5.2&6 - Dynamical Systems

Section 5.2 page 278 and Section 5.6

Dynamical Systems

A n x n matrix

x_0 initial vector in $\mathbb{R}^n$

$$\boxed{x_t = A x_{t-1}}$$

$$x_0 \xrightarrow{A} x_1 \xrightarrow{A} x_2 \xrightarrow{A} x_3 \xrightarrow{A} x_4, \ \ldots$$

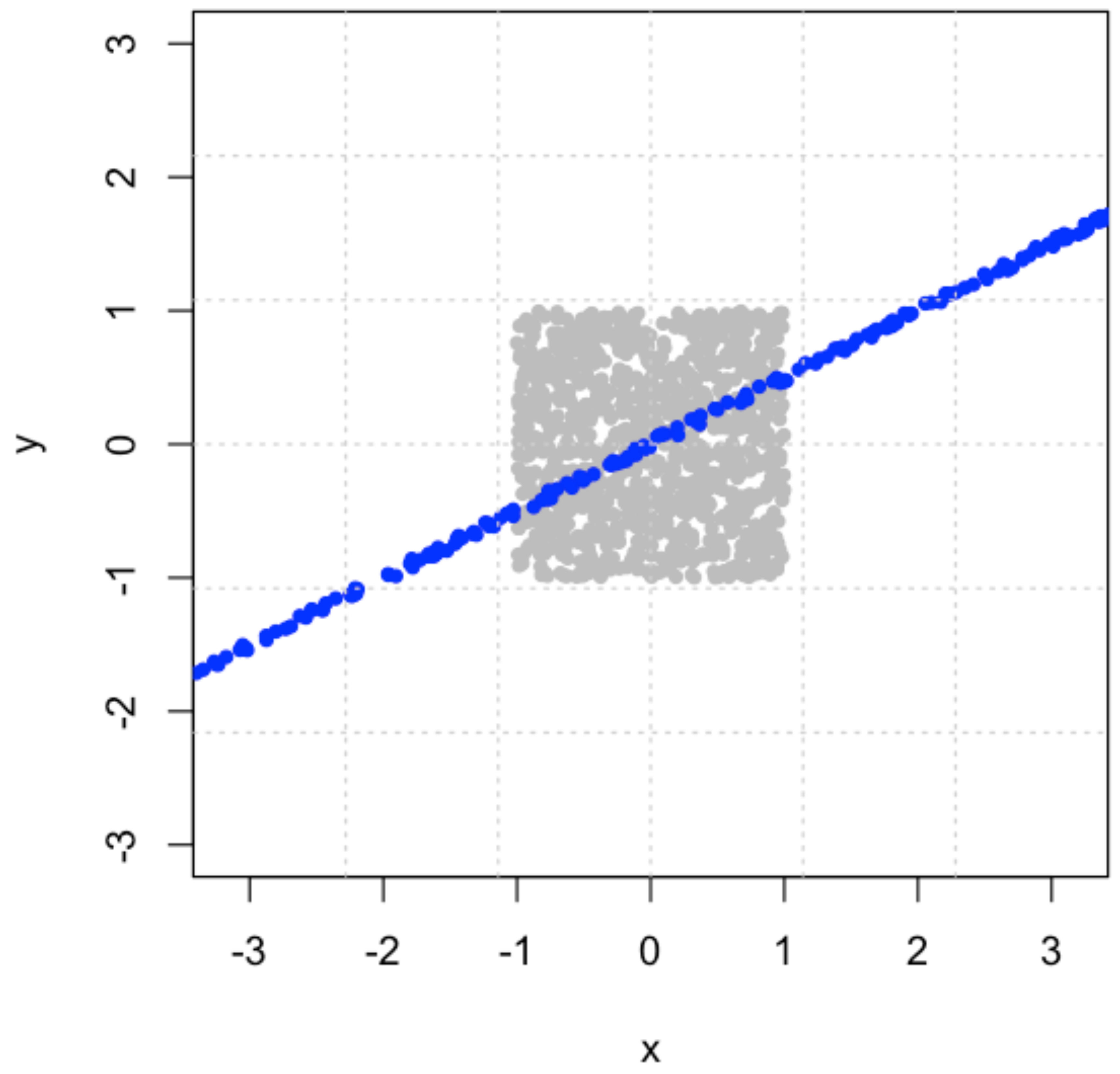
$$\boxed{x_t = A^t x_0}$$

sequence of vectors

$$x_0, \quad x_1 = A x_0, \quad x_2 = A x_1, \quad x_3 = A x_2, \ldots$$

Example

$$A = \frac{1}{30}\begin{bmatrix} 31 & 4 \\ 2 & 29 \end{bmatrix}$$



Example 1

$$A = \frac{1}{30}\begin{bmatrix} 31 & 4 \\ 2 & 29 \end{bmatrix}$$

Apply A: $A x_0 = A(-3 v_1 + v_2) = -3 A v_1 + A v_2$

