1.4. Matrix Equations

3 important points of view on the same problem

All three are solved by row reducing

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ -4 & 2 & 0 & -20 \\ 2 & 1 & -2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

1.1-1.2. Linear System of Equations

$$\left\{\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 3x_3 & = & 9 \\ -4x_1 & + & 2x_2 & & & = & -20 \\ 2x_1 & + & x_2 & - & 2x_3 & = & 0 \end{array}\right\}$$

1.3. Vector Equation

$$x_1 \begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 9 \\ -20 \\ 0 \end{bmatrix}$$

1.4. Matrix Equation

$$\begin{bmatrix} 1 & 2 & 3 \\ -4 & 2 & 0 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ -20 \\ 0 \end{bmatrix}$$

3 Representations of the Same Problem

1. Solve the following system of equations

$$\left\{\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 3x_3 & = & 9 \\ -4x_1 & + & 2x_2 & & & = & -20 \\ 2x_1 & + & x_2 & - & 2x_3 & = & 0 \end{array}\right\}$$

Augmented Matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ -4 & 2 & 0 & -20 \\ 2 & 1 & -2 & 0 \end{array}\right] \xrightarrow[\text{Operations}]{\text{Elementary Row}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Reduced Row Echelon Form RREF

Unique solution: (4,-2,3)

2. Solve the vector equation

$$x_1 \underbrace{\begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix}}_{\mathbf{u}} + x_2 \underbrace{\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}}_{\mathbf{v}} + x_3 \underbrace{\begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}}_{\mathbf{w}} = \underbrace{\begin{bmatrix} 9 \\ -20 \\ 0 \end{bmatrix}}_{\mathbf{b}}.$$

- Write **b** as a <u>linear combination</u> of **u**, **v**, **w**.
- There are a <u>unique</u> set of weights 4, -2, 3 so that **b** = 4 **u** - 2 **v** + 3 **w**.
- **b** is in the <u>span</u> of **u**, **v**, **w**.

3. Solve the Matrix Equation

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ -4 & 2 & 0 \\ 2 & 1 & -2 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 9 \\ -20 \\ 0 \end{bmatrix}}_{\mathbf{b}}$$

The <u>solution</u> to the matrix equation **A x = b** is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix}$$

Matrix Equation

Matrix notation is just a shorthand for the vector notation

$$\underset{}{x_1} \underbrace{\begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix}}_{\mathbf{u}} + x_2 \underbrace{\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}}_{\mathbf{v}} + x_3 \underbrace{\begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}}_{\mathbf{w}} = \underbrace{\begin{bmatrix} 9 \\ -20 \\ 0 \end{bmatrix}}_{\mathbf{b}}.$$

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ -4 & 2 & 0 \\ 2 & 1 & -2 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 9 \\ -20 \\ 0 \end{bmatrix}}_{\mathbf{b}}$$

- The question is:

  Can we write **b** as a <u>linear combination</u> of the columns of **A**?

- Or, is **b** is in the <u>span</u> of the columns of **A**?

- Can we multiply **A** by some vector **x** to get **b**?

The Matrix-Vector Product: $\mathbf{A}\,\mathbf{x}$

Two perspectives:

1. Linear combination of the columns of A:

$$\begin{bmatrix} 1 & -2 & 3 \\ 1 & 0 & 1 \\ 2 & 4 & -2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \\ -2 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + 4\begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix} + (-2)\begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -11 \\ 1 \\ 26 \end{bmatrix}$$

This is the best way to **interpret** Ax

2. Dot product of the rows of A with x:

$$\begin{bmatrix} 1 & -2 & 3 \\ 1 & 0 & 1 \\ 2 & 4 & -2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 1\cdot 3 + (-2)\cdot 4 + 3\cdot(-2) \\ 1\cdot 3 + 0\cdot 4 + 1\cdot(-2) \\ 2\cdot 3 + 4\cdot 4 + -2\cdot(-2) \end{bmatrix} = \begin{bmatrix} -11 \\ 1 \\ 26 \end{bmatrix}$$

