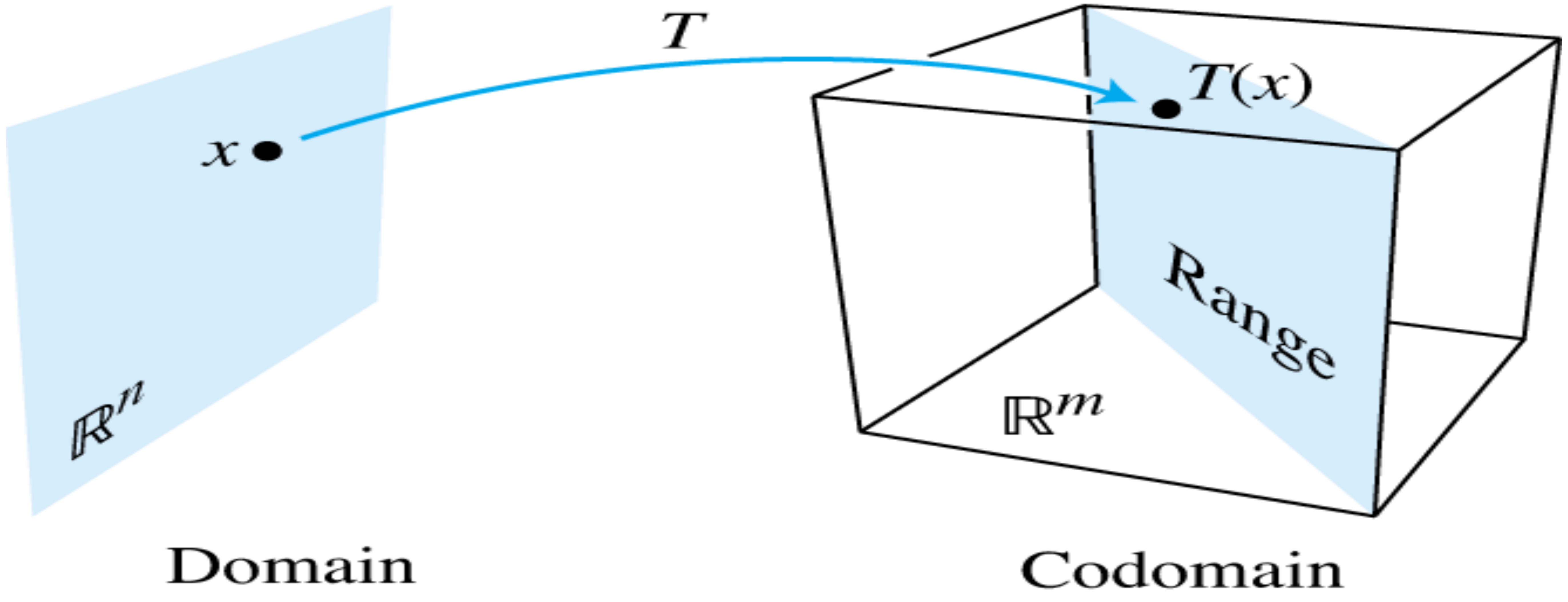
1.8. Linear Transformations

Transformations

A **transformation** is a *function*

$$T : \mathbb{R}^n \to \mathbb{R}^m$$

that assigns to each $\mathbf{x} \in \mathbb{R}^n$ a vector $T(\mathbf{x}) \in \mathbb{R}^m$



- The **domain** is $\mathbb{R}^n$
- The **codomain** is $\mathbb{R}^m$
- The **range** (aka image) is the subset of the codomain $\mathbb{R}^m$ that is mapped to

Matrix Transformations

A **matrix transformation** is a transformation given by multiplication by a matrix.

Eg 1. $\mathbf{A} = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 2 & -1 & -1 & 1 \end{bmatrix}$ and T(**x**) = **A x**

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 2 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 + 0x_2 - x_3 + 2x_4 \\ 2x_1 - x_2 - x_3 + x_4 \end{bmatrix}$$

This is a transformation from $\mathbf{R}^4$ to $\mathbf{R}^2$

Eg 2. $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \\ 2 & 3 \end{bmatrix}$ and T(**x**) = **A x**

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 1 & -1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1 - x_2 \\ 2x_1 + 3x_2 \end{bmatrix}$$

This is a transformation from $\mathbf{R}^2$ to $\mathbf{R}^3$

Matrix Transformations

A **matrix transformation** is a transformation given by multiplication by a matrix.

