Convergence to dominant eigenvector

dominant eigenvector

A $\underline{n \times n}$

$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n|$

$v_1 \quad v_2 \quad v_3 \quad \cdots \quad v_n$ basis of eigenvectors

$$x_0 = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + \cdots + c_n\vec{v}_n$$

$$\begin{aligned} x_t = A^t x_0 &= c_1 A^t v_1 + c_2 A^t v_2 + \cdots + c_n A^t v_n \\ &= c_1 \lambda_1^t v_1 + c_2 \lambda_2^t v_2 + \cdots + c_n \lambda_n^t v_n \\ &= \lambda_1^t \left(c_1 v_1 + c_2 \underbrace{\left(\frac{\lambda_2}{\lambda_1}\right)^t}_{\to 0} v_2 + \cdots + \underbrace{\left(\frac{\lambda_n}{\lambda_1}\right)^t}_{\to 0} v_n \right) \end{aligned}$$

$$\boxed{\lim_{t\to\infty} A^t x_0 = \lambda_1^t c_1 v_1}$$

$$x_0 \xrightarrow{A} Ax_0 = x_1 \xrightarrow{A} Ax_1 = x_2 \xrightarrow{A} Ax_2 = x_3 \quad \cdots \quad \text{limit?}$$

Fibonacci Example

14.5 in Handbook (c.f., PS 8.4)

$$\boxed{f_{n+1} = f_n + f_{n-1}}$$

$$F = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

0	1	2	3	4	5	6	7	8	9	10	11
$\begin{bmatrix}0\\1\end{bmatrix}$	$\begin{bmatrix}1\\1\end{bmatrix}$	$\begin{bmatrix}1\\2\end{bmatrix}$	$\begin{bmatrix}2\\3\end{bmatrix}$	$\begin{bmatrix}3\\5\end{bmatrix}$	$\begin{bmatrix}5\\8\end{bmatrix}$	$\begin{bmatrix}8\\13\end{bmatrix}$	$\begin{bmatrix}13\\21\end{bmatrix}$	$\begin{bmatrix}21\\34\end{bmatrix}$	$\begin{bmatrix}34\\55\end{bmatrix}$	$\begin{bmatrix}55\\89\end{bmatrix}$	$\begin{bmatrix}89\\144\end{bmatrix}$

$21 + 34 = 55$

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 5 \\ 3+5 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} f_{n-1} \\ f_n \end{bmatrix} = \begin{bmatrix} f_n \\ f_{n-1}+f_n \end{bmatrix} = \begin{bmatrix} f_n \\ f_{n+1} \end{bmatrix}$$

Closed form

evals

$$\det\begin{bmatrix} -\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} = (-\lambda)(1-\lambda) - 1 = \lambda^2 - \lambda - 1$$

$$\lambda = \frac{1 \pm \sqrt{1^2 - 4(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

$$\phi = \frac{1+\sqrt{5}}{2} = 1.618\ldots \qquad v_1 = \begin{bmatrix} 1 \\ \phi \end{bmatrix}$$

$$\bar{\phi} = \frac{1-\sqrt{5}}{2} = -.618\ldots \qquad v_2 = \begin{bmatrix} 1 \\ \bar{\phi} \end{bmatrix}$$

Golden ratio

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ \phi \end{bmatrix} + \frac{-1}{\sqrt{5}}\begin{bmatrix} 1 \\ \bar{\phi} \end{bmatrix}$$

$$\begin{bmatrix} f_n \\ f_{n+1} \end{bmatrix} = F^n \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{5}}\phi^n \begin{bmatrix} 1 \\ \phi \end{bmatrix} + \frac{-1}{\sqrt{5}}\bar{\phi}^n \begin{bmatrix} 1 \\ \bar{\phi} \end{bmatrix}$$

$$* \quad \boxed{f_n = \frac{1}{\sqrt{5}}\phi^n - \frac{1}{\sqrt{5}}\bar{\phi}^n} \quad *$$

