Daily vocabulary: Matrix form $Ax = b$; matrix-vector product; continuing: vector equations and span

Class Discussion

1. Compute the following matrix-vector products:

(a) $\begin{bmatrix} 1 & 0 & -2 \\ 2 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} =$

(b) $\begin{bmatrix} | & | & | & | \\ v_1 & v_2 & v_3 & v_4 \\ | & | & | & | \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 5 \\ 2 \end{bmatrix} =$

2. Write the following system of equations in vector form and in matrix form ($Ax = b$).

Linear System

$$\left\{ \begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 13 \\ x_1 & & & + & x_3 & = & 7 \\ -x_1 & + & x_2 & & & = & -5 \\ 2x_1 & + & x_2 & + & x_3 & = & 6 \end{array} \right\}$$

Vector Equation

Matrix Equation

To solve any of these three problems, reduce the corresponding system of equations. This can be done by hand, using R, or in WolframAlpha. We will do it in R in class

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 13 \\ 1 & 0 & 1 & 7 \\ -1 & 1 & 0 & -5 \\ 2 & 1 & 1 & 6 \end{array} \right] \longrightarrow$$

How many solutions does this have? 0 1 ∞. Answer the question in terms of the linear system, the vector equation, and the matrix equation:

Linear System

Vector Equation

Matrix Equation

3. (Discuss) Can we find another vector d so that $Ax = d$ has no solution (for this same matrix A)?

4. Do the vectors $\{\vec{\mathbf{u}}, \vec{\mathbf{v}}, \vec{\mathbf{w}}\}$, below, span all of $\mathbb{R}^3$?

If not, find a vector that is not in their span and describe all of the vectors that are in the span.

$$\vec{\mathbf{u}} = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \vec{\mathbf{v}} = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}, \vec{\mathbf{w}} = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}. \qquad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

5. Restate question 4 as

(a) A matrix $Ax = \vec{\mathbf{b}}$ question

(b) A question about the corresponding linear system of equations

6. Checkpoint:

Do these vectors span R^3 ?

$$\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -10 \\ 11 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 1 & 1 \\ -2 & 2 & -10 & 2 \\ 3 & 1 & 11 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Do these vectors span R^4 ?

$$\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Do these vectors span R^5 ?

$$\begin{bmatrix} 1 \\ 1 \\ 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 1 \\ 1 \\ 4 \end{bmatrix}$$

7. Four matrices are row reduced here for you.

$$A = \begin{bmatrix} 2 & 1 & -1 & 3 & 6 \\ 1 & 2 & 1 & 2 & 5 \\ 1 & 2 & 1 & 1 & 4 \\ -3 & 1 & 4 & 0 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 4 & 1 & 1 & 1 \\ 1 & 0 & 2 & 1 \\ 1 & 2 & 1 & 1 \\ -3 & 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 1 & 3 & 2 & 1 & 4 \\ 1 & 2 & 2 & -1 & 1 & 5 \\ 1 & 2 & 1 & -2 & 0 & 4 \\ -3 & 1 & 0 & 2 & 0 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 1 \\ -3 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Match the following statements below with the matrices A, B, C, D above.

(i) The columns of the matrix span $\mathbb{R}^4$ and reach every vector in a unique way.

(ii) The columns of the matrix span $\mathbb{R}^4$ and reach each vector in infinitely many ways.

(iii) The columns of the matrix do not span $\mathbb{R}^4$, but every vector in their span can be reached in a unique way.

(iv) The columns of the matrix do not span $\mathbb{R}^4$, but every vector in their span can be reached in infinitely many ways.

8. Leftover from Wednesday

In each example below there are 3 vectors, $\vec{\mathbf{u}}, \vec{\mathbf{v}}, \vec{\mathbf{w}}$, in $\mathbb{R}^3$. Describe the span of the vectors. A useful row reduction has been done for you in each case.

(a) $\vec{\mathbf{u}} = \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}, \vec{\mathbf{v}} = \begin{bmatrix} 3 \\ 12 \\ -9 \end{bmatrix}, \vec{\mathbf{w}} = \begin{bmatrix} -2 \\ -8 \\ 6 \end{bmatrix}$ $\qquad \begin{bmatrix} 1 & 3 & -2 \\ 4 & 12 & -8 \\ -3 & -9 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(b) $\vec{\mathbf{u}} = \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}, \vec{\mathbf{v}} = \begin{bmatrix} 3 \\ 12 \\ -9 \end{bmatrix}, \vec{\mathbf{w}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\qquad \begin{bmatrix} 1 & 3 & 1 \\ 4 & 12 & 1 \\ -3 & -9 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

(c) $\vec{\mathbf{u}} = \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}, \vec{\mathbf{v}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{\mathbf{w}} = \begin{bmatrix} 2 \\ 5 \\ -2 \end{bmatrix}$ $\qquad \begin{bmatrix} 1 & 1 & 2 \\ 4 & 1 & 5 \\ -3 & 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

(d) $\vec{\mathbf{u}} = \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}, \vec{\mathbf{v}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{\mathbf{w}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $\qquad \begin{bmatrix} 1 & 1 & 1 \\ 4 & 1 & 2 \\ -3 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

9. Time permitting:

(a) Is the $\vec{\mathbf{0}}$ vector in the span of the vectors below? If it is, is there a unique combination of the vectors that get to the 0 vector?

$$\vec{\mathbf{v}}_1 = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \vec{\mathbf{v}}_2 = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} \ \vec{\mathbf{v}}_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

check:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 4 & 5 & 1 & 0 \\ 7 & 8 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & -6 & -8 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 3 & 4 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

(b) Is the $\vec{\mathbf{0}}$ vector in the span of the vectors below? If it is, is there a unique combination of the vectors that get to the 0 vector?

$$\vec{\mathbf{v}}_1 = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \vec{\mathbf{v}}_2 = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} \ \vec{\mathbf{v}}_3 = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$$

check:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -3 & -6 & 0 \\ 0 & -6 & -12 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

(c) What can you say about the *homogeneous* system of equations $Ax = \vec{\mathbf{0}}$