Eg 3. $\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 \\ 4 & -1 & 3 \\ 2 & 3 & 1 \end{bmatrix}$ and T(**x**) = **A x**

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 & -1 \\ 4 & -1 & 3 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 - x_3 \\ 4x_1 - x_2 + 3x_3 \\ 2x_1 + 3x_2 + x_3 \end{bmatrix}$$

This is a transformation from $\mathbf{R}^3$ to $\mathbf{R}^3$

Eg 4. $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and T(**x**) = **A x**

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1 + x_2 \end{bmatrix}$$

This is a transformation from $\mathbf{R}^2$ to $\mathbf{R}^2$

Linear Transformations

A **transformation** is a function

$$T : \mathbb{R}^n \to \mathbb{R}^m$$

The domain is $\mathbb{R}^n$
and the codomain is $\mathbb{R}^m$

that assigns to each $\mathbf{x} \in \mathbb{R}^n$ a vector $T(\mathbf{x}) \in \mathbb{R}^m$

A transformation is **linear** if for all vectors **u** and **v** and all scalars *c*

(1) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$

(2) $T(c\,\mathbf{u}) = c\,T(\mathbf{u})$

(3) $T\left(\mathbf{0}_n\right) = \mathbf{0}_m$

(4) $T\left(c\,\mathbf{u} + d\,\mathbf{v}\right) = c\,T(\mathbf{u}) + d\,T(\mathbf{v})$

Matrix transformations are linear

$\mathbf{A}\,(\mathbf{u} + \mathbf{v}) = \mathbf{A}\,\mathbf{u} + \mathbf{A}\,\mathbf{v}$
$\mathbf{A}\,(c\,\mathbf{u}) = c\,\mathbf{A}\,\mathbf{u}$

Only one the following transformations is linear. Which is it?

(a) $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x-y \\ y+z \end{bmatrix}$ $\quad \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+y \\ x-y \\ y+z \end{bmatrix}$

A matrix transformation! So linear.

(b) $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y+1 \\ y+2 \\ x-1 \end{bmatrix}$ Try the **0** vector

$$T\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0+0+1 \\ 0+2 \\ 0-1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \quad \text{X}$$

(c) $T\left(\begin{bmatrix} x \\ y \\ x \end{bmatrix}\right) = \begin{bmatrix} xy \\ yz \end{bmatrix}$ Try the **0** vector

$$T\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0\cdot 0 \\ 0\cdot 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \checkmark$$

$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$

$T(c\,\mathbf{u}) = c\,T(\mathbf{u})$

....

$T(\mathbf{0}) = \mathbf{0}$

$T(c\mathbf{u}+d\mathbf{v}) = cT(\mathbf{u})+dT(\mathbf{u})$

$$T\left(3\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right) = T\left(\begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}\right) = \begin{bmatrix} 3\cdot 6 \\ 6\cdot 9 \end{bmatrix} = \begin{bmatrix} 18 \\ 54 \end{bmatrix} \quad \text{X}$$

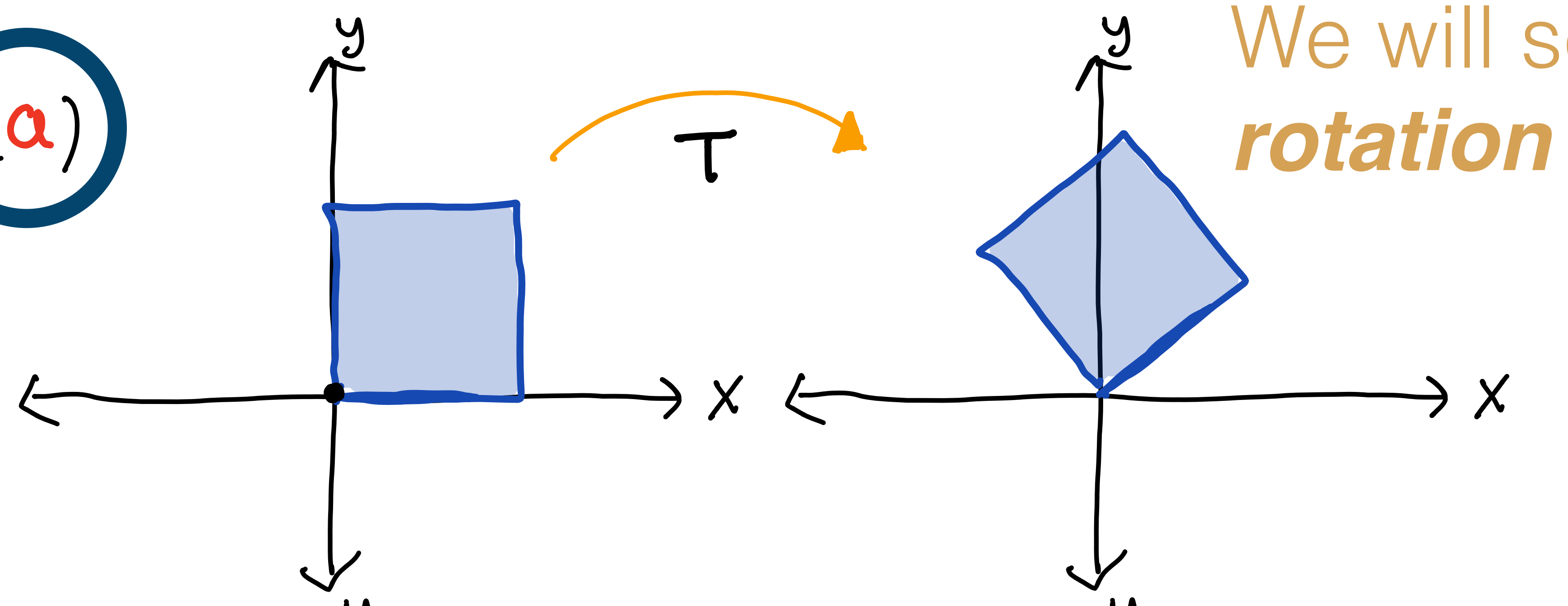
$$3\,T\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right) = 3\begin{bmatrix} 1\cdot 2 \\ 2\cdot 3 \end{bmatrix} = 3\begin{bmatrix} 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 6 \\ 18 \end{bmatrix}$$

$$T\left(\underset{\mathbf{u}}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} + \underset{\mathbf{v}}{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}\right) = T\left(\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 2\cdot 3 \\ 3\cdot 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \end{bmatrix} \quad \text{X}$$

$$T\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) + T\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 1\cdot 1 \\ 1\cdot 1 \end{bmatrix} + \begin{bmatrix} 1\cdot 2 \\ 2\cdot 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

