# Chapter 1

# Introduction and Motivation

## 1.1 Preliminary Remarks

This book is concerned with dynamic noncooperative game theory. In a nutshell, game theory involves multi-person decision making; it is *dynamic* if the order in which the decisions are made is important, and it is *noncooperative* if each person involved pursues his or her[1] own interests which are partly conflicting with others'.

A considerable part of everything that has been written down, whether it is history, literature or a novel, has as its central theme a conflict situation—a collision of interests. Even though the notion of "conflict" is as old as mankind, the scientific approach has started relatively recently, in the years around 1930, with, as a result, a still growing stream of scientific publications. We also see that more and more scientific disciplines devote time and attention to the analysis of conflicting situations. These disciplines include (applied) mathematics, economics, engineering, aeronautics, sociology, politics and mathematical finance.

It is relatively easy to delineate the main ingredients of a conflict situation: an individual has to make a decision and each possible decision leads to a different outcome or result, which are valued differently by that individual. This individual may not be the only one who decides about a particular outcome; a series of decisions by several individuals may be necessary. If all these individuals value the possible outcomes differently, the germs for a conflict situation are there.

The individuals involved, also called *players* or *decision makers*, or simply *persons*, do not always have complete control over the outcome. Sometimes there are uncertainties which influence the outcome in an unpredictable way. Under

[1] Without any preference to sexes, a decision maker, in this book, is most times referred to as a "he". It could equally well be a "she".

Table 1.1: The place of dynamic game theory.

| | One player | Many players |
|---|---|---|
| Static | Mathematical programming | (Static) game theory |
| Dynamic | Optimal control theory | Dynamic (and/or differential) game theory |

such circumstances, the outcome is (partly) based on data not yet known and not determined by the other players' decisions. Sometimes it is said that such data is under the control of "nature", or "God", and that every outcome is caused by the joint or individual actions of human beings and nature.

The established names of "game theory" (developed from approximately 1930) and "theory of differential games" (developed from approximately 1950, parallel to that of optimal control theory) are somewhat unfortunate. "Game theory", especially, appears to be directly related to parlor games; of course it is, but the notion that it is only related to such games is far too restrictive. The term "differential game" became a generally accepted name for games where differential equations play an important role. Nowadays the term "differential game" is also being used for other classes of games for which the more general term "dynamic game" would be more appropriate.

The applications of "game theory" and the "theory of differential games" mainly deal with economic and political conflict situations, worst-case designs and also modeling of war games. However, it is not only the applications in these fields that are important; equally important is the development of suitable concepts to describe and understand conflict situations. It turns out, for instance, that the role of information—what one player knows relative to others—is very crucial in such problems.

Scientifically, dynamic game theory can be viewed as a child of the parents game theory and optimal control theory.[2] Its character, however, is much more versatile than that of its parents, since it involves a dynamic decision process evolving in (discrete or continuous) time, with more than one decision maker, each with his own cost function and possibly having access to different information. This view is the starting point behind the formulation of "games in extensive form", which started in the 1930s through the pioneering work of Von Neumann, which culminated in his book with Morgenstern (Von Neumann and Morgenstern, 1947), and then made mathematically precise by Kuhn

[2] In almost all analogies there is a deficiency; a deficiency in the present analogy is that the child is as old as one of his parents—optimal control theory. For the relationship between these theories and the theory of mathematical programming, see Table 1.1.

(1953), all within the framework of "finite" games. The general idea in this formulation is that a game evolves according to a road or tree structure, where at every crossing or branching a decision has to be made as how to proceed.

In spite of this original set-up, the evolution of game theory has followed a rather different path. Most research in this field has been, and is being, concentrated on the normal or strategic form of a game. In this form all possible sequences of decisions of each player are set out against each other. For a two-player game this results in a matrix structure. In such a formulation dynamic aspects of a game are completely suppressed, and this is the reason why game theory is classified as basically "static" in Table 1.1. In this framework emphasis has been more on (mathematical) existence questions, rather than on the development of algorithms to obtain solutions.

Independently, control theory gradually evolved from Second World War servomechanisms, where questions of solution techniques and stability were studied. Then followed Bellman's "dynamic programming" (Bellman, 1957) and Pontryagin's "maximum principle" (Pontryagin et al., 1962), which spurred the interest in a new field called optimal control theory. Here the concern has been on obtaining optimal (i.e., minimizing or maximizing) solutions and developing numerical algorithms for one-person single-objective dynamic decision problems. The merging of the two fields, game theory and optimal control theory, which leads to even more concepts and to actual computation schemes, has achieved a level of maturity, which the reader will hopefully agree with after he/she goes through this book.

At this point, at the very beginning of the book, where many concepts have yet to be introduced, it is rather difficult to describe how dynamic game theory evolved in time and what the contributions of relevant references are. We therefore defer such a description until later, to the "notes" section of each chapter (except the present), where relevant historical remarks are included.

## 1.2 Preview on Noncooperative Games

A clear distinction exists between two-player (or, equivalently, two-person) zero-sum games and the others. In a zero-sum game, as the name implies, the sum of the cost functions of the players is identically zero. Mathematically speaking, if $u^i$ and $L^i$ denote, respectively, the decision variable and the cost function of the $i$th player (to be written $\mathbf{P}i$), then $\sum_{i=1}^{2} L^i(u^1, u^2) \equiv 0$ in a zero-sum game. If this sum is, instead, equal to a nonzero constant (independent of the decision variables), then we talk about a "constant-sum" game which can, however, easily be transformed to a zero-sum game through a simple translation without altering the essential features of the game. Therefore, constant-sum games can be treated within the framework of zero-sum games, without any loss of generality, which we shall choose to do in this book.

A salient feature of two-person zero-sum games that distinguishes them from other types of games is that they do not allow for any cooperation between the players, since, in a two-person zero-sum game, what one player gains incurs a

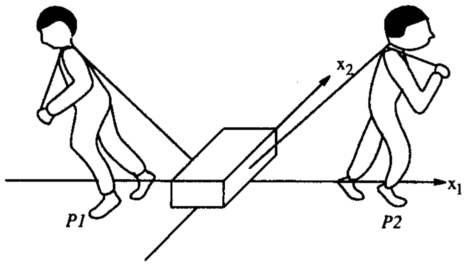



Figure 1.1: The rope-pulling game.

loss to the other player. However, in other games, such as two-player nonzero-sum games (wherein the quantity $\sum_{i=1}^{2} L^i(u^1, u^2)$ is not a constant) or three- or more-player games, the cooperation between two or more players may lead to their mutual advantage.

**Example 1.1** A point object (with mass one) can move in a plane which is endowed with the standard $(x_1, x_2)$-coordinate system. Initially, at $t = 0$, the point mass is at rest at the origin. Two unit forces act on the point mass; one is chosen by **P1**, the other by **P2**. The directions of these forces, measured counter-clockwise with respect to the positive $x_1$-axis, are determined by the players and are denoted by $u^1$ and $u^2$, respectively; they may in general be time-varying. At time $t = 1$, **P1** wants to have the point mass as far in the negative $x_1$-direction as possible, i.e., he wants to minimize $x_1(1)$, whereas **P2** wants it as far in the positive $x_1$-direction as possible, i.e., he wants to maximize $x_1(1)$, or equivalently, to minimize $-x_1(1)$. (See Fig. 1.1.) The "solution" to this zero-sum game follows immediately; each player pulls in his own favorite direction, and the point mass remains at the origin—such a solution is known as the *saddle-point* solution.

We now alter the formulation of the game slightly, so that, in the present set-up, **P2** wishes to move the point mass as far in the negative $x_2$-direction as possible, i.e., he wants to minimize $x_2(1)$. **P1**'s goal is still to minimize $x_1(1)$. This new game is clearly nonzero-sum. The equations of motion for the point mass are

$$\begin{aligned} \ddot{x}_1 &= \cos(u^1) + \cos(u^2), & \dot{x}_1(0) &= x_1(0) = 0; \\ \ddot{x}_2 &= \sin(u^1) + \sin(u^2), & \dot{x}_2(0) &= x_2(0) = 0. \end{aligned}$$

Let us now consider the pair of decisions $\{u^1 \equiv \pi, u^2 \equiv -\pi/2\}$ with the corresponding values of the cost functions being $L^1 = x_1(1) = -\frac{1}{2}$ and $L^2 = x_2(1) = -\frac{1}{2}$. If **P2** sticks to $u^2 \equiv -\pi/2$, the best thing for **P1** to do is to choose $u^1 \equiv \pi$; any other choice of $u^1(t)$ will yield an outcome which is greater than $-\frac{1}{2}$. Analogously, if **P1** sticks to $u^1 \equiv \pi$, **P2** does not have a better choice than $u^2 \equiv -\pi/2$. Hence, the pair $\{u^1 \equiv \pi, u^2 \equiv -\pi/2\}$ exhibits an equilibrium behavior, and this kind of a solution, where one player cannot improve his outcome by altering his decision unilaterally, is called a *Nash equilibrium solution*, or shortly, a *Nash solution*.

If both players choose $u^1 \equiv u^2 \equiv 5\pi/4$, however, then the cost values become $L^1 = L^2 = -\frac{1}{2}\sqrt{2}$, which are obviously better, for both players, than the costs incurred under the Nash solution. But, in this case, the players have to cooperate. If, for instance, **P2** would stick to his choice $u^2 \equiv 5\pi/4$, then **P1** can improve upon his outcome by playing $u^1 \equiv c$, where $c$ is a constant with $\pi \leq c < 5\pi/4$. **P1** is better off, but **P2** is worse off! Therefore, the pair of strategies $\{u^1 \equiv 5\pi/4, u^2 \equiv 5\pi/4\}$ cannot be in equilibrium in a noncooperative mode of decision making, since it requires some kind of faith (or even negotiation), and thereby cooperation, on part of the players. If this is allowable, then the said pair of strategies—known as a *Pareto-optimal solution*—stands out as a reasonable equilibrium solution for the game problem (which is called a *cooperative game*), since it features the property that no other joint decision of the players can improve the performance of at least one of them, without degrading the performance of the other. □

In this book we shall deal only with noncooperative games. The reasons for such a seeming limitation are twofold. Firstly, cooperative games[3] can, in general, be reduced to optimal control problems by determining a single cost function to be optimized by all players, which suppresses the "game" aspects of the problem. Secondly, the size of this book would have increased considerably by inclusion of a complete discussion on cooperative games.

### Actions and strategies

Heretofore, we have safely talked about "decisions" made by the players, without being very explicit about what a decision really is. This will be made more precise now in terms of information available to each player. In particular, we shall distinguish between *actions* (also called controls) on the one hand and *strategies* (or, equivalently, decision rules) on the other.

If an individual has to decide about what to do the next day, and the options are fishing and going to work, then a strategy is: "if the weather report early tomorrow morning predicts dry weather, then I will go fishing, otherwise I will go to my office". This is a *strategy* or *decision rule*: what actually will be done depends on quantities not yet known and not controlled by the decision maker; the decision maker cannot influence the course of the events further, once he has fixed his strategy. Any consequence of such a strategy, after the unknown quantities are realized, is called an *action*. In a sense, a constant strategy (such as an irrevocable decision to go fishing without any restrictions or reservations) coincides with the notion of action.

In the example above, the alternative actions are to go fishing and to go to work, and the action to be implemented depends on information (the weather report) which has to be known at the time it is carried out. In general, such information can be of different types. It can, for instance, comprise the previous

[3] What we have in mind here is the class of cooperative games without side payments. If side payments are allowed, we enter the territory of (cooperative) games in characteristic function form, which is an altogether different topic. (See Owen (1968, 1982) and Vorob'ev (1977).)

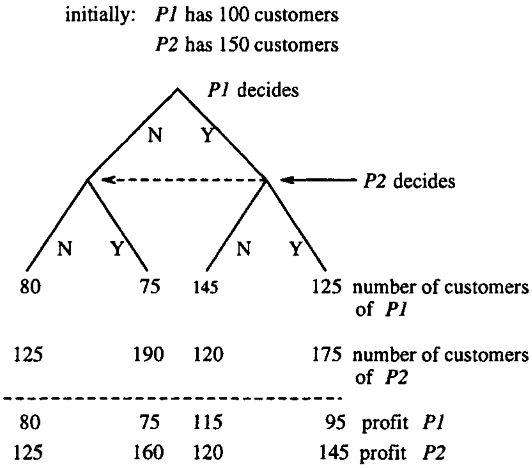



Figure 1.2: One-period advertising game.

actions of all the other players. As an example, consider the following sequence of actions: if he is nice to me, I will be kind to him; if he is cool, I will be cool, etc. The information can also be of a stochastic nature, such as in the fishing example. Then, the actual decision (action) is based on data not yet known and not controlled by other players, but instead determined by "nature". The next example will now elucidate the role of information in a game situation and show how it affects the solution. The information here is deterministic; "nature" does not play a role.

**Example 1.2** This example aims at a game-theoretic analysis of the advertising campaigns of two competing firms during two periods of time. Initially, each firm is allotted a certain number of customers; **P1** has 100 and **P2** has 150 customers. Per customer, each firm makes a fixed profit per period, say one dollar. Through advertisement, a firm can increase the number of its customers (some are stolen away from the competitor and others come from outside) and thereby its profit. However, advertising costs money; an advertising campaign for one period costs thirty dollars (for each firm). The figures in this example may not be very realistic; it is, however, only the ratio of the data that matters, not the scale with respect to which it is expressed.

First, we consider the one-period version of this game and assume that the game ends after the first period is over. Suppose that **P1** has to decide first whether he should advertise (Yes) or not (No), and subsequently **P2** makes his choice. The four possible outcomes and the paths that lead to those outcomes are depicted in Fig. 1.2, in the form of a tree diagram. At every branching, a decision has to be made as how to proceed. The objective of each firm is to maximize its profit (for this one-period game). The "best" decisions, in this case, can be found almost by inspection; whatever **P1** does ($Y$ or $N$), **P2** will always advertise ($Y$), since that is more profitable to him. **P1**, realizing this, has

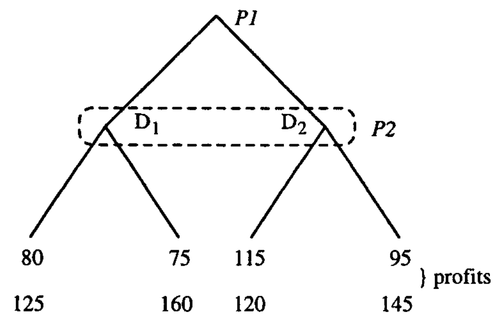



Figure 1.3: **P2** chooses independently of **P1**'s action.

to choose between $Y$ (with profit 95) and $N$ (with profit 75), and will therefore choose $Y$. Note that, at the point when **P2** makes his decision, he knows **P1**'s choice (action) and therefore **P2**'s choice depends, in principle, on what **P1** has done. **P2**'s best strategy can be described as: "if **P1** chooses $N$, then I choose $Y$; if **P1** chooses $Y$, then I choose $Y$", which, in this case is a *constant strategy.*

This one-period game will now be modified slightly so that **P1** and **P2** make their decisions simultaneously, or, equivalently, independently of each other. The information structure associated with this new game is depicted in Fig. 1.3; a dashed curve encircling the points $D_1$ and $D_2$ has been drawn, which indicates that **P2** cannot distinguish between these two points. In other words, **P2** has to arrive at a decision without knowing what **P1** has actually done. Hence, in this game, the strategy of **P2** has a different domain of definition, and it can easily be verified that the pair $\{Y, Y\}$ provides a Nash solution, in the sense described before in Example 1.1, yielding the same profits as in the previous game.

We now extend the game to two periods, with the objective of the firms being maximization of their respective cumulative profits (over both periods). The complete game is depicted in Fig. 1.4 without the information sets; we shall, in fact, consider three different information structures in the sequel. First, the order of the actions will be taken as **P1** – **P2** – **P1** – **P2**, which means that at the time of his decision each player knows the actions previously taken. Under the second information structure to be considered, the order of the actions is (**P1**, **P2**)-(**P1**, **P2**), which means that during each period the decisions are made independently of each other, but that for the decisions of the second period the (joint) actions of the first period are known. Finally, as a third case, it will be assumed that there is no order in the actions at all; **P1** has to decide on what to do during both periods without any prior knowledge of what **P2** will be doing, and vice versa. We shall be looking for Nash solutions in all three games.

## The P1 – P2 – P1 – P2 information structure

The solution is obtained by working backward in time (à la "dynamic programming"). For the decision during the second period, **P2** knows to which point ($D_7 - D_{14}$) the game has proceeded. From each of these points he chooses $Y$

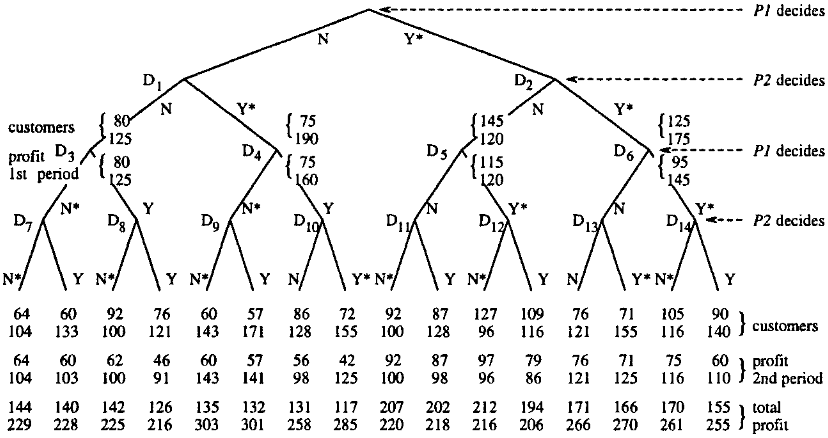



Figure 1.4: The two-period advertising game, without the information sets.

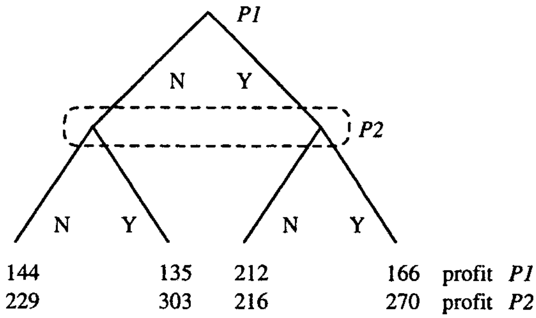
or $N$, depending on which decision leads to higher profit. These decisions are denoted by an asterisk in Fig. 1.4. At the time of his decision during the second period, **P1** knows that the game has proceeded to one of the points $D_3 - D_6$. He can also guess precisely what **P2** will do after he himself has made his decision (the underlying assumption being that **P2** will behave rationally). Hence, **P1** can determine what will be best for him. At point $D_3$, for instance, decision $N$ leads to a profit of 144 (for **P1**) and $Y$ leads to 142; therefore, **P1** will choose $N$ if the game would be at $D_3$. The optimal decisions for **P1** at the second period are indicated by an asterisk also. We continue this way (in retrograde time) until the vertex of the tree is reached; all best actions are designated by an asterisk in Fig. 1.4. As a result, the actual game will evolve along a path passing through the points $D_2$, $D_6$, and $D_{14}$, and the cumulative profits for **P1** and **P2** will be 170 and 261, respectively.

## The (P1, P2)-(P1, P2) information structure

During the last period, both players know that the game has evolved to one of the points $D_3 - D_6$, upon which information the players will have to base their actions. This way, we have to solve four "one-period games", one for each of the points $D_3 - D_6$. The reader can easily verify that the optimal (Nash) solutions associated with these games are:

| information (starting point) | action P1 | action P2 | profit P1 | profit P2 |
|---|---|---|---|---|
| $D_3$ | $N$ | $N$ | 144 | 229 |
| $D_4$ | $N$ | $N$ | 135 | 303 |
| $D_5$ | $Y$ | $N$ | 212 | 216 |
| $D_6$ | $N$ | $Y$ | 166 | 270 |

The profit pairs corresponding to these solutions can be attached to the respective points $D_3 - D_6$, after which a one-period game is left to be solved. The game tree is:




It can be verified that both players should play $Y$ during the first period, since a unilateral deviation from this decision leads to a smaller profit for both players. The realized cumulative profits of **P1** and **P2** under this information scheme are therefore 166 and 270, respectively.

## The no-information case

Both players have four possible choices: $NN$, $NY$, $YN$ and $YY$, where the first (respectively, second) symbol refers to the first (respectively, second) period. Altogether there are $4 \times 4 = 16$ possible pairs of realizable profits, which can be written in a matrix form:

| **P1** chooses \ **P2** chooses | $NN$ | $NY$ | $YN$ | $YY$ |
|---|---|---|---|---|
| $NN$ | 144<br>229 | 140<br>228 | 135<br>303 | 132<br>301 |
| $NY$ | 142<br>225 | 126<br>216 | 131<br>258 | 117<br>285 |
| $YN$ | 207<br>220 | 202<br>218 | 171<br>266 | 166<br>270 |
| $YY$ | 212<br>216 | 194<br>206 | 170<br>261 | 155<br>255 |

If **P1** chooses $YN$ and **P2** chooses $YY$, then the profits are 166 and 270, respectively. If any one of the players deviates from this decision, he will be worse off. (If, for example, **P1** deviates, his other profit options are 132, 117 and 155, each one being worse than 166.) Therefore $\{(YN), (YY)\}$ is the Nash solution under the "no-information"-scheme. $\square$

Game problems of this kind will be studied extensively in this book, specifically in Chapter 3 which is devoted to *finite* nonzero-sum games, i.e., games in which each player has only a finite number of possible actions available to him. Otherwise, a game is said to be *infinite*, such as the one treated in Example 1.1. In Chapter 3, we shall also elucidate the reasons why the realized profits are, in general, dependent on the information structure.

The information structure (**P1**, **P2**)-(**P1**, **P2**) in Example 1.2 is called a *feedback* information structure; during each period the players know exactly to which point ("state") the game has evolved and that information is *fed back* into their strategies, which then leads to certain actions. This structure is sometimes referred to as *closed-loop*, though later on in this book a distinction between feedback and closed-loop will be made. The no-information case is referred to as an *open-loop* information structure.

Not every game is well-defined with respect to every possible information structure; even though Example 1.2 was well-defined under both a feedback and an open-loop information structure, this is not always true. To exemplify this situation, we provide, in the sequel, two specific games. The game to be formulated in Example 1.3 does not make much sense under an open-loop information structure, and the one of Example 1.4 can only be played under the open-loop structure.

**Example 1.3** "*The lady in the lake.*" A lady is swimming in a circular lake with a maximum speed $v_l$. A man who wishes to intercept the lady, and who has not mastered swimming, is on the side of the lake and can run along the shore with a maximum speed $v_m$. The lady does not want to stay in the lake forever and wants eventually to escape. If, at the moment she reaches the shore, the man cannot intercept her, she "wins" the game since on land she can run faster than the man. Note that open-loop solutions (i.e., solutions corresponding to the open-loop information structure) do not make sense here (at least not for the man). It is reasonable to assume that each player will react immediately to what his/her opponent has done and hence the feedback information structure, whereby the current position of man and lady are fed back, is more appropriate. □

**Example 1.4** "*The princess and the monster.*" This game is played in complete darkness. Just like the previous example, this one is also of the pursuit-evasion type. Here, however, the players do not know where their opponent is, unless they bump into each other (or come closer to each other than a given distance $\epsilon$); that moment in fact defines the termination of the game. The velocities of both players are bounded and their positions are confined to a bounded area. The monster wants to bump into the princess as soon as possible, whereas the princess wants to push this moment as far into the future as possible. Since the players cannot react to each other's actions or positions, the information structure is open-loop. □

The solution to Example 1.3 will be given in Chapter 8; for more information on Example 1.4, see Foreman (1977). This latter example also leads, rather naturally, to the concept of "mixed strategies"; the optimal strategies in the "princess and monster" game cannot be deterministic, as were the strategies in Examples 1.1 and 1.2, for instance. For, if the monster would have a deterministic optimal strategy, then the princess would be able to calculate this strategy; if, in addition, she would know the monster's initial position, this would enable her to determine the monster's path and thereby to choose for herself an appropriate strategy so that she can avoid him forever. Therefore, an optimal strategy for the monster (if it exists) should dictate random actions, so that his trajectory cannot be predicted by the princess. Such a strategy is called *mixed*. Equilibria in mixed strategies will be discussed throughout this book.

### What is optimal?

In contrast to optimal control problems (one-player games) where optimality has an unambiguous meaning, in multi-person decision making, optimality, in itself, is not a well-defined concept. Heretofore we have considered the Nash equilibrium solution, which is a specific form of "optimality". We also briefly discussed Pareto-optimality. There exist yet other kinds of optimality in nonzero-sum games. Two of them will be introduced now by means of the matrix game encountered in the "no-information"-case of Example 1.2. The equilibrium strategies in the "Nash" sense were $YN$ for **P1** and $YY$ for **P2**. Suppose now that **P2** is a careful and defensive player and wants to protect himself against any irrational behavior on the part of the other player. If **P2** sticks to $YY$, the worst that can happen to him is that **P1** chooses $YY$, in which case **P2**'s profit becomes 255 instead of 270. Under such a defensive attitude **P2** might play $YN$, since then his profit is at least 258. This strategy (or, equivalently, action in this case) provides **P2** with a lower bound for his earnings and the corresponding solution concept is called *minimax*. The player who adopts this solution concept basically solves a zero-sum game, even though the original game might be nonzero-sum.

Yet another solution concept is the one that involves a hierarchy in decision making: one of the players, say **P1**, declares and announces his strategy before the other player chooses his strategy and he (i.e., **P1**) is in a position to enforce this strategy. In the matrix game of the previous paragraph, if **P1** says "I will play $YY$" and irrevocably sticks to it (by the rules of the game we assume that cheating is not possible), then the best **P2** can do is to choose $YN$, in which case **P1**'s profit becomes 170 instead of 166 which was obtainable under the Nash solution concept. Such games in which one player (called the *leader*) declares his strategy first and enforces it on the other player (called the *follower*) are referred to as *Stackelberg games*. They will be discussed in Chapter 3 for finite action sets, and in Chapters 4 and 7 for infinite action sets.

### Static versus dynamic

As the last topic of this section, we shall attempt to answer the question: "When is a game called dynamic and when is it static?" So far, we have talked rather loosely about these terms; there is, in fact, no uniformly accepted separating line between static games on the one hand and dynamic games on the other. We shall choose to call a game *dynamic* if at least one player is allowed to use a strategy that depends on previous actions. Thus, the games treated in Example 1.2 with the information schemes **P1** – **P2** – **P1** – **P2** and (**P1**, **P2**)-(**P1**, **P2**) are dynamic. The third game of Example 1.2, with the "no-information" scheme, should be called static, but, by an abuse of language, such a game is often also called dynamic. The reason is that the players act more than once and thus time plays a role. A game in which the players act only once and independently of each other is definitely called a *static* game. The game displayed in Fig. 1.3 is clearly static.

## 1.3 Outline of the Book

The book comprises eight chapters and three appendices. The present (first) chapter serves the purpose of introducing the reader to the contents of the book, and to the conventions and terminology adopted. The next three chapters constitute Part I of the text, and deal with finite static and dynamic games and infinite static games.

Chapter 2 discusses the existence, derivation and properties of saddle-point equilibria in pure, mixed and behavioral strategies for two-person zero-sum finite games. It is in this chapter that the notions of normal and extensive forms of a dynamic game are introduced, and the differences between actions and strategies are delineated. Also treated is the class of two-person zero-sum finite games in which a third (chance) player with a fixed mixed strategy affects the outcome.

Chapter 3 extends the results of Chapter 2 to nonzero-sum finite games under basically two different types of equilibrium solution concepts, viz. the Nash solution and the Stackelberg (hierarchical) solution. The impact of dynamic information on the structure of these equilibria is thoroughly investigated, and in this context the notion of prior and delay commitment modes of play are elucidated.

Chapter 4 deals with static infinite games of both the zero-sum and nonzero-sum type. In this context it discusses the existence, uniqueness and derivation of (pure and mixed) saddle-point, Nash and Stackelberg equilibria, as well as the consistent conjectural variations equilibrium. It provides explicit solutions for some types of games, with applications in microeconomics.

The remaining four chapters constitute Part II of the book, for which Chapter 5 provides a general introduction. It introduces the class of infinite dynamic games to be studied in the remaining chapters, and also gives some background material on optimal control theory. Furthermore, it makes the notions of "representations of strategies on given trajectories" and "time consistency" precise.

The major portion of Chapter 6 deals with the derivation and properties of Nash equilibria with prescribed fixed duration under different types of deterministic information patterns, in both discrete and continuous time. It also presents as a special case saddle-point equilibria in such dynamic games, with important applications in worst-case controller designs (such as $H^\infty$-optimal control).

Chapter 7 discusses the derivation of global and feedback Stackelberg equilibria for the class of dynamic games treated in Chapter 6, and also the relationship with the theory of incentives.

Finally, Chapter 8 deals with the class of zero-sum differential games for which the duration is not fixed *a priori* —the so-called pursuit-evasion games— and under the feedback information pattern. It first presents some necessary and sufficient conditions for the saddle-point solution of such differential games, and then applies these to pursuit-evasion games with specific structures so as to obtain some explicit results.

