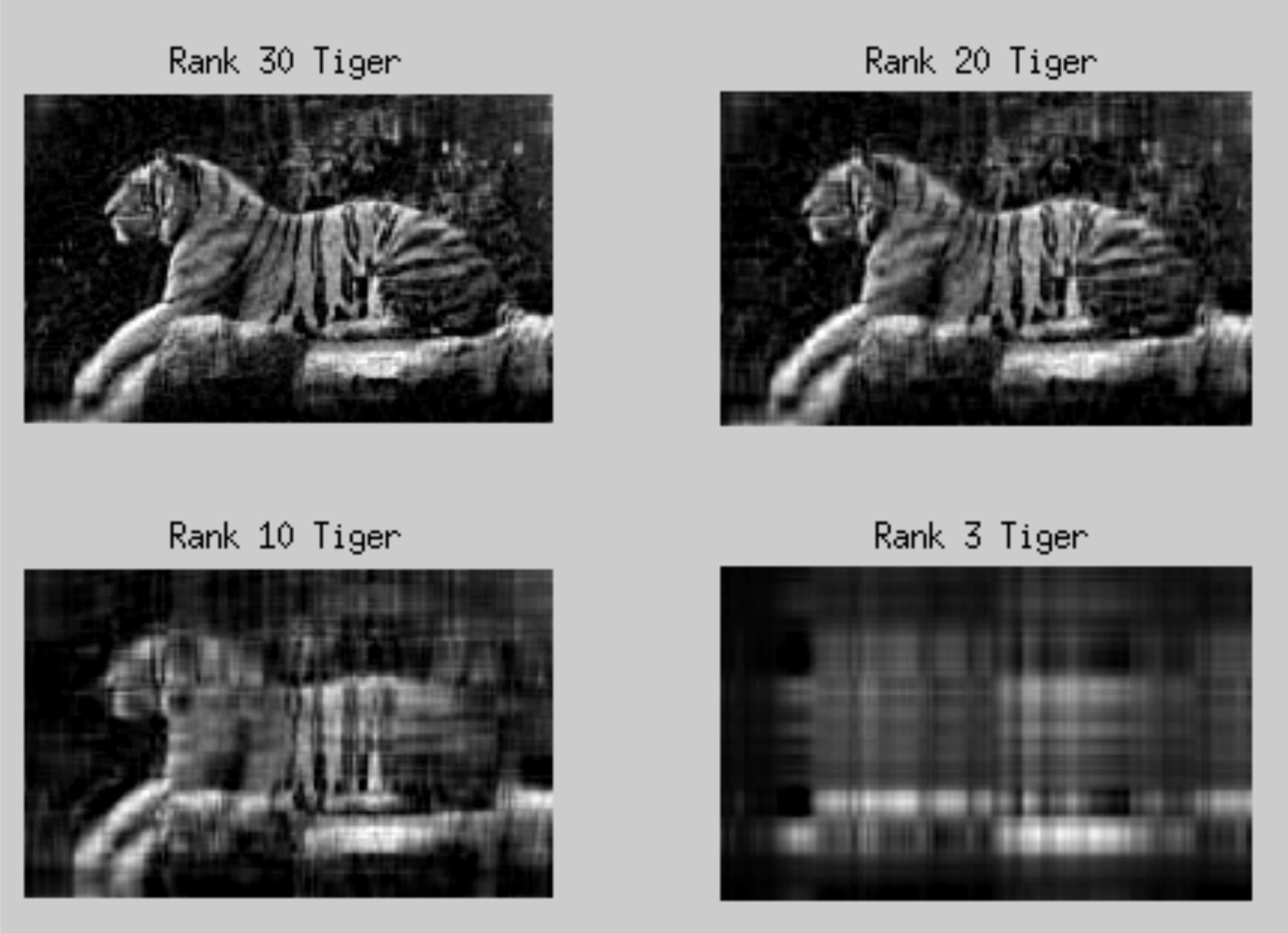
# Diagonalizing Symmetric Matrices

# Teaser: Our Final Application

In a future video, we will use **singular value decomposition** of a matrix to perform **image compression**.

