6.1. Orthogonality

The Inner Product (aka, Dot Product)

Def. The ***inner product*** (or **dot product**) of two vectors is given by:

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n, \quad \Rightarrow \quad \mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v} = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$= \boxed{u_1 v_1 + u_2 v_2 + \cdots + u_n v_n} \in \mathbb{R}$$

It is a ***scalar product***: turns two vectors **u**, **v** into a scalar **u • v**

The Inner Product (aka, Dot Product)

Has nice algebraic properties that are easy to verify from the definition

THEOREM 1

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in $\mathbb{R}^n$, and let c be a scalar. Then

a. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

b. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$

c. $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v})$

d. $\mathbf{u} \cdot \mathbf{u} \geq 0$, and $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$

Properties (b) and (c) can be combined several times to produce the following useful rule:

$$(c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p) \cdot \mathbf{w} = c_1(\mathbf{u}_1 \cdot \mathbf{w}) + \cdots + c_p(\mathbf{u}_p \cdot \mathbf{w})$$

Geometry: Length and Distance

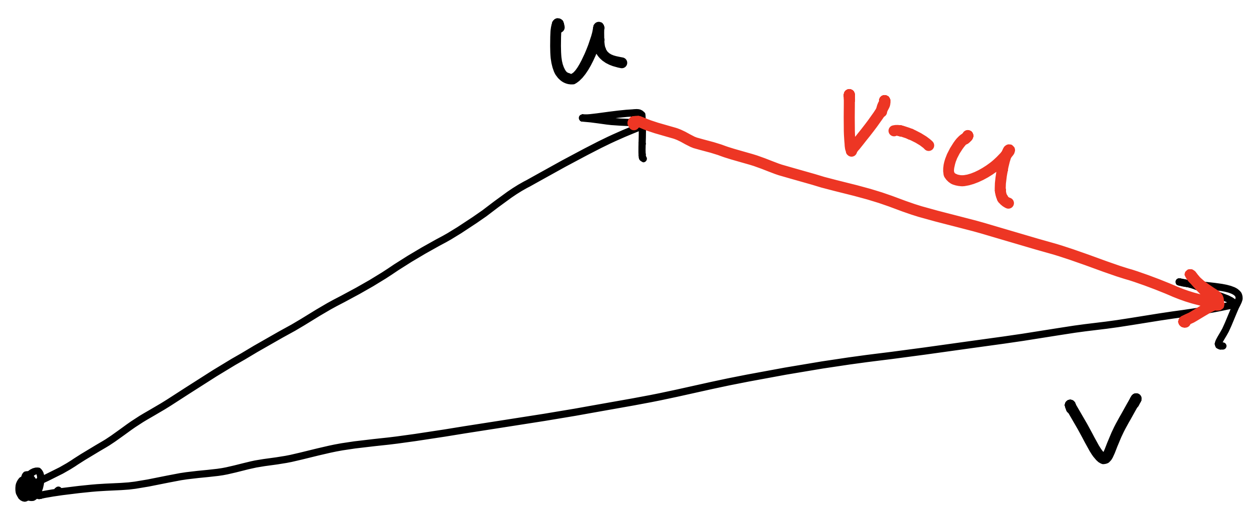
Def. The ***length*** of a vector can be computed using a dot product:

$$\|\vec{u}\| = \sqrt{\vec{u} \bullet \vec{u}} = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}$$

"Pythagorean distance"

in other words: $\boxed{\vec{u} \bullet \vec{u} = \|\vec{u}\|^2}$

Def. The ***distance*** between two vectors is the length of the vector between them



