$A^{-1} = \frac{1}{\det(A)} [\ \ ]$

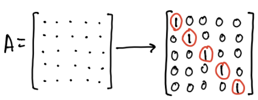
Determinants.

If A is an $n \times n$ matrix, then $\det(A)$ is a number. What does it tell us?

$\boxed{\det(A) \neq 0} \iff \boxed{A \text{ is invertible}}$

Invertible Matrix Theorem Revisited

If A is an nxn matrix, then the following statements are equivalent



- A is invertible
- $RREF(A) = I_n$
- A has a pivot in every row
- A has a pivot in every column
- $T(x) = Ax$ is one-to-one
- $T(x) = Ax$ is onto
- The columns of A span $\mathbb{R}^n$
- The columns of A are linearly independent
- $Ax = b$ has exactly one solution for all $b \in \mathbb{R}^n$
- $Ax = 0$ has only the 0 solution
- The columns of A are a basis of $\mathbb{R}^n$
- $Col(A) = \mathbb{R}^n$
- $\dim(Col(A)) = \mathbb{R}^n$
- $rank(A) = n$
- $Nul(A) = \{0\}$
- $nullity(A) = 0$
- $\boxed{\det(A) \neq 0}$

Determinant of a 4x4 Matrix

$$\det\left(\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{bmatrix}\right) =$$

each summand has exactly one entry from each row and each column

$$a_{1,4}\,a_{2,3}\,a_{3,2}\,a_{4,1} - a_{1,3}\,a_{2,4}\,a_{3,2}\,a_{4,1} - a_{1,4}\,a_{2,2}\,a_{3,3}\,a_{4,1} + a_{1,2}\,a_{2,4}\,a_{3,3}\,a_{4,1} +$$
$$a_{1,3}\,a_{2,2}\,a_{3,4}\,a_{4,1} - a_{1,2}\,a_{2,3}\,a_{3,4}\,a_{4,1} - a_{1,4}\,a_{2,3}\,a_{3,1}\,a_{4,2} + a_{1,3}\,a_{2,4}\,a_{3,1}\,a_{4,2} +$$
$$a_{1,4}\,a_{2,1}\,a_{3,3}\,a_{4,2} - a_{1,1}\,a_{2,4}\,a_{3,3}\,a_{4,2} - \boxed{a_{1,3}\,a_{2,1}\,a_{3,4}\,a_{4,2}} + a_{1,1}\,a_{2,3}\,a_{3,4}\,a_{4,2} +$$
$$a_{1,4}\,a_{2,2}\,a_{3,1}\,a_{4,3} - a_{1,2}\,a_{2,4}\,a_{3,1}\,a_{4,3} - a_{1,4}\,a_{2,1}\,a_{3,2}\,a_{4,3} + a_{1,1}\,a_{2,4}\,a_{3,2}\,a_{4,3} +$$
$$a_{1,2}\,a_{2,1}\,a_{3,4}\,a_{4,3} - a_{1,1}\,a_{2,2}\,a_{3,4}\,a_{4,3} - a_{1,3}\,a_{2,2}\,a_{3,1}\,a_{4,4} + a_{1,2}\,a_{2,3}\,a_{3,1}\,a_{4,4} +$$
$$a_{1,3}\,a_{2,1}\,a_{3,2}\,a_{4,4} - a_{1,1}\,a_{2,3}\,a_{3,2}\,a_{4,4} - a_{1,2}\,a_{2,1}\,a_{3,3}\,a_{4,4} + a_{1,1}\,a_{2,2}\,a_{3,3}\,a_{4,4}$$

$4 \cdot 3 \cdot 2 \cdot 1$

$4! =$ 24 summands

$n \times n$ $\quad n!$

Determinant Computations:

1. 2×2 determinants are easy to compute:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

2. Compute the following determinant using diagonals, row reduction, and cofactor expansion $\begin{vmatrix} 1 & 2 & 4 \\ 0 & 2 & 8 \\ 3 & 1 & 2 \end{vmatrix}$.

   (a) Diagonals

$$\left|\begin{matrix} 1 & 2 & 4 \\ 0 & 2 & 8 \\ 3 & 1 & 2 \end{matrix}\right| \begin{matrix} 1 & 2 \\ 0 & 2 \\ 3 & 1 \end{matrix} = (1\cdot 2\cdot 2) + (2\cdot 8\cdot 3) + (4\cdot 0\cdot 1) - (3\cdot 2\cdot 4) - (1\cdot 8\cdot 1) - (2\cdot 0\cdot 2)$$

$$= 4 + 48 + 0 - 24 - 8 - 0$$

$$= 52 - 32$$

$$= 20$$

$\boxed{\text{important:}}$ $\boxed{\text{this does not work for matrices larger than } 3\times 3}$

