Dynamical Systems: If A is an $n \times n$ matrix, and $\vec{x}_0 \in \mathbb{R}^n$ we can create a *dynamical system* of the form

$$\vec{x}_0, \quad \vec{x}_1, \quad \vec{x}_2, \quad \vec{x}_3, \quad \ldots \qquad \text{where} \quad \vec{x}_{t+1} = A\vec{x}_t \quad \text{(recursive definition)}$$

Many applications are done this way, with t typically representing time: population dynamics, weather models, financial models, historical processes, pharmacology, ...

Note: we get to $\vec{x}_k$ by applying A to $\vec{x}_0$ k times. Thus: $\vec{x}_k = A^k x_0$.

1. **Today's CheckPoint**. A dynamical system is defined by the recursive rule $x_{k+1} = A\vec{x}_k$ with matrix A and initial value $\vec{x}_0$:

$$A = \begin{bmatrix} 97/100 & 3/55 \\ -4/55 & 123/100 \end{bmatrix}, \qquad \vec{x}_0 = \begin{bmatrix} 1 \\ 15 \end{bmatrix}$$

Can generate some values

$$\underset{\vec{x}_0}{\begin{bmatrix} 1 \\ 15 \end{bmatrix}}, \quad \underset{\vec{x}_1}{\begin{bmatrix} 1.7 \\ 16.7 \end{bmatrix}}, \quad \underset{\vec{x}_2}{\begin{bmatrix} 2.41 \\ 18.55 \end{bmatrix}}, \quad \underset{\vec{x}_3}{\begin{bmatrix} 3.137 \\ 20.567 \end{bmatrix}}, \quad \underset{\vec{x}_4}{\begin{bmatrix} \quad \\ \quad \end{bmatrix}}, \quad \ldots \quad \underset{\vec{x}_k}{\begin{bmatrix} \quad \\ \quad \end{bmatrix}}, \quad \ldots$$

A "closed formula" is a formula for the kth term that can be computed using k and doesn't require us to compute all of the values up to $\vec{x}_k$ first.

The *eigenvectors* and *eigenvalues* of A are what we need:

$$\lambda_1 = 1.1 \quad \lambda_2 = 0.9$$
$$\vec{v}_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

step 1: Write $\vec{x}_0$ as a linear combination of the eigenbasis (how?):

① augment and row reduce
② use the inverse of $\begin{bmatrix} 1 & 3 \\ 4 & 1 \end{bmatrix}$
③ Guess and check

$$\begin{bmatrix} 1 \\ 15 \end{bmatrix} = \underline{4} \begin{bmatrix} 1 \\ 4 \end{bmatrix} + \underline{-1} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

step 2: Apply A k times:

$$\vec{x}_k = \begin{bmatrix} a_k \\ b_k \end{bmatrix} = A^k \vec{x}_0 = A^k \begin{bmatrix} 1 \\ 15 \end{bmatrix} = \underline{4}\, A^k \begin{bmatrix} 1 \\ 4 \end{bmatrix} + \underline{(-1)}\, A^k \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \underline{4\,(1.1)^k} \begin{bmatrix} 1 \\ 4 \end{bmatrix} + \underline{-(.9)^k} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Thus,

$$a_k = 4\,(1.1)^k - 3\,(.9)^k$$

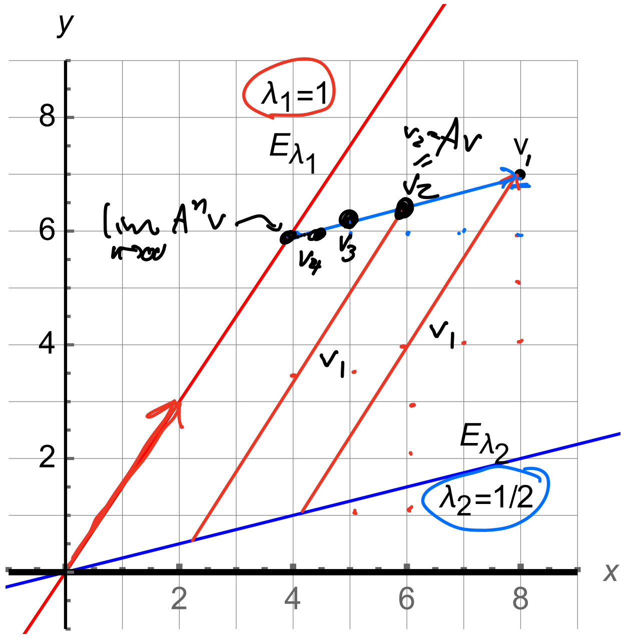
$$b_k = 16\,(1.1)^k - (.9)^k$$

Alternatively: diagonalize: $A = PDP^{-1} = \begin{bmatrix} 1 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1.1 & 0 \\ 0 & .9 \end{bmatrix} \left(-\frac{1}{11}\right) \begin{bmatrix} 1 & -3 \\ -4 & 1 \end{bmatrix}$

$$A^k \begin{bmatrix} 1 \\ 15 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1.1^k & 0 \\ 0 & .9^k \end{bmatrix} \underbrace{\left(-\frac{1}{11}\right) \begin{bmatrix} 1 & -3 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 15 \end{bmatrix}} = \begin{bmatrix} 1 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1.1^k & 0 \\ 0 & .9^k \end{bmatrix} \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 4(1.1)^k \\ -(.9)^k \end{bmatrix} = 4\,(1.1)^k \begin{bmatrix} 1 \\ 4 \end{bmatrix} - (.9)^k \begin{bmatrix} 3 \\ 1 \end{bmatrix} \text{ as above}$$

2. A is a matrix with eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 1/2$ and corresponding eigenspaces shown below. The vector $\vec{v}$ is shown on the plot at position $(8, 7)$. Using the eigeninformation, compute the position of $A\vec{v}$ and plot it on the graph. Compute it exactly if possible; if not, estimate it.



Av

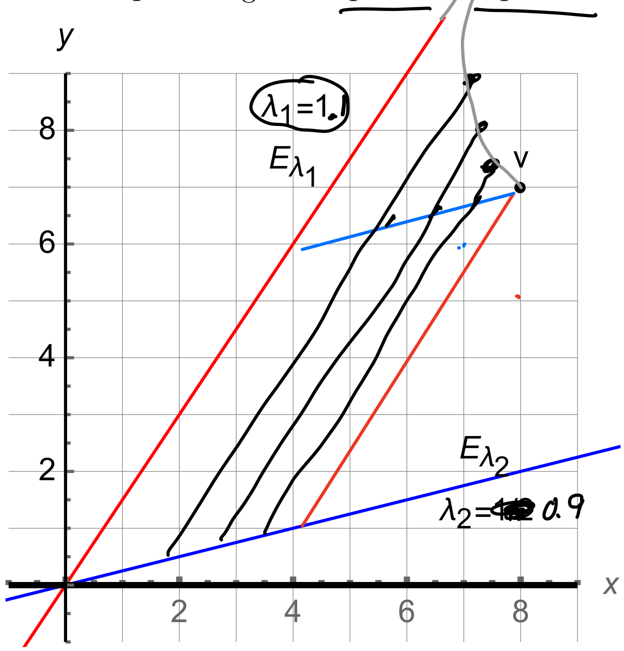
$V = V_1 + V_2$

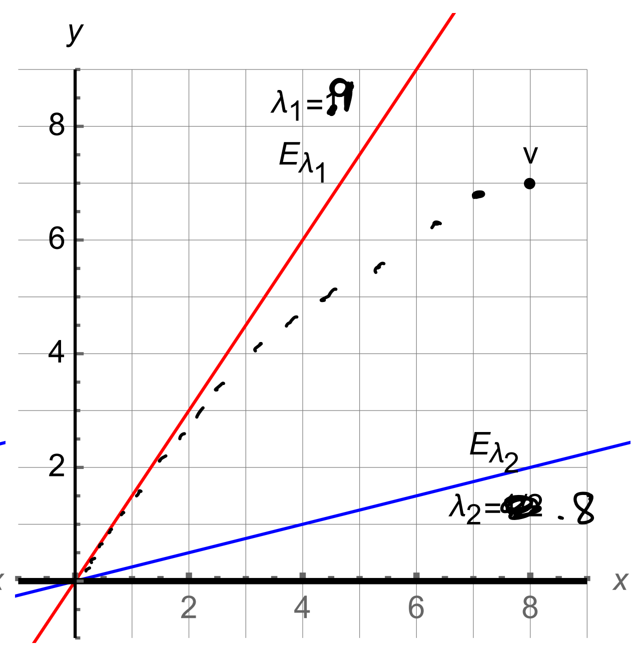
$Av = Av_1 + Av_2$

$= V_1 + \frac{1}{2} V_2$

On this same problem, where will $A^2\vec{v}$ be?

3. What if λ_1 is changed to $\lambda_1 = 1.1$ or $\lambda_1 = 0.9$?





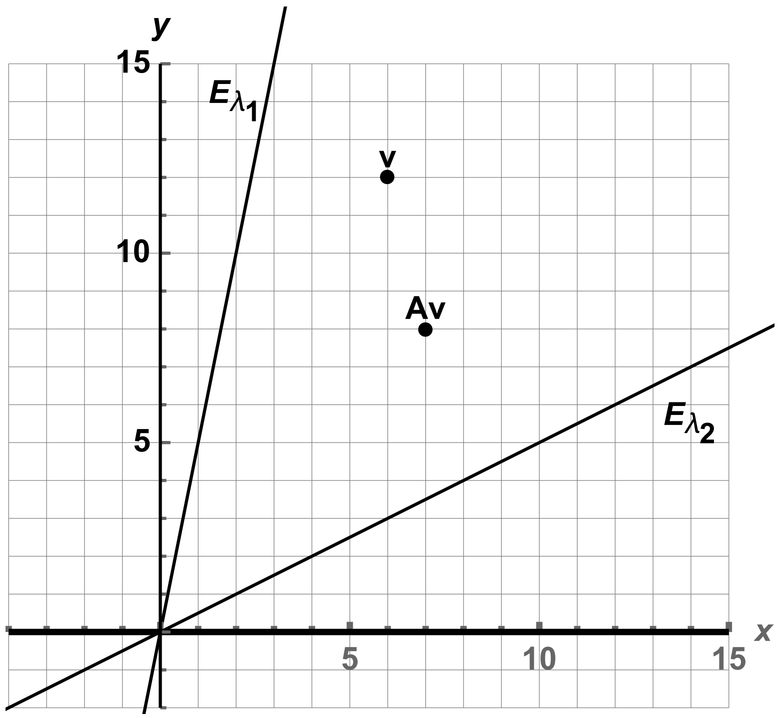
4. A is a matrix that sends $\vec{\mathsf{v}}$ to $\mathsf{A}\vec{\mathsf{v}}$ as shown in the plot below with its two eigenspaces E_{λ_1} and E_{λ_2}.

   (a) Estimate, as accurately as possible from the given information, the eigenvalues $\lambda_1 =$ ____ and $\lambda_2 =$ ____.

   (b) Indicate on the plot above where $\mathsf{A}^2\vec{\mathsf{v}}$ will be.

   (c) What happens in the limit: $\lim_{n\to\infty} \mathsf{A}^n\vec{\mathsf{v}}$?

   (d) If $\mathsf{A}^n\vec{\mathsf{v}} = \begin{bmatrix} x_n \\ y_n \end{bmatrix}$ what happens to the ratio x_n/y_n as n grows larger and larger?



5. **Dominant Eigenvectors** It is very common for an $n \times n$ matrix A to have an eigenvalue λ_1 that is bigger, in absolute value, than all of the other eigenvalues.

$$A \qquad |\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n|$$

$$x = c_1 v_1 + c_2 v_2 + c_3 v_3 + \cdots + c_n v_n$$

$$A^n x = c_1 \lambda_1^n v_1 + c_2 \lambda_2^n v_2 + c_3 \lambda_3^n v_3 + \cdots + c_n \lambda_n^n v_n$$

$$= \lambda_1^n \left[c_1 v_1 + \underbrace{c_2 \left(\frac{\lambda_2}{\lambda_1}\right)^n v_2 + c_3 \left(\frac{\lambda_3}{\lambda_1}\right)^n v_3 + \cdots + c_n \left(\frac{\lambda_n}{\lambda_1}\right)^n v_n}_{\to 0} \right]$$

$$\approx \lambda_1^n c_1 v_1$$

converges to the direction of the dominant eigenvector

6. Look at the Northern Spotted Owl example on R: