Thursday, October 6

1. Welcome!
2. HW 4 due tonight - ish
   ↳ problem 5 can kick to next hw
3. Looking ahead
   ↳ topic / pairings by next week (Moodle submit)
   ↳ hw 5 will go up Friday / weekend
   ↳ hw 5 due Tuesday, 10/18 (tues. before fall break)
4. Questions?
5. Kruskal's!
6. Outro
   ↳ small work: topics

Now that we've defined Kruskal's algorithm and proven it's correct, let's prove that a tree generated this way creates an optimal solution to the integer program that we described last time. First, let's review how that program is defined.

Minimum spanning tree program (which we immediately reformat)

$$\begin{cases} \min & \sum c_e x_e \\ \text{s.t.} & \sum x_e = |V| - 1 \\ & \sum\limits_{e \in H} x_e \le |H| - 1 \quad \forall H \subsetneq V \\ & x_e \ge 0 \end{cases} \rightarrow \left(\begin{array}{l} \text{for a vector} \\ v \in \mathbb{R}^A \\ \text{and } B \subseteq A \\ v(B) = \sum\limits_{b \in B} v_b \end{array} \right) \rightarrow (P) \begin{cases} \min & c^T x \\ \text{s.t.} & x(E) = |V| - 1 \\ & x(\gamma(S)) \le |S| - 1 \quad (\star) \\ & \qquad \forall S,\ \emptyset \ne S \subset V \\ & x_e \ge 0 \end{cases}$$

$\gamma(S)$ → edges contained in S

Note that a characteristic vector for any spanning tree is a feasible solution to this program, so the cost of a MST is an *upper bound* for the optimal value of this program.

Why?

we can achieve an MST cost, can we do better?

Possibly unsurprisingly, this upper bound is actually equality.

Theorem. *Let x^* be the characteristic vector of a minimum spanning tree. Then x^* is an optimal solution of (P).*

This proof is going to be pretty involved, so let's break it into some steps.

Step 1: *Write (P) in an equivalent form, (P'), that's easier to work with.*

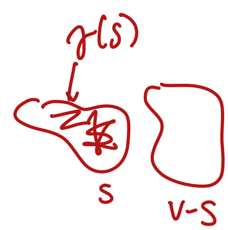
For $A \subseteq E$, let $\kappa(A)$ be the number of components of the graph (V, A). We define

$$(P') \begin{cases} \min & c^T x \\ \text{s.t.} & x(E) = |V| - 1 \\ & x(A) \le |V| - \kappa(A) \quad (\star\star) \text{ *!} \\ & \qquad \forall A,\ A \subset E \\ & x_e \ge 0 \end{cases}$$

Claim: these program have the same feasible solutions.

$(**) \Rightarrow (*)$ Let $A = \gamma(S)$ Then

$$x(\gamma(S)) \le |V| - \underbrace{\kappa(\gamma(S))}_{\ge |V| - |S| + 1} \le |V| - (|V| - |S| + 1) = |S| - 1$$



$(*) \Rightarrow (**)$ let $A \subseteq E$, $S_1 \dots S_k$ the components of (V, A)

$$x(A) \le \sum_{i=1}^{k} x(\gamma(S_i)) \overset{*}{\le} \sum_{i=1}^{k} |S_i| - 1 = |V| - \underbrace{k}_{\kappa(A)}$$

Step 2: *Build the dual program of (P'), which we'll call (D').*

Our strategy for showing x^* is optimal is to build a feasible solution for the dual and show it satisfies complementary slackness. That means we're gonna need the dual.

$$(P') \to \begin{cases} \max & -c^T x \\ \text{s.t.} & x(E) = |V| - 1 \\ & x(A) \leq |V| - \kappa(A) \\ & \qquad \forall A,\ A \subset E \\ & x_e \geq 0 \end{cases} \xrightarrow{\textit{Dual}} (D') \begin{cases} \max & \sum\limits_{A \subseteq E} (|V| - \kappa(A)) y_A \\ \text{s.t.} & \sum_{A \ni e} y_A \geq -c_e \quad \forall e \in E \\ & y_A \geq 0 \quad \forall A \subset E \end{cases}$$

y_E free!

Why?

$$\begin{array}{cc} & \text{edges} \\ A \subset E & \begin{bmatrix} & \\ & \end{bmatrix} \end{array} \begin{matrix} \leq \\ \\ = \end{matrix} \begin{bmatrix} |V| - \kappa(A) \\ \vdots \\ |V| - 1 \end{bmatrix}$$

Step 3: *Build y^* out of x^* and show it's dual feasible.*

Since we proved Kruskal's algorithm yields a minimum spanning tree, we can assume $x*$ is built from Kruskal's.

Let $\{e_1, e_2, \ldots e_m\}$ be the order in which Kruskal's processes the edges. Define $R_i = \{e_1 \ldots e_i\}$ and build y^* as follows (recall, y is indexed by subsets of the edges.)

$$y^*_A = \begin{cases} 0 & A \neq R_i \ \forall i \\ c_{e_{i+1}} - c_{e_i} & A = R_i \text{ for } i \in [1, 2, \ldots m-1] \\ -c_{e_m} & A = R_m \end{cases}$$

Claim: y^* is a feasible solution for (D').

Step 4: *x^* and y^* satisfy complementary slackness.*

Recall that the complementary slackness condition states that if any variable is nonzero, its associated constraint must be tight (go back and check the proof for weak duality).

- If $x_e^* > 0$,

- If $y_A^* > 0$, then $A = R_i$ for some i. Consider the corresponding constraint in (P'),

$$x(R_i) \le |V| - \kappa(R_i)$$

  Note that $x(R_i) = |T \cap R_i|$, or the state of the Kruskal's forest after we process edge i.

  Claim: $|T \cap R_i| + \kappa(R_i) = |V|$.

  Which proves the equality is tight and completes the proof.