Tuesday, November 15

1. Welcome!
2. HW 7 due tonight, let me know if you need more time.
   ↳ also get in touch if you haven't submitted hw6
3. transitioning to project mode this week
4. questions?
5. hamiltonian graphs
6. outro
   ↳ get on projects!

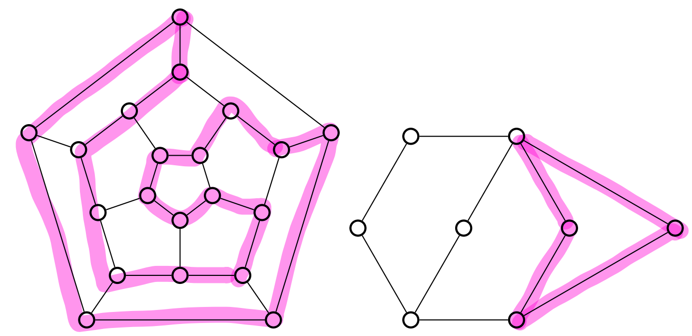
no office hrs 11/21, 11/22

Math 494: Discrete Optimization 11/15

Welcome to our last week of new content in DiscOp! There are so many more things we could talk about (looking at you, set coverings, graph coloring, knapsack, matroids, etc), but we're just limited on time. We'll close our discussion with probably the most famous combinatorial optimization problem: finding a route for a traveling salesperson. It's related to a difficult graph feasibility problem, which is where we'll start.

Definition. *A graph G is* hamiltonian *if it contains a cycle which contains all the vertices.*

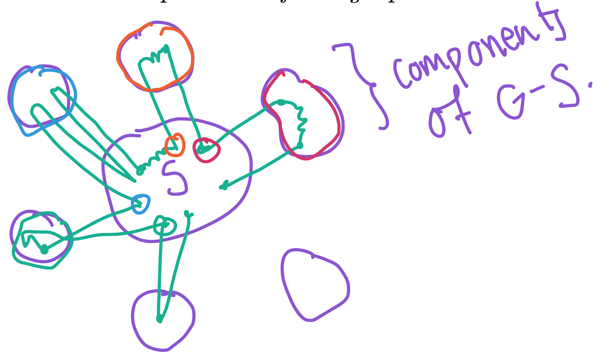
Example: Determine which of the graphs below are Hamiltonian. If the graph is Hamiltonian, draw a hamiltonian cycle. If not, describe a substructure that prevents a hamiltonian cycle.



So a natural question might be if there are necessary or sufficient conditions (or, hopefully, both!) for having a Hamiltonian cycle. There's some necessary conditions, some sufficient conditions, but none that are both. Let's talk about one of each.

Example: Explain why the following theorem is true.

Claim. *If G is hamiltonian, then $k(G - S) \leq |S|$ for all $S \subseteq V$, where $k(H)$ is the number of connected components of the graph H.*



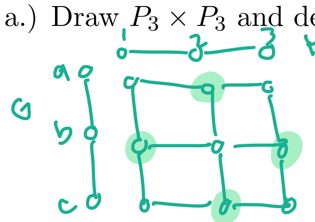
match components the vertex in S where the cycle leaves the comp. for the last time.

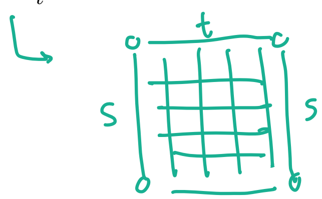
Example: The cartesian product of two graphs, $G \times H$ is the graph with vertex set $V(G \times H) = V(G) \times V(H)$ and edge set

ordered pairs of vertices

$$E(G \times H) = \{(u, v) \sim (w, x) : (\underbrace{u = w}_{\text{in } G} \wedge \underbrace{v \sim x}_{\text{in } H}) \vee (\underbrace{u \sim w}_{\text{in } G} \wedge \underbrace{v = x}_{\text{in } H})\}$$

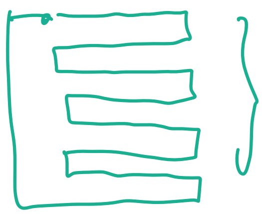
a.) Draw $P_3 \times P_3$ and describe $P_s \times P_t$.





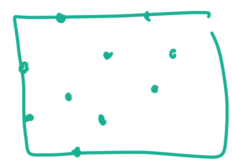
b.) When is $P_s \times P_t$ hamiltonian? Prove your answer.

at least one of s,t must be even!



even dimension

if both odd



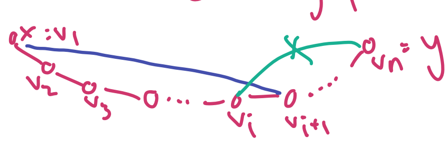
delete alternating vertices

So we've got necessary, how about sufficient?

Example: Prove the following claim.

Claim. *If* $|V| \geq 3$ *and* $\deg(v) + \deg(u) \geq n$ *for all* $u \nsim v \in V$, *then* G *is Hamiltonian.*

Pf: assume for sake of contradiction that G has this property but is not Hamiltonian. add edges to G until we can no longer do so while preserving G being not Hamiltonian. This graph is not complete, so $\exists\, x, y$ st $x \nsim y$. Note, $G + xy$ is hamiltonian, so there exists an xy path in G that containing all the vertices



if $x \sim v_{i+1}$, then $y \nsim v_i$

otherwise hamiltonian cycle.

