Dynamical Systems: If A is an $n \times n$ matrix, and $\vec{x}_0 \in \mathbb{R}^n$ we can create a *dynamical system* of the form

$$\vec{x}_0, \quad \vec{x}_1, \quad \vec{x}_2, \quad \vec{x}_3, \quad \ldots \qquad \text{where} \quad \vec{x}_{t+1} = \mathsf{A}\vec{x}_t \quad \text{(recursive definition)}$$

Many applications are done this way, with t typically representing time: population dynamics, weather models, financial models, historical processes, pharmacology, ...

Note: we get to $\vec{x}_k$ by applying A to $\vec{x}_0$ k times. Thus: $\vec{x}_k = \mathsf{A}^k x_0$.

1. **Today's CheckPoint**. A dynamical system is defined by the recursive rule $x_{k+1} = \mathsf{A}\vec{x}_k$ with matrix A and initial value $\vec{x}_0$:

$$\mathsf{A} = \begin{bmatrix} 97/100 & 3/55 \\ -4/55 & 123/100 \end{bmatrix}, \qquad \vec{x}_0 = \begin{bmatrix} 1 \\ 15 \end{bmatrix}$$

Can generate some values

$$\underset{\vec{x}_0}{\begin{bmatrix} 1 \\ 15 \end{bmatrix}}, \quad \underset{\vec{x}_1}{\begin{bmatrix} 1.7 \\ 16.7 \end{bmatrix}}, \quad \underset{\vec{x}_2}{\begin{bmatrix} 2.41 \\ 18.55 \end{bmatrix}}, \quad \underset{\vec{x}_3}{\begin{bmatrix} 3.137 \\ 20.567 \end{bmatrix}}, \quad \underset{\vec{x}_4}{\begin{bmatrix} \quad \\ \quad \end{bmatrix}}, \quad \cdots \quad \underset{\vec{x}_k}{\begin{bmatrix} \quad \\ \quad \end{bmatrix}}, \quad \cdots$$

A "closed formula" is a formula for the kth term that can be computed using k and doesn't require us to compute all of the values up to $\vec{x}_k$ first.

The *eigenvectors* and *eigenvalues* of A are what we need:

$$\lambda_1 = 1.1 \quad \lambda_2 = 0.9$$
$$\vec{v}_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

step 1: Write $\vec{x}_0$ as a linear combination of the eigenbasis (how?):

$$\begin{bmatrix} 1 \\ 15 \end{bmatrix} = \_\_ \begin{bmatrix} 1 \\ 4 \end{bmatrix} + \_\_ \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

step 2: Apply A k times:

$$\vec{x}_k = \begin{bmatrix} a_k \\ b_k \end{bmatrix} = \mathsf{A}^k \vec{x}_0 = \mathsf{A}^k \begin{bmatrix} 1 \\ 15 \end{bmatrix} = \_\_\ \mathsf{A}^k \begin{bmatrix} 1 \\ 4 \end{bmatrix} + \_\_\ \mathsf{A}^k \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \_\_ \quad \begin{bmatrix} 1 \\ 4 \end{bmatrix} + \_\_ \quad \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

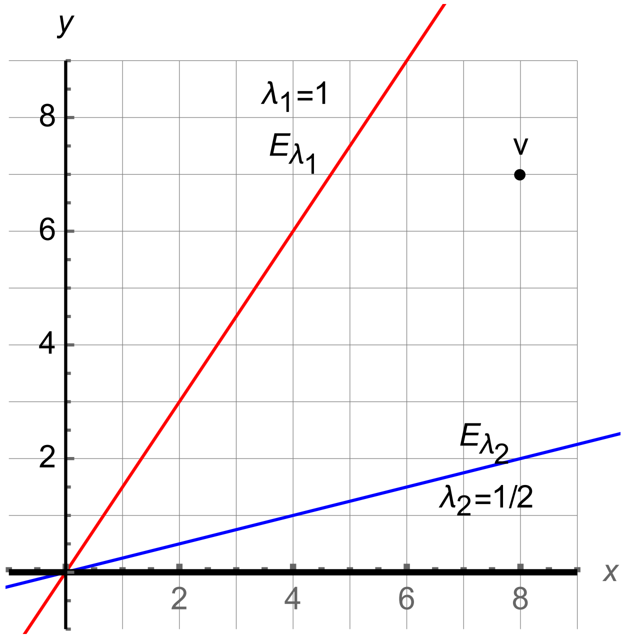
Thus,

$a_k =$

$b_k =$

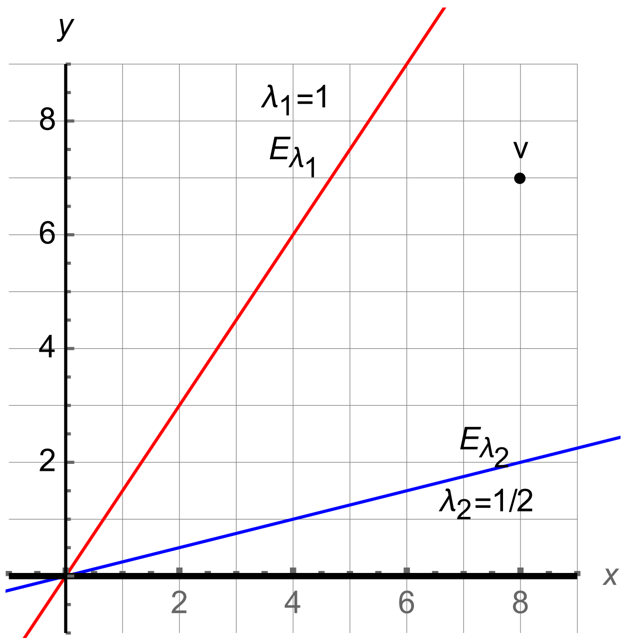
Alternatively: diagonalize: $\mathsf{A} = PDP^{-1} =$