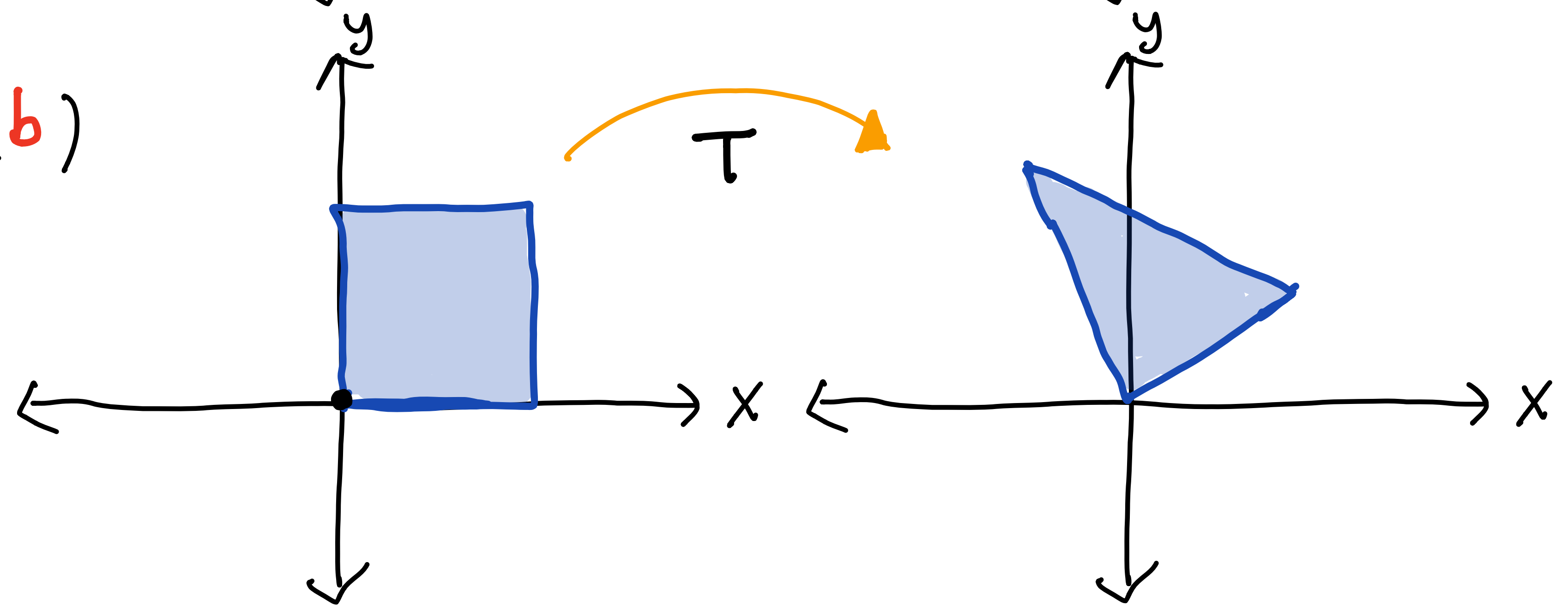
Below we see how a transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ transforms the points in the unit square. Two of these are not linear. The other 4 are linear. Which are nonlinear?

(a)