This implies that for every vertex in $\{v_2, v_3 \dots v_n\}$ that x is adj to, $\exists$ vertex in $\{v_1, v_2, \dots v_{n-1}\}$ that y is not adj. to

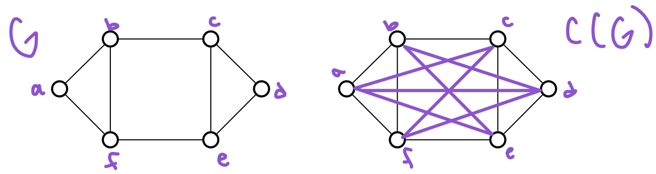
Space for continued proof...

this means $\deg(y) \le \underbrace{n-1}_{\text{all}} - \underbrace{\deg(x)}_{\text{forbidden by } x}$

so $\deg(x) + \deg(y) \le n-1$. ↯.

This actually can help us quite a bit! We'll use it in the following idea about *closure*.

Example: The *closure* of a graph G, denoted $C(G)$ is the graph formed by adding edges between nonadjacent vertices whose degree sum is at least n. Draw the closure of the graph below.



Example: First, determine how the claim below differs from the one we just proved. Second, prove the claim.

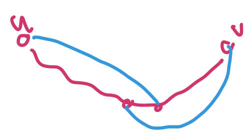
↳ need all nonadj pairs to have the deg sum $\ge n$

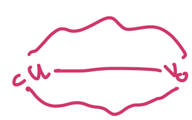
only one pair

Claim. *Let u and v be nonadjacent vertices in a graph G of order n such that* $\deg(u) + \deg(v) \ge n$*. Then $G + uv$ is Hamiltonian if and only if G is Hamiltonian.*

Pf: $(\Leftarrow)$ yea of course. if G is, certainly $G+uv$ will be!

$(\Rightarrow)$ s/p $G+uv$ is hamiltonian. Then this path exists in G. There must be a $\boxed{\text{crossover}}$, otherwise ↯ degree sum, so G is Hamiltonian.





In some sense, this means we can add edges between vertices with high total degree sum and not affect the hamiltonicity (and possibly even make it easier to search for). We can build an algorithm for this! The algorithm will search for the longest paths it can make.

Algorithm 1 LongPath(G, x)

Finds a long path containing x in G. We will add to both sides of P, like a path!

Initialize $u \leftarrow x$, $v \leftarrow x$, $P = (x)$

while $\exists w \sim u$ such that $w \notin P$ **do**

  add w to the front of P

  $u \leftarrow w$

end while

extend path left as far as possible

while $\exists w \sim v$ such that $w \notin P$ **do**

  add w to the back of P

  $v \leftarrow w$

end while

extend path right as far as possible

$P = (u \cdots x \cdots v)$

w must be in the path!

for all $w \sim v$ **do**

  if $u \sim w^+$ **then**

    Construct a cycle $C \leftarrow P + vw - ww^+ + uw^+$

    if C is Hamiltonian **then**

      Great! You're done!

    end if

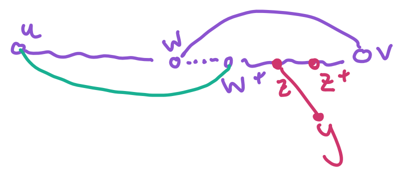
    Find $z \in C$ such that $z \sim y$ and $y \notin C$.

    Convert $C + zy$ into a path P from y to z^+

    $u \leftarrow y$, $v \leftarrow z^+$

  end if

end for



$(y\ z \cdots w^+ u \cdots w\ v \cdots z^+)$

Example: Prove the claim below.

Claim. *Let u and v be nonadjacent vertices in a graph G of order n such that* $\deg(u) + \deg(v) \geq n$. *Then the LongPath algorithm will return a hamiltonian cycle.*

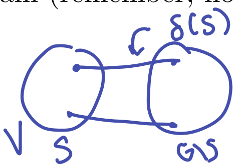
Pf: let P be the last path found by the algorithm. for each $w \sim v$, $u \nsim w^+$, otherwise crossover found and alg. continues.

then $\deg(u) + \deg(v) \leq \ell(P)$ ← length of path P.

If you're interested, this algorithm can run (with nice data structures) in $O(n^2)$ time. And maybe this makes us hopeful! An algorithm that finds a hamiltonian cycle in polynomial time sounds great! Unfortunately, while it's great (and correct) for fairly dense graphs, it's not guaranteed to work in general.

So this is going to be a persistent problem. We're working so hard on finding hamiltonian cycles because they're going to be feasible solutions for our traveling salesperson problem. If we want to use a linear program, we're gonna need an initial feasible solution, which we've already argued is gonna be hard to find. Let's give it a shot anyway.

Example: Develop an integer feasibility program (remember, no objective function) for finding a hamiltonian cycle.



$$\begin{cases} \text{s.t.} \quad \sum_{e \sim v} x_e = 2 & \forall v \in V \\ \sum_{e \in \delta(S)} x_e \geq 2 & \forall S \subsetneq V, \ S \neq \emptyset \\ x_e \in \{0,1\} \end{cases}$$

So as our final item for today, let's define the traveling salesperson problem.

The traveling salesperson problem considers a finite list of locations and an associated cost (travel time) between each pair of locations. Find a minimum cost hamiltonian cycle in this graph.

Example: Write out the linear program for TSP. Denote c_{uv} to be the cost of traveling between u and v.

$$\begin{cases} \min \sum c_{uv} x_{uv} \\ \text{st} \quad \sum_{e \sim v} x_e = 2 & \forall v \in V \\ \sum_{e \in \delta(S)} x_e \geq 2 & \forall S \subseteq V, \ S \neq 0 \\ x_e \in \{0,1\} \end{cases}$$

Note, TSP requires every that every possible pair of locations be travel between-able (real word). How could we possibly forbid this?

really high edge cost!

If we had a magical way to solve the TSP problem could we use it to solve a general hamiltonicity problem?

make present edges of G cost low, non present edges high