Assignment #3

Sub. .... Date ....

$\theta \to$ model parameters , $y \to$ observed variables, $\phi \to$ fn may depend on other known variable

$$y(i) = \phi^T(i)\,\theta^0 \qquad \text{(least square)} \qquad \#1$$

Need to get θ to minimize least squares loss function

$$V(\theta,t) = \frac{1}{2}\sum_{i=1}^{t}\left(y(i) - \phi^T(i)\,\theta^2\right)$$

$\because$ y linear in θ^0 & least squares criterion is quadratic

$$V(\theta,t) = \frac{1}{2}\sum_{i=1}^{t}\varepsilon^2(i) = \frac{1}{2}E^TE = \frac{1}{2}\|E\|^2 \qquad \#2$$

where $E = Y - \hat{Y} = Y - \phi\theta$
& $\varepsilon(i) = y(i) - \hat{y}(i) = y(i) - \phi^T(i)\,\theta$

$$2V(\theta,t) = E^TE = Y^TY - Y^T\phi\theta - \theta^T\phi^TY + \theta^T\phi^T\phi\theta \qquad \#3$$

as we consider minimal for θ & $\phi^T\phi$ is non singular & minimus is unique

To find minimum by completing the square

$$\therefore 2V(\theta,t) = Y^T(I - \phi(\phi^T\phi)^{-1}\phi^T)Y + (\theta - (\phi^T\phi)^{-1}\phi^TY)^T\phi^T\phi(\theta - (\phi^T\phi)^{-1}\phi^TY)$$

$$\frac{\partial V}{\partial\theta} = \frac{1}{2}\left(0 + \phi^T\phi(\theta - (\phi^T\phi)^{-1}\phi^TY) + (\phi^T\phi)^T(\theta - (\phi^T\phi)^{-1}\phi^TY)\right)$$

$$= \phi^T\phi\theta - \phi^T\phi(\phi^T\phi)^{-1}\phi^TY \qquad \#4$$

$$\frac{\partial^2 V}{\partial\theta^2} = \phi^T\phi \qquad \#5 \quad \Rightarrow \text{non negative}$$

$\therefore$ local minimum

Let $\frac{\partial V}{\partial\theta} = 0$

$$\theta = (\phi^T\phi)^{-1}\phi^TY \quad \text{(local minimum)} \qquad \#6$$

Recursive least square

∵ $\phi^T\phi$ is non singular

$$P(t) = (\phi^T\phi)^{-1} \longrightarrow \theta(t) = P(t)\phi^T Y$$

$$\phi^T Y = \Sigma\phi(i)y(i) \longrightarrow \theta(t) = P(t)\,\Sigma\phi(i)y(i)$$

∴ $\sum_{i=1}^{t}\phi(i)y(i) = \phi(t)y(t) + P(t-1)^{-1}\theta(t-1)$

∴ $\theta(t) = P(t)\left[\phi(t)y(t) + P(t-1)^{-1}\theta(t-1)\right]$ #1 #

∴ $P(t)^{-1} = \phi(t)\phi^T(t) + P(t-1)^{-1}$

∴ $P(t-1)^{-1} = P(t)^{-1} - \phi(t)\phi^T(t)$ #2

from 1 & 2 ∴ $\theta(t) = P(t)\phi(t)\left[y(t) - \phi^T(t)\theta(t-1)\right] + \theta(t-1)$

∴ $\theta(t) = K(t)\varepsilon(t) + \theta(t-1)$

where $K(t) = P(t)\phi(t)$ gain, $\varepsilon(t) = y(t) - \phi^T(t)\hat{\theta}(t-1)$ error

∴ $P(t) = \left[\underbrace{\phi(t)}_{B}\underbrace{\phi^T(t)}_{D} + \underbrace{P(t-1)^{-1}}_{A}\right]^{-1}$ $C \rightarrow I$

from matrix inverse lemma $(A+BCD)^{-1} = A^{-1} - A^{-1}B(C^{-1}+DA^{-1}B)^{-1}DA^{-1}$

∴ $P(t) = P(t-1) - P(t-1)\phi(t)\left(I + \phi^T(t)(P(t-1))(\phi(t))\right)^{-1}\phi^T(t)P(t-1)$ #

∴ $K = P(t)\phi(t)$

∴ $K = P(t-1)\phi(t)\left[I - \left[I + \phi(t)^T P(t-1)\phi(t)\right]^{-1}\phi(t)^T P(t-1)\phi(t)\right]$

∴ $K = P(t-1)\phi(t)\left(I + \phi(t)^T P(t-1)\phi(t)\right)^{-1}$ #