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Section 4.2: Inverse Transforms and Transforms of Derivatives

In this section, we will begin to see how the Laplace Transform may be used to solve certain types of equations

We begin by discussing the inverse of the Laplace Transform

Let $F(s)$ denote the Laplace Transform of some function $f(t)$, i.e. $\mathcal{L}\{f(t)\}=F(s)$. Then we call $f(t)$ the inverse Laplace Transform of $F(s)$ and write $f(t)=\mathcal{L}^{-1}\{F(s)\}$

Theorem (Basic Inverse Transforms):

(i) $1=\mathcal{L}^{-1}\{\frac{1}{s}\}$

(ii) $t^n=\mathcal{L}^{-1}\{\frac{n!}{s^{n+1}}\}$, $n=1,2,3,\ldots$

(iii) $e^{at}=\mathcal{L}^{-1}\{\frac{1}{s-a}\}$

(iv) $\sin kt=\mathcal{L}^{-1}\{\frac{k}{s^2+k^2}\}$

(v) $\cos kt=\mathcal{L}^{-1}\{\frac{s}{s^2+k^2}\}$

(vi) $\sinh kt=\mathcal{L}^{-1}\{\frac{k}{s^2-k^2}\}$

(vii) $\cosh kt=\mathcal{L}^{-1}\{\frac{s}{s^2-k^2}\}$

4.3.2 Example: Evaluate $\mathcal{L}^{-1}\{\frac{1}{s^4}\}$

$$\mathcal{L}^{-1}\{\frac{1}{s^4}\}=\frac{6}{6}\mathcal{L}^{-1}\{\frac{1}{s^4}\}=\frac{1}{6}\mathcal{L}^{-1}\{\frac{6}{s^4}\}=\frac{1}{6}\mathcal{L}^{-1}\{\frac{3!}{s^4}\}=\frac{1}{6}t^3.$$

Example: Evaluate $\mathcal{L}^{-1}\{\frac{1}{s^2+16}\}$

$$\mathcal{L}^{-1}\{\frac{1}{s^2+16}\}=\frac{4}{4}\mathcal{L}^{-1}\{\frac{1}{s^2+16}\}=\frac{1}{4}\mathcal{L}^{-1}\{\frac{4}{s^2+16}\}=\frac{1}{4}\sin 4t$$

Like the Laplace transform, the inverse Laplace transform is linear

$$\mathcal{L}^{-1}\{\alpha F(s)+\beta G(s)\} = \alpha\mathcal{L}^{-1}\{F(s)\} + \beta\mathcal{L}^{-1}\{G(s)\}$$

where F and G are transforms of some functions f and g.

Example: Evaluate $\mathcal{L}^{-1}\left\{\frac{3s-7}{s^2+9}\right\}$

$$\mathcal{L}^{-1}\left\{\frac{3s-7}{s^2+9}\right\} = \mathcal{L}^{-1}\left\{\frac{3s}{s^2+9}\right\} - \mathcal{L}^{-1}\left\{\frac{7}{s^2+9}\right\}$$

$$= 3\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} - \frac{7}{3}\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\} = 3\cos 3t - \frac{7}{3}\sin 3t$$

Partial fraction decomposition will be a useful tool as we solve inverse Laplace Transform problems

Example: Evaluate $\mathcal{L}^{-1}\left\{\frac{1}{s^2+s-20}\right\}$

$$\frac{1}{(s+5)(s-4)} = \frac{A}{s+5} + \frac{B}{s-4}$$

$$1 = A(s-4) + B(s+5) \qquad \frac{1}{9} = B$$

$$-\frac{1}{9} = A$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+5)(s-4)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{9}\left(\frac{1}{s-4} - \frac{1}{s+5}\right)\right\}$$

$$= \frac{1}{9}\left(\mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+5}\right\}\right)$$

$$= \frac{1}{9}e^{4t} - \frac{1}{9}e^{-5t}$$

Transforms of Derivatives

Now we're going to return to the original Laplace transform (not the inverse) to see how it behaves w.r.t. derivatives

Suppose $f'(t)$ is continuous for $t \geq 0$. Then

$$\mathcal{L}\{f'(t)\}=\int_0^\infty e^{-st}f'(t)\,dt \overset{\text{integration by parts}}{=} e^{-st}f(t)\Big|_0^\infty + s\int_0^\infty e^{-st}f(t)\,dt$$

$$= -f(0) + s\mathcal{L}\{f(t)\} \quad \text{assuming } e^{-st}f(t)\to 0 \text{ as } t\to\infty.$$

i.e.

