1 & 1 & 1 & -1 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 1 & 1 & 1 & 6 \\ 1 & 2 & 0 & 1 & 4 \\ -1 & 2 & 1 & 0 & 2 \\ -1 & 2 & 0 & -1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 1 & 1 & 1 & -1 \\ 1 & 2 & 0 & 1 & -4 \\ -1 & 2 & 1 & 0 & -4 \\ -1 & 2 & 0 & -1 & -4 \\ 1 & 1 & 1 & -1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 1 & 1 & 1 & 3 \\ 1 & 2 & 0 & 1 & 2 \\ -1 & 2 & 1 & 0 & 1 \\ -1 & 2 & 0 & -1 & 0 \\ 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$