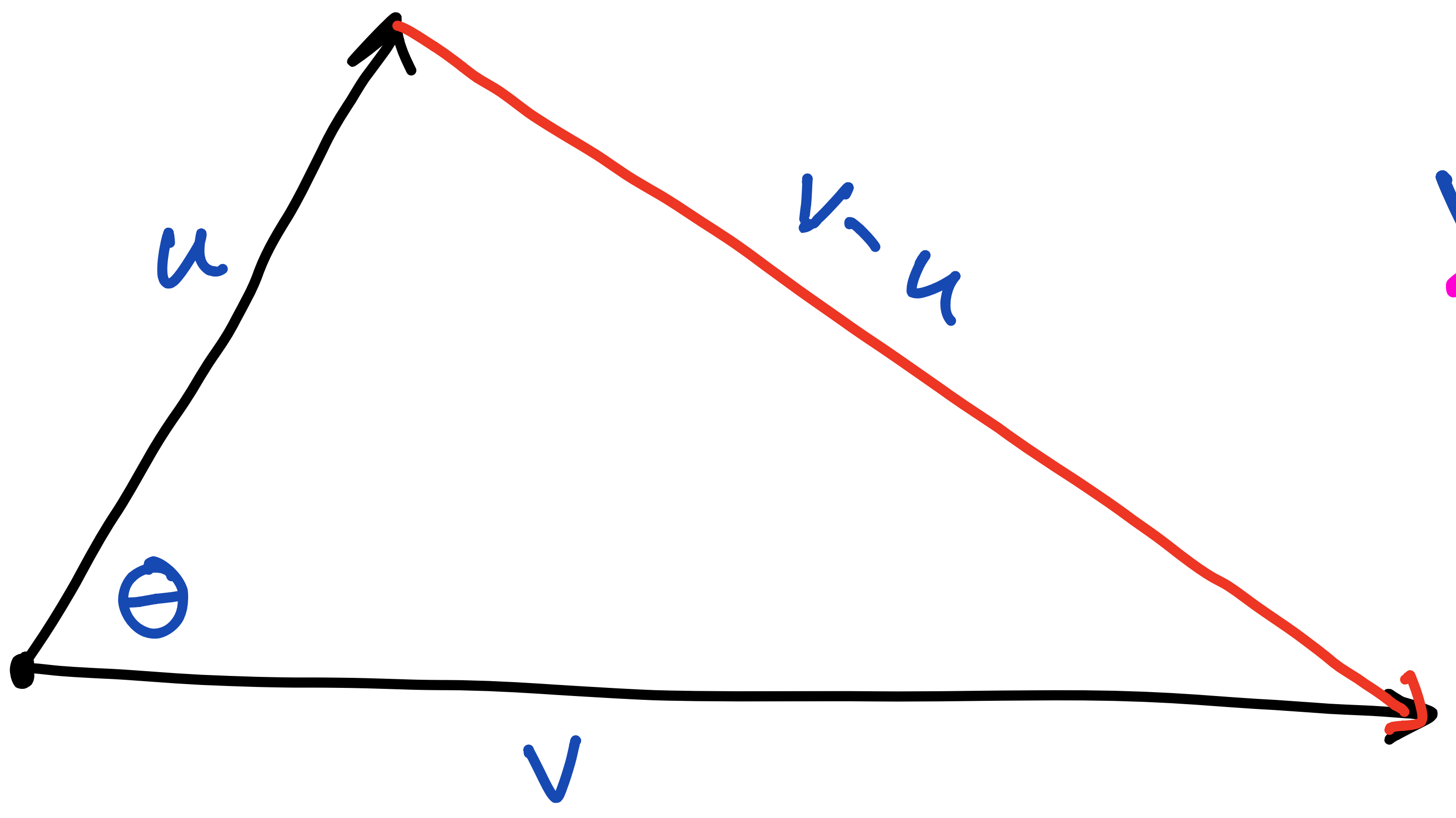
$$\vec{u} + (\vec{v} - \vec{u}) = \vec{v}$$

$$\begin{aligned} \operatorname{dist}(\vec{u}, \vec{v}) &= \|\vec{v} - \vec{u}\| \\ &= \sqrt{(v_1 - u_1)^2 + (v_2 - u_2)^2 + \cdots (v_n - u_n)^2} \end{aligned}$$

The Angle Between 2 Vectors

The ***angle*** between two vectors can be computed using a dot product:

