Section 1.1: Introduction to Differential Equations

derivatives

What is a derivative?

1. The instantaneous rate of change of a quantity
2. The slope of the tangent line for a function f at each x.
3. The limit $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ $\forall x$

Calculus class:

Given a function $y = f(x)$, find the derivative

Using appropriate rules

$$\frac{dy}{dx} = f'(x)$$

Like the initial function y

$\frac{dy}{dx}$ is a function of x.

Example: Let $y = e^{4x^2}$. Find $\frac{dy}{dx}$

By the chain rule,

$$\frac{dy}{dx} = 8x e^{4x^2} = 8xy$$

In DE's course.

Example: Identify a function y that satisfies

$$\frac{dy}{dx} = 8xy$$

dependent.

independent variable

$$y = e^{4x^2}$$

A <u>differential equation</u> is an equation that contains the derivative of one or more function w.r.t. one or more independent variables.

- Ordinary differential equations (ODE) are DEs where all derivatives are w.r.t. a single independent variable
- Partial differential equations (PDE) are DEs where derivatives are taken w.r.t. more than one independent variable.

Examples

- $\frac{dy}{dx} = 8xy$ order 1 (ODE)
- $y''(x) + 2y'(x) = 3x + y(x)$ (ODE)

order 2

- $2\dfrac{dx}{dt} + \dfrac{dy}{dt} = 3x + y$ ← order 1 (ODE)

$x = x(t)$ $y = y(t)$ $x(t)$ $y(t)$

- $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ ← Harmonic equations (PDE)

order 2

The order of a differential equation is the highest derivative that appears in equation

Example:

- $\dfrac{d^5y}{dx^5} + \dfrac{d^3y}{dx^3} + y = 0$ Linear ODE

$a_5(x)=1$, $a_3(x)=1$, $a_0(x)=1$

Order 5

- $y'' + (y')^3 + 2y = 0$ Order 2

Nonlinear ODE

An ODE is linear if it has the form

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

Example:

Linear: $(x^2+1)\boxed{y''} + (2x+\sin(x))\boxed{y'}$

$+ (x^2+2)\boxed{y} = 0$

Nonlinear

$$y' + x\sin(y) = 0$$

A <u>solution</u> to a differential equation is a function that satisfies the equation

Example: $\boxed{y = e^{4x^2}}$ is a solution to $\frac{dy}{dx} = 8xy$

Verify: $\frac{dy}{dx} = \boxed{8xe^{4x^2}}$ $\Big|$ $\boxed{8x(e^{4x^2})}$ ✓

Example: $y = x^2e^{5x}$ is a solution to

$$y'' - 10y' + 25y = 2e^{5x}$$

Verify: $y' = 2xe^{5x} + 5x^2e^{5x}$

$y'' = 2e^{5x} + 10xe^{5x} + 10xe^{5x} + 25x^2e^{5x}$

$$(2e^{5x} + \cancel{20xe^{5x}} + \boxed{25x^2e^{5x}}) - 10(\cancel{2xe^{5x}} + \boxed{5x^2e^{5x}})$$

$$+ \boxed{25(x^2e^{5x})} = 2e^{5x}$$