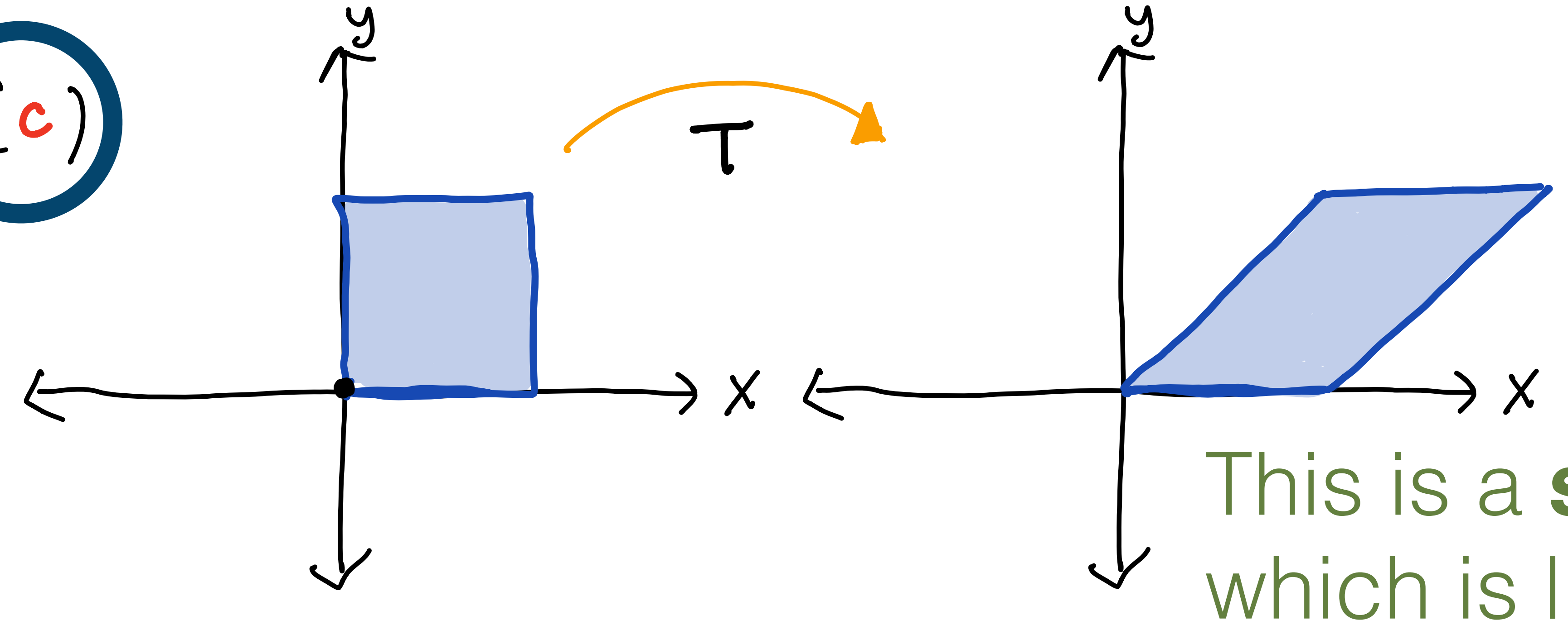


We will see that ***rotation*** is linear

(b)

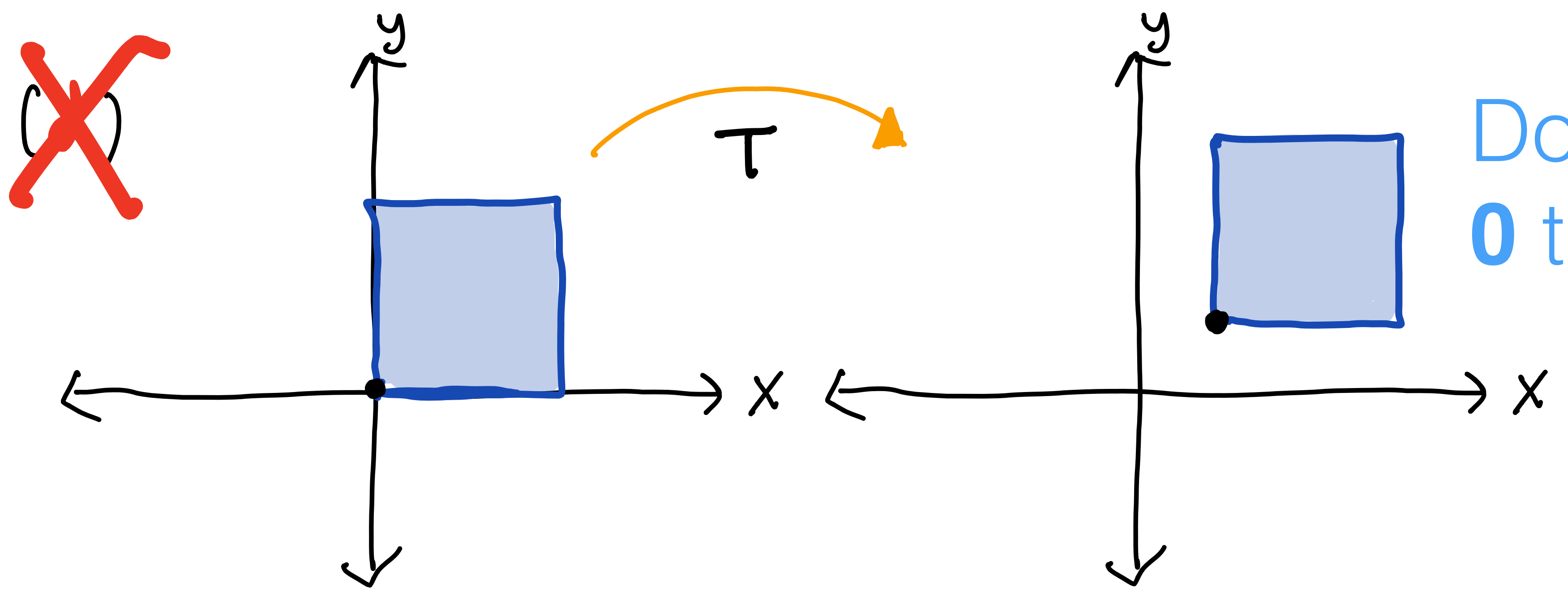


(c)



