Tuesday, Sept 27

1. Welcome!
2. Topics deadline will move, list up tomorrow!
3. Homework 1 looked good!
   - ↳ biggest thing: proofs
   - ↳ check solutions
4. Questions?
5. Duality!
6. Outro
   - ↳ Small work:

We finish up our discussion of linear programs with possibly the coolest idea about them: duality. We'll start with an example which motivates the remainder of our discussion.

Example: Consider the linear program below. Without solving it, can you upper bound what the objective might be?

$$(P)\begin{cases} \max & 2x_1 + 3x_2 \\ \text{s. t.} & 4x_1 + 8x_2 \le 12 \\ & 2x_1 + x_2 \le 3 \\ & 3x_1 + 2x_2 \le 4 \\ & x_i \ge 0 \end{cases}$$

$2x_1 + 3x_2 \le 4x_1 + 8x_2 \le 12$

$2x_1 + 3x_2 \le \frac{1}{2}(4x_1 + 8x_2) \le \frac{1}{2}(12) = 6$

$$\begin{array}{rl} & 4x_1 + 8x_2 \le 12 \\ + & 2x_1 + x_2 \le 3 \\ \hline \frac{1}{3} & (6x_1 + 9x_2 \le 15) \\ & 2x_1 + 3x_2 \le \underline{\underline{5}} \end{array}$$

Ok, so wait a minute... we're trying to minimize the upper bound, which is built out of a certain number of copies of 12, 3, and 4. However, we still need it to be an upper bound, which means the number of copies of x_1 and x_2 needs to be something too...

$$D\begin{cases} \min & 12y_1 + 3y_2 + 4y_3 \\ & 4y_1 + 2y_2 + 3y_3 \ge 2 \\ & 8y_1 + y_2 + 2y_3 \ge 3 \\ & y_i \ge 0 \end{cases}$$

Whoa, a wild linear program appeared! Not the same, but definitely related! This new program provides an upper bound on P, while P provides a lower bound on D. Why?

because if D bounds P from above, just flip perspective!

And why do the y_i need to be positive?

if y_i is a scalar on constraints of P, $-y_i$ flips constraints.

It turns out that for any *primal* program P, we can write a *dual* program D with the following rules. Suppose $A \in \mathbb{R}^{m\times n}$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$. Then

	Max Primal P	Min Dual D
variables	$x_1 \dots x_n$	$y_1 \dots y_m$
matrix	A	A^T
right side	b	c
objective	$\max c^T x$	$\min b^T y$
contraints	ith constraint $\leq$	$y_i \geq 0$
	ith constraint $\geq$	$y_i \leq 0$
	ith constraint $=$	$y_i \in \mathbb{R}$
	$x_i \geq 0$	ith constraint $\geq$
	$x_i \leq 0$	ith constraint $\leq$
	$x_i \in \mathbb{R}$	ith constraint $=$

Example: Find the dual program to the program below.

$$\begin{cases} \max & 12x_1 + 26x_2 + 20x_3 \\ \text{s.t.} & \begin{bmatrix} 1 & 2 & 1 \\ 4 & 6 & 5 \\ 2 & -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{matrix} \geq \\ \leq \\ = \end{matrix} \begin{bmatrix} -2 \\ 2 \\ 13 \end{bmatrix} \\ & x_1, x_3 \geq 0,\ x_2 \text{ free} \end{cases}$$

$$\begin{cases} \min & -2y_1 + 2y_2 + 13y_3 \\ \text{st} & \begin{bmatrix} 1 & 4 & 2 \\ 2 & 6 & -1 \\ 1 & 5 & -3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \begin{matrix} \geq \\ = \\ \geq \end{matrix} \begin{bmatrix} 12 \\ 26 \\ 20 \end{bmatrix} \end{cases}$$

$$y_1 \leq 0,\ y_2 \geq 0\quad y_3 \in \mathbb{R}$$

The part of that table we haven't totally justified is the equality $\leftrightarrow$ free variable idea. Let's sketch that out now.

