Nonlinear and Switched Systems: Geometric Motion Planning, Non-monotonic Lyapunov Functions and Input-to-State Stability

Final Exam

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Overview

1. Motion planning via geometric approach
   - Review
   - Motion planning for systems with drift
   - Extensions

2. Stability analysis via non-monotonic Lyapunov function
   - Review
   - Show ISS via almost Lyapunov functions
   - Show ISS via higher order derivatives of Lyapunov functions

Motion planning via geometric approach

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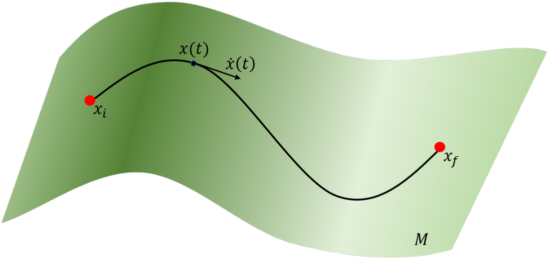
Motion planning problem

Given a system

$$\dot{x} = f(x, u) \tag{1}$$

and two points $x_i, x_f \in M$, find a control $u^*(t)$ that steers the system from x_i to x_f in T units of time.



Literature review

- Motion planning has been widely studied (see, e.g., [Laumond, 1998], [LaValle, 2006])
- One of the early control papers which addresses the issue of motion planning for non-holonomic systems is [Brockett, 1982], where motion planning is stated as a sub-Riemannian geodesic problem. See also the monograph [Jean, 2014] for a recent survey of this line of work.
- Other motion planning methods include but are not limited to LQR-tree method [Tedrake et al., 2010], sum-of-square techniques [Majumdar and Tedrake, 2013]. motion primitives [Murphey, 2006] [Woodruff and Lynch, 2017], random sampling-based [Karaman and Frazzoli, 2011], graph-based [Kuffner et al., 2003] and optimization-based approaches [Dai et al., 2014], etc.

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Challenges

Difficulties in motion planning

- non-holonomic dynamics,
- drift,
- constraints on the inputs/states.

The geometric approach proposed in our work can address all three difficulties.

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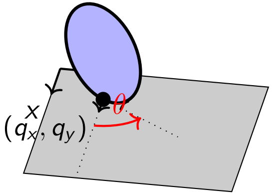
Core idea

"Deforming" a curve in order to make it feasible.



$$\underbrace{\begin{pmatrix} \dot{q}_x \\ \dot{q}_y \\ \dot{\theta} \end{pmatrix}}_{\dot{x}} = \underbrace{\begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}}_{f_1} u_1 + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{f_2} u_2. \qquad (2)$$

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"Deforming" a curve in order to minimize its "length".

- What is "length"? Answer: Riemannian metric – encodes dynamics, constraints, etc.
- How to "deform"? Answer: Homotopies – achieved by solving PDEs.

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Riemannian metric

- A Riemannian metric on M is a family of positive definite matrices $G(x), x \in M$.
- A curve on M has length $\mathcal{L} = \int_0^T \sqrt{\dot{x}^\top G(x)\dot{x}}dt$ w.r.t. Riemannian metric G. e.g., identity matrix $G(x) \equiv I$ gives usual Euclidean length.
- Consider the driftless system affine in control

$$\dot{x} = F(x)u \tag{3}$$

  To encode non-holonomic constraints, we set

$$G(x) = (\bar{F}(x)^{-1})^\top D\bar{F}(x)^{-1},$$

  where $D = \mathrm{diag}(\underbrace{\lambda, \cdots, \lambda}_{n-m}, \underbrace{1, \cdots, 1}_{m})$ for some large $\lambda > 0$ and $\bar{F}(x) = (F_c(x)|F(x)) \in \mathbb{R}^{n\times n}$ so that it is full rank.
- $\dot{x}^\top G(x)\dot{x} \approx \left(\left|\mathbf{P}_{\mathcal{F}}\dot{x}\right|^2 + \lambda\left|\mathbf{P}_{\mathcal{F}^\perp}\dot{x}\right|^2\right)$

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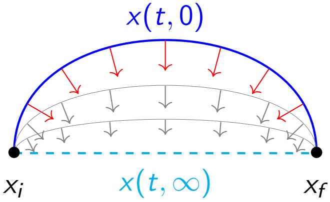
Geometric heat flow (GHF)

For the driftless system (3),

$$\frac{\partial x(t,s)}{\partial s} = \nabla_{\dot{x}(t,s)}\dot{x}(t,s) \tag{4}$$

yields a curve of minimal length when $s \to \infty$ [Jost, 1995].



$$\nabla_f g := \frac{df}{dt} + \sum_{i,j,k} \Gamma^k_{ij} f_i g_j e^k,$$

$$\Gamma^i_{jk}(x) := \frac{1}{2}\sum_l (G^{-1})_{il}\left(\frac{\partial G_{lj}}{\partial x_k} + \frac{\partial G_{lk}}{\partial x_j} - \frac{\partial G_{jk}}{\partial x_l}\right)$$

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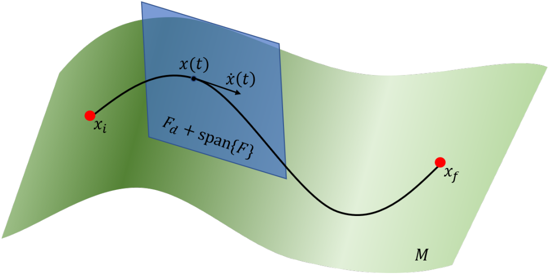
System with affine controls

$$\dot{x} = F_d(x) + F(x)u \tag{5}$$



Assumption A

Both $F_d(x), F(x)$ are assumed to be at least C^2, Lipschitz with constants L_1, L_2 respectively;
$F(x)$ is of rank m almost everywhere on M.

Affine geometric heat flow

Taking inspiration from the GHF, we introduce AGHF as below:

$$\frac{\partial x(t,s)}{\partial s} = \nabla_{\dot{x}(t,s)} \left(\dot{x}(t,s) - F_d\right) + r(x(t,s), \dot{x}(t,s)) \tag{6}$$

where

$$r(x,\dot{x}) = G^{-1} \left(\left(\frac{\partial F_d}{\partial x}\right)^{\top} G(\dot{x} - F_d) + \frac{1}{2} \begin{pmatrix} (\dot{x} - F_d)^{\top} \frac{\partial G}{\partial x_1} F_d \\ \vdots \\ (\dot{x} - F_d)^{\top} \frac{\partial G}{\partial x_n} F_d \end{pmatrix} \right)$$

Geometric interpretation

$$\frac{\partial x(t,s)}{\partial s} = \nabla_{\dot{x}(t,s)} (\dot{x}(t,s) - F_d) + r(x(t,s), \dot{x}(t,s)) \tag{6}$$

- $\nabla_{\dot{x}} (\dot{x} - F_d)$ is the covariant derivative of $\dot{x} - F_d$ in the direction $\dot{x}$, which updates the curve in the direction of decreasing its "curvature" and hence minimizing the "length".
- $-r(x, \dot{x})$ is the scaled gradient of the point-wise map

$$P_f : M \to \mathbb{R} : x \mapsto \langle F_d(x) - f, F_d(x) \rangle$$

with $f = \dot{x}$ and $\langle f, h \rangle := f^\top G(x) h$. This map reaches its minimal value when $F_d(x)$ is aligned with $\dot{x}$.
- $\mathcal{L} = \int_0^T g(x(t), \dot{x}(t))^{\frac{1}{2}} dt$ where $g(x, \dot{x}) \approx \left(\left| \mathbf{P}_{\mathcal{F}} (\dot{x} - F_d(x)) \right|^2 + \lambda \left| \mathbf{P}_{\mathcal{F}^\perp} (\dot{x} - F_d(x)) \right|^2 \right)$.

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Convergence of AGHF

Our AGHF minimizes the action functional

$$\mathcal{A}(x(\cdot)) := \frac{1}{2}\int_0^T (\dot{x} - F_d(x))^\top G(x)(\dot{x} - F_d(x))dt \tag{7}$$

Lemma

Let $x^*(t)$ be a steady-state solution of the AGHF (6). Then $x^*(t)$ is an extremal curve for $\mathcal{A}$ in (7). Furthermore, $\mathcal{A}$ decreases along the solutions of the AGHF; i.e. if $x(t,s)$ is such a solution, then $\frac{d}{ds}\mathcal{A}(x(\cdot,s)) \leq 0$, and equality holds only if $x(\cdot,s)$ is an extremal curve for $\mathcal{A}$.

Algorithm

Step 1: Encode system dynamics into the Riemannian metric G;

Step 2: Solve the AGHF (6) with boundary conditions

$$x(0,s) = x_i, x(T,s) = x_f \quad \forall s \geq 0$$

and an initial condition

$$x(t,0) = y(t), \quad t \in [0,T]$$

for some $y(\cdot) \in \mathcal{X}'$;

Step 3: Evaluate

$$u(t) := F(x(t,s_{\max}))^{\dagger}(\dot{x}(t,s_{\max}) - F_d(x(t,s_{\max})). \qquad (8)$$

Output: The control $u(t)$ obtained in (8) is our solution to the motion planning problem. When integrating (5) with initial state x_i and input $u(t)$, the integrated path $\tilde{x}(t)$ approximately ends with $\tilde{x}(T) \approx x_f$.

