4.3. Basis of a Subspace

Basis

Definition

A subset $S \subseteq \mathbb{R}^n$ is a **subspace** when

1. if $\vec{u}, \vec{v} \in S$ then $\vec{u} + \vec{v} \in S$.
2. if $\vec{u} \in S$ and $c \in \mathbb{R}$ then $c\vec{u} \in S$.

Definition

A set of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ is a **basis** for a subspace $S \subseteq \mathbb{R}^n$ when

1. $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ are linearly independent
2. $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ span S

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Idea:

1. can describe all of the vectors in S
2. as efficiently as possible

1. The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$, **span** S if every $\mathbf{v}$ in S can be written as a linear combination

$$\mathbf{v} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \cdots + c_k \mathbf{v}_k$$ for some weights $c_1, c_2, \ldots, c_k$ in $\mathbb{R}$.

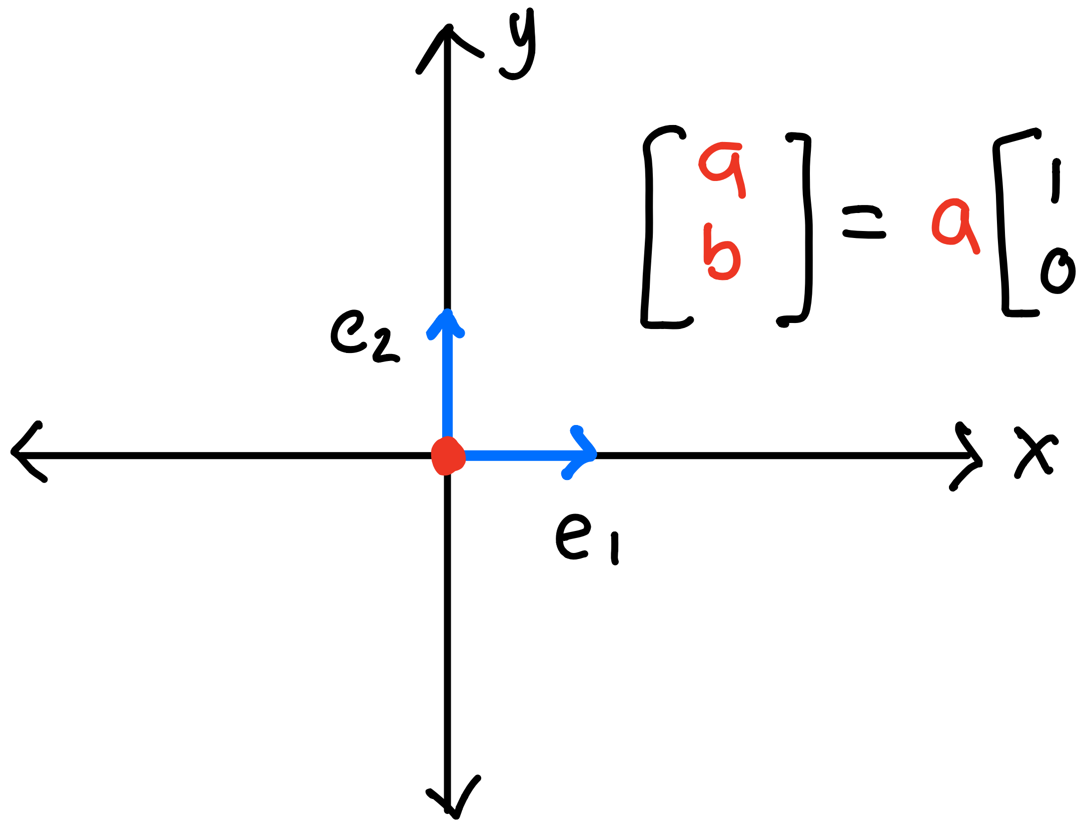
2. The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$, are **linearly independent** if the only combination that gets to the zero vector is

$$\mathbf{0} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \cdots + c_k \mathbf{v}_k$$ with $c_1 = c_2 = \cdots = c_k = 0$.