Law of Cosines



$$(v-u)\cdot(v-u) = u\cdot u + v\cdot v - 2\|u\|\|v\|\cos(\theta)$$

$$\cancel{v\cdot v} + \cancel{u\cdot u} - \cancel{2}\,u\cdot v = \cancel{u\cdot u} + \cancel{v\cdot v} - \cancel{2}\,\|u\|\|v\|\cos(\theta)$$

$$\boxed{u\cdot v = \|u\|\|v\|\cos(\theta)}$$

$$\|v-u\|^2 = \|u\|^2 + \|v\|^2 - 2\|u\|\|v\|\cos(\theta)$$

$$\boxed{\cos(\theta) = \frac{u\cdot v}{\|u\|\|v\|}} \qquad \boxed{\theta = \cos^{-1}\left(\frac{u\cdot v}{\|u\|\|v\|}\right)}$$

Examples

1. Find the length of each of the following vectors

   (a) $\vec{v} = \begin{bmatrix} -5 \\ 12 \end{bmatrix}$
   $$\|\vec{v}\| = \sqrt{\vec{v}\cdot\vec{v}} = \sqrt{25+144} = \sqrt{169} = 13.$$

   (b) $\vec{w} = \begin{bmatrix} 4 \\ 0 \\ -4 \\ 2 \end{bmatrix}$
   $$\|\vec{w}\| = \sqrt{\vec{w}\cdot\vec{w}} = \sqrt{16+0+16+4} = \sqrt{36} = 6.$$

2. Find the angle between the vectors $\vec{v}$ and $\vec{w}$.

   (a) $\vec{v} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \vec{w} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$
   $$\cos\theta = \frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\,\|\vec{w}\|} = \frac{6-6}{\|\vec{v}\|\,\|\vec{w}\|} = 0 \qquad \text{so} \quad \theta = \pi/2$$

   (b) $\vec{v} = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \vec{w} = \begin{bmatrix} 3 \\ 0 \\ 1 \\ -2 \end{bmatrix}$
   $$\cos\theta = \frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\,\|\vec{w}\|} = \frac{6+0+1+0}{\sqrt{4+9+1}\sqrt{9+1+4}} = \frac{7}{14} = \frac{1}{2} \qquad \text{so} \quad \theta = \pi/3$$