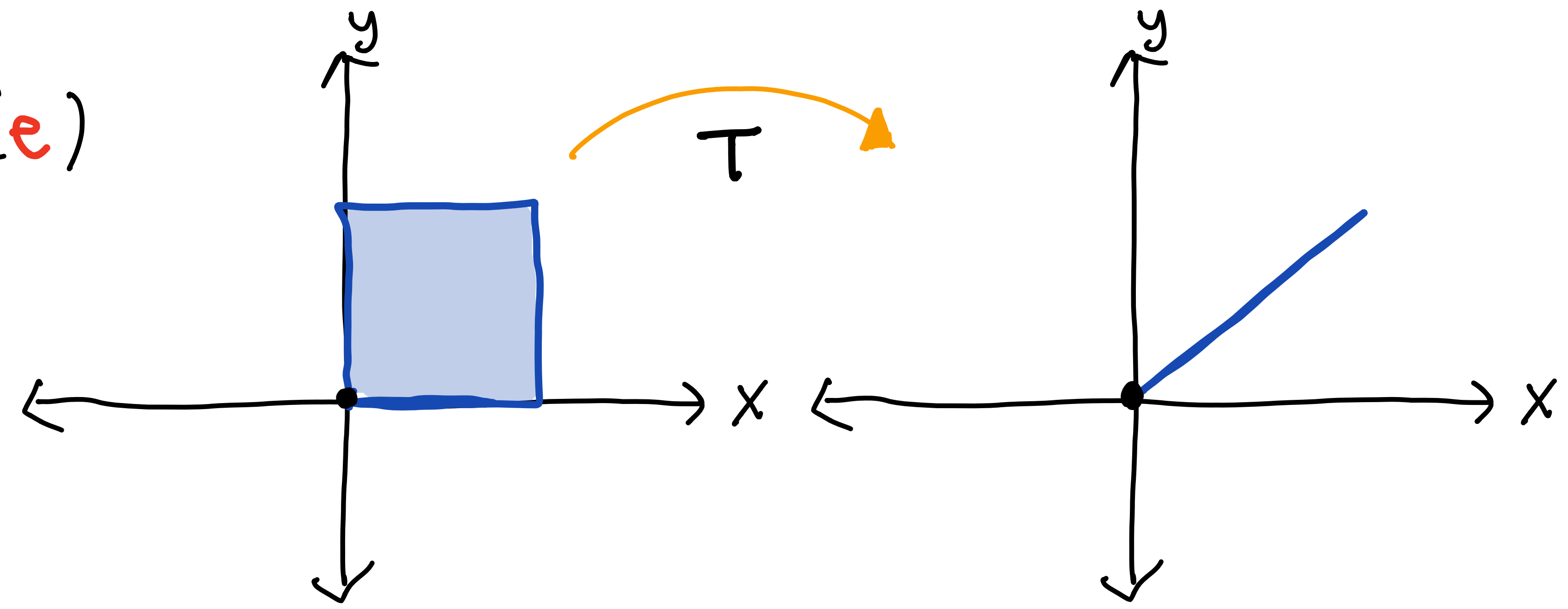
This is a **shear**, which is linear

~~(d)~~

