Problem 1.

(a).

Proof. We know that $L_n : C[(A,B)] \mapsto \mathbb{P}^n$ is defined by

$$L_n := \sum_{k=0}^{n} \phi_{n,k}(x_k),$$

$\phi_{n,k}$ is the k-th Lagrange basis function associated to the nodes $x_0, \ldots, x_n$. By the definition of the operator norm, we have that

$$||L_n||_{op} = \sup_{f \in C([a,b])} \frac{\|L_n f\|_\infty}{\|f\|_\infty} = \sup_{||f||_\infty = 1} ||\sum_{k=0}^{n} \phi_{n,k}(x_k)||_\infty.$$

First we will show that

$$\|L_n\|_{op} \leq \max_{x\in[a,b]} \sum_{k=0}^{n} |\phi_{n,k}(x)|,$$

we fix f such that $\|f\| = 1$ and $x \in [a,b]$ then we have

$$|\sum_{k=0}^{n} \phi_{n,k}(x_k) f(x_k)| \leq \sum_{k=0}^{n} |\phi_{n,k}(x)| \leq \max_{x\in[a,b]} \sum_{k=0}^{n} |\phi_{n,k}(x)|$$

then since $x \in [a.b]$ is arbitrarily fixed, we have

$$\max |\sum_{k=0}^{n} \phi_{n,k}(x_k) f(x_k)| = ||\sum_{k=0}^{n} \phi_{n,k}(x_k) f(x_k)||_\infty \leq \max_{x\in[a,b]} \sum_{k=0}^{n} |\phi_{n,k}(x)|.$$

Moreover, $\|f\| = 1$ is also arbitrarily fixed and we take the sup here,

$$\|L_n\|_{op} = \sup_{\|f\|_\infty = 1} ||\sum_{k=0}^{n} \phi_{n,k}(x_k) f(x_k)||_\infty \leq \max_{x\in[a,b]} \sum_{k=0}^{n} |\phi_{n,k}(x)|,$$

the first step is done.

The next step is ton prove the opposite direction s.t.

$$\|L_n\|_{op} \geq \max_{x\in[a,b]} \sum_{k=0}^{n} |\phi_{n,k}(x)|.$$

Note that $\sum_{k=0}^{n} |\phi_{n,k}(x)|$ is continuous in $[a,b]$ since it's the absolute value of a polynomial. Hence there exists $\xi \in [a,b]$ where $\sum_{k=0}^{n} |\phi_{n,k}(x)|$ attains the maximum in $[a,b]$.
Define $f(x_i) = 0$ if $i \in \{0, \cdots n\}$ s.t. $\phi_{n,i}(\xi) = 0$ and $f(x_i) = \frac{|\phi_{n,i}(\xi)|}{\phi_{n,i}(\xi)}$. Obviously the interpolation of

the function between these nodes satisfies $f \in C([a,b])$ and $\|f\| = 1$. Therefore we have

$$\begin{aligned}\max_{x\in[a,b]}\sum_{k=0}^{n}|\phi_{n,k}(x)| &= \sum_{k=0}^{n}|\phi_{n,k}(\xi)| \\ &= \sum_{\phi_{n,k}\neq 0}|\phi_{n,k}(\xi)| \\ &= \sum_{\phi_{n,k}\neq 0}\phi_{n,k}(\xi)\frac{|\phi_{n,k}(\xi)|}{\phi_{n,k}(\xi)} \\ &= \sum_{\phi_{n,k}\neq 0}\phi_{n,k}(\xi)f(x_k) = \sum_{k=0}^{n}\phi_{n,k}(\xi)f(x_k)\end{aligned}$$

hence,

$$\max_{x\in[a,b]}\sum_{k=0}^{n}|\phi_{n,k}(x)| \leq \sup_{||f||_\infty=1}\|\sum_{k=0}^{n}\phi_{n,k}(\xi)f(x_k)\|_\infty = \sup_{||f||_\infty=1}\|\sum_{k=0}^{n}\phi_{n,k}(\xi)f(x_k)\|_\infty = ||L_n||_{op}.$$

This step is done.
Therefore, we proved

$$\|L_n\|_{op} = \max_{x\in[a,b]}\sum_{k=0}^{n}|\phi_{n,k}(x)|,$$

$\|L_n\|_{op}$ is the Lebesgue constant of the corresponding Lagrange interpolation.

□

(b).

We'll show

$$\|S_n\| = \int_{-1}^{1}\Big|\sum_{k=0}^{n}a_kP_k(x)\Big|dx,$$

which is the Lebesgue constants of the Legendre truncation, where P_k are the Legendre polynomials.

(c).

Proof. From the Quadrature Formula,

$$Q_n(f) = \sum_{k=0}^{n}\omega_k f(x_k),$$

also we have

$$\|Q_n\| = \sup_{f\in C([a,b])}\frac{|Q_n(f)|}{\|f\|_\infty} = \sup_{\|f\|=1}|Q_nf|.$$

First we will show that $||Q_n|| \leq \sum_{k=0}^{n}|\omega_k|$, we fix $f \in [a,b], \|f\| = 1$ then we have:

$$|Q_n(f)| = |\sum_{k=0}^{n}\omega_k f(x_k)| \leq \sum_{k=0}^{n}|\omega_k f(x_k)| = \sum_{k=0}^{n}|\omega_k||f(x_k)| \leq^* \sum_{k=0}^{n}|\omega_k|.$$

"$\leq *$" is because $\|f\|_\infty = 1$, also we can take the sup s.t.

$$||Q_n||_\infty \leq \sum_{k=0}^{n} |\omega_k|.$$

The first step is done.

The next step is to prove the opposite direction s.t. $||Q_n||_\infty \geq \sum_{k=0}^{n} |\omega_k|$. Let $f(x_k) = \frac{|\omega_k|}{\omega_k}$, the interpolation of the function between these nodes satisfies $f \in C([a,b])$ and $\|f\| = 1$. Therefore we have

$$\sum_{k=0}^{n} |\omega_k| \leq \sum_{k=0}^{n} \omega_k \frac{|\omega_k|}{\omega_k} = \sum_{k=0}^{n} \omega_k f(x_k) \leq |\sum_{k=0}^{n} \omega_k f(x_k)|.$$

Taking the sup and then we have:

$$\sum_{k=0}^{n} |\omega_k| \leq \sup \sum_{k=0}^{n} |\omega_k| \leq \sup_{||f||_\infty = 1} |\sum_{k=0}^{n} \omega_k f(x_k)| = \sup_{||f||_\infty = 1} |Q_n(f)| = ||Q_n(f)||_\infty.$$

Combined the two steps above we have

$$||Q_n||_\infty = \sum_{k=0}^{n} |\omega_k|.$$

Furthermore, we want to show that $\sum_{k=0}^{n} |\omega_k| = \sum_{n=0}^{k} | \int_a^b \phi_{n,k}(x)dx|$, from the class we know that we have

$$\sum_{k=0}^{n} \omega_k f(x_k) = \int_a^b f(x)\omega(x)dx = \int_a^b f(x)dx$$

is exact when $f(x)$ is a polynomial of degree less than $2n+1$. For $\phi_{n,k}(x)$ of degree n, we have

$$\sum_{k=0}^{n} \omega_k \phi_{n,k}(x_k) = \int_a^b \phi_{n,k}(x)dx,$$

also we know that $\phi_{n,k}(x_k) = 1$ and $\phi_{n,k}(x_j) = 0, j \neq k$, hence we have

$$\omega_k = \int_a^b \phi_{n,k}(x)dx.$$

Therefore,

$$\|Q_n\| = \sum_{k=0}^{n} |\omega_k| = \sum_{k=0}^{n} \Big| \int_a^b \phi_{n,k}(x)dx \Big|.$$

□

Problem 2.

Proof. First we have $\beta'_{n,k}(t) = n(\beta_{n-1,k-1}(t) - \beta_{n-1,k}(t))$ since

$$\begin{aligned}
\beta'_{n,k}(t) &= \frac{d}{dt}\left(\frac{n!}{(n-k)!k!}(1-t)^{n-k}t^k\right) \\
&= -(n-k)\frac{n!}{(n-k)!k!}(1-t)^{n-k-1}t^k + k\frac{n!}{(n-k)!k!}(1-t)^{n-k}t^{k-1} \\
&= n\left(-\frac{(n-1)!}{(n-1-k)!k!}(1-t)^{n-k-1}t^k + \frac{n!}{(n-k)!(k-1)!}(1-t)^{n-k}t^{k-1}\right) \\
&= n\left(\beta_{n-1,k-1}(t) - \beta_{n-1,k}(t)\right).
\end{aligned}$$

Besides we find that $\beta_{n-1,-1}(t) = \beta_{n-1,n}(t) = 0$. Then we find the velocity i.e. derivative of the Bézier curve:

$$\begin{aligned}
B'(t) &= \sum_{k=0}^{n} P_k \beta'_{n,k}(t) \\
&= n \sum_{k=0}^{n} P_k \left(\beta_{n-1,k-1}(t) - \beta_{n-1,k}(t)\right) \\
&= n \sum_{k=0}^{n} P_k \beta_{n-1,k-1}(t) - n \sum_{k=0}^{n-1} P_k \beta_{n-1,k}(t) \\
&= n \sum_{k=0}^{n-1} P_{k+1} \beta_{n-1,k}(t) - n \sum_{k=0}^{n-1} P_k \beta_{n-1,k}(t) \\
&= n \sum_{k=0}^{n-1} (P_{k+1} - P_K) \beta_{n-1,k}(t), 0 \le t \le 1
\end{aligned}$$

$\square$

Problem 3.

(a).

Proof. First we write the function:

$$f(x) = a_{n+1} \left[x^{n+1} - \sum_{i=0}^{n} \frac{-a_i}{a_{n+1}} x^i \right].$$

Then we aim at finding the minimax polynomial $p_n \in \mathbb{P}^n$ for f on the interval $[-1, 1]$. Let $p_n = \sum_{i=0}^{n} b_i x^i$. Then we have

$$f(x) - p_n(x) = a_{n+1} \left[x^{n+1} - \sum_{i=0}^{n} \frac{b_i - a_i}{a_{n+1}} x^i \right].$$

Recall(Theorem): Suppose that $n \ge 0$. The polynomial $p_n \in \mathbb{P}^n$ defined by

$$p_n(x) = x^{n+1} - 2^{-n} T_{n+1} \ , \ x \in [-1, 1]$$

is the minimax approximation of degree n to the function $x \mapsto x^{n+1}$ on the interval $[-1, 1]$, where T_{n+1} is the Chebyshev Polynomial of degree $n + 1$.

Given three properties of Chebyshev Polynomials

(i). For $n \ge 1$, T_n is a polynomial in x of degree n on the interval $[-1, 1]$, with leading coefficient $2^{n-1} x^n$;

(ii). $|T_n(x)| \le 1$ for all $x \in [-1, 1]$ and all $n \ge 0$;

(iii). For $n \geq 1, T_n(x) = \pm 1$, alternatively at $n+1$ points $x_i = \cos(\frac{i\pi}{n}), i = 0, 1, \cdots, n$.

From these properties,

$$x^{n+1} - p_n(x) = 2^{-n} T_{n+1}(x)$$

does not exceed 2^{-n} in the interval $[-1,1]$, and attains this value at the $n+2$ points $x_i = \cos(\frac{i\pi}{n+1}), i = 0, 1, \cdots, n+1$. Therefore, by the Chebyshev Oscillation Theorem, p_n is the unique minimax polynomial approximation from $\mathbb{P}^n$ to the function $x \mapsto x^{n+1}$ on the interval $[-1,1]$.

Therefore, by the theorem above, the $||\cdot||_\infty$ norm of

$$a_{n+1}\left[x^{n+1} - \sum_{i=0}^{n} \frac{b_i - a_i}{a_{n+1}} x^i\right]$$

attains its minimum value when

$$a_{n+1}\left[x^{n+1} - \sum_{i=0}^{n} \frac{b_i - a_i}{a_{n+1}} x^i\right] = x^{n+1} - 2^{-n} T_{n+1}.$$

Therefore, the required minimax polynomial of f is

$$p_n(x) = f(x) - a_{n+1} 2^{-n} T_{n+1}(x).$$

$\square$

(b).

We are going to find the least squares approximation polynomials of degrees $n = 0, 1, 2, 3, 4$ for $f(x) = |x|$ on the interval $(-1, 1)$ with respect to the weight function $w(x) \equiv 1$. Here we deal the current problem with the characteristic property of p_n, where p_n is the least squares approximation of f from $\mathbb{P}^n$, then the error $f - p_n$ must be orthogonal to $\mathbb{P}^n$,

$$\int_{-1}^{1} (f(x) - p_n(x)) x^i dx = 0, i = 0, \cdots, n.$$

(n=0): Let $p_0(x) = c$ we need find a constant c such that $|x| - c$ is orthogonal to all constants:

$$\int_{-1}^{1} (f(x) - c) dx = 0 \implies c \int_{-1}^{1} dx = \int_{-1}^{1} |x| dx$$

$$\implies p_0(x) = \frac{1}{2} \int_{-1}^{1} |x| dx = \frac{1}{2}$$

(n=1): Let $p_1(x) = ax + b$, we need

$$\int_{-1}^{1} (f(x) - ax - b) dx = 0,$$

$$\int_{-1}^{1} (f(x) - ax - b) x dx = 0.$$

implies:

$$\begin{cases} \int_{-1}^{1} |x| dx = a \int_{-1}^{1} x dx + b \int_{-1}^{1} 1 dx \\ \int_{-1}^{1} x|x| dx = a \int_{-1}^{1} x^2 dx + b \int_{-1}^{1} x dx \end{cases} \implies \begin{cases} a = 0 \\ b = \frac{1}{2} \end{cases}$$

