Average Dwell-Time Bounds for ISS and Integral ISS of Switched Systems using Lyapunov Functions

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Overview

Introduction

Background

- Over the last couple of decades, switched systems have received a great deal of attentions due to their significance both in theory and in practical applications [Liberzon, 2003, Lin and Antsaklis, 2009].
- Generally, a switched system consists of a family of subsystems and a rule that governs the switching between them.
- A canonical approach for stability analysis of such systems is to develop characterizations based on the stability properties of the individual subsystems, and the effects observed at switching times.

Literature review

- For dynamical systems with inputs, the notion of input-to-state stability (ISS) [Sontag, 1989], or its close variant integral ISS (iISS) [Angeli et al., 2000], provide an elegant method to quantify the performance in control related applications.
- ISS or iISS for switched systems under arbitrary switching were studied in the literature [Mancilla-Aguilar and García, 2001, Haimovich and Mancilla-Aguilar, 2018].
- ISS for switched systems under slow switching were also studied in [Vu et al., 2007, Müller and Liberzon, 2012, Zhang and Tanwani, 2019].
- An iISS-related property for switched systems under slow switching was recently studied in [Russo et al., 2020].
- Stability analysis of switched systems is also related to stability analysis of impulsive systems [Hespanha et al., 2008] and hybrid systems [Cai and Teel, 2009, Noroozi et al., 2017].

Contribution of this work

- Lyapunov conditions for iISS switched systems under slow switching were not studied in the literature;
- In most of the aforementioned work, in order for the switched system to be ISS, the decay rate of the subsystems needs to be linear. This assumption is invalid when dealing with iISS subsystems.
- Our work provides a set of very general assumptions such that the switched system is ISS/iISS under slow switching, and a formula for computing a lower bound on the average dwell-time for the switching signal.

Switching signals and switched systems

Switching signals

- Mode index set: $\mathsf{P} \subset \mathbb{N}$ is a set of either finite or infinite cardinality.
- Let $\bar{\Sigma}$ denote the collection of all right-continuous, piece-wise constant functions from $\mathbb{R}_{\geqslant 0}$ to P.
- For any $\sigma \in \bar{\Sigma}$, denote $\mathcal{T}(\sigma) := \{t > 0 : \sigma(t) \neq \sigma(t^-)\}$.
- Introduced in [Hespanha and Morse, 1999], a switching signal $\sigma \in \bar{\Sigma}$ satisfies the average dwell-time (ADT) condition with parameters $\tau_a > 0$, $N_0 \geqslant 0$ if

$$\forall t_2 \geqslant t_1 \geqslant 0 : \quad N_\sigma(t_1, t_2) \leqslant N_0 + \frac{t_2 - t_1}{\tau_a}, \tag{1}$$

  where $N_\sigma(t_1, t_2) := |(t_1, t_2] \cap \mathcal{T}(\sigma)|$. The set $\Sigma(\tau_a, N_0) \subset \bar{\Sigma}$ denotes the collection of such switching signals.

Switched systems

The dynamics of the switched system is given by

$$\dot{x}(t) = f_{\sigma(t)}(x(t), \omega(t)) \qquad \text{if } t \notin \mathcal{T}(\sigma), \tag{2a}$$

$$x(t) = g_{\sigma(t^-),\sigma(t)}(x(t^-), \omega(t)) \text{ if } t \in \mathcal{T}(\sigma). \tag{2b}$$

- The functions $f_p : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n, p \in \mathrm{P}$ are the continuous flow maps, and
- the functions $g_{q,p} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n, (p, q) \in \mathrm{P} \times \mathrm{P}$ are the discrete jump maps.
- The input $\omega : \mathbb{R}_{\geqslant 0} \to \mathbb{R}^m$ is locally essentially bounded on $\mathbb{R}_{\geqslant 0}$ and bounded on $\mathcal{T}(\sigma)$.

Stability definitions for switched system

Definition 1

A switched system (2) is uniformly input-to-state stable (ISS) over $\Sigma \subset \bar{\Sigma}$ if there exist $\beta \in \mathcal{KL}, \gamma_1, \gamma_2 \in \mathcal{K}$ such that

$$|x(t; x_0, \omega, \sigma)| \leqslant \beta(|x_0|, t) + \gamma_1\Big(\operatorname*{ess\,sup}_{s \in [0,t] \setminus \mathcal{T}(\sigma)} |\omega(s)|\Big) + \gamma_2\Big(\sup_{s \in [0,t] \cap \mathcal{T}(\sigma)} |\omega(s)|\Big) \tag{3}$$

for all $t \geqslant 0, x_0 \in \mathbb{R}^n$ and $\sigma \in \Sigma$.

Definition 2

A switched system (2) is uniformly integral input-to-state stable (iISS) over $\Sigma \subset \bar{\Sigma}$ if there exist $\alpha_0 \in \mathcal{K}_\infty, \beta \in \mathcal{KL}, \gamma_1, \gamma_2 \in \mathcal{K}$ such that

$$\alpha_0(|x(t; x_0, \omega, \sigma)|) \leqslant \beta(|x_0|, t) + \int_0^t \gamma_1(|\omega(s)|) ds + \sum_{s \in [0,t] \cap \mathcal{T}(\sigma)} \gamma_2(|\omega(s)|) \tag{4}$$

for all $t \geqslant 0, x_0 \in \mathbb{R}^n$, and $\sigma \in \Sigma$.

Why slow switching?

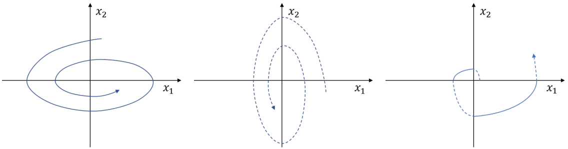
- The switched system is ISS/iISS under arbitrary switching if it is uniformly ISS/iISS over $\bar{\Sigma}$; the switched system is ISS/iISS under slow switching if there exist $\tau_a > 0, N_0 \geqslant 1$ such that it is uniformly ISS/iISS over $\Sigma(\tau_a, N_0)$.
- When all the subsystems of a switched system are ISS/iISS, it may not be ISS/iISS under arbitrary switching.
- The switched system may not be ISS/iISS under slow switching either.

Why slow switching? (contd.)

- Consider the system given by

$$\dot{x} = A_p x + B_p u, \quad p = 1, 2, \tag{5}$$

$$A_1 = \begin{pmatrix} -0.1 & -1 \\ 2 & -0.1 \end{pmatrix}, \; A_2 = \begin{pmatrix} -0.1 & -2 \\ 1 & -0.1 \end{pmatrix}. \tag{6}$$



(a) Mode 1 only (b) Mode 2 only (c) An unstable switched case

Figure: Solution trajectories of (5) when $u \equiv 0$. Solid curve for $\sigma = 1$ and dashed curve for $\sigma = 2$.

- The subsystems are both ISS, but the switched system is not ISS under arbitrary switching. However, the switched system can be shown ISS under slow switching.

Is ADT slow switching good enough?

- Consider the system studied in [Russo et al., 2020], given by

$$\dot{x} = \frac{1}{1+|x|^2} A_p x + u, \quad p = 1, 2 \tag{7}$$

  where A_1, A_2 are given by (6).
- The subsystems are both iISS.
- The vector fields of the subsystems of (7) look similar to the ones of the subsystems of (5), but the magnitude is scaled. The larger $|x|$, the smaller $|\dot{x}|$.
- By picking x_0 far away enough from the origin, $|\dot{x}|$ will be small enough and hence we can always find a switching signal with sufficiently large dwell-time such that the unforced solution trajectory of (7) with initial state x_0 is the same as the one in Fig. 1c and diverges.
- The switched system is not iISS under ADT switching for any τ_a.
- We would like to derive sufficient conditions in nonlinear setting such that the switched system inherits the iISS/ISS property from its subsystems under slow switching.

Main results

Assumptions

Assumption 1

There exist $\mathcal{C}^1$ Lyapunov functions $V_p : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}, p \in \mathrm{P}$, satisfying the conditions:

(L1) For some $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$, we have

$$\underline{\alpha}(|x|) \leqslant V_p(x) \leqslant \overline{\alpha}(|x|), \quad \forall x \in \mathbb{R}^n, p \in \mathrm{P}. \tag{8}$$

(L2) For some positive definite function α and $\gamma \in \mathcal{K}$, we have

$$\left\langle \frac{\partial}{\partial x} V_p(x), f_p(x, \omega) \right\rangle \leqslant -\alpha(V_p(x)) + \gamma(|\omega|) \quad \forall x \in \mathbb{R}^n, \omega \in \mathbb{R}^m, p \in \mathrm{P}. \tag{9}$$

(L3) For some $\chi \in \mathcal{K}_\infty, \rho \in \mathcal{K}$, we have

$$V_p(g_{q,p}(x, \omega)) \leqslant \chi(V_q(x)) + \rho(|\omega|) \quad \forall x \in \mathbb{R}^n, \omega \in \mathbb{R}^m, p, q \in \mathrm{P}. \tag{10}$$

Assumptions (contd.)

