Vector Spaces: The set of n-dimensional vectors $\mathbb{R}^n = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \middle| a_1, a_2, \ldots, a_n \in \mathbb{R} \right\}$ forms a vector space.

A set V is a *vector space* if it satisfies the following rules for all vectors $\vec{u}, \vec{v}, w \in \mathsf{V}$ and all scalars $c, d \in \mathbb{R}$.

1. $\vec{u} + \vec{v} \in \mathsf{V}$,
2. $c\vec{u} \in \mathsf{V}$,
3. $\vec{v} + \mathbf{0} = \mathbf{0} + \vec{v} = \vec{v}$,
4. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$,
5. $(\vec{u} + \vec{v}) + w = \vec{u} + (\vec{v} + w)$,
6. $\vec{u} + (-\vec{u}) = \mathbf{0}$,
7. $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$,
8. $(c + d)\vec{u} = c\vec{u} + d\vec{u}$,
9. $(cd)\vec{u} = c(d\vec{u})$,
10. $1\vec{u} = \vec{u}$,
11. $0\vec{u} = \mathbf{0}$.

Subspaces: We are interested in subsets $S \subseteq \mathbb{R}^n$ that are also vector spaces. These are called *subspaces.* To be a subspace, S must satisfy three properties:

0. $\mathbf{0} \in S$ (contains the zero vector)
1. If $\vec{u}, \vec{v} \in S$ then $\vec{u} + \vec{v} \in S$ (closed under addition)
2. If $\vec{u} \in S$ and $c \in \mathbb{R}$, then $c\vec{u} \in S$ (closed under scalar multiplication).

From these two properties, we can also derive: (3) $c\vec{u} + d\vec{v} \in S$ (closed under linear combinations).

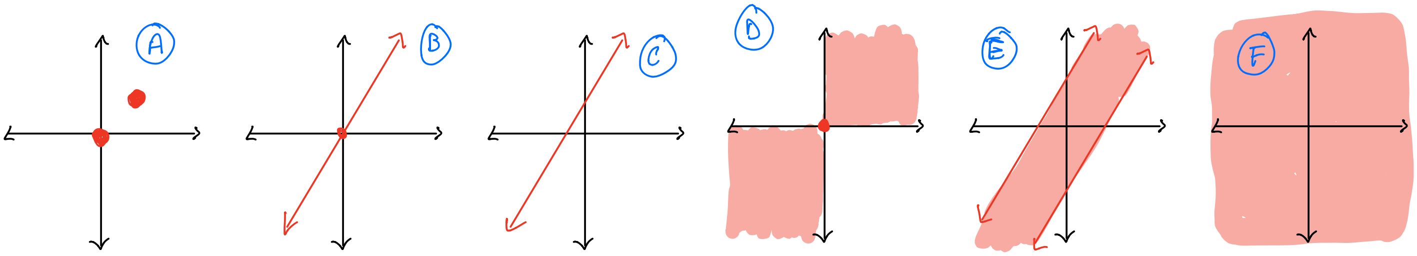
Three Important Examples

A. The set of solutions to $\mathsf{A}x = \mathbf{0}$ forms a subspace: $S = \{ \vec{x} \mid \mathsf{A}\vec{x} = \mathbf{0} \}$

B. The set of solutions to $\mathsf{A}x = \vec{b}$ (with $\vec{b} \neq \mathbf{0}$) does *not* form a subspace: $S = \{ \vec{x} \mid \mathsf{A}\vec{x} = \vec{b} \}$

C. The span of a set of vectors forms a subspace: $S = \mathsf{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} = \{x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n \mid x_i \in \mathbb{R}\}$

1. Which of the following subsets of $\mathbb{R}^2$ are subspaces?



2. Describe the possible subspaces of $\mathbb{R}^3$ from smallest to largest?

3. Which of the following subsets of $\mathbb{R}^3$ are subspaces?

$$A = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \;\middle|\; x_1 x_2 x_3 \geq 0 \right\}, \qquad B = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \;\middle|\; x_1^2 + x_2^2 + x_3^2 \leq 1 \right\}, \qquad C = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \;\middle|\; 5x_1 - 2x_2 - x_3 = 0 \right\}.$$

(a) For the ones that are not subspaces, give a specific example of vectors that break rules 1 and 2 above.

(b) For those that are, show that the rules above are satisfied for any vectors $\vec{u}, \vec{v}$ and any scalar c.

4. Decide if the following subsets are subspaces. As in the previous examples, if it is not, give a specific example of vectors that break rules 1 and 2 above, and if it is, show that the rules above are satisfied for any vectors $\vec{u}, \vec{v}$ and any scalar c.

(a) $S = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \;\middle|\; b = 2a, \; c = 2b, \; d = 2c \right\} \subseteq \mathbb{R}^4$

(b) $S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \;\middle|\; x_3 = \max(x_1, x_2) \right\} \subseteq \mathbb{R}^3$

(c) Here $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation and T is the set of *fixed points*:

$S = \{\vec{x} \in \mathbb{R}^n \mid T(\vec{x}) = \vec{x}\} \subseteq \mathbb{R}^n$

(d) Here $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation and S is the image:

$S = \{\vec{y} \in \mathbb{R}^m \mid T(\vec{x}) = \vec{y}\text{for some } \vec{x} \in \mathbb{R}^n\} \subseteq \mathbb{R}^m$

(e) We can encode polynomials $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ with vectors as follows:

$$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \qquad = \qquad \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

For example, if we consider the set $\mathcal{P}_3$ of polynomials of degree 3 or less, here are some examples of their encoding as 4 dimensional vectors in $\mathbb{R}^4$:

$$1+2x+3x^2+4x^3 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \qquad x = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \qquad 2-5x^2-3.14x^3 = \begin{bmatrix} 2 \\ 0 \\ -5 \\ 3.14 \end{bmatrix}, \qquad 1+x-x^2+x^3 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}.$$

Notice that the polynomials in $\mathcal{P}_n$, of degree n or less, correspond to $\mathbb{R}^{n+1}$.

Observe that polynomial addition is the same as the corresponding vector addition:

$$\begin{array}{rcl} p(x) & = & 1+2x+3x^2+4x^3 \\ q(x) & = & 1+\ x-\ x^2+\ x^3 \\ \hline p(x)+q(x) & = & 2+3x+2x^2+5x^3 \end{array} \quad \text{in } \mathcal{P}_3 \qquad \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 2 \\ 5 \end{bmatrix} \quad \text{in } \mathbb{R}^4.$$

and scalar multiplication of polynomials is the same as scalar multiplication of vectors:

$$2p(x) = 2+4x+6x^2+8x^3 \quad \text{in } \mathcal{P}_3 \qquad 2\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 6 \\ 8 \end{bmatrix} \quad \text{in } \mathbb{R}^4.$$

Thus $\mathcal{P}_n$ is really the same as $\mathbb{R}^{n+1}$. The only difference is cosmetic. We say that $\mathcal{P}_n$ and $\mathbb{R}^{n+1}$ are *isomorphic* vector spaces.

Here are two subsets of $\mathcal{P}_4$. Decide if they are subspaces. In each case, if it is not a subspace, give examples using specific polynomials to show that one of the rules is broken, and if it is a subspace, show that the subspace rule holds for any two polynomials $p(x), q(x)$ and any constant $c \in \mathbb{R}$.

i. $U = \{p(x) \in \mathcal{P}_4 \mid p(1) = 0\}$

ii. $V = \{p(x) \in \mathcal{P}_4 \mid p(0) = 1\}$