$$\begin{cases} \max c^T x \\ \text{st} \quad Ax = b \\ \quad x \geq 0 \end{cases} \to \begin{cases} \max & c^T x \\ \text{st} & Ax \leq b \\ & -Ax \leq -b \\ & x \geq 0 \end{cases} \to \begin{cases} \max & c^T x \\ \text{st} & \begin{bmatrix} A \\ -A \end{bmatrix} x \leq \begin{bmatrix} b \\ -b \end{bmatrix} \\ & x \geq 0 \end{cases}$$

$Ax \geq b \to -Ax \leq -b$

$Ax \leq b$

$$\xrightarrow{\text{dual!}} \begin{cases} \min & \begin{bmatrix} b^T & -b^T \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \\ & \begin{bmatrix} A^T & -A^T \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \geq c \\ & u, v \geq 0 \end{cases} \xrightarrow[y = u - v]{\text{let}} \begin{cases} \min & b^T y \\ & A^T y \geq c \\ & y \in \mathbb{R}^n \end{cases}$$

Example: Consider the linear program below.

a. Construct the dual program.

b. There are five basic solutions (not all feasible) for both programs. Find each one (you may want to use SAGE or graph them out) and find the objective at each.

$$(P)\begin{cases} \max & 3x_1 + 2x_2 \\ \text{s. t.} & 2x_1 + x_2 \leq 10 \\ & x_2 \leq 6 \\ & x_i \geq 0 \end{cases}$$

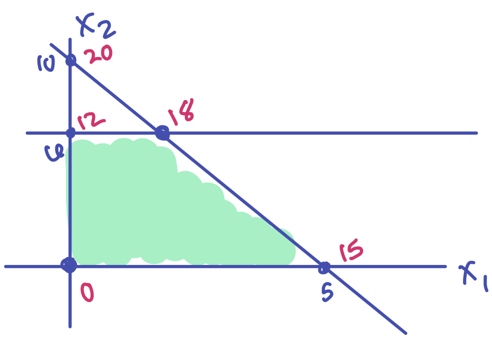
$$D\begin{cases} \min & 10y_1 + 6y_2 \\ \text{st} & 2y_1 \geq 3 \\ & y_1 + y_2 \geq 2 \\ & y_i \geq 0 \end{cases}$$

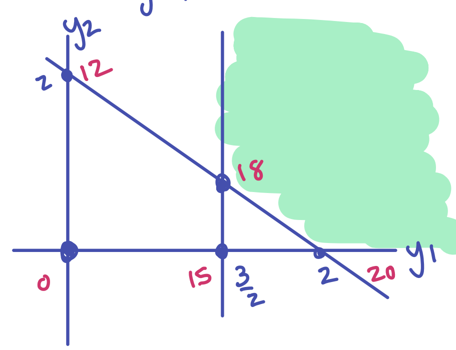
SEF

$$2x_1 + x_2 + s_1 = 10$$
$$x_2 + s_2 = 6$$
$$x_i, s_i \geq 0$$

$$2y_1 - s_1 = 3$$
$$y_1 + y_2 - s_2 = 2$$
$$y_i, s_i \geq 0$$

Graph





Primal	Feasible	Value	Feasible	D val
(0,0,10,6)	y	0	N	(0,0,−3,−2)
(0,1,4,0)	y	12	N	(0,2,−3,0)
(5,0,0,6)	y	15	N	$(\frac{3}{2}, 0, 0, -\frac{1}{2})$
(2,6,0,0)	y	18	y	$(\frac{3}{2}, \frac{1}{2}, 0, 0)$
(0,10,0,−4)	N	20	y	(2,0,1,0)

Example: One of the first programs we wrote had to do with meeting the percent daily value cheaply. The values of three vitamins is shown below for three foods, and I've written the primal program below.