Based on Assumption 1, we introduce the function $\psi : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ as

$$\psi(t) := \min_{s \in [0,t]} \{\alpha(s) + c(t - s)\} \tag{11}$$

where $c > 0$ is some constant.
The second assumption relates to lower bound on ADT appearing in [Zhang and Tanwani, 2019].

Assumption 2

The supremum

$$\zeta^* := \sup_{s>0} \int_s^{\chi(s)} \frac{1}{\psi(r)} dr \tag{12}$$

is finite, where χ comes from (L3) in Assumption 1 and ψ is defined in (11).

Main theorem

Theorem 1

Consider the switched system (2) *and suppose that both Assumption 1 and Assumption 2 hold. Then the system* (2) *is uniformly iISS over* $\Sigma(\tau_a, N_0)$ *for any* $\tau_a > \zeta^*, N_0 \geqslant 0$. *In addition, if in (L2)* $\alpha \in \mathcal{K}_\infty$, *then the system* (2) *is uniformly ISS over* $\Sigma(\tau_a, N_0)$ *for any* $\tau_a > \zeta^*, N_0 \geqslant 0$.

Special cases

- When $\alpha(s) = \lambda s$ and $\chi(s) = \mu s$ for some $\lambda, \mu > 0$, Theorem 1 implies that the switched system is uniformly ISS when $\tau_a > \frac{\ln \mu}{\lambda}$, same as in [Vu et al., 2007].
- On the other hand, when there is a common Lyapunov function and the switches do not introduce jumps to the states, we have $\chi(s) = s$. Consequently it can be concluded from (12) that $\zeta^* = 0$ and hence Theorem 1 implies that the switched system is iISS/ISS over arbitrary switching, known in [Mancilla-Aguilar and García, 2001].

Sketch of proof

- The key idea to prove Theorem 1 is to model the switched system with ADT switching signals as a special type of hybrid system in the framework of [Goebel et al., 2012], and show that this hybrid system is ISS/iISS using Lyapunov characterizations.
- The hybrid ISS/iISS Lyapunov function used is

$$V(x, \sigma, \tau) := \varphi^{-1}(e^{2c\zeta\tau}\varphi(V_\sigma(x))) \tag{13}$$

where

$$\varphi(s) := \begin{cases} \exp\left(\int_1^s \frac{2c}{\psi(r)} dr\right) & s > 0, \\ 0 & s = 0. \end{cases} \tag{14}$$

Case study: switched bilinear system

Switched bilinear system

- Consider the switched system with Hurwitz matrices $A_p, p \in \mathrm{P} = \{1, \cdots N\}$:

$$\dot{x} = A_\sigma x + uBx \qquad \text{if } t \notin \mathcal{T}(\sigma), \tag{15a}$$

$$x = Dx^- + uEx^- \qquad \text{if } t \in \mathcal{T}(\sigma), \tag{15b}$$

- All subsystems are iISS (see, e.g., [Chaillet et al., 2014]) by picking the iISS-Lyapunov function $V_p(x)$ as

$$V_p(x) := \ln(1 + x^\top M_p x) \tag{16}$$

where M_p are the solutions of

$$M_p A_p + A_p^\top M_p + I = 0. \tag{17}$$

Verifying Assumption 1

- The condition (L1) ("sandwich" bounds on V_p) is satisfied with $\underline{\alpha}(|x|) := \ln(1 + \min_{p\in P} \underline{\sigma}(M_p)|x|^2)$, $\bar{\alpha}(|x|) := \ln(1 + \max_{p\in P} \bar{\sigma}(M_p)|x|^2)$.
- For the continuous flow, it can be shown that

$$\left\langle \frac{\partial}{\partial x} V_p(x), f_p(x,u) \right\rangle \leqslant -\frac{(e^{V_p} - 1)}{\bar{\sigma}(M_p)e^{V_p}} + \frac{\bar{\sigma}(M_p B)}{\underline{\sigma}(M_p)}|u|.$$

  Hence the condition (L2) is satisfied with $\alpha(s) = \frac{1-e^{-s}}{\sigma_{\max}}$, $\gamma(s) = c_1 s$, where $\sigma_{\max} = \max_{p\in P} \bar{\sigma}(M_p)$, $c_1 = \max_{p\in P} \frac{\bar{\sigma}(M_p B)}{\underline{\sigma}(M_p)}$.
- For the discrete jump, let $\mu > 0$ be such that $x^\top D^\top M_p D x \leqslant \mu x^\top M_q x$ for all $x \in \mathbb{R}^n$ and $p, q \in P$. It can be computed that

$$V_p(g_{q,p}(x,u)) \leqslant \ln\left(1 + \mu\left(e^{V_q(x)} - 1\right)\right) + \ln\left(1 + \frac{2\bar{\sigma}(E^\top M_p D)|u| + \bar{\sigma}(E^\top M_p E)|u|^2}{\mu\underline{\sigma}(M_q)}\right)$$

