2.2. Matrix Inverse

Calculate these two products

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 8 & 2 \end{bmatrix} \begin{bmatrix} 10 & -6 & 1 \\ -2 & 1 & 0 \\ -7 & 5 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3 x 3 identity matrix $\mathbf{I}$

$$\begin{bmatrix} 10 & -6 & 1 \\ -2 & 1 & 0 \\ -7 & 5 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 8 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

They are ***inverses*** of each other

Inverse Matrices

Definition

A square $n \times n$ matrix A is **invertible** when there exists an $n \times n$ matrix C such that

$$CA = I \qquad \text{and} \qquad AC = I$$

where I is the $n \times n$ identity matrix. We call C the **inverse** of A, and we write $C = A^{-1}$.

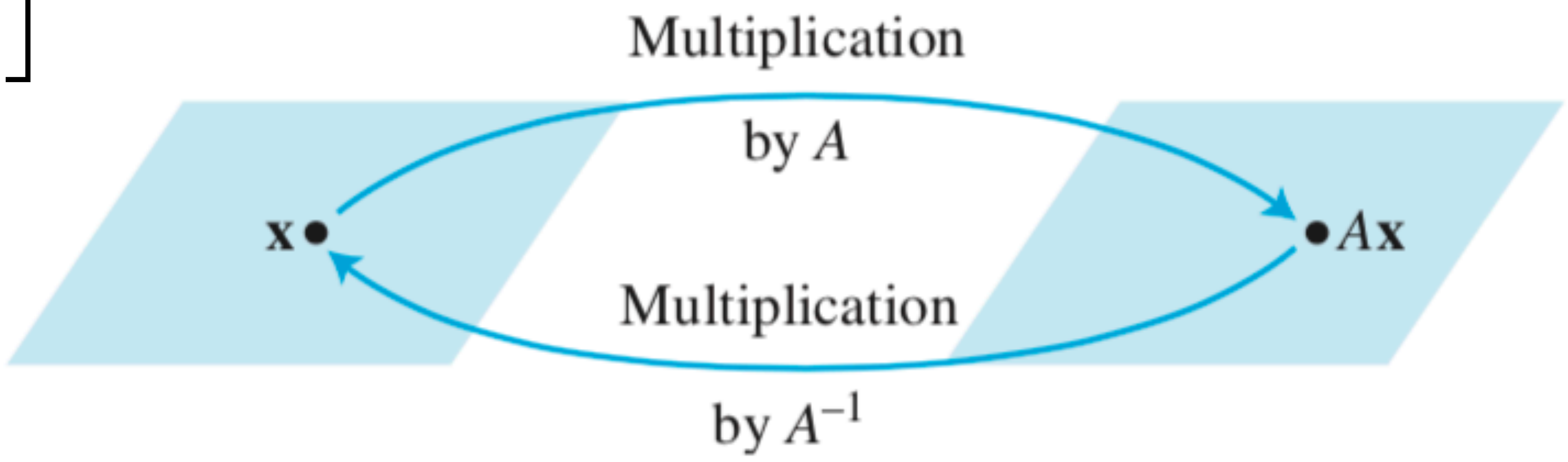
$$\underbrace{\begin{bmatrix} 10 & -6 & 1 \\ -2 & 1 & 0 \\ -7 & 5 & -1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 8 & 2 \end{bmatrix}}_{A^{-1}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{I}$$

Inverse Linear Transformation

$$\begin{bmatrix} 10 & -6 & 1 \\ -2 & 1 & 0 \\ -7 & 5 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 8 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 8 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} 10 & -6 & 1 \\ -2 & 1 & 0 \\ -7 & 5 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$



How are they useful?

Solving Vector Equations

If $\mathbf{A}x = y$ then $x = \mathbf{A}^{-1} y$

Solution to $\mathbf{A}x = b$, is $x = \mathbf{A}^{-1} b$

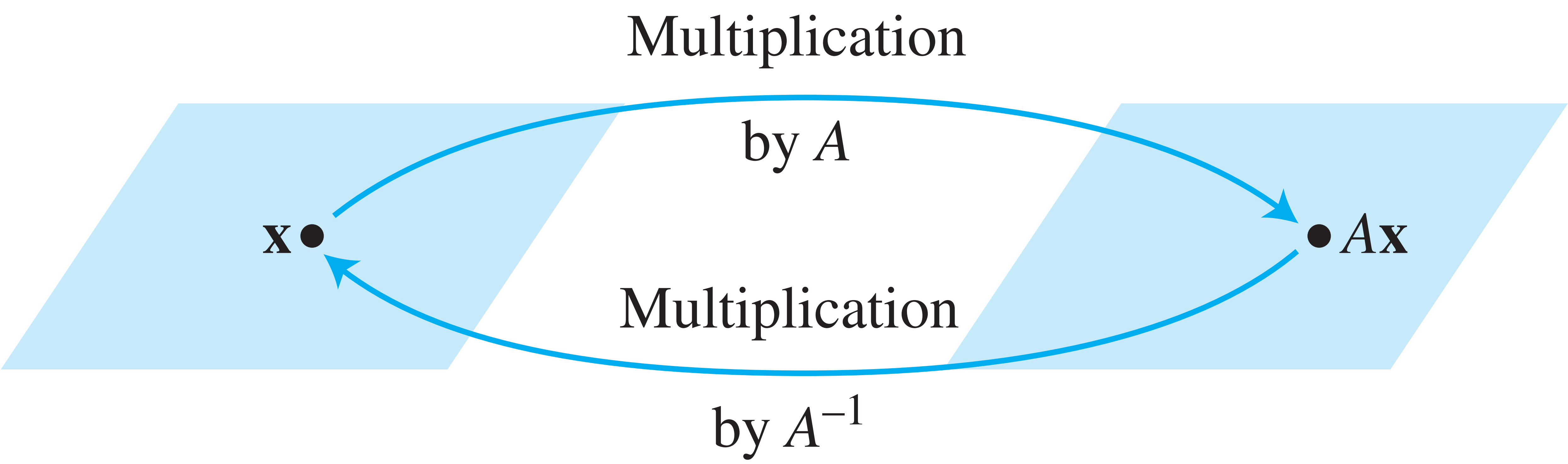


FIGURE 2 A^{-1} transforms $A\mathbf{x}$ back to $\mathbf{x}$.

Solving Matrix Equations

$$\mathbf{A}\,\mathbf{B} = \mathbf{C}$$
$$\mathbf{A}^{-1}(\mathbf{AB}) = \mathbf{A}^{-1}\mathbf{C}$$
$$(\mathbf{A}^{-1}\mathbf{A})\mathbf{B} = \mathbf{A}^{-1}\mathbf{C}$$
$$\mathbf{I}\,\mathbf{B} = \mathbf{A}^{-1}\mathbf{C}$$
$$\mathbf{B} = \mathbf{A}^{-1}\mathbf{C}$$

Example

$$\begin{bmatrix} 10 & -6 & 1 \\ -2 & 1 & 0 \\ -7 & 5 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 8 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solve $\mathbf{A}\mathbf{x} = \mathbf{b}$

$$\begin{bmatrix} 10 & -6 & 1 \\ -2 & 1 & 0 \\ -7 & 5 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$$

$$\boxed{\mathbf{A}\mathbf{x} = \mathbf{b} \qquad \Rightarrow \qquad \mathbf{x} = \mathbf{A}^{-1}\mathbf{b}}$$

Solve:

$$\mathbf{x} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 8 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

Check

$$\begin{bmatrix} 10 & -6 & 1 \\ -2 & 1 & 0 \\ -7 & 5 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$$

Finding 2x2 Inverses

A formula worth memorizing.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = (ad-bc)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So we get a general formula for the inverse of a 2x2 matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad-bc$$

$$\begin{bmatrix} 3 & 2 \\ 10 & 5 \end{bmatrix}^{-1} = -\frac{1}{5}\begin{bmatrix} 5 & -2 \\ -10 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 2/5 \\ 2 & -3/5 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 3 \\ 6 & -9 \end{bmatrix}^{-1} = \text{DNE}$$

Finding 3x3 Inverses

Don't memorize!

Is there a similar formula for 3 x 3 matrices?

$$\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix}^{-1} = \frac{1}{det(A)} \begin{bmatrix} a_{2,2}a_{3,3} - a_{2,3}a_{3,2} & a_{1,3}a_{3,2} - a_{1,2}a_{3,3} & a_{1,2}a_{2,3} - a_{1,3}a_{2,2} \\ a_{2,3}a_{3,1} - a_{2,1}a_{3,3} & a_{1,1}a_{3,3} - a_{1,3}a_{3,1} & a_{1,3}a_{2,1} - a_{1,1}a_{2,3} \\ a_{2,1}a_{3,2} - a_{2,2}a_{3,1} & a_{1,2}a_{3,1} - a_{1,1}a_{3,2} & a_{1,1}a_{2,2} - a_{1,2}a_{2,1} \end{bmatrix}$$

$$det(A) = -a_{1,3}a_{2,2}a_{3,1} + a_{1,2}a_{2,3}a_{3,1} + a_{1,3}a_{2,1}a_{3,2} - a_{1,1}a_{2,3}a_{3,2} - a_{1,2}a_{2,1}a_{3,3} + a_{1,1}a_{2,2}a_{3,3}$$

Finding Inverses

$$\begin{bmatrix} 10 & -6 & 1 \\ -2 & 1 & 0 \\ -7 & 5 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 8 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

ALGORITHM: $[A \mid I] \longrightarrow [I \mid A^{-1}]$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 3 & 2 & 0 & 1 & 0 \\ 3 & 8 & 2 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 5 & -1 & -3 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & -1 & 7 & -5 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -7 & 5 & -1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 8 & -5 & 1 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -7 & 5 & -1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 10 & -6 & 1 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -7 & 5 & -1 \end{array}\right]$$

The Inverse of a Product

If $\mathbf{A}$ is invertible and has inverse $\mathbf{A}^{-1}$ and if $\mathbf{B}$ is invertible and has inverse $\mathbf{B}^{-1}$

What is the inverse of the product $\mathbf{AB}$?

$$\mathbf{AB}\,\mathbf{B}^{-1}\mathbf{A}^{-1} = \mathbf{A}\,\mathbf{I}\,\mathbf{A}^{-1} = \mathbf{A}\,\mathbf{A}^{-1} = \mathbf{I}$$

So $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$

The shoes and socks property