Daily vocabulary: parametric form of solution to $Ax = \vec{\mathbf{b}}$. Comparing the solutions to $Ax = \vec{\mathbf{0}}$, $Ax = \vec{\mathbf{b}}$, $Ax = \vec{\mathbf{c}}$.

Warm Up

dot product

$(-1)(1) + (2)(0) + (2)(-1) + (-1)(2) + (1)(1) = -4$

1. Fill in the missing entry

$$\begin{bmatrix} 2 & -2 & -3 & 2 & -3 \\ 2 & -3 & 2 & 1 & -3 \\ -1 & 2 & 2 & -1 & 1 \\ -3 & -2 & -2 & 1 & -3 \\ 3 & 2 & -1 & 3 & 1 \\ -2 & 0 & 2 & -3 & -3 \\ -1 & -3 & 2 & -3 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ -1 \\ -4 \\ 0 \\ 3 \\ -5 \\ -6 \end{bmatrix}$$

compute using dot product

2. "Compute" the product:

$$\begin{bmatrix} | & | & | & | \\ v_1 & v_2 & v_3 & v_4 \\ | & | & | & | \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 5 \\ 2 \end{bmatrix} =$$

interpret as linear comb of columns of A

$= 3v_1 + -1v_2 + 5v_3 + 2v_2$

linear comb of columns of A

3. If $A\vec{\mathbf{v}} = \vec{\mathbf{b}}$ and $A\vec{\mathbf{w}} = \vec{\mathbf{d}}$ then what are the following matrix-vector products?

$A(2\vec{\mathbf{v}}) = 2A\vec{v}$ $\quad A(\vec{\mathbf{v}}+\vec{\mathbf{w}}) = A\vec{v} + A\vec{w}$ $\quad A(5\vec{\mathbf{v}}+3\vec{\mathbf{w}}) = 5A\vec{v} + 3Aw$

$A(c\vec{v}) = cA\vec{v}$

4. A system equations of the form $Ax = \vec{\mathbf{0}}$ is a *homogeneous* system of equation (when the right-hand side is 0). Otherwise, it is called a nonhomogeneous system of equations. Here are two homogeneous systems of equations and their row reductions. *Describe the solution to each of them.*

(a) $\begin{bmatrix} 1 & 2 & 1 \\ 4 & 5 & 1 \\ 7 & 8 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 4 & 5 & 1 & 0 \\ 7 & 8 & -1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & -6 & -8 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 3 & 4 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(b) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -3 & -6 & 0 \\ 0 & -6 & -12 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 - x_3 = 0$

$x_2 + 2x_3 = 0$

x_3 free

$$\boxed{\begin{aligned} x_1 &= x_3 \\ x_2 &= -2x_3 \\ x_3 &= \text{free} \end{aligned}} \quad \begin{aligned} &= t \\ &= -2t \\ &= t \end{aligned}$$

(c) Discuss: What can you say, in general, about homogeneous systems of equations? Eg., how many solutions can they have? What can you say about the corresponding vector equation?

parametriz:
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}$$

Parametric Solutions

5. Here is a matrix problem $\mathsf{A}x = \vec{\mathbf{b}}$. Write out the corresponding system of equations and the corresponding vector problem.

$$\begin{bmatrix} 1 & 1 & -1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 12 \\ 2 \\ 15 \\ 3 \end{bmatrix}$$

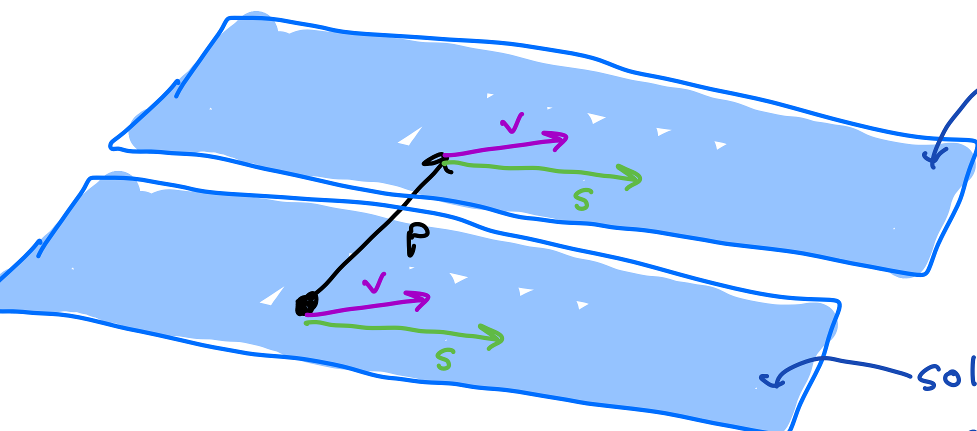
(a) All three problems are solved by "augmenting and row reducing". Use the row reduction to write down the parametric solution to $\mathsf{A}x = \vec{\mathbf{b}}$.

$$\left[\begin{array}{ccccc|c} 1 & 1 & -1 & 1 & 1 & 12 \\ 1 & 0 & 1 & 1 & 0 & 2 \\ 1 & 1 & -1 & 0 & 2 & 15 \\ 0 & 0 & 0 & -1 & 1 & 3 \end{array}\right] \longrightarrow \begin{array}{cccccc} x_1 & x_2 & x_3 \downarrow & x_4 & x_5 \downarrow & \\ \hline \end{array} \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 1 & 5 \\ 0 & 1 & -2 & 0 & 1 & 10 \\ 0 & 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 = 5 - x_3 - x_5$
$x_2 = 10 + 2x_3 - x_5$
$x_3 = \text{free} = s$
$x_4 = -3 + x_5$
$x_5 = \text{free} = t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 5 - s - t \\ 10 + 2s - t \\ 0 + s \\ -3 + t \\ 0 + t \end{bmatrix} = \underbrace{\begin{bmatrix} 5 \\ 10 \\ 0 \\ -3 \\ 0 \end{bmatrix}}_{p} + s \underbrace{\begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{u} + t \underbrace{\begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{v} \qquad s, t \in \mathbb{R}$$

(b) Draw a picture of the geometry of this solution: It is a subset of $\mathbb{R}^5$. Is it the empty set, a point, a line, a plane, a circle, a 3-dimensional space, a sphere, or something else?



solution to $Ax = b$

$= \{ \vec{p} + \underbrace{s\vec{u} + t\vec{v}}_{\text{homogeneous solution}} \mid s, t \in \mathbb{R} \}$

particular solution

solution to $Ax = 0$

$= \{ \underbrace{s\vec{u} + t\vec{v}}_{\text{homogeneous } (Ax=0) \text{ solution}} \mid s, t \in \mathbb{R} \}$

(c) Write down the parametric solution to $\mathsf{A}x = \vec{\mathbf{0}}$ (the corresponding homogenous equations).

$$s \underbrace{\begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{u} + t \underbrace{\begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{v} \qquad s, t \in \mathbb{R}$$

(d) Discuss the relation between your answers to part (a) and part (c).

parallel planes in $\mathbb{R}^5$

6. True-False. Continued with the same problem.

$$\mathsf{A} = \begin{bmatrix} 1 & 1 & -1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}, \quad \vec{\mathbf{b}} = \begin{bmatrix} 12 \\ 2 \\ 15 \\ 3 \end{bmatrix}, \qquad \left[\begin{array}{ccccc|c} 1 & 1 & -1 & 1 & 1 & 12 \\ 1 & 0 & 1 & 1 & 0 & 2 \\ 1 & 1 & -1 & 0 & 2 & 15 \\ 0 & 0 & 0 & -1 & 1 & 3 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 1 & 5 \\ 0 & 1 & -2 & 0 & 1 & 10 \\ 0 & 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Here are some true and false questions to discuss. See if you can also agree on *why* they are T or F.

(a) T F $\mathsf{A}x = \vec{\mathbf{d}}$ has a solution for all right-hand sides $\vec{\mathbf{d}} \in \mathbb{R}^4$

it might not have a solution

(b) T F If $\mathsf{A}x = \vec{\mathbf{d}}$ has a solution, then it has infinitely many solutions.

has 2 free variables, so if consistent then ∞ solutions

(c) T F If $\vec{\mathbf{p}}$ and $\vec{\mathbf{q}}$ are both solutions to $\mathsf{A}x = \vec{\mathbf{b}}$, then $\vec{\mathbf{p}} - \vec{\mathbf{q}}$ is a solution to $\mathsf{A}x = \vec{\mathbf{0}}$.

$A(p-q) = Ap - Aq = b - b = \vec{0}$ ✓

(d) T F If $\vec{\mathbf{p}}$ and $\vec{\mathbf{q}}$ are both solutions to $\mathsf{A}x = \vec{\mathbf{b}}$, then $\frac{1}{2}\vec{\mathbf{p}} + \frac{1}{2}\vec{\mathbf{q}}$ is a solution to $\mathsf{A}x = \vec{\mathbf{b}}$.

$A(\frac{1}{2}p + \frac{1}{2}q) = \frac{1}{2}Ap + \frac{1}{2}Aq = \frac{1}{2}b + \frac{1}{2}b = b$ ✓

7. Consider the matrix equation $\mathsf{A}x = \vec{\mathbf{b}}$:

$$A = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 0 & 3 & -1 \\ -1 & 1 & -5 & 1 \\ 1 & 2 & -1 & 1 \end{bmatrix}, \qquad \text{and} \qquad \vec{\mathbf{b}} = \begin{bmatrix} 2 \\ 4 \\ -6 \\ 4 \end{bmatrix},$$

Create an R markdown file R to do the row-reduction. Here are R commands for defining A and $\vec{\mathbf{b}}$ and the augmented matrix:

```
A = cbind(c(1,1,-1,1),c(1,0,1,2),c(1,3,-5,-1),c(-1,-1,1,1))
b = c(2,4,-6,4)
Ab = cbind(A,b)
A
Ab
```

Remember that you need to include pracma to get rref:

```
require('pracma')
```

(a) Row reduce the matrix using rref. Write out the solution to $\mathsf{A}x = \vec{\mathbf{b}}$ in parametric form.

(b) Find *two* specific, and different, solutions to $Ax = b$. Call them x_1 and x_2. Name them x1 and x2 in R. Show that they are solutions to $\mathsf{A}x = \vec{\mathbf{b}}$ by multiplying by A. The matrix-vector product in R is funny. You use the following syntax to do it.

```
A %*% x1
```

(c) Show, with a matrix-vector product, that $x_h = x_1 - x_2$ is a solution to the homogeneous equation $\mathsf{A}x = \vec{\mathbf{0}}$.

(d) Are any of these vectors solutions to $\mathsf{A}x = \vec{\mathbf{b}}$? Confirm with a matrix vector product:

i. $x_1 + x_2$

ii. $2x_1$

iii. $\frac{1}{2}x_1 + \frac{1}{2}x_2$

iv. $x_1 + 2021x_h$