Def. If A is an $n \times n$ matrix, then a *nonzero* vector $\vec{v} \in \mathbb{R}^n$ is an *eigenvector* of A if

$$A\vec{v} = \lambda\vec{v}, \qquad \text{for some } \lambda \in \mathbb{R}^n.$$

The scalar λ is the *eigenvalue* corresponding to $\vec{v}$. Observe that

- A does not change the direction of A (except that it may flip it over when $\lambda < 0$). It rescales $\vec{v}$.
- $A\vec{v}$ is on the same line as $\vec{v}$.

Examples:

1. Are any of the vectors below eigenvectors for the matrix A?

$$A = \begin{bmatrix} 3 & -5 & 7 \\ -4 & 7 & 2 \\ -14 & 25 & -6 \end{bmatrix} \qquad \vec{u} = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

$\boxed{\lambda = -3}$ $\boxed{\lambda = 5}$ NOT

$$\boxed{A\underbrace{\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}}_{\vec{u}} = \begin{bmatrix} 6 \\ 3 \\ -3 \end{bmatrix} = (-3)\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}} \qquad \boxed{A\underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{\vec{v}} = \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} \qquad \boxed{A\underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}_{\vec{w}} = \begin{bmatrix} 10 \\ -2 \\ -20 \end{bmatrix} \overset{?}{=} ?\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}$$

2. Given that λ is an eigenvalue of A, how do we find the corresponding eigenvectors?

This derivation is important!

$\boxed{\text{solve}}$ $\boxed{A\vec{v} = \lambda\vec{v}}$

$$A\vec{v} - \lambda\vec{v} = \vec{0}$$

$$A\vec{v} - \lambda I_n\vec{v} = \vec{0}$$

$$\underbrace{(A - \lambda I_n)}\vec{v} = \vec{0}$$

$$\boxed{v \in \text{Nul}(A - \lambda I)}$$

$$E_5 = \text{Nul}(A - 5I) = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = \left\{ s\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \middle| s \in \mathbb{R} \right\}$$

$$\underbrace{\begin{bmatrix} 3 & -5 & 7 \\ -4 & 7 & 2 \\ -14 & 25 & -6 \end{bmatrix}}_{A} - \underbrace{\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}}_{5I} = \begin{bmatrix} 3-5 & -5 & 7 \\ -4 & 7-5 & 2 \\ -14 & 25 & -6-5 \end{bmatrix} = \begin{bmatrix} -2 & -5 & 7 \\ -4 & 2 & 2 \\ -14 & 25 & -11 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

3. (Checkpoint 5.1) Given that 4 and -3 are eigenvalues of $\mathsf{A} = \begin{bmatrix} 3 & 2 \\ \cancel{4}\,3 & -2 \end{bmatrix}$ find its eigenvectors.

$\boxed{\lambda = 4}$

$$\begin{bmatrix} 3-4 & 2 \\ 3 & -2-4 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 3 & -6 \end{bmatrix} \to \begin{array}{cc} x_1 & s \\ \end{array} \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

$$E_4 = \operatorname{span}\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} \right) = \left\{ s \begin{bmatrix} 2 \\ 1 \end{bmatrix} \,\middle|\, s \in \mathbb{R} \right\}$$

$\boxed{\lambda = -3}$

$$\begin{bmatrix} 3+3 & 2 \\ 3 & -2+3 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1/3 \\ 0 & 0 \end{bmatrix}$$

$$E_{-3} = \operatorname{span}\left(\begin{bmatrix} -1/3 \\ 1 \end{bmatrix} \right) = \operatorname{span}\left(\begin{bmatrix} -3 \\ 1 \end{bmatrix} \right) = \left\{ t \begin{bmatrix} -3 \\ 1 \end{bmatrix} \,\middle|\, t \in \mathbb{R} \right\}$$

4. Given that $\vec{v}$ is an eigenvector of the matrix A of eigenvalue λ.

(a) Show that $5\vec{v}$ is an eigenvector of A. What is its eigenvalue?

$$A(5v) = 5Av = 5\lambda v = \lambda(5v)$$

so $5v$ is an eigenvector of eigenvalue λ

(b) Show that $\vec{v}$ is an eigenvector of A^2. What is its eigenvalue?

$$A^2 v = A(Av) = A(\lambda v) = \lambda Av = \lambda \lambda v = \lambda^2 v$$

so v is an eigenvector of eigenvalue λ^2

(c) (tricky) Show that $\vec{v}$ is an eigenvector of A^{-1}. What is its eigenvalue?

$$Av = \lambda v$$
$$A^{-1}Av = A^{-1}(\lambda v)$$
$$Iv = \lambda A^{-1} v$$
$$v = \lambda A^{-1} v$$
$$\frac{1}{\lambda} v = A^{-1} v$$
$$\boxed{A^{-1} v = \frac{1}{\lambda} v}$$

v is an eigenvector of eigenvalue $1/\lambda$

(d) If $\vec{v}$ is an eigenvector for A with eigenvalue 3 and $\vec{w}$ is an eigenvector for A with eigenvalue 5, is $\vec{v} + \vec{w}$ is an eigenvector of A. If so, what is its eigenvalue?

$$Av = 3v$$
$$Aw = 5w$$

$$A(v+w) = Av + Aw = 3v + 5w \neq \lambda(v+w)$$

$\boxed{\text{not an eigenvector}}$

5. $\mathsf{A} = \begin{bmatrix} -2 & -1 & 1 \\ -3 & -2 & 3 \\ -3 & -1 & 2 \end{bmatrix}$ has eigenvalues $\lambda = -2, -1, 1$. Use the information below to describe the eigenspaces.

