Basis: basis for a vector space (or subspace) V is a set of vectors $\mathcal{B} = \{\vec{\mathsf{v}}_1, \ldots, \vec{\mathsf{v}}_n\}$ such that

1.
2.

Alternatively, for each vector $\vec{\mathsf{v}} \in V$, we can write

$$\vec{\mathsf{v}} = c_1\vec{\mathsf{v}}_1 + c_2\vec{\mathsf{v}}_2 + \cdots + c_n\vec{\mathsf{v}}_n, \qquad \text{for } \underline{\text{unique}} \text{ weights } c_1, c_2, \ldots, c_n \in \mathbb{R}.$$

These weights are the *coordinates* of $\vec{\mathsf{v}}$ with respect to the basis $\mathcal{B}$ often written as

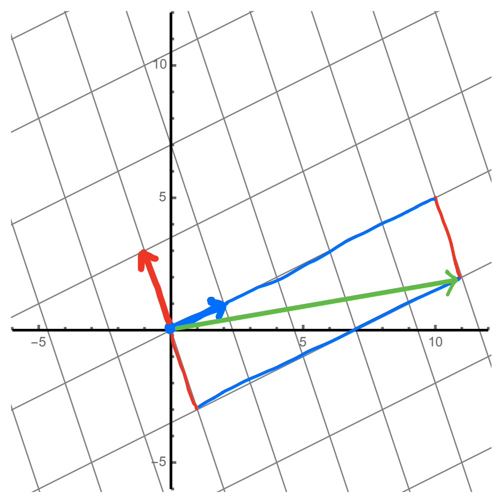
$$\vec{\mathsf{v}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}_{\mathcal{B}} = c_1\vec{\mathsf{v}}_1 + c_2\vec{\mathsf{v}}_2 + \cdots + c_n\vec{\mathsf{v}}_n.$$

Examples

1. Consider the basis $\mathcal{B} = \{\vec{\mathsf{u}}, \vec{\mathsf{v}}\}$ and the vector $\vec{\mathsf{b}} \in \mathbb{R}^2$.

   (a) Give the coordinates of the green vector $\vec{\mathsf{b}}$ in the standard basis $\mathcal{S} = \{\mathbf{e}_1, \mathbf{e}_2\}$ and in the basis $\mathcal{B}$.

$$\mathcal{B} = \left\{ \vec{\mathsf{u}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \vec{\mathsf{v}} = \begin{bmatrix} -1 \\ 3 \end{bmatrix} \right\}$$



   (b) Observe that the algebraic way to find these coordinates is to augment and row reduce!

$$\left[\begin{array}{cc|c} 2 & -1 & 11 \\ 1 & 3 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 2 & -1 & 11 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & -7 & 7 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & 1 & -1 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & -1 \end{array}\right]$$

   (c) Make the change-of-basis matrix B, find B^{-1}, and demonstrate that these matrices take us back and forth between $[\vec{\mathsf{b}}]_{\mathcal{S}}$ and $[\vec{\mathsf{b}}]_{\mathcal{B}}$.

2. Here is a basis of $\mathbb{R}^3$: $\qquad \mathcal{B} = \left\{ \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$

   (a) The vector below is given in $\mathcal{B}$ coordinates. Give its standard coordinates:

   $$\vec{v} = \begin{bmatrix} 5 \\ 4 \\ 0 \end{bmatrix}_{\mathcal{B}}$$

   (b) The vector below is given in standard $\mathcal{S}$ coordinates. Describe *two* ways to find its $\mathcal{B}$ coordinates.

   $$\mathsf{w} = \begin{bmatrix} 12 \\ 6 \\ 24 \end{bmatrix}_{\mathcal{S}}$$

3. Color theory. We often describe colors using mixtures of Red, Green, and Blue. We will think of this as our standard color basis. An alternative, is to describe the colors in terms of Cyan, Magenta, and Yellow. Below we express CMY in terms of RGB.

$$RGB = \left\{ R = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, G = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}, \qquad CMY = \left\{ C = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, M = \begin{bmatrix} .80 \\ .25 \\ .50 \end{bmatrix}, Y = \begin{bmatrix} 1.0 \\ 0.9 \\ 0.1 \end{bmatrix} \right\},$$

   Olive uses 0.7 red, 0.7 green, and 0.4 blue. Use the information below to give the coordinates of olive in RGB and CMY.

```
     cyan magenta yellow olive
[1,]    0    0.80    1.0   0.7
[2,]    1    0.25    0.9   0.7
[3,]    1    0.50    0.1   0.4
```

