5.2. Eigenvalues

Eigenvalues and Eigenvectors

eigenvalue $\lambda \in \mathbb{R}$

$$\mathbf{A}\mathrm{v} = \lambda \mathrm{v}$$

An nxn matrix

An **eigenvector**

Given an eigenvector find the eigenvalue

Given that v is an eigenvector of A, find the corresponding eigenvalue.

$$A = \begin{bmatrix} -4 & -3 & -3 \\ 2 & 3 & 1 \\ 4 & 2 & 4 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$$

Multiply:

$$A\mathbf{v} = \begin{bmatrix} -4 & -3 & -3 \\ 2 & 3 & 1 \\ 4 & 2 & 4 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix} = - \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$$

so the eigenvalue is -1.

Given an eigenvalue find the eigenvectors

Given that λ = **2** is an eigenvalue of the matrix A, find all of the eigenvectors of eigenvalue 2.

$$A = \begin{bmatrix} -4 & -3 & -3 \\ 2 & 3 & 1 \\ 4 & 2 & 4 \end{bmatrix}$$

Key idea: $\boxed{A\vec{v} = \lambda\vec{v} \iff (A-\lambda I)\vec{v} = \vec{0} \iff \vec{v} \in Nul(A-\lambda I)}$

$$A - 2I_3 = \begin{bmatrix} -6 & -3 & -3 \\ 2 & 1 & 1 \\ 4 & 2 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 1 & 1 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1/2 & 1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$E_2 = \left\{ s\begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix} \;\middle|\; s,t \in \mathbb{R} \right\} = \text{span}\left\{ \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \right\}$$

How to find an eigenvalue without an eigenvector

If we have neither the eigenvectors nor the eigenvalues of $\mathbf{A}$, then here is a way to find the eigenvalues

Again: $A\vec{v} = \lambda\vec{v} \iff (A-\lambda I)\vec{v} = \vec{0} \iff \vec{v} \in \mathrm{Nul}(A-\lambda I)$

$\iff$ $A-\lambda I$ is not invertible $\iff$ $\underbrace{\det(A-\lambda I)}=0$

$\boxed{\text{This is an } n^{\text{th}} \text{ degree polynomial in variable } \lambda}$

called the "characteristic polynomial"

This means:

- A has n eigenvalues
- possibly with repeats (multiplicity)
- possibly complex

Example

Find the eigenvalues and eigenvectors of A $A = \begin{bmatrix} -2 & 2 \\ 4 & 5 \end{bmatrix}$

1. Eigenvalues

$$\det \begin{bmatrix} -2-\lambda & 2 \\ 4 & 5-\lambda \end{bmatrix} = (-2-\lambda)(5-\lambda)-8$$

$$= -10+2\lambda-5\lambda+\lambda^2-8$$

$$= \lambda^2-3\lambda-18$$

$$= (\lambda-6)(\lambda+3)$$

$$\boxed{\lambda=6} \quad \boxed{\lambda=-3}$$

2. Eigenvectors

$\lambda=6$

$$\begin{bmatrix} -8 & 2 \\ 4 & -1 \end{bmatrix} \to \begin{bmatrix} 4 & -1 \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1/4 \\ 0 & 0 \end{bmatrix}$$

$$4\begin{bmatrix} 1/4 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$E_6 = \text{span}\left(\begin{bmatrix} 1 \\ 4 \end{bmatrix}\right)$$

$\lambda=-3$

$$\begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

$$E_{-3} = \text{span}\left(\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right)$$

3. Check

$$\begin{bmatrix} -2 & 2 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 24 \end{bmatrix} = 6\begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 2 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \end{bmatrix} = -3\begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Example

Find the eigenvalues of the matrices below.