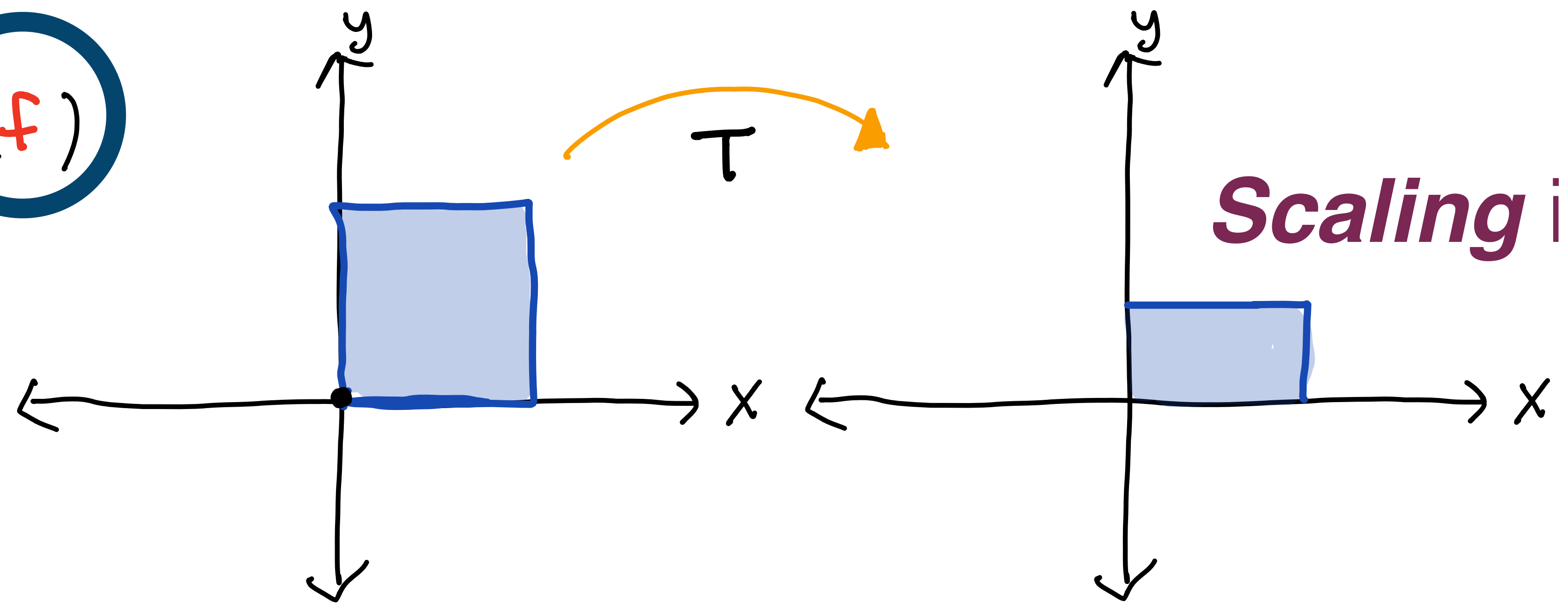


Does not send **0** to **0**!

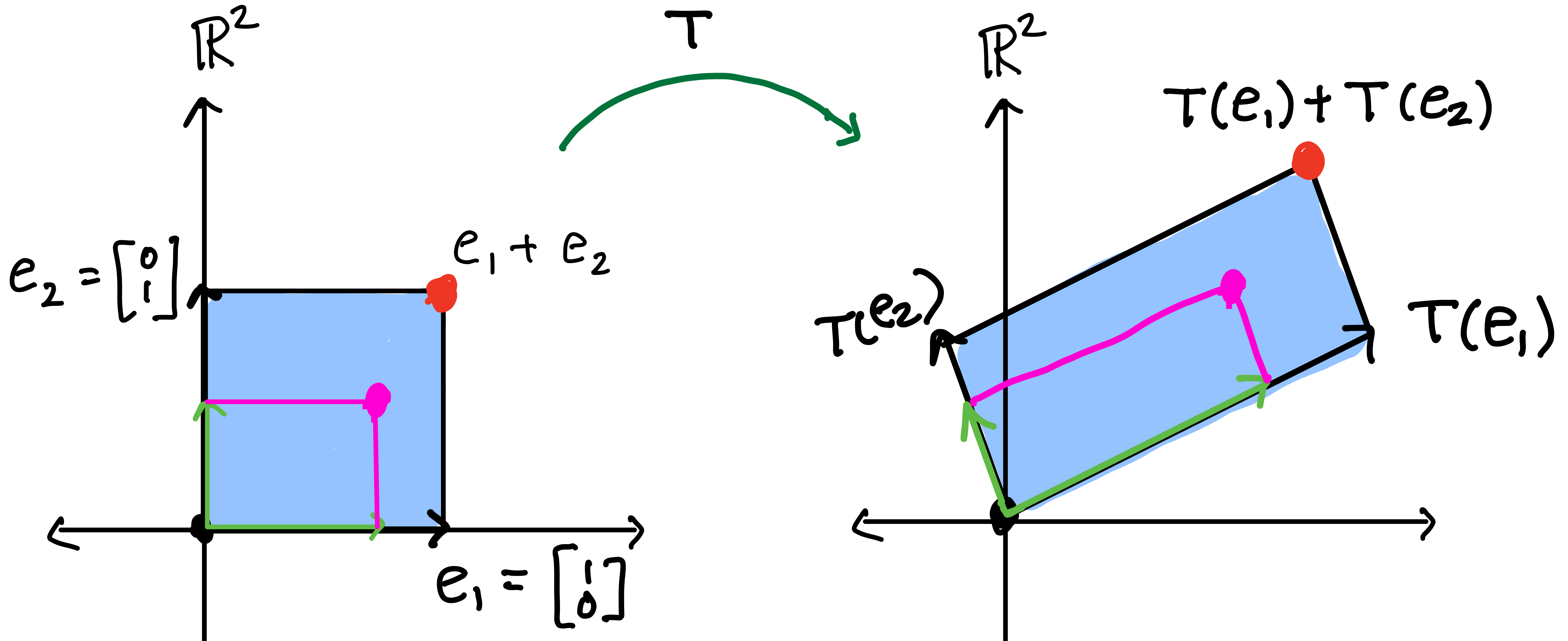
(e)



(f)