This is the best way to **compute** Ax

Example 1 Solve the matrix equation **A x = b** below.

$$\underbrace{\begin{bmatrix} 1 & -2 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\vec{\mathbf{x}}} = \underbrace{\begin{bmatrix} 9 \\ 3 \\ -5 \\ 2 \end{bmatrix}}_{\vec{\mathbf{b}}}$$

This is the same as writing **b** as a linear combination of the columns of **A**.

$$\vec{x} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$$ is the <u>unique</u> solution

CHECK:

$$\begin{bmatrix} 1 & -2 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2\cdot 1 - 3\cdot(-2) + 1 \\ 2\cdot 1 - 3\cdot 0 + 1 \\ 2\cdot(-1) - 3\cdot 1 + 0 \\ 2\cdot 2 - 3\cdot 1 + 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \\ -5 \\ 2 \end{bmatrix}$$

<u>Augment and row reduce</u>:

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 9 \\ 1 & 0 & 1 & 3 \\ -1 & 1 & 0 & -5 \\ 2 & 1 & 1 & 2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -2 & 1 & 9 \\ 0 & 2 & 0 & -6 \\ 0 & -1 & 1 & 4 \\ 0 & 5 & -1 & -16 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -2 & 1 & 9 \\ 0 & 1 & 0 & -3 \\ 0 & -1 & 1 & 4 \\ 0 & 5 & -1 & -16 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -2 & 1 & 9 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -2 & 1 & 9 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -2 & 0 & 8 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Example 2 Solve the matrix equation **A x** = **b** below.

$$\underbrace{\begin{bmatrix} 1 & -2 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\vec{\mathbf{x}}} = \underbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}}_{\vec{\mathbf{b}}}$$

- **A x** = **b** does not have a solution
- **b** cannot be written as a linear combination of the columns of **A**
- **b** is not in the span of the columns of **A**
- The following system of equations is inconsistent

$$\left\{ \begin{array}{rcrcrcl} x_1 & - & 2x_2 & + & x_3 & = & 1 \\ x_1 & & & + & x_3 & = & 2 \\ -x_1 & + & x_2 & & & = & 3 \\ 2x_1 & + & x_2 & + & x_3 & = & 4 \end{array} \right\}$$

<u>Augment and row reduce</u>:

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 1 & 0 & 1 & 2 \\ -1 & 1 & 0 & 3 \\ 2 & 1 & 1 & 4 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Inconsistent 0 = 1

Example 3 Solve the matrix equation **A x** = **b** below.

$$\underbrace{\begin{bmatrix} 1 & 1 & -1 & 2 \\ -2 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{\vec{\mathbf{x}}} = \underbrace{\begin{bmatrix} 6 \\ -4 \\ 4 \end{bmatrix}}_{\vec{\mathbf{b}}}$$

- **A x** = **b** has infinitely many solutions
- **b** can be written as a linear combination of the columns of **A** in infinitely many ways
- **b** is in the span of the columns of **A**

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 + x_4 \\ 3 - 2x_4 \\ -2 + x_4 \\ x_4 \end{bmatrix}$$

examples: $\underset{x_4=0}{\begin{bmatrix} 1 \\ 3 \\ -2 \\ 0 \end{bmatrix}}$ $\underset{x_4=1}{\begin{bmatrix} 2 \\ 1 \\ -1 \\ 1 \end{bmatrix}}$ $\underset{x_4=100}{\begin{bmatrix} 101 \\ -197 \\ 98 \\ 100 \end{bmatrix}}$

Augment and row reduce:

Free ↓

pivot

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 6 \\ -2 & 0 & 1 & 1 & -4 \\ 1 & 1 & 0 & 1 & 4 \end{array}\right] \longrightarrow \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 2 & 3 \\ 0 & 0 & 1 & -1 & -2 \end{array}\right] \end{array}$$