$$\mathcal{L}\{f'(t)\}=sF(s)-f(0)$$

Similarly,

$$\mathcal{L}\{f''(t)\}=\int_0^\infty e^{-st}f''(t)\,dt \overset{\text{integration by parts}}{=} e^{-st}f'(t)\Big|_0^\infty + s\int_0^\infty e^{-st}f'(t)\,dt$$

$$= -f'(0) + s\mathcal{L}\{f'(t)\}$$

$$= -f'(0) + s(sF(s)-f(0))$$
$$= -f'(0) + s^2F(s) - sf(0)$$
$$= s^2F(s)-sf(0)-f'(0)$$

Theorem (Transforms of a Derivative): If $f, f', \ldots f^{(n-1)}$ are continuous on $[0,\infty)$ and are of exponential order and if $f^{(n)}(t)$ is piecewise continuous on $[0,\infty)$, then

$$\mathcal{L}\{f^{(n)}(t)\}=s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)-\ldots-f^{(n-1)}(0)$$

where $F(s)=\mathcal{L}\{f(t)\}$

Given an initial value problem

$$a_n\frac{d^ny}{dt^n}+a_{n-1}\frac{d^{n-1}y}{dx^{n-1}}+\ldots+a_1\frac{dy}{dx}+a_0y=g(t)$$
$$y(0)=y_0,\ y'(0)=y_1,\ y''(0)=y_2,\ \ldots,\ y^{(n-1)}(0)=y_{n-1}$$

The above theorem applied to this problem yeilds

$\frac{1}{s-1} - \frac{s+1}{s^2+9}$ $\qquad$ $\cancel{s^2}+9-(\cancel{s^2}+\cancel{s}-\cancel{s}-1)$

$= 10$ ✓

$a_n\mathcal{L}\{\frac{d^ny}{dx^n}\} + a_{n-1}\mathcal{L}\{\frac{d^{n-1}y}{dx^{n-1}}\} + \dots + a_0\mathcal{L}\{y\} = \mathcal{L}\{g(t)\}$

and by substituting in initial conditions, the left-hand side becomes a function of $Y(s)$

Example: Solve $2\frac{dy}{dt} + y = 0$ subject to $y(0) = -3$

$\mathcal{L}\{2\frac{dy}{dt} + y\} = \mathcal{L}\{0\}$ $\qquad$ we could also use integrating factor to solve this equation

$2\mathcal{L}\{\frac{dy}{dt}\} + \mathcal{L}\{y\} = 0$

distribute 2 $\quad 2(sY(s) - y(0)) + Y(s) = 0$

$2sY(s) + 6 + Y(s) = 0$

$(2s+1)Y(s) = -6$

$Y(s) = \frac{-6}{2s+1}$

$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{\frac{3}{2s+1}\} = \frac{-6}{2}\mathcal{L}^{-1}\{\frac{1}{s+\frac{1}{2}}\}$

$y(t) = -3e^{-\frac{1}{2}t}$

Example: $\frac{d^2y}{dx^2} + 9y = e^t$ $\quad y(0) = 0 \quad y'(0) = 0$

$\mathcal{L}\{\frac{d^2y}{dx^2} + 9y\} = \mathcal{L}\{e^t\}$

$\mathcal{L}\{\frac{d^2y}{dx^2}\} + 9\mathcal{L}\{y\} = \mathcal{L}\{e^t\}$

$s^2Y(s) - sy(0) - y'(0) + 9Y(s) = \frac{1}{s-1}$

$s^2Y(s) + 9Y(s) = \frac{1}{s-1}$

$(s^2+9)Y(s) = \frac{1}{s-1}$ $\qquad$ $\frac{1}{(s-1)(s^2+9)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+9}$

$Y(s) = \frac{1}{(s-1)(s^2+9)}$ $\qquad$ $1 = A(s^2+9) + (Bs+C)(s-1)$

$Y(s) = \frac{1}{10}(\frac{1}{s-1} - \frac{s+1}{s^2+9})$ $\qquad$ at $s=1 \quad 1 = A(10)$

$Y(s) = \frac{1}{10}(\frac{1}{s-1} - \frac{s}{s^2+9} - \frac{1}{s^2+9})$ $\qquad$ $A = \frac{1}{10}$

what is the benefit of transform? we don't need to use variation of parameters for example to find a particular solution

$y(t) = \frac{1}{10}(e^t - \cos 3t - \frac{1}{3}\sin 3t)$ $\qquad$ $As^2 + Bs^2 = 0 \quad (A+B)s^2 = 0$

$\Rightarrow B = -\frac{1}{10}$

$A9 - C = 1$

$\frac{9}{10} - C = 1 \qquad C = -\frac{1}{10}$