Convergence to dominant eigenvector

A

$\lambda_1 \quad \lambda_2 \quad \lambda_3 \quad \quad \quad \lambda_n$

$v_1 \quad v_2 \quad v_3 \quad \cdots \quad v_n$

Fibonacci Example

14.5 in Handbook (c.f., PS 8.4)

$$F = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\begin{array}{cccccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ \begin{bmatrix} 0 \\ 1 \end{bmatrix} & \begin{bmatrix} 1 \\ 1 \end{bmatrix} & \begin{bmatrix} 1 \\ 2 \end{bmatrix} & \begin{bmatrix} 2 \\ 3 \end{bmatrix} & \begin{bmatrix} 3 \\ 5 \end{bmatrix} & \begin{bmatrix} 5 \\ 8 \end{bmatrix} & \begin{bmatrix} 8 \\ 13 \end{bmatrix} & \begin{bmatrix} 13 \\ 21 \end{bmatrix} & \begin{bmatrix} 21 \\ 34 \end{bmatrix} & \begin{bmatrix} 34 \\ 55 \end{bmatrix} & \begin{bmatrix} 55 \\ 89 \end{bmatrix} & \begin{bmatrix} 89 \\ 144 \end{bmatrix} \end{array}$$

Complex Eigenvectors

Diagonalized over the complex numbers

Rental Car Matrix

$$\begin{bmatrix} .85 & .30 & .35 \\ .09 & .60 & .05 \\ .06 & .10 & .60 \end{bmatrix}$$

$$\lambda_1 = 1.0, \quad \lambda_2 = .525 + .037i, \quad \lambda_3 = .525 - .037i$$

$$\begin{bmatrix} .95 \\ .24 \\ .20 \end{bmatrix} \quad \begin{bmatrix} .62 + .00i \\ -.31 - .46i \\ -.31 + .46i \end{bmatrix} \quad \begin{bmatrix} .62 - .00i \\ -.31 + .46i \\ -.31 - .46i \end{bmatrix}$$

$$\begin{bmatrix} .85 & .30 & .35 \\ .09 & .60 & .05 \\ .06 & .10 & .60 \end{bmatrix} = \underbrace{\begin{bmatrix} .95 & .62+.00i & .62-.00i \\ .24 & -.31-.46i & -.31+.46i \\ .20 & -.31+46i & -.31-.46i \end{bmatrix}}_{P} \begin{bmatrix} 1.0 & 0 & 0 \\ 0 & .35-.04i & 0 \\ 0 & 0 & .35+.04i \end{bmatrix} \underbrace{\begin{bmatrix} .72 & .72 & .72 \\ .26 - .01i & -.55 + .53i & -.55 - .56i \\ .26 + .01i & -.55 - .53i & -.55 + .56i \end{bmatrix}}_{P^{-1}}$$

Complex Eigenvectors

Rental Car Matrix

$$\begin{bmatrix} .85 & .30 & .35 \\ .09 & .60 & .05 \\ .06 & .10 & .60 \end{bmatrix}$$

$$\lambda_1 = 1.0, \quad \lambda_2 = .525 + .037i, \quad \lambda_3 = .525 - .037i$$

$$\begin{bmatrix} .95 \\ .24 \\ .20 \end{bmatrix} \quad \begin{bmatrix} .62 + .00i \\ -.31 - .46i \\ -.31 + .46i \end{bmatrix} \quad \begin{bmatrix} .62 - .00i \\ -.31 + .46i \\ -.31 - .46i \end{bmatrix}$$