(a) $\mathsf{A} + 2\mathbf{I}_3 = \begin{bmatrix} 0 & -1 & 1 \\ -3 & 0 & 3 \\ -3 & -1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

(b) $\mathsf{A} + \mathbf{I}_3 = \begin{bmatrix} -1 & -1 & 1 \\ -3 & -1 & 3 \\ -3 & -1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(c) $\mathsf{A} - \mathbf{I}_3 = \begin{bmatrix} -3 & -1 & 1 \\ -3 & -3 & 3 \\ -3 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

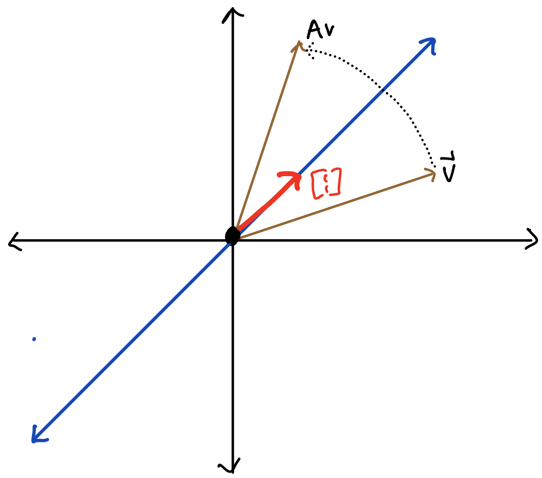
6. $\mathsf{B} = \begin{bmatrix} -4 & 9 & -3 \\ -6 & 11 & -3 \\ -12 & 18 & -4 \end{bmatrix}$ has eigenvalues $\lambda = 2, -1$. Use the information below to describe the eigenspaces.

(a) $\mathsf{B} - 2\mathbf{I}_3 = \begin{bmatrix} -6 & 9 & -3 \\ -6 & 9 & -3 \\ -12 & 18 & -6 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{3}{2} & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

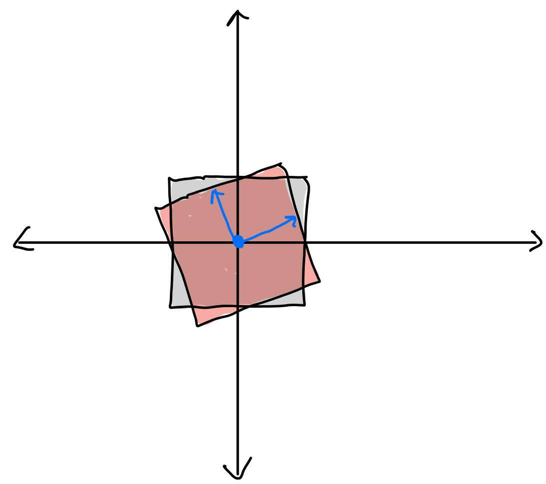
(b) $\mathsf{B} + \mathbf{I}_3 = \begin{bmatrix} -3 & 9 & -3 \\ -6 & 12 & -3 \\ -12 & 18 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$

7. Here we describe some matrix transformations geometrically. Try to find any eigenvectors. That is, find vectors that are only rescaled (and whose angles does not change). IIf you find some, what are the corresponding eigenvalues? You can do this just by thinking about the geometry (you don't need to make the matrix and solve). There might not be any!

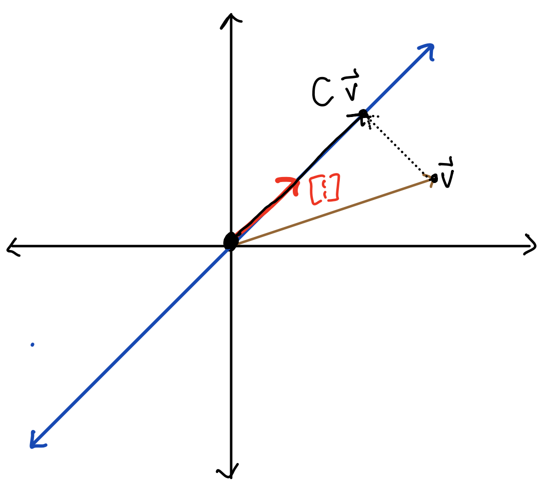
(a) A reflects over the line $y = x$.



(b) B rotates by $\theta = \pi/6$ radians counter clockwise.



(c) C projects onto the line $y = x$.



(d) $\mathsf{D} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ is the shear shown below: