Financial complexity and systemic stability

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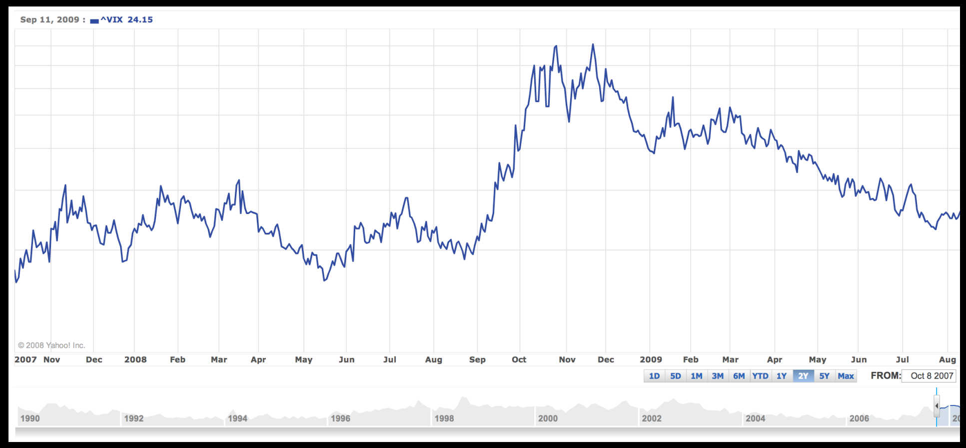
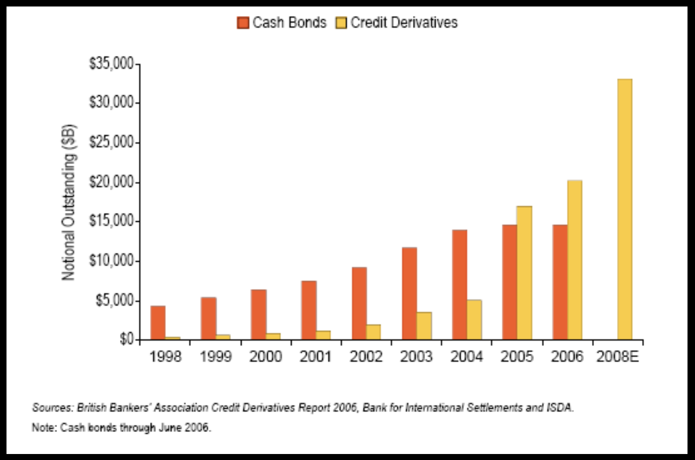
- The financial innovation spiral and systemic risk
  - When the diversity of financial instruments increases (e.g. derivatives) and financial markets approach completeness, systemic instability arises, even in an ideal world
- Instability of risk measures and portfolio management
  - When the size of portfolio increases, empirical estimates of risk develops instability. Accounting for the market impact removes the instability

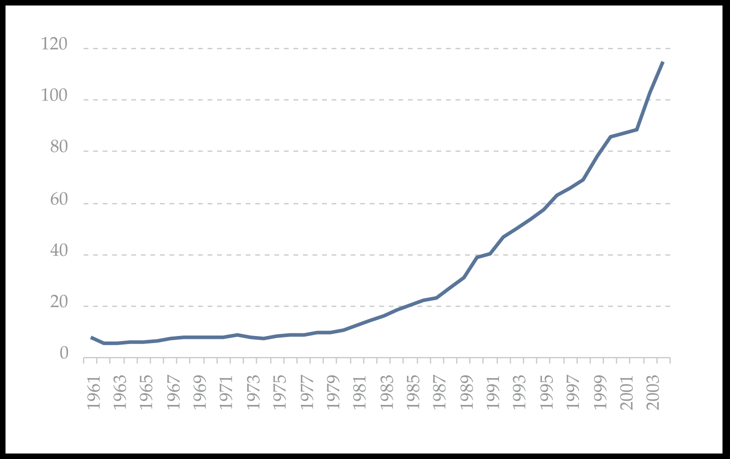
2007/8 crisis: Bad guys or bad theories?



Financial instability: price volatility (CBOE-VIX index)



Volumes traded within financial industry (e.g. credit derivatives)



Size of financial markets/GDP (Iceland 1970-2003)

Financial systemic risk: asymmetric information, networks, contagion, etc...

Can this happen even without market imperfections, misaligned incentives, toxic assets, contagion effects, etc?

The financial innovation spiral

(Merton and Bodie 2005)

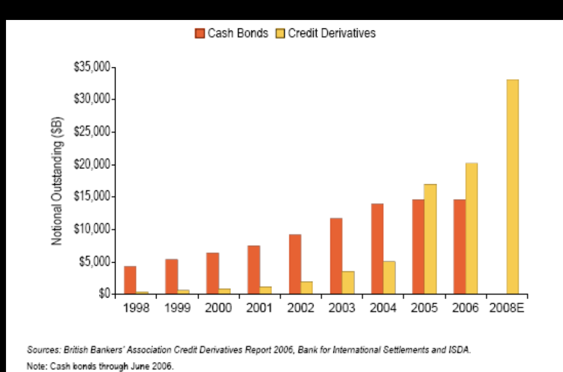
"As products such as futures, options, swaps, and securitized loans become standardized [...] the producers (typically, financial intermediaries) trade in these new markets and volume expands; increased volume reduces marginal transaction costs and thereby makes possible further implementation of more new products and trading strategies by intermediaries, which in turn leads to still more volume [...] and so on it goes, spiraling toward the theoretically limiting case of zero marginal transactions costs and dynamically complete markets."

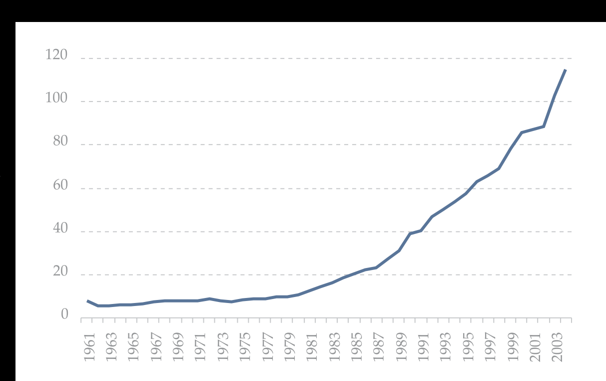
"When particular transaction costs or behavioral patterns produce large departures from the predictions of the ideal frictionless neoclassical equilibrium for a given institutional structure, new institutions tend to develop that partially offset the resulting inefficiencies. In the longer run, after institutional structures have had time to fully develop, the predictions of the neoclassical model will be approximately valid for asset prices and resource allocations."

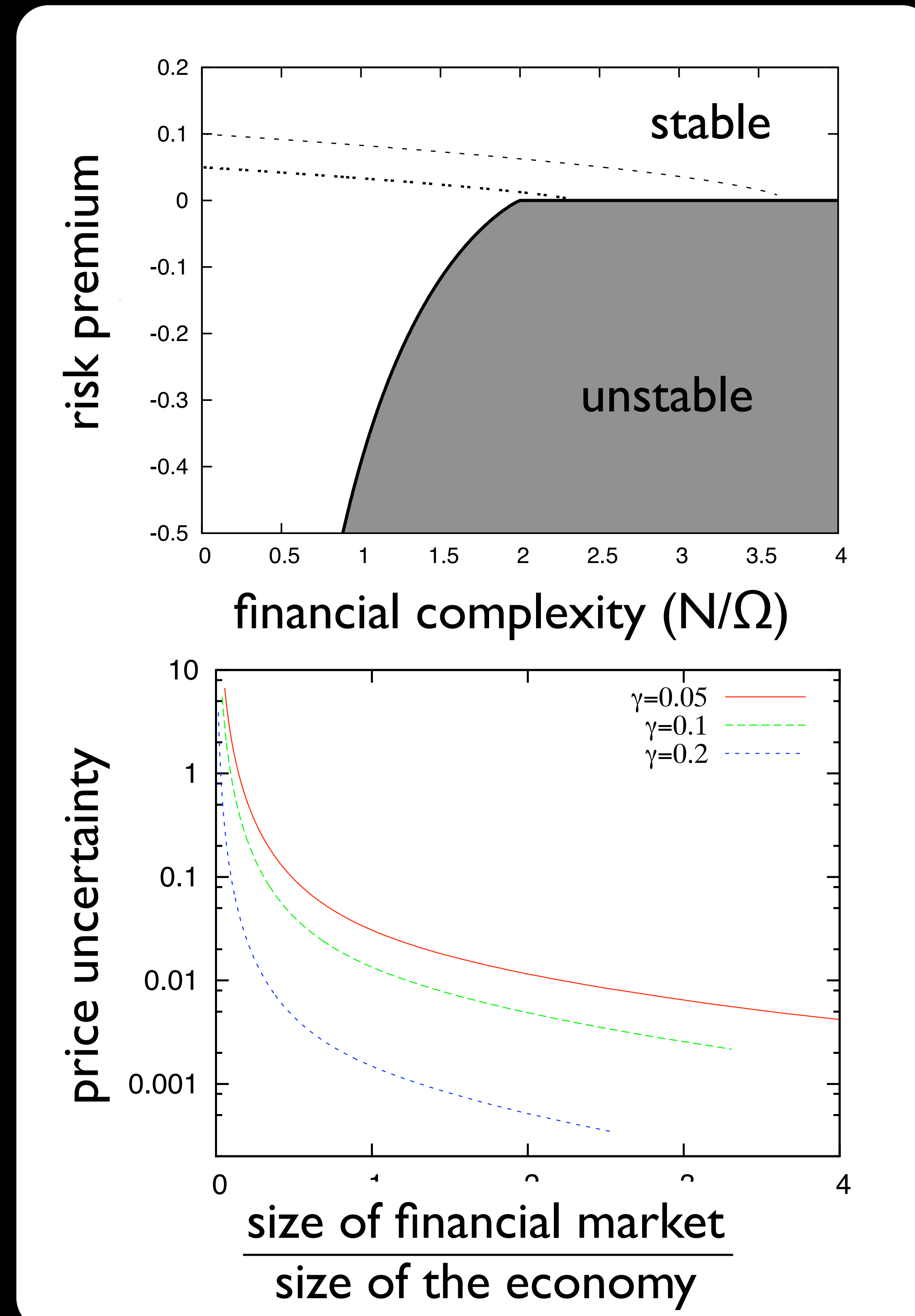
(see also R. J. Shiller, "The Subprime Solution" 2008)

Main result

- As markets approach completeness:
  - allocations become more and more unstable
  - The size of the financial market grows unbounded wrt the ``real economy”
- Stability vs size diagram

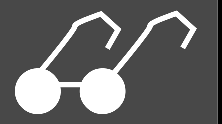
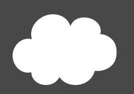






Intuition: Asset allocation in a risky world

- Tomorrow: rain or sun?
  wait and buy sunglasses or umbrella
  Inefficient, if e.g. tomorrow
  price of sunglasses > price of umbrella

		
	Yes	No
	No	Yes

- Contingent commodity markets:
  markets and prices, open today for
  (sunglasses if rain), (sunglasses if sun), (umbrella if rain), (umbrella if sun)
  Today: shopping in contingency commodity markets
  Tomorrow: delivery and consumption

- Optimal allocation under perfect competition

What if contingent commodity markets do not exist?

- Financial market: 1 riskless B_t and 1 risky S_t assets
  Today $B_0=S_0=1$
  Tomorrow $B_1=1$, $S_1=1+u$ if sun, $S_1=1-d$ if rain

- I want to have C^{rain} euros to buy an umbrella if it rains and C^{sun} euros to buy sunglasses if it is sunny. Can I do that? How much does it cost?

- Yes! Buy a portfolio z_B units of B and z_S units of S such that

$$z_B + (1+u)z_S = C^{\mathrm{sun}}$$
$$z_B + (1-d)z_S = C^{\mathrm{rain}}$$

- How much does it cost?

$$C_0 = z_B + z_S = \frac{d}{u+d}C^{\mathrm{sun}} + \frac{u}{u+d}C^{\mathrm{rain}} = E_q[C_{t=1}]$$

- This can be done for any contingent claim C^w. Independent of probability!

- Assumptions:
  i) perfect competition
  ii) full information
  iii) no-arbitrage: $ud>0$
  iv) complete market: what if there are three states? (e.g. sun, cloud, rain)

An equilibrium economy: N assets, Ω states

The market (returns)

demand

Investors

$$\Leftarrow \begin{array}{c} z_1 \\ \vdots \\ z_k \\ \vdots \\ z_N \end{array} \Leftarrow \begin{array}{ccccc} r_1^1 & \cdots & r_1^\omega & \cdots & r_1^\Omega \\ \vdots & \ddots & \vdots & & \vdots \\ r_k^1 & \cdots & r_k^\omega & \cdots & r_k^\Omega \\ \vdots & & \vdots & \ddots & \vdots \\ r_N^1 & \cdots & r_N^\omega & \cdots & r_N^\Omega \end{array} \Leftarrow$$

Financial industry (banks)

$$\Downarrow$$

$$\max_{\vec{z}\geq 0} E\left[u\left(c(\vec{z})\right)\right]$$

$$r_{N+1}^1 \quad \cdots \quad r_{N+1}^\omega \quad \cdots \quad r_{N+1}^\Omega$$

financial innovation

$$N, \Omega \to \infty, \qquad n = \frac{N}{\Omega}$$

Optimizing consumers

Solution of optimal consumption problem:

$$\max_{\vec{z}\geq 0} E\left[u\left(c(\vec{z})\right)\right] \qquad c^\omega = \frac{w_0 + \sum_i z_i r_i^\omega}{p^\omega}$$

w_0 = initial wealth
p^ω = price of goods in state ω

First order conditions:

$$\frac{\partial}{\partial z_i} E_\pi\left[u\left(c^\omega\right)\right] = \sum_\omega \pi^\omega \frac{u'(c^\omega)}{p^\omega} r_i^\omega \quad \begin{cases} = 0 & \Leftrightarrow \quad z_i > 0 \\ < 0 & \Leftrightarrow \quad z_i = 0 \end{cases}$$

i) investors select the assets which are traded (z_i>0) and those who are not (z_i=0)

ii) they determine the Equivalent Martingale Measure (EMM)

$$q^\omega = \pi^\omega \frac{u'(c^\omega)}{Q p^\omega}, \qquad Q = \sum_\omega \pi^\omega \frac{u'(c^\omega)}{p^\omega}$$

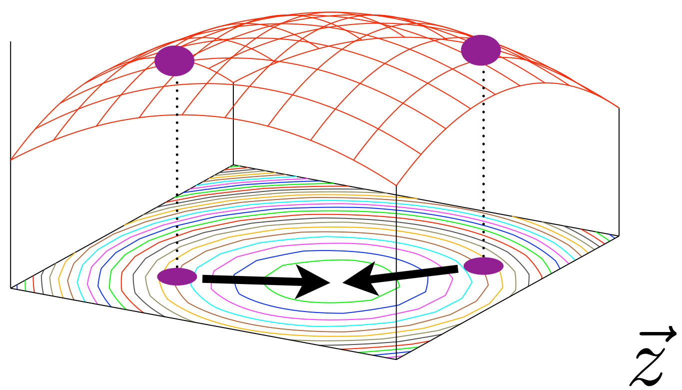
A creative financial sector

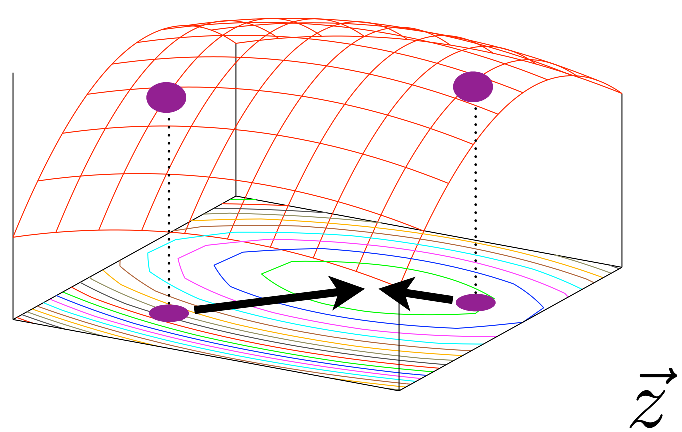
- Financial instruments are drawn at random from a probability distribution with

$$E_\pi\left[r_i\right] = \sum_\omega \pi^\omega r_i^\omega = -\frac{\epsilon}{\Omega}, \qquad \mathrm{Var}\left[r_i\right] = \frac{1}{\Omega}, \quad i = 1, \ldots, N$$

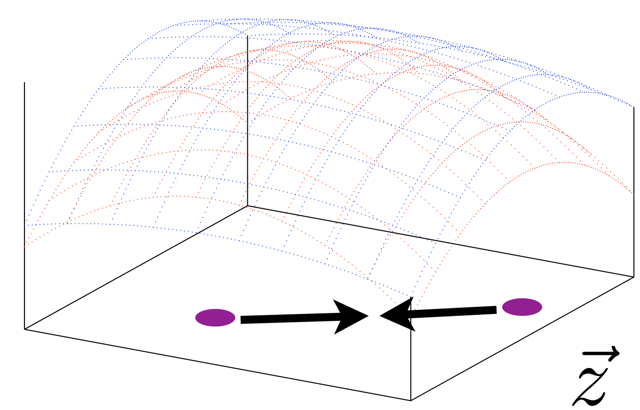
- Key variables:
  - financial complexity: $n = N/\Omega$
  - risk premium: ε
- Note: Successful innovations ($z_i > 0$) are not independent draws

Theory: intuition





Average on samples



Two approximate solutions converge to the same point which depends on the sample

Two approximate solutions attract each other

Characterizing equilibrium

- Market completeness:

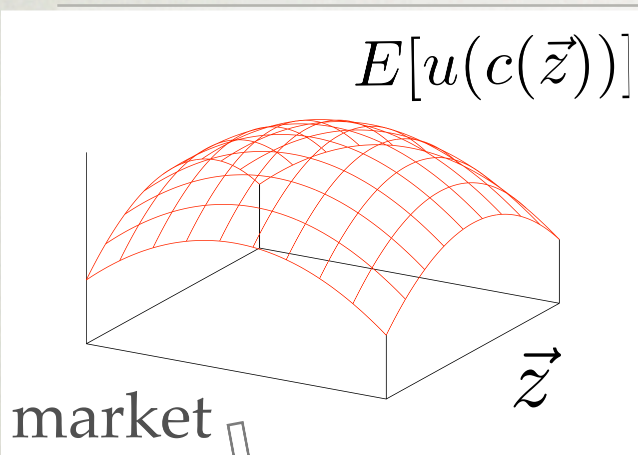
$$\phi = \frac{\text{number of traded assets}}{\text{number of states (\quad)}}$$

- Susceptibility: how much do portfolios allocation z responds to price uncertainty

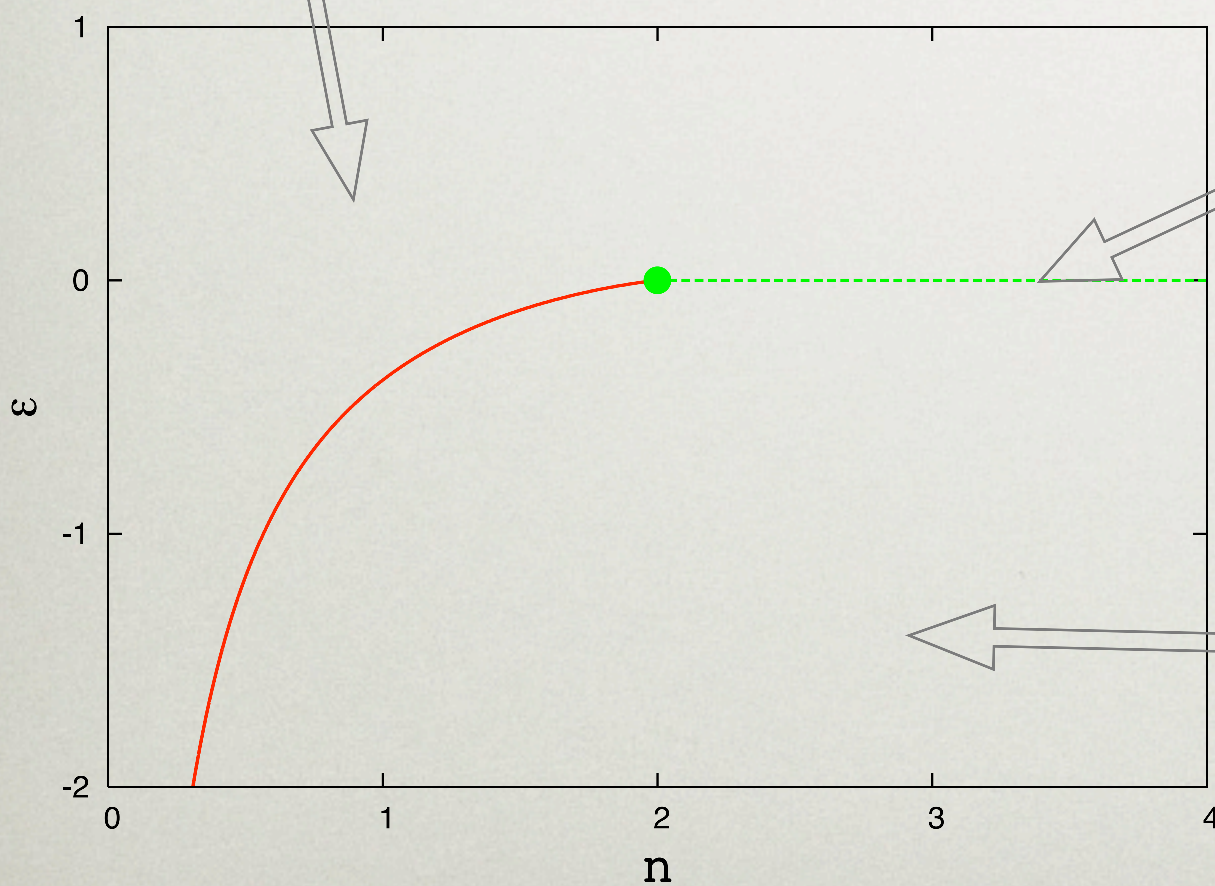
$$\chi = \frac{\delta z}{\delta p}$$

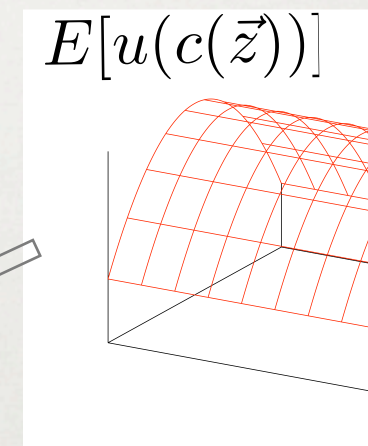
Intuition: landscape E[u]

- Ω free variables (z_i>0), Ω constraints
  ε<0 ⇒ unstable directions can appear (arbitrages)



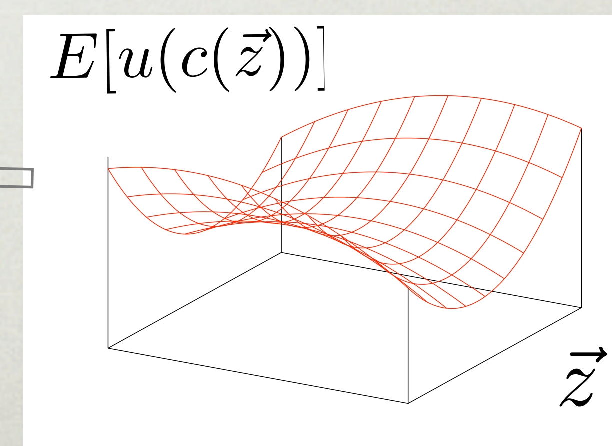
stable market





complete market

$$\phi = 1$$
$$\chi = \frac{\delta z}{\delta p} \to \infty$$



unstable market
(arbitrages)

A competitive Financial Industry

- Part of the risk of a new instrument can be hedged buying existing instruments
- Residual risk

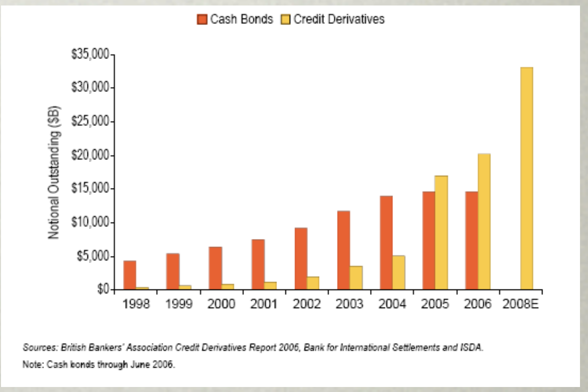
$$\Sigma = \min_{\vec{u}} \mathrm{Var}\left[r^{\omega}_{\mathrm{new}} - \sum_i v_i r^{\omega}_i \right] = 1 - \phi$$

- In competitive market, risk premium vanishes as $\phi \to 1$ e.g. Mean Variance profit function

$$\Rightarrow \quad \epsilon = \frac{\gamma}{2}(1-\phi)$$

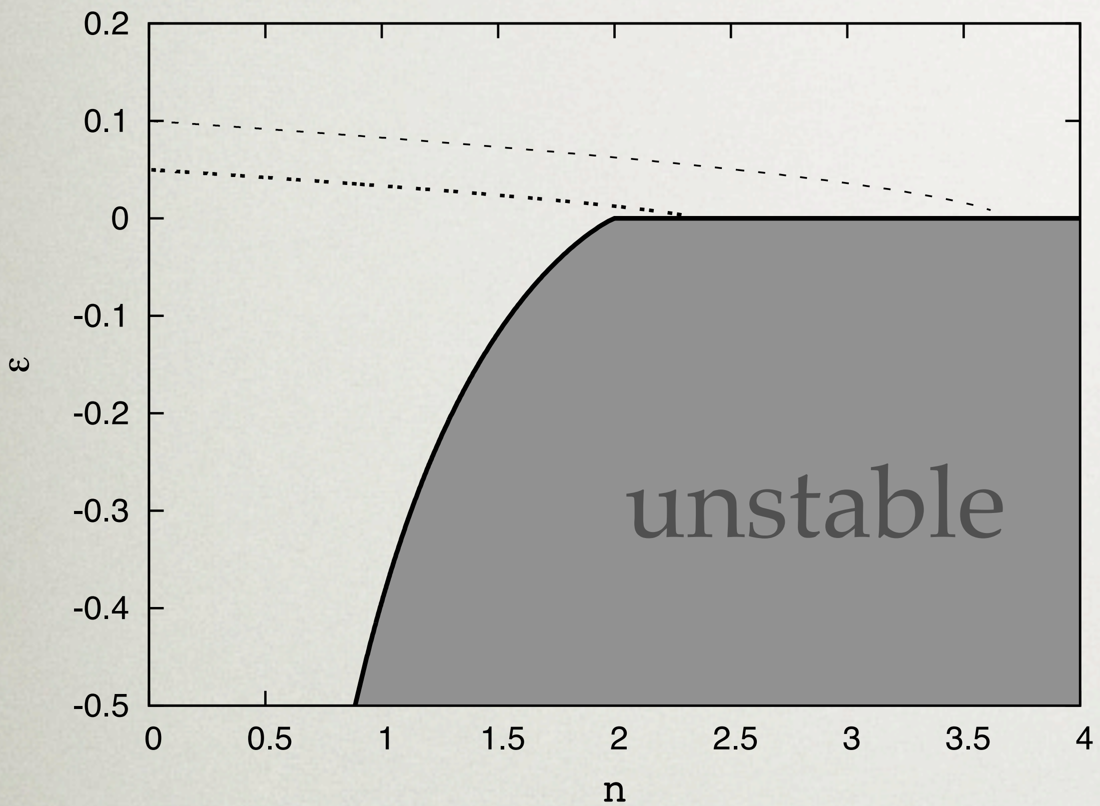
- The weights of portfolios used to hedge each instrument diverges as $\phi \to 1$

$$\sum_i v_i^2 = \frac{\phi}{1-\phi}$$

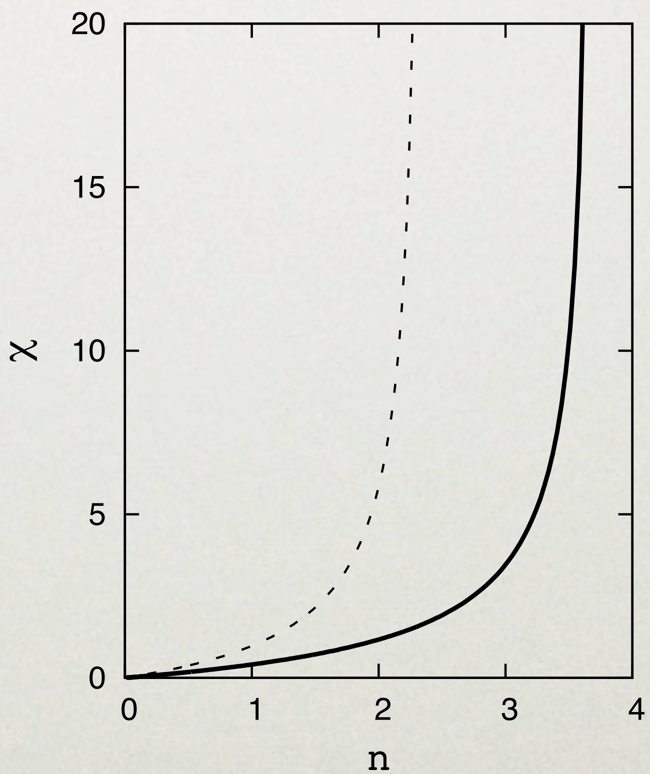


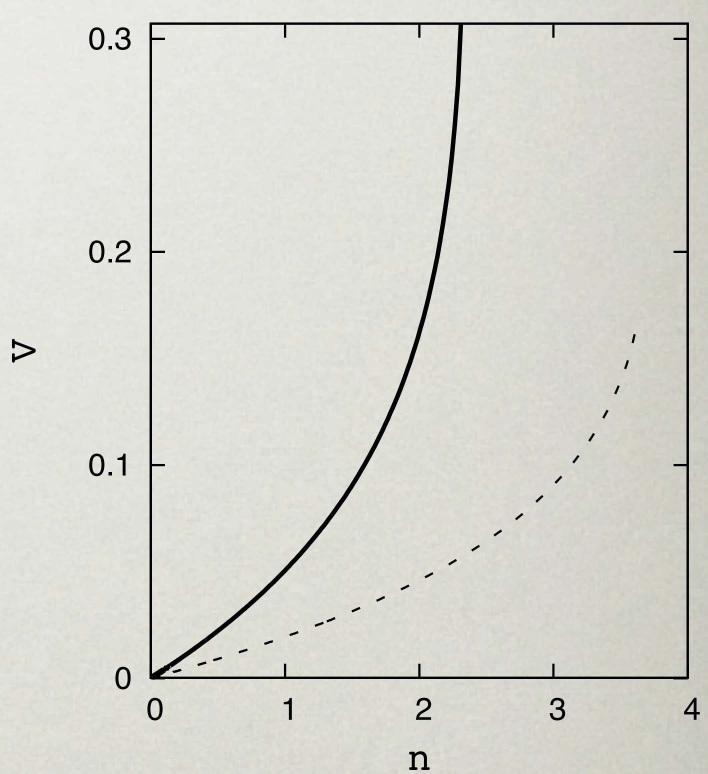
- Susceptibility in the interbank market also diverges

Mean variance banks $\epsilon = \frac{\gamma}{2}\Sigma$



Consumer market:
infinite susceptibility, finite volume





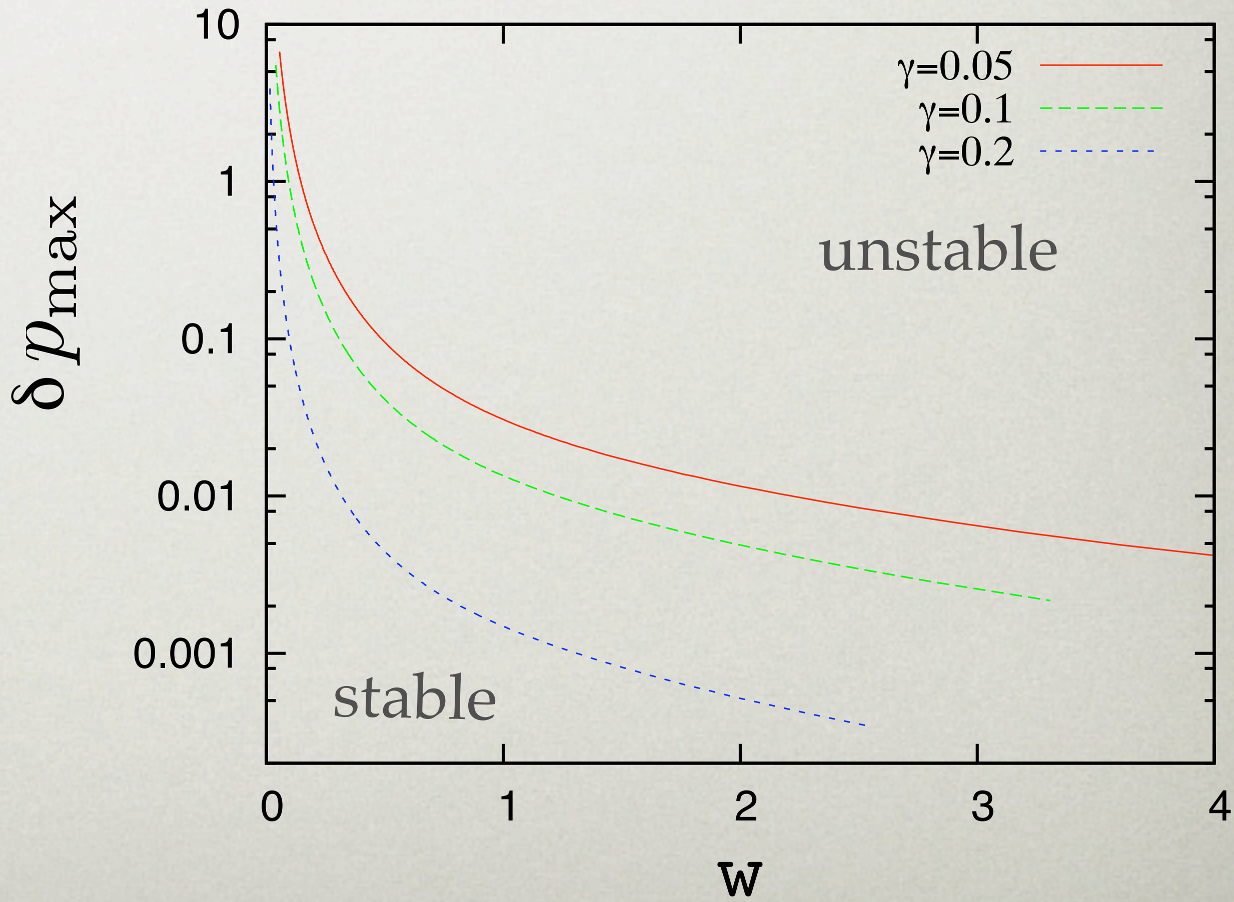
Interbank market:
both susceptibility and volumes diverge as $\phi \to 1$

Stability and the size of financial markets

- Relative size of financial markets ≈ volume of trading for hedging one unit of a new asset $w = \sqrt{\sum_i v_i^2}$
- Financial stability: → price uncertainty $\frac{\delta z}{z} = \frac{1}{z}\frac{\delta z}{\delta p}\delta p = \frac{\chi}{z}\delta p \ll 1$

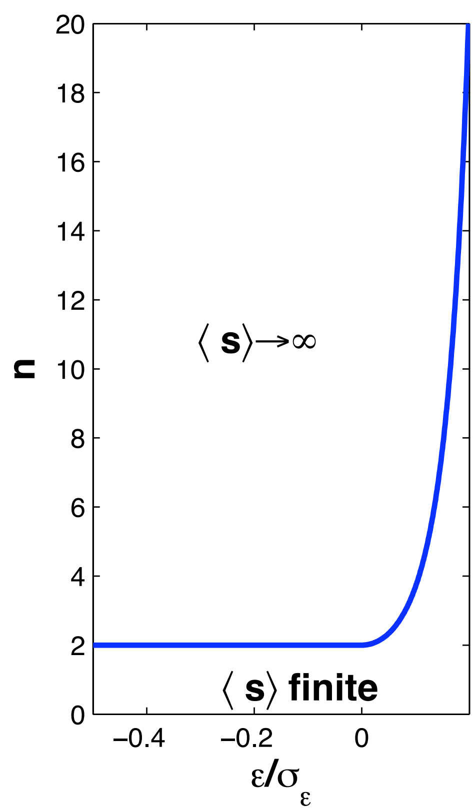
$$\delta p_{\max} = \frac{z}{\chi}$$

- Stability diagram on a given trajectory in (n,ε) plane.

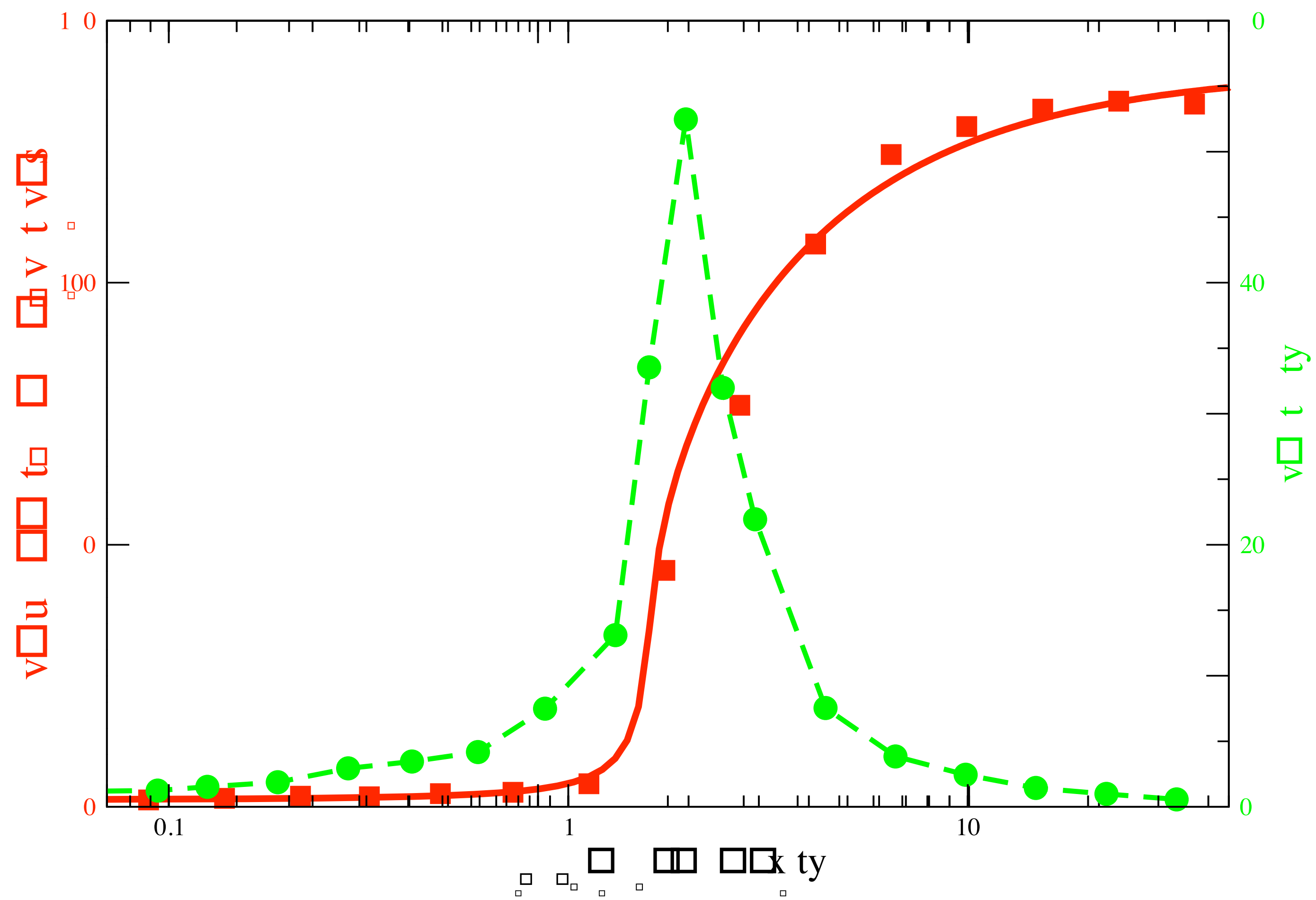


A model of illiquid market: N derivatives, one underlying

Phase transition
from supply limited
to demand limited



Susceptibility $\chi \to \infty$
instability in underlying market



(F. Caccioli, M. Marsili, P. Vivo EPJB 2009)

Conclusions (I)

- The proliferation of financial instruments, even in an ideal world (perfect competition and full information), leads to systemic instability
  - Complete markets lie on a critical line with infinite susceptibility
  - A competitive financial sector is expected to converge to this singularity
  - The volume generated by banks to hedge financial instruments they sell diverges as markets approach completeness
  - Learning to invest optimally is hard (Brock, Hommes, Wagener 2006)
- The larger (and more complex) the financial market is, the more price indeterminacy is problematic
  - Institution should grow in size with financial complexity
  - Quantitative measure of financial stability based on price indeterminacy and relative size of financial sector?

II- Instability in portfolio management and market impact

- Instability of risk measures for large portfolios
- Market dynamics: “empirical arbitrages” and market impact
  Liquidity risk (Acerbi & Scandolo 2007)
- Example:
  large regularized optimal portfolio under Expected Shortfall
- Evidence for market impact

Portfolio management and execution

management

Valuation based on current prices

execution

buy/sell q_1 shares of asset 1

buy/sell q_2 shares of asset 2

.

.

.

buy/sell q_N shares of asset N

Liquidation faces market impact

Portfolio optimization should account for market impact

Optimal Portfolios

- Portfolio is a linear combination $\vec{w} \cdot \vec{x}$ with weights w_i that fulfill the constraint $\sum_i w_i = 1$
- Given a risk functional $F(\vec{w} \cdot \vec{x})$
- Portfolio weights are chosen such that the risk $R(\vec{w}) = \langle F(\vec{w} \cdot \vec{x}) \rangle_{p(\vec{x})}$ is minimized.
- In practice, expectation is taken on empirical returns x_i^k for assets i=1, ...,N at k=1, ...,T time points.

E.g. Markowitz

- Risk = variance $F(x) = \frac{1}{2}x^2$ $\left[\text{assume} \quad \langle \vec{x} \rangle_{p(\vec{x})} = 0\right]$

$$\min_{\vec{w}} \left\langle \frac{1}{2}(\vec{w}\vec{x})^2 \right\rangle_{p(\vec{x})} \quad \textbf{s.t.} \quad \sum_i w_i = 1$$

- Optimal solution: $w_i^* = \frac{\sum_j \sigma_{ij}^{-1}}{\sum_{j,k} \sigma_{kj}^{-1}}$

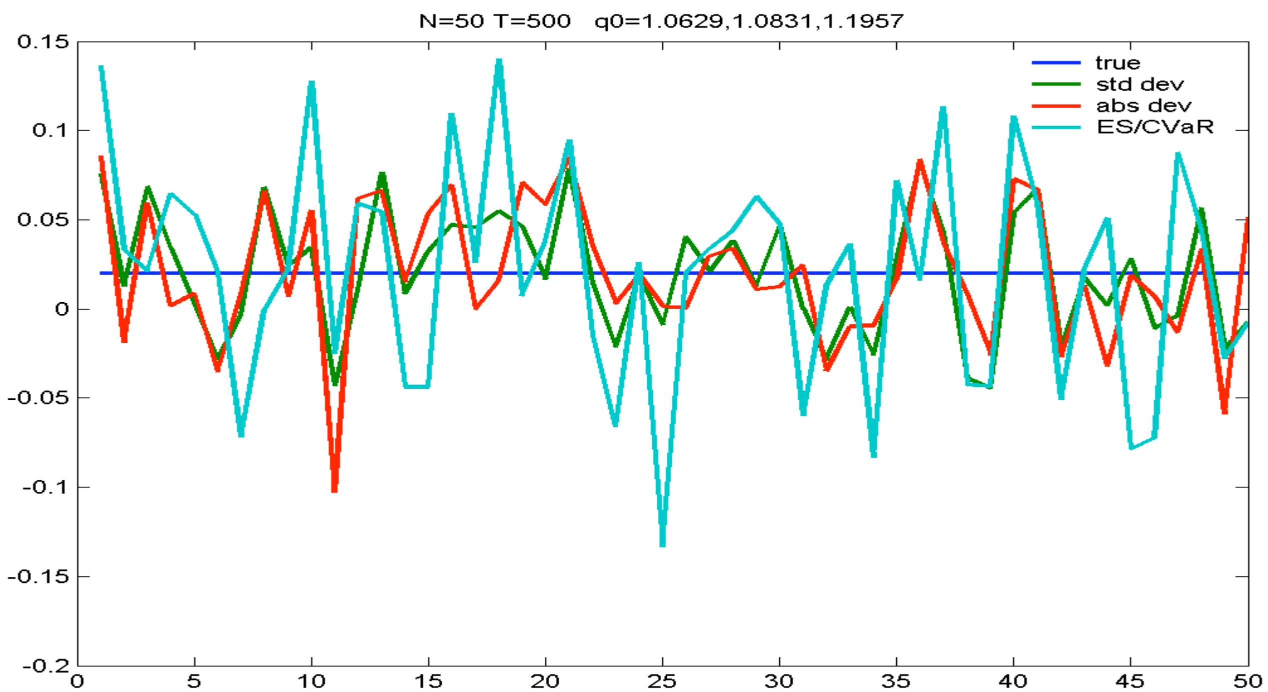
- with covariance $\sigma_{ij} = \langle x_i x_j \rangle$
- In practice, have to use: $\hat{\sigma}_{ij} = \frac{1}{T} \sum_k x_i^{(k)} x_j^{(k)}$

Other risk measures

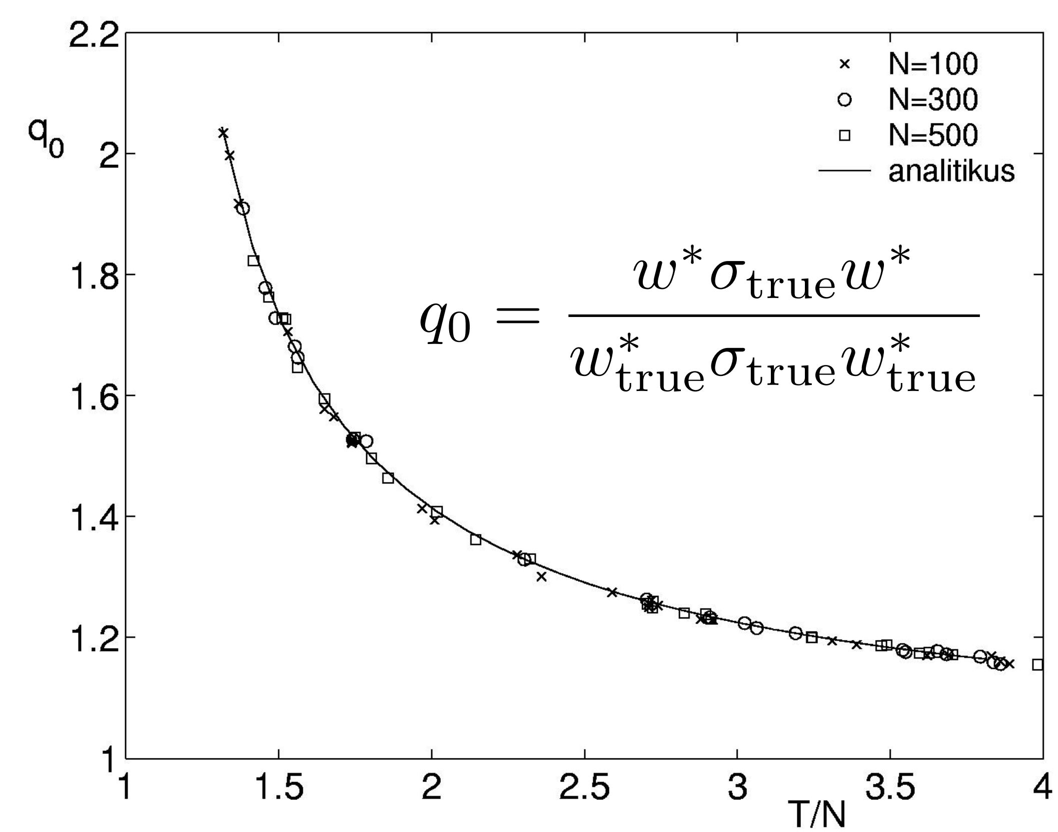
- Mean Absolute Deviation (MAD)
- Value at Risk (VaR): high quantile - threshold below which a given percentage of the weight of the profit-loss distribution resides. NOT CONVEX.
- Expected Shortfall (ES): the conditional average over a high quantile: $F(\vec{w}\vec{x}) = \vec{w}\vec{x}\ \ \theta(\vec{w}\vec{x} - \alpha_\beta)$
- Maximal Loss (ML): the extreme case of ES, the optimal combination of the worst outcomes.
- ***Coherent risk measures***: monotonic, sub-additive, positive homogeneous, and translationally invariant.
- *ES and ML are coherent.*
  VaR, ES, and ML are downside risk measures.

Instability of risk measures

- For large portfolios the weights are far from optimal.
- Weights fluctuate due to sampling errors (T samples).



- Estimation error diverges at a critical value of the ratio N/T.
  Measure: ratio of estimated vs true

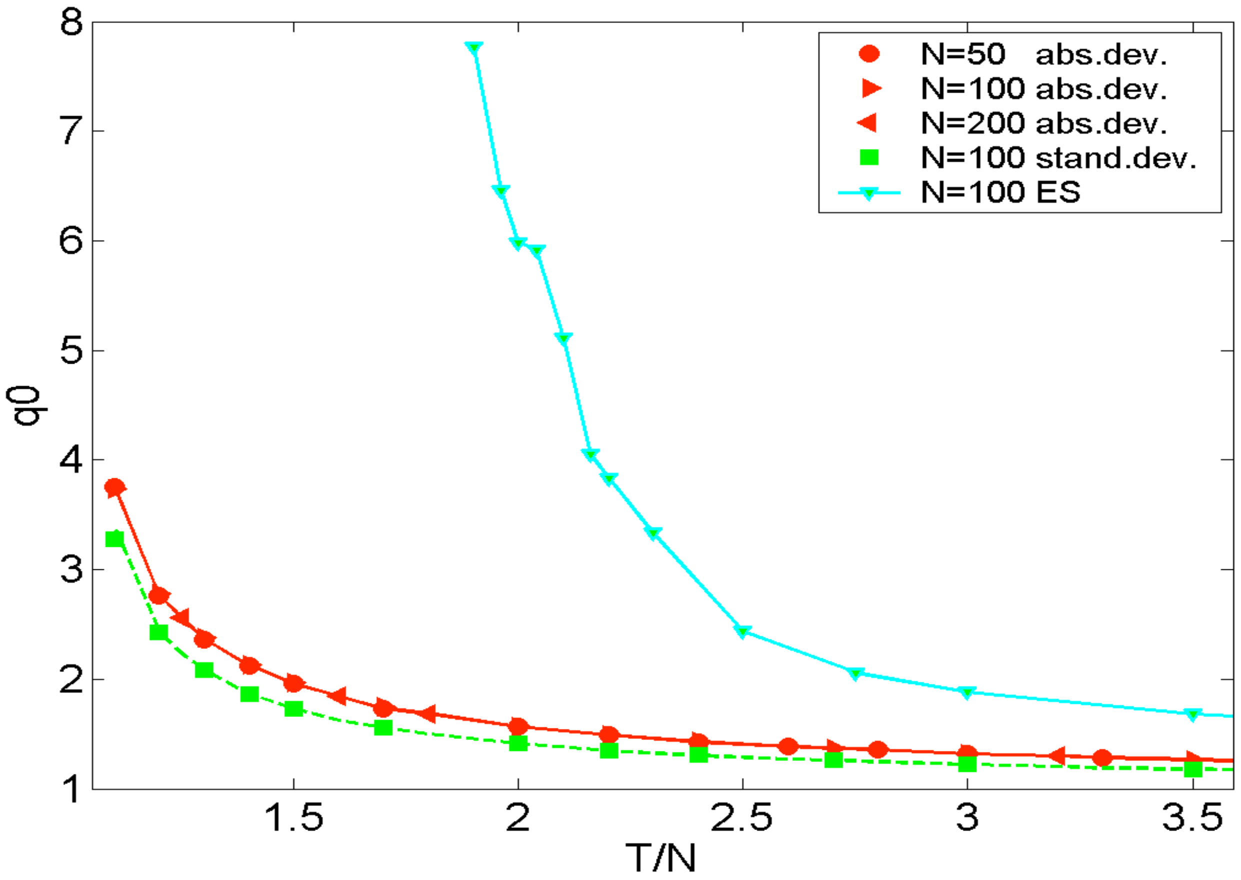


$$q_0 = \frac{w^* \sigma_{\text{true}} w^*}{w^*_{\text{true}} \sigma_{\text{true}} w^*_{\text{true}}}$$

Figures courtesy of I. Kondor

Instability is robust

- Found across ***all tested risk measures***
- Persists when linear constraints are added



Kondor and coworkers
2003-2008

Figure courtesy of I. Kondor

Machine learning angle

(Still & Kondor 2010)

- Replacing the true covariance matrix with the empirical average corresponds to minimizing the empirical risk, rather then the actual risk.
- For large portfolios, this is done in a regime in which there is not enough data to ensure that small empirical risk guarantees small actual risk.
- Instability is related to overfitting!
- In this 'thermodynamic' regime, where $N \to \infty; T \to \infty$ but N/T does not go to zero, ***portfolio optimization needs to be regularized***.

Regularizing: e.g. ES + L_2 norm

PO is equivalent to:

$$\min_{\{w_i\},\{\xi_k\},\epsilon} \left[\frac{1}{T}\sum_{k=1}^{T}\xi_k + \nu\epsilon\right]$$

$$\text{s.t.} \quad \vec{w}\vec{x}^{(k)} + \epsilon + \xi_k \geq 0; \quad \xi_k; \geq 0$$

$$\sum_i w_i = 1.$$

(Rockafellar & Uryasev 2000)

Regularized PO:

$$\min_{\{w_i\},\{\xi_k\},\epsilon} \left[\frac{1}{2}\|\vec{w}\|^2 + C\left(\frac{1}{T}\sum_{k=1}^{T}\xi_k + \nu\epsilon\right)\right]$$

$$\text{s.t.} \quad -\vec{w}\vec{x}^{(k)} \leq \epsilon + \xi_k;$$

$$\xi_k \geq 0; \quad \epsilon \geq 0;$$

$$\sum_i w_i = 1.$$

(Still & Kondor 2010)

- Algorithm that is similar to Support Vector Regression
- 1/C ~ diversification pressure
- Why L_2? How to choose the norm?

Origin of the instability

- "Empirical arbitrage":

$$\begin{aligned} \vec{w}_0 \cdot \vec{x}_t &= \lambda > 0, \quad \forall t = 1, \dots, T \\ \vec{w}_0 \cdot \vec{1} &= 0 \end{aligned}$$

- If N is large enough, w_0 exists
- Translation property of risk measures

$$F(\vec{w} \cdot \vec{x} + \lambda) = F(\vec{w} \cdot \vec{x}) - \lambda$$

- When evaluating this empirically, optimal portfolio is given by $\lambda \to \infty$ (buy infinite amount of w_0)

Market impact

- Liquidation moves prices (specially in tail events)
- Realized prices in liquidating a position $\vec{w}$

$$\vec{p}_{\rm est} = \vec{p}_{\rm now} + \vec{x} - \vec{\psi}(\vec{w}) \qquad \text{e.g.} \quad \vec{\psi}(\vec{w}) = \eta \vec{w}$$

- Cash flow: $\begin{aligned} c &= \vec{w} \cdot \vec{p}_{\rm est} \\ &= \vec{w} \cdot \vec{p}_{\rm now} + \vec{w} \cdot \vec{x} - \eta \vec{w} \cdot \vec{\psi}(\vec{w}) \end{aligned}$

- Risk: $F(\vec{w} \cdot \vec{x} - \vec{w} \cdot \vec{\psi}) = F(\vec{w} \cdot \vec{x}) + \vec{w} \cdot \vec{\psi}(\vec{w})$

- Impact has same effect as regularizing

$$\text{e.g.} \quad \vec{\psi}(\vec{w}) = \eta \vec{w} |\vec{w}|^{p-2} \quad \Rightarrow \quad L_p \;\; \text{norm}$$

E.g. ES + linear impact

PO + linear impact:

$$\min_{\{w_i\},\{\xi_k\},\epsilon} \left[\frac{1}{T} \sum_{k=1}^{T} \xi_k + \nu\epsilon \right]$$

$$\text{s.t.} \quad \vec{w}\vec{x}^{(k)} - \eta||w||^2 + \epsilon + \xi_k \geq 0; \quad \xi_k; \geq 0$$
$$\sum_i w_i = 1.$$

$$\Longleftrightarrow$$

L_2 Regularized PO:

$$\min_{\{w_i\},\{\xi_k\},\epsilon} \left[\frac{1}{2}\|\vec{w}\|^2 + C\left(\frac{1}{T} \sum_{k=1}^{T} \xi_k + \nu\epsilon \right) \right]$$

$$\text{s.t.} \quad -\vec{w}\vec{x}^{(k)} \leq \epsilon + \xi_k;$$
$$\xi_k \geq 0; \quad \epsilon \geq 0;$$
$$\sum_i w_i = 1.$$

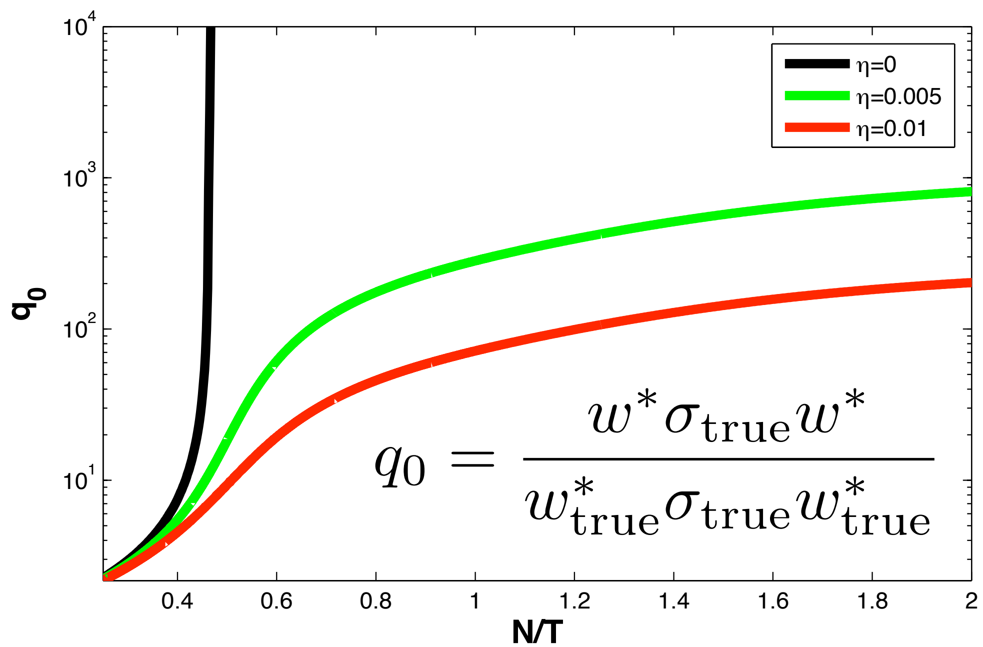
(Still & Kondor 2010)

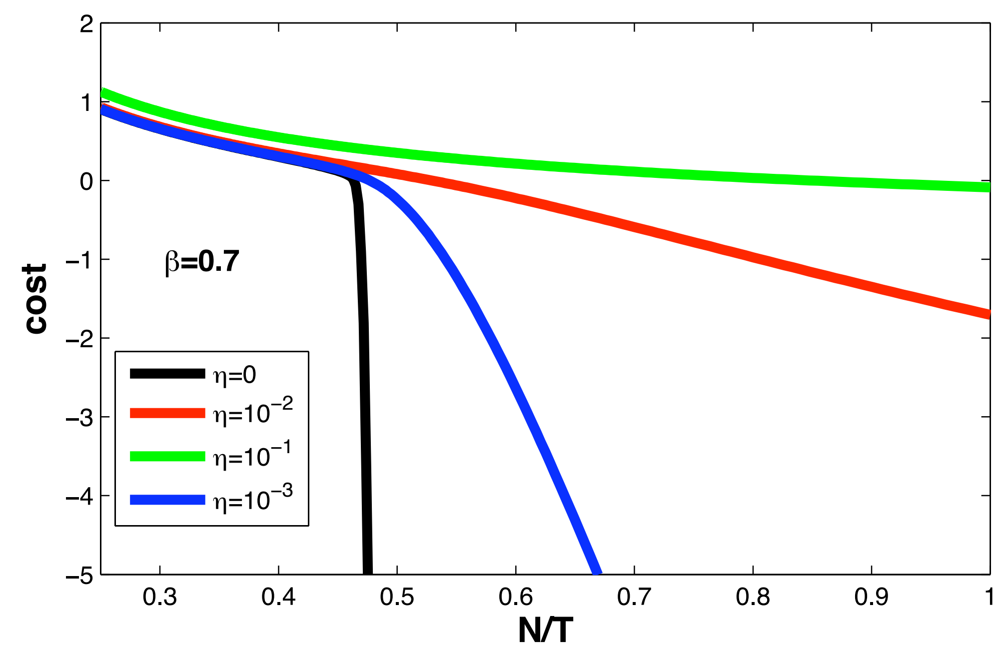
- Equivalent for $\eta\nu = \dfrac{1}{2C}$

Does this change things?

