Tuesday, Sept 20

1. Welcome!
2. Masking
3. Quiz
4. Math Talk: Prof. Ian Whitehead (Swarthmore) 12:00 - 1:00 pm, 9/22, OLRI 100 } Polyhedral Packing
5. questions?
6. Simplex pt 2
7. Outro
   ↳ finish any unfinished simplex

Welcome to week four of the semester and another day of working through the simplex method! I know we left a few loose ends last time, so let's tie everything together.

$\begin{cases} \text{min/max} & c^T x \\ \text{st} & Ax \le b \end{cases} \xrightarrow{\text{SEF}} \begin{cases} \max & c^T x \\ \text{st} & Ax = b \\ & x \ge 0 \end{cases} \longrightarrow n \overset{m}{\begin{bmatrix} & A & \end{bmatrix}} \begin{bmatrix} x \\ x^{\pm} \\ s \end{bmatrix} = \begin{bmatrix} b \end{bmatrix}$

- slack
- unbnd var.
- max

SEF raises to m columns

$Ax \le b$ defines polyhedron
- intersecting half spaces
- extreme points
  ↳ no conv. comb

‖

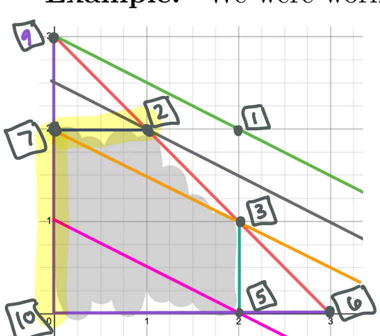
vertices: $d^T y > d^T x$ $\forall y \neq x$, $\exists d$

want to solve an LP? find optimal → optimal values occur @ vertices ⟷ vertices = basic feasible soln: n lin ind. tight constraints.
- if $m > n$, will have $m-n$ free (nonbasic) variables
  ↳ these $= 0$

$\xrightarrow{\text{how to find all?}}$ Pick any n col. (variables) of A
- if $n \times n$ submatrix B is nonsing., it's a basis.
  $x = B^{-1} b$
  ↑ value of basic sol.
- lift x (an n vector) to y (an m vector) by setting all nonbasic var $= 0$
- is $y \ge 0$? basic feasible.

So, in theory, we could do the following: do all the work to find every single vertex (which means checking $\binom{m}{n}$ ~~vertices~~ matrices for nonsingularity and then possibly inverting that many matrices as well) and then checking the objective function for each of them.

Example: We were working through this problem last time.



$$\begin{cases} \max & \frac{1}{2}x_1 + x_2 \\ \text{s.t.} & x_1 + x_2 \le 3 \\ & x_1 \le 2 \\ & x_2 \le 2 \\ & x_i \ge 0 \end{cases} \xrightarrow{SEF} \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}$$

$\frac{1}{2}x_1 + x_2 \rightarrow [\frac{1}{2}, 1, 0, 0, 0]$

$\boxed{1}$ $\{1,2,3\}$: $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ $B^{-1}b = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 \\ 2 \\ -1 \\ 0 \\ 0 \end{bmatrix}$

$\boxed{2}$ $\{1,2,4\}$: $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

$\boxed{3}$ $\{1,2,5\}$: $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$\boxed{4}$ $\{1,3,4\}$: $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ not a basis (x_1)

$\boxed{5}$ $\{1,3,5\}$: $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \rightarrow \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \\ 2 \end{bmatrix}$

$\boxed{6}$ $\{1,4,5\}$ $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} \rightarrow \begin{bmatrix} 3 \\ 0 \\ 0 \\ -1 \\ 2 \end{bmatrix}$

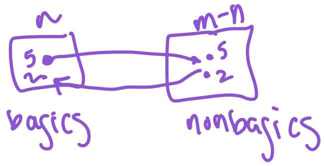
$\boxed{7}$ $\{2,3,4\}$ $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ 2 \\ 1 \\ 2 \\ 0 \end{bmatrix}$

$\boxed{8}$ $\{2,3,5\}$ $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ not a basis (x_2)

$\boxed{9}$ $\{2,4,5\}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ 3 \\ 0 \\ 2 \\ -1 \end{bmatrix}$

$\boxed{10}$ $\{3,4,5\}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ 0 \\ 3 \\ 2 \\ 2 \end{bmatrix}$

This is bad news, in that it's a ton of work. Instead, we'll process the vertices systematically!



swap out basic variables one @ a time & so that our objective gets better

Ok, so how do we do this in general? It's the same sort of steps that we were working on earlier, with the finding how much we can increase a certain variable, but we'll condense the work immensely. We'll do this in two forms, one that uses matrices and one that uses equations. Confusingly, both are sometimes called tableau. Either method will be available to you to solve programs by hand. You'll quickly notice that we won't do a lot of these by hand.