Complex Eigenvectors

Diagonalized over the complex numbers

Rental Car Matrix

$$\begin{bmatrix} .85 & .30 & .35 \\ .09 & .60 & .05 \\ .06 & .10 & .60 \end{bmatrix}$$

$$\lambda_1 = 1.0, \quad \lambda_2 = .525 + .037i, \quad \lambda_3 = .525 - .037i$$

$$\begin{bmatrix} .95 \\ .24 \\ .20 \end{bmatrix} \quad \begin{bmatrix} .62 + .00i \\ -.31 - .46i \\ -.31 + .46i \end{bmatrix} \quad \begin{bmatrix} .62 - .00i \\ -.31 + .46i \\ -.31 - .46i \end{bmatrix}$$

$$\begin{bmatrix} .85 & .30 & .35 \\ .09 & .60 & .05 \\ .06 & .10 & .60 \end{bmatrix} = \underbrace{\begin{bmatrix} .95 & .62+.00i & .62-.00i \\ .24 & -.31-.46i & -.31+.46i \\ .20 & -.31+46i & -.31-.46i \end{bmatrix}}_{P} \begin{bmatrix} 1.0 & 0 & 0 \\ 0 & .35-.04i & 0 \\ 0 & 0 & .35+.04i \end{bmatrix} \underbrace{\begin{bmatrix} .72 & .72 & .72 \\ .26-.01i & -.55+.53i & -.55-.56i \\ .26+.01i & -.55-.53i & -.55+.56i \end{bmatrix}}_{P^{-1}}$$

Complex Eigenvectors

Rental Car Matrix

$$\begin{bmatrix} .85 & .30 & .35 \\ .09 & .60 & .05 \\ .06 & .10 & .60 \end{bmatrix}$$

$$\lambda_1 = 1.0, \qquad \lambda_2 = .525 + .037i, \qquad \lambda_3 = .525 - .037i$$

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$$\begin{bmatrix} .85 & .30 & .35 \\ .09 & .60 & .05 \\ .06 & .10 & .60 \end{bmatrix} = \underbrace{\begin{bmatrix} .95 & .00 & .62 \\ .24 & -.46 & -.31 \\ .20 & .46 & -.31 \end{bmatrix}}_{P} \begin{bmatrix} \boxed{1.0} & 0 & 0 \\ 0 & .525 & -.037 \\ 0 & .037 & .525 \end{bmatrix} \underbrace{\begin{bmatrix} .72 & .72 & .72 \\ .03 & -1.06 & 1.12 \\ .51 & -1.10 & -1.10 \end{bmatrix}}_{P^{-1}}$$

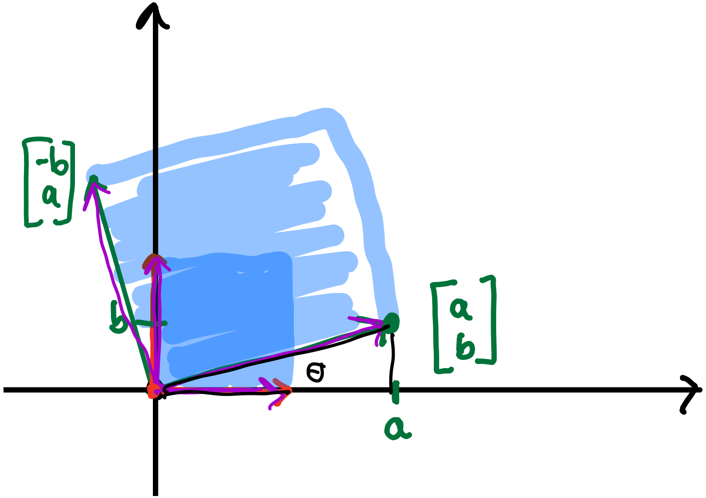
$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

$$\lambda = a \pm bi$$

Rotation-Dilation Matrices

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

$$\boxed{\begin{aligned} \lambda_1 &= a + bi \\ \lambda_2 &= a - bi \end{aligned}}$$