Each chapter (with the exception of Chapter 1) starts with an introduction section which summarizes its contents, and therefore we have kept the descriptions above rather brief. Following Chapter 8 are the three appendices, two of which (Appendices A and B) provide some background material on sets, vector spaces, matrices, optimization theory and probability theory to the extent to which these notions are utilized in the book. The third appendix, on the other hand, presents some theorems which are used in Chapters 3 and 4. The book ends with a list of references, a table that indicates the page numbers of the Corollaries, Definitions, Examples, Lemmas, Propositions, Remarks and Theorems appearing in the text and an index.

## 1.4 Conventions, Notation and Terminology

Each chapter of the book is divided into sections, and occasionally sections are divided into subsections. Section 2.1 refers to Section 1 of Chapter 2, and Section 8.5.2 refers to subsection 2 of Section 8.5. The following items appear in this book and they are numbered per chapter, such as Prop. 3.7 referring to the seventh proposition of the third chapter:

| | | |
|---|---|---|
| Theorem (abbreviated to Thm.) | Corollary | Lemma |
| Definition (abbreviated to Def.) | Problem | Equation |
| Figure (abbreviated to Fig.) | Property | Example |
| Proposition (abbreviated to Prop.) | Remark | |

Unlike the numbering of other items, equation numbers appear within parentheses, such as equation (2.5) which refers to the fifth numbered equation in Chapter 2.

References to bibliographical sources (listed alphabetically at the end of the book) are made according to the Harvard system (i.e., by name(s) and date).

The following abbreviations and symbols are adopted in the book, unless stated otherwise in specific contexts:

| | |
|---|---|
| RHS | right-hand side |
| LHS | left-hand side |
| w.p. $\alpha$ | with probability $\alpha$ |
| LP | linear programming |
| $=$ | equality sign |

| | |
|---|---|
| $\triangleq$ | defined by |
| $\Box$ | end of proof, remark, example, etc. |
| $\parallel$ | parallel to |
| $\operatorname{sgn}(x) = \begin{cases} +1, & \text{if } x > 0 \\ \text{undefined}, & \text{if } x = 0 \\ -1, & \text{if } x < 0 \end{cases}$ | |
| $\partial\Lambda$ | boundary of the set $\Lambda$ |
| $\mathbf{P}i$ | player $i$ |
| $\mathbf{P}(\mathbf{E})$ | pursuer (evader) |
| $L^i$ | cost function(al) of $\mathbf{P}i$ for a game in extensive form |
| $J^i$ | cost function(al) of $\mathbf{P}i$ for a game in normal form |
| $N$ | number of players |
| $\mathbf{N}$ | $\triangleq \{1,\ldots,N\}$ players' set; $i \in \mathbf{N}$ |
| $u^i$ | action (decision, control) variable of $\mathbf{P}i$; $u^i \in U^i$ |
| $\gamma^i$ | strategy (decision law) of $\mathbf{P}i$; $\gamma^i \in \Gamma^i$ |
| $\eta^i$ | information available to $\mathbf{P}i$ |
| $K$ | number of stages (levels) in a discrete-time game |
| $\mathbf{K}$ | $\triangleq \{1,\ldots,K\}$; $k \in \mathbf{K}$ |
| $[0,T]$ | time interval on which a differential game is defined: $t \in [0,T]$ |
| $x$ | state variable; $x(t)$ in continuous time and $x_k$ in discrete time |
| $\dot{x}(t)$ | time-derivative of $x(t)$ |
| $V$ | value of zero-sum game (in pure strategies) |
| $\bar{V}$ | upper value |
| $\underline{V}$ | lower value |
| $V_{\mathrm{m}}$ | value of a zero-sum game in mixed strategies |
| $\mathbf{R}$ | real line |
| $\mathbf{R}^n$ | $n$-dimensional Euclidean space |
| $\lvert x\rvert$ | Euclidean norm of a finite-dimensional vector $x$, i.e., $\{x'x\}^{1/2}$ |
| $\lVert x\rVert$ | norm of a vector $x$ in an infinite-dimensional space. |

Other symbols or abbreviations used with regard to sets, unions, summation, matrices, optimization, random variables, etc., are introduced in Appendices A and B.

A convention that we have adopted throughout the text (unless stated otherwise) is that in nonzero-sum games all players are cost-minimizers, and in two-person zero-sum games **P1** is the minimizer and **P2** is the maximizer. In two-person matrix games **P1** is the row-selector and **P2** is the column-selector. The word "optimality" is used in the text rather freely for different equilibrium solutions when there is no ambiguity in the context; optimum quantities (such as strategies, controls, etc.) are generally identified by an asterisk (*).