Problem 1.

(a).

Proof. First we will show the existence of QR factorization. $A \in \mathbb{R}^{n*m}$ is a full column rank matrix, denote its columns by $a_1, a_2, \cdots, a_m \in \mathbb{R}^n$. However, a proof of existence is provided by the Gram-Schmidt Algorithm (The algorithm follows Algorithm 7.1. in Trefethen-Bau's Numerical Linear Algebra). The Gram-Schmidt Algorithm generates orthonormal vectors $q_1, \cdots, q_m$ s.t. for $j \in [1, m]$ and $j \in \mathbb{N}^+$,

$$a_j = \sum_{k=1}^{j} r_{kj} q_k = \sum_{k=1}^{m} r_{kj} q_k,$$

for $k > j, r_{kj} = 0$. Therefore, $A = QR$ exists with $R := \{r_{ij}\}_{i,j=1}^m$ which is an $m * m$ upper-triangular matrix, and Q is an $n * m$ matrix with orthonormal column vectors $q_1, \cdots, q_m$. If we choose $r_{jj} > 0$ in GS Algorithm, then R has positive diagonal entries. Note that the GS Algorithm only fails when v_j is zero and thus cannot be normalized to produce q_j, which contradicts to A has full rank.

Then we'll show the uniqueness of QR factorization. Also by GS Algorithm we can say any QR factorization should satisfy the following:

i. $q_j = \frac{a_j - \sum_{i=1}^{j-1} r_{ij} q_i}{r_{jj}}$;

ii. $r_{ij} = q_i^T a_j$;

iii. $|r_{jj}| = \left|\left|a_j - \sum_{i=1}^{j-1} r_{ij} q_i\right|\right|_2$.

Since A is full column rank and R has positive diagonal entries thus $r_{jj} > 0$, at each successive step j, r_{ij} and q_j is only determined by the formulas above. Therefore, $A = QR$ is uniquely determined. □

(b).

(b.1). First we'll show similar matrices share the same collection of eigenvalues.

Proof. Suppose A and B are similar matrices so we can write $\Theta A \Theta^{-1} = B$, this implies that $A = \Theta^{-1} B \Theta$.
Let $Ax = \lambda x$ s.t. x is an eigenvector of A with eigenvalue λ, then

$$\Theta B \Theta^{-1} x = \lambda x \implies B \Theta^{-1} x = \lambda \Theta^{-1} x,$$

$\Theta^{-1} x$ is an eigenvector of B with eigenvalue λ. Hence, Similar matrices share the same collection of eigenvalues. □

(b.2). Then we'll show that even in exact arithmetic, there is no general procedure to construct Θ for any given diagonalizable A, such that $\Theta A \Theta^{-1}$ is diagonal.

Proof. If it's possible to find such a general procedure, this means that we have to find all the roots of a polynomial in a general procedure with finite sequence of elementary operations. It's not possible to give an exact solution, the best way is approximate the roots with some iterative method. □

Problem 2.

(a).

The Givens Rotation matrix is of the form

$$G = \begin{bmatrix} 1 & & & \dots & & & 0 \\ & \ddots & & & & & \\ & & c & \dots & -s & & \\ \vdots & & \vdots & & \vdots & & \vdots \\ & & s & \dots & c & & \\ & & & & & \ddots & \\ 0 & & & \dots & & & 1 \end{bmatrix},$$

this matrix is orthogonal. Suppose we have a vector

$$\begin{bmatrix} x_1 \\ \vdots \\ x_i \\ \vdots \\ x_j \\ \vdots \\ x_n \end{bmatrix}$$

Applying the Givens matrix to this vector we have

$$\begin{bmatrix} 1 & & & \dots & & & 0 \\ & \ddots & & & & & \\ & & c & \dots & -s & & \\ \vdots & & \vdots & & \vdots & & \vdots \\ & & s & \dots & c & & \\ & & & & & \ddots & \\ 0 & & & \dots & & & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_i \\ \vdots \\ x_j \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ \vdots \\ r \\ \vdots \\ 0 \\ \vdots \\ x_n \end{bmatrix}$$

where $r = \sqrt{x_i^2 + x_j^2}$ and c and s can be computed as

$$c = \frac{x_i}{r} = \cos\theta, s = \frac{-x_j}{r} = \sin\theta.$$

G only affect the i-th and j-th row. So we can use a product of Givens Rotation Matrices to reduce $A \in \mathbb{R}^{m*n}$ to an upper triangular matrix R:

$$G_{m,n-1} \cdot \dots \cdot G_{3,1}G_{2,1}A = R.$$

Note that $G_{i,j}$ means the Givens Rotation eliminating the i-th entry in the j-th column of A, thus we can write $A = QR$ such $Q = G_{2,1}^T G_{3,1}^T \cdot \cdots \cdot G_{m,n-1}^T$. The algorithm is based on the Givens Rotations which are orthogonal. Using a sequence of Givens Rotations and the given matrix can be transformed to an upper triangular matrix.

It is easy to see that $O(n^2)$ rotations is required and each rotation cost $O(n)$ operations, thus the asymptotic complexity of the algorithm is $O(n^3)$. Therefore, the asymptotic complexity of the algorithm is same with the Householder method.

(b).

Recall : Householder Transformation

$$H = I - 2\frac{vv^T}{v^T v},$$

note that we have I is identity matrix and "$H = H^T$", "$H^2 = I$" thus H is orthogonal and symmetric. Given $a \in \mathbb{R}^n$ is a vector, pick $v \in \mathbb{R}^n$, then applying Householder Transformation with $v = \begin{cases} a + ||a||e_1 \text{ If } a_1 \geq 0 \\ a - ||a||e_1 \text{ If } a_1 < 0 \end{cases}$, a_1 is the first entry of vector a

$$\implies Ha = (I - 2\frac{vv^T}{v^T v}) \cdot a = \begin{bmatrix} * \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

The QR decomposition is then accomplished by eliminating all entries below the diagonal by using the appropriate householder transformation on the first column of the submatrix $A_{k:n,k:m} \in \mathbb{R}^{(n-k+1)*(m-k+1)} \subseteq A \in \mathbb{R}^{n*m}$ with $k = 1, \ldots, \min(m, n)$. So we can write each iteration as

$$\begin{aligned} A_{k:n,k:m} &= HA_{k:n,k:m} \\ &= A_{k:n,k:m]} - 2\frac{vv^T}{v^T v} A_{k:n,k:m} \\ &= A_{k:n,k:m} - 2\frac{v}{||v||}\left(\frac{v^T}{||v||} A_{k:n,k:m}\right). \end{aligned}$$

Therefore, the Householder algorithm is:
For $k = 1, \cdots, m$,
Consider the matrix $A := A_{k:n,k:m}$, $a_k = A_{k:n,k}$,
$v_k = v_k(a_k) = \begin{cases} p_k a_k + ||p_k a_k||e_k \text{ If } a_{k_k} \geq 0 \\ p_k a_k - ||p_k a_k||e_k \text{ If } a_{k_k} < 0 \end{cases}$,
a_k is the k-th column of matrix A, $p_k x = (e_k e_k^T + \cdots + e_n e_n^T)x$
$A := A_{k:n,k:m} - 2\frac{v}{||v||}\left(\frac{v^T}{||v||} A_{k:n,k:m}\right)$.

Here we take the advantage of band lower band such that we can take the column of the matrix a_k to only where it's non-zero. The upper band does not affect our factorization to form an upper triangular matrix.

Therefore, the modified algorithm is
For $k = 1, \cdots, m$,

Let $kl = \min\{k+p, n\}$,
Consider the matrix $A := A_{k:kl,k:m}$, $a_k = A_{k:n,kl}$,
$v_k = v_k(a_k) = \begin{cases} p_k a_k + ||p_k a_k|| e_k \text{ If } a_{k_k} \geq 0 \\ p_k a_k - ||p_k a_k|| e_k \text{ If } a_{k_k} < 0 \end{cases}$,
a_k is the k-th column of matrix A, $p_k x = (e_k e_k^T + \cdots + e_n e_n^T)x$
$A := A_{k:kl,k:m} - 2\frac{v}{||v||}\left(\frac{v^T}{||v||} A_{k:kl,k:m}\right)$.
It can efficiently handle the band matrix case.

(c).

If we construct a Householder matrix Q with $QAQ^T = B$, A is square and assuming it is full column rank, the B we get is full again thus not a Hessenberg matrix.
If we want to form a Hessenberg matrix we need to leaves the i-th rows of k-th column unchanged, if $A \in \mathbb{R}^{n*n}$ then $k = \{1, \cdots, n\}, i \leq k$. Construct a matrix

$$Q_1 := \begin{bmatrix} 1 & 0 \\ 0 & H \end{bmatrix},$$

such $H \in \mathbb{R}^{(n-1)*(n-1)}$ is a Householder matrix , it's obvious that Q is orthogonal. When Q_1 is multiplied on the left side of A, it only forms linear combinations of rows 2 to n and put 0 into the 3 to n rows of the first column. Then after Q_1^T is multiplied on the right of $Q_1 A$, it leaves the first column unchanged. Repeat this step $n-2$ times, then we have

$$Q_{n-2}Q_{n-3}\cdots Q_2 Q_1 A Q_1^T Q_2^T \cdots Q_{n-3}^T Q_{n-2}^T = B$$

$$\implies QAQ^T = B,\ B \text{ is a Hessenberg matrix.}$$

Since A is symmetric, QAQ^T is also symmetric, and any symmetric Hessenberg matrix is tridiagonal.

