Review of Coordinates.

1. Here are two bases of $\mathbb{R}^3$:

$$\mathcal{B} = \left\{ \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}, \qquad \mathcal{S} = \left\{ \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\},$$

Fill in the blanks.

$$\begin{bmatrix} 5 \\ -3 \\ 4 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} \end{bmatrix}_{\mathcal{S}}$$

$$\begin{bmatrix} 6 \\ 4 \\ 3 \end{bmatrix}_{\mathcal{S}} = \begin{bmatrix} \end{bmatrix}_{\mathcal{B}}$$

$$\mathbf{e}_2 = \begin{bmatrix} \end{bmatrix}_{\mathcal{S}} = \begin{bmatrix} \end{bmatrix}_{\mathcal{B}}$$

$$\vec{v}_2 = \begin{bmatrix} \end{bmatrix}_{\mathcal{S}} = \begin{bmatrix} \end{bmatrix}_{\mathcal{B}}$$

Dimension. The *dimension* of a vector space V (including subspaces, which are vector spaces), denoted $\dim(V)$ is the number of vectors in a basis of V.

Key points:

-

-

Examples:

1. $\dim(\mathbb{R}) =$ 2. $\dim(\mathbb{R}^2) =$ 3. $\dim(\mathbb{R}^3) =$ 4. $\dim(\mathbb{R}^n) =$ 5. $\dim(\{\mathbf{0}\}) =$

6. If L is a line through the origin in $\mathbb{R}^3$ and P is a plane through the origin in $\mathbb{R}^3$ then

   $\dim(L) =$ $\dim(P) =$

7. Find the dimension of $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5, \vec{v}_6, \vec{v}_7\} \subseteq \mathbb{R}^5$.
   To do so, we make $S = \mathsf{Col}(\mathsf{A})$ and row reduce A.

$$\mathsf{A} = \begin{bmatrix} | & | & | & | & | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 & \vec{v}_5 & \vec{v}_6 & \vec{v}_7 \\ | & | & | & | & | & | & | \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 2 & 0 & 1 & 0 & -3 \\ 0 & 1 & -3 & 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

8. Also find a basis of the null space of the matrix above:

9. The dimension of the column space $\mathsf{Col}(\mathsf{A})$ is called the *rank* of the matrix A and the dimension of the null space $\mathsf{Nul}(\mathsf{A})$ is called the *nullity* of the matrix A. Find the rank and the nullity of the following matrix. Give bases of $\mathsf{Col}(\mathsf{A})$ and $\mathsf{Nul}(\mathsf{A})$.

$$\begin{bmatrix} 1 & 1 & 3 & 1 & 2 \\ -2 & 1 & -3 & -1 & -9 \\ -1 & 1 & -1 & 2 & -3 \\ 4 & 1 & 9 & 3 & 13 \\ -2 & 3 & -1 & 2 & -10 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 & 0 & 3 \\ 0 & 1 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 2 \\ -1 & 1 & 3 \\ 2 & 1 & 1 \\ 4 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

For a general matrix A:

$\mathsf{rank}(\mathsf{A}) =$

$\mathsf{nullity}(\mathsf{A}) =$

Rank-Nullity Theorem: If A is an $m \times n$ matrix then $\mathsf{rank}(A) + \mathsf{nullity}(A) =$

Invertible Matrices: An $n \times n$ matrix A is invertible   if and only if $\mathsf{rank}(\mathsf{A}) =$ if and only if $\mathsf{nullity}(\mathsf{A}) =$

Discussion

1. (a) Find the rank and nullity of the following matrices

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 1 & 1 \\ -1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 & 1 & -2 & 3 & 0 \\ -2 & 1 & -1 & -2 & -2 & -5 \\ -1 & 1 & 2 & -9 & 2 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 2 & 1 & 1 \\ 0 & 1 & 0 & -1 & 1 & -2 \\ 0 & 0 & 1 & -3 & 1 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 3 & 5 & -2 & 4 \\ 1 & 2 & 4 & -1 & 3 \\ 2 & 1 & 5 & 1 & 3 \\ 2 & 0 & 4 & 2 & 2 \\ -1 & 1 & -1 & -2 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 2 & 1 & 1 \\ 0 & 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 1 & 1 \\ -2 & 1 & -1 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \\ -2 & 3 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(b) An $m \times n$ matrix has *full row rank* if $\mathsf{rank}(A) = m$ and it has *full column rank* if $\mathsf{rank}(A) = n$. Do any of the matrices above have full row rank or full column rank?

(c) What do you know if a matrix has full row rank?

(d) What do you know if a matrix has full column rank?

2. Here is a basis $\mathcal{B}$ of $\mathbb{R}^4$ and two vectors $\vec{u}$ and $\vec{w}$ in $\mathbb{R}^4$.

$$\mathcal{B} = \left\{ \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix} \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_4 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\} \qquad \vec{w} = \begin{bmatrix} 2 \\ -1 \\ 3 \\ 2 \end{bmatrix}_\mathcal{B}, \qquad \vec{u} = \begin{bmatrix} 3 \\ 1 \\ 1 \\ 2 \end{bmatrix}_\mathcal{S}.$$

```
A = cbind(c(1,2,-1,0),c(1,1,1,1),c(1,1,0,0),c(0,1,1,1))
```

(a) The vector $\vec{w}$ is given in $\mathcal{B}$. Express $\vec{w}$ in standard coordinates.