$= -3(.9) v_1 + (1.1) v_2$

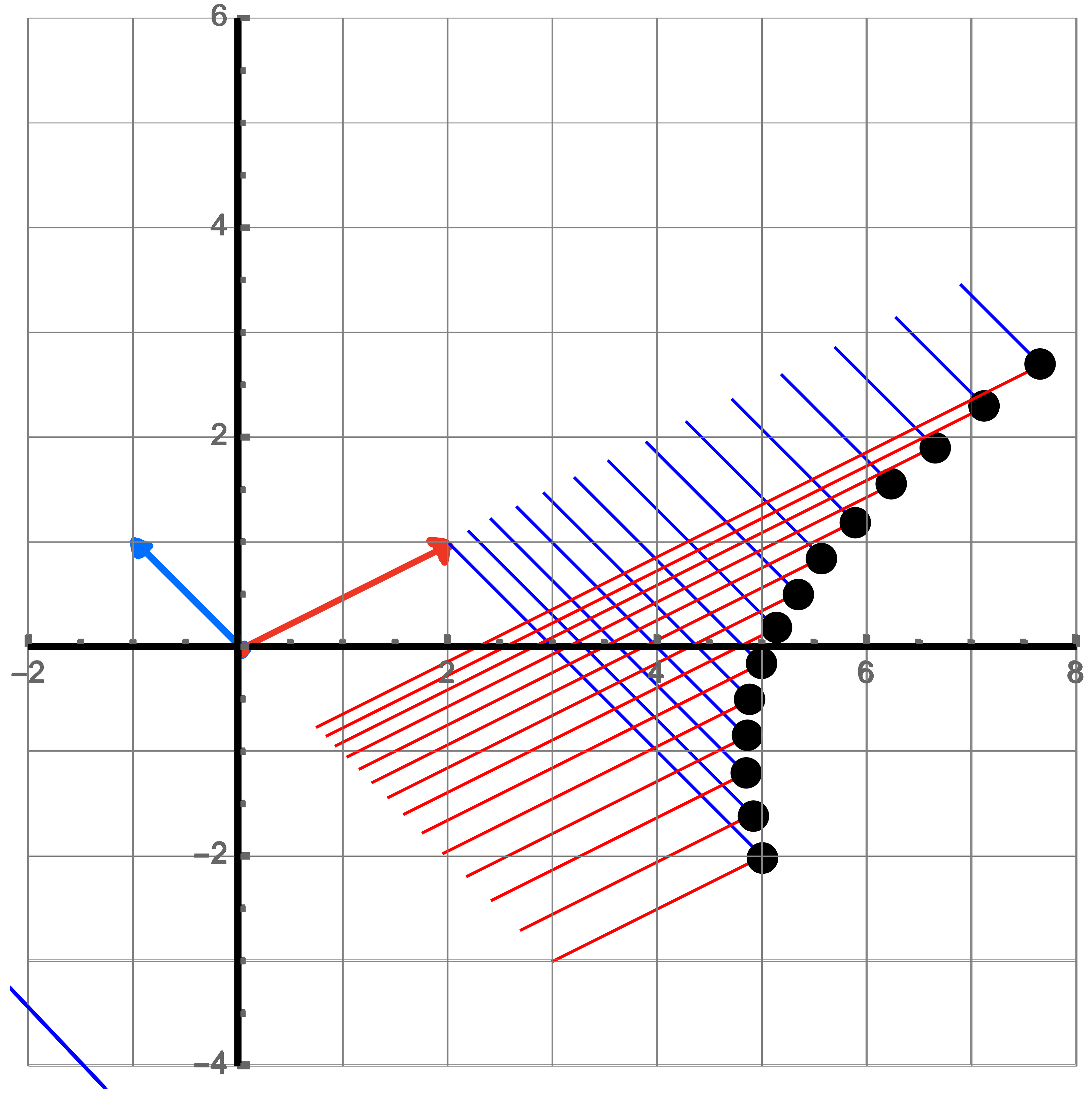
$\det(A - \lambda I) = \cdots = \lambda^2 - 2\lambda + \frac{99}{100} = (\lambda - \frac{9}{10})(\lambda - \frac{11}{10})$

$\boxed{\lambda = \frac{9}{10} = .9}$ $\boxed{\lambda = \frac{11}{10} = 1.1}$

$v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $P = \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix}$

$x_0 = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$ Express in terms of Eigenvectors

$\begin{bmatrix} 5 \\ -2 \end{bmatrix} = \underline{-3} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \underline{1} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}$