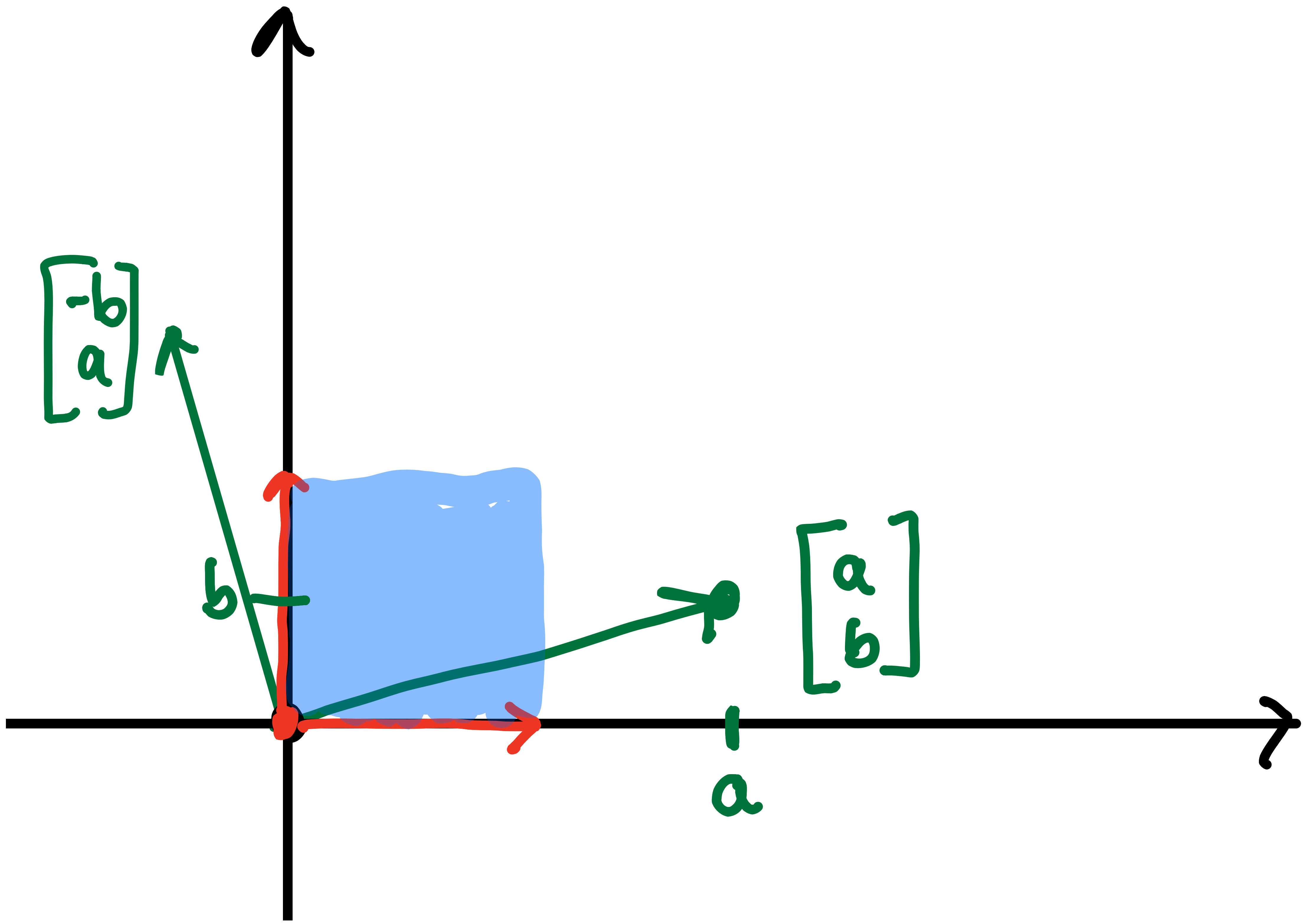
$$\begin{bmatrix} .85 & .30 & .35 \\ .09 & .60 & .05 \\ .06 & .10 & .60 \end{bmatrix} = \underbrace{\begin{bmatrix} .95 & .00 & .62 \\ .24 & -.46 & -.31 \\ .20 & .46 & -.31 \end{bmatrix}}_{P} \begin{bmatrix} 1.0 & 0 & 0 \\ 0 & .525 & -.037 \\ 0 & .037 & .525 \end{bmatrix} \underbrace{\begin{bmatrix} .72 & .72 & .72 \\ .03 & -1.06 & 1.12 \\ .51 & -1.10 & -1.10 \end{bmatrix}}_{P^{-1}}$$