General solution

$$\boxed{\begin{aligned} x_1 &= 1 + x_4 \\ x_2 &= 3 - 2x_4 \\ x_3 &= -2 + x_4 \\ x_4 &= \text{free} \end{aligned}}$$

Example 4 Describe all vectors **b** such that A **x** = **b** is consistent

$$\begin{bmatrix} 1 & -1 & 7 \\ 3 & 0 & 12 \\ 2 & 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

- The columns of **A** do not span $\mathbf{R}^3$
- Only those that satisfy the condition below

<u>Augment and row reduce</u>:

$$\left[\begin{array}{ccc|c} 1 & -1 & 7 & b_1 \\ 3 & 0 & 12 & b_2 \\ 2 & 3 & -1 & b_3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1 & 7 & b_1 \\ 0 & 3 & -9 & -3b_1 + b_2 \\ 0 & 5 & -15 & -2b_1 + b_3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1 & 7 & b_1 \\ 0 & 1 & -3 & -b_1 + \frac{1}{3} b_2 \\ 0 & 5 & -15 & -2b_1 + b_3 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & -1 & 7 & b_1 \\ 0 & 1 & -3 & -b_1 + \frac{1}{3} b_2 \\ 0 & 0 & 0 & \underbrace{5b_1 - \frac{5}{3} b_2 - 2b_1 + b_3} \end{array}\right]$$

$$3b_1 - \frac{5}{3} b_2 + b_3 = 0$$

Example 5 Do the columns of these vectors span $\mathbf{R}^4$? Uniquely?

Transformed $\vec{b}$

$$\left[\begin{array}{c|c} A & \begin{matrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{matrix} \end{array}\right] \rightarrow \left[\begin{array}{c|c} \underbrace{\begin{matrix} \blacksquare & * & * & * & * \\ 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 & \blacksquare \end{matrix}}_{REF(A)} & \begin{matrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{matrix} \end{array}\right]$$

- Never get a 0 = 1 row, no matter what **b** is.
- Always consistent
- Can get to all vectors in $\mathbf{R}^4$
- The columns span $\mathbf{R}^4$
- Free variables so do not get to the vectors uniquely: no pivot in the third column
- A has a pivot in every row

Transformed $\vec{b}$

$$\left[\begin{array}{c|c} A & \begin{matrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{matrix} \end{array}\right] \rightarrow \left[\begin{array}{c|c} \begin{matrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 \end{matrix} & \begin{matrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{matrix} \end{array}\right]$$

REF(A)

- No pivot in 4th row
- There will be some vectors you cannot get to (see Example 4)
- Columns do not span $\mathbf{R}^4$
- Free variables: if you can get to a vector, you will not get there uniquely

Matrix Spanning Theorem

THEOREM 4

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Let A be an $m \times n$ matrix. Then the following statements are logically equivalent. That is, for a particular A, either they are all true statements or they are all false.

a. For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.

b. Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.

c. The columns of A span $\mathbb{R}^m$.

d. A has a pivot position in every row.

$$\left[\begin{array}{c|c} & b_1 \\ A & b_2 \\ & b_3 \\ & b_4 \end{array}\right] \longrightarrow \left[\underbrace{\begin{matrix} \blacksquare & * & * & * & * \\ 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 & \blacksquare \end{matrix}}_{\text{REF}(A)} \;\middle|\; \begin{matrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{matrix}\right]$$

You Try

Do these vectors span R^3 ?

$$\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -10 \\ 11 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 1 & 1 \\ -2 & 2 & -10 & 2 \\ 3 & 1 & 11 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Do these vectors span R^4 ?

$$\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Do these vectors span R^5 ?

$$\begin{bmatrix} 1 \\ 1 \\ 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 1 \\ 1 \\ 4 \end{bmatrix}$$