$\longrightarrow$

```
     cyan magenta yellow     olive
[1,]    1       0      0 0.2073034
[2,]    0       1      0 0.2921348
[3,]    0       0      1 0.4662921
```

4. $A = \begin{bmatrix} 1 & 1 & 3 & 1 & 2 \\ -2 & 1 & -3 & -1 & -9 \\ -1 & 1 & -1 & 2 & -3 \\ 4 & 1 & 9 & 3 & 13 \\ -2 & 3 & -1 & 2 & -10 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 0 & 3 \\ 0 & 1 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. $\quad \mathsf{Nul}(A) = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = s \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 2 \\ 0 \\ -1 \\ 1 \end{bmatrix} \middle| s, t \in \mathbb{R} \right\}$

(a) Here is our "standard" basis of the **null space**. Give the coordinates of $\vec{\mathsf{v}}$ in this basis.

$$\mathcal{B}_{\text{Nul}} = \left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}, \qquad \vec{\mathsf{v}} = \begin{bmatrix} -1 \\ -11 \\ 5 \\ 3 \\ -3 \end{bmatrix}.$$

(b) Here our basis of the **column space** of this matrix. Give the coordinates of $\vec{\mathsf{v}}_3$ and $\vec{\mathsf{v}}_5$ in this basis.

$$\mathcal{B}_{\text{Col}} = \left\{ \vec{\mathsf{v}}_1 = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 4 \\ -2 \end{bmatrix}, \vec{\mathsf{v}}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 3 \end{bmatrix}, \vec{\mathsf{v}}_4 = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \\ 2 \end{bmatrix} \right\}, \qquad \vec{\mathsf{v}}_3 = \begin{bmatrix} 3 \\ -3 \\ -1 \\ 9 \\ -1 \end{bmatrix}, \vec{\mathsf{v}}_5 = \begin{bmatrix} 2 \\ -9 \\ -3 \\ 13 \\ -10 \end{bmatrix}$$

(c) Are the vectors $\mathsf{w}_1 = [6, -17, 0, 29, -11]^T$ or $\mathsf{w}_2 = [5, 10, 3, 12, 7]^T$ in $\mathsf{Col}(A)$? If so, what are their coordinates in the basis?

$$\left[\begin{array}{ccc|cc} 1 & 1 & 1 & 6 & 5 \\ -2 & 1 & -1 & -17 & 10 \\ -1 & 1 & 2 & 0 & 3 \\ 4 & 1 & 3 & 29 & 12 \\ -2 & 3 & 2 & -11 & 7 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 5 & 0 \\ 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

5. Let $\mathcal{S} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ be the standard basis of $\mathbb{R}^4$, and define $\mathcal{B} = \{\mathsf{w}_1, \mathsf{w}_2, \mathsf{w}_3, \mathsf{w}_4\}$ where

$$\mathsf{w}_1 = \mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3+\mathbf{e}_4 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \qquad \mathsf{w}_2 = \mathbf{e}_2+\mathbf{e}_3+\mathbf{e}_4 = \begin{bmatrix}0\\1\\1\\1\end{bmatrix}, \qquad \mathsf{w}_3 = \mathbf{e}_3+\mathbf{e}_4 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix} \qquad \mathsf{w}_4 = \mathbf{e}_4 = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}.$$

(a) Write down the change-of-basis matrix B that changes between $\mathcal{B}$ and $\mathcal{S}$.

(b) Enter your matrix B in `R`.

(c) Use your matrix B to give $\vec{\mathsf{v}} = \begin{bmatrix}1\\2\\3\\4\end{bmatrix}_{\mathcal{B}}$ in standard coordinates.

(d) Use augmentation and row reduction to express $\mathsf{w} = \begin{bmatrix}1\\2\\2\\5\end{bmatrix}_{\mathcal{S}}$ in $\mathcal{B}$-coordinates.

(e) Use B^{-1} to express $\mathsf{w} = \begin{bmatrix}1\\2\\2\\5\end{bmatrix}_{\mathcal{S}}$ in $\mathcal{B}$-coordinates.

(f) Use B^{-1} to convert your answer to part (c) back to $\mathcal{B}$-coordinates

(g) Express $\mathbf{e}_2$ in $\mathcal{B}$ coordinates. Compare your answer with the second column of B^{-1} and discuss what you see.