Equations

basis! $\{3,4,5\}$

basic feasible

$$\begin{aligned} s_1 &= \boxed{3} - x_1 - x_2 \\ s_2 &= 2 - x_1 \\ s_3 &= \boxed{2} - x_2 \\ \hline z &= \tfrac{1}{2} x_1 + x_2 \end{aligned}$$

obj. fnctn value @ that spot.

pivot x_2 here!

$$\begin{aligned} s_1 &= 3 - x_1 - (2 - s_3) = 1 - x_1 + s_3 \\ s_2 &= 2 - x_1 \\ x_2 &= 2 - s_3 \\ \hline z &= \tfrac{1}{2} x_1 + (2 - s_3) = 2 + \tfrac{1}{2} x_1 - s_3 \end{aligned}$$

$$\begin{aligned} x_1 &= 1 - s_1 + s_3 \\ s_2 &= 2 - (1 - s_1 + s_3) = 1 + s_1 - s_3 \\ x_2 &= 2 - s_3 \\ \hline z &= 2 + \tfrac{1}{2}(1 - s_1 + s_3) - s_3 \\ &= 2.5 - \tfrac{1}{2} s_1 - \tfrac{1}{2} s_3 \end{aligned}$$

- coeff on variables means done!

Matrix

z (always 1)

$-c$

current obj. val

$$\left[\begin{array}{c|ccccc|c} 1 & -\frac{1}{2} & -1 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 1 & 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 & 2 \end{array}\right]$$

A $\quad$ b

$$\begin{array}{l} R_1 + R_4 \\ R_2 - R_4 \\ \\ \\ \end{array} \left[\begin{array}{c|ccccc|c} 1 & -\frac{1}{2} & 0 & 0 & 0 & 1 & 2 \\ \hline 0 & 1 & 0 & 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 & 2 \end{array}\right]$$

$$\begin{array}{l} R_1 + \frac{1}{2} R_2 \\ \\ R_3 - R_2 \\ \\ \end{array} \left[\begin{array}{c|ccccc|c} 1 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 2.5 \\ \hline 0 & 1 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 2 \end{array}\right]$$

Example: This is the linear program we put on hold last time. Solve this program by pivoting equations.

$$\begin{cases} \max & x_1 + 2x_2 \\ \text{s.t.} & x_1 + x_2 \le 5 \\ & x_1 \le 4 \\ & -x_1 + x_2 \le 1 \\ & x_i \ge 0 \end{cases}$$

$x_1 + x_2 + s_1 = 5$

$x_1 + s_2 = 4$

$-x_1 + x_2 + s_3 = 1$

$$\begin{aligned} s_1 &= 5 - x_1 - x_2 \\ s_2 &= 4 - x_1 \\ s_3 &= 1 + x_1 - x_2 \\ \hline z &= x_1 + 2x_2 \end{aligned} \quad \xrightarrow[\text{R3}]{\text{pivot}} \quad x_2 \text{ @ R3} \quad \begin{aligned} s_1 &= 4 - 2x_1 + s_3 \\ s_2 &= 4 - x_1 \\ x_2 &= 1 + x_1 - s_3 \\ \hline z &= 2 + 3x_1 - 2s_3 \end{aligned}$$

$$\xrightarrow{\text{pivot}} \quad x_1 \text{ @ R1} \quad \begin{aligned} x_1 &= 2 - \tfrac{1}{2}s_1 + \tfrac{1}{2}s_3 \\ s_2 &= 2 + \tfrac{1}{2}s_1 - \tfrac{1}{2}s_3 \\ x_2 &= 3 - \tfrac{1}{2}s_1 + s_3 \\ \hline z &= 8 - \tfrac{3}{2}s_1 - \tfrac{1}{2}s_3 \end{aligned}$$

* practice? do it @ x_1 to start.

