Complex Eigenvalues

Complex eigenvalues come in conjugate pairs

$$\lambda_1 = a+bi \qquad\qquad \lambda_2 = a-bi = \overline{\lambda_1}$$

Theorem: Let A be a coefficient matrix for a system of differential equations and let $\vec{k_1}$ be an eigenvector corresponding to eigenvalue λ_1. Then

$$\vec{x} = c_1 \vec{k_1} e^{\lambda_1 t} + c_2 \overbrace{\overline{\vec{k_1}}}^{\text{complex conjugate vector}} e^{\underbrace{\overline{\lambda_1}}_{=\lambda_2} t}$$

Solve :

$$\left.\begin{aligned} \frac{dx}{dt} &= 4x+5y \\ \frac{dy}{dt} &= -2x+6y \end{aligned}\right\} \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ -2 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\det(A-\lambda I) = \det\left(\begin{bmatrix} 4-\lambda & 5 \\ -2 & 6-\lambda \end{bmatrix}\right)$$

$$= (4-\lambda)(6-\lambda) - (-10)$$

$$= 24 - 10\lambda + \lambda^2 + 10$$

$$= \lambda^2 - 10\lambda + 34$$

$$\lambda = \frac{10 \pm \sqrt{(-10)^2 - 4(1)(34)}}{2} = \frac{10}{2} \pm \frac{\sqrt{-36}}{2}$$

$$= 5 \pm 3i$$

$\lambda_1 = 5 + 3i$

$$\left[\begin{array}{cc|c} a-\lambda & b & 0 \\ c & d-\lambda & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 4-(5+3i) & 5 & 0 \\ -2 & 6-(5+3i) & 0 \end{array}\right]$$

$$= \left[\begin{array}{cc|c} -1-3i & 5 & 0 \\ -2 & 1-3i & 0 \end{array}\right]$$

$(-1-3i)(-1+3i)$

$= (-1)^2 - 3i$

$+3i - (3i)^2$

$= 1 + 9$

$$\xrightarrow{R_1 \to (-1+3i)R_1} \left[\begin{array}{cc|c} 10 & 5(-1+3i) & 0 \\ -2 & 1-3i & 0 \end{array}\right]$$

$$\xrightarrow{R_1 \to \frac{1}{10}R_1} \left[\begin{array}{cc|c} 1 & \frac{1}{2}(-1+3i) & 0 \\ -2 & 1-3i & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 + 2R_1} \left[\begin{array}{cc|c} 1 & \frac{1}{2}(-1+3i) & 0 \end{array}\right.$$

$$\left[0 \quad 0 \quad | \; 0\right]$$

k_2 is free

$k_1 + \left(\frac{-1}{2} + \frac{3}{2}i\right)k_2 = 0$

$\vec{K} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$

$k_1 = \left(\frac{1}{2} - \frac{3}{2}i\right)k_2$

any nonzero constant

$$= \begin{bmatrix} \left(\frac{1}{2} - \frac{3}{2}i\right)k_2 \\ k_2 \end{bmatrix} = k_2 \begin{bmatrix} \frac{1}{2} - \frac{3}{2}i \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 \begin{bmatrix} \frac{1}{2} - \frac{3}{2}i \\ 1 \end{bmatrix} e^{(5+3i)t} + C_2 \begin{bmatrix} \frac{1}{2} + \frac{3}{2}i \\ 1 \end{bmatrix} e^{(5-3i)t}.$$

$$\vec{K_1} = \begin{bmatrix} K_1 \\ \frac{1}{\left(\frac{-1}{2} + \frac{3}{2}i\right)} K_1 \end{bmatrix}$$

Theorem: Let $\lambda_1 = a + bi$ be a complex eigenvalue of coefficient matrix A. Let

$$\vec{b_1} = \mathrm{Re}(\vec{K_1}), \text{ and } \vec{b_2} = \mathrm{Im}(\vec{K_1})$$

where $\vec{K_1}$ is eigenvector corresponding to λ_1. Then

$$\vec{x} = c_1 \left[\vec{b_1} \cos(bt) - \vec{b_2} \sin(bt)\right] e^{at}$$

$$+ c_2\left[\vec{b_2}\cos(bt) + \vec{b_1}\sin(bt)\right]e^{at}$$

is a (real-valued) solution to the system

Example (continued)

$$\vec{K} = \begin{bmatrix} \frac{1}{2} - \frac{3}{2}i \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{3}{2} \\ 0 \end{bmatrix} i$$

$$\vec{b_1} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \qquad \vec{b_2} = \begin{bmatrix} -\frac{3}{2} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1\left(\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}\cos(3t) - \begin{bmatrix} -\frac{3}{2} \\ 0 \end{bmatrix}\sin(3t)\right)e^{5t}$$

$$+ c_2\left(\begin{bmatrix} -\frac{3}{2} \\ 0 \end{bmatrix}\cos(3t) + \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}\sin(3t)\right)e^{5t}$$

Real-repeated eigenvalue case.