  Hence the condition (L3) is satisfied with $\chi(s) = \ln\left(1 + \mu\left(e^s - 1\right)\right)$, $\rho(s) = \ln(1 + c_2 s + c_3 s^2)$, where $c_2 = \frac{2\max_{p\in P} \bar{\sigma}(E^\top M_p D)}{\mu \min_{q\in P} \underline{\sigma}(M_q)}$, $c_3 = \frac{2\max_{p\in P} \bar{\sigma}(E^\top M_p E)}{\mu \min_{q\in P} \underline{\sigma}(M_p)}$.

Verifying Assumption 2

$$\alpha(s) = \frac{1 - e^{-s}}{\sigma_{\max}}, \chi(s) = \ln\left(1 + \mu\left(e^s - 1\right)\right).$$

Because α is globally Lipschitz, we can let $\psi = \alpha$ by setting c sufficiently large. To check Assumption 2, it is computed by (12) that

$$\begin{aligned}
\zeta^* &= \sup_{s>0} \int_s^{\chi(s)} \frac{1}{\psi(r)} dr = \sup_{s>0} \int_s^{\chi(s)} \frac{\sigma_{\max} e^r}{(e^r - 1)} dr \\
&= \sigma_{\max} \sup_{s>0} \ln(e^r - 1)\Big|_s^{\ln(\mu(e^s-1)+1)} \\
&= \sigma_{\max} \sup_{s>0} \ln \frac{(\mu(e^s - 1) + 1) - 1}{e^s - 1} \\
&= \sigma_{\max} \ln \mu
\end{aligned}$$

Hence by Theorem 1, the system (15) is uniformly iISS over $\Sigma(\tau_a, N_0)$ for all $\tau_a > \sigma_{\max} \ln \mu$, $N_0 \geqslant 0$.

Generalization

The same arguments also work for switched bilinear systems with resets and stable subsystems in general, in the form

$$\dot{x} = A_\sigma x + \sum_{j=1}^{m_c} B_{\sigma,j} x \omega_j + C_\sigma \omega, \qquad \text{if } t \notin \mathcal{T}(\sigma), \tag{18a}$$

$$x = D_{(\sigma^-,\sigma)} x^- + \sum_{k=1}^{m_d} E_{(\sigma^-,\sigma),k} x^- \omega_k + F_{(\sigma^-,\sigma)} \omega \qquad \text{if } t \in \mathcal{T}(\sigma), \tag{18b}$$

Proposition 2

Consider the switched system (18) with finite modes and assume that A_p are Hurwitz for all $p \in \mathrm{P}$ such that there exist positive definite symmetric matrices $M_p, Q_p \in \mathbb{R}^{n \times n}$ and $A_p^\top M_p + M_p A_p + Q_p = 0$ hold. Let $\lambda := \min_{p \in \mathrm{P}} \frac{\underline{\sigma}(Q_p)}{\overline{\sigma}(M_p)}$ and $\mu > 0$ be such that $x^\top D_{(p,q)}^\top M_p D_{(p,q)} x \leqslant \mu x^\top M_q x$ for all $x \in \mathbb{R}^n$ and $p, q \in \mathrm{P}$. Then (18) is uniformly iISS over $\Sigma(\tau_a, N_0)$ for all $\tau_a > \frac{\ln \mu}{\lambda}$ and $N_0 \geqslant 0$.

Conclusion

Conclusion

- Within the context of stability of switched systems under slow switching, our work provides lower bounds on ADT which guarantee ISS or iISS properties.
- In particular, we considered the case where each subsystem is ISS (resp. iISS). It is seen that switched bilinear systems indeed fall within the framework studied in this paper, and we have provided conditions under which slowly switching bilinear systems are seen to be iISS.
- Among several possible extensions, it is desirable to adopt this framework under relaxed hypotheses which allow for destabilizing effect of disturbances in continuous dynamics, and a wider class of supply functions associated with individual subsystems.
- Since iISS property has found utility in analyzing interconnections of dynamical systems [Arcak et al., 2002], one can also use the results of this paper to analyse stability of interconnections of switched systems under slow switching.

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