6.2. Orthogonal Sets

Example 1

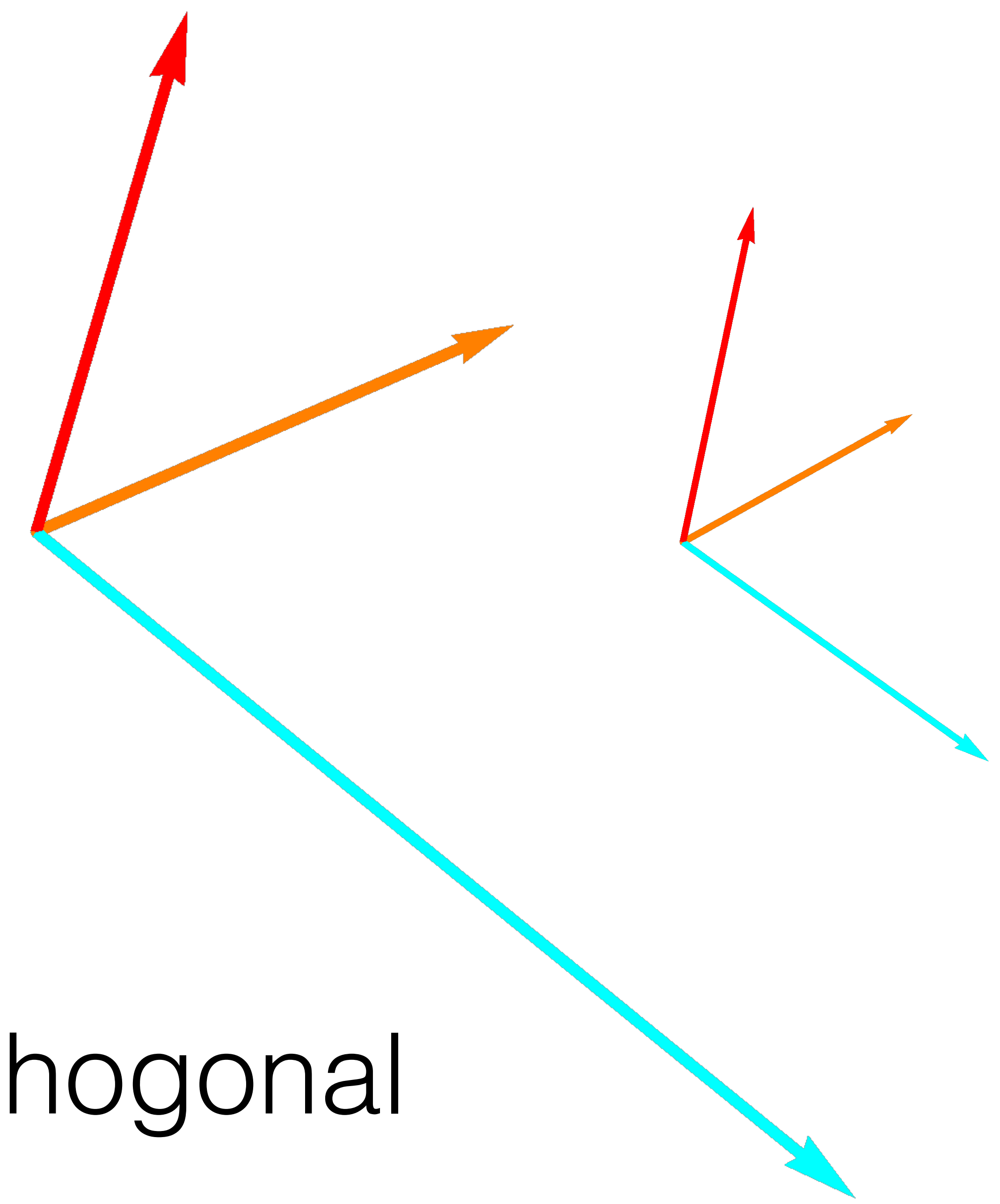
Compute the dot product between <u>all</u> the vectors in this set:

$$\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \\ 2 \end{bmatrix} \right\} \qquad A = \begin{bmatrix} 1 & 1 & 3 & -1 \\ 1 & 2 & 1 & -1 \\ -1 & 3 & 1 & 1 \\ -1 & 4 & 3 & 2 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 2 & 3 & 4 \\ 3 & 1 & 1 & 3 \\ -1 & -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 & -1 \\ 1 & 2 & 1 & -1 \\ -1 & 3 & 1 & 1 \\ -1 & 4 & 3 & 2 \end{bmatrix} = \begin{bmatrix} \mathbf{4} & -4 & 0 & -5 \\ -4 & 30 & 20 & 8 \\ 0 & 20 & 20 & 3 \\ -5 & 8 & 3 & 7 \end{bmatrix}$$

Example 2

Compute the dot product between <u>all</u> the vectors in this set:



$$\mathcal{S} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

$$A^T A = \begin{bmatrix} 0 & 1 & 1 \\ -2 & -1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & -2 & 1 \\ 1 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Mutually orthogonal

$$\begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ -2/\sqrt{6} & -1/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 0 & -2/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & -1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

orthonormal

$$AA^T = I$$

$$A^{-1} = A^T$$

Orthogonal Sets

If $A = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_k \\ | & | & & | \end{bmatrix}$ Then

$$A^T A = \begin{bmatrix} — & v_1 & — \\ — & v_2 & — \\ & \vdots & \\ — & v_k & — \end{bmatrix} \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_k \\ | & | & & | \end{bmatrix} = \begin{array}{cc} & j \\ i & \begin{bmatrix} & \vdots & \\ \cdots & \bigcirc & \\ & & \end{bmatrix} \end{array}$$