For 2×2 matrices with repeated eigenvalue there is only 1 linearly independent eigenvector

From exponential sub for systems

$\vec{k} e^{\lambda t}$ ($\vec{k}$ eigenvector corresponding to λ)

is solution

Guess (motivated by section 3.2)

$\vec{k} t e^{\lambda t}$

is a solution. Check if it works:

$$\frac{d}{dt}\left(\vec{k} t e^{\lambda t}\right) = \begin{bmatrix} \frac{d}{dt}\left(k_1 t e^{\lambda t}\right) \\ \frac{d}{dt}\left(k_2 t e^{\lambda t}\right) \end{bmatrix}$$

$$= \begin{bmatrix} k_1 e^{\lambda t} + k_1 \lambda t e^{\lambda t} \\ k_2 e^{\lambda t} + k_2 \lambda t e^{\lambda t} \end{bmatrix}$$

$$= \vec{k} e^{\lambda t} + \lambda \vec{k} t e^{\lambda t}$$

LHS. RHS

$$\underbrace{\vec{k} e^{\lambda t} + \underline{\lambda \vec{k} t e^{\lambda t}}} = \underline{A \vec{k} t e^{\lambda t}}$$

"$te^{\lambda t}$" terms $\quad \lambda \vec{k}\, te^{\lambda t} = A\vec{k}\, te^{\lambda t}$

"$e^{\lambda t}$" terms $\quad \vec{k} e^{\lambda t} = \vec{0} e^{\lambda t}.$

$\vec{k} = \vec{0}$

Problem because by assumption $\vec{k}$ is eigenvector and eigenvectors cannot be $\vec{0}$.

New guess: $\vec{k}\, te^{\lambda t} + \vec{p} e^{\lambda t}$

some other vector.

"Right hand side" of system is

$$A(\vec{k}\, te^{\lambda t} + \vec{p} e^{\lambda t})$$

$$= A\vec{k}\, te^{\lambda t} + A\vec{p} e^{\lambda t}$$

"Left hand side" of system is

$$\frac{d}{dt}(\vec{k}\, te^{\lambda t} + \vec{p} e^{\lambda t}) = \vec{k} e^{\lambda t} + \lambda \vec{k}\, te^{\lambda t} + \lambda \vec{p} e^{\lambda t}$$

$$\vec{k}e^{\lambda t} + \lambda \vec{k}\, t e^{\lambda t} + \lambda \vec{p} e^{\lambda t} = A\vec{k}\, t e^{\lambda t} + A\vec{p} e^{\lambda t}$$

"$te^{\lambda t}$" terms

$$\lambda \vec{k}\; t e^{\lambda t} = A\vec{k}\; t e^{\lambda t}$$

$$(\lambda \vec{k} = A\vec{k})$$

k is an eigenvector corresponding to eigenvalue λ.

"$e^{\lambda t}$" terms

$$\vec{k}e^{\lambda t} + \lambda \vec{p} e^{\lambda t} = A\vec{p} e^{\lambda t}$$

$$(\vec{k} + \lambda \vec{p} = A\vec{p})$$

$I\vec{p} = \vec{p}$

$$(A - \lambda I)\vec{p} = \vec{k} \quad *$$

matrix scalar

$\cancel{(A - \lambda)\vec{p}}$

particular vector

We can solve for $\vec{p}$ using using row reduction on $[A - \lambda I \mid \vec{k}]$

Example: $\frac{dx}{dt} = -6x + 5y$

$\frac{dy}{dt} = -5x + 4y$

$\lambda = -1$ is an eigenvalue of multiplicity 2.

Eigenvector

$$\left[\begin{array}{cc|c} -6-(-1) & 5 & 0 \\ -5 & 4-(-1) & 0 \end{array}\right] = \left[\begin{array}{cc|c} -5 & 5 & 0 \\ -5 & 5 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 \to -\frac{1}{5}R_1} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ -5 & 5 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 + 5R_1} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\vec{k} = c\begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad c \in \mathbb{R}$$

(I'll choose $c=1$)

Find $\vec{p}$

$$\left[A - \lambda I \,|\, \vec{k}\right] = \left[\begin{array}{cc|c} -5 & 5 & 1 \\ -5 & 5 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \to -\frac{1}{5}R_1} \left[\begin{array}{cc|c} 1 & -1 & -\frac{1}{5} \\ -5 & 5 & 1 \end{array}\right] \xrightarrow{R_2 \to R_2 + 5R_1} \left[\begin{array}{cc|c} 1 & -1 & -\frac{1}{5} \\ 0 & 0 & 0 \end{array}\right]$$

p_2 is free.

$$-p_2 = -\frac{1}{5} - p_1$$
$$p_2 = \frac{1}{5} + p_1$$

$$p_1 - p_2 = -\frac{1}{5}$$
$$\Rightarrow p_1 = -\frac{1}{5} + p_2$$

$$\vec{p} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{5} + p_2 \\ p_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{5} \\ 0 \end{bmatrix} + \begin{bmatrix} p_2 \\ p_2 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{5} \\ 0 \end{bmatrix} + p_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

[I'll choose

$p_2 = 0$] $\quad \vec{p} = \begin{bmatrix} -\frac{1}{5} \\ 0 \end{bmatrix}$

$$\vec{x} = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + C_2 \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-t} + \begin{bmatrix} -\frac{1}{5} \\ 0 \end{bmatrix} e^{-t} \right)$$

In general for repeated eigenvalues

$$\vec{x} = c_1 \vec{k} e^{\lambda t} + c_2 (t\vec{k} + \vec{p}) e^{\lambda t}$$

$$= \underbrace{e^{\lambda t}}_{\text{exponential}} \underbrace{(c_1 \vec{k} + c_2 t\vec{k} + c_2 \vec{p})}_{\text{vector equation of a line}}$$

$$(c_1\vec{k} + c_2\vec{p}) + c_2 t \vec{k}$$