Daily vocabulary: augmented matrix, elementary row operations, consistent, unique, RREF, REF

Class Discussion

1. What row operation that is performed in each case?

(a) $\left[\begin{array}{cccc|c} 2 & 1 & -4 & 0 & 6 \\ 7 & 7 & 14 & 21 & -7 \\ -1 & 1 & 0 & -5 & -1 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 2 & 1 & -4 & 0 & 6 \\ 1 & 1 & 2 & 3 & -1 \\ -1 & 1 & 0 & -5 & -1 \end{array}\right] \leftarrow \frac{1}{7}R_2$

(b) $\left[\begin{array}{cccc|c} 2 & 1 & -4 & 0 & 6 \\ 1 & 1 & 2 & 3 & -1 \\ -1 & 1 & 0 & -5 & -1 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 1 & 2 & 3 & -1 \\ 2 & 1 & -4 & 0 & 6 \\ -1 & 1 & 0 & -5 & -1 \end{array}\right]$

(c) $\left[\begin{array}{cccc|c} 1 & 1 & 2 & 3 & -1 \\ 2 & 1 & -4 & 0 & 6 \\ -1 & 1 & 0 & -5 & -1 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 1 & 2 & 3 & -1 \\ 0 & -1 & -8 & -6 & 8 \\ -1 & 1 & 0 & -5 & -1 \end{array}\right] \leftarrow R_2 + -2R_1$

2. Which of the following augmented matrices are in row-echelon form (REF)?

$$A = \left[\begin{array}{ccccc|c} 7 & 5 & 1 & 4 & 5 & 0 \\ 0 & 2 & 1 & 1 & 3 & 11 \\ 0 & 0 & 6 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \qquad B = \left[\begin{array}{cccc|c} 1 & 1 & 2 & 3 & -1 \\ 0 & -1 & -8 & -6 & 8 \\ 0 & 0 & 0 & -14 & 14 \end{array}\right]$$

$$\cancel{C} = \left[\begin{array}{ccc|c} 3 & 0 & 0 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 1 & 0 & 3 \end{array}\right] \qquad \cancel{D} = \left[\begin{array}{ccc|c} 0 & 1 & 5 & 1 \\ 0 & 2 & 11 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

3. Which of the following augmented matrices are in reduced row-echelon Form (RREF)?

$$\cancel{A} = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 4 \\ 0 & 1 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \qquad C = \left[\begin{array}{cccccc|c} 1 & 2 & 0 & 5 & 0 & 100 & 10 \\ 0 & 0 & 1 & 3 & 0 & 9 & 8 \\ 0 & 0 & 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$B = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 3 & 1 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 1 & -1 \end{array}\right] \qquad \cancel{D} = \left[\begin{array}{ccc|c} 1 & 0 & 2 & 5 \\ 0 & 2 & 1 & 3 \end{array}\right]$$

$$E = \left[\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

4. In solving a linear system of equations, one student found the answer $x_1 = 5, x_2 = 0, x_3 = 2, x_4 = 1$. Another student found the answer $x_1 = 1, x_2 = 2, x_3 = 3, x_4 = -4$. Which of the statements below can possibly be true?

always

(a) Those are the only two answers.

(b) At least one of the students made a mistake.

$\underline{0}, \underline{1}, \underline{\infty}$ solutions

(c) There are infinitely many other correct answers.

5. You can tell how many solutions a system has from its Row Echelon Form (REF) or its RREF.

   In each of the examples below, the augmented matrix of a system of equations has been row reduced. (i) Does the system of equations have 0, 1, or ∞ many solutions? (ii) Which systems are <u>consistent</u> and which are <u>inconsistent</u>. (iii) Each of these sets of solutions are in 3-dimensional space. Describe the possible geometry of the equations in terms of intersecting planes.

$$A \longrightarrow \begin{array}{c} \begin{matrix} x & y & z & \end{matrix} \\ \left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}, \qquad B \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{array}\right], \qquad C \longrightarrow \begin{array}{c} \begin{matrix} & & \text{free} & \end{matrix} \\ \left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}.$$

exactly one

$x + 2y - z = 4 \Rightarrow x = 1$
$y + z = 3 \Rightarrow y = 2$
$z = 1$

0 solutions

$0x + 0y + 0z = 1$
$0 = 1$

∞ solutions

6. Solve the following linear system of equations:

$$\left\{ \begin{array}{rcrcrcrcr} x_1 & + & 2x_2 & + & 3x_3 & + & 3x_4 & = & 11 \\ 2x_1 & + & 2x_2 & + & 6x_3 & + & 8x_4 & = & 26 \\ -3x_1 & + & 2x_2 & + & x_3 & - & 7x_4 & = & -9 \end{array} \right\}.$$

done on video

Day 2: Breakout Discussion

1. At each step below decide which row operation is being done and write it out (for example, "scale $R2$ by 3" or "swap $R1$ and $R3$" or "replace $R3$ with $R3 + \frac{1}{5}R2$."

$$\left[\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ 2 & 1 & 4 & 1 \\ -3 & -1 & -7 & -2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 1 & 4 & 1 \\ -3 & -1 & -7 & -2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -1 & 2 & 1 \\ -3 & -1 & -7 & -2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -1 & 2 & 1 \\ 0 & 2 & -4 & -2 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -2 & -1 \\ 0 & 2 & -4 & -2 \end{array}\right] \longrightarrow \underset{\text{REF}}{\boxed{\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]}} \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

   (a) Decide if the final matrix is in RREF

   (b) Underline the first matrix in the process that is in REF

   (c) Write out the equations that correspond to the last matrix. Let the variables be x, y, z.

   (d) Write out the general solution.

$$\left\{\begin{array}{l} x_1 + 3x_3 = 1 \\ x_2 - 2x_3 = -1 \end{array}\right\}$$

$$\boxed{\begin{array}{l} x_1 = 1 - 3x_3 \\ x_2 = -1 + 2x_3 \\ x_3 = \text{free} \end{array}}$$ General solution

2. In each case below give an example of an augmented matrix with a 3×3 coefficient matrix that is in row-echelon form and has the solutions described below.

   (a) Has exactly one solution: $x = 2, y = 7, z = 3$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

   (b) Has no solutions

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \leftarrow \text{inconsistency}$$

   (c) Has infinitely many solutions, and one of those solutions is $x = 3, y = 1, z = 1$.

one possibility $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow$ general solution $\begin{array}{l} x = 4 - z \\ y = 2 - z \\ z = \text{free} \end{array}$

if $z = 1$ then we get
$x = 3$
$y = 1$
$z = 1$

3. Give the general solution to the system of equations whose RREF is shown below. The variables are x_1, x_2, x_3, x_4, x_5.

$$\left[\begin{array}{ccccc|c} 1 & 0 & -2 & 0 & 1 & 10 \\ 0 & 1 & 3 & 0 & 2 & -4 \\ 0 & 0 & 0 & 1 & -3 & 3 \end{array}\right] \Longrightarrow \left\{\begin{array}{rcl} x_1 - 2x_3 + x_5 & = & 10 \\ x_2 + 3x_3 + 2x_5 & = & -4 \\ x_4 - 3x_5 & = & 3 \end{array}\right\}$$

$$\boxed{\begin{array}{l} x_1 = 10 + 2x_3 - x_5 \\ x_2 = -4 - 3x_3 - 2x_5 \\ x_3 = \text{free} \\ x_4 = 3 + 3x_5 \\ x_5 = \text{free} \end{array}}$$

General solution

4. You wish to find all possible parabolas $f(x) = a + bx + cx^2$ that pass through the three points $(1,3), (3,11), (2,4)$. (i) Write down the linear system of equations for this problem. (ii) Create the augmented matrix. (iii) Row reduce!