Row Space. The *row space*, $\mathsf{Row}(\mathsf{A})$, is the span of the rows of A.

1. Consider the row space:

$$S = \mathsf{Row}\left(\begin{bmatrix} 0 & 1 & 5 \\ 2 & -6 & -32 \\ -2 & 8 & 42 \end{bmatrix}\right) = \mathsf{span}\left\{\begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ -6 \\ 8 \end{bmatrix}, \begin{bmatrix} 5 \\ -32 \\ 42 \end{bmatrix}\right\} \subseteq \mathbb{R}^3.$$

How do row operations change the row space?

$$\begin{bmatrix} 0 & 1 & 5 \\ 2 & -6 & -32 \\ -2 & 8 & 42 \end{bmatrix} \to \begin{bmatrix} 2 & -6 & -32 \\ 0 & 1 & 5 \\ -2 & 8 & 42 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & -16 \\ 0 & 1 & 5 \\ -2 & 8 & 42 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & -16 \\ 0 & 1 & 5 \\ 0 & 2 & 10 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & -16 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

Key Points:

2. Here is the transpose of the matrix above row reduced. How does this compare to the row space above?

$$\begin{bmatrix} 0 & 2 & -2 \\ 1 & -6 & 8 \\ 5 & -32 & 42 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Application to finding bases.

3. Let $S \subseteq \mathbb{R}^4$ be the span of the vectors below. Find the dimension of S and find a basis for S.

$$S = \mathsf{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ -3 \\ -9 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 5 \\ 13 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(a) <u>Method 1</u>: put the vectors of S into the columns of a matrix A and row reduce:

$$A = \begin{bmatrix} 1 & 1 & -1 & 2 & 1 \\ 1 & -1 & -3 & 4 & 0 \\ 2 & 1 & -3 & 5 & 0 \\ 4 & -1 & -9 & 13 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -2 & 3 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) <u>Method 2</u>: put the vectors of S into the columns of a matrix A and row reduce:

$$A^T = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 1 & -1 & 1 & -1 \\ -1 & -3 & -3 & -9 \\ 2 & 4 & 5 & 13 \\ 1 & 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & \frac{7}{3} \\ 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(c) Is the vector $\vec{v}$ in S?

$$\vec{v} = \begin{bmatrix} 5 \\ -9 \\ 12 \\ -12 \end{bmatrix}$$

(d) $\dim(S) = \mathsf{rank}(\mathsf{A}) = \mathsf{rank}(A^T)$.

You Try!

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 8 & 0 & 0 \\ 1 & 1 & 2 & 0 & 2 \\ 0 & 2 & -6 & 1 & 3 \\ 0 & 2 & -6 & 2 & 4 \\ 2 & 0 & 10 & 0 & 2 \\ 2 & 0 & 10 & -2 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 5 & 0 & 1 \\ 0 & 1 & -3 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{A}^T = \begin{bmatrix} 1 & 1 & 0 & 0 & 2 & 2 \\ -1 & 1 & 2 & 2 & 0 & 0 \\ 8 & 2 & -6 & -6 & 10 & 10 \\ 0 & 0 & 1 & 2 & 0 & -2 \\ 0 & 2 & 3 & 4 & 2 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & -1 & 1 & 3 \\ 0 & 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Fill in the blanks:

a) rank($\mathbf{A}$) = 3

b) nullity($\mathbf{A}$) = 2

c) rank($\mathbf{A}^T$) = 3

d) nullity($\mathbf{A}^T$) = 3

i) Col($\mathbf{A}$) is a 3 dimensional subspace of $\mathbb{R}^6$

ii) Nul($\mathbf{A}$) is a 2 dimensional subspace of $\mathbb{R}^5$

iii) Row($\mathbf{A}$) is a 3 dimensional subspace of $\mathbb{R}^5$

iv) Null($\mathbf{A}^T$) is a 3 dimensional subspace of $\mathbb{R}^6$