Orthogonality

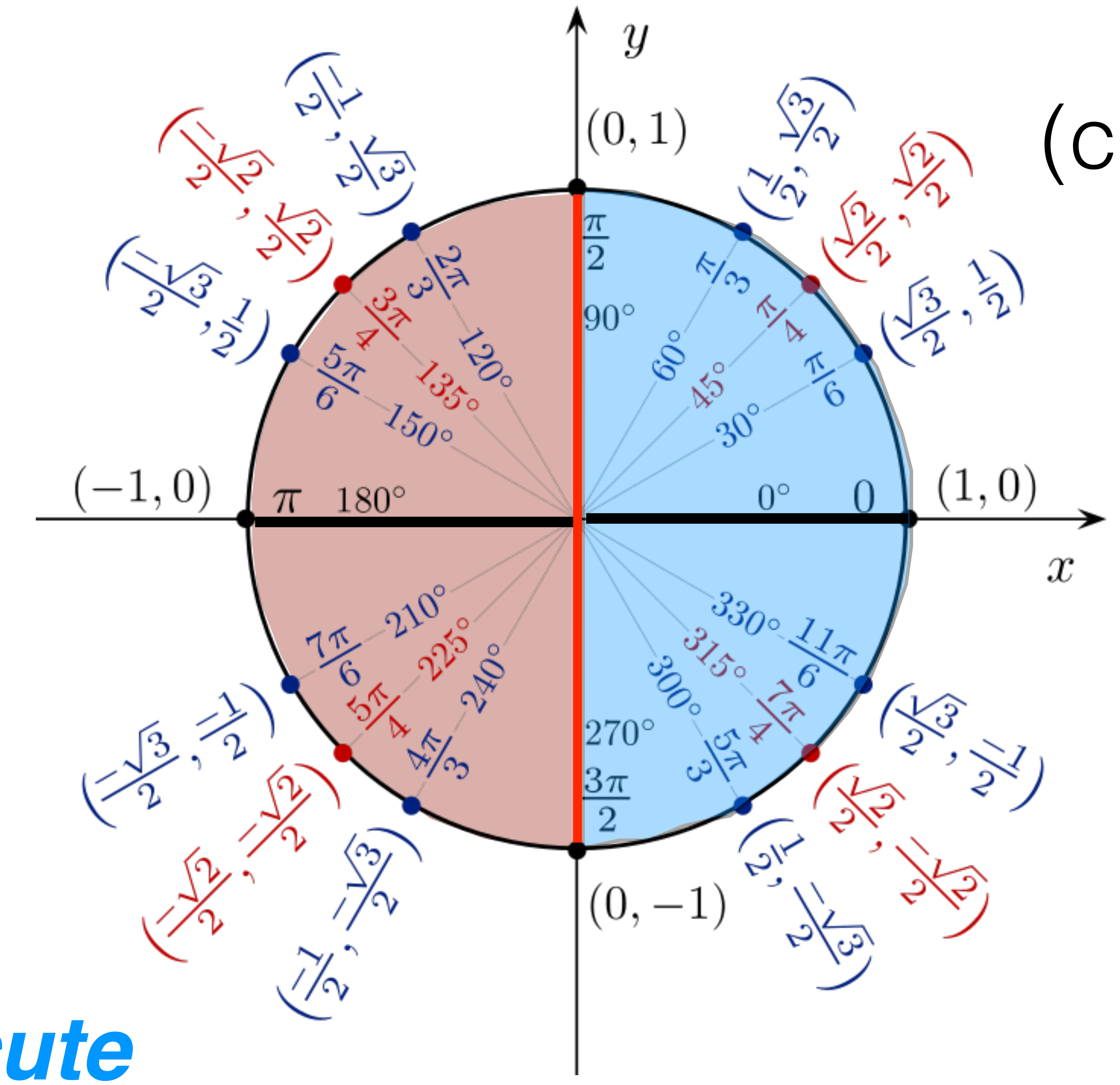
$$\boxed{u \cdot v = \|u\| \|v\| \cos(\theta)}$$

Def. The vectors are ***orthogonal*** when **u·v** = 0.

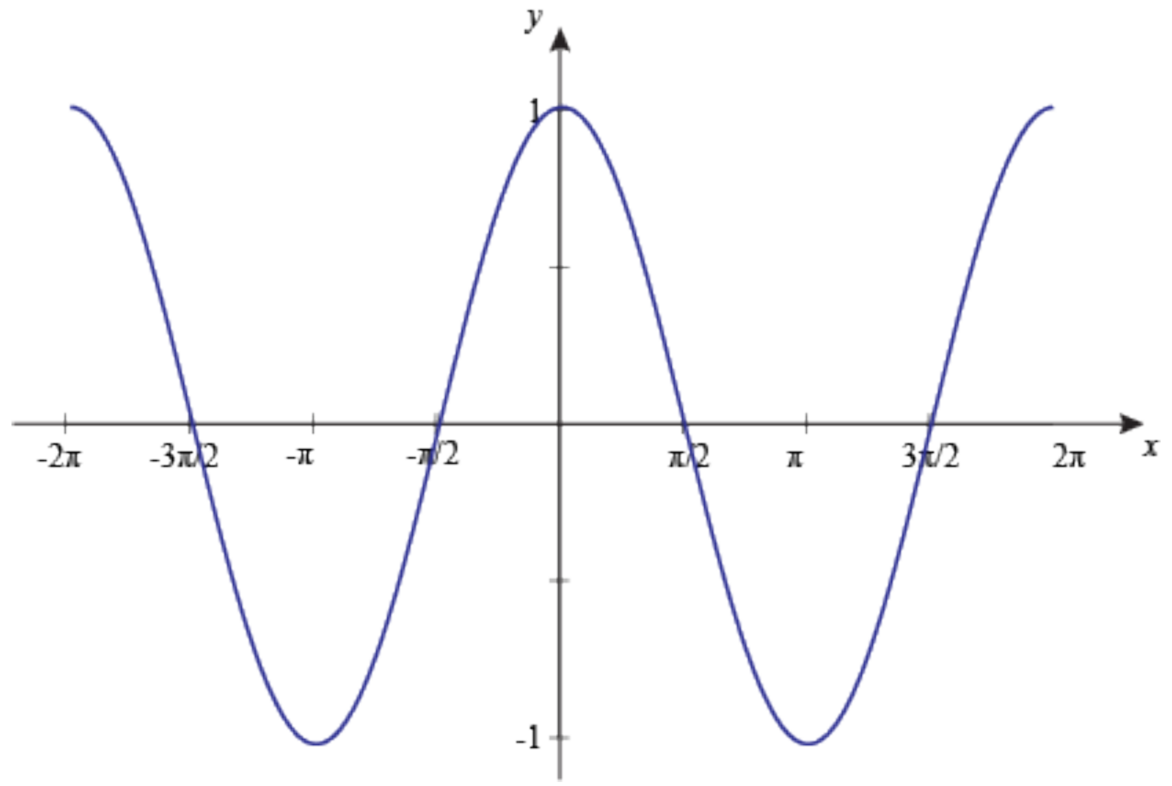
$$\boxed{u \cdot v = 0} \iff \boxed{\theta = \frac{\pi}{2} \ (90^\circ)}$$

