Row Space. The *row space*, Row(A), is the span of the rows of A.

1. Consider the row space:

$$S = \text{Row}\left(\begin{bmatrix} 0 & 1 & 5 \\ 2 & -6 & -32 \\ -2 & 8 & 42 \end{bmatrix}\right) = \text{span}\left\{\begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ -6 \\ -32 \end{bmatrix}, \begin{bmatrix} -2 \\ 8 \\ 42 \end{bmatrix}\right\} \subseteq \mathbb{R}^3.$$

How do row operations change the row space?

$$\underset{A_1}{\boxed{\begin{bmatrix} 0 & 1 & 5 \\ 2 & -6 & -32 \\ -2 & 8 & 42 \end{bmatrix}}} \longleftrightarrow \underset{A_2}{\begin{bmatrix} 2 & -6 & -32 \\ 0 & 1 & 5 \\ -2 & 8 & 42 \end{bmatrix}} \xleftrightarrow{\frac{1}{2}} \underset{A_3}{\begin{bmatrix} 1 & -3 & -16 \\ 0 & 1 & 5 \\ -2 & 8 & 42 \end{bmatrix}} \longleftrightarrow \begin{bmatrix} 1 & -3 & -16 \\ 0 & 1 & 5 \\ 0 & 2 & 10 \end{bmatrix} \longleftrightarrow \begin{bmatrix} 1 & -3 & -16 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix} \longleftrightarrow \boxed{\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix}}$$

row3 ← row3+2 row1

$$\mathcal{B} = \left\{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix}\right\}$$

0's-1's std basis property

Key Points:

- row operations do not change the row space
- Row(A) = Row(RREF(A))
- Basis for Row(A) is the set of nonzero rows in RREF(A)
- dim(Row(A)) = # pivots(A) = rank(A)

row operations mess up the column space

The tell us about relations among the columns

$$C_3 = (-1)C_1 + 5C_2$$

2. Here is the transpose of the matrix above row reduced. How does this compare to the row space above?

$$\begin{bmatrix} 0 & 2 & -2 \\ 1 & -6 & 8 \\ 5 & -32 & 42 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Application to finding bases. Done on the board

3. Let $S \subseteq \mathbb{R}^4$ be the span of the vectors below. Find the dimension of S and find a basis for S.

$$S = \mathsf{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ -3 \\ -9 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 5 \\ 13 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(a) <u>Method 1</u>: put the vectors of S into the columns of a matrix A and row reduce:

$$A = \begin{bmatrix} 1 & 1 & -1 & 2 & 1 \\ 1 & -1 & -3 & 4 & 0 \\ 2 & 1 & -3 & 5 & 0 \\ 4 & -1 & -9 & 13 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -2 & 3 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) <u>Method 2</u>: put the vectors of S into the ~~columns~~ rows! of a matrix A and row reduce:

$$A^T = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 1 & -1 & 1 & -1 \\ -1 & -3 & -3 & -9 \\ 2 & 4 & 5 & 13 \\ 1 & 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & \frac{7}{3} \\ 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(c) Is the vector $\vec{\mathsf{v}}$ in S?

$$\vec{\mathsf{v}} = \begin{bmatrix} 5 \\ -9 \\ 12 \\ -12 \end{bmatrix}$$

(d) $\dim(S) = \mathsf{rank(A)} = \mathsf{rank}(A^T)$.

You Try!

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 8 & 0 & 0 \\ 1 & 1 & 2 & 0 & 2 \\ 0 & 2 & -6 & 1 & 3 \\ 0 & 2 & -6 & 2 & 4 \\ 2 & 0 & 10 & 0 & 2 \\ 2 & 0 & 10 & -2 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 5 & 0 & 1 \\ 0 & 1 & -3 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{c} 6 \\ \mathbf{A}^T = \begin{bmatrix} 1 & 1 & 0 & 0 & 2 & 2 \\ -1 & 1 & 2 & 2 & 0 & 0 \\ 8 & 2 & -6 & -6 & 10 & 10 \\ 0 & 0 & 1 & 2 & 0 & -2 \\ 0 & 2 & 3 & 4 & 2 & 0 \end{bmatrix} \\ 5 \end{array} \longrightarrow \begin{array}{cccccc} & & & s & t & u \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array} \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & -1 & 1 & 3 \\ 0 & 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Fill in the blanks:

a) rank($\mathbf{A}$) = 3 i) Col($\mathbf{A}$) is a 3 dimensional subspace of $\mathbb{R}^6$

b) nullity($\mathbf{A}$) = 2 ii) Nul($\mathbf{A}$) is a 2 dimensional subspace of $\mathbb{R}^5$

c) rank($\mathbf{A}^T$) = 3 iii) Row($\mathbf{A}$) is a 3 dimensional subspace of $\mathbb{R}^5$

d) nullity($\mathbf{A}^T$) = 3 iv) Null($\mathbf{A}^T$) is a 3 dimensional subspace of $\mathbb{R}^6$