Example: Solve the linear program below by using the matrix method.

$$\max\{10 + c^T x : Ax = b, x \geq 0\}$$

$$A = \begin{bmatrix} 1 & 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 1 & -1/2 & 0 \\ 0 & 3/2 & 0 & 1/2 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 5 \\ 1 \\ 9 \end{bmatrix} \quad c = \begin{bmatrix} 0 \\ 2 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{c|ccccc|c} 1 & 0 & -2 & 0 & 1 & 0 & 10 \\ \hline 0 & 1 & 1/2 & 0 & 1/2 & 0 & 5 \\ 0 & 0 & 1/2 & 1 & -1/2 & 0 & 1 \\ 0 & 0 & 3/2 & 0 & 1/2 & 1 & 9 \end{array}\right] \xrightarrow[\text{@ } R_2]{\text{pivot } x_2} \left[\begin{array}{c|ccccc|c} 1 & 0 & 0 & 4 & -1 & 0 & 14 \\ \hline 0 & 1 & 0 & -1 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 & -1 & 0 & 2 \\ 0 & 0 & 0 & -3 & \boxed{2} & 1 & 6 \end{array}\right]$$

$$\xrightarrow[\text{@ } R3]{\text{pivot } x_4} \left[\begin{array}{c|ccccc|c} 1 & 0 & 0 & 5/2 & 0 & 1/2 & 17 \\ \hline 0 & 1 & 0 & 1/2 & 0 & -1/2 & 1 \\ 0 & 0 & 1 & 1/2 & 0 & \frac{1}{2} & 3 \\ 0 & 0 & 0 & -3/2 & 1 & 1/2 & 5 \end{array}\right]$$

Example: We're not going to do the simplex method on this question, but rather discuss how we could get up to the starting line of the simplex method. Consider the linear program below.

$$\begin{cases} \max & 6x_1 + x_2 \\ & 2x_1 + x_2 \leq 18 \\ \text{s. t.} & x_2 \leq 8 \\ & x_1 + x_2 \geq 10 \\ & x_1, x_2 \geq 0 \end{cases}$$

This is going to need some work to bring it to standard equality form. Go ahead and do that first.

$$\begin{aligned} 2x_1 + x_2 + s_1 &= 18 \\ x_2 + s_2 &= 8 \\ x_1 + x_2 - s_3 &= 10 \end{aligned} \longrightarrow \begin{bmatrix} 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 18 \\ 8 \\ 10 \end{bmatrix}$$

When we work through the simplex method, we need an initial basic feasible solution. Why can't we do this right away, and how can we fix it?

add an artificial variable to last eq.

$$\begin{bmatrix} 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \\ s_3 \\ a_1 \end{bmatrix}$$

Example: The following linear program is (spoiler) unbounded. Put it into either form, attempt to simplex, and identify where things go awry.

$$\max\{c^T x : Ax = b, x \geq 0\}$$

$$A = \begin{bmatrix} -2 & 4 & 1 & 0 & 1 \\ -3 & 7 & 0 & 1 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad c = \begin{bmatrix} -1 \\ 3 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{aligned} x_3 &= 1 + 2x_1 - 4x_2 - x_5 \\ x_4 &= 3 + 3x_1 - 7x_2 - x_5 \\ \hline z &= -x_1 + 3x_2 + x_5 \end{aligned}$$

pivot $\rightarrow$ x_5 @ R_1

$$\begin{aligned} x_5 &= 1 + 2x_1 - 4x_2 - x_3 \\ x_4 &= 3 + 3x_1 - 7x_2 - (1 + 2x_1 - 4x_2 - x_3) \\ \hline z &= -x_1 + 3x_2 + (1 + 2x_1 - 4x_2 - x_3) \end{aligned}$$

rewrite $\rightarrow$

$$\begin{aligned} x_5 &= 1 + 2x_1 - 4x_2 - x_3 \\ x_4 &= 2 + x_1 - 3x_2 + x_3 \\ \hline z &= x_1 - x_2 - x_3 \end{aligned}$$

but no bounds on how much we can increase it!

we want to pivot x_1

Certificate!

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}$$

Generalizing: can you summarize the sort of of thing you're looking for, either in matrix or equation form, that indicates an unbounded program?