4.5. Dimension

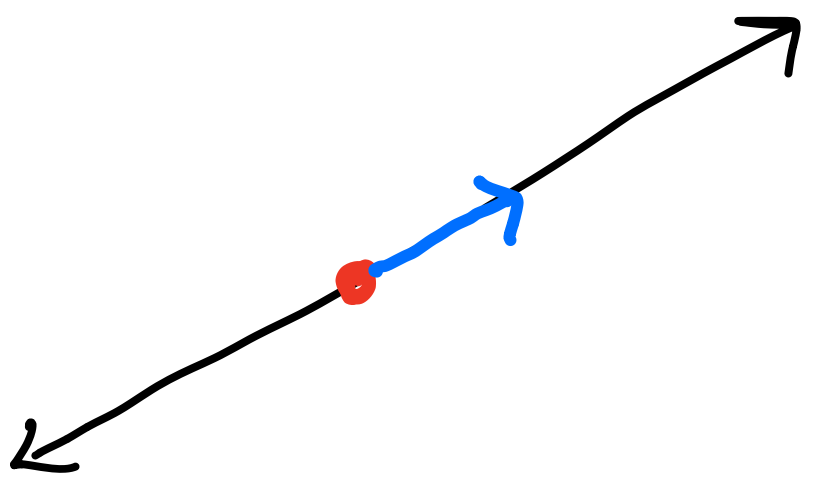
Dimension

Definition: The **dimension** of a vector space (or subspace) is the number of vectors in a basis of that space.

Examples

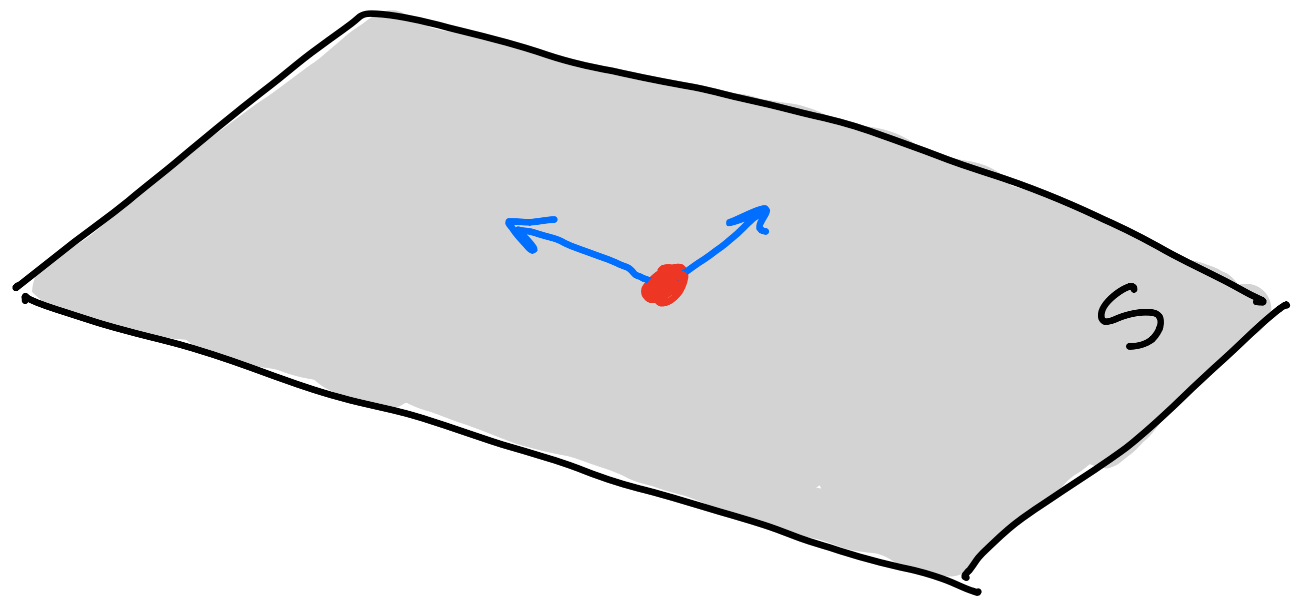
$\dim(\mathbb{R}^n) = n$ **Basis**: $\mathcal{S} = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$

