Section 2.3: First Order Linear Equations and Integrating Factors

First order linear equation

$$\frac{dy}{dx} + \underline{p(x)}\,y = \underline{g(x)}$$

Not necessarily linear

Linear in y (in the dependent variable)

Examples

- $\frac{dy}{dx} + \underbrace{(x+5)}_{p(x)}y = \underbrace{x^2+2x+2}_{g(x)}$
- $x\frac{dy}{dx} + x^2y = 3x+2$

  $\hookrightarrow \frac{dy}{dx} + xy = 3 + \frac{2}{x}$

Nonexample

- $y\frac{dy}{dx} + xy = 2x+3$
- $\left(\frac{dy}{dx}\right)^2 + (x+3)y = 2$

$\left(\frac{dy}{dx}\right)^2 \leftarrow$ 1st derivative & square

$\frac{d^2y}{dx^2} \leftarrow$ 2nd derivative

$$\frac{dy}{dx} = \frac{d}{dx}(y)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{d}{dx}(y)\right)$$
$$= \frac{d^2}{dx^2}(y)$$

Product Rule:

$$(uv)' = u'v + u \cdot v'$$

Motivation

"reverse" product rule.

$$\int \frac{du}{dx}v + u\frac{dv}{dx}\, dx \stackrel{?}{=} \int \frac{d}{dx}(uv)\, dx$$

$$= uv + C$$

Steps for Solving using an Integrating factor

1. Write equation in the form
$$\frac{dy}{dx} + p(x)y = g(x)$$
2. Find integrating factor if not given.
3. Multiply both sides of equation by integrating factor.
4. Apply reverse product rule on L.H.S
5. Integrate both sides w.r.t. x.

Example: $\frac{dy}{dx} + 3x^2y = 5x^2$
using integrating factor e^{x^3}

$$e^{x^3}\left(\frac{dy}{dx} + 3x^2 y\right) = e^{x^3}(5x^2)$$

$$e^{x^3}\frac{dy}{dx} + 3x^2 e^{x^3} y = 5x^2 e^{x^3}$$

$$\frac{d}{dx}\left(e^{x^3} y\right) = 5x^2 e^{x^3}$$

reverse product rule.

$$\int \frac{d}{dx}\left(e^{x^3} y\right) dx = \int \underbrace{5x^2 e^{x^3} \, dx}$$

$$e^{x^3} y = \frac{5}{3} e^{x^3} + C$$

$$\frac{d}{dx}\left(e^{x^3}\right) = 3x^2 e^{x^3}$$

$$y = \frac{5}{3} + Ce^{-x^3}$$

$$\int 5x^2 e^{x^3} \, dx :$$

Let $u = x^3$ $\quad \frac{du}{dx} = 3x^2$

$$\sim \frac{1}{3} du = x^2 dx$$

$$\int 5x^2 e^{x} \, dx = \int 5e^{x} \, x^2 \, dx$$

$$= \int 5e^{u} \cdot \frac{1}{3} du$$

$$= \int \frac{5}{3} e^{u} \, du$$

$$= \frac{5}{3} \int e^{u} \, du = \frac{5}{3} e^{u} + C$$

$$= \frac{5}{3} e^{x^3} + C$$