Section 5.1: An Introduction to Systems of Equations

Lotka-Volterra Model.

Example (Predator-Prey Model)

Rabbits eat only vegetation and are eaten only by foxes.

Foxes eat only rabbits and have no predators

Let $x(t)$ denote the fox population at time t

$y(t)$ denote the rabbit population at time t.

When there are no foxes

$$\frac{dy}{dt} = c_1 y \qquad c_1 > 0$$

(rabbits experience uninhibited growth)

When there are foxes

$$(*) \qquad \frac{dy}{dt} = c_1 y - c_2 \cdot \underbrace{xy} \qquad c_2 > 0$$

When there are no rabbits

$$\frac{dx}{dt} = -C_3 x \qquad C_3 > 0$$

When rabbits are added

$$\frac{dx}{dt} = -C_3 x + C_4 xy \qquad C_4 > 0$$

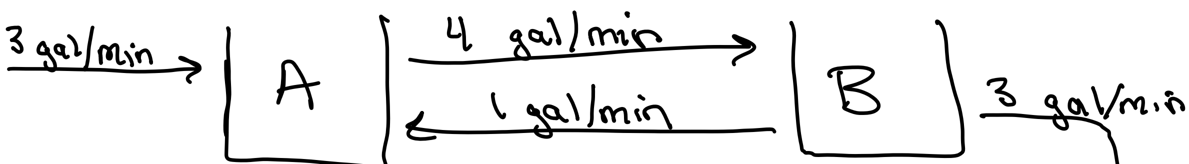
We get

system of equations

$$\begin{cases} \dfrac{dx}{dt} = -C_3 x + C_4 xy \\[2ex] \dfrac{dy}{dt} = C_1 y - C_2 xy \end{cases}, \quad C_1, C_2, C_3, C_4 > 0$$

Nonlinear systems ← possible project topic

$$\frac{dx}{dt} = \left(\begin{matrix}\text{Rate that}\\ \text{salt enters}\end{matrix}\right) - \left(\begin{matrix}\text{Rate that}\\ \text{salt exits}\end{matrix}\right)$$



lbs of salt/min

$$\frac{dx}{dt} = \cancel{\left(3\ \text{gal}/\text{min}\right)\left(0\ \text{lb}/\text{gal}\right)} + \left(1\ \text{gal}/\text{min}\right)\left(y\ \text{lb}/50\ \text{gal}\right) - \left(4\ \text{gal}/\text{min}\right)\left(x\ \text{lb}/50\ \text{gal}\right)$$

$$= \frac{1}{50}(y - 4x)$$

$$\frac{dy}{dt} = \frac{4}{50}(x-y)$$

$$\frac{dx}{dt} = \frac{1}{50}(y-4x)$$
$$\frac{dy}{dx} = \frac{4}{50}(x-y)$$

In general, a first order linear system of equations has the form

$$\frac{dx}{dt} = c_1 x + c_2 y$$
$$\frac{dy}{dt} = c_3 x + c_4 y \qquad c_1, c_2, c_3, c_4 \in \mathbb{R}$$

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \underbrace{\overset{2\times 2}{\begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}}}_{\text{coefficient matrix}} \underbrace{\overset{2\times 1}{\begin{bmatrix} x \\ y \end{bmatrix}}}_{\text{dependent variables}} = \begin{bmatrix} c_1 x + c_2 y \\ c_3 x + c_4 y \end{bmatrix}$$

Shorthand notation $\vec{x}\,' = A\vec{x}$

contains derivatives w.r.t. single independent variable

dependent variable.