Example 1b

$$A = \frac{1}{30}\begin{bmatrix} 31 & 4 \\ 2 & 29 \end{bmatrix}$$

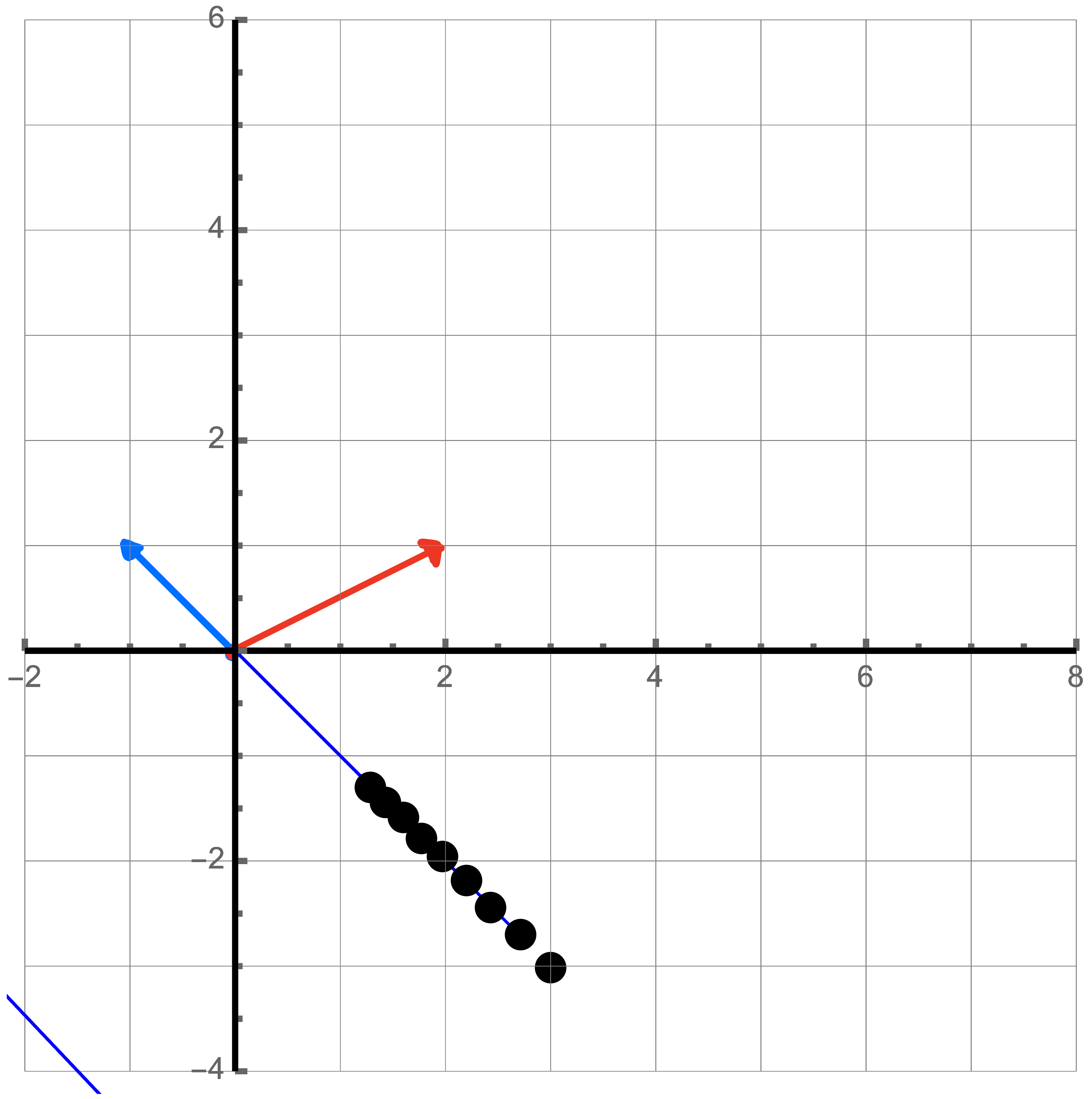
$$\det(A - \lambda I) = \cdots = \lambda^2 - 2\lambda + \frac{99}{100} = \left(\lambda - \frac{9}{10}\right)\left(\lambda - \frac{11}{10}\right)$$

$$\boxed{\lambda = \frac{9}{10} = .9} \qquad \boxed{\lambda = \frac{11}{10} = 1.1}$$

$$v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad P = \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix}$$

Express in terms of Eigenvectors

$$\begin{bmatrix} 3 \\ 3 \end{bmatrix} = -3\begin{bmatrix} -1 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$$



Example 2

$\boxed{\lambda_1 = 1.05}$ $\lambda_2 = 0.8$

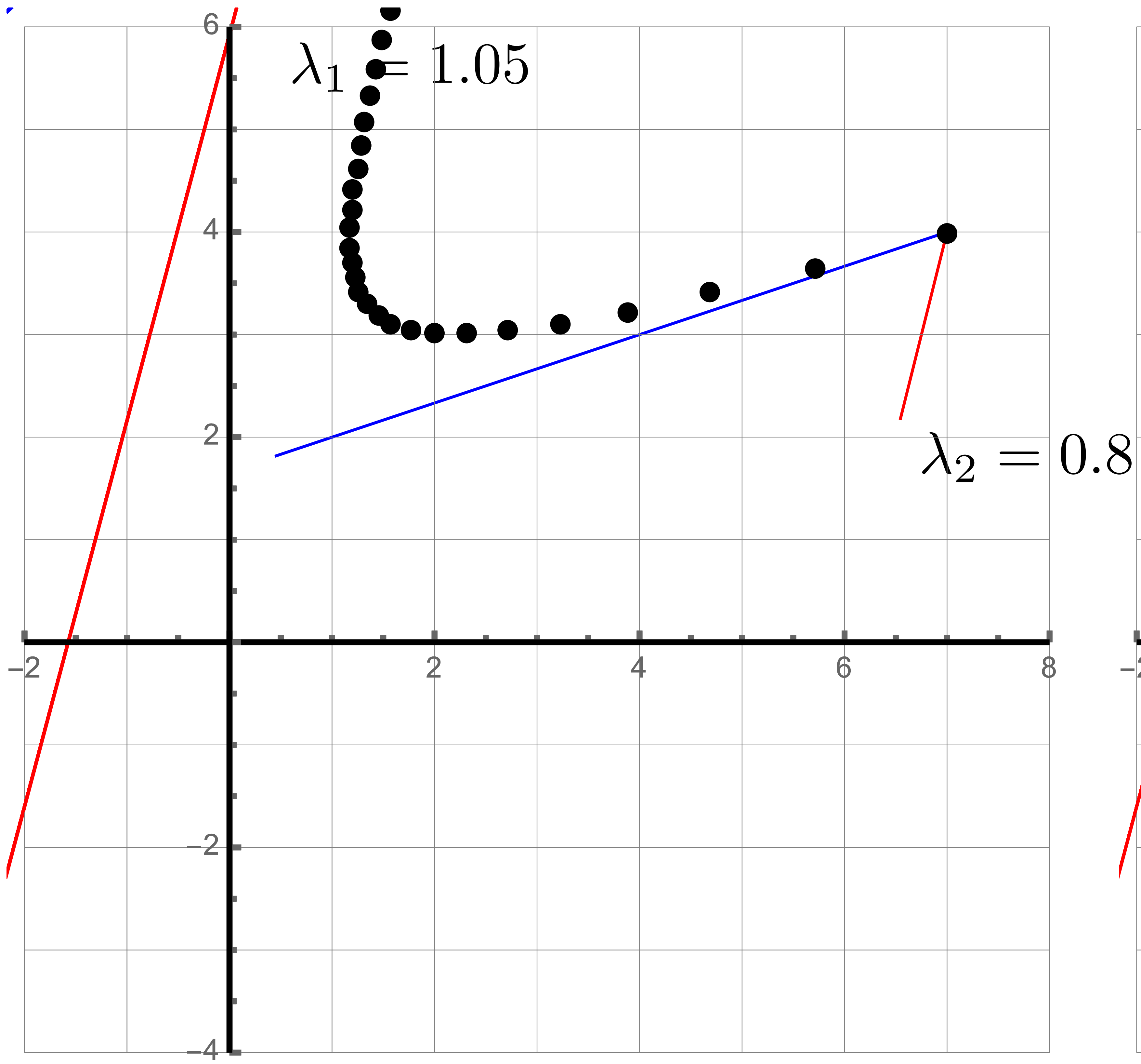
$\vec{\mathbf{v}}_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ $\vec{\mathbf{v}}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