(i.e., no redundancies among the vectors).

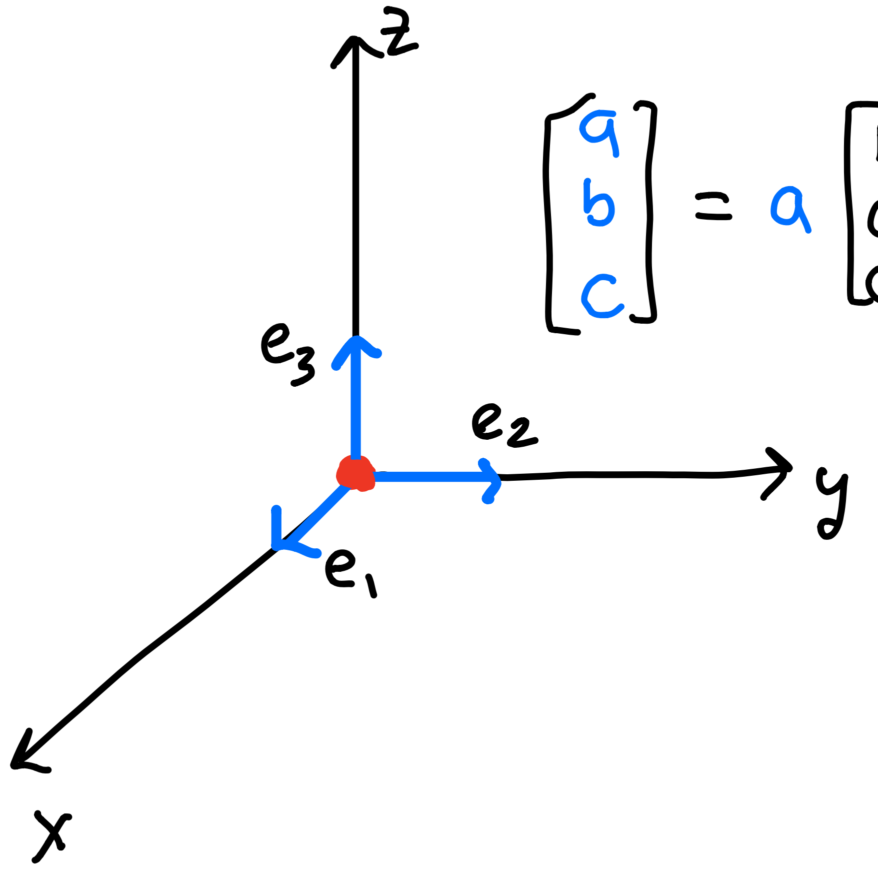
Examples

Standard Basis for $\mathbb{R}^2$

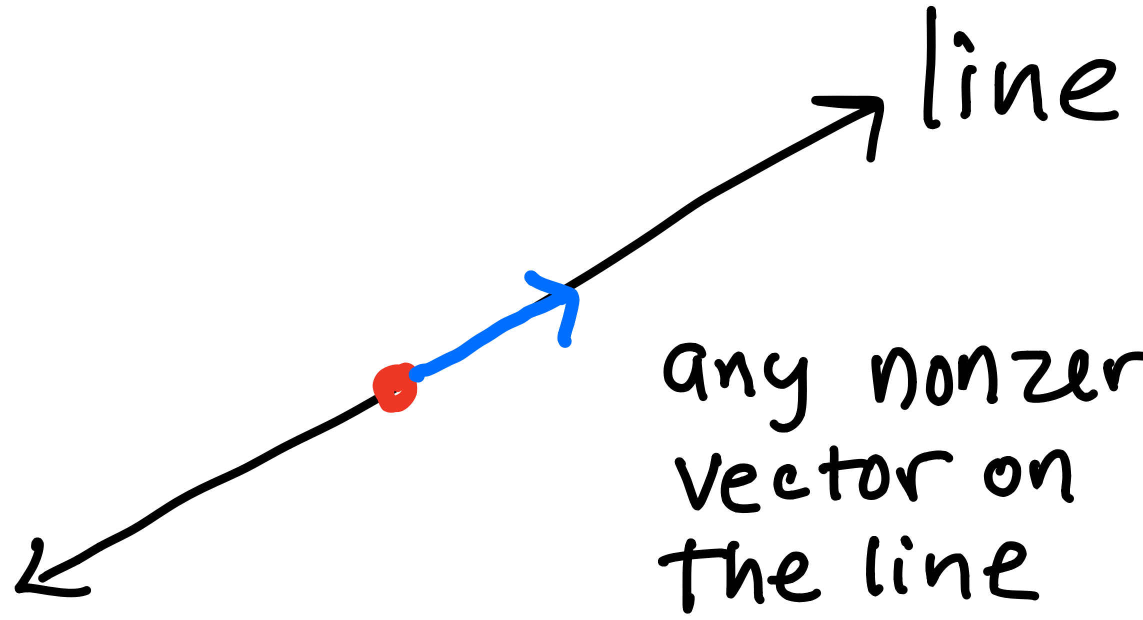


$$\begin{bmatrix} a \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Standard Basis for $\mathbb{R}^3$