   (b) Row Reduction

replacement doesnt change determinant

Factor scalar out of a row

replacement: no change

$$\begin{vmatrix} 1 & 2 & 4 \\ 0 & 2 & 8 \\ 3 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 4 \\ 0 & 2 & 8 \\ 0 & -5 & -10 \end{vmatrix} = 2\begin{vmatrix} 1 & 2 & 4 \\ 0 & 1 & 4 \\ 0 & -5 & -10 \end{vmatrix} = 2\begin{vmatrix} 1 & 2 & 4 \\ 0 & 1 & 4 \\ 0 & 0 & 10 \end{vmatrix}$$

$$= 2\cdot 1\cdot 1\cdot 10 = 20$$

$$\det\begin{bmatrix} 5 & - & - & - \\ 0 & 3 & - & - \\ 0 & 0 & 8 & - \\ 0 & 0 & 0 & 2 \end{bmatrix} = 5\cdot 3\cdot 8\cdot 2$$

$\boxed{\text{determinant of a triangular matrix is product of diagonal}}$

Notes:

- swapping rows (not shown here) multiplies the determinant by (-1).
- for larger matrices this is the computationally "easiest" way to compute the determinant
- same amount of work as row reducing to see if it is invertible.

(c) Cofactor Expansion

$\boxed{\text{cofactor expansion}}$ 3 ways $\quad \begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$

$\boxed{\text{row 1:}}$

$$\begin{vmatrix} 1 & 2 & 4 \\ 0 & 2 & 8 \\ 3 & 1 & 2 \end{vmatrix} = +(1)\begin{vmatrix} 2 & 8 \\ 1 & 2 \end{vmatrix} - (2)\begin{vmatrix} 0 & 8 \\ 3 & 2 \end{vmatrix} + (4)\begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix}$$

$$= +(4-8) - 2(0-24) + 4(0-6)$$

$$= -4 + 48 - 24 = 20$$

$\boxed{\text{col 1:}}$

$$\begin{vmatrix} 1 & 2 & 4 \\ 0 & 2 & 8 \\ 3 & 1 & 2 \end{vmatrix} = +(1)\begin{vmatrix} 2 & 8 \\ 1 & 2 \end{vmatrix} - (0)\begin{vmatrix} 2 & 4 \\ 1 & 2 \end{vmatrix} + (3)\begin{vmatrix} 2 & 4 \\ 2 & 8 \end{vmatrix}$$

$$= +(4-8) - 0 + 3(16-8)$$

$$= -4 + 24 = 20$$

$\boxed{\text{row 2:}}$

$$\begin{vmatrix} 1 & 2 & 4 \\ 0 & 2 & 8 \\ 3 & 1 & 2 \end{vmatrix} = -(0)\begin{vmatrix} 2 & 4 \\ 1 & 2 \end{vmatrix} + (2)\begin{vmatrix} 1 & 4 \\ 3 & 2 \end{vmatrix} - (8)\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}$$

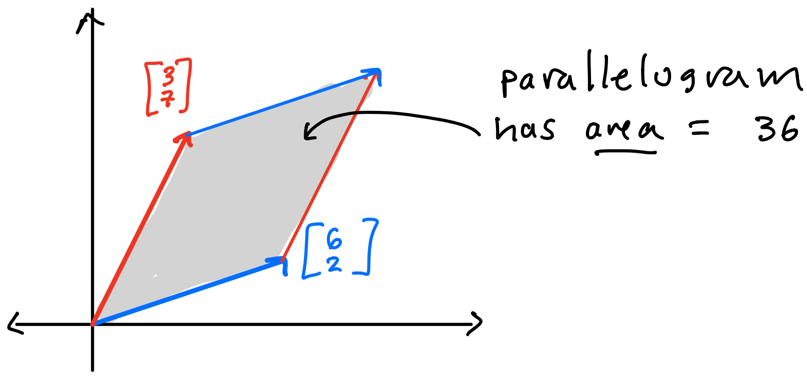
$$= -0 + 2(2-12) - 8(1-6)$$

$$= -20 + 40 = 20$$

3. **But what does the value of the determinant mean?**

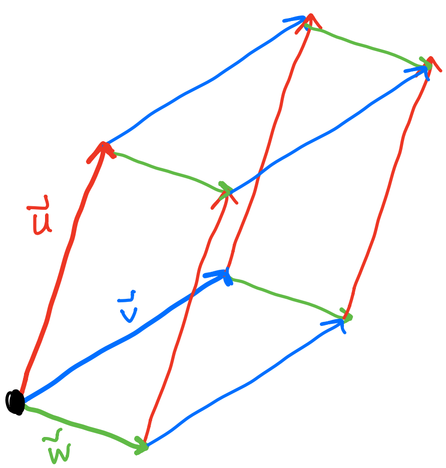
(a) 2×2:

$$\begin{vmatrix} 3 & 6 \\ 7 & 2 \end{vmatrix} = 3\cdot 2 - 6\cdot 7 = 6 - 42 = -36$$