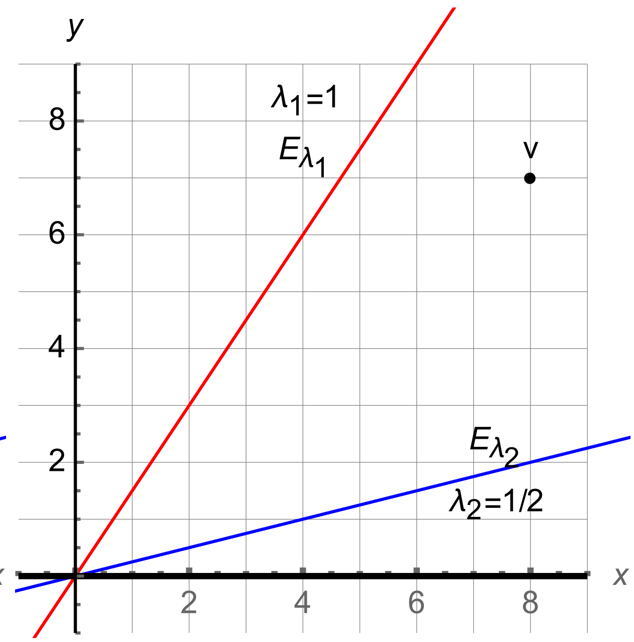
2. A is a matrix with eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 1/2$ and corresponding eigenspaces shown below. The vector $\vec{v}$ is shown on the plot at position $(8, 7)$. Using the eigeninformation, compute the position of $A\vec{v}$ and plot it on the graph. Compute it exactly if possible; if not, estimate it.



On this same problem, where will $A^2\vec{v}$ be?

3. What if λ_1 is changed to $\lambda_1 = 1.1$ or $\lambda_1 = 0.9$?





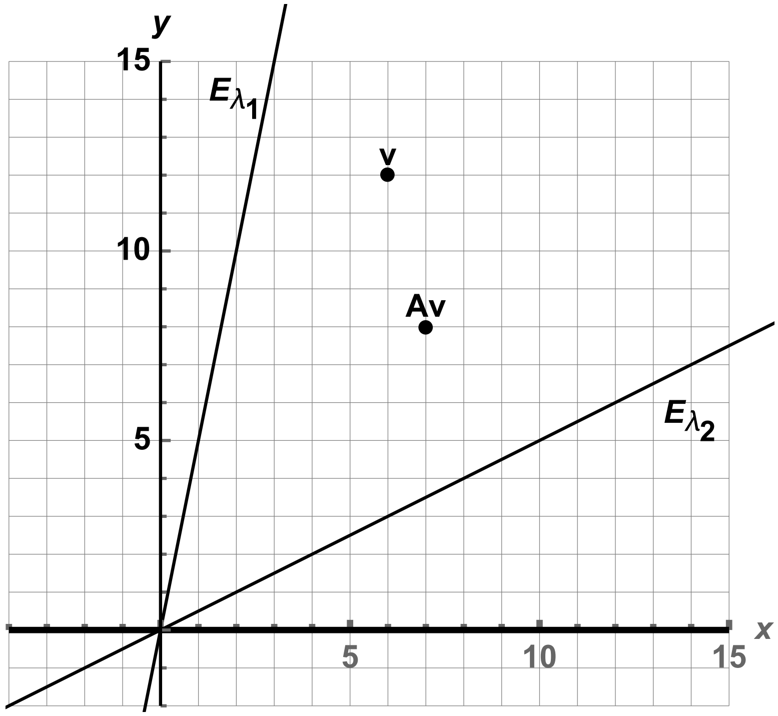
4. A is a matrix that sends $\vec{\mathsf{v}}$ to $\mathsf{A}\vec{\mathsf{v}}$ as shown in the plot below with its two eigenspaces E_{λ_1} and E_{λ_2}.

(a) Estimate, as accurately as possible from the given information, the eigenvalues $\lambda_1 =$ ____ and $\lambda_2 =$ ____.

(b) Indicate on the plot above where $\mathsf{A}^2\vec{\mathsf{v}}$ will be.

(c) What happens in the limit: $\lim_{n\to\infty} \mathsf{A}^n\vec{\mathsf{v}}$?

(d) If $\mathsf{A}^n\vec{\mathsf{v}} = \begin{bmatrix} x_n \\ y_n \end{bmatrix}$ what happens to the ratio x_n/y_n as n grows larger and larger?



5. **Dominant Eigenvectors** It is very common for an $n \times n$ matrix A to have an eigenvalue λ_1 that is bigger, in absolute value, than all of the other eigenvalues.

$$A \qquad |\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n|$$

$$x = c_1 v_1 + c_2 v_2 + c_3 v_3 + \cdots + c_n v_n$$

$$A^n x = c_1 \lambda_1^n v_1 + c_2 \lambda_2^n v_2 + c_3 \lambda_3^n v_3 + \cdots + c_n \lambda_n^n v_n$$

$$= \lambda_1^n \Big[c_1 v_1 + \underbrace{c_2 \left(\frac{\lambda_2}{\lambda_1}\right)^n v_2 + c_3 \left(\frac{\lambda_3}{\lambda_1}\right)^n v_3 + \cdots + c_n \left(\frac{\lambda_n}{\lambda_1}\right)^n v_n}_{\to 0} \Big]$$

$$\approx \lambda_1^n c_1 v_1$$

converges to the direction of the dominant eigenvector

6. Look at the Northern Spotted Owl example on R: