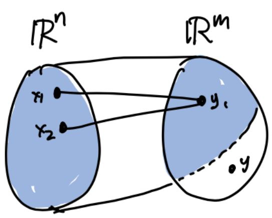
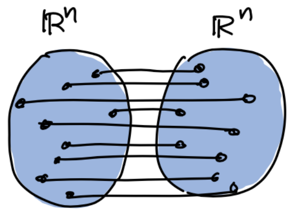
Matrix Inverses

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is invertible if and only if it is both *one-to-one* and *onto*. What does it say about n and m, pivots, and RREF?



To be invertible can't have this



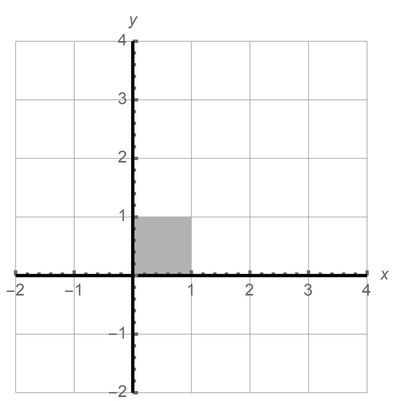
must be one-to-one and onto

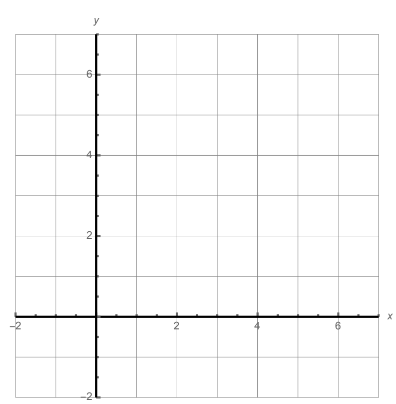
1. There is a nice formula for the inverse of a 2×2 matrix that you should know: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} =$

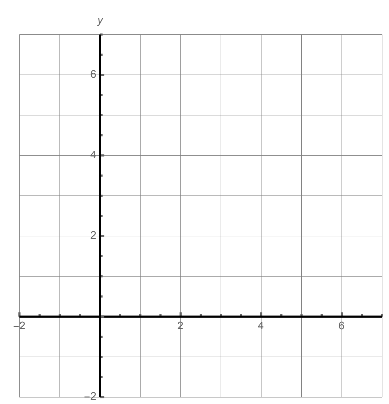
   One of these matrices is invertible and one is not find the invertible one and find its inverse. What can you say about the columns of the one that is not invertible?

   $\begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix}^{-1} =$ $\qquad\qquad$ $\begin{bmatrix} 2 & 2 \\ 5 & 6 \end{bmatrix}^{-1} =$

   Where does the unit square map in each of these transformations?







2. The most computationally efficient way to find the inverse of a 3×3 or larger matrix, is to augment with the identity matrix and row reduce. For example, to find the inverse of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$ we do the following.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

Check:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix} \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix} =$$

Use the inverse to solve $Ax = b$.

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Compare with:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 5 & 3 & 2 \\ 1 & 0 & 8 & 3 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & -3 & 0 \\ 0 & -2 & 5 & 2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & -1 & 2 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & -2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} & 2 & 0 & 7 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & -2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 19 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

3. Properties of matrix inverses. If A and B are invertible matrices.

   (a) $AA^{-1} =$

   (b) $A^{-1}A =$

   (c) Is AB invertible? If so, what is $(AB)^{-1} =$

   (d) Is $A + B$ invertible? If so, what is $(A + B)^{-1} =$

   Hint: on the last one, consider the following example $\begin{bmatrix} 2 & 2 \\ 5 & 6 \end{bmatrix} + \begin{bmatrix} 2 & -2 \\ 5 & -6 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 10 & 0 \end{bmatrix}$

4. Given that A, B, C are invertible matrices, solve the following matrix equation for X

$$C(A + X)B^{-1} = I$$

5. Given the matrices and vectors below

```
A = cbind(c(1,2,1),c(2,5,0),c(3,3,8))
B = cbind(c(1,-1,2),c(3,1,1),c(-1,-1,2))
b = c(1,2,3)
```

   (a) Use `solve` to solve $Ax = b$
   (b) Also use `solve` to find A^{-1} and compute $x = A^{-1}b$.
   (c) Show that $(AB)^{-1} = B^{-1}A^{-1}$.