	A	C	K	\$
Apples	60	26	6	1.53
Bananas	3	33	1	0.37
Cucumbers	2	7	12	0.18

$$\begin{cases} \overset{\max}{\cancel{\min}}\ -1.53x_1 \cancel{+} 0.37x_2 \cancel{+} 0.18x_3 \\ \text{s.t.} \quad 60x_1 + 3x_2 + 2x_3 \geq 100 \\ 26x_1 + 33x_2 + 7x_3 \geq 100 \\ 6x_1 + x_2 + 12x_3 \geq 100 \\ x_i \geq 0 \end{cases}$$

y_i : cost/% daily value.

Construct the dual program and *interpret* what the variables mean.

$$\begin{cases} \min\ 100y_1 + 100y_2 + 100y_3 \\ \text{st} \quad 60y_1 + 26y_2 + 6y_3 \geq -1.53 \\ 3y_1 + 33y_2 + y_3 \geq -0.37 \\ 2y_1 + 7y_2 + 12y_3 \geq -0.18 \\ y_i \leq 0 \end{cases} \rightarrow \begin{cases} \max\ 100y_1 + 100y_2 + 100y_3 \\ \min\ -100y_1 - 100y_2 - 100y_3 \\ 60y_1 + 26y_2 + 6y_3 \leq 1.53 \\ 3y_1 + 33y_2 + y_3 \leq 0.37 \\ 2y_1 + 7y_2 + 12y_3 \leq 0.18 \\ y_i \geq 0 \end{cases}$$

Contextualize the dual program.

Pill makers problem: s/p I want to maximize how much I sell a vitamin pill for that meets all values yet is cheaper than the foods itself.

So we've got a way to form a dual program, and we're pretty sure each bounds the other. Let's make that formal with the idea of *weak* duality.

Theorem (4.1)**.** *Let P and D be a primal-dual pair with standard notation of A, b and c. Let $\bar{x}$ and $\bar{y}$ be feasible solutions to P and D, respectively. Then*

- ☆! $c^T\bar{x} \leq b^T\bar{y}$, *and*
- *if $c^T\bar{x} = b^T\bar{y}$, then $\bar{x}$ and $\bar{y}$ are optimal solutions to P and D, respectively.*

Proof: Suppose P is the primal.

$$P \begin{cases} \max\ c^T x \\ \text{st} \\ A_i x \leq b_i & \forall i \in R_1 \\ A_i x \geq b_i & \forall i \in R_2 \\ A_i x = b_i & \forall i \in R_3 \\ x_j \geq 0 & \forall j \in C_1 \\ x_j \leq 0 & \forall j \in C_2 \\ x_j \text{ free} & \forall j \in C_3 \end{cases} \qquad D \begin{cases} \min\ b^T y \\ \mathrm{col}_j(A)^T y \geq c_j & \forall j \in C_1 \\ \mathrm{col}_j(A)^T y \leq c_j & \forall j \in C_2 \\ \mathrm{col}_j(A)^T y = c_j & \forall j \in C_3 \\ y_i \geq 0 & \forall i \in R_1 \\ y_i \leq 0 & \forall i \in R_2 \\ y_i \text{ free} & \forall i \in R_3 \end{cases}$$

row i of matrix A

space to most likely continue the proof...

$$P_S = \begin{cases} \max \; c^T x \\ \text{st} \;\; Ax + s = b \; * \\ \quad s_i \geq 0 \quad i \in R_1 \\ \quad s_i \leq 0 \quad i \in R_2 \\ \quad s_i = 0 \quad i \in R_3 \\ \quad x_j \geq 0 \quad j \in C_1 \\ \quad x_j \leq 0 \quad j \in C_2 \\ \quad x_j \text{ free} \quad j \in C_3 \end{cases} \qquad D_W = \begin{cases} \min \; b^T y \\ \text{st} \;\; A^T y + w = c \\ \quad w_j \leq 0 \quad j \in C_1 \\ \quad w_j \geq 0 \quad j \in C_2 \\ \quad w_j = 0 \quad j \in C_3 \\ \quad y_i \geq 0 \quad i \in R_1 \\ \quad y_i \leq 0 \quad i \in R_2 \\ \quad y_i \text{ free} \quad i \in R_3 \end{cases}$$

if $\bar{x}$ is a feas. sol for P, then $\bar{x}, \bar{s} = b - A\bar{x}$ is a solution for P_S. Similarly with $\bar{y}$ creating a feas. soln for D_W with $\bar{y}, \bar{w} = c - A^T\bar{y}$

aside

$(x^T y)^T = y^T x$

$A^T y + w = c$

$(A^T y)^T = (c-w)^T$

$y^T A = (c-w)^T$

$$y^T b = y^T(Ax + s) = y^T A x + y^T s = (c-w)^T x + y^T s = c^T x - \underbrace{\overbrace{\overbrace{w^T x}^{\leq 0} + \overbrace{y^T s}^{\geq 0}}^{\geq 0}}_{\geq 0}$$

$$w^T x = \underbrace{\sum_{C_1} w_j x_j}_{\leq 0} + \underbrace{\sum_{C_2} w_j x_j}_{\leq 0} + \underbrace{\sum_{C_3} w_j x_j}_{= 0}$$

$$y^T s = \underbrace{\sum_{R_1} y_i s_i}_{\geq 0} + \underbrace{\sum_{R_2} y_i s_i}_{\geq 0} + \underbrace{\sum_{R_3} y_i s_i}_{= 0}$$

so $c^T x \leq y^T b$

and if $c^T x = y^T b$

then must be optimal.

Let's talk a bit about the second condition, basically when we have optimal solutions for both. This brings up the idea of *complementary slackness.* Specifically, we have the following:

If variable $x_i > 0$ in P, constraint i in D is tight.

Why? What does this mean?

if we're working with an optimal solution, x, y. Then $\sum x_i w_i = 0$ and $\sum y_i s_i = 0$. Because of the signs in $w^T x$ and $y^T s$, each summand must be $= 0$.

This means if $x_i \neq 0$, then $w_i = 0$, so constraint i is tight in the dual.

Question: why is this called *weak* duality? Doesn't this give us everything we need?

we needed to have feasable soln for both

Can we have a feas. soln for the primal, then build a feas. sol. for the dual?

So it looks like we'll need something stronger to get to a truly equivalent version, or essentially one that talks about feasibility. Fortunately, we do have just that.

Theorem (4.3)**.** *Let P and D be a primal-dual pair with standard notation of A, b and c. If there exists an optimal solution of P, then there exists an optimal solution of D. Moreover, these programs have the same optimal objective value.*

Pf: we'll prove for SEF of the primal.

$P = \max\{c^T x : Ax = b, x \geq 0\}$. Note, $D = \min\{b^T y : A^T y \geq c, y \text{ free}\}$

Simplex method terminates @ basis B. $A_B = A$ restricted to B col. $A_N =$ col. of A of nonbasic var. We'll rewrite P as

$$\begin{aligned} \max\ & z = \bar{y}^T b + \bar{c}^T x \\ \text{st}\ & x_B + A_B^{-1} A_N x_N = A_B^{-1} b \end{aligned} \qquad \text{where } \bar{y} = (A_B^{-1})^T c_B, \quad \bar{c} = c^T - \bar{y}^T A$$

if x^* is optimal, and $z = \bar{y}^T b$. We cl: this $\bar{y}$ is dual feasible.

Note that since simplex terminated, then $\bar{c} \leq 0$, thus $c^T - \bar{y}^T A \leq 0$ and $c^T \leq \bar{y}^T A$ and $A^T \bar{y} \geq c$. Plugging $\bar{y}$ into the dual yields the same objective, so by weak duality, optimal!

Example: Fill in the following table with either "possible" or "impossible". Explain your reasoning!

Dual ↓ Primal →	Optimal Solution	Unbounded	Infeasible
Optimal Solution	yes! Strong duality	X	X
Unbounded	X impossible	X impossible	possible
Infeasible	X	possible	possible