Section 4.1: Definition of Laplace Transform

In Section 3.1, 3.2, and 3.3

We focus on equations

$$ay'' + by' + cy = f(x)$$

f is in the nullspace of some polynomial in D
(We could find an annihilator which was a polynomial in D)

Electrical Circuits

$$Lq'' + Rq' + \frac{q}{C} = E(t)$$

impressed voltage.
May have discontinuities

Laplace Transform: Let f be a function defined $t \geq 0$. Then the integral

$$\mathcal{L}\{t\} = \int_0^\infty e^{-st} f(t)\,dt.$$

a function in s.

is called the Laplace Transform of f, assuming the integral converges

Notation: lower case letter - input
capital letter - output

$$\mathcal{L}\{f(t)\} = F(s) \quad , \mathcal{L}\{g(t)\} = G(s)$$

$$\mathcal{L}\{y(t)\} = Y(s)$$

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st} \cdot 1 \, dt = \int_0^\infty e^{-st} \, dt$$

$$= \lim_{b\to\infty} \int_0^b e^{-st} \, dt$$

$$= \lim_{b\to\infty} -\frac{1}{s} e^{-st} \Big|_0^b$$

$$= \lim_{b\to\infty} -\frac{1}{s}\left(e^{-sb} - e^{-s\cdot 0}\right)$$

$$= -\frac{1}{s}\left(\cancel{0} - e^{0}\right) \qquad s > 0$$

$$= \frac{1}{s} \qquad s > 0$$

$$f(t) = 1 \qquad F(s) = \frac{1}{s} \qquad s > 0$$

Example: $\mathcal{L}\{e^{7t}\} = \frac{1}{s-7}$

$$\mathcal{L}\{e^{7t}\} = \int_0^\infty e^{-st} \cdot e^{7t} \, dt \qquad e^{-s+7}$$

$$e^{-(s-7)}$$

$$= \int^\infty e^{-(s-7)t} \, dt$$

$^{\cup}s$

$$= -\frac{1}{s-7} e^{-(s-7)t}\Big|_0^\infty$$

$$= -\frac{1}{s-7}(0-1) \qquad s>7$$

$$= \frac{1}{s-7} \qquad s>7$$

$$= \int_0^\infty e^{(-s+7)t}\, dt$$

$$= \frac{1}{-s+7} e^{(-s+7)t}\Big|_0^\infty$$

$$-s+7<0$$
$$-s<-7 \Rightarrow s>7$$

$$= \frac{1}{-s+7}(0-1) = \frac{-1}{-s+7} = \frac{1}{s-7}$$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a} \qquad s>a$$

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \qquad n=0,1,2,3,\dots$$

$$\mathcal{L}\{t\} = \int_0^\infty te^{-st}\, dt = \overbrace{t\cdot \frac{-1}{s} e^{-st}\Big|_0^\infty}^{\to 0} + \underbrace{\frac{1}{s}\int_0^\infty e^{-st}\, dt}$$

$u=t \qquad dv=e^{-st}$

$du=1\,dt \qquad v=-\frac{1}{s}e^{-st}$

$$= 0 + \frac{1}{s}\mathcal{L}\{1\}^{*}$$
$$= \frac{1}{s}\left(\frac{1}{s}\right) = \frac{1}{s^2}$$

Example: Evaluate $\mathcal{L}\{\cos(9t)\}$

$$\mathcal{L}\{\cos 9t\} = \int_0^\infty e^{-st}\cos(9t)\,dt$$

$u = \cos(9t)$
$du = -9\sin(9t)\,dt$
$dv = e^{-st}\,dt$
$v = -\frac{1}{s}e^{-st}$

$$= -\frac{1}{s}e^{-st}\cos(9t)\Big|_0^\infty - \frac{9}{s}\int_0^\infty e^{-st}\sin(9t)\,dt$$

$u = \sin(9t)$
$du = 9\cos(9t)\,dt$
$dv = e^{-st}\,dt$
$v = -\frac{1}{s}e^{-st}$

$$= -\frac{1}{s}e^{-st}\cos(9t)\Big|_0^\infty$$
$$- \frac{9}{s}\left(-\frac{1}{s}e^{-st}\sin(9t)\Big|_0^\infty + \frac{9}{s}\int_0^\infty e^{-st}\cos(9t)\,dt\right)$$

$$= -\frac{1}{s}e^{-st}\cos(9t)\Big|_0^\infty - \frac{9}{s}\left(-\frac{1}{s}e^{-st}\sin(9t)\Big|_0^\infty\right.$$
$$\left. + \frac{9}{s}\mathcal{L}\{\cos(9t)\}\right)$$

$$= -\frac{1}{s}(0-1) - \frac{9}{s}\left(-\frac{1}{s}(0+0) + \frac{9}{s}\mathcal{L}\{\cos(9t)\}\right)$$

$$= \frac{1}{s} - \frac{81}{s^2}\mathcal{L}\{\cos(9t)\}$$

$$\mathcal{L}\{\cos(9t)\} = \frac{1}{s} - \frac{81}{s^2}\mathcal{L}\{\cos(9t)\}$$

$$\left(1 + \frac{81}{s^2}\right)\mathcal{L}\{\cos(9t)\} = \frac{1}{s}$$

$$\mathcal{L}\{\cos(9t)\} = \frac{1}{s} \cdot \frac{1}{(1 + 81/s^2)}$$

$$= \frac{s}{s^2 + 81}$$

$$\mathcal{L}\{\cos(at)\} = \frac{s}{s^2+a^2} \qquad s > 0$$

$$\mathcal{L}\{\cos(9t)\} + \frac{81}{s^2}\mathcal{L}\{\cos(9t)\} = \frac{1}{s}$$

$$\mathcal{L}\{\sin(at)\} = \frac{a}{s^2+a^2} \qquad s > 0$$

Laplace Transform is Linear!

$$\mathcal{L}\{af(t) + g(t)\} = \mathcal{L}\{af(t)\} + \mathcal{L}\{g(t)\}$$

$$= a\mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\}$$

for any constant a and functions f and g as long as the Laplace transforms of f and g exist.

Example: Find $\mathcal{L}\{3t^4 + \cos(2t)\}$

$$\mathcal{L}\{3t^4 + \cos(2t)\} = 3\mathcal{L}\{t^4\} + \mathcal{L}\{\cos(2t)\}$$

$$= 3\left(\frac{4!}{s^5}\right) + \frac{s}{s^2+4}$$

$s > 0$