$v_i \cdot v_j$

Matrix of all dot products

$v_i \cdot v_j$

Orthogonal Sets

Def. A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is ***orthogonal*** if $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ for all $i \neq j$

The set is ***orthonormal*** if also $\mathbf{v}_i \cdot \mathbf{v}_i = 1$ for all i. That is, $||\mathbf{v}_i|| = 1.$

- If the columns of n x n matrix **A** are orthogonal, then $\mathbf{A}^T\mathbf{A}$ = diagonal
- If the columns of n x n matrix **A** are orthonormal, then $\mathbf{A}^T\mathbf{A} = \mathbf{I}$

Def. An *n* x *n* matrix with orthonormal columns is an ***orthogonal matrix***

Orthogonal Sets are Nice

- given an orthogonal set $\{v_1, v_2, \ldots, v_k\}$
- suppose you have a dependence relation
- dot both sides with v_i
- do this for all i

#1 They are linearly independent

$$\vec{0} = c_1 V_1 + c_2 V_2 + c_3 V_3 + c_4 V_4 + c_5 V_5$$

$$0 = V_4 \cdot \vec{0} = c_1 \underbrace{(V_4 \cdot V_1)}_{0} + c_2 \underbrace{(V_4 \cdot V_2)}_{0} + c_3 \underbrace{(V_4 \cdot V_3)}_{0} + c_4 \underbrace{(V_4 \cdot V_4)}_{\|V_4\|^2} + c_5 \underbrace{(V_4 \cdot V_5)}_{0}$$

$$0 = c_4 \|V_4\|^2$$

$$\Rightarrow \quad c_4 = 0$$

Orthogonal Sets are Nice

- to find the coordinates $c_1, c_2, \ldots, c_n$
- dot both sides with v_i
- do this for all i

#2 Nice way to find coordinates of vectors in the span.

$$w = c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 + c_5 v_5$$

$$v_4 \cdot w = c_1 \underbrace{(v_4 \cdot v_1)}_{0} + c_2 \underbrace{(v_4 \cdot v_2)}_{0} + c_3 \underbrace{(v_4 \cdot v_3)}_{0} + c_4 \underbrace{(v_4 \cdot v_4)}_{\|v_4\|^2} + c_5 \underbrace{(v_4 \cdot v_5)}_{0}$$

$$v_4 \cdot w = c_4 \, v_4 \cdot v_4$$

$$\Rightarrow c_4 = \frac{v_4 \cdot w}{v_4 \cdot v_4}$$

Coordinates in an orthogonal basis

Theorem

Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ be an **orthogonal basis** for $\mathbb{R}^n$. For any $\vec{w} \in \mathbb{R}^n$, we have

$$\vec{w} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$$

where

$$c_i = \frac{\vec{w} \cdot \vec{v}_i}{\vec{v}_i \cdot \vec{v}_i} = \frac{\vec{w} \cdot \vec{v}_i}{\|\vec{v}_i\|^2}.$$

Note: this formula for c_i only works when we have an **orthogonal basis**!

An **orthogonal basis** satisfies $\vec{v}_i \cdot \vec{v}_j = 0$ when $i \neq j$

When you do not have an orthogonal basis you: augment and row reduce

Example

(a) Show that vectors below form an orthogonal basis of $\mathbb{R}^n$

(b) Express the vector w as a linear combination of the basis vectors

$$\left\{ \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}, \begin{bmatrix}1\\1\\-1\\-1\end{bmatrix}, \begin{bmatrix}1\\-1\\-1\\1\end{bmatrix} \right\} \qquad \begin{bmatrix}1&1&1&1\\1&-1&1&-1\\1&1&-1&-1\\1&-1&-1&1\end{bmatrix}\begin{bmatrix}1&1&1&1\\1&-1&1&-1\\1&1&-1&-1\\1&-1&-1&1\end{bmatrix} = \begin{bmatrix}4&0&0&0\\0&4&0&0\\0&0&4&0\\0&0&0&4\end{bmatrix}$$

$$\vec{w} = \begin{bmatrix}1\\2\\3\\4\end{bmatrix} = \frac{\vec{w}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + \frac{\vec{w}\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2}\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix} + \frac{\vec{w}\cdot\vec{v}_3}{\vec{v}_3\cdot\vec{v}_3}\begin{bmatrix}1\\1\\-1\\-1\end{bmatrix} + \frac{\vec{w}\cdot\vec{v}_4}{\vec{v}_4\cdot\vec{v}_4}\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}$$

$$= \frac{10}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + \frac{-2}{4}\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix} + \frac{-4}{4}\begin{bmatrix}1\\1\\-1\\-1\end{bmatrix} + \frac{0}{4}\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix} = \frac{5}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + \frac{-1}{2}\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix} - \begin{bmatrix}1\\1\\-1\\-1\end{bmatrix}$$

You Try

One of the sets below is an orthogonal basis of $\mathbb{R}^3$

(a) Figure out which one it is
(b) And write w as a linear combination of the basis vectors

$$\mathcal{A} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\} \qquad \mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \right\}$$

$$\vec{w} = \begin{bmatrix} -3 \\ 1 \\ -13 \end{bmatrix}$$