expansion factor

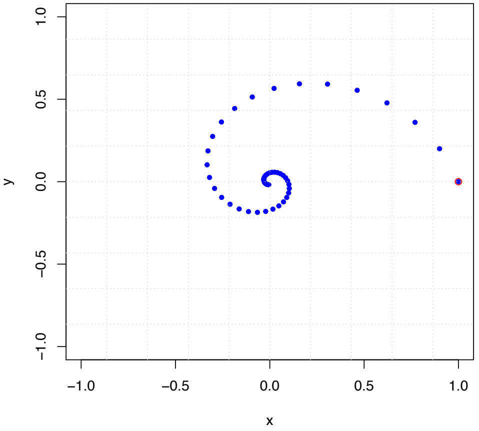
$|\lambda| = \sqrt{a^2 + b^2}$ Modulus of λ growth rate/dilation factor

$\theta = \tan^{-1}\left(\frac{b}{a}\right)$ Argument of λ angle of rotation

Examples

$A = \begin{bmatrix} .9 & -.2 \\ .2 & .9 \end{bmatrix}$ $|\lambda| = \sqrt{.9^2 + .2^2} = .92$

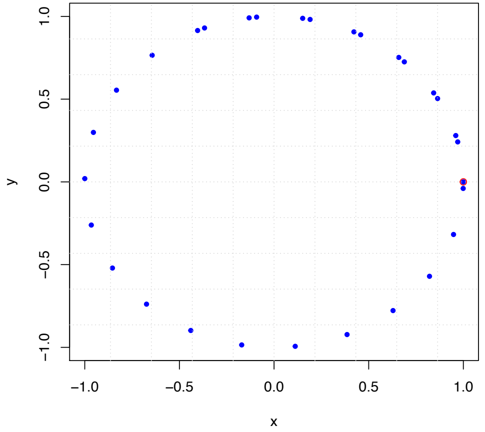
$\boxed{\lambda = .9 \pm .2i}$ $\theta = \tan^{-1}\left(\frac{.2}{.9}\right) = .21 \text{ rad} = 12.5°$



$B = \begin{bmatrix} .96 & -.28 \\ .28 & .96 \end{bmatrix}$ $|\lambda| = \sqrt{.96^2 + .28^2} = \boxed{1.00}$

$\theta = \tan^{-1}\left(\frac{.28}{.96}\right) = .28 \text{ rad} = \boxed{16.3°}$

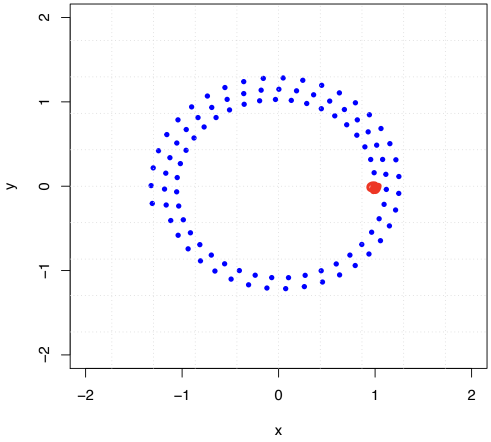
$\lambda = .96 \pm .28i$



$C = \begin{bmatrix} .99 & -.16 \\ .16 & .99 \end{bmatrix}$ $|\lambda| = \sqrt{.99^2 + .16^2} = 1.002$