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On convergence guarantees for motion planning

Theorem

Consider the system (5) and let $x_i, x_f \in \mathbb{R}^n$. Assume that the motion planning problem from x_i to x_f is feasible and that Assumption A is met. Then there exists $C > 0$ such that for any $\lambda > 0$, there exists an open set $\Omega_\lambda \subseteq \mathcal{X}'$ (with respect to $\|\cdot\|_{AC}$) so that as long as the initial curve $y \in \Omega_\lambda$, the integrated path $\tilde{x}(t)$ from our algorithm with sufficiently large $s_{\max}$ has the property that

$$|\tilde{x}(T) - x_f| \leq \sqrt{\frac{3TC}{\lambda}} \exp\left(\frac{3T}{2}(L_2^2 T + L_1^2 C)\right). \tag{9}$$

Remarks

$$|\tilde{x}(T) - x_f| \leq \sqrt{\frac{3TC}{\lambda}} \exp\left(\frac{3T}{2}(L_2^2 T + L_1^2 C)\right) \tag{9}$$

- Our algorithm gives a solution to the relaxed motion planning problem:

Relaxed motion planning problem

Given $x_i, x_f \in \mathbb{R}^n$, $T > 0, \epsilon > 0$, find an integrable u (potentially continuous u) such that the corresponding solution of (5) with initial condition $x(0) = x_i$ satisfies $|x(T) - x_f| \leq \epsilon$.

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Example: unicycle with constant linear velocity

$$\underbrace{\begin{pmatrix} \dot{q}_x \\ \dot{q}_y \\ \dot{\theta} \end{pmatrix}}_{\dot{x}} = \underbrace{\begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}}_{F_d} + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{F} u. \tag{10}$$

Figure: Two different scenarios for parallel parking. Both start with $x_i = (0, 0, 0)^\top$. Left: $x_f = (0, 1, 0)^\top$. Right: $x_f = (0, 1, 2\pi)^\top$

Example: dynamic unicycle

$$\underbrace{\begin{pmatrix} \dot{q}_x \\ \dot{q}_y \\ \dot{\theta} \\ \dot{u}_1 \\ \dot{u}_2 \end{pmatrix}}_{\dot{x}} = \underbrace{\begin{pmatrix} u_1 \cos\theta \\ u_1 \sin\theta \\ u_2 \\ 0 \\ 0 \end{pmatrix}}_{F_d} + \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}}_{F} \underbrace{\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}}_{v} \tag{11}$$

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Extensions

Our algorithm can also be modified to tackle variants of the basic motion planning problems, including but not limited to state constraints, input constraints, holonomic constraints, indefinite boundary conditions and free terminal time.

State constraints (obstacles)

- Suppose $\Omega \subset M$ is the set of obstacles that the system should avoid. Design a barrier function $b : M \backslash \Omega \to \mathbb{R}$ such that
  1. $b(x)$ is positive and differentiable for all $x \in M \backslash \Omega$
  2. $b(x) \to \infty$ as $x \to \partial\Omega$,
  3. $b(x) = 1$ when x is far away from Ω.
- Let the new Riemannian metric be

$$G(x) = b(x)(\bar{F}(x)^{-1})^{\top} D\bar{F}(x)^{-1},$$

- Intuitively, $G(x)$ becomes larger when x is close to obstacles. Since our algorithm minimizes the "length" of the curve, AGHF deforms the curve such that it avoids obstacles while still satisfying the dynamic constraints.

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Input constraints

For the general control system with input constraints

$$\begin{aligned} \dot{x} &= f(x, u) \\ l(x(t), u(t)) &\geq 0, \quad \forall t \in [0, T], \end{aligned} \tag{12}$$

- Define $\dot{u} = v$ and also define the augmented state $y = \begin{pmatrix} x \\ u \end{pmatrix} \in \mathbb{R}^{n+m}$, then we can set

$$\dot{y} = \begin{pmatrix} \dot{x} \\ \dot{u} \end{pmatrix} = \underbrace{\begin{pmatrix} f(x, u) \\ 0 \end{pmatrix}}_{F_d} + \underbrace{\begin{pmatrix} 0 \\ I_{m \times m} \end{pmatrix}}_{F} v, \tag{13}$$

  which is a system with affine control and drift, similar to (5).
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Holonomic constraints

- Holonomic constraints can be written as $q_i(x) = 0, \quad i = 1, 2, \cdots, k$, which, after differentiating with respect to time, give $\nabla q_i(x) \cdot \dot{x} = 0, \quad i = 1, 2, \cdots, k$.
- An affine control system with drift and holonomic cosntraints can be written as

$$\begin{aligned} \dot{x} &= F_d(x) + F(x)u, \\ H(x)\dot{x} &= 0. \