Rotation-Dilation Matrices

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

$\lambda_1 = a + bi$

$\lambda_2 = a - bi$

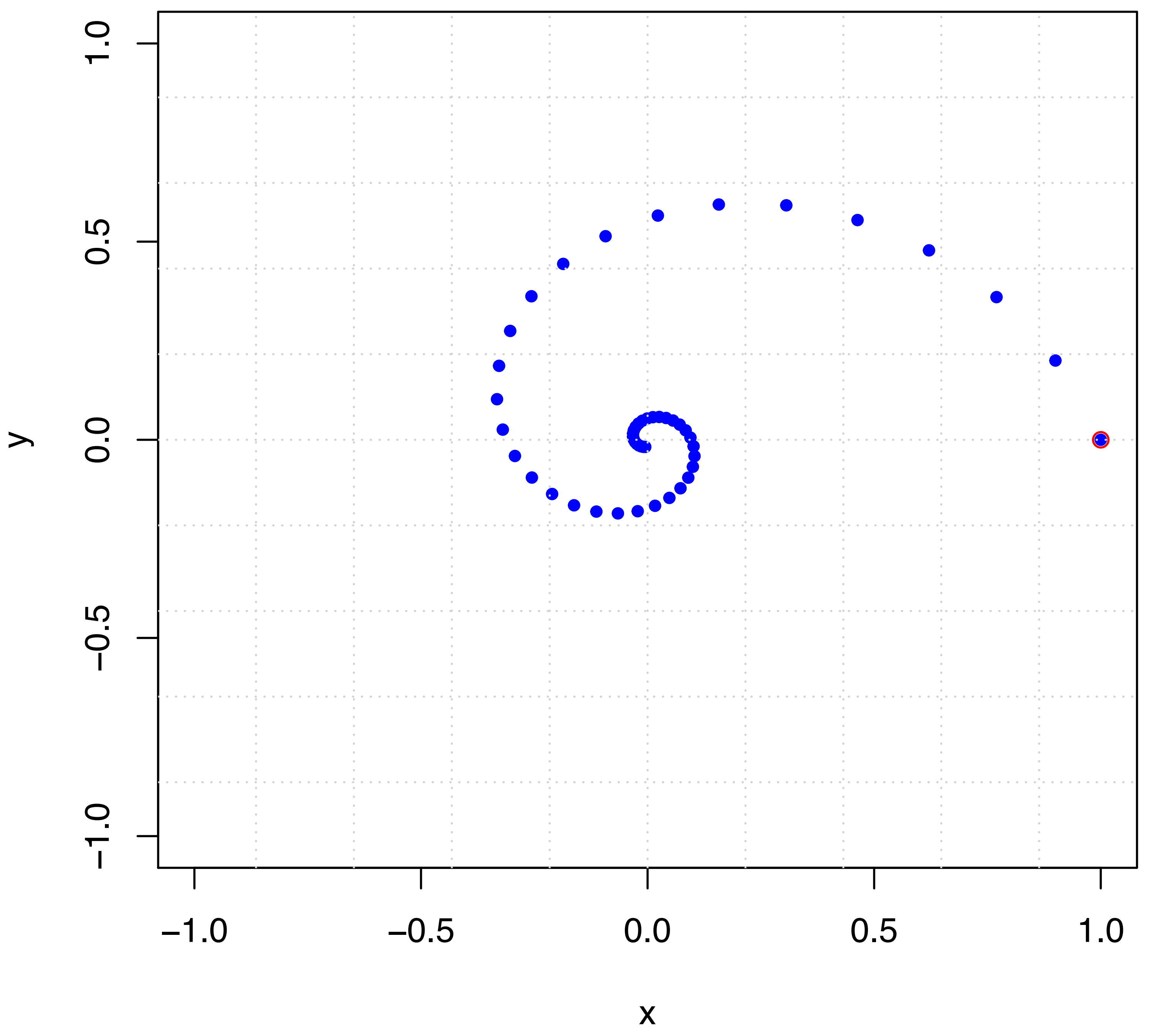


Examples

$A = \begin{bmatrix} .9 & -.2 \\ .2 & .9 \end{bmatrix}$ $|\lambda| = \sqrt{.9^2 + .2^2} = .92$