$\theta = \tan^{-1}\left(\frac{.16}{.99}\right) = .16 \text{ rad} = 9.2°$

$\lambda = .99 + .16i$



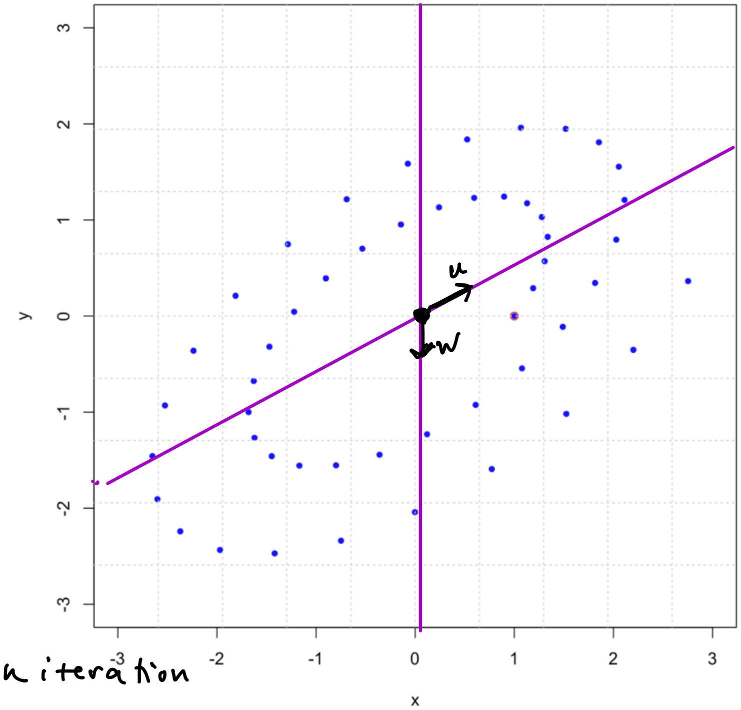
More Generally

$$A = \begin{bmatrix} 1.19 & -0.39 \\ 0.29 & 0.78 \end{bmatrix}$$

$\lambda_1 = .98 + .26i$ $\qquad \lambda_1 = .98 - .26i$

$v_1 = \underbrace{\begin{bmatrix} 0.75 \\ 0.41 \end{bmatrix}}_{u} + \underbrace{\begin{bmatrix} 0.00 \\ -.52 \end{bmatrix}}_{w} i \qquad v_1 = \begin{bmatrix} 0.75 \\ 0.41 \end{bmatrix} - \begin{bmatrix} 0.00 \\ -.52 \end{bmatrix} i$

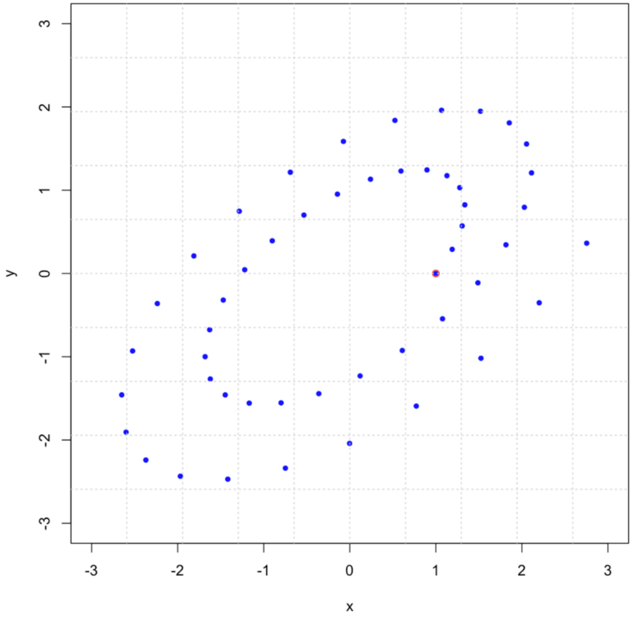
$$\begin{bmatrix} 1.19 & -.39 \\ 0.29 & .78 \end{bmatrix} = \underbrace{\begin{bmatrix} .00 & 0.75 \\ -.52 & 0.41 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} .98 & -.26 \\ .26 & .98 \end{bmatrix}}_{\text{rotation-dilation}} \begin{bmatrix} P^{-1} \end{bmatrix}$$