(b) The vector $\vec{u}$ is given in standard $\mathcal{S}$. Express $\vec{u}$ in $\mathcal{B}$ coordinates.

3. The following problem is on the next problem set.

In practice, we change bases because problems are computationally easier in another coordinate system or because we learn something by looking at a problem from a different point of view. The following example illustrates this with ideas that arises both in chemistry and computer graphics. Below is the tetrahedral molecule methane, CH_4 along with the coordinates of its atoms
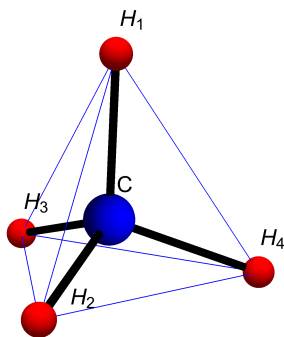
$$C = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, H_1 = \begin{bmatrix} 0 \\ 0 \\ \frac{3}{2\sqrt{6}} \end{bmatrix}, H_2 = \begin{bmatrix} -\frac{1}{2\sqrt{3}} \\ -\frac{1}{2} \\ -\frac{1}{2\sqrt{6}} \end{bmatrix}, H_3 = \begin{bmatrix} -\frac{1}{2\sqrt{3}} \\ \frac{1}{2} \\ -\frac{1}{2\sqrt{6}} \end{bmatrix}, H_4 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ 0 \\ -\frac{1}{2\sqrt{6}} \end{bmatrix}$$

(a) Find a dependence relation among the vectors H_1, H_2, H_3, H_4. Hint: add them together (you can do it "by hand" by just looking at the sum).

(b) We can see visually that $\mathcal{M} = \{H_1, H_2, H_3\}$ is a basis of $\mathbb{R}^3$, which we will call the *tetrahedral basis*. You can see from the plot that these vectors are linearly independent (not all on the same plane). Give the coordinates of each of the vectors H_1, H_2, H_3, H_4 in the $\mathcal{M}$ basis (for H_4 you will need to use part a).

(c) In chemistry and physics, we are interested in symmetry operations. These are linear transformations such that the atom looks the same after the transformation as it did before. For example one such operation is rotation r by 120° around the H_4 axis. This rotation sends H_1 to H_3, H_3 to H_2, and H_2 to H_1. Give the matrix of r in the $\mathcal{M}$ basis. The columns should be the result of applying the symmetry operation to each of the basis vectors and then expressing the answer in the $\mathcal{M}$ basis.

$$r = \begin{array}{c} \\ H_1 \\ H_2 \\ H_3 \end{array} \overset{\begin{array}{ccc} H_1 & H_2 & H_3 \end{array}}{\begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}}_{\mathcal{M}}$$

(d) Show, by multiplying by hand, that your matrix r sends H_1 to H_3, H_3 to H_2, H_2 to H_1, *and* H_4 to H_4.

(e) Another symmetry operation is a rotation s by 180° around the axis that passes through the midpoint between H_1 and H_2 and the midpoint between H_3 and H_4. This rotation exchanges H_1 and H_2 and exchanges H_3 and H_4. Find the matrix of s in the $\mathcal{M}$ basis. By hand, apply it to each of the four hydrogen atoms and show that they go to the right place.

(f) Now we will convert our matrix for r to standard coordinates. You will do this in R Studio. The vectors H_1, H_2, H_3, H_4 are given to you under the PS6 link on the handbook. Here is the recipe.

 i. Enter the change of basis matrix T that converts from the tetrahedral basis $\mathcal{M}$ to the standard basis $\mathcal{S}$ and compute its inverse that converts from the standard basis back to $\mathcal{M}$. .

 ii. Enter the matrix of the rotation r from part c above.

 iii. Compute the matrix of $[r]_\mathcal{S}$ in the standard basis by computing the matrix product below. Notice that, working from right to left, it first converts from standard coordinates to $\mathcal{M}$ coordinates. Then it does the rotation in $\mathcal{M}$ coordinates. Then it converts the answer back to standard coordinates.

$$[r_4]_\mathcal{S} = \underbrace{\begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \end{array} \overset{\begin{array}{ccc} H_1 & H_2 & H_3 \end{array}}{\left(\begin{array}{c} \text{convert the} \\ \text{answer} \\ \text{back to} \\ \text{standard} \\ \text{coordinates} \end{array}\right)}}_{\mathcal{M} \to \mathcal{S}} \underbrace{\begin{array}{c} \\ H_1 \\ H_2 \\ H_3 \end{array} \overset{\begin{array}{ccc} H_1 & H_2 & H_3 \end{array}}{\left(\begin{array}{c} \text{perform the} \\ \text{rotation in} \\ \mathcal{M}\text{-coordinates} \end{array}\right)}} \underbrace{\begin{array}{c} \\ H_1 \\ H_2 \\ H_3 \end{array} \overset{\begin{array}{ccc} e_1 & e_2 & e_3 \end{array}}{\left(\begin{array}{c} \text{convert from} \\ \text{standard} \\ \text{coordinates} \\ \mathcal{M} \text{ coordinates} \end{array}\right)}}_{\mathcal{S} \to \mathcal{M}}.$$