$\theta = \tan^{-1}\left(\frac{.2}{.9}\right) = .21 \text{ rad} = 12.5°$

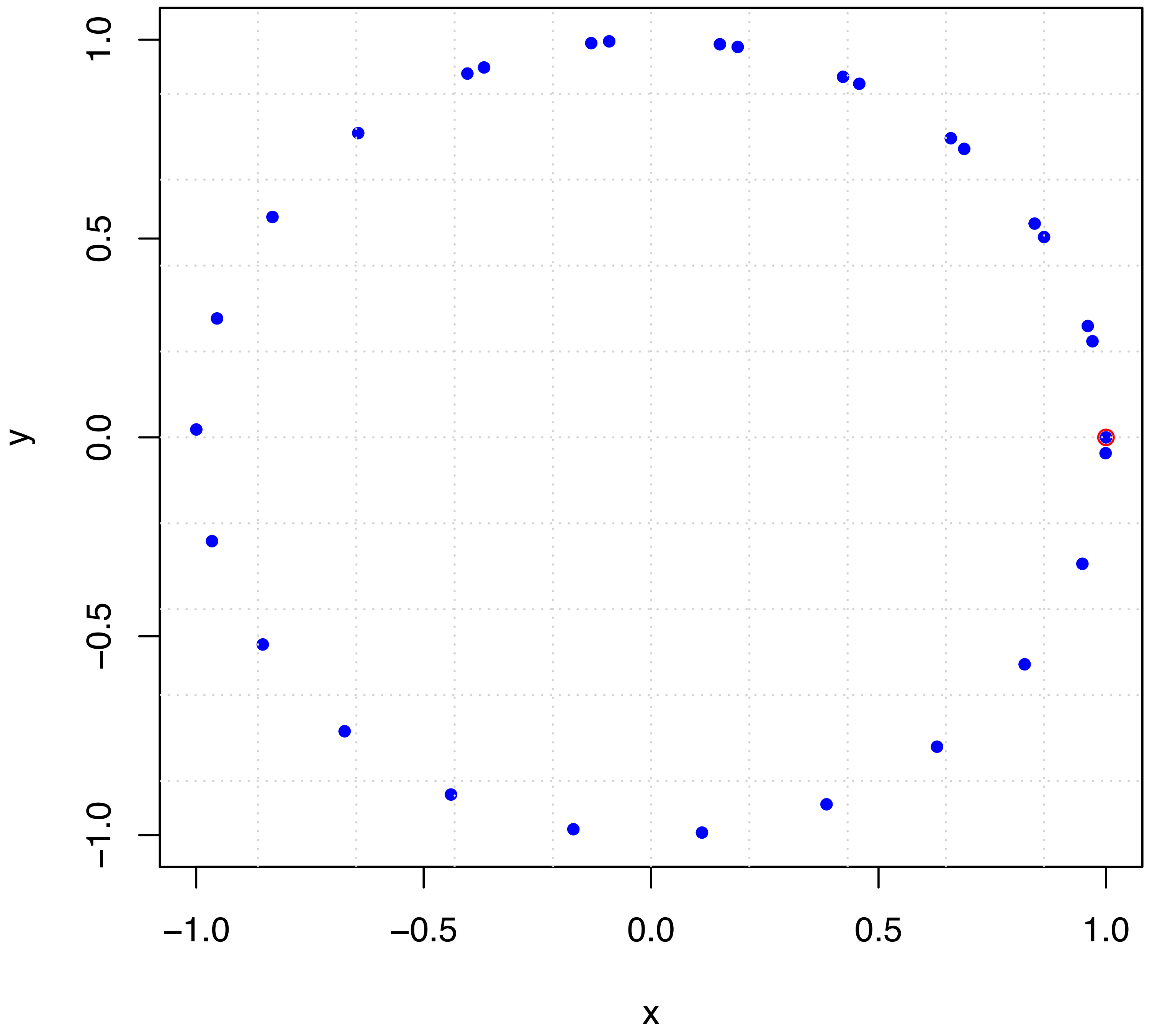
$\lambda = .9 \pm .2i$



$B = \begin{bmatrix} .96 & -.28 \\ .28 & .96 \end{bmatrix}$ $|\lambda| = \sqrt{.96^2 + .28^2} = 1.00$