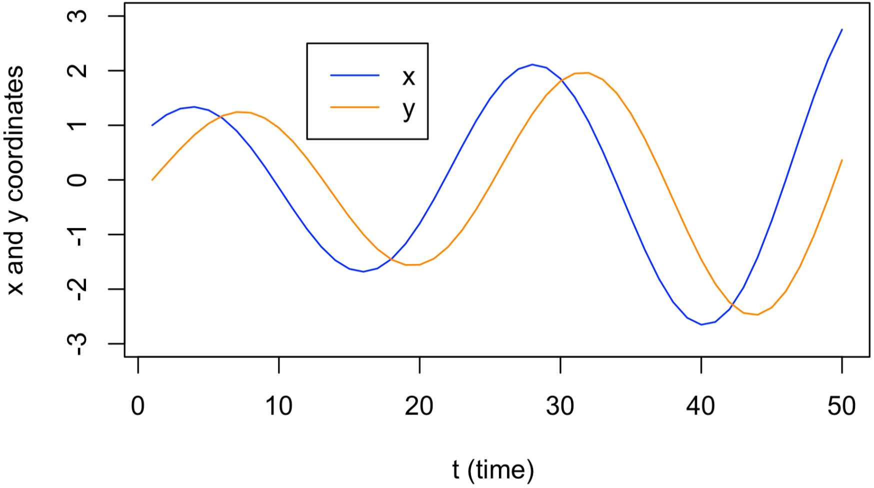


$|\lambda_1| = \sqrt{.98^2 + .26^2} = 1.0139$ grows by 1.39% each iteration

$\theta = \tan^{-1}\left(\frac{.26}{.98}\right) = .239 \text{ rad} = 13.696^\circ$

Sketch the xy-coordinates of the trajectory





Checkpoint Question for Today

3. Checkpoint: $A = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$.

$$\lambda = \frac{3 \pm \sqrt{3}\,i}{2}$$

$$v = \underbrace{\begin{bmatrix} -1 \\ 2 \end{bmatrix}}_{u} \pm \underbrace{\begin{bmatrix} \sqrt{3}/2 \\ 0 \end{bmatrix}}_{w} i$$

$$\begin{vmatrix} 1-\lambda & -1 \\ 1 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda)+1 = 2-3\lambda+\lambda^2+1 = \lambda^2-3\lambda+3$$

$$\lambda = \frac{3 \pm \sqrt{9-4\cdot 3}}{2} = \frac{3 \pm \sqrt{-3}}{2} = \frac{3}{2} \pm \frac{\sqrt{3}}{2} i \approx 1.5 \pm .866 i$$

$$\|\lambda\| = \sqrt{\left(\tfrac{3}{2}\right)^2 + \left(\tfrac{\sqrt{3}}{2}\right)^2} = \sqrt{\tfrac{9}{4}+\tfrac{3}{4}} = \sqrt{\tfrac{12}{4}} = \sqrt{3} \approx 1.73$$

Grows by 73% each iteration!

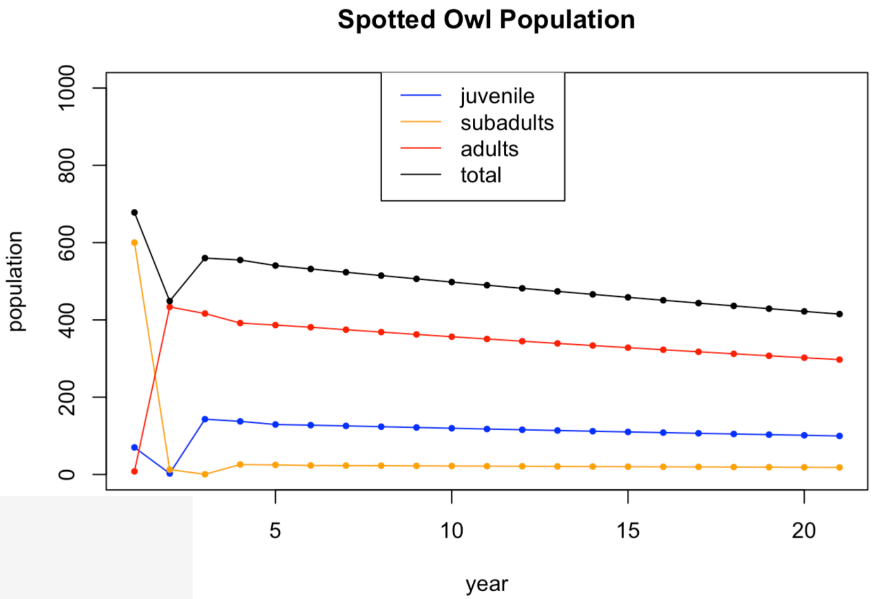
$$\theta = \arctan\left(\frac{\sqrt{3}/2}{3/2}\right) = \arctan\left(\frac{\sqrt{3}}{3}\right) = \pi/6 \approx .524 \text{ rad} = 30^\circ$$