Therefore, $p_1(x) = \frac{1}{2} = p_0(x)$

(n=2): Let $p_2(x) = ax^2 + bx + c$, we need

$$\int_{-1}^{1}(f(x) - ax^2 - bx - c)dx = 0,$$
$$\int_{-1}^{1}(f(x) - ax^2 - bx - c)xdx = 0,$$
$$\int_{-1}^{1}(f(x) - ax^2 - bx - c)x^2dx = 0.$$

which implies:

$$\begin{cases} \int_{-1}^{1}|x|dx = a\int_{-1}^{1}x^2dx + b\int_{-1}^{1}xdx + c\int_{-1}^{1}1dx \\ \int_{-1}^{1}x|x|dx = a\int_{-1}^{1}x^3dx + b\int_{-1}^{1}x^2dx + c\int_{-1}^{1}xdx \\ \int_{-1}^{1}x^2|x|dx = a\int_{-1}^{1}x^4dx + b\int_{-1}^{1}x^3dx + c\int_{-1}^{1}x^2dx \end{cases} \implies \begin{cases} \frac{2}{3}a + 2c = 1 \\ b = 0 \\ \frac{2}{5}a + \frac{2}{3}c = \frac{1}{2} \end{cases}$$

Therefore we have $\begin{cases} a = \frac{15}{16} \\ b = 0 \\ c = \frac{3}{16} \end{cases}$, $p_2(x) = \frac{15}{16}x^2 + \frac{3}{16}$.

(n=3): Let $p_2(x) = ax^3 + bx^2 + cx + d$, we need

$$\int_{-1}^{1}(f(x) - ax^3 - bx^2 - cx - d)dx = 0,$$
$$\int_{-1}^{1}(f(x) - ax^3 - bx^2 - cx - d)xdx = 0,$$
$$\int_{-1}^{1}(f(x) - ax^3 - b^2x - cx - d)x^2dx = 0,$$
$$\int_{-1}^{1}(f(x) - ax^3 - b^2x - cx - d)x^3dx = 0.$$

which implies:

$$\begin{cases} \int_{-1}^{1}|x|dx = a\int_{-1}^{1}x^3dx + b\int_{-1}^{1}x^2dx + c\int_{-1}^{1}xdx + d\int_{-1}^{1}1dx \\ \int_{-1}^{1}x|x|dx = a\int_{-1}^{1}x^4dx + b\int_{-1}^{1}x^3dx + c\int_{-1}^{1}x^2dx + d\int_{-1}^{1}xdx \\ \int_{-1}^{1}x^2|x|dx = a\int_{-1}^{1}x^5dx + b\int_{-1}^{1}x^4dx + c\int_{-1}^{1}x^3dx + d\int_{-1}^{1}x^2dx \\ \int_{-1}^{1}x^3|x|dx = a\int_{-1}^{1}x^6dx + b\int_{-1}^{1}x^5dx + c\int_{-1}^{1}x^4dx + d\int_{-1}^{1}x^3dx \end{cases} \implies \begin{cases} \frac{2}{3}b + 2d = 1 \\ \frac{2}{5}a + \frac{2}{3}c = 0 \\ \frac{2}{5}b + \frac{2}{3}d = \frac{1}{2} \\ \frac{2}{7}a + \frac{2}{5}c = 0 \end{cases}$$

Therefore, we have $\begin{cases} a = 0 \\ b = \frac{15}{16} \\ c = 0 \\ d = \frac{3}{16} \end{cases}$, $p_3(x) = \frac{15}{16}x^2 + \frac{3}{16}$.