$\theta = \tan^{-1}\left(\frac{.28}{.96}\right) = .28 \text{ rad} = 16.3°$

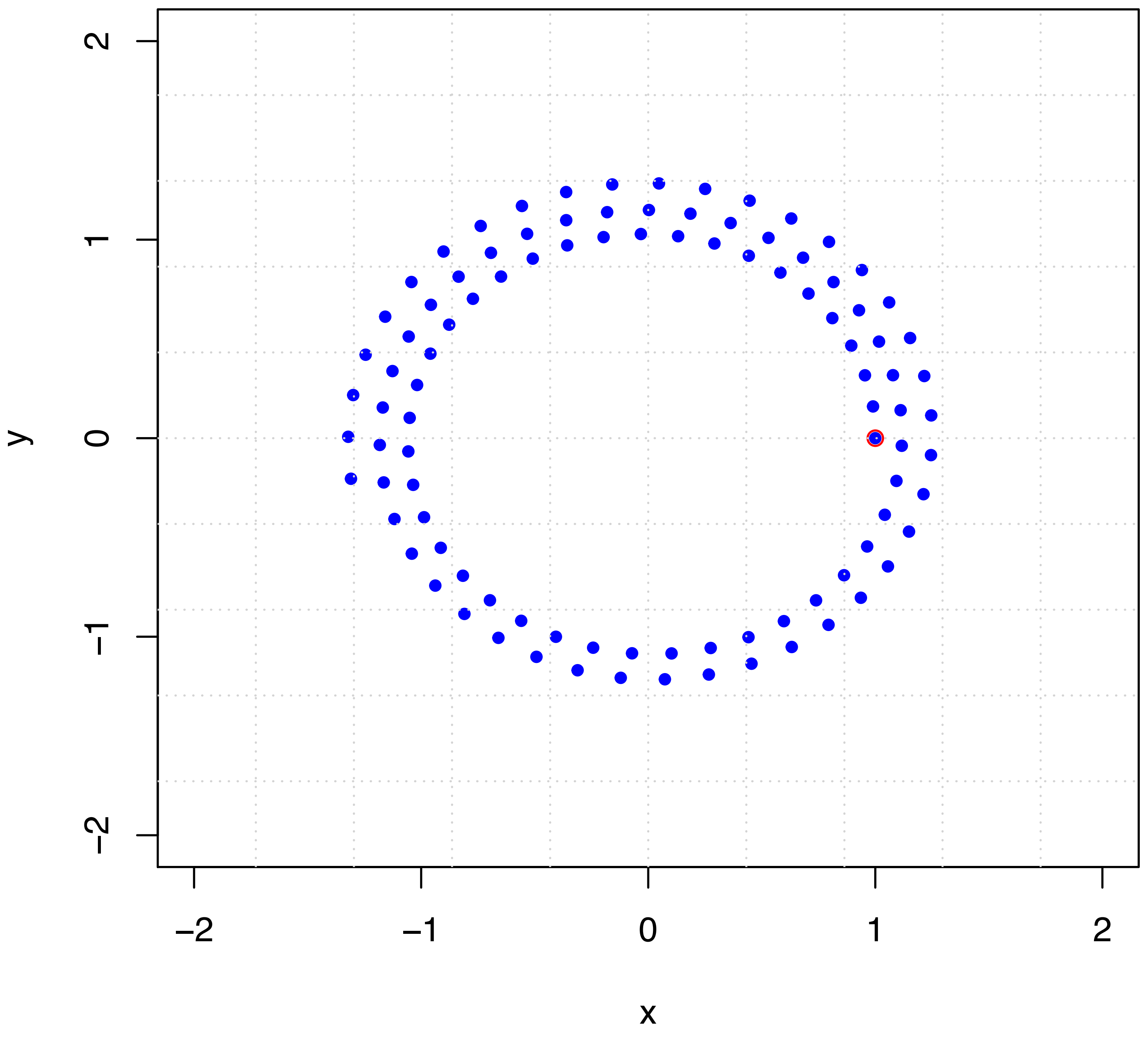
$\lambda = .96 \pm .28i$



$C = \begin{bmatrix} .99 & -.16 \\ .16 & .99 \end{bmatrix}$ $|\lambda| = \sqrt{.99^2 + .16^2} = 1.002$