But first, we need to talk about **eigensystems of symmetric matrices**.



# Symmetric Matrices

**DEFINITION**: A square matrix $\mathbf{A}$ is **symmetric** if $\mathbf{A}^T = \mathbf{A}$

$$\begin{bmatrix} 34 & 19 & 11 \\ 19 & 52 & 29 \\ 11 & 29 & 82 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

$$\begin{bmatrix} 13 & -2 & 0 & -2 & -4 & 2 \\ -2 & 7 & 6 & 0 & -4 & -4 \\ 0 & 6 & 13 & 2 & -10 & -9 \\ -2 & 0 & 2 & 14 & 2 & 2 \\ -4 & -4 & -10 & 2 & 14 & 4 \\ 2 & -4 & -9 & 2 & 4 & 18 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 & 5 & 6 \\ 3 & 1 & 4 & 0 \\ 5 & 4 & 1 & 7 \\ 6 & 0 & 7 & 0 \end{bmatrix}$$

$$a_{i,j} = a_{j,i} \quad \text{for all } i, j$$

# Warm Up

In RStudio, enter the commands to the right to create a random 3 x 3 matrix and find its eigenvalues.

Execute this block a few times. How often do you get complex eigenvalues?

````
```{r}
require(pracma)
```

```{r}
# make a matrix with random entries
(A = rand(3,3))

# find its eigenvalues
eigen(A)$values

```
````

Next, enter the commands to the right to create a random **symmetric** 3 x 3 matrix and find its eigenvalues.

Execute this block a few times. How often do you get complex eigenvalues?

````
```{r}
# make a random symmetric matrix B
A = rand(3,3)
(B = A + t(A))

# find its eigenvalues
eigen(B)$values
```
````

# 5x5 Matrices: Random vs Random Symmetric

```
```{r}
require(pracma)


for (i in 1:5) {
  A = rand(5,5)
  print(eigen(A)$values)
  print('----------')
}


```
```

```
[1]  2.20913174+0.0000000i  0.53212124+0.0000000i
[3]  0.32374341+0.1834463i  0.32374341-0.1834463i
[5] -0.08273106+0.0000000i
[1] "----------"
[1]  3.00875131 -0.42441293  0.39513924  0.08491735
[5] -0.02234662
[1] "----------"
[1]  2.4072169+0.0000000i  0.5662187+0.2744568i
[3]  0.5662187-0.2744568i -0.4148545+0.0000000i
[5]  0.2441052+0.0000000i
[1] "----------"
[1]  3.0827445+0.0000000i  0.7534657+0.0000000i
[3] -0.5613591+0.0000000i -0.0694400+0.5167479i
[5] -0.0694400-0.5167479i
[1] "----------"
[1]  2.5337608+0.0000000i -0.3849859+0.0000000i
[3]  0.3206170+0.1723825i  0.3206170-0.1723825i
[5]  0.3050158+0.0000000i
[1] "----------"
```

5 x 5 matrices often have complex eigenvalues.

5 x 5 symmetric matrices always have real eigenvalues.

```
```{r}
for (i in 1:5) {
  A = rand(5,5)
  B = t(A) + A
  print(eigen(B)$values)
  print('----------')
}
```
```

```
[1]  4.8309288  1.1423534  0.0852732 -0.6718585
[5] -1.0576892
[1] "----------"
[1]  4.9307332  1.0937184 -0.4950239 -1.0225289
[5] -2.0028119
[1] "----------"
[1]  5.5413419  1.2639945  0.5713014 -0.3621521
[5] -0.9294614
[1] "----------"
[1]  3.7089876  1.0141206  0.4958937 -0.4985107
[5] -0.8022475
[1] "----------"
[1]  5.13166269  1.58954052  0.83822512 -0.04628437
[5] -1.29946327
[1] "----------"
```

# What about the Eigenvectors?

A **symmetric matrix** always has **real eigenvalues**!
What is special about the **eigenvectors?**

```
```{r}
# make a random symmetric matrix B
A = rand(3,3)
(B = A + t(A))

# find its eigenvalues
eigen(B)$values

# find its eigenvectors
(vec = eigen(B)$vectors)

# find their dot products
t(vec[,1])  %*% vec[,2]
t(vec[,1])  %*% vec[,3]
t(vec[,2])  %*% vec[,3]
```
```

A **symmetric matrix** always has **orthogonal eigenvectors**!

```
           [,1]       [,2]        [,3]
[1,] -0.6651539 -0.4821164  0.57020535
[2,] -0.4484953 -0.3525919 -0.82129833
[3,] -0.5970111  0.8020242 -0.01830093
             [,1]
[1,] 5.551115e-17
             [,1]
[1,] 1.162265e-16
             [,1]
[1,] 1.058181e-16
```

# Example: A symmetric matrix and its eigensystem

## Real Eigenvalues

$$\begin{bmatrix} 34 & 19 & 11 \\ 19 & 52 & 29 \\ 11 & 29 & 82 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 34+38+33 \\ 19+104+87 \\ 11+58+246 \end{bmatrix} = \begin{bmatrix} 105 \\ 210 \\ 315 \end{bmatrix} = 105 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 34 & 19 & 11 \\ 19 & 52 & 29 \\ 11 & 29 & 82 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -34-19+11 \\ -19-52+29 \\ -11-29+82 \end{bmatrix} = \begin{bmatrix} -42 \\ -42 \\ 42 \end{bmatrix} = 42 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 34 & 19 & 11 \\ 19 & 52 & 29 \\ 11 & 29 & 82 \end{bmatrix} \begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 170-76+11 \\ 95-208+29 \\ 55-116+82 \end{bmatrix} = \begin{bmatrix} 105 \\ -84 \\ 21 \end{bmatrix} = 21 \begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix}$$

## Orthogonal Eigenvectors

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix} = 0$$

# Symmetric Matrices are Orthogonally Diagonalizable

**THEOREM:** Symmetric Matrices

1) Have **real** eigenvalues and **real** eigenvectors (not complex)!
2) Are **diagonalizable** (algebraic multiplicity = geometric multiplicity)
3) Have ***orthogonal*** eigenvectors

*i.e.*, symmetric matrices are **Orthogonally Diagonalizable** $\boxed{A = PDP^T}$

$$\begin{bmatrix} 34 & 19 & 11 \\ 19 & 52 & 29 \\ 11 & 29 & 82 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 5 \\ 2 & -1 & -4 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 105 & 0 & 0 \\ 0 & 42 & 0 \\ 0 & 0 & 21 \end{bmatrix} \begin{bmatrix} 1/14 & 2/14 & 3/14 \\ -1/3 & -1/3 & 1/3 \\ 5/42 & -4/42 & 1/42 \end{bmatrix}$$

$$\boxed{= \begin{bmatrix} 1/\sqrt{14} & -1/\sqrt{3} & 5/\sqrt{42} \\ 2/\sqrt{14} & -1/\sqrt{3} & -4/\sqrt{42} \\ 3/\sqrt{14} & 1/\sqrt{3} & 1/\sqrt{42} \end{bmatrix} \begin{bmatrix} 105 & 0 & 0 \\ 0 & 42 & 0 \\ 0 & 0 & 21 \end{bmatrix} \begin{bmatrix} 1/\sqrt{14} & 2/\sqrt{14} & 3/\sqrt{14} \\ -1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \\ 5/\sqrt{42} & -4/\sqrt{42} & 1/\sqrt{42} \end{bmatrix}}$$

# Symmetric Matrices have Orthogonal Eigenvectors

Here is a proof that the eigenvectors of a symmetric matrix are orthogonal. Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be eigenvectors with distinct eigenvalues $\lambda_1$ and $\lambda_2$. Give a justification of each step.

| | | | justification |
|---|---|---|---|
| $\lambda_1(\mathbf{v}_1 \cdot \mathbf{v}_2)$ | $=$ | $(\lambda_1 \mathbf{v}_1) \cdot \mathbf{v}_2$ | linearity of the dot product |
| | $=$ | $(A\mathbf{v}_1) \cdot \mathbf{v}_2$ | $Av_1 = \lambda_1 v_1$ |
| | $=$ | $(A\mathbf{v}_1)^T \mathbf{v}_2$ | $w \cdot v = w^T v$ |
| | $=$ | $\mathbf{v}_1^T A^T \mathbf{v}_2$ | by the order reversing property of the transpose: $(AB)^T = B^T A^T$ |
| | $=$ | $\mathbf{v}_1^T A \mathbf{v}_2$ | $A^T = A$ |
| | $=$ | $\mathbf{v}_1^T(\lambda_2 \mathbf{v}_2)$ | $Av_2 = \lambda_2 v_2$ |
| | $=$ | $\lambda_2(\mathbf{v}_1 \cdot \mathbf{v}_2)$. | $w^T v = w \cdot v$ |

Thus $(\lambda_1 - \lambda_2)(\mathbf{v}_1 \cdot \mathbf{v}_2) = 0$, so $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$.

# Spectral Decomposition of a 2x2 Symmetric Matrix

$$A = \begin{bmatrix} | & | \\ \mathbf{u}_1 & \mathbf{u}_2 \\ | & | \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} -\mathbf{u}_1^\top - \\ -\mathbf{u}_2^\top - \end{bmatrix}$$

$$= \begin{bmatrix} | & | \\ \mathbf{u}_1 & \mathbf{u}_2 \\ | & | \end{bmatrix} \begin{bmatrix} -\lambda_1 \mathbf{u}_1^\top - \\ -\lambda_2 \mathbf{u}_2^\top - \end{bmatrix}$$

$$= \begin{bmatrix} u_{11} & u_{21} \\ u_{12} & u_{22} \end{bmatrix} \begin{bmatrix} \lambda_1 u_{11} & \lambda_1 u_{12} \\ \lambda_2 u_{21} & \lambda_2 u_{22} \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 u_{11}^2 + \lambda_2 u_{21}^2 & \lambda_1 u_{11} u_{12} + \lambda_2 u_{21} u_{22} \\ \lambda_1 u_{11} u_{12} + \lambda_2 u_{21} u_{22} & \lambda_1 u_{12}^2 + \lambda_2 u_{22}^2 \end{bmatrix}$$

$$= \lambda_1 \begin{bmatrix} u_{11}^2 & u_{11} u_{12} \\ u_{11} u_{12} & u_{12}^2 \end{bmatrix} + \lambda_2 \begin{bmatrix} u_{21}^2 & u_{21} u_{22} \\ u_{21} u_{22} & u_{22}^2 \end{bmatrix}$$

$$= \lambda_1 \begin{bmatrix} u_{11} \\ u_{12} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} \end{bmatrix} + \lambda_2 \begin{bmatrix} u_{21} \\ u_{22} \end{bmatrix} \begin{bmatrix} u_{21} & u_{22} \end{bmatrix}$$

$$= \lambda_1 \begin{bmatrix} | \\ \mathbf{u}_1 \\ | \end{bmatrix} \begin{bmatrix} -\mathbf{u}_1^\top - \end{bmatrix} + \lambda_2 \begin{bmatrix} | \\ \mathbf{u}_2 \\ | \end{bmatrix} \begin{bmatrix} -\mathbf{u}_2^\top - \end{bmatrix}$$

$$A = \lambda_1 \mathbf{u}_1 \mathbf{u}_1^\top + \lambda_2 \mathbf{u}_2 \mathbf{u}_2^\top$$