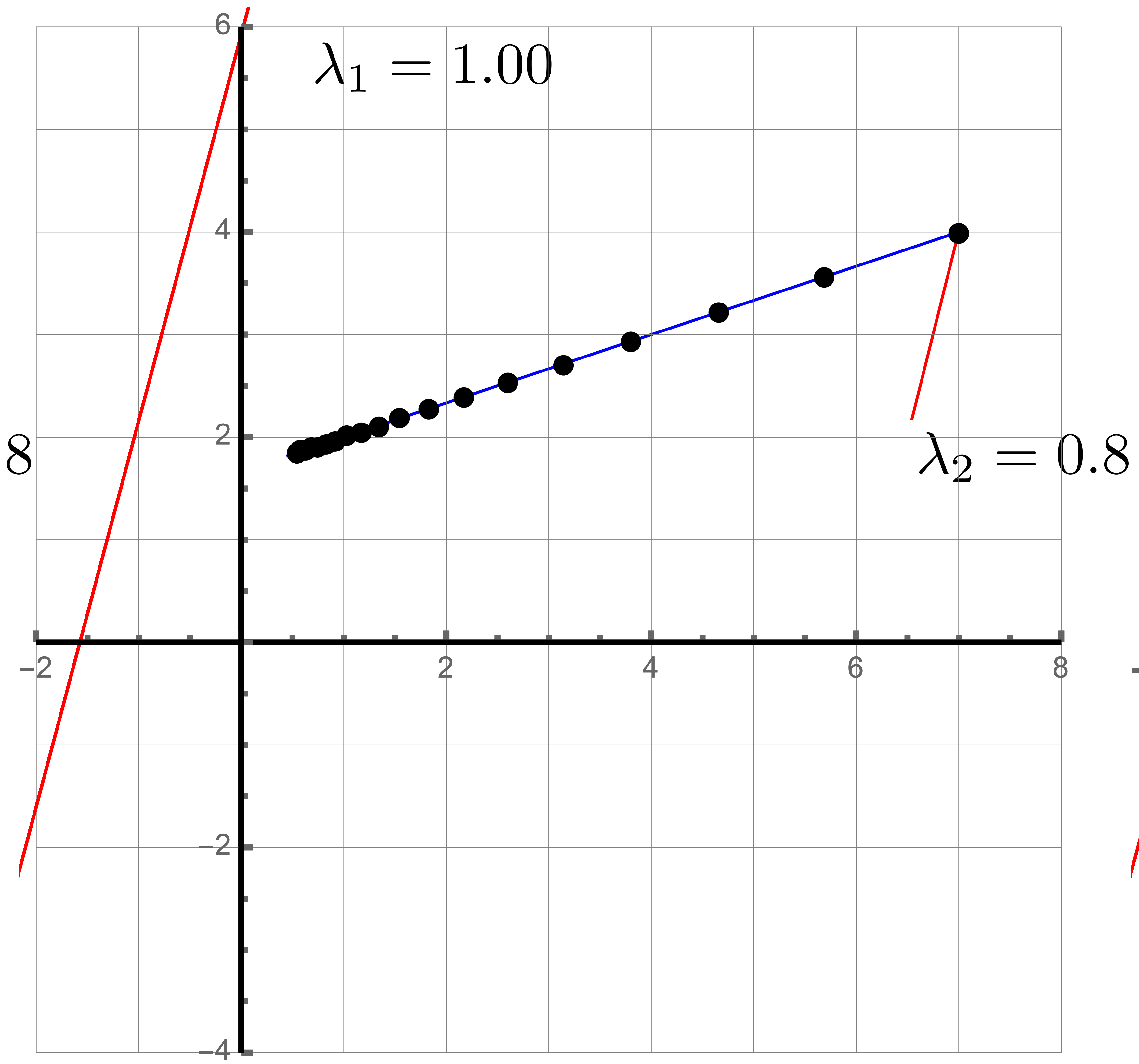
$\boxed{\lambda_1 = 1.00}$ $\lambda_2 = 0.8$