$\theta = \tan^{-1}\left(\frac{.16}{.99}\right) = .16 \text{ rad} = 9.2°$

$\lambda = .99 + .16i$

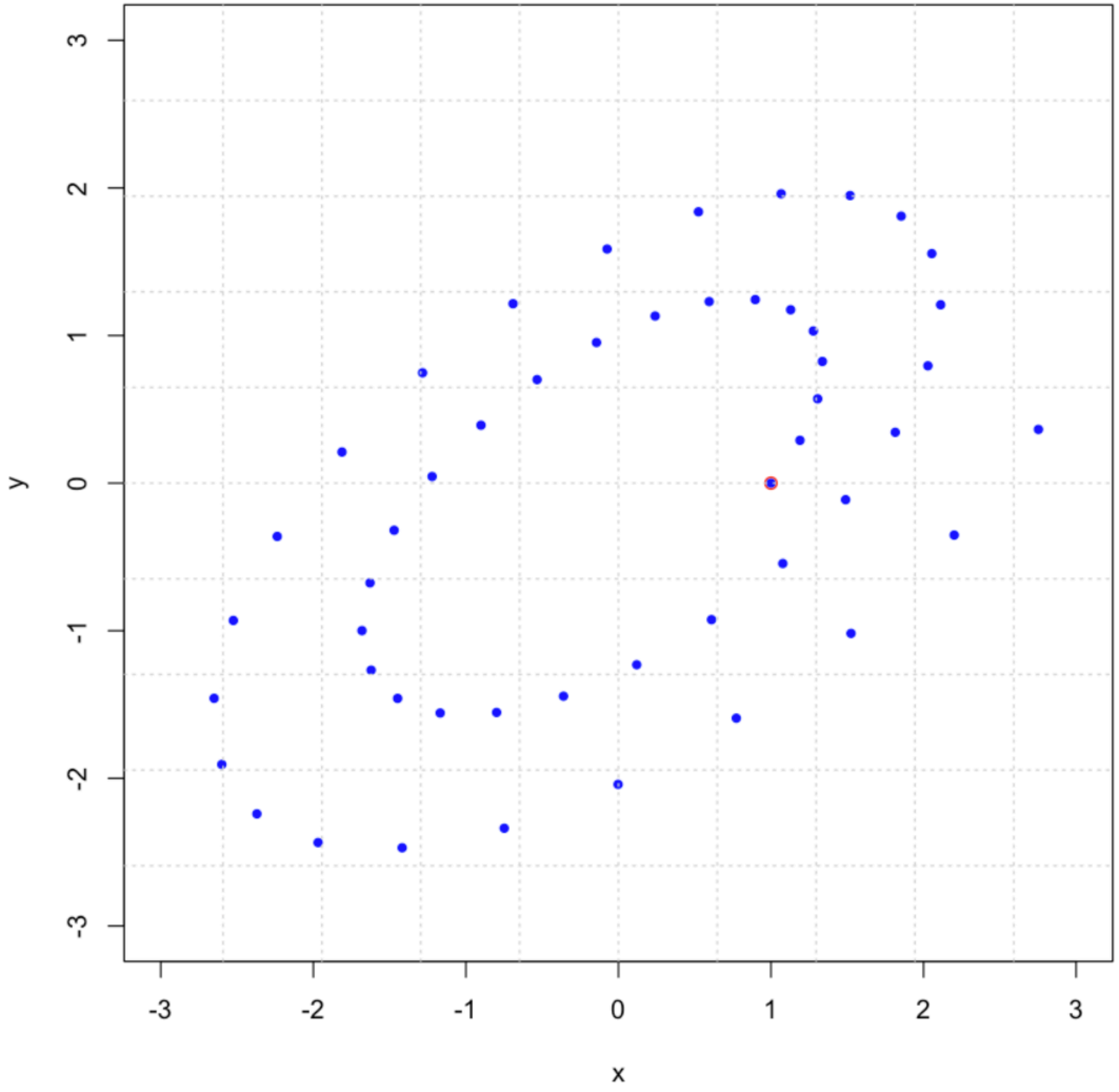


More Generally

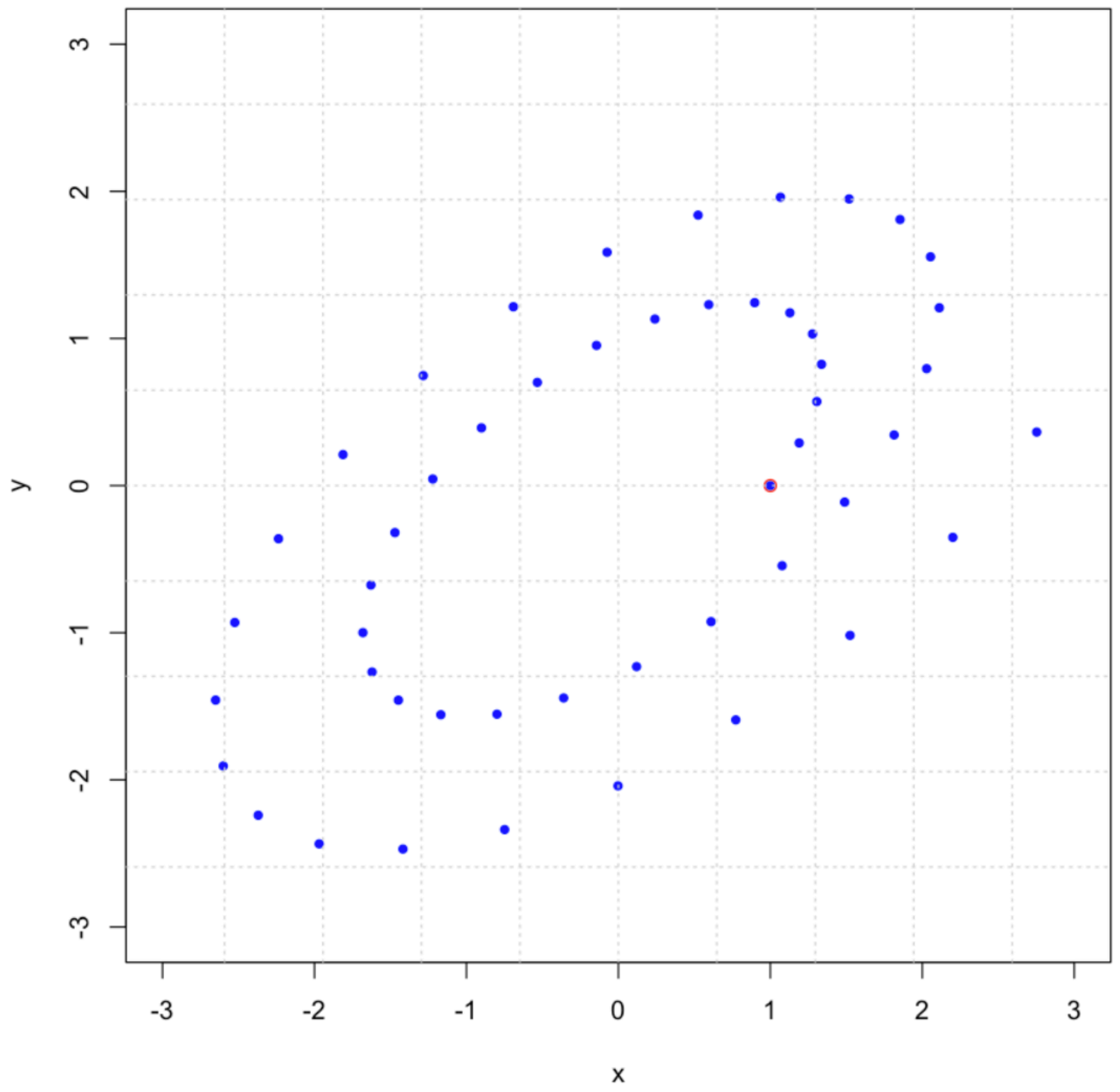
$$A = \begin{bmatrix} 1.19 & -0.39 \\ 0.29 & 0.78 \end{bmatrix}$$