Scaling is linear



$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$

$T(c\,\mathbf{u}) = c\,T(\mathbf{u})$

$T(\mathbf{0}) = \mathbf{0}$

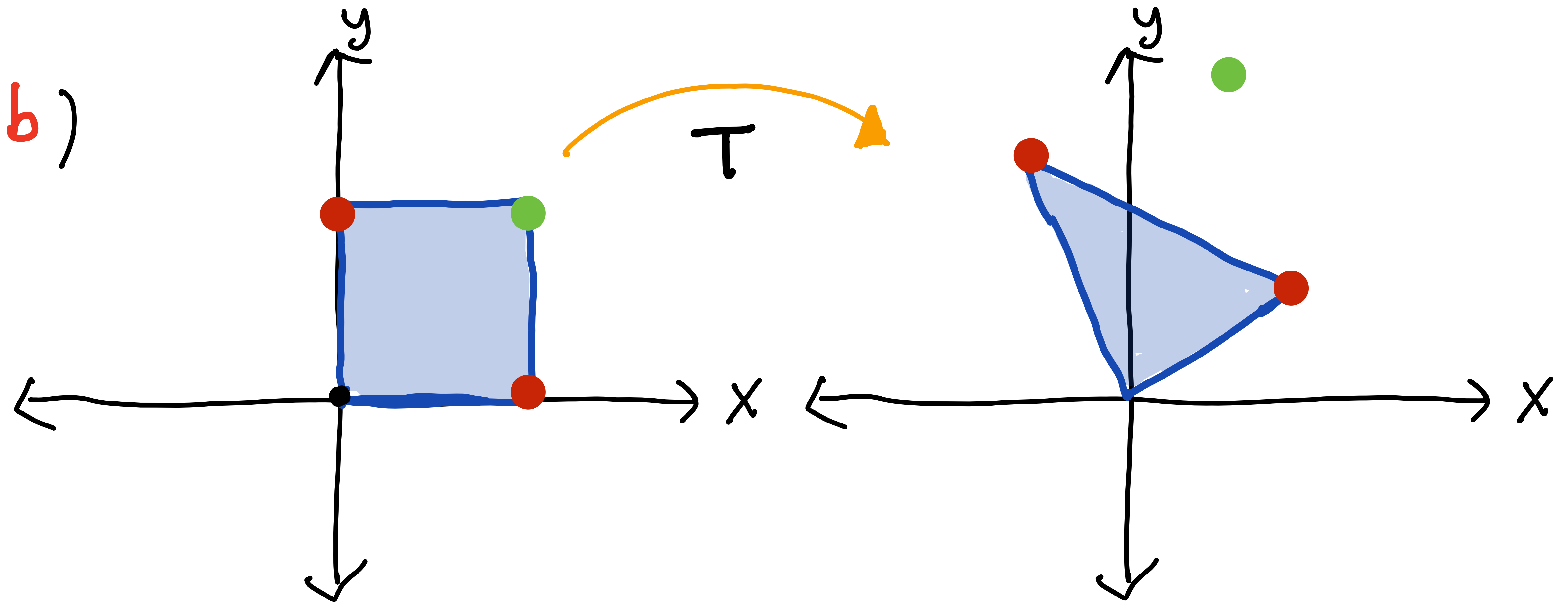
$T(c\mathbf{u}+d\mathbf{v}) = cT(\mathbf{u})+dT(\mathbf{u})$