Problem 3.

(a).

Proof. Let x be any arbitrary eigenvector of A corresponds to eigenvalue λ, then we have $Ax = \lambda x$. Moreover, $x^T A x = x^T \lambda x = x^T x \lambda$. Since A is positive definite, for all nonzero x, $x^T A x > 0$, also $x^T x = ||x||^2 > 0$, thus $\lambda > 0$ always holds. Therefore, all eigenvalues of A are positive. □

(b).

Definition: A minor of a matrix A is the determinant of some smaller square matrix, cut down from A by removing one or more of its rows or columns. A minor of A of order k is principal if it is obtained by deleting $n-k$ rows and the $n-k$ columns with the same numbers.

Proof. Let B be any arbitrary submatrix of square matrix A, form a vector x with zeros in the position corresponding to the rows which have been deleted. Then we have $x^T A x = y^T B y$, y is the vector consisting of the elements of x which were not deleted. Since A is positive definite, $x^T A x > 0 \implies y^T B y > 0$, thus B is also positive definite. Therefore, by (a), all eigenvalues of B are positive, also the determinant is the product of the eigenvalues, the principle minors of A are positive, and an LU decomposition of A exists. □

(c).

Proof. By (b), we know that $A = LU$ exists. Assuming that

$$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} & \cdots & \cdots & \cdots & \cdots & \cdots & u_{1n} \\ 0 & u_{22} & u_{23} & \cdots & \cdots & \cdots & \cdots & \cdots & u_{2n} \\ 0 & 0 & u_{33} & \cdots & \cdots & \cdots & \cdots & \cdots & u_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & u_{nn} \end{bmatrix}.$$

Also we know that D is the diagonal matrix consisting of the diagonal entries of U,

$$D = \begin{bmatrix} u_{11} & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & u_{22} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & u_{33} & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & u_{nn} \end{bmatrix}.$$

Observing these two matrices we know that there exists a matrix M s.t. $U = DM$, thus we can write

$$M = \begin{bmatrix} 1 & \frac{u_{12}}{u_{11}} & \frac{u_{13}}{u_{11}} & \cdots & \cdots & \cdots & \cdots & \cdots & \frac{u_{1n}}{u_{11}} \\ 0 & 1 & \frac{u_{23}}{u_{22}} & \cdots & \cdots & \cdots & \cdots & \cdots & \frac{u_{2n}}{u_{22}} \\ 0 & 0 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \frac{u_{3n}}{u_{33}} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \end{bmatrix}.$$

Therefore, we have $A = LU = LDM$. Since A is also a symmetric matrix then $A = A^T$,

$$\implies LDM = M^T D^T L^T =^* M^T D L^T$$

" $=^*$ " is because D is a diagonal matrix. By the uniqueness of LU factorization, $L = M^T$, hence $A = LDL^T$. □

(d).

Remark: Since $A = LU$, by (b), the determinants of all submatrices of A are positive, and the diagonal elements of L are 1, hence we know the diagonal entries of U are al positive, the diagonal entries of D is positive too.

Proof. It's obvious that there exists Λ such that $\Lambda^2 = D$, Λ is also a diagonal matrix with diagonal entries $\sqrt{u_{ii}}, i = 1, \cdots, n$.
Since L^T is an upper triangular matrix with unit diagonal entries, Λ is diagonal matrix with positive diagonal entries, hence there exists $R := \Lambda L^T$ thus R is upper triangular with positive diagonal entries. Moreover,

$$A = LDL^T = L\Lambda\Lambda L^T = R^T R,$$

hence the Cholesky factorization exists. □

(e).

Proof. By (d), there exists Cholesky factorization $A = R^T R$ with R is upper triangular with positive diagonal entries and R^T is lower triangular with positive diagonal entries. a_{ij} is the entry of A in i-th row and j-th column, r_{ij} is the entry of R in i-th row and j-th column, then we have

$$a_{ij} = \sum_{k=1}^{j} r_{ki} r_{kj}, \text{s.t. } 1 \le i, j \le n.$$

If $i = j$, then we have

$$a_{jj} = \sum_{k=1}^{j} r_{kj}^2 \implies r_{jj} = \sqrt{a_{jj} - \sum_{k=1}^{i-1} r_{kj}^2}$$

Since $r_{jj} > 0$, then $a_{jj} > \sum_{k=1}^{i-1} r_{kj}^2 \implies \forall\, 1 \le k \le i-1, a_{jj} > r_{kj}^2$, also we have $r_{11}^2 = a_{11}$,

$$r_{ij}^2 \le a_{jj},\ 1 \le i, j \le n$$

□

(f).

