Daily vocabulary: Matrix form $Ax = b$; matrix-vector product; continuing: vector equations and span

Class Discussion

1. Compute the following matrix-vector products:

(a) $\begin{bmatrix} 1 & 0 & -2 \\ 2 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \\ 6 \end{bmatrix}$

(b) $\begin{bmatrix} | & | & | & | \\ v_1 & v_2 & v_3 & v_4 \\ | & | & | & | \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 5 \\ 2 \end{bmatrix} = 3\vec{v}_1 + (-1) v_2 + 5\vec{v}_3 + 2\vec{v}_4$

2. Write the following system of equations in vector form and in matrix form ($Ax = b$).

Linear System

$$\left\{ \begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 13 \\ x_1 & & & + & x_3 & = & 7 \\ -x_1 & + & x_2 & & & = & -5 \\ 2x_1 & + & x_2 & + & x_3 & = & 6 \end{array} \right\}$$

Vector Equation

$$x_1 \begin{bmatrix} 1 \\ 1 \\ -1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 13 \\ 7 \\ -5 \\ 6 \end{bmatrix}$$

Matrix Equation

$$\begin{bmatrix} 1 & -2 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 13 \\ 7 \\ -5 \\ 6 \end{bmatrix}$$

To solve any of these three problems, reduce the corresponding system of equations. This can be done by hand, using R, or in WolframAlpha. We will do it in R in class

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 13 \\ 1 & 0 & 1 & 7 \\ -1 & 1 & 0 & -5 \\ 2 & 1 & 1 & 6 \end{array} \right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{array} \right] \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}$$

How many solutions does this have? 0 1 ∞. Answer the question in terms of the linear system, the vector equation, and the matrix equation:

Linear System Vector Equation Matrix Equation

$$2 \begin{bmatrix} 1 \\ 1 \\ -1 \\ 2 \end{bmatrix} - 3 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix} + 5 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 13 \\ 7 \\ -5 \\ 6 \end{bmatrix}$$

3. (Discuss) Can we find another vector d so that $Ax = d$ has no solution (for this same matrix A)?

yes, because of the row of 0's in RREF(A)

4. Do the vectors $\{\vec{\mathbf{u}}, \vec{\mathbf{v}}, \vec{\mathbf{w}}\}$, below, span all of $\mathbb{R}^3$?

If not, find a vector that is not in their span and describe all of the vectors that are in the span.

$$\vec{\mathbf{u}} = \begin{bmatrix}1\\4\\7\end{bmatrix}, \vec{\mathbf{v}} = \begin{bmatrix}2\\5\\8\end{bmatrix}, \vec{\mathbf{w}} = \begin{bmatrix}3\\6\\9\end{bmatrix}. \qquad \begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix} \longrightarrow \begin{bmatrix}1&0&-1\\0&1&2\\0&0&0\end{bmatrix}$$

which b's make this consistent

$$\left[\begin{array}{ccc|c}1&2&3&b_1\\4&5&6&b_2\\7&8&9&b_3\end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c}1&2&3&b_1\\0&-3&-6&-4b_1+b_2\\0&-6&-12&-7b_1+b_3\end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c}1&2&3&b_1\\0&-3&-6&-4b_1+b_2\\0&0&0&-2(-4b_1+b_2)-7b_1+b_3\end{array}\right]$$

$$= b_1 - 2b_2 + b_3$$

$Ax=b$ is consistent if $b_1 - 2b_2 + b_3 = 0$ $\quad \begin{bmatrix}1\\1\\1\end{bmatrix}$

inconsistent $b_1 - 2b_2 + b_3 \neq 0$ $\quad \begin{bmatrix}1\\1\\2\end{bmatrix}$

5. Restate question 4 as

(a) A matrix $Ax = \vec{\mathbf{b}}$ question

$$\boxed{\left\{ \begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix} \,\middle|\, b_1 - 2b_2 + b_3 = 0 \right\}}$$ span

(b) A question about the corresponding linear system of equations

6. Checkpoint:

Do these vectors span R^3 ?

yes

$$\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -10 \\ 11 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\begin{array}{c} \begin{bmatrix} 1 & 3 & 1 & 1 \\ -2 & 2 & -10 & 2 \\ 3 & 1 & 11 & 3 \end{bmatrix} \end{array} \begin{array}{l} b_1 \\ b_2 \\ b_3 \end{array} \longrightarrow \begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{array}{l} b_1' \\ b_2' \\ b_3' \end{array}$$

Do these vectors span R^4 ?

no

$$\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{array}{l} b_1 \\ b_2 \\ b_3 \\ b_4 \end{array} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Do these vectors span R^5 ?

$$\begin{bmatrix} 1 \\ 1 \\ 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 1 \\ 1 \\ 4 \end{bmatrix} \rightarrow \begin{bmatrix} & & & \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

4 vectors cannot span $\mathbb{R}^5$

$*$ $\boxed{\Rightarrow \text{need at least } n \text{ vectors to span } \mathbb{R}^n}$

always have free vars if [illegible]

