4.2. Null Space and Column Space

Subspaces

Def: A **subspace** is a subset S of $\mathbb{R}^n$ that satisfies the following properties:

For all vectors **u** and **v** in **S**:

1. **u** + **v** is in **S** ⟵ *closed* under addition
2. *c* **u** is in **S** for each scalar *c* in $\mathbb{R}$ ⟵ *closed* under scalar multiplication
3. **0** is in **S**
4. *c* **u** + *d* **v** is in **S**

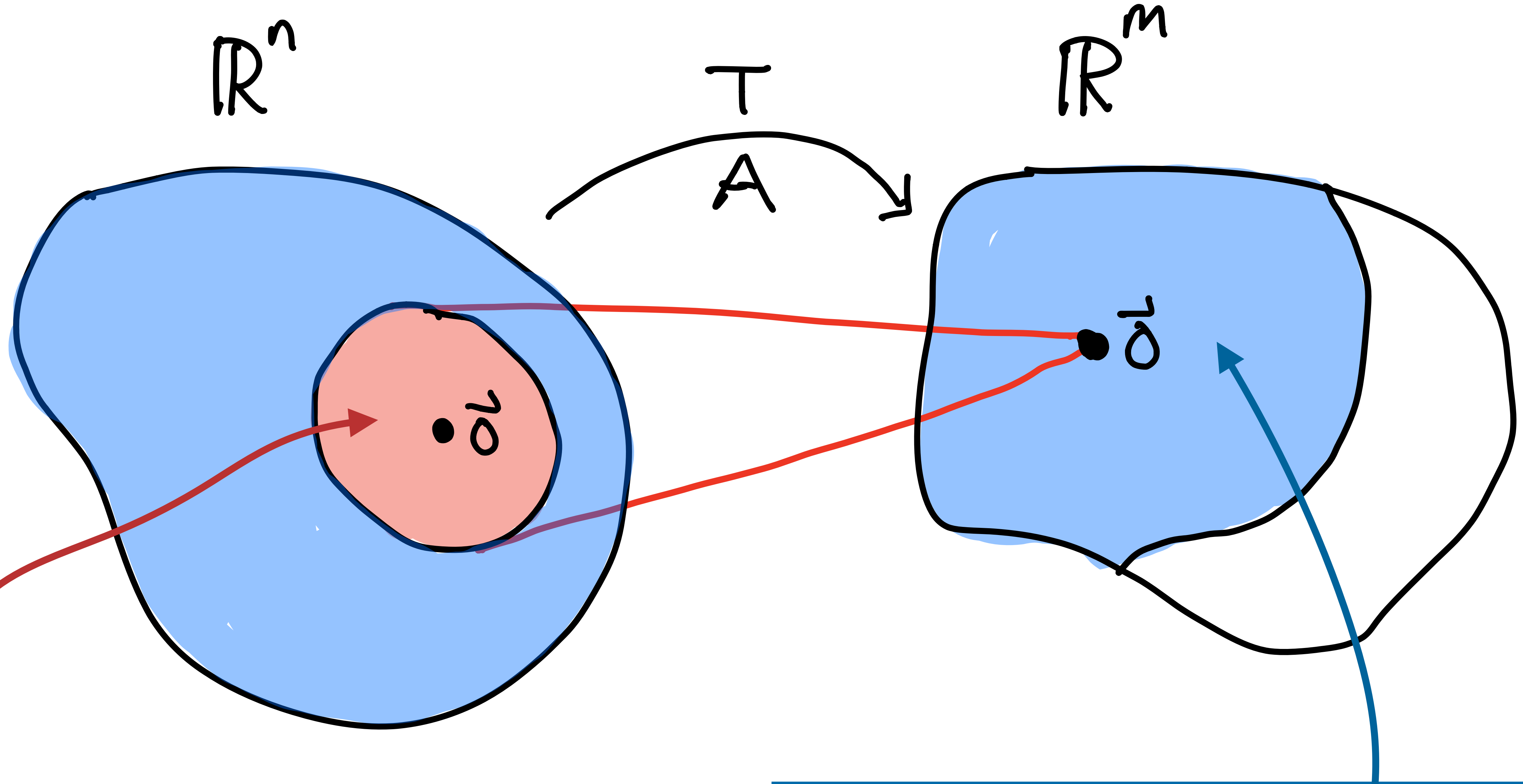
Linear Transformation

A linear transformation

$$T : \mathbb{R}^n \to \mathbb{R}^m$$

With matrix $T(x) = \mathbf{A}\,x$

1. $T(x+y) = T(x) + T(y)$
2. $T(cx) = cT(x)$
3. $T(0) = 0$



Range
Image

Column
Space

Kernel

Null
Space

$$\mathrm{Ker}(T) = \{\vec{x} \in \mathbb{R}^n \mid T(\vec{x}) = \vec{0}\}$$
$$\mathrm{Nul}(A) = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0}\}$$

$$\mathrm{Im}(T) = \{T(\vec{x}) \mid \vec{x} \in \mathbb{R}^n\}$$
$$\mathrm{Col}(A) = \{A\vec{x} \mid \vec{x} \in \mathbb{R}^n\}$$

The vectors that map to 0

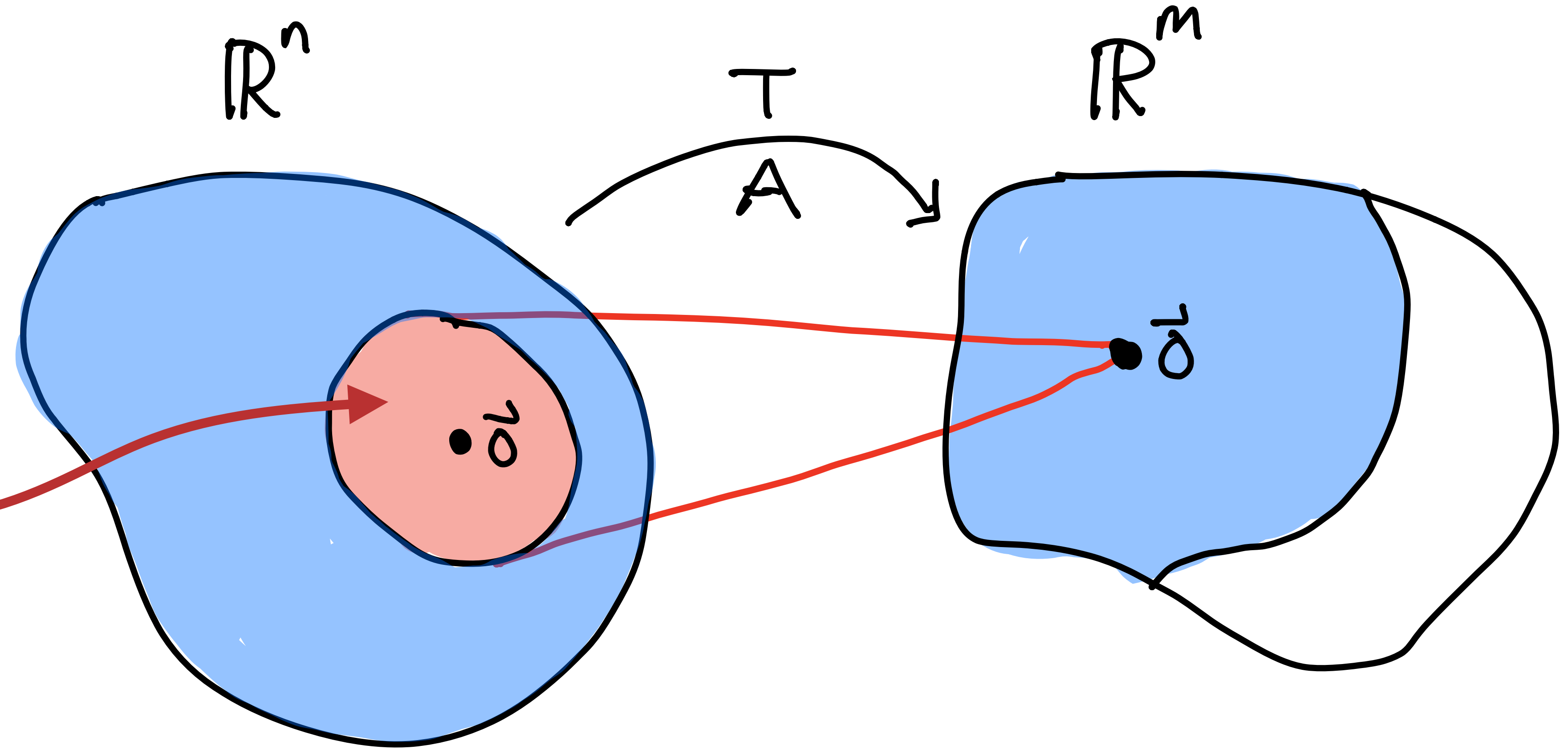
The vectors that get mapped to

$$\mathrm{Col}(A) = \{b \in \mathbb{R}^m \mid Ax = b \text{ for at least one } x \in \mathbb{R}^n\}$$

The Null Space is a Subspace

$T : \mathbb{R}^n \to \mathbb{R}^m$ with matrix $T(x) = \mathbf{A}\,x$

$\boxed{\mathrm{Nul}(A) = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0}\} \text{ is a subspace of } \mathbb{R}^n}$



Proof.

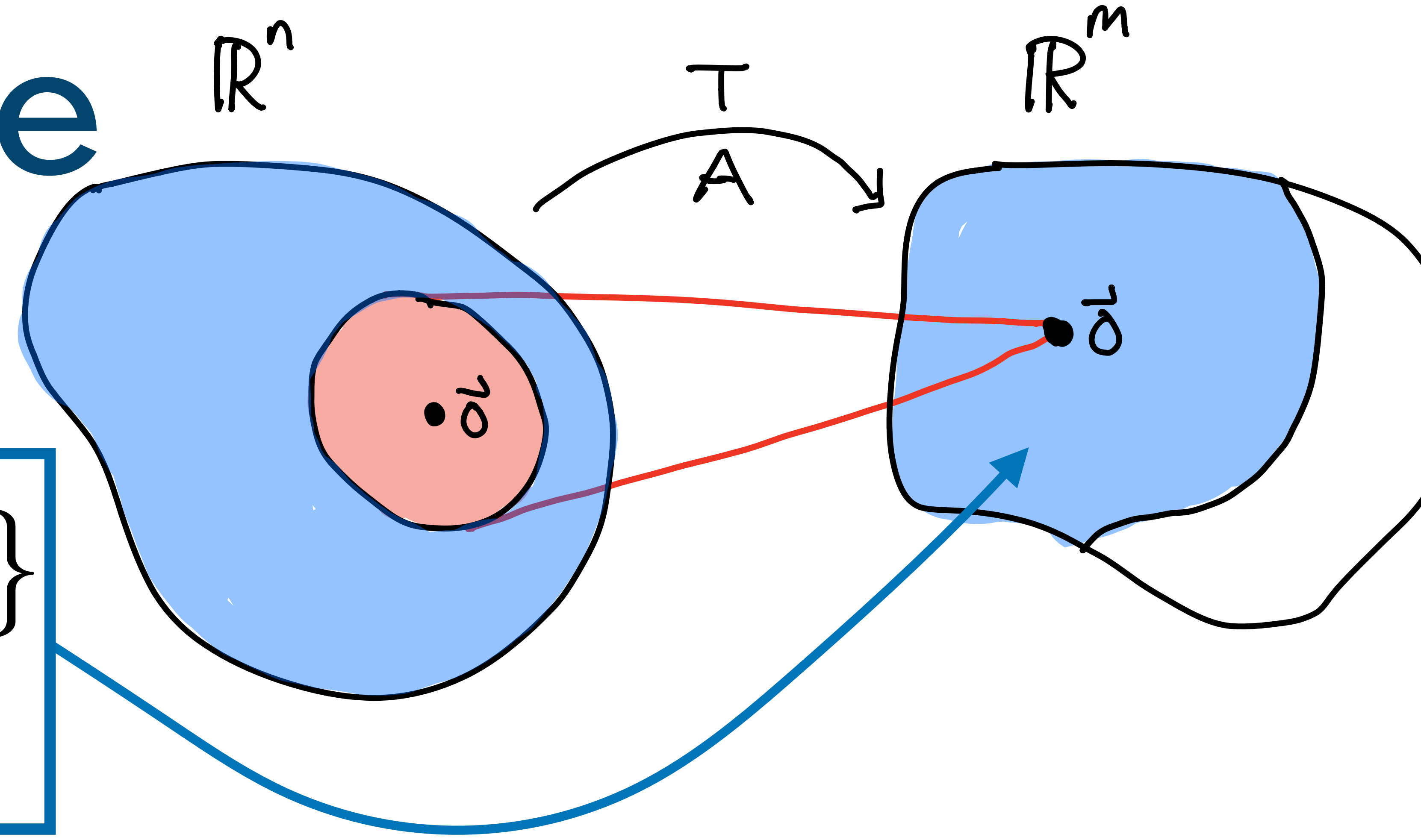
① Let $u, v \in \mathrm{Nul}(A)$, so that $Au = \vec{0}$ and $Av = \vec{0}$.

Then $A(u+v) = Au + Av = \vec{0} + \vec{0} = \vec{0}$, so $u+v \in \mathrm{Nul}(A)$

② Let $u \in \mathrm{Nul}(A)$ so that $Au = 0$, and let $c \in \mathbb{R}$.

Then $A(cu) = cAu = c \cdot \vec{0} = \vec{0}$, so $cu \in \mathrm{Nul}(A)$ ∎

The Column Space is a Subspace



$T : \mathbb{R}^n \to \mathbb{R}^m$ with matrix $T(x) = \mathbf{A}\,x$

$$\mathrm{Col}(A) = \{b \in \mathbb{R}^m \mid Ax = b \text{ for at least one } x \in \mathbb{R}^n\}$$
is a subspace of $\mathbb{R}^m$

Proof.

① Let $b_1, b_2 \in \mathrm{Col}(A)$. There exist $x_1, x_2 \in \mathbb{R}^n$ so that $Ax_1 = b_1$ and $Ax_2 = b_2$.
Then $A(x_1 + x_2) = Ax_1 + Ax_2 = b_1 + b_2$, so $b_1 + b_2 \in \mathrm{Col}(A)$.

② Let $\vec{b} \in \mathrm{Col}(A)$ and $c \in \mathbb{R}$. There exists $\vec{x} \in \mathbb{R}^n$ so that $A\vec{x} = \vec{b}$.
Then $A(c\vec{x}) = cA\vec{x} = c\vec{b}$, so $c\vec{b} \in \mathrm{Col}(A)$.

Is v in the Null Space?

The vectors that map to **0**

$$\boxed{\mathrm{Nul}(A) = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0}\}}$$

Are the vectors **v** or **w**, below, in the **null space** of the matrix **A**?

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 1 & -1 & 0 & 0 \\ 2 & 1 & 0 & 3 \\ 4 & -1 & 1 & 4 \end{bmatrix}$$

$$\mathbf{v} = \begin{bmatrix} 2 \\ 5 \\ 4 \\ 15 \end{bmatrix}$$

$$\mathbf{w} = \begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

$$\mathbf{Av} = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 1 & -1 & 0 & 0 \\ 2 & 1 & 0 & 3 \\ 4 & -1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 4 \\ 15 \end{bmatrix} = \begin{bmatrix} 56 \\ -3 \\ 54 \\ 67 \end{bmatrix}$$

No

$$\mathbf{Aw} = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 1 & -1 & 0 & 0 \\ 2 & 1 & 0 & 3 \\ 4 & -1 & 1 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Yes

Is v in the Column Space?

The vectors that get mapped to

$$\mathrm{Col}(A) = \{A\vec{x} \mid \vec{x} \in \mathbb{R}^n\}.$$
$$\mathrm{Col}(A) = \{b \in \mathbb{R}^m \mid Ax = b \text{ for at least one } x \in \mathbb{R}^n\}$$

Are the vectors **v** or **w**, below, in the **column space** of the matrix **A**?

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 1 & -1 & 0 & 0 \\ 2 & 1 & 0 & 3 \\ 4 & -1 & 1 & 4 \end{bmatrix}$$

$$\mathbf{v} = \begin{bmatrix} 2 \\ 5 \\ 4 \\ 15 \end{bmatrix}$$

$$\mathbf{w} = \begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 3 & 2 \\ 1 & -1 & 0 & 0 & 5 \\ 2 & 1 & 0 & 3 & 4 \\ 4 & -1 & 1 & 4 & 15 \end{array}\right] \longrightarrow \begin{array}{c} \text{free variable (column 4)} \\ \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

Yes

one possible **x**

$$\mathbf{x} = \begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 3 & -1 \\ 1 & -1 & 0 & 0 & -1 \\ 2 & 1 & 0 & 3 & -1 \\ 4 & -1 & 1 & 4 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

No

inconsistency

Column Space is the Span of the Columns

$$\mathrm{Col}(A) = \{A\vec{x} \mid \vec{x} \in \mathbb{R}^n\}.$$

$$\mathrm{Col}(A) = \{b \in \mathbb{R}^m \mid Ax = b \text{ for at least one } x \in \mathbb{R}^n\}$$

$$A x = \begin{bmatrix} | & | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 \\ | & | & | & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_1 \vec{v}_1 + x_2 \vec{v}_2 + x_3 \vec{v}_3 + x_4 \vec{v}_4$$

Linear combination of the columns of **A**

$$\mathrm{Col}(A) = \text{ span of the columns of } A$$

Is v in the Column Space?

The vectors that get mapped to

$$\mathrm{Col}(A) = \{A\vec{x} \mid \vec{x} \in \mathbb{R}^n\}.$$
$$\mathrm{Col}(A) = \{b \in \mathbb{R}^m \mid Ax = b \text{ for at least one } x \in \mathbb{R}^n\}$$
$$\mathrm{Col}(A) = \text{ span of the columns of } A$$

Is the vector **v** in the **column space** of the matrix **A**?

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 1 & -1 & 0 & 0 \\ 2 & 1 & 0 & 3 \\ 4 & -1 & 1 & 4 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 4 \end{bmatrix}$$

Yes! it is one of the columns.

$$A \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = v$$

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 3 \\ 0 & 0 & 0 \\ 4 & -1 & 5 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

No! The columns of A can never combine to get anything other than zero in the 3rd coordinate.

You Try!

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 6 \\ -1 & 2 & 0 & 1 \\ 2 & 2 & 0 & 10 \\ 1 & 1 & 1 & 6 \\ 1 & 3 & 1 & 10 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Find a nonzero vector u that is in the null space of **A**
(b) Find a nonzero vector v that is the in column space of **A**