Matrix Inverses

$$A = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

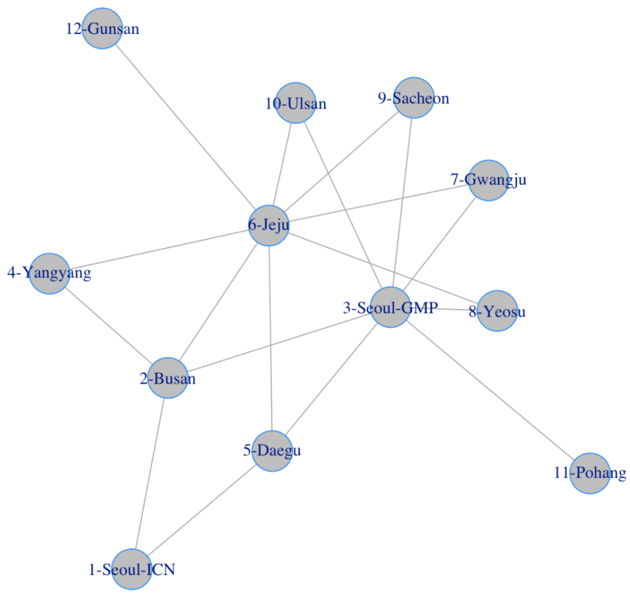
If A is a square $(n \times n)$ matrix then the following statements are equivalent; that is, if one is true, then they all are true, and if one is false, then they all are false.

- A is invertible
- $RREF(A) = I_n$
- A has a pivot in every row
- A has a pivot in every column
- $T(x) = Ax$ is one-to-one
- $T(x) = Ax$ is onto
- The columns of A span $\mathbb{R}^n$
- The columns of A are linearly independent
- $Ax = b$ has exactly one solution for all $b \in \mathbb{R}^n$
- $Ax = 0$ has only the 0 solution

Networks

We've seen a few problems that ask you to interpret the meaning of matrix multiplication: the rental car problem, the rain-sunshine matrix, and the graves and pottery matrix. Here is another problem in which the goal is to interpret the meaning of matrix multiplication.

Here is a graph of the network of domestic airline flights in Korea. They are not located geographically in the plane, but there is a connection if there is a direct flight from one airport to the other.



$$A = \begin{array}{c|c} & \begin{array}{cccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \end{array} \\ \begin{array}{r} 1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12 \end{array} & \begin{bmatrix} 0&1&0&0&1&0&0&0&0&0&0&0\\ 1&0&1&1&0&1&0&0&0&0&0&0\\ 0&1&0&0&1&0&1&1&1&1&1&0\\ 0&1&0&0&0&1&0&0&0&0&0&0\\ 1&0&1&0&0&1&0&0&0&0&0&0\\ 0&1&0&1&1&0&1&1&1&1&0&1\\ 0&0&1&0&0&1&0&0&0&0&0&0\\ 0&0&1&0&0&1&0&0&0&0&0&0\\ 0&0&1&0&0&1&0&0&0&0&0&0\\ 0&0&1&0&0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0 \end{bmatrix} \end{array} \begin{bmatrix} 1\\1\\1\\1\\1\\1\\1\\1\\1\\1\\1\\1 \end{bmatrix} = \begin{bmatrix} \ \\ \ \end{bmatrix}$$

1. The matrix to the right is the *adjacency matrix* of the network. By looking at the connections in the network, decide when this matrix has a 1 and when it has a 0

2. Multiply A by the all 1s vector v as seen above. Do a few entries by hand and the rest in R (or all by hand). You can find the matrix as part of the exam 1 practice problems. What do the entries of the vector Av tell us about the network?

3. Square the matrix, i.e., compute A^2. This amounts to dotting the rows of A with the columns of A. Compute the 6-3 and the 2-2 entry of A^2 by hand (see the back)and do the rest in R. Decide what the entries of A^2 mean in terms of the network.

4. Compute A^2v, where again v is the all 1s matrix. What do the entries of the vector A^2v tell us about the network?

5. Is $A^T = A$? That is, is A equal to its transpose? What does that say about the matrix. When $A = A^T$ we say that the matrix is *symmetric*.

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