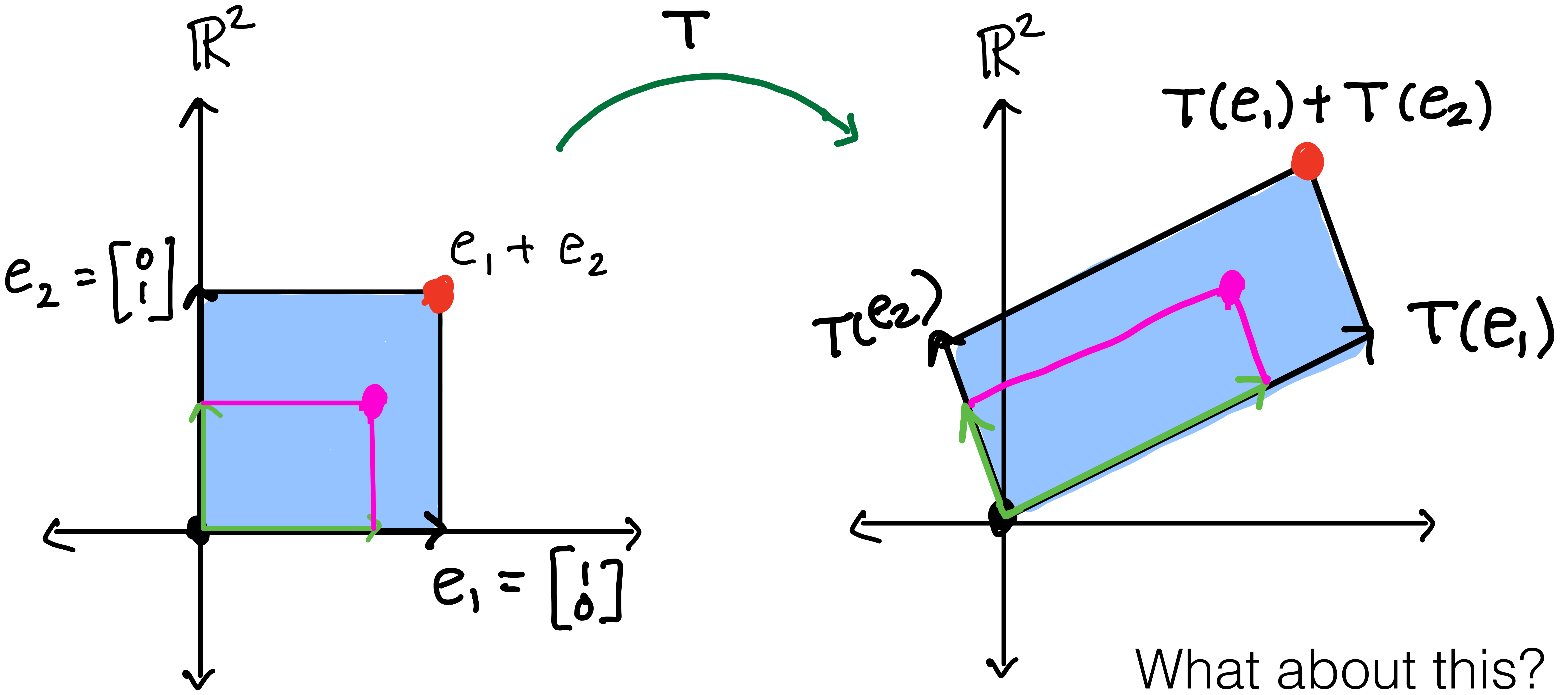
This is not possible

$$T(e_1+e_2) = T(e_1) + T(e_2)$$

$$T\left(\tfrac{3}{4}e_1 + \tfrac{1}{2}e_2\right) = \tfrac{3}{4}T(e_1) + \tfrac{1}{2}T(e_2)$$

(b)





$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$

$T(c\,\mathbf{u}) = c\,T(\mathbf{u})$

$T(\mathbf{0}) = \mathbf{0}$

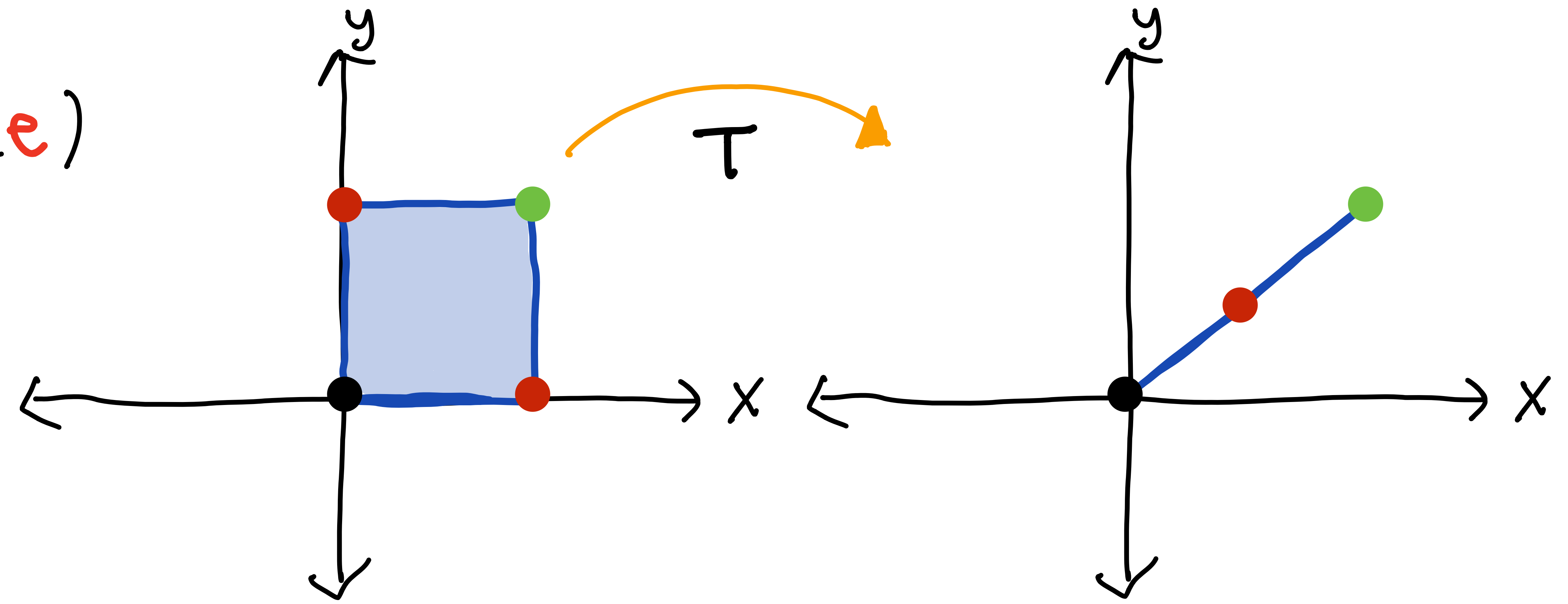
$T(c\mathbf{u}+d\mathbf{v}) = cT(\mathbf{u})+dT(\mathbf{u})$

$$T(e_1+e_2) = T(e_1) + T(e_2)$$

$$T\left(\tfrac{3}{4}e_1 + \tfrac{1}{2}e_2\right) = \tfrac{3}{4}T(e_1) + \tfrac{1}{2}T(e_2)$$

What about this?

(e)



You Try!

Are any of these linear transformations?

$$T\left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\right) = \begin{bmatrix} x+y \\ z^2 \\ z-w \end{bmatrix}$$

$$R\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ x_3 \\ x_1 \end{bmatrix}$$

$$S\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$U\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} 1-x_2+x_3 \\ 2-x_3+x_4 \\ x_1-x_2+x_3-4x_4 \end{bmatrix}$$