rotates by 30°

$$\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} = \underbrace{\begin{array}{cc} w & u \\ \left[\begin{matrix} \sqrt{3}/2 & -1 \\ 0 & 2 \end{matrix}\right] \end{array}}_{P} \underbrace{\begin{bmatrix} 3/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 3/2 \end{bmatrix}}_{\text{rotation-dilation}} \underbrace{\left(\frac{1}{\sqrt{3}} \begin{bmatrix} 2 & 1 \\ 0 & \sqrt{3}/2 \end{bmatrix}\right)}_{P^{-1}}$$

Northern Spotted Owl

$$\begin{bmatrix} j_{n+1} \\ s_{n+1} \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0.33 \\ 0.18 & 0 & 0 \\ 0 & 0.71 & 0.94 \end{bmatrix} \begin{bmatrix} j_n \\ s_n \\ a_n \end{bmatrix}$$




```
## eigen() decomposition
## $values
## [1]  0.9835927+0.0000000i -0.0217964+0.2059185i -0.0217964-0.2059185i
##
## $vectors
##                 [,1]                   [,2]                   [,3]
## [1,] 0.31754239+0i  0.6820937+0.0000000i  0.6820937+0.0000000i
## [2,] 0.05811107+0i -0.0624124-0.5896338i -0.0624124+0.5896338i
## [3,] 0.94646180+0i -0.0450520+0.4256233i -0.0450520-0.4256233i
```

```
> Mod(eigen(A)$values)
[1] 0.9835927 0.2070688 0.2070688
> Arg(eigen(A)$values)
[1]  0.000000  1.676253 -1.676253

> Arg(eigen(A)$values)/(2*pi)*360
[1]   0.00000  96.04223 -96.04223
```

1. What does this matrix do? $A = \begin{bmatrix} 0.9 & -0.2 \\ 0.2 & 0.9 \end{bmatrix}$

$$\det\left(\begin{bmatrix} 0.9-\lambda & -0.2 \\ 0.2 & 0.9-\lambda \end{bmatrix}\right) = (0.9-\lambda)^2 + 0.04 = \lambda^2 - 1.8\lambda + 0.85.$$

$$\lambda = \frac{1.8 \pm \sqrt{1.8^2 - 4(0.85)}}{2} = \frac{1.8 \pm \sqrt{-.16}}{2} = \frac{1.8 \pm 0.4i}{2} = 0.9 \pm 0.2i.$$

$|\lambda| = \sqrt{.9^2 + .2^2} = .92$

spirals in decreasing by 8% each time

$\theta = \tan^{-1}(.2/.9) = .2186 \text{ rad} = 12.5^\circ$

2. What about this one? $B = \begin{bmatrix} 1.19 & -0.38 \\ 0.29 & 0.78 \end{bmatrix}$

```
## eigen() decomposition
## $values
## [1] 0.985+0.2611034i 0.985-0.2611034i
##
## $vectors
##                   [,1]              [,2]
## [1,] 0.7531030+0.0000000i 0.7531030+0.0000000i
## [2,] 0.4062793-0.5174679i 0.4062793+0.5174679i
```

This one is done above