(n=4): Let $p_2(x) = ax^4 + bx^3 + cx^2 + dx + e$, we need

$$\int_{-1}^{1} (f(x) - ax^4 - bx^3 - cx^2 - dx - e)dx = 0,$$
$$\int_{-1}^{1} (f(x) - ax^4 - bx^3 - cx^2 - dx - e)xdx = 0,$$
$$\int_{-1}^{1} (f(x) - ax^4 - bx^3 - cx^2 - dx - e)x^2dx = 0,$$
$$\int_{-1}^{1} (f(x) - ax^4 - bx^3 - cx^2 - dx - e)x^3dx = 0,$$
$$\int_{-1}^{1} (f(x) - ax^4 - bx^3 - cx^2 - dx - e)x^4dx = 0.$$

which implies:

$$\begin{cases} \int_{-1}^1 |x|dx = a\int_{-1}^1 x^4dx + b\int_{-1}^1 x^3dx + c\int_{-1}^1 x^2dx + d\int_{-1}^1 xdx + d\int_{-1}^1 1dx \\ \int_{-1}^1 x|x|dx = a\int_{-1}^1 x^5dx + b\int_{-1}^1 x^4dx + c\int_{-1}^1 x^3dx + d\int_{-1}^1 x^2dx + d\int_{-1}^1 xdx \\ \int_{-1}^1 x^2|x|dx = a\int_{-1}^1 x^6dx + b\int_{-1}^1 x^5dx + c\int_{-1}^1 x^4dx + d\int_{-1}^1 x^3dx + d\int_{-1}^1 x^2dx \\ \int_{-1}^1 x^3|x|dx = a\int_{-1}^1 x^7dx + b\int_{-1}^1 x^6dx + c\int_{-1}^1 x^5dx + d\int_{-1}^1 x^4dx + d\int_{-1}^1 x^3dx \\ \int_{-1}^1 x^4|x|dx = a\int_{-1}^1 x^8dx + b\int_{-1}^1 x^7dx + c\int_{-1}^1 x^6dx + d\int_{-1}^1 x^5dx + d\int_{-1}^1 x^4dx \end{cases} \implies \begin{cases} \frac{2}{5}a + \frac{2}{3}c + 2e = 1 \\ \frac{2}{5}b + \frac{2}{3}d = 0 \\ \frac{2}{7}a + \frac{2}{5}c + \frac{2}{3}e = \frac{1}{2} \\ \frac{2}{7}b + \frac{2}{5}d = 0 \\ \frac{2}{9}a + \frac{2}{7}c + \frac{2}{5}e = \frac{1}{3} \end{cases}$$

Therefore, we have $\begin{cases} a = -\frac{105}{128}, \\ c = -\frac{105}{64}, \\ e = \frac{15}{128} \end{cases}$, $p_4(x) = -\frac{105}{128}x^4 + \frac{105}{64} + \frac{15}{128}$.

Problem 4.

Proof. First we prove the existence of polynomial q of degree at most $2n+1$. Suppose that $n \geq 1$, we consider two auxiliary polynomials:

$$L_{i,0}(x) = [L_i(x)]^2[1 - 2L_i'(x_i)(x - x_i)] \tag{1}$$

and

$$L_{i,1} = [L_i(x)]^2(x - x_i) \tag{2}$$

with

$$L_i(x) = \prod_{j=1, i\neq j}^{n-1} \frac{x - x_j}{x_i - x_j},$$

we can easily find that $L_{i,0}$ and $L_{i,1}$ are both polynomials of degree at most $2n+1$. It's easy to see that $L_{i,0}(x_j) = L_{i,1}(x_j) = 0$ whenever $i, j \in \{0, 1, \cdots, n\}$ and $i \neq j$. Moreover,

$$L_{i,0}'(x) = 2[1 - 2L_i'(x_i)(x - x_i)]L_i'(x)L_i(x) + [1 - 2L_i'(x_i)][L_i(x)]^2 \tag{3}$$

$$L_{i,1}'(x) = 2L_i'(x)L_i(x)(x - x_i) + [L_i(x)]^2. \tag{4}$$

which implies $L_{i,0}'(x_j) = L_{i,1}'(x_j) = 0$ whenever $i, j \in \{0, 1, \cdots, n\}$ and $i \neq j$. Note that $L_{i,0}(x_j) = 1$ and $L_{i,1}'(x_j) = 1$.
Therefore, let

$$q(x) = \sum_{i=0}^{n} [L_{i,0}(x)f(x_i) + L_{i,1}(x)f'(x_i)],$$

we proved the existence.

Note: Existence of Hermite interpolation polynomial is obtained by constructing the Lagrange form of Hermite polynomial.