# Spectral Decomposition of an n x n Symmetric Matrix

$$\underset{\text{symmetric}}{\begin{bmatrix} A \end{bmatrix}} = \overset{\text{orthogonal}}{\begin{bmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_n \\ | & | & & | \end{bmatrix}} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \begin{bmatrix} — u_1^T — \\ — u_2^T — \\ \vdots \\ — u_n^T — \end{bmatrix}$$

$$= \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T + \cdots + \lambda_n u_n u_n^T$$

$$= \lambda_1 \begin{bmatrix} | \\ u_1 \\ | \end{bmatrix} \begin{bmatrix} — u_1^T — \end{bmatrix} + \lambda_2 \begin{bmatrix} | \\ u_2 \\ | \end{bmatrix} \begin{bmatrix} — u_2^T — \end{bmatrix} + \cdots + \lambda_n \begin{bmatrix} | \\ u_n \\ | \end{bmatrix} \begin{bmatrix} — u_n^T — \end{bmatrix}$$

# Example: Spectral Decomposition

$$\begin{bmatrix} 34 & 19 & 11 \\ 19 & 52 & 29 \\ 11 & 29 & 82 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{14} & -1/\sqrt{3} & 5/\sqrt{42} \\ 2/\sqrt{14} & -1/\sqrt{3} & -4/\sqrt{42} \\ 3/\sqrt{14} & 1/\sqrt{3} & 1/\sqrt{42} \end{bmatrix} \begin{bmatrix} 105 & 0 & 0 \\ 0 & 42 & 0 \\ 0 & 0 & 21 \end{bmatrix} \begin{bmatrix} 1/\sqrt{14} & 2/\sqrt{14} & 3/\sqrt{14} \\ -1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \\ 5/\sqrt{42} & -4/\sqrt{42} & 1/\sqrt{42} \end{bmatrix}$$

$$= 105 \begin{bmatrix} \frac{1}{\sqrt{14}} \\ \frac{2}{\sqrt{14}} \\ \frac{3}{\sqrt{14}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{14}} & \frac{2}{\sqrt{14}} & \frac{3}{\sqrt{14}} \end{bmatrix} + 42 \begin{bmatrix} \frac{-1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} + 21 \begin{bmatrix} \frac{5}{\sqrt{42}} \\ \frac{-4}{\sqrt{42}} \\ \frac{1}{\sqrt{42}} \end{bmatrix} \begin{bmatrix} \frac{5}{\sqrt{42}} & \frac{-4}{\sqrt{42}} & \frac{1}{\sqrt{42}} \end{bmatrix}$$

$$= 105 \begin{bmatrix} 1/14 & 2/14 & 3/14 \\ 2/14 & 4/14 & 6/14 \\ 3/14 & 6/14 & 9/14 \end{bmatrix} + 42 \begin{bmatrix} 1/3 & 1/3 & -1/3 \\ 1/3 & 1/3 & -1/3 \\ -1/3 & -1/3 & 1/3 \end{bmatrix} + 21 \begin{bmatrix} 25/42 & -20/42 & 5/42 \\ -20/42 & 16/42 & -4/42 \\ 5/42 & -4/42 & 1/42 \end{bmatrix}$$

**A** has been written as a sum of 3 rank 1 matrices

# You Try

Consider the symmetric matrix

$$A = \begin{bmatrix} -20 & 38 & 8 \\ 38 & 7 & 46 \\ 8 & 46 & 22 \end{bmatrix}$$

- Use RStudio to find the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ and the corresponding orthonormal eigenbasis $\mathsf{u}_1, \mathsf{u}_2, \mathsf{u}_3$.
- Create the rank 1 matrices $\lambda_1 \mathsf{u}_1 \mathsf{u}_1^\top$ and $\lambda_2 \mathsf{u}_2 \mathsf{u}_2^\top$ and $\lambda_3 \mathsf{u}_3 \mathsf{u}_3^\top$
- Add these rank 1 matrices together to confirm that

$$A = \lambda_1 \mathsf{u}_1 \mathsf{u}_1^\top + \lambda_2 \mathsf{u}_2 \mathsf{u}_2^\top + \lambda_3 \mathsf{u}_3 \mathsf{u}_3^\top$$

Submit your three rank 1 matrices and their sum. (R Output is fine)