(Analytic calculation for gaussian returns; replica trick)

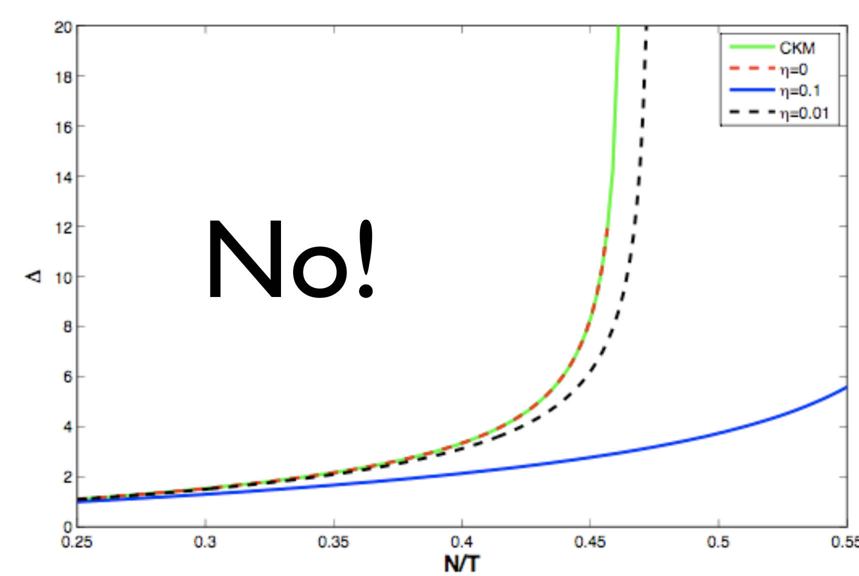
- Linear impact/L_2: Divergence disappears ∀ η>0





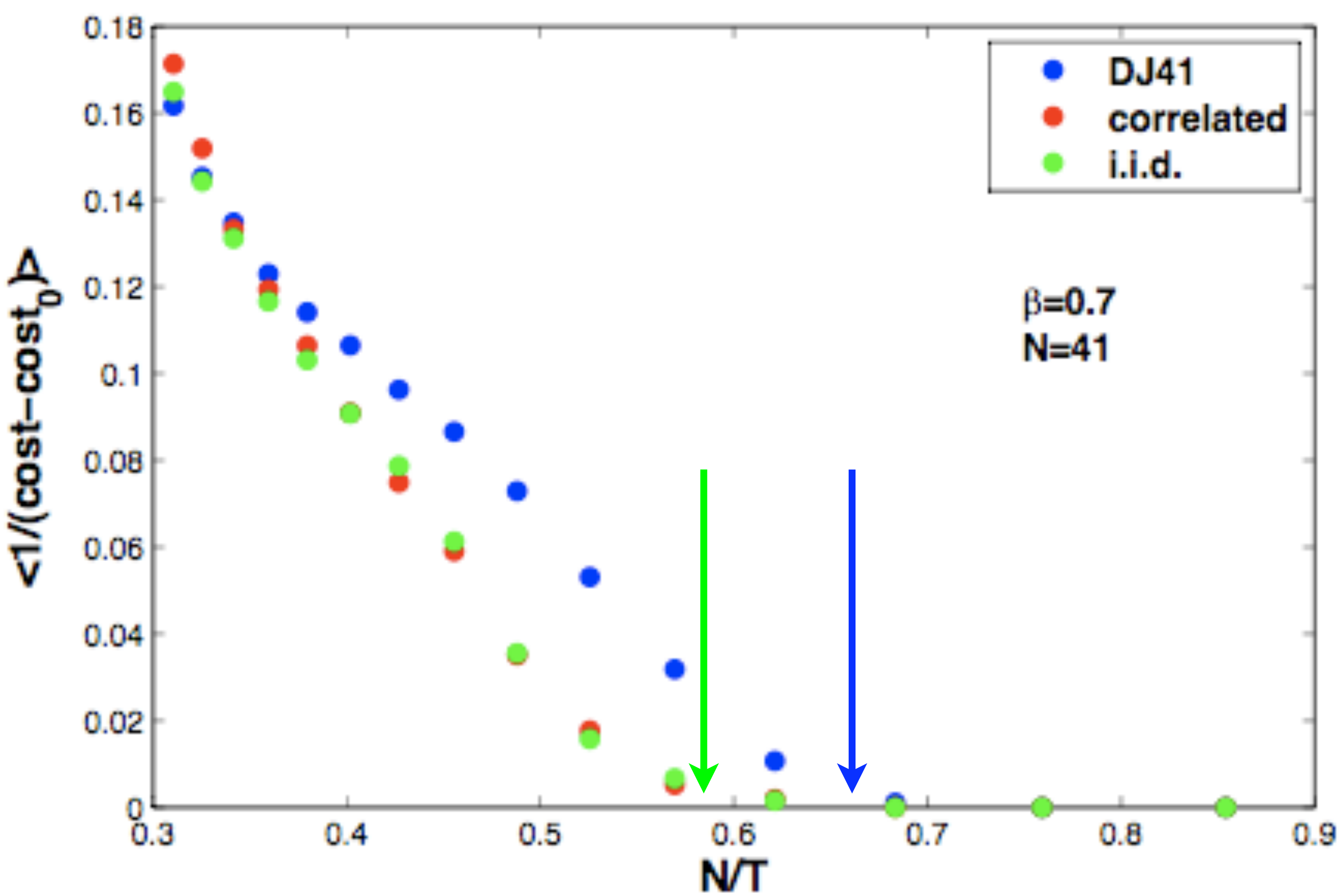
- Bid-ask spread ⇔ L_1

$$\vec{\psi}(\vec{w}) = \eta \, \text{sign} \vec{w}$$



Evidence of market impact

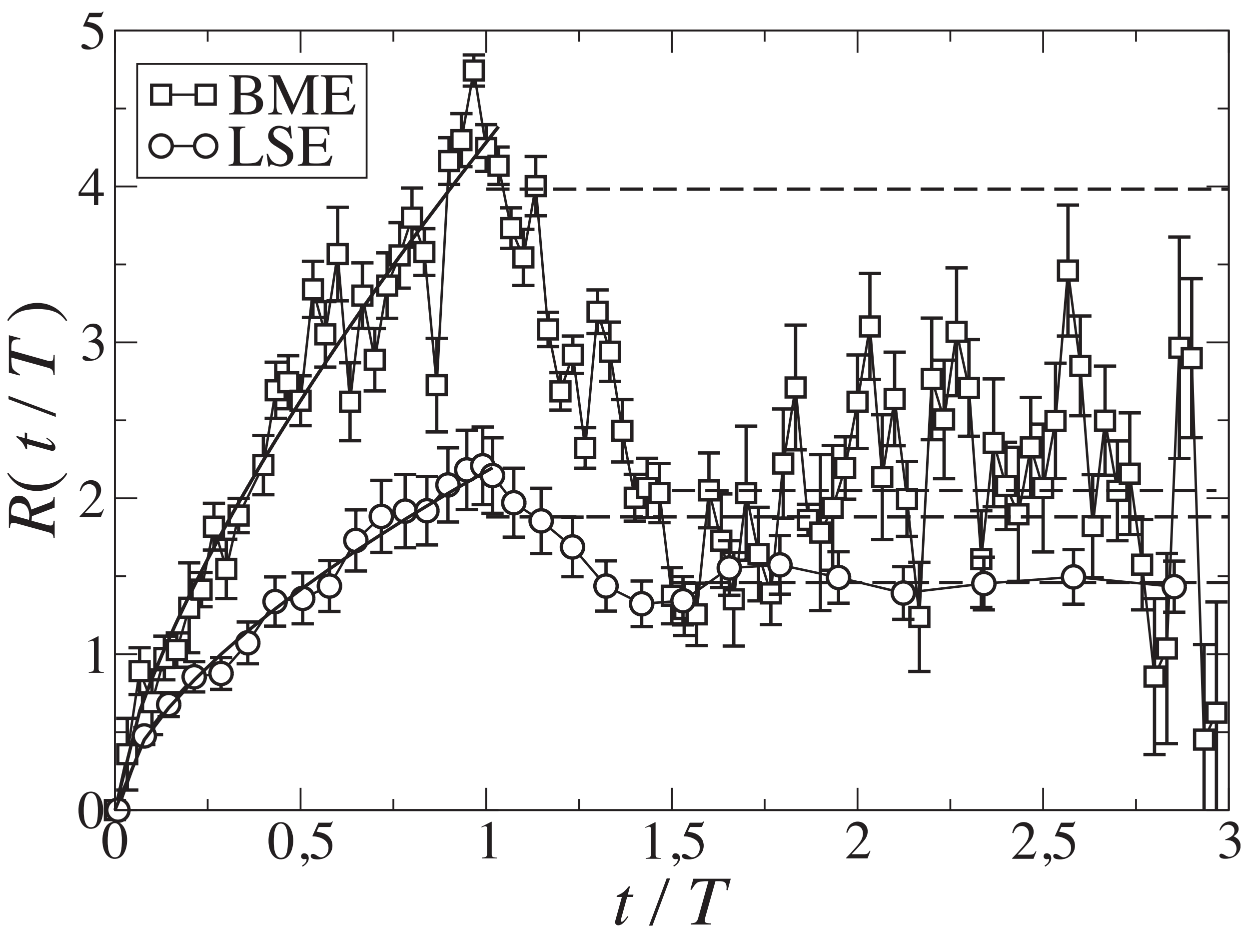
If market impact exist, empirical arbitrages should be less frequent in real data than in iid samples



Singularity is shifted!

The shape of market impact

Moro et al. Phys. Rev. E 80 (2009)



$$\Delta p \approx \sigma_T \sqrt{\frac{q}{V_T}}$$ ⇨ $L_{3/2}$ norm

(see also A. Ferraris Deutsche Bank Dec. 2008)

R. Almgreen (Risk 2005)

$$\Delta p \propto q^{3/5}$$

⇨ $L_{8/5}$ norm

Summary

- For large portfolios, classical portfolio theory is inadequate, because
  - empirical risk is not a good measure of actual risk
  - 'optimal' weights fluctuate wildly, estimation error diverges
- PO needs to be regularized!
- Regularization results from considering market impact
- Empirical evidence for market impact

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