By (a)-(d), we have

$$A = R^T R = R^T \Lambda L^T = R^T \sqrt{D} L^T$$

If we want to obtain an algorithm with $(*)$ and $(**)$, we need to know the first column of R^T, we can find that $\sqrt{D}$ is diagonal matrix , L^T is the upper triangular matrix with diagonal entries 1, hence $\sqrt{D}L^T$ is diagonal with diagonal entries $\sqrt{u_{ii}}, 1 \le i \le n$. Therefore, $(R^T)_1 = \frac{1}{\sqrt{u_{11}}}(A)_1$.
Then by two iterative steps,

$$r_{jj}(R^\top)_j = A_j - \sum_{k=1}^{j-1} r_{kj}(R^T)_k =: v$$

$$(R^T)_j = \frac{1}{\sqrt{v_j}} v$$

We can compute each column of R^T, combining these column and taking transpose, then we get the Cholesky factor R.

(g).

Proof. At j-th step of the aforementioned algorithm is to compute the j-th row of R. Hence we only need to be compute the last $n-j+1$ components of v in $(*)$. To get each component of v there's $2j$ multiplications except the first one in the aforementioned algorithm. Totally number of multiplication is $\sum_{j=1}^{n}(2j-1)(n-j+1) \sim \frac{1}{6}n^3 + O(n^2)$, which is twice as efficient as the Gaussian elimination. $\square$

Problem 4.

(a).

Proof. Since x_m satisfies the recurrent relation, thus we have

$$\begin{aligned} x_m - x &= \omega b + (I - \omega A)x_{m-1} - x \\ &= \omega Ax + (I - \omega A)x_{m-1} - x \\ &= (I - \omega A)x_{m-1} - (I - \omega A)x \\ &= (I - \omega A)(x_{m-1} - x) \end{aligned}$$

Assuming that $||I - \omega A|| \leq 1$, there exists some $\beta \in (0, 1)$ s.t.

$$\begin{aligned} ||x_m - x|| &\leq ||I - \omega A||\ ||x_{m-1} - x|| \\ &\leq \beta||x_{m-1} - x|| \\ &\leq \beta^2||x_{m-2} - x|| \\ &\leq \beta^3||x_{m-3} - x|| \leq \cdots \\ &\leq \beta^m||x_0 - x|| \to 0 \text{ as } m \to +\infty. \end{aligned}$$

$\square$

(b).

Assuming A is diagonalizable, we can write $A = Q^{-1}DQ$ with Q is invertible, $||Q|| = 1$ and D is also diagonalizable.

$$\begin{aligned} I - \omega A &= QQ^{-1} - \omega Q^{-1}DQ \\ &= Q(I - \omega D)Q^{-1} \\ \implies ||I - \omega A|| &\leq ||Q||\ ||I - \omega D||\ ||Q^{-1}|| \\ &=^{(1)} ||I - \omega D|| \end{aligned}$$

Remark (1): Q is invertible with unit norm thus $||Q|| = ||Q^{-1}|| = 1$.
Note the entries of D are all eigenvalues of A, denoted by λ_i, $I - \omega D$ is still a diagonal matrix with entries $1 - \omega\lambda_i$.
Remark (2): The norm of any diagonal matrix is the max of the absolute value of the entries.

$$||I - \omega D|| = \max\{|1 - \omega\lambda_i|\} = \max\{|1 - \omega\lambda_1|, |1 - \omega\lambda_n|\}$$

(c).

Here we need a trick:

$$\max\{|a|,|b|\} = \max \frac{|a|+|b|+||a|-|b||}{2}$$

Applying this trick into the estimated derived in (b), we have

$$||I-\omega A|| \leq \max\{|1-\omega\lambda_1|,|1-\omega\lambda_n|\} \leq \max \frac{|1-\omega\lambda_1|+|1-\omega\lambda_n|+||1-\omega\lambda_1|-|1-\omega\lambda_n||}{2}.$$

We can see the minimum appears when $|1-\omega\lambda_1|-|1-\omega\lambda_n|=0$, so $\omega=\frac{2}{\lambda_1+\lambda_n}$, we get a optimized choice of ω.

Remark:

I have worked this assignment with David Knapik, Luke Steverango, Ralph Sarkis, Carl Perreault-Lafleur and Kabilan Sriranjan .