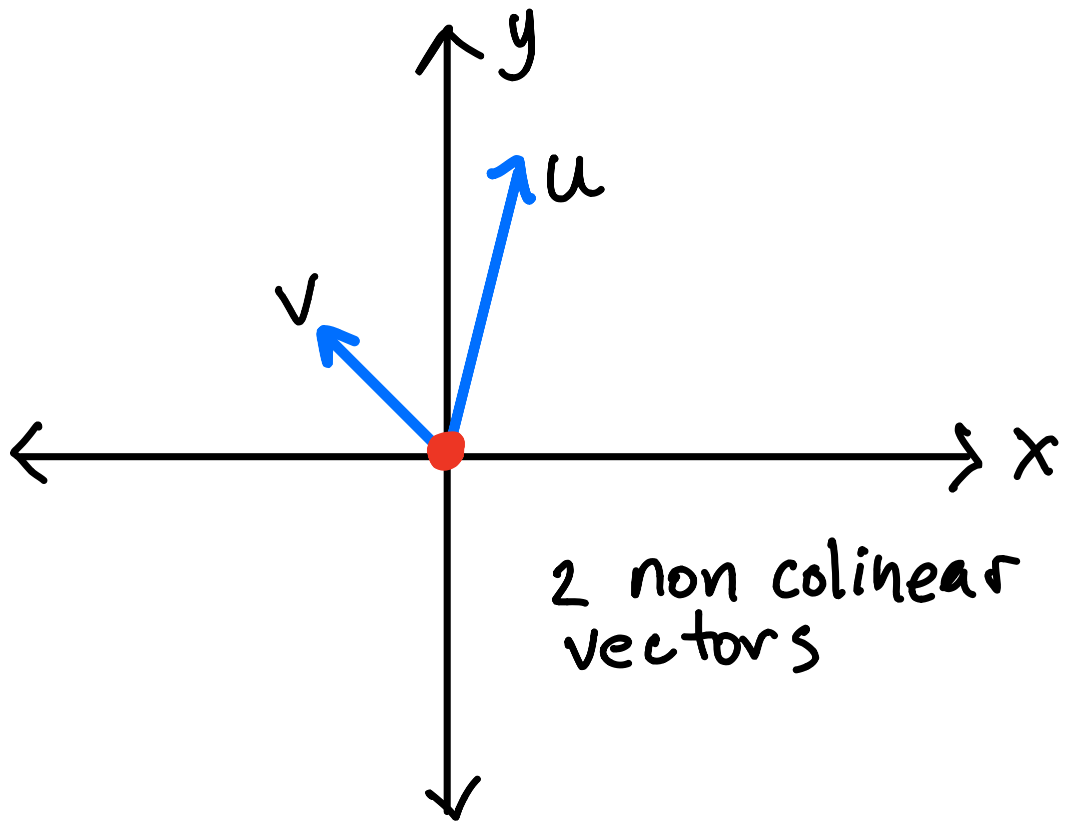


$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$



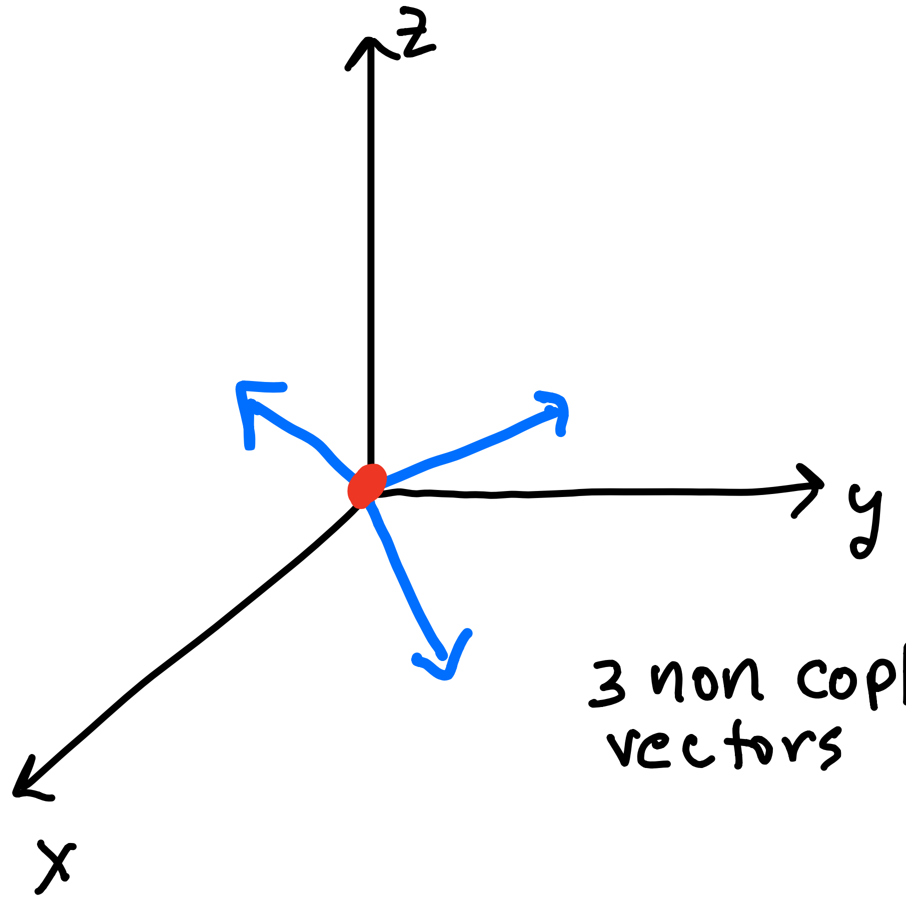
any nonzero vector on the line

Basis for $\mathbb{R}^2$



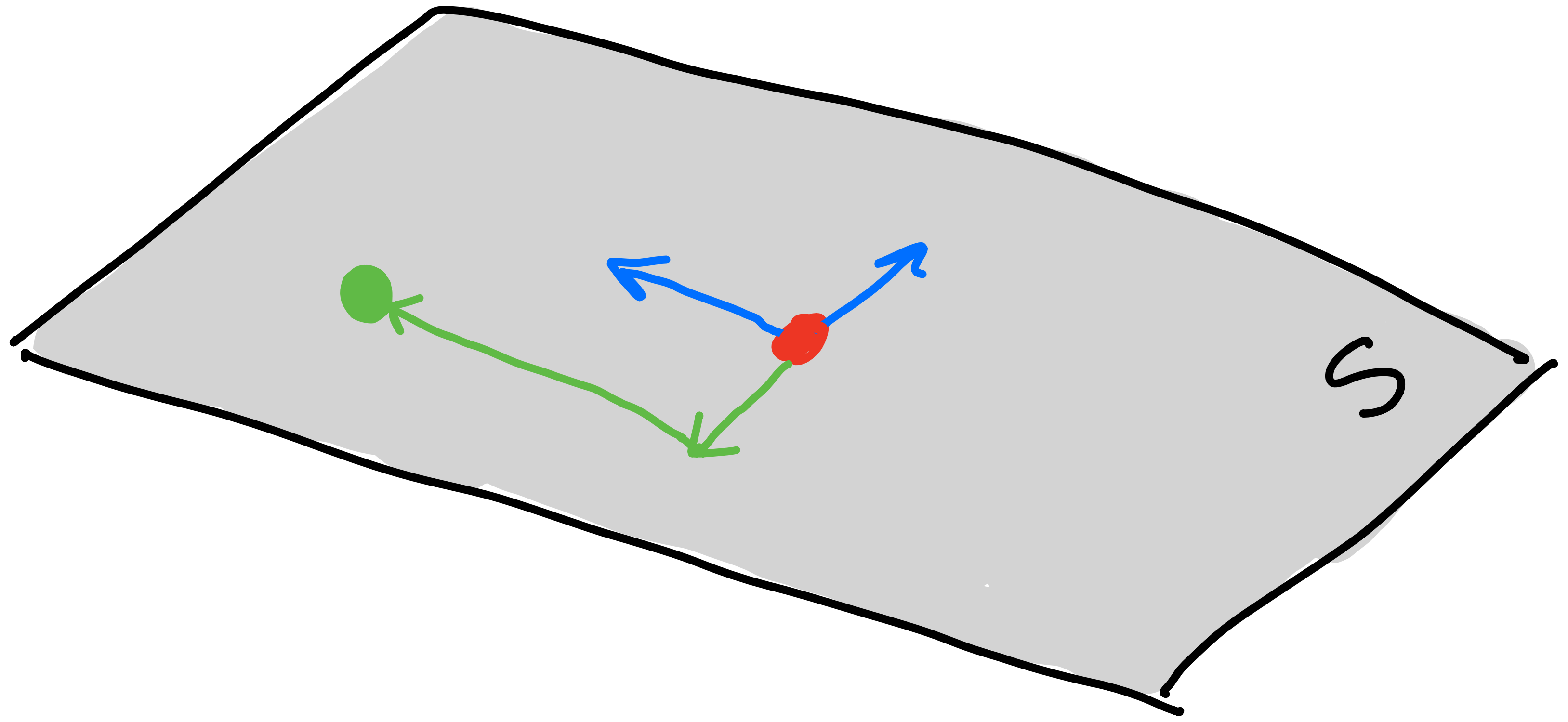
2 non colinear vectors

Basis for $\mathbb{R}^3$



3 non coplanar vectors

Plane in $\mathbb{R}^3$



Null Space Basis

$$\mathrm{Nul}(A) = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0}\}$$

$$\mathbf{A} = \begin{bmatrix} 3 & 1 & 1 & 4 & 8 \\ 3 & 1 & 1 & 4 & 8 \\ -1 & 4 & 0 & -5 & -6 \\ 0 & 4 & 0 & -4 & -4 \\ 2 & -1 & 1 & 5 & 8 \\ 2 & 1 & 2 & 5 & 9 \end{bmatrix} \longrightarrow \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & s & t \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

span

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = s \underbrace{\begin{bmatrix} -1 \\ 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}}_{u_1} + t \underbrace{\begin{bmatrix} -2 \\ 1 \\ -3 \\ 0 \\ 1 \end{bmatrix}}_{u_2}$$

parametric solution to $Ax=0$

Basis

$$\mathcal{B} = \left\{ \begin{bmatrix} -1 \\ 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

linearly independent

$$s \begin{bmatrix} -1 \\ 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ -3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The *standard* null space basis

ASIDE: Pivot Columns are Linearly Independent

Key idea: row operations do not change relations among columns, since they do the same thing to each column.

