Section 2.3: First Order Linear Equations and Integrating Factors

First order linear equation

$$\frac{dy}{dx} + \underline{p(x)}\,y = \underline{g(x)}$$

Linear in $\underline{y}$ (in the dependent variable)

Not necessarily linear

Examples

- $\frac{dy}{dx} + \underbrace{(x+5)}_{p(x)}y = \underbrace{x^2+2x+2}_{g(x)}$

- $x\frac{dy}{dx} + x^2y = 3x+2$

  $\hookrightarrow \frac{dy}{dx} + xy = 3 + \frac{2}{x}$

Nonexample

- $y\frac{dy}{dx} + xy = 2x+3$

- $\left(\frac{dy}{dx}\right)^2 + (x+3)y = 2$

$\left(\frac{dy}{dx}\right)^2$ ← 1st derivative & square

$\frac{d^2y}{dx^2}$ ← 2nd derivative

$$\frac{dy}{dx} = \frac{d}{dx}(y)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{d}{dx}(y)\right)$$
$$= \frac{d^2}{dx^2}(y)$$

Product Rule:

$$(uv)' = u'v + u \cdot v'$$

Motivation

"reverse" product rule.

$$\int \frac{du}{dx}v + u\frac{dv}{dx}\, dx \overset{?}{=} \int \frac{d}{dx}(uv)\, dx$$

$$= uv + C$$

Steps for Solving using an Integrating factor

1. Write equation in the form
$$\frac{dy}{dx} + p(x)y = g(x)$$
2. Find integrating factor if not given.
3. Multiply both sides of equation by integrating factor.
4. Apply reverse product rule on L.H.S
5. Integrate both sides w.r.t. x.

Example: $\frac{dy}{dx} + 3x^2y = 5x^2$
using integrating factor e^{x^3}

$$e^{x^3}\left(\frac{dy}{dx} + 3x^2 y\right) = e^{x^3}(5x^2)$$

$$e^{x^3}\frac{dy}{dx} + 3x^2 e^{x^3} y = 5x^2 e^{x^3}$$

$$\frac{d}{dx}\left(e^{x^3} y\right) = 5x^2 e^{x^3}$$

reverse product rule.

$$\int \frac{d}{dx}\left(e^{x^3} y\right) dx = \int 5x^2 e^{x^3}\, dx$$

$$\frac{d}{dx}\left(e^{x^3}\right) = 3x^2 e^{x^3}$$

$$e^{x^3} y = \frac{5}{3} e^{x^3} + C$$

$$y = \frac{5}{3} + Ce^{-x^3}$$

$\int 5x^2 e^{x^3}\, dx$:

Let $u = x^3$ $\qquad \frac{du}{dx} = 3x^2$

$\sim \frac{1}{3} du = x^2\, dx$

$$\int 5x^2 e^x \, dx = \int 5e^x \, x^2 \, dx$$

$$= \int 5e^u \cdot \tfrac{1}{3} \, du$$

$$= \int \tfrac{5}{3} e^u \, du$$

$$= \tfrac{5}{3} \int e^u \, du = \tfrac{5}{3} e^u + C$$

$$= \tfrac{5}{3} e^{x^3} + C$$

How does this process work? How do we come up with the correct choice of integrating factor so that we can reverse product rule?

Given $\quad y' + p(x)y = g(x)$

μ integrating factor

Goal: Find some function $\mu(x)$ such that we can apply reverse product rule to

$$\mu(x)y' + \mu(x)p(x)y$$

i.e. $\mu(x)y' + \mu(x)p(x)y = \frac{d}{dx}\left(\mu(x)y\right)$

This means $\frac{d}{dx}(\mu(x)) = \mu(x) \cdot p(x)$

$$\mu(x) = e^{\int p(x)\,dx}$$

$$\frac{d}{dx}(\mu(x)) = \frac{d}{dx}\left(e^{\int p(x)\,dx}\right)$$

$$= e^{\int p(x)\,dx} \cdot \frac{d}{dx}\left(\int p(x)\,dx\right)$$

$$= e^{\int p(x)\,dx}\, p(x)$$

$$= \mu(x)\, p(x) \quad \checkmark$$

Example: Solve

$$x^2 \frac{dy}{dx} + 4xy = 3x^3 \quad \text{subject to } y(1) = \frac{3}{2}$$

Divide by x^2

$$\frac{dy}{dx} + \frac{4}{x} y = 3x$$

Find integrating factor.

$$e^{\int \frac{4}{x}dx} = e^{4\ln|x|} = x^4$$

$$e^{4\ln|x| + c}$$

$$Ae^{4\ln|x|} \qquad A = e^c$$

Multiply through

$$x^4 \frac{dy}{dx} + 4x^3 y = 3x^5$$

Reverse Product Rule

$$\frac{d}{dx}(x^4 y) = 3x^5$$

Integrate w.r.t. x

$$\int \frac{d}{dx}(x^4 y)\, dx = \int 3x^5\, dx$$

$$x^4 y = \frac{1}{2}x^6 + C$$

$$y = \frac{1}{2}x^2 + \frac{c}{x^4}$$

$$\frac{3}{2} = y(1) = \frac{1}{2}(1)^2 + \frac{c}{1}$$

$C = 1$

$$y = \frac{1}{2}x^2 + \frac{1}{x^4}$$