$$\boxed{u \cdot v > 0} \iff \boxed{0 \leq \theta < \frac{\pi}{2}}$$ ***Acute***

$$\boxed{u \cdot v < 0} \iff \boxed{\frac{\pi}{2} < \theta \leq \pi}$$ ***Obtuse***



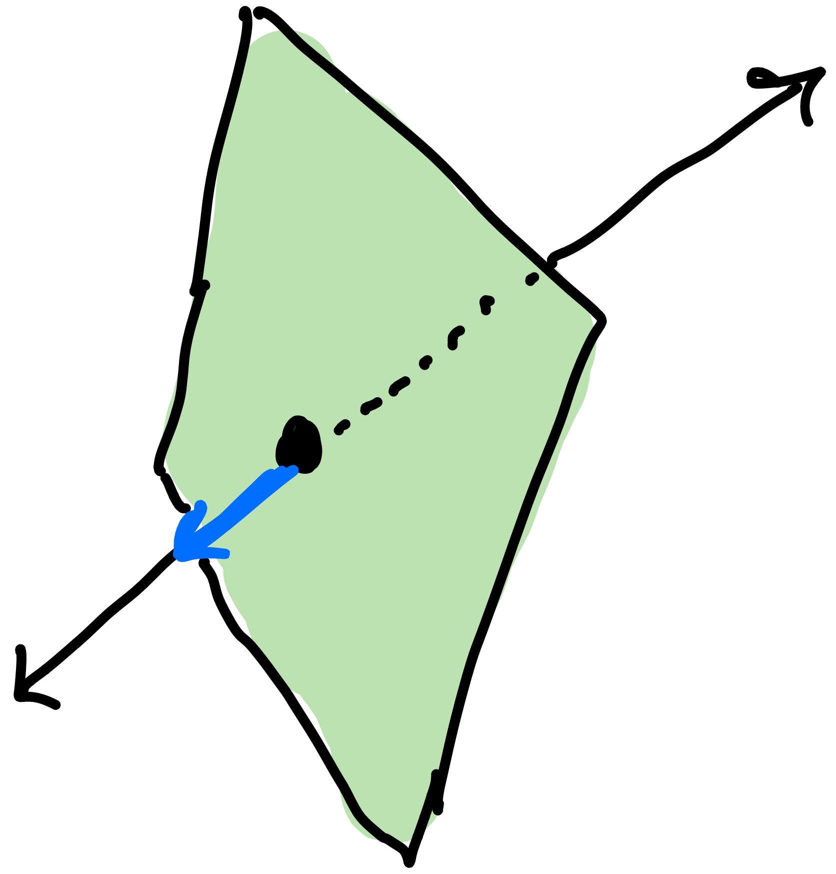
(cos(θ), sin(θ))



Example

Pick any vector **w** in $\mathbf{R}^n$. Let V be the set of all vectors that are orthogonal to **w**.