$\dim(\text{line through } \mathbf{0}) = 1$ **Basis**: any nonzero vector on the line



$\dim(\text{plane through } \mathbf{0}) = 2$ **Basis**: any two non-colinear vectors on the plane

$\dim(\{\mathbf{0}\}) = 0$ **Basis**: none



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Examples

$\dim(\mathrm{Col}(A)) =$ Number of pivots in A **Basis**: pivot columns of A

$\dim(\mathrm{Nul}(A)) =$ Number of free variables in A **Basis**: parametric basis of Nul(A)

$$\mathbf{A} = \begin{bmatrix} 3 & 1 & 1 & 4 & 8 \\ 3 & 1 & 1 & 4 & 8 \\ -1 & 4 & 0 & -5 & -6 \\ 0 & 4 & 0 & -4 & -4 \\ 2 & -1 & 1 & 5 & 8 \\ 2 & 1 & 2 & 5 & 9 \end{bmatrix} \longrightarrow \begin{array}{ccccc} x_1 & x_2 & x_3 & s & t \end{array} \begin{bmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = s \underbrace{\begin{bmatrix} -1 \\ 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}}_{u_1} + t \underbrace{\begin{bmatrix} -2 \\ 1 \\ -3 \\ 0 \\ 1 \end{bmatrix}}_{u_2}$$

parametric solution to $Ax=0$

Dimension

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If a vector space V has a basis of n vectors, then every basis of V must consist of exactly n vectors.

Proof.

Suppose that we have two bases of V:

$$\mathcal{B}_1 = \{ v_1, v_2, v_3, \dots, v_n \}$$

$$\mathcal{B}_2 = \{ w_1, w_2, w_3, \dots, w_m \}$$

Express each $\mathbf{v}_i$ in the $\mathbf{w}$ basis:

$$v_i = a_{1i} w_1 + a_{2i} w_2 + a_{3i} w_3 + \cdots + a_{mi} w_m$$

Make the change of basis matrix:

$$\begin{array}{c|c} & \begin{array}{ccccc} v_1 & v_2 & v_3 & & v_n \end{array} \\ \begin{array}{c} w_1 \\ w_2 \\ w_3 \\ \vdots \\ w_m \end{array} & \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \\ a_{m1} & a_{m2} & a_{m3} & & a_{mn} \end{bmatrix} \end{array} \xrightarrow[\text{reduce}]{\text{row}} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots \end{bmatrix}$$

pivot in every column since v's are independent

pivot in every row since v's span

$\Rightarrow$ SQUARE $\Rightarrow$ $m = n$

Example

The set $\mathbf{F}$ of Fibonacci vectors in $\mathbb{R}^5$ is given below. Find $\dim(\mathbf{F})$.

$$\mathbf{F} = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \;\middle|\; a_3 = a_1 + a_2,\ a_4 = a_2 + a_3,\ a_5 = a_3 + a_4 \right\} \subseteq \mathbb{R}^5$$

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_1 + a_2 \\ a_1 + 2a_2 \\ 2a_1 + 3a_2 \end{bmatrix} = a_1 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 2 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \\ 3 \end{bmatrix} \Rightarrow F = \operatorname{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \\ 3 \end{bmatrix} \right\}$$

$$\Rightarrow \dim(F) = 2$$

You Try!

The vector space $\mathbf{S}$ is the span of the vectors below.

$$\mathbf{S} = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ -2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ 5 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ -4 \\ 2 \\ 2 \end{bmatrix} \right\}$$

Find the dimension of $\mathbf{S}$.