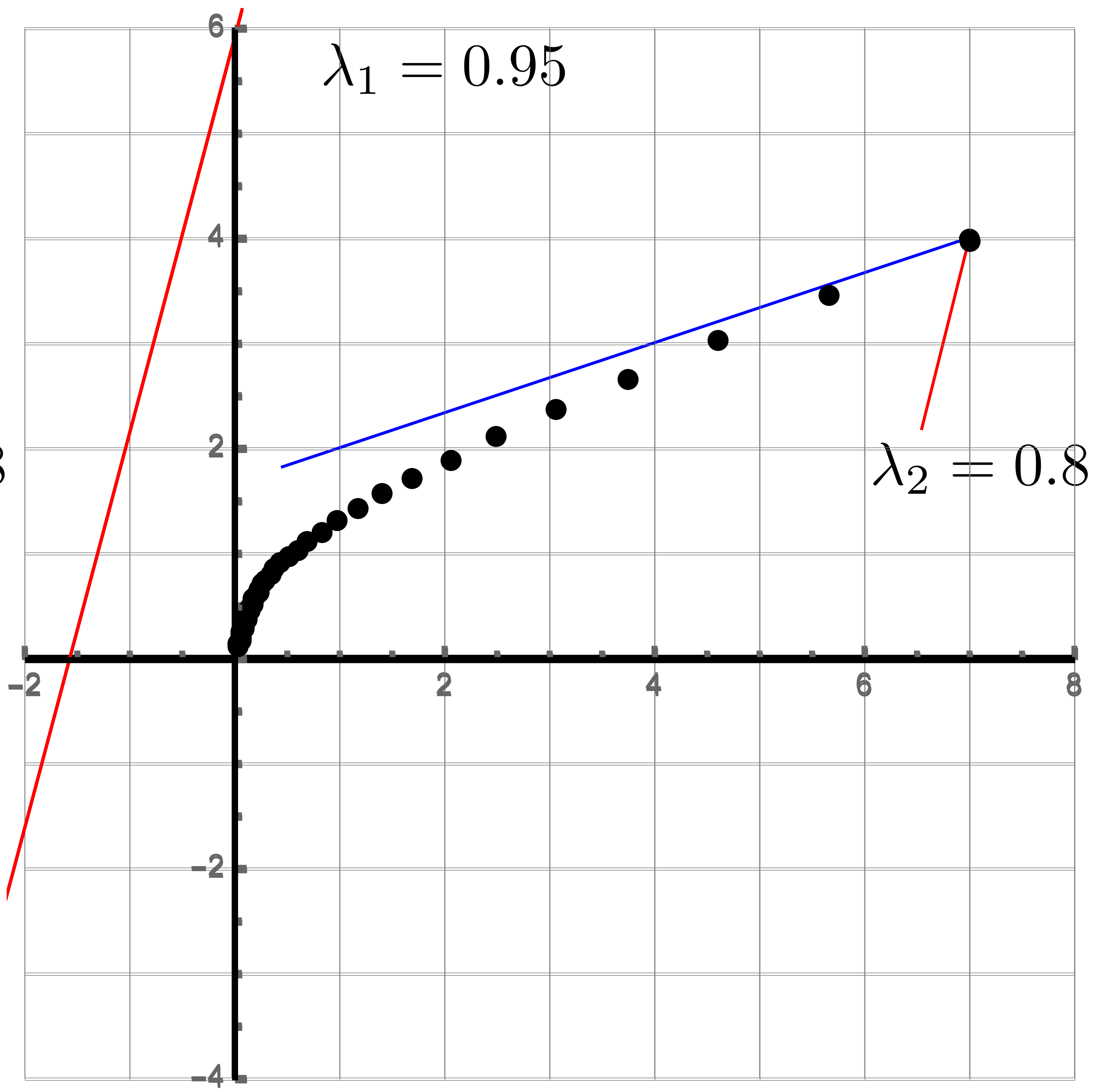
$\vec{\mathbf{v}}_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ $\vec{\mathbf{v}}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$\boxed{\lambda_1 = 0.95}$ $\lambda_2 = 0.8$

$\vec{\mathbf{v}}_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ $\vec{\mathbf{v}}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$







Dominant Eigenvectors

A $\quad |\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n|$

$$x = c_1 v_1 + c_2 v_2 + c_3 v_3 + \cdots + c_n v_n$$

$$A^n x = c_1 \lambda_1^n v_1 + c_2 \lambda_2^n v_2 + c_3 \lambda_3^n v_3 + \cdots + c_n \lambda_n^n v_n$$

$$= \lambda_1^n \Big[c_1 v_1 + \underbrace{c_2 \left(\frac{\lambda_2}{\lambda_1}\right)^n v_2 + c_3 \left(\frac{\lambda_3}{\lambda_1}\right)^n v_3 + \cdots + c_n \left(\frac{\lambda_n}{\lambda_1}\right)^n v_n}_{\to 0} \Big]$$