$$\begin{array}{cc} & \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ A = & \begin{bmatrix} 1 & 1 & -1 & 2 & 1 \\ 1 & -1 & -3 & 4 & 0 \\ 2 & 1 & -3 & 5 & 0 \\ 4 & -1 & -9 & 13 & 1 \end{bmatrix} \end{array} \longrightarrow \begin{array}{cc} & \begin{array}{ccccc} u_1 & u_2 & u_3 & u_4 & u_5 \end{array} \\ U = & \begin{bmatrix} 1 & 0 & -2 & 3 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

u_1, u_2, u_5 are linearly independent

Easy to see

$u_3 = -2u_1 + u_2$

$u_4 = 3u_1 - u_2$

$\Rightarrow$ v_1, v_2, v_5 are linearly independent

$$\underset{v_3}{\begin{bmatrix} -1 \\ -3 \\ -3 \\ -9 \end{bmatrix}} = -2 \underset{v_1}{\begin{bmatrix} 1 \\ 1 \\ 2 \\ 4 \end{bmatrix}} + \underset{v_2}{\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}}$$

$$\underset{v_4}{\begin{bmatrix} 2 \\ 4 \\ 5 \\ 13 \end{bmatrix}} = 3 \underset{v_1}{\begin{bmatrix} 1 \\ 1 \\ 2 \\ 4 \end{bmatrix}} - \underset{v_2}{\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}}$$

RREF(A) reveals hidden relationships in A

Column Space Basis

$$\mathrm{Col}(A) = \{A\vec{x} \mid \vec{x} \in \mathbb{R}^n\}.$$
$$\mathrm{Col}(A) = \{b \in \mathbb{R}^m \mid Ax = b \text{ for at least one } x \in \mathbb{R}^n\}$$
$$\mathrm{Col}(A) = \text{ span of the columns of } A$$

$$\mathbf{A} = \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{bmatrix} 3 & 1 & 1 & 4 & 8 \\ 3 & 1 & 1 & 4 & 8 \\ -1 & 4 & 0 & -5 & -6 \\ 0 & 4 & 0 & -4 & -4 \\ 2 & -1 & 1 & 5 & 8 \\ 2 & 1 & 2 & 5 & 9 \end{bmatrix} \end{array} \longrightarrow \begin{array}{c} \begin{array}{ccccc} u_1 & u_2 & u_3 & u_4 & u_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$v_4 = v_1 - v_2 + 2v_3$

$v_5 = 2v_1 - v_2 + 3v_3$

redundant

$\mathcal{B} = \{v_1, v_2, v_3\}$ basis of pivot columns

WARNING
Use the pivot vectors in A not in rref(A). Those matrices do not have the same column space.

Basis of a Span of Vectors

Q: Find a basis for:

$$S = \textbf{span}\left\{ \begin{array}{c} v_1 \\ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 4 \end{bmatrix} \end{array}, \begin{array}{c} v_2 \\ \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \end{array}, \begin{array}{c} v_3 \\ \begin{bmatrix} -1 \\ -3 \\ -3 \\ -9 \end{bmatrix} \end{array}, \begin{array}{c} v_4 \\ \begin{bmatrix} 2 \\ 4 \\ 5 \\ 13 \end{bmatrix} \end{array}, \begin{array}{c} v_5 \\ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \end{array} \right\} \subset \mathbb{R}^4$$

Put them in the columns of a matrix and row reduce

$$A = \begin{bmatrix} 1 & 1 & -1 & 2 & 1 \\ 1 & -1 & -3 & 4 & 0 \\ 2 & 1 & -3 & 5 & 0 \\ 4 & -1 & -9 & 13 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -2 & 3 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Note that S = Col(A)

A common *wrong* answer is to use the pivot columns of rref(A)

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Basis of pivot columns: $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

You Try

Q: Find a basis for Nul(A) and Col(A) if

$$A = \begin{array}{c} \begin{array}{cccc} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \end{array} \\ \begin{bmatrix} 1 & 2 & 5 & -1 \\ 0 & 1 & 2 & -1 \\ 2 & 2 & 6 & 0 \\ 1 & 1 & 3 & 0 \\ 1 & 1 & 3 & 0 \end{bmatrix} \end{array} \longrightarrow \begin{array}{c} \begin{array}{cccc} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 & \mathbf{u}_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$