(b) 3×3:

$$\begin{vmatrix} | & | & | \\ u & v & w \\ | & | & | \end{vmatrix} = \text{(signed) volume of the parallelepiped}$$



Determinants

1. Here is a row reduction of a matrix.

   (a) Find its determinant by the row reduction method.

$$\begin{vmatrix} 0 & 2 & -1 \\ 1 & 5 & -10 \\ -4 & 0 & 65 \end{vmatrix} = - \begin{vmatrix} 1 & 5 & -10 \\ 0 & 2 & -1 \\ -4 & 0 & 65 \end{vmatrix} = - \begin{vmatrix} 1 & 5 & -10 \\ 0 & 2 & -1 \\ 0 & 20 & 25 \end{vmatrix} = -5 \begin{vmatrix} 1 & 5 & -10 \\ 0 & 2 & -1 \\ 0 & 4 & 5 \end{vmatrix} = -5 \begin{vmatrix} 1 & 5 & -10 \\ 0 & 2 & -1 \\ 0 & 0 & 7 \end{vmatrix}$$

$$= -5(1 \cdot 2 \cdot 7)$$

$$= -70$$

   (b) Find the same determinant using cofactor expansion along the first row.

$$0 \begin{vmatrix} 5 & -10 \\ 0 & 65 \end{vmatrix} - 2 \begin{vmatrix} 1 & -10 \\ -4 & 65 \end{vmatrix} + (-1) \begin{vmatrix} 1 & 5 \\ -4 & 0 \end{vmatrix}$$

$$= 0 \quad -2(65-40) - (1 \cdot 0 + 20) = -70$$

2. What is this determinant? $\begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 6 & 7 & 8 & 9 \\ 0 & 0 & 8 & 7 & 6 \\ 0 & 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 0 & 3 \end{vmatrix} = 1 \cdot 6 \cdot 8 \cdot 5 \cdot 3 = 720$

3. Use row operations to find this determinant

$$\begin{vmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{vmatrix} = (-1)^3 \begin{vmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{vmatrix} = (-1)^3$$

4. Compute this determinant. Hint look at columns 6 and 7.

$$\begin{vmatrix} -4 & -1 & -3 & -5 & 4 & 2 & 4 & 1 & 3 & -1 \\ 4 & 6 & 2 & -2 & -5 & 1 & 2 & 5 & -3 & 5 \\ -5 & 6 & -2 & 2 & 5 & 0 & 0 & -4 & 3 & 0 \\ -4 & 1 & -1 & -4 & 3 & -2 & -4 & -6 & 1 & -1 \\ -1 & -2 & 2 & -6 & -5 & 3 & 6 & 2 & -2 & -1 \\ 5 & 5 & -4 & -5 & 4 & 1 & 2 & -6 & 4 & -4 \\ -6 & -5 & 4 & -3 & -6 & -4 & -8 & 5 & -2 & -6 \\ 2 & -5 & 2 & 1 & 5 & -6 & -12 & 2 & -2 & 3 \\ -5 & 0 & 1 & 0 & -6 & 3 & 6 & -6 & 3 & 3 \\ -2 & 0 & -1 & 0 & 2 & -2 & -4 & 1 & 5 & 3 \end{vmatrix} = 0$$

v_6 v_7

$v_7 = 2v_6$

Eigenvalues: time permitting, we will discuss eigenvalues together. Then try these problems.

1. Find the characteristic polynomial and eigenvalues of the following matrices.

   (a) $\mathsf{A} = \begin{bmatrix} 5 & 4 \\ 2 & -2 \end{bmatrix}$

$$A - \lambda I_2 = \begin{bmatrix} 5 & 4 \\ 2 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 2 & -2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 5-\lambda & 4 \\ 2 & -2-\lambda \end{bmatrix}$$

$$\det\left(\begin{bmatrix} 5-\lambda & 4 \\ 2 & -2-\lambda \end{bmatrix}\right) = (5-\lambda)(-2-\lambda) - 8$$

$$= -10 + 2\lambda - 5\lambda + \lambda^2 - 8$$

$$= \lambda^2 - 3\lambda - 18 \quad \leftarrow \text{quadratic formula}$$

$$= (\lambda - 6)(\lambda + 3)$$

$\boxed{\lambda = 6}$ $\boxed{\lambda = -3}$ only possible eigenvalues

   (b) $\mathsf{B} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 5 & 4 \\ 1 & 2 & -2 \end{bmatrix}$

2. Show that these are eigenvectors of the matrices above by multiplying

   (a) $\mathsf{v}_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \mathsf{v}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

   (b) $\vec{\mathsf{w}}_1 = \begin{bmatrix} 20 \\ -8 \\ 1 \end{bmatrix}, \vec{\mathsf{w}}_2 = \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix}, \vec{\mathsf{w}}_3 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$.