2.1. Matrix Multiplication

First: Addition and Scalar Multiplication

Addition: A + B

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & -3 & 5 \\ 2 & -4 & 6 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 8 \\ 6 & 1 & 12 \end{bmatrix}$$

- Add entry wise
- Must be the same size

Scalar Multiplication: *c*A

$$-2\begin{bmatrix} 1 & -2 & 3 \\ -5 & 8 & -13 \end{bmatrix} = \begin{bmatrix} -2 & 4 & -6 \\ 10 & -16 & 26 \end{bmatrix}$$

- rescale every entry

Matrix Multiplication

Dot product

$$[\text{Row}] \cdot \begin{bmatrix} \text{C} \\ \text{o} \\ \text{l} \\ \text{u} \\ \text{m} \\ \text{n} \end{bmatrix}$$

$$\begin{bmatrix} 3 & -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \\ 2 \\ 4 \end{bmatrix} = 3 \cdot 5 + (-2) \cdot 1 + 0 \cdot 2 + 1 \cdot 4 = 17$$

$$\mathbf{A}\,\mathbf{B} = \begin{bmatrix} 3 & 2 & 0 \\ 1 & -1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 & -1 \\ 2 & 1 & 2 & 1 \\ 0 & 1 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 5 & 10 & -1 \\ -1 & 1 & 3 & -1 \\ 5 & 6 & 15 & 4 \end{bmatrix}$$

Dot the rows of **A** with the columns of **B**

- Number of columns of **A** must be equal to Number of rows of **B**

Examples

1. Fill in the missing entries

$$\mathbf{AB} = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 1 & 3 & 1 & 1 \\ -2 & 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 & 0 & 2 \\ 1 & 0 & -2 & 1 & 1 \\ 3 & -2 & 1 & 2 & 1 \\ 4 & 1 & 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 5 & 0 & 2 & 3 \\ 12 & 0 & -4 & \mathbf{7} & 7 \\ 9 & \mathbf{-2} & -5 & 8 & 1 \end{bmatrix}$$

2. Multiply? Does not make sense. Inner dimensions do not match.

$$\begin{bmatrix} 1 & 1 & 2 & -1 \\ 2 & 0 & 2 & 1 \\ 1 & 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 0 & 4 \\ 1 & -1 & 1 & 0 \\ 2 & 2 & 3 & 1 \end{bmatrix}$$

3. Multiply: $AB = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} -3 & 5 \\ -10 & 12 \end{bmatrix}$ $\qquad$ $BA = \begin{bmatrix} 1 & 3 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 0 & -2 \end{bmatrix}$

$\boxed{\text{Not commutative}}$

Why?

$$AB = \begin{bmatrix} 3 & 2 & 0 \\ 1 & -1 & 1 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 & -1 \\ 2 & 1 & 2 & 1 \\ 0 & 1 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 5 & 10 & -1 \\ -1 & 1 & 3 & -1 \\ 5 & 6 & 15 & 4 \\ 3 & 3 & 7 & 1 \\ 1 & 2 & 5 & 0 \end{bmatrix}$$

B

$$\begin{bmatrix} 1 & 1 & 2 & -1 \\ 2 & 1 & 2 & 1 \\ 0 & 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ 14 \\ 15 \end{bmatrix}$$