That is, $V = \{ v \in \mathbb{R}^n \mid w \cdot v = 0 \}$



Prove that V is a subspace.

- $w \cdot \vec{0} = \vec{0}$, so $\vec{0} \in V$
- if $v_1, v_2 \in V$ then $w \cdot v_1 = 0$ and $w \cdot v_2 = 0$ so

$$w \cdot (v_1 + v_2) = w \cdot v_1 + w \cdot v_2 = 0 + 0 = 0$$

- If $v \in V$, then $w \cdot v = 0$ and

$$w \cdot (cv) = c(w \cdot v) = c\,0 = 0$$

Orthogonal Complement

Definition

Let W be a subspace of $\mathbb{R}^n$. The **orthogonal complement** of W is the set

$$W^\perp = \{\vec{v} \in \mathbb{R}^n \mid \vec{v} \bullet \vec{w} = 0 \quad \text{for all } \vec{w} \in W\}.$$

$\mathbf{W}^\perp$ is pronounced: "W perp"

Show that $W^\perp$ is a subspace of $\mathbb{R}^n$

For any $\vec{w} \in W$:

- If $\vec{u}, \vec{v} \in W^\perp$ then $\vec{u} + \vec{v} \in W^\perp$ $\longrightarrow$ $(\vec{u} + \vec{v}) \bullet \vec{w} = \vec{u} \bullet \vec{w} + \vec{v} \bullet \vec{w} = 0 + 0 = 0$
- If $\vec{v} \in W^\perp$ and $c \in \mathbb{R}$ then $c\vec{v} \in W^\perp$ $\longrightarrow$ $c\vec{u} \bullet \vec{w} = c(\vec{u} \bullet \vec{w}) = c \cdot 0 = 0$

Orthogonal Complement

Let $\vec{w}_1 = \begin{bmatrix} 1 \\ 1 \\ -2 \\ -3 \\ 2 \end{bmatrix}, \quad \vec{w}_2 = \begin{bmatrix} 5 \\ 3 \\ 2 \\ 4 \\ 1 \end{bmatrix}, \quad \vec{w}_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

and let $W = \operatorname{span}(\vec{w}_1, \vec{w}_2, \vec{w}_3)$

Then $\vec{v} \in W^\perp \iff \vec{w}_1 \cdot \vec{v} = \vec{w}_2 \cdot \vec{v} = \vec{w}_3 \cdot \vec{v} = 0$

$$\iff \underbrace{\begin{bmatrix} 1 & 1 & -2 & -3 & 2 \\ 5 & 3 & 2 & 4 & 1 \\ 2 & 1 & 1 & 1 & 1 \end{bmatrix}}_{A} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\iff \vec{v} \in \operatorname{Nul}(A)$$

Nul(A) is orthogonal to Row(A)!

where Row(**A**) is the span of the rows of **A**

$$\begin{bmatrix} 1 & 1 & -2 & -3 & 2 \\ 5 & 3 & 2 & 4 & 1 \\ 2 & 1 & 1 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & -7/2 & 7/2 \\ 0 & 1 & 0 & 11/2 & -9/2 \\ 0 & 0 & 1 & 5/2 & -3/2 \end{bmatrix}$$

$$W^\perp = \operatorname{Nul}(A) = \operatorname{span}\left(\begin{bmatrix} 7/2 \\ -11/2 \\ -5/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -7/2 \\ 9/2 \\ 3/2 \\ 0 \\ 1 \end{bmatrix} \right)$$

Check that these basis vectors are orthogonal to **w**$_1$, **w**$_2$, **w**$_3$

You Try!

Give a basis for the orthogonal complement of the space W

$$W = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ -3 \\ 1 \\ 1 \end{bmatrix} \right\}$$