7. Four matrices are row reduced here for you.

$$A = \begin{bmatrix} 2 & 1 & -1 & 3 & 6 \\ 1 & 2 & 1 & 2 & 5 \\ 1 & 2 & 1 & 1 & 4 \\ -3 & 1 & 4 & 0 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 4 & 1 & 1 & 1 \\ 1 & 0 & 2 & 1 \\ 1 & 2 & 1 & 1 \\ -3 & 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 1 & 3 & 2 & 1 & 4 \\ 1 & 2 & 2 & -1 & 1 & 5 \\ 1 & 2 & 1 & -2 & 0 & 4 \\ -3 & 1 & 0 & 2 & 0 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 1 \\ -3 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

too tall to span

Match the following statements below with the matrices A, B, C, D above.

no row 0's no free var

C (i) The columns of the matrix span $\mathbb{R}^4$ and reach every vector in a unique way.

B (ii) The columns of the matrix span $\mathbb{R}^4$ and reach each vector in infinitely many ways.

D (iii) The columns of the matrix do not span $\mathbb{R}^4$, but every vector in their span can be reached in a unique way.

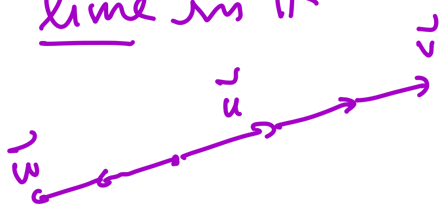
A (iv) The columns of the matrix do not span $\mathbb{R}^4$, but every vector in their span can be reached in infinitely many ways.

8. Leftover from Wednesday

In each example below there are 3 vectors, $\vec{\mathbf{u}}, \vec{\mathbf{v}}, \vec{\mathbf{w}}$, in $\mathbb{R}^3$. Describe the span of the vectors. A useful row reduction has been done for you in each case.

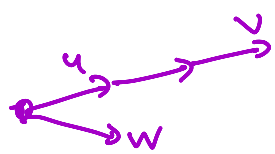
(a) $\vec{\mathbf{u}} = \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}, \vec{\mathbf{v}} = \begin{bmatrix} 3 \\ 12 \\ -9 \end{bmatrix}, \vec{\mathbf{w}} = \begin{bmatrix} -2 \\ -8 \\ 6 \end{bmatrix}$ $\qquad$ $\begin{bmatrix} 1 & 3 & -2 \\ 4 & 12 & -8 \\ -3 & -9 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

line in $\mathbb{R}^3$



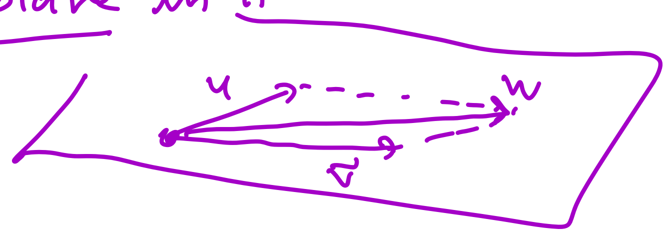
(b) $\vec{\mathbf{u}} = \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}, \vec{\mathbf{v}} = \begin{bmatrix} 3 \\ 12 \\ -9 \end{bmatrix}, \vec{\mathbf{w}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\qquad$ $\begin{bmatrix} 1 & 3 & 1 \\ 4 & 12 & 1 \\ -3 & -9 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

plane in $\mathbb{R}^3$



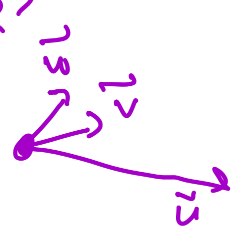
(c) $\vec{\mathbf{u}} = \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}, \vec{\mathbf{v}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{\mathbf{w}} = \begin{bmatrix} 2 \\ 5 \\ -2 \end{bmatrix}$ $\qquad$ $\begin{bmatrix} 1 & 1 & 2 \\ 4 & 1 & 5 \\ -3 & 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

plane in $\mathbb{R}^3$



(d) $\vec{\mathbf{u}} = \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}, \vec{\mathbf{v}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{\mathbf{w}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $\qquad$ $\begin{bmatrix} 1 & 1 & 1 \\ 4 & 1 & 2 \\ -3 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

all of $\mathbb{R}^3$



not on a plane

9. Time permitting: See Monday

(a) Is the $\vec{\mathbf{0}}$ vector in the span of the vectors below? If it is, is there a unique combination of the vectors that get to the 0 vector?

$\vec{\mathbf{v}}_1 = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \vec{\mathbf{v}}_2 = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} \vec{\mathbf{v}}_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

check:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 4 & 5 & 1 & 0 \\ 7 & 8 & -1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & -6 & -8 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 3 & 4 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

(b) Is the $\vec{\mathbf{0}}$ vector in the span of the vectors below? If it is, is there a unique combination of the vectors that get to the 0 vector?

$\vec{\mathbf{v}}_1 = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \vec{\mathbf{v}}_2 = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} \vec{\mathbf{v}}_3 = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$

check:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -3 & -6 & 0 \\ 0 & -6 & -12 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

(c) What can you say about the *homogeneous* system of equations $Ax = \vec{\mathbf{0}}$