Def. If A is an $n \times n$ matrix, then a *nonzero* vector $\vec{v} \in \mathbb{R}^n$ is an *eigenvector* of A if

$$A\vec{v} = \lambda\vec{v}, \qquad \text{for some } \lambda \in \mathbb{R}^n.$$

The scalar λ is the *eigenvalue* corresponding to $\vec{v}$.

Computations:

1. If you have an eigenvector $\vec{v}$ for a matrix A, how do you find the eigenvalue?

$$\vec{v} = \begin{bmatrix}1\\2\\3\end{bmatrix} \qquad A = \begin{bmatrix}1 & -2 & 3\\2 & -4 & 6\\2 & -10 & 12\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}6\\12\\18\end{bmatrix} = 6\begin{bmatrix}1\\2\\3\end{bmatrix} \qquad \boxed{\lambda=6}$$

2. If you have an eigenvalue λ for a matrix A, how do you find the eigenvectors?

e.g., $\lambda = 3$ is another eigenvalue, find the eigenspace:

$\boxed{\lambda=3}$

$$\begin{bmatrix}1-3 & -2 & 3\\2 & -4-3 & 6\\2 & -10 & 12-3\end{bmatrix} = \begin{bmatrix}-2 & -2 & 3\\2 & -7 & 6\\2 & -10 & 9\end{bmatrix} \to \begin{bmatrix}-2 & -2 & 3\\0 & -9 & 9\\0 & -12 & 12\end{bmatrix} \to \begin{bmatrix}-2 & -2 & 3\\0 & 1 & -1\\0 & 0 & 0\end{bmatrix}$$

$$\to \begin{bmatrix}-2 & 0 & 1\\0 & 1 & -1\\0 & 0 & 0\end{bmatrix} \to \begin{bmatrix}1 & 0 & -1/2\\0 & 1 & -1\\0 & 0 & 0\end{bmatrix} \qquad E_3 = N(A-3I) = \text{span}\left(\begin{bmatrix}+1/2\\+1\\1\end{bmatrix}\right)$$

$$= \text{span}\left(\begin{bmatrix}1\\2\\2\end{bmatrix}\right)$$

Check $A\begin{bmatrix}1\\2\\2\end{bmatrix} = 3\begin{bmatrix}1\\2\\2\end{bmatrix}$

3. If you don't have the eigenvalues or eigenvectors.

3×3 characteristic polynomial

$$\det\left(\begin{bmatrix}1-\lambda & -2 & 3\\2 & -4-\lambda & 6\\2 & -10 & 12-\lambda\end{bmatrix}\right) = \cdots = -\lambda^3 + 9\lambda^2 - 18\lambda = -\lambda(\lambda^2 - 9\lambda + 18)$$

$$= -\lambda(\lambda-3)(\lambda-6) = 0$$

$$\boxed{\lambda=0}, \boxed{\lambda=3}, \boxed{\lambda=6}$$

4. Find the other eigenvectors:

$\boxed{\lambda=6}$

$$\begin{bmatrix}1-6 & -2 & 3\\2 & -4-6 & 6\\2 & -10 & 12-6\end{bmatrix} = \begin{bmatrix}-5 & -2 & 3\\2 & -10 & 6\\2 & -10 & 6\end{bmatrix} \to \cdots \to \begin{bmatrix}1 & 0 & -1/3\\0 & 1 & -2/3\\0 & 0 & 0\end{bmatrix} \qquad E_6 = \text{span}\left(\begin{bmatrix}1/3\\2/3\\1\end{bmatrix}\right) = \text{span}\left(\begin{bmatrix}1\\2\\3\end{bmatrix}\right)$$

$\boxed{\lambda=0}$

$$\begin{bmatrix}1 & -2 & 3\\2 & -4 & 6\\2 & -10 & 12\end{bmatrix} \to \cdots \to \begin{bmatrix}1 & 0 & -1\\0 & 1 & -1\\0 & 0 & 0\end{bmatrix} \qquad E_0 = \text{span}\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right)$$

5. What does it mean if $\lambda = 0$ is an eigenvalue?

$\boxed{\text{A is invertible}} \iff \boxed{\lambda = 0 \text{ is } \underline{\text{NOT}} \text{ an eigenvalue}}$

Examples: Today's checkpoint: Find the eigenvalues of the matrices below.

1. $\mathsf{A} = \begin{bmatrix} -7 & -10 \\ 5 & 8 \end{bmatrix}$

$$\det(A-\lambda I_2) = \begin{vmatrix} -7-\lambda & -10 \\ 5 & 8-\lambda \end{vmatrix} = \boxed{(-7-\lambda)(8-\lambda)+50}$$
$$= -56-\lambda+\lambda^2+50$$
$$= \lambda^2-\lambda-6$$
$$= (\lambda-3)(\lambda+2) \qquad \boxed{\lambda = 3,-2}$$

quadratic formula

2. $\mathsf{B} = \begin{bmatrix} 4 & 0 & 0 \\ 3 & -7 & -10 \\ 17 & 5 & 8 \end{bmatrix}$

$$\det(B-\lambda I_3) = \begin{vmatrix} 4-\lambda & 0 & 0 \\ 3 & -7-\lambda & -10 \\ 17 & 5 & 8-\lambda \end{vmatrix} = (4-\lambda)\begin{vmatrix} -7-\lambda & -10 \\ 5 & 8-\lambda \end{vmatrix} + 0\,|\;\;| + 0\,|\;\;|$$
$$= (4-\lambda)\overbrace{(\lambda-3)(\lambda+2)} = -\lambda^3+5\lambda^2+2\lambda-24$$
$$\boxed{\lambda = 4, 3, -2}$$

$$= \left|\begin{array}{ccc|cc} 4-\lambda & 0 & 0 & 4-\lambda & 0 \\ 3 & -7-\lambda & -10 & 3 & -7-\lambda \\ 17 & 5 & 8-\lambda & 17 & 5 \end{array}\right| = (4-\lambda)(-7-\lambda)(8-\lambda) - (4-\lambda)\,5\cdot 10$$
$$= (4-\lambda)\Big[\underbrace{(-7-\lambda)(8-\lambda) - 5\cdot 10}_{\text{same as above}}\Big]$$
$$= (4-\lambda)(\lambda-3)(\lambda+2)$$

3. Here are a few more characteristic polynomials.

(a) Another 2×2

$$(-\lambda)(1-\lambda)-1 = -\lambda+\lambda^2-1$$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \qquad f_A(\lambda) = \begin{vmatrix} 0-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = \cancel{(1-\lambda)^2-1} = \cancel{1-2\lambda+\lambda^2-1} = \boxed{\lambda^2-\lambda-1}$$

$$\lambda = \frac{+1 \pm \sqrt{1+4}}{2} = \boxed{\frac{1\pm\sqrt{5}}{2}}$$

(b) Rental car problem

$$M = \begin{bmatrix} 0.85 & 0.30 & 0.35 \\ 0.09 & 0.60 & 0.05 \\ 0.06 & 0.10 & 0.60 \end{bmatrix}, \qquad f_M(\lambda) = \boxed{-\lambda^3+2.05\lambda^2-1.327\lambda+0.277} = \boxed{(\lambda-1)}\underbrace{(\lambda^2-1.05\lambda+0.277)}$$

$$\lambda = \frac{1.05 \pm \sqrt{1.05^2-4\cdot 1\cdot .277}}{2} = \frac{1.05\pm\sqrt{-.0055}}{2} = \boxed{\frac{1.05\pm .037\,i}{2}}$$

(c) A 4 x 4 example:

$$B = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 5 & 8 & -3 & -5 \\ -2 & 0 & 2 & 2 \\ 5 & 6 & -3 & -3 \end{bmatrix}, \qquad f_B(\lambda) = \boxed{\lambda^4-7\lambda^3+12\lambda^2+4\lambda-16} = (\lambda-4)(\lambda-2)^2(\lambda+1).$$

$\lambda = 4$

$\lambda = 2, 2$ (multiplicity 2)

$\lambda = -1$

(d) How many eigenvalues can an $n\times n$ matrix have?

has n eigenvalues

possibly with multiplicity >1 and possibly complex

4. Find bases for the eigenspaces of these matrices

(a) $A = \begin{bmatrix} -10 & 6 \\ -18 & 11 \end{bmatrix}$ has characteristic polynomial

$$f_A(\lambda) = \begin{vmatrix} -10-\lambda & 6 \\ -18 & 11-\lambda \end{vmatrix} = (-10-\lambda)(11-\lambda)+108 = -110-\lambda+\lambda^2+108 = \lambda^2-\lambda-2 = (\lambda-2)(\lambda+1).$$

Use the information below to describe the eigenspaces.

i. $A - 2\mathbf{I}_2 = \begin{bmatrix} -12 & 6 \\ -18 & 9 \end{bmatrix} \to \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} \to \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix}$ $\quad E_2 = \text{span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$

ii. $A + \mathbf{I}_2 = \begin{bmatrix} -9 & 6 \\ -18 & 12 \end{bmatrix} \to \begin{bmatrix} 3 & -2 \\ 3 & -2 \end{bmatrix} \to \begin{bmatrix} 3 & -2 \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2/3 \\ 0 & 0 \end{bmatrix}$ $\quad E_{-1} = \text{span}\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right)$

(b) $B = \begin{bmatrix} -2 & -1 & 1 \\ -3 & -2 & 3 \\ -3 & -1 & 2 \end{bmatrix}$ has characteristic polynomial $p(\lambda) = -\lambda^3 - 2\lambda^2 + \lambda + 2 = -(\lambda+2)(\lambda+1)(\lambda-1)$.

Use the information below to describe the eigenspaces.

i. $B + 2\mathbf{I}_3 = \begin{bmatrix} 0 & -1 & 1 \\ -3 & 0 & 3 \\ -3 & -1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad E_{-2} = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$

ii. $B + \mathbf{I}_3 = \begin{bmatrix} -1 & -1 & 1 \\ -3 & -1 & 3 \\ -3 & -1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad E_{-1} = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right)$

iii. $B - \mathbf{I}_3 = \begin{bmatrix} -3 & -1 & 1 \\ -3 & -3 & 3 \\ -3 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad E_1 = \text{span}\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right)$

(c) $C = \begin{bmatrix} -4 & 9 & -3 \\ -6 & 11 & -3 \\ -12 & 18 & -4 \end{bmatrix}$ has characteristic polynomial $p(\lambda) = -\lambda^3 + 3\lambda^2 - 4 = -(\lambda-2)^2(\lambda+1)$.

i. $C - 2\mathbf{I}_3 = \begin{bmatrix} -6 & 9 & -3 \\ -6 & 9 & -3 \\ -12 & 18 & -6 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{3}{2} & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad E_2 = \text{span}\left(\begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}\right)$

ii. $C + \mathbf{I}_3 = \begin{bmatrix} -3 & 9 & -3 \\ -6 & 12 & -3 \\ -12 & 18 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$ $\quad E_{-1} = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}\right)$

5. Diagonalize the matrices on the previous page

(a) $\mathsf{A} = \begin{bmatrix} -10 & 6 \\ -18 & 11 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}$

$E_2 = \text{span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} \right)$

$E_{-1} = \text{span}\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix} \right)$

(b) $\mathsf{B} = \begin{bmatrix} -2 & -1 & 1 \\ -3 & -2 & 3 \\ -3 & -1 & 2 \end{bmatrix} = \overbrace{\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}}^{P} \overbrace{\begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{D} \overbrace{\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ -1 & 1 & 0 \end{bmatrix}}^{P^{-1}}$ ← Found using WolframAlpha

$E_{-2} = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)$

$E_{-1} = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right)$

$E_1 = \text{span}\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right)$

(c) $\mathsf{C} = \begin{bmatrix} -4 & 9 & -3 \\ -6 & 11 & -3 \\ -12 & 18 & -4 \end{bmatrix} = \overbrace{\begin{bmatrix} 3 & 1 & 1 \\ 2 & 0 & 1 \\ 0 & 2 & 2 \end{bmatrix}}^{P} \overbrace{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}}^{D} \overbrace{\begin{bmatrix} 1/3 & 0 & -1/6 \\ 2/3 & -1 & 4/6 \\ -2/3 & 1 & 1/3 \end{bmatrix}}^{P^{-1}}$ ← Found using WolframAlpha

$E_2 = \text{span}\left(\begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right)$

$E_{-1} = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right)$