Then we prove the uniqueness. Assume there's another polynomials p satisfies the given condition. Let $d(x) = q(x) - p(x)$ such that

$$d(x_j) = 0, d'(x_j) = 0, j = 0, 1, \cdots, n.$$

This implies that $d(x) = 0$ has $n + 1$ distinct solutions.

Recall(Rolle's Theorem): If $f \in C[a, b]$ and f' exists on $[a, b]$, and if $f(a) = f(b) = 0$, there exists a number $\xi \in (a, b)$ such that $f'(\xi) = 0$.

By Rolle's theorem we conclude that $d'(x) = 0$ has $2n + 1$ distinct solutions, which means that $d'(x)$ is identically 0, so $d(x)$ is a constant function, $d(x_j) = 0, j = 0, 1, \cdots, n$ implies that $d(x) \equiv 0$. Therefore, $p(x) \equiv q(x)$, the proof of uniqueness done.

Furthermore, we prove that if $f \in C^{2n+2}([a, b])$, then for any $x \in [a, b]$, there exists $\xi \in [a, b]$ such that

$$f(x) - q(x) = \frac{(x - x_0)^2 \cdots (x - x_n)^2}{(2n + 2)!} f^{(2n+2)}(\xi),$$

which is the error of Hermite interpolation.
It's obvious that the equality is trivial when $x = x_j$ for all $j = 0, 1, \cdots, n$. Hence now we need to consider $x \in [a, b]$ but $x \neq x_j, i = 0, 1, \cdots, n$.

For each x, we define a function

$$\Phi(t) = f(t) - q(t) - \frac{f(x) - q(x)}{\prod_{i=0}^{n}(x - x_i)^2} \prod_{i=0}^{n}(t - x_i)^2 = f(t) - q(t) - \frac{f(x) - q(x)}{W(x)} W(t).$$

Where $W(x) = \prod_{i=0}^{n}(x - x_i)^2$.

Note that $\Phi(x_j) = 0, i = 0, 1, \cdots, n$ and $\Phi(x) = 0$. Applying Rolle's Theorem there are $n + 1$ solutions of $\Phi'(t) = 0$ distinct from $\{x_0, x_1, \cdots, x_n, x\}$, also $\Phi'(x_j) = 0, j = 0, 1, \cdots, n$, thus there exists $2n + 2$ distinct roots of $\Phi'(t)$. We apply Rolle's Theorem repeatedly, we find eventually

$$\Phi^{(2n+2)}(\xi) = 0, \xi \in [a, b],$$

hence we have

$$0 = f^{(2n+2)}(\xi) - \frac{f(x) - q(x)}{W(x)} W^{2n+2}(\xi).$$

Note that $q^{2n+2}(\xi) \equiv 0$ since it's a polynomial of degree $2n + 1$. Moreover, we need to show $W^{(2n+2)}(t) = (2n + 2)!$. We can write $W(t) = t^{2n+2} + r(t)$ where $r(t)$ is a polynomial of degree $2n + 1$, it is easy to see the result. Therefore we have

$$f(x) - q(x) = \frac{(x - x_0)^2 \cdots (x - x_n)^2}{(2n + 2)!} f^{(2n+2)}(\xi).$$

$\square$

Problem 5.

(a).

Proof. Define a function $c = (c_0, c_1, \cdots, c_n) \in \mathbb{R}^{n+1} \mapsto E(c_0, c_1, \cdots, c_n)$ by

$$E(c) = ||f-q||_p = \left(\int_a^b |f-q|^p dx\right)^{\frac{1}{p}} \text{ where } q(x) = c_0 + c_1 x + \cdots + c_n x^n.$$

First we will show that E is continuous for each point $(c_0, c_1, \cdots, c_n) \in \mathbb{R}^{n+1}$.
Pick any $\delta = (\delta_0, \delta_1, \cdots, \delta_n) \in \mathbb{R}^{n+1}$ such that $|\delta_i| \leq \delta, i = 0, \cdots, n$, define $p = \delta_0 + \delta_1 x + \cdots + \delta_n x^n$ then

$$\begin{aligned}
|E(c+\delta) - E(c)| &= \big| ||f-q-p||_p - ||f-q||_p \big| \\
&\leq ||p||_p \text{ by Reverse Triangle Inequality} \\
&= \left(\int_a^b |\delta_0 + \delta_1 x + \cdots + \delta_n x^n|^p dx\right)^{\frac{1}{p}} \\
&\leq \left(\int_a^b |\delta_0|^p dx + \int_a^b |\delta_1 x|^p dx + \cdots + \int_a^b |\delta_n x^n|^p dx\right)^{\frac{1}{p}} \\
&\leq \left(\int_a^b \delta^p dx + \int_a^b \delta^p |x|^p dx + \cdots + \int_a^b \delta^p |x^n|^p dx\right)^{\frac{1}{p}} \\
&= \delta \left(\int_a^b 1 dx + \int_a^b |x|^p dx + \cdots + \int_a^b |x^n|^p dx\right)^{\frac{1}{p}}
\end{aligned}$$