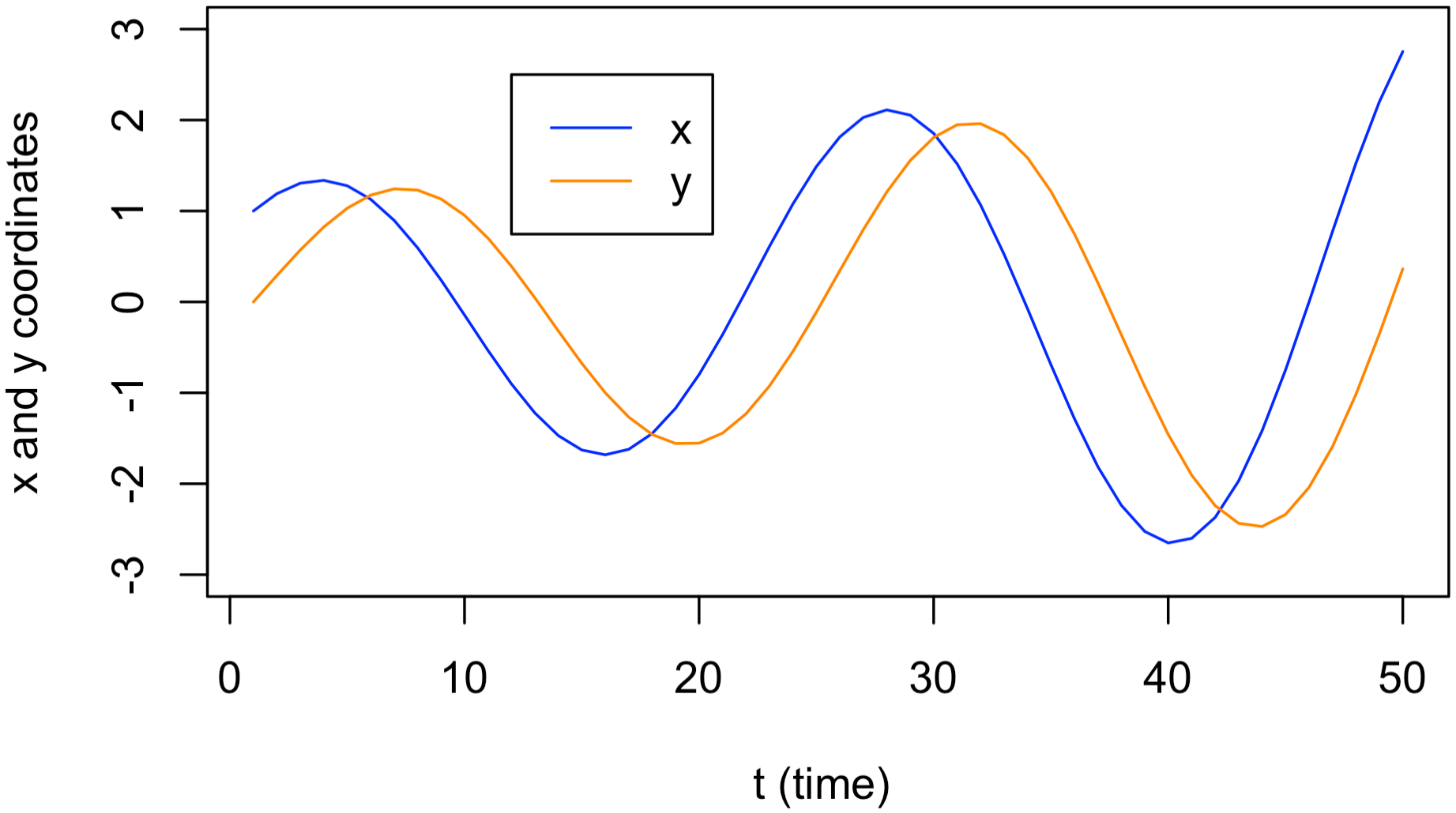
$\lambda_1 = .98 + .26i$ $\quad$ $\lambda_1 = .98 - .26i$

$v_1 = \begin{bmatrix} 0.75 \\ 0.41 \end{bmatrix} + \begin{bmatrix} 0.00 \\ -.52 \end{bmatrix} i$ $\quad$ $v_1 = \begin{bmatrix} 0.75 \\ 0.41 \end{bmatrix} - \begin{bmatrix} 0.00 \\ -.52 \end{bmatrix} i$



Sketch the xy-coordinates of the trajectory





Checkpoint Question for Today

3. Checkpoint: $\mathsf{A} = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$.

$$\begin{vmatrix} 1-\lambda & -1 \\ 1 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda)+1 = 2-3\lambda+\lambda^2+1 = \lambda^2-3\lambda+3$$

