Section 6.2: PDEs and the Heat Equation

Partial differential equation - a differential equation where derivatives are w.r.t. more than one independent variable.

$u(x,t)$

$u_x(x,t)$ $\quad \frac{\partial u}{\partial x}$

(partial derivatives in x variable)

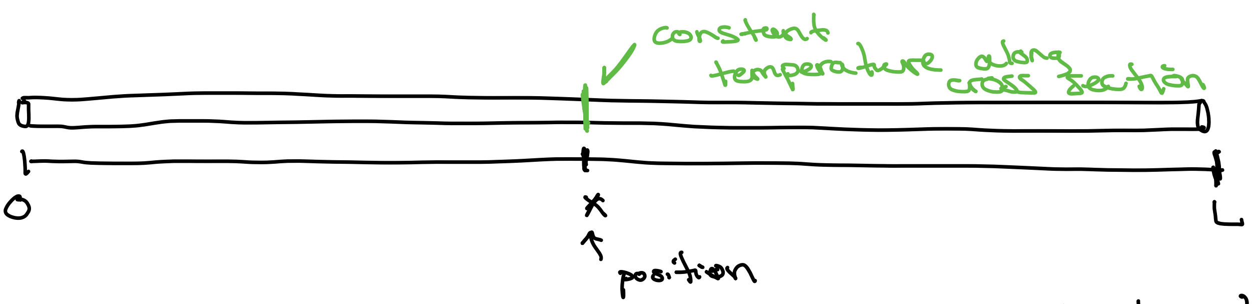
$u_t(x,t)$ $\quad \frac{\partial u}{\partial t}$

(partial derivatives in t variable

$u(x,y,t)$

$u(x_1, x_2, x_3, \dots, x_n, t)$

To build the build the one dimensional heat equation.

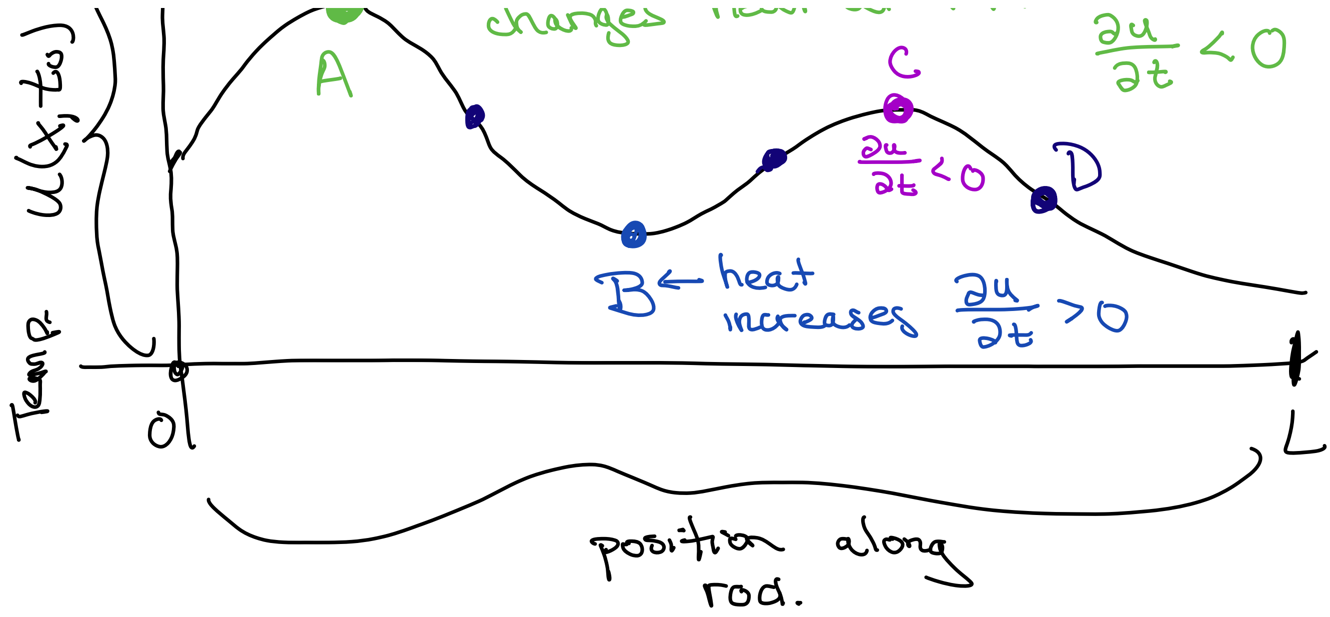


temperature at time t and position x

Time slice of $u(x,t)$ at time $t = t_0$

as time

heat at A will decrease



Heat Equation

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$$

inflection points along $u(x, t_0)$ for fixed t_0.

↑ thermal diffusivity constant

$$u_t = \alpha u_{xx}$$

$$\dot{u} = \alpha \nabla^2 u$$

Nabla

$$u_t = \alpha \Delta u$$

Laplace

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

$$\nabla^2 := \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

$$= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

initial data.

Initial Condition - specify a condition for $t=0$.

tells us
how heat is distributed along the rod at time 0.

$$u(x,0)=f(x)$$

Boundary Conditions

Dirichlet Boundary Conditions - explicitly prescribe values of u on the boundary (end points) of rod.

$$u(0,t)=c_1 \qquad u(L,t)=c_2$$

(For example, keeping left and right tips dipped in water baths of temp. c_1 and c_2.)

Neumann Boundary Condition

$$u_x(0,t)=c_1 \quad , \quad u_x(L,t)=c_2$$

(For example, $u_x(0,t)=u_x(L,t)=0$

corresponds to pipe with insolated endpoints)

Robin boundary condition

Example at position L.

$$u(L,t) + u_x(L,t) = 0 \quad \text{for all } t$$

Example: $u_1(x,t) = e^{-t}\sin x$ solves the heat equation $\underbrace{\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}}$ specific $\alpha = 1$

$$\frac{\partial u_1}{\partial t} = -e^{-t}\sin x$$

$$\frac{\partial u_1}{\partial x} = e^{-t}\cos x \qquad \frac{\partial^2 u_1}{\partial x^2} = -e^{-t}\sin x$$

$$-e^{-t}\sin x = -e^{-t}\sin x \quad \checkmark$$

$u_2(x,t) = e^{-4t}\sin(2x)$ solves $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} = -4e^{-4t}\sin(2x) \quad \checkmark$$

Heat equation is linear!

$$u(x,t) = c_1 u_1(x,t) + c_2 u_2(x,t)$$
$$= c_1 e^{-t}\sin x + c_2 e^{-4t}\sin(2x)$$

also satisfies

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

Add boundary and initial conditions:

Dirichlet Boundary Condition

$$u(0,t) = u(\pi,t) = 0$$
$$u(x,0) = 30\sin x - 4\sin(2x)$$

$$c_1 = 30 \quad , \quad c_2 = -4$$

$$u(x,t) = 30e^{-t}\sin x - 4e^{-4t}\sin(2x)$$

Often we need an infinite series of "building block" functions to solve a boundary value problem.

$$u(x,t) = \sum_{i=1}^{\infty} c_n u_n(x,t)$$

exponential times sine or cosine function