Let $|x^i| \leq M_i, x \in [a,b]$ for $i = 0, \cdots, n$ which means we can bound every $\int_a^b |x^i|^p dx$ since polynomials in a fixed interval exist the upper bound. Let $M := \max_{i=0,\cdots,n} \{M_i\}$, then

$$\begin{aligned}
(*) &\leq \delta[M^p(b-a) + M^p(b-a) + M^p(b-a) + \cdots]^{\frac{1}{p}} \\
&= \delta M[(n+1)(b-a)]^{\frac{1}{p}} \\
&= \epsilon
\end{aligned}$$

Therefore, we conclude that $|E(c+\delta c) - E(c)| \leq \epsilon$ for all $\delta = (\delta_0, \delta_1, \cdots, \delta_n) \in \mathbb{R}^{n+1}$ such that $|\delta_i| \leq \delta, i = 0, \cdots, n$ where

$$\delta = \frac{\epsilon}{M[(n+1)(b-a)]^{\frac{1}{p}}}.$$

Hence E is continuous.

Let $K = \{c \in \mathbb{R}^{n+1} | E(c) \leq ||f||_p + 1\}$. Obviously K is non-empty since $0 \in K$ and closed for K is the pre-image of $[0, ||f||_p + 1]$ under the continuous map $E : \mathbb{R}^{n+1} \to \mathbb{R}$. Then we will show that K is bounded.

Since

$$||q||_p \leq ||q-f||_p + ||f||_p = ||q-f||_p + ||f||_p,$$

we have

$$||c||_p = \kappa ||q||_p \leq \kappa E(c) + \kappa ||f||_p \leq \kappa + 2\kappa ||f||_p$$

if $c \in K$ thus we can say K is bounded. Moreover, E is continuous so E attains its lower bound on K. If $c* \in \mathbb{R}^{n+1} \backslash K$,

$$E(c*) \le E(0) = ||f||_p < ||f||_p + 1 < E(c).$$

Therefore, there exists $c^* \in \mathbb{R}^{n+1}$ such that $E(c^*) = \inf_{c \in K} E(c)$ i.e. for any $f \in C([a,b])$, there exists $g_n \in \mathbb{P}_n$ such that

$$\|f - g_n\|_p = \inf_{q \in \mathbb{P}_n} \|f - q\|_p.$$

We called g_n the minimax polynomial. □

(b).

Proof. To prove the uniqueness we need two more theorems.

Theorem 1. (De la Vallée Poussin's Theorem)
Let $f \in C([a,b])$ and $p \in \mathbb{P}^n$. Suppose that there exist $n+2$ points $x_0 < \cdots < x_{n+1}$ in the interval $[a,b]$, such that $f(x_i) - p(x_i)$ and $f(x_{i+1}) - p(x_{i+1})$ have opposite signs, for $i = 1, \cdots, n$. Then,

$$\min_{p \in \mathbb{P}^n} ||f - q||_p \ge \min_{i=0c,1,\cdots,n+1} |f(x_i) - p(x_i)|$$

Theorem 2. (Chebyshev's Oscillation)
$p \in \mathbb{P}^n$ is a minimax polynomial for $f \in [a,b]$ if and only if $f - p$ takes the value $\pm||f - p||_p$ with alternate changes of sign, at least $n+2$ times in $[a,b]$.

Remark: We proved the De la Vallée Poussin's Theorem and Chebyshev's Oscillation Theorem with infinity norm but they are also hold for p-norm. We don't discuss the details here. Also, we only need Theorem 2 to prove the uniqueness of minimax polynomial but Theorem 1 is essential to the proof of Theorem 2.