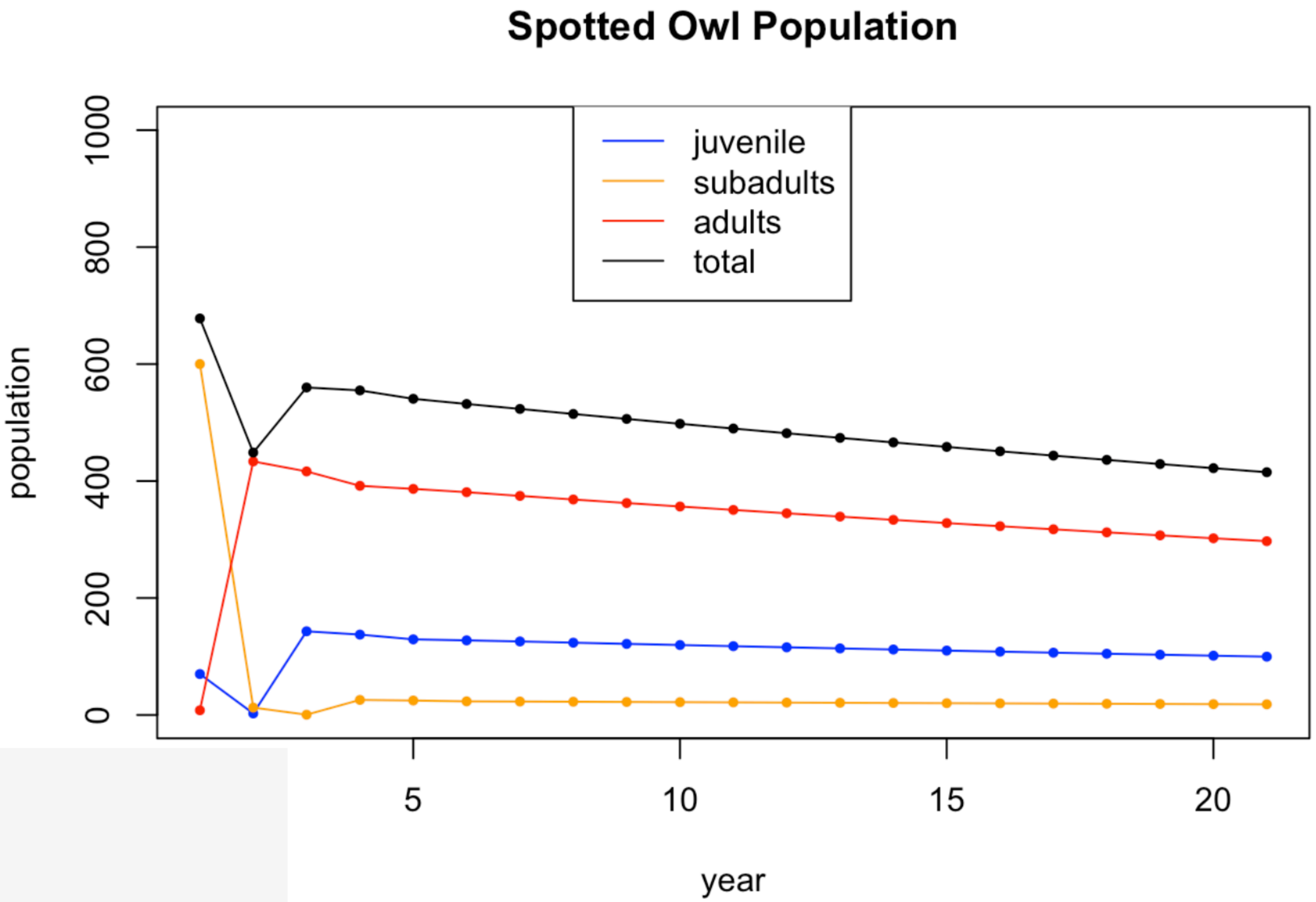
$$\lambda = \frac{3 \pm \sqrt{9-4\cdot 3}}{2} = \frac{3 \pm \sqrt{-3}}{2} = \frac{3}{2} \pm \frac{\sqrt{3}}{2} i \approx 1.5 \pm .866 i$$

$$\|\lambda\| = \sqrt{\left(\tfrac{3}{2}\right)^2 + \left(\tfrac{\sqrt{3}}{2}\right)^2} = \sqrt{\tfrac{9}{4}+\tfrac{3}{4}} = \sqrt{\tfrac{12}{4}} = \sqrt{3} \approx 1.73$$

$$\theta = \arctan\left(\frac{\sqrt{3}/2}{3/2}\right) = \arctan\left(\frac{\sqrt{3}}{3}\right) = \pi/6 \approx .524 \text{ rad} = 30^\circ$$

Northern Spotted Owl

$$\begin{bmatrix} j_{n+1} \\ s_{n+1} \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0.33 \\ 0.18 & 0 & 0 \\ 0 & 0.71 & 0.94 \end{bmatrix} \begin{bmatrix} j_n \\ s_n \\ a_n \end{bmatrix}$$




```
## eigen() decomposition
## $values
## [1]  0.9835927+0.0000000i -0.0217964+0.2059185i -0.0217964-0.2059185i
##
## $vectors
##                 [,1]                   [,2]                   [,3]
## [1,] 0.31754239+0i  0.6820937+0.0000000i  0.6820937+0.0000000i
## [2,] 0.05811107+0i -0.0624124-0.5896338i -0.0624124+0.5896338i
## [3,] 0.94646180+0i -0.0450520+0.4256233i -0.0450520-0.4256233i
```

```
> Mod(eigen(A)$values)
[1] 0.9835927 0.2070688 0.2070688
> Arg(eigen(A)$values)
[1]  0.000000  1.676253 -1.676253

> Arg(eigen(A)$values)/(2*pi)*360
[1]   0.00000  96.04223 -96.04223
```

1. What does this matrix do? $\mathsf{A} = \begin{bmatrix} 0.9 & -0.2 \\ 0.2 & 0.9 \end{bmatrix}$

$$\det\left(\begin{bmatrix} 0.9-\lambda & -0.2 \\ 0.2 & 0.9-\lambda \end{bmatrix}\right) = (0.9-\lambda)^2+0.04 = \lambda^2-1.8\lambda+0.85.$$

$$\lambda = \frac{1.8 \pm \sqrt{1.8^2-4(0.85)}}{2} = \frac{1.8 \pm \sqrt{-.16}}{2} = \frac{1.8 \pm 0.4i}{2} = 0.9 \pm 0.2i.$$

2. What about this one? $\mathsf{A} = \begin{bmatrix} 1.19 & -0.38 \\ 0.29 & 0.78 \end{bmatrix}$

```
## eigen() decomposition
## $values
## [1] 0.985+0.2611034i 0.985-0.2611034i
##
## $vectors
##                     [,1]                [,2]
## [1,] 0.7531030+0.0000000i 0.7531030+0.0000000i
## [2,] 0.4062793-0.5174679i 0.4062793+0.5174679i
```