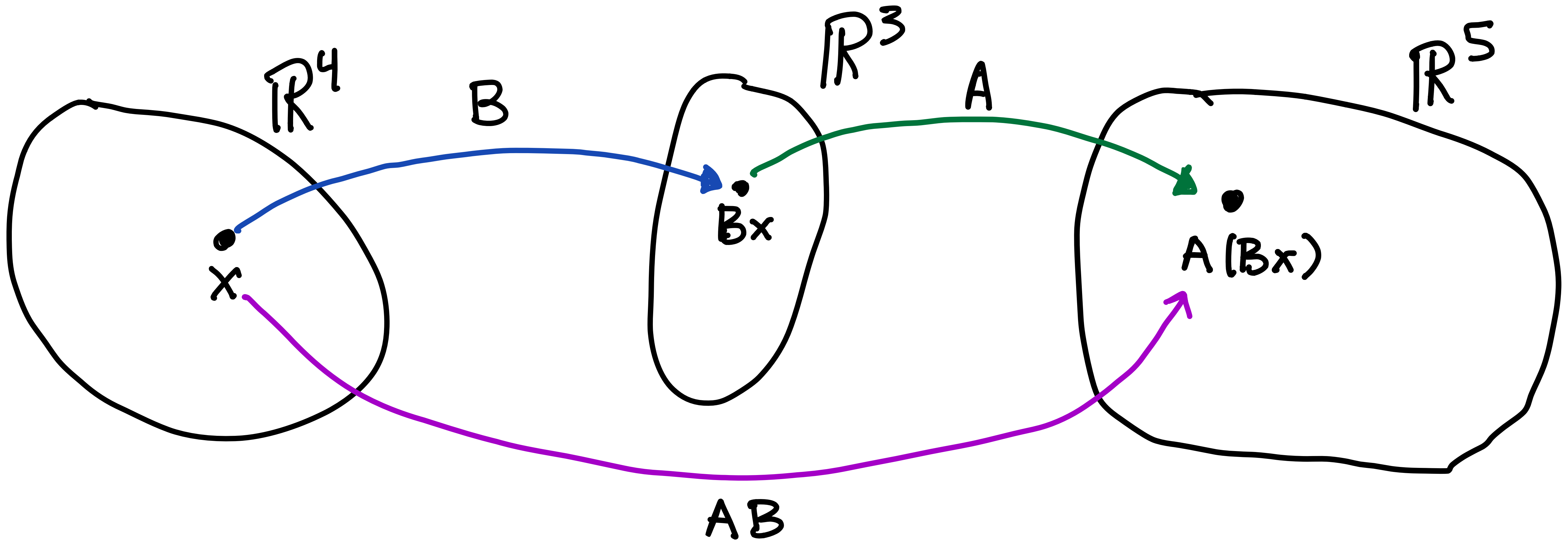
A

$$\begin{bmatrix} 3 & 2 & 0 \\ 1 & -1 & 1 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 14 \\ 15 \end{bmatrix} = \begin{bmatrix} 43 \\ 6 \\ 78 \\ 34 \\ 20 \end{bmatrix}$$

Matrix multiplication corresponds to **composition** of transformations

AB

$$\begin{bmatrix} 7 & 5 & 10 & -1 \\ -1 & 1 & 3 & -1 \\ 5 & 6 & 15 & 4 \\ 3 & 3 & 7 & 1 \\ 1 & 2 & 5 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 43 \\ 6 \\ 78 \\ 34 \\ 20 \end{bmatrix}$$



Transpose

Exchange rows and columns

$$A^T = \begin{bmatrix} 3 & 2 & 0 \\ 1 & -1 & 1 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 3 & 1 & 1 & 1 & 1 \\ 2 & -1 & 2 & 1 & 0 \\ 0 & 1 & 3 & 1 & 1 \end{bmatrix}$$

$$B^T = \begin{bmatrix} 1 & 1 & 2 & -1 \\ 2 & 1 & 2 & 1 \\ 0 & 1 & 3 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 2 & 2 & 3 \\ -1 & 1 & 1 \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} 7 & 5 & 10 & -1 \\ -1 & 1 & 3 & -1 \\ 5 & 6 & 15 & 4 \\ 3 & 3 & 7 & 1 \\ 1 & 2 & 5 & 0 \end{bmatrix}^T = \begin{bmatrix} 7 & -1 & 5 & 3 & 1 \\ 5 & 1 & 6 & 3 & 2 \\ 10 & 3 & 15 & 7 & 5 \\ -1 & -1 & 4 & 1 & 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 3 & 2 & 0 \\ 1 & -1 & 1 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 & -1 \\ 2 & 1 & 2 & 1 \\ 0 & 1 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 5 & 10 & -1 \\ -1 & 1 & 3 & -1 \\ 5 & 6 & 15 & 4 \\ 3 & 3 & 7 & 1 \\ 1 & 2 & 5 & 0 \end{bmatrix}$$

$$\underset{(AB)^T}{\begin{bmatrix} 7 & -1 & 5 & 3 & 1 \\ 5 & 1 & 6 & 3 & 2 \\ 10 & 3 & 15 & 7 & 5 \\ -1 & -1 & 4 & 1 & 0 \end{bmatrix}} = \underset{B^T}{\begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 2 & 2 & 3 \\ -1 & 1 & 1 \end{bmatrix}} \underset{A^T}{\begin{bmatrix} 3 & 1 & 1 & 1 & 1 \\ 2 & -1 & 2 & 1 & 0 \\ 0 & 1 & 3 & 1 & 1 \end{bmatrix}}$$

$$\boxed{(AB)^T = B^T A^T}$$

order reversing