Now we prove the uniqueness of the minimax polynomial. Assuming that $q_n \in \mathbb{P}^n$ is also a minimax polynomial for f, and g_n and q_n are distinct. Then we have

$$E_n(f) = ||f - g_n||_p = ||f - q_n||_p = \inf_{q \in \mathbb{P}_n} \|f - q\|_p.$$

Hence by Triangle Inequality,

$$\begin{aligned}
||f - \frac{1}{2}(g_n + q_n)||_p &= ||\frac{1}{2}(f - g_n) + \frac{1}{2}(f - q_n)||_p \\
&\le ||\frac{1}{2}(f - g_n)||_p + ||\frac{1}{2}(f - q_n)||_p \\
&= E_n(f)
\end{aligned}$$

Therefore, $\frac{1}{2}(g_n + q_n)$ is also a minimax polynomial for f on $[a,b]$. By Theorem 2, there exists a sequence of $n+2$ points $x_i, i = 0, \cdots, n+1$, at which

$$|f(x_i) - \frac{1}{2}(g_n(x_i) + q_n(x_i))| = E_n(f), \ i = 0, \cdots, n+1.$$

This implies

$$f(x_i) - g_n(x_i) = f(x_i) - q_n(x_i), \ i = 0, \cdots, n+1,$$

then we can say $g_n = q_n$ at $n+2$ distinct points. Moreover, g_n and q_n are both polynomials of degree at most n, they are identically equal which contradicts to our assumption thus the minimax polynomial is unique. □

(c).

Proof. By Weierstrass Approximation Theorem, for the fixed $\epsilon_0 > 0$, there exists a positive integer N, for $n \geq N$ we have

$$||f - g_n||_\infty = \max_{x\in[a,b]} |g_n - f| < \epsilon_0.$$

Then, for $n \geq N$

$$||g_n - f||_p = \left(\int_a^b |g_n - f|^p dx\right)^{\frac{1}{p}} < \epsilon_0(b-a)^{\frac{1}{p}} = \epsilon,$$

where $\epsilon > 0$ and $\epsilon_0 = \frac{\epsilon}{(b-a)^{\frac{1}{p}}}$. Therefore, g_n converges to f in the p-norm as $n \to \infty$. □

(d).

Now we design an algorithm to compute the minimax polynomial.
(1). For a given function $f(x)$ on an interval $[a, b]$, specify the degree of interpolating polynomial;

$*$ To initialize the the algorithm we need a set of $n+2$ points in the interval $[a, b]$. We use Chebyshev nodes here since the Chebyshev nodes will not lead to Runge Phenomenon and minimize the interpolation error.

(2). Computing the $n + 2$ Chebyshev nodes $x_0, \cdots, x_{n+1}$

$*$ Let the polynomial which pass through the Chebyshev nodes be:

$$P_n(X) = b_0 + b_1x + \cdots + b_nx^n$$

where $b_i, i = 0, \cdots, n$ are coefficients. Then we need to enforce the oscillation criteria on this polynomial since we want the error between the polynomial and the function f to oscillate alternatively at the Chebyshev nodes.

(3). Solve the system of equations:

$$b_0 + b_1x_i + b_2x_i^2 + \cdots + b_nx_i^n + (-1)^iE = f(x_i), i = 0, 1, \cdots, n$$

to get $b_0, \cdots, b_n, E$.

$*$ We have now enforced the oscillation criteria but note that the error E is not necessarily the extrema of the error function. By Theorem 2, the error function alternate sign in $n + 2$ points and from the Intermediate Value Theorem, there are $n + 1$ roots for the error function. We can compute the roots using any numerical method and consider the $n + 2$ intervals

$$[a, z_1], [z_0, z_1], \cdots, [z_{n-1}, z_n], [z_n, b]$$

where $z_0, \cdots, z_n$ are $n+1$ roots. For each interval above, we find the point at which the error functions attains its extrema. We can do this by differentiating the error function and locating the minimum or maximum in each interval. If the minimum or maximum doesn't exist, we compute the value of the error at the two end points and take the one with the largest absolute value. This provides us with a new set of points $x_0^*, \cdots, x_{n+1}^*$

(4). Compute the extremes of $E_n(f)$. This will give a new set of control points $x_0^*, \cdots, x_{n+1}^*$.

$*$ This new set of points will be used in the second step of iteration. We continue the iteration until a stopping criteria is met. Now we find the stopping criteria. We now evaluate the error at these control points. Let $E_m = \min_i |E_i|$ and $E_m = \max_i |E_i|$. As we converge and approach the minimax polynomial, the difference between the old and new set of control points is minimized. Hence a reasonable stopping criteria is to stop the iteration when $E_m \approx E_M$.

(5). If the stopping criteria is hold then stop the iteration. Otherwise, repeat step 3 to find new control points.

Remark:

I have worked this assignment with David Knapik and Luke Steverango.