$$A = \begin{bmatrix} -2 & -1 & 1 \\ -3 & -2 & 3 \\ -3 & -1 & 2 \end{bmatrix} \quad \det\begin{pmatrix} -2-\lambda & -1 & 1 \\ -3 & -2-\lambda & 3 \\ -3 & -1 & 2-\lambda \end{pmatrix} = \cdots = -\lambda^3 - 2\lambda^2 + \lambda + 2$$

$$= -(\lambda+2)(\lambda+1)(\lambda-1)$$

The only eigenvalues of **A** are 2, -1, 1

$$B = \begin{bmatrix} -4 & 9 & -3 \\ -6 & 11 & -3 \\ -12 & 18 & -4 \end{bmatrix} \quad \det\begin{pmatrix} -4-\lambda & 9 & -3 \\ -6 & 11-\lambda & -3 \\ -12 & 18 & -4-\lambda \end{pmatrix} = \cdots = -\lambda^3 + 3\lambda^2 - 4 = -(\lambda-2)^2(\lambda+1)$$

The only eigenvalues of **B** are 2, -1

2 has multiplicity 2

$$C = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 4 & 3 & 0 & 0 \\ -2 & 1 & 7 & 0 \\ 3 & 2 & 5 & 1 \end{bmatrix} \quad \det\begin{bmatrix} -2-\lambda & 0 & 0 & 0 \\ 4 & 3-\lambda & 0 & 0 \\ -2 & 1 & 7-\lambda & 0 \\ 3 & 2 & 5 & 1-\lambda \end{bmatrix} = (-2-\lambda)(3-\lambda)(7-\lambda)(1-\lambda)$$

$$\lambda = -2, 3, 7, 1$$

The 0 Eigenspace is the Null Space

$$\mathbf{E}_\lambda = \{ \mathbf{v} \in \mathbb{R}^n \mid \mathbf{A}\,\mathbf{v} = \lambda\,\mathbf{v} \} = \mathbf{Nul}\,(\mathbf{A} - \lambda\,\mathbf{I}).$$

$$\mathbf{E}_0 = \{ \mathbf{v} \in \mathbb{R}^n \mid \mathbf{A}\,\mathbf{v} = 0\,\mathbf{v} = \mathbf{0} \} = \mathbf{Nul}\,(\mathbf{A}).$$

Remember that **A** is invertible if and only if it does not have a nonzero vector in Nul(A)

Thus **A** is invertible if and only if there is no eigenvector of eigenvalue 0

Thus **A** is invertible if and only if 0 is not an eigenvalue

Eigenvalue Equivalences

Theorem

The following are equivalent for an $n \times n$ matrix A

1. λ is an eigenvalue of A
2. $(A - \lambda I)\vec{x} = \vec{0}$ has a nontrivial solution
3. $\mathrm{Nul}(A - \lambda I) \neq \{\vec{0}\}$
4. $A - \lambda I$ is not invertible
5. $\det(A - \lambda I) = 0$

Notice what this says if λ = 0 is an eigenvalue

1. 0 is an eigenvalue of A
2. A x = 0 has a nontrivial solution
3. Nul(A) is not {0}
4. A is not invertible
5. det(A) = 0

Invertible Matrix Theorem Revisited

If A is an *n*x*n* matrix, then the following statements are equivalent

$$A = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

- A is invertible
- $RREF(A) = I_n$
- A has a pivot in every row
- A has a pivot in every column
- $T(x) = Ax$ is one-to-one
- $T(x) = Ax$ is onto
- The columns of A span $\mathbb{R}^n$
- The columns of A are linearly independent
- $Ax = b$ has exactly one solution for all $b \in \mathbb{R}^n$
- $Ax = 0$ has only the 0 solution
- The columns of A are a basis of $\mathbb{R}^n$
- $Col(A) = \mathbb{R}^n$
- $\dim(Col(A)) = \mathbb{R}^n$
- $rank(A) = n$
- $Nul(A) = \{0\}$
- $nullity(A) = 0$
- $\det(A) \neq 0$
- $\lambda = 0$ is not an eigenvalue of A

You Try!

Find the eigenvalues of the matrices below.

$$A = \begin{bmatrix} -7 & -10 \\ 5 & 8 \end{bmatrix}$$

$$B = \begin{bmatrix} 4 & 0 & 0 \\ 3 & -7 & -10 \\ 17 & 5 & 8 \end{bmatrix}$$