$$\approx \lambda_1^n c_1 v_1$$

converges to the direction of the dominant eigenvector

Example 2

$\boxed{\lambda_1 = 1.05}$ $\lambda_2 = 0.8$

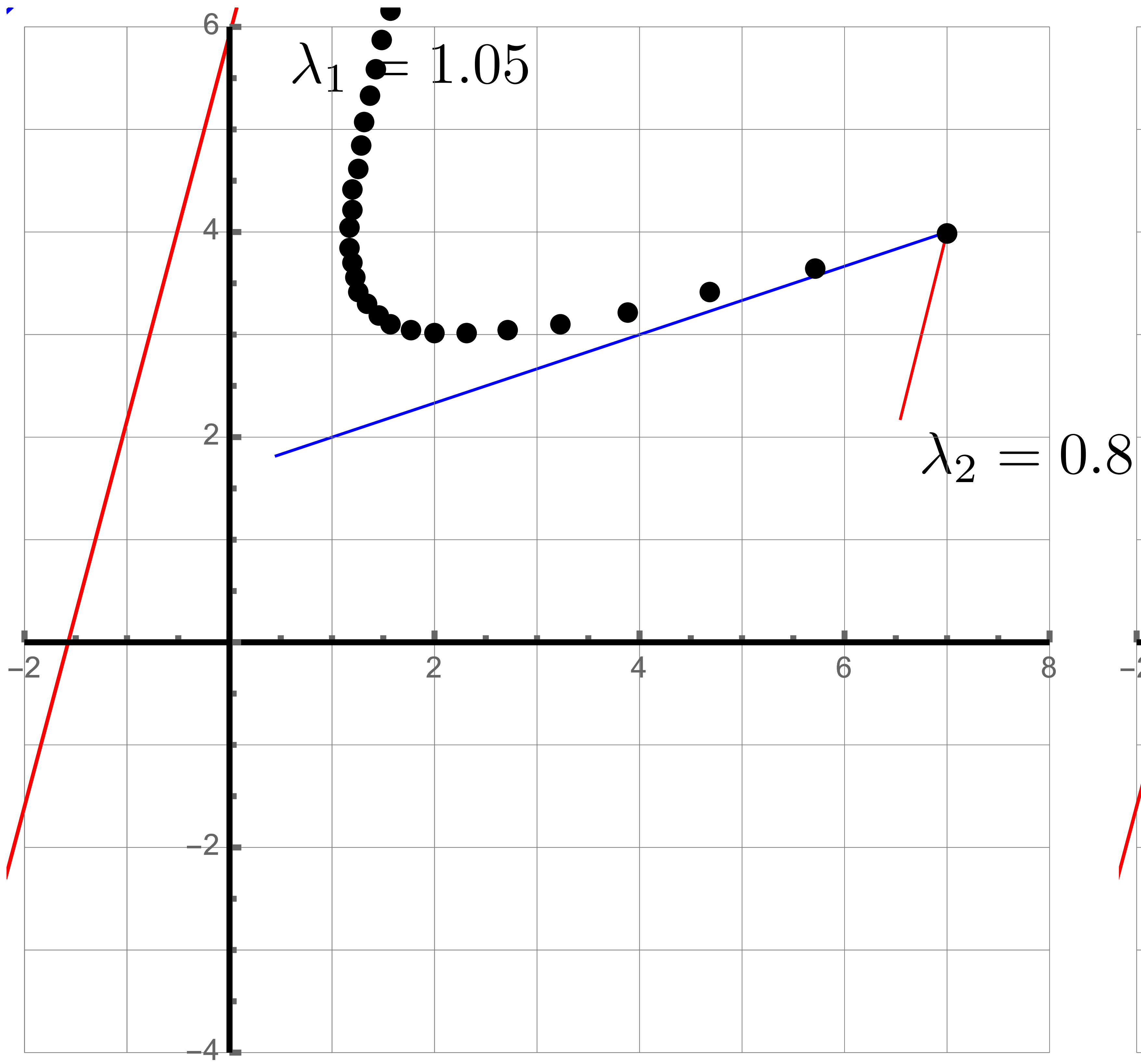
$\vec{\mathbf{v}}_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ $\vec{\mathbf{v}}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

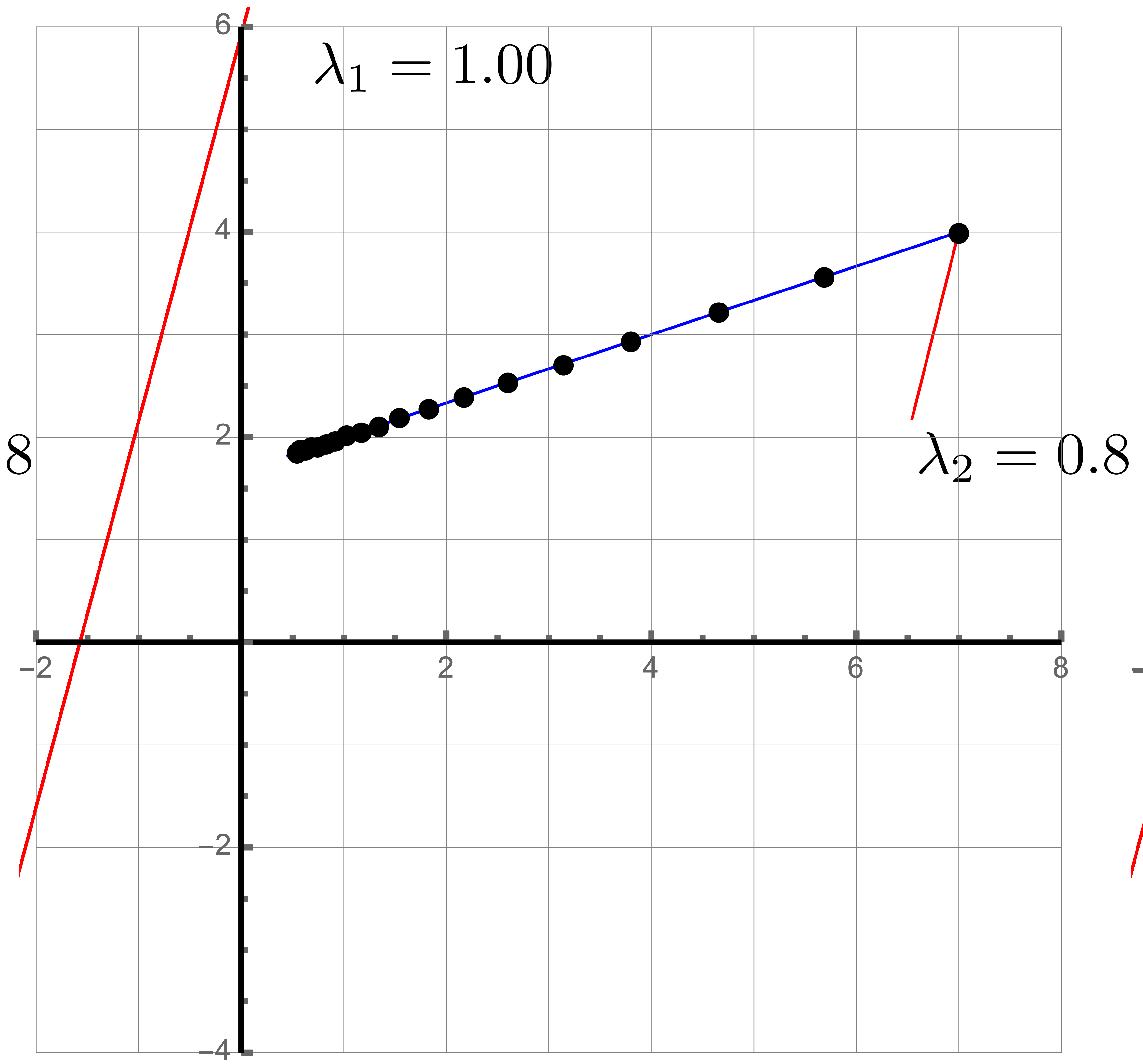
$\boxed{\lambda_1 = 1.00}$ $\lambda_2 = 0.8$

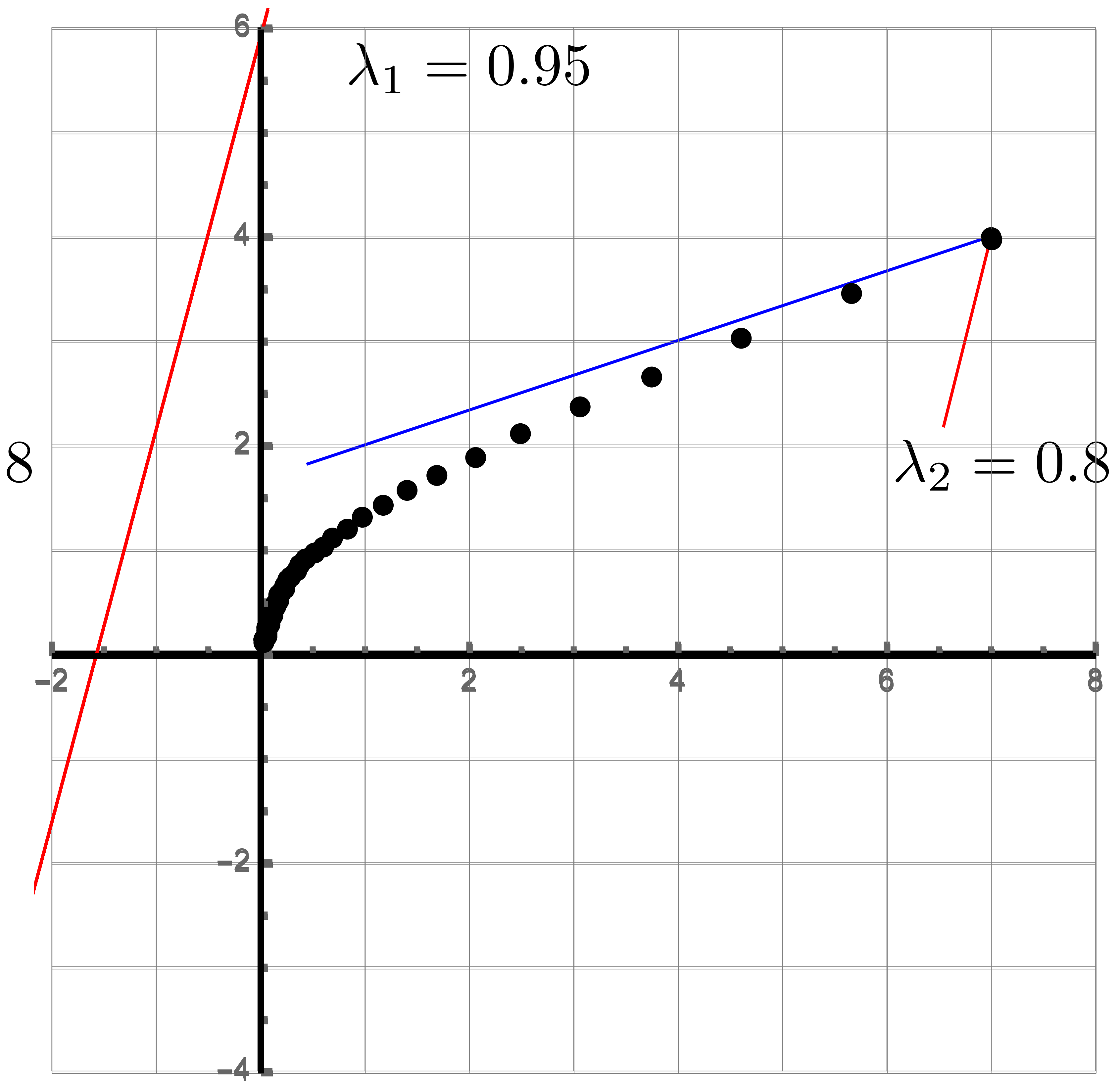
$\vec{\mathbf{v}}_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ $\vec{\mathbf{v}}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$\boxed{\lambda_1 = 0.95}$ $\lambda_2 = 0.8$

$\vec{\mathbf{v}}_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ $\vec{\mathbf{v}}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$







Rental Car Problem

```
P = cbind(c(.97,.00,.03),c(.05,.90,.05),c(.10,.05,.85))
P
```

```
##      [,1] [,2] [,3]
## [1,] 0.97 0.05 0.10
## [2,] 0.00 0.90 0.05
## [3,] 0.03 0.05 0.85
```

```
eigen(A)
```

```
## eigen() decomposition
## $values
## [1] 1.0000000 0.9109902 0.8090098
##
## $vectors
##           [,1]       [,2]       [,3]
## [1,] 0.9658343  0.7659409 -0.3672241
## [2,] 0.1159001 -0.6279213 -0.4479414
## [3,] 0.2318002 -0.1380196  0.8151656
```

```
v = c(20,50,30)
for (i in 1:100){
  v = P %*% v
}
v
```

```
##           [,1]
## [1,] 73.524745
## [2,]  8.827356
## [3,] 17.647900
```

```
vecs = eigen(A)$vectors
v = vecs[,1]
v /sum(v)
```

```
## [1] 0.73529412 0.08823529 0.17647059
```

Predator-Prey Example

Deep in the redwood forests of California, dusky-footed wood rats provide up to 80% of the diet for the spotted owl, the main predator of the wood rat. Example 1 uses a linear dynamical system to model the physical system of the owls and the rats. (Admittedly, the model is unrealistic in several respects, but it can provide a starting point for the study of more complicated nonlinear models used by environmental scientists.)

Dynamical System:

$$\begin{bmatrix} \mathsf{O}_{k+1} \\ \mathsf{R}_{k+1} \end{bmatrix} = \begin{bmatrix} 1/2 & 2/5 \\ -1/5 & 7/5 \end{bmatrix} \begin{bmatrix} \mathsf{O}_k \\ \mathsf{R}_k \end{bmatrix} = \begin{bmatrix} 1/2\,\mathsf{O}_k + 2/5\,\mathsf{R}_k \\ -1/5\,\mathsf{O}_k + 7/5\,R_k \end{bmatrix}$$

With initial value:

$$\begin{bmatrix} \mathsf{O}_0 \\ \mathsf{R}_0 \end{bmatrix} = \begin{bmatrix} 11 \\ 8 \end{bmatrix}$$

Predator-Prey Example

$P = \begin{bmatrix} 1 & 4 \\ 2 & 1 \end{bmatrix}$

Recursive Formula

$$\begin{bmatrix} O_{k+1} \\ R_{k+1} \end{bmatrix} = \begin{bmatrix} 1/2 & 2/5 \\ -1/5 & 7/5 \end{bmatrix} \begin{bmatrix} O_k \\ R_k \end{bmatrix}$$

Initial Value

$$\begin{bmatrix} O_0 \\ R_0 \end{bmatrix} = \begin{bmatrix} 11 \\ 8 \end{bmatrix}$$

Eigensystem

$$\boxed{\lambda_1 = \frac{13}{10} \quad \lambda_2 = \frac{3}{5} \quad v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad v_2 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}}$$

① Express x_0 in the eigenbasis

$$\begin{bmatrix} 11 \\ 8 \end{bmatrix} = \underline{3} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \underline{2} \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

② Apply A to the eigendecomposition

$$\begin{bmatrix} O_k \\ R_k \end{bmatrix} = A^k \begin{bmatrix} 11 \\ 8 \end{bmatrix} = 3 A^k \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2 A^k \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} O_k \\ R_k \end{bmatrix} = A^k \begin{bmatrix} 11 \\ 8 \end{bmatrix} = 3 (1.3)^k \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2 (.6)^k \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

"Closed" Formula

$$\boxed{\begin{aligned} O_k &= 3 (1.3)^k + 8 (.6)^k \\ R_k &= 6 (1.3)^k + 2 (.6)^k \end{aligned}}$$

You Try!

The matrix $\mathbf{A} = \left[\begin{array}{cc} 97/110 & 3/55 \\ -4/55 & 123/110 \end{array} \right]$ has eigenvalues: $\lambda_1 = 1.1$ $\lambda_2 = 0.9$

and eigenvectors: $\vec{\mathbf{v}}_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ $\vec{\mathbf{v}}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

If we start with $\vec{\mathbf{x}}_0 = \begin{bmatrix} 1 \\ 15 \end{bmatrix}$ and generate the sequence $\vec{\mathbf{x}}_0, \vec{\mathbf{x}}_1, \vec{\mathbf{x}}_2, \vec{\mathbf{x}}_3, \ldots$

by the recursive rule: $\vec{\mathbf{x}}_{k+1} = \mathbf{A}\vec{\mathbf{x}}_k$

Then use the methods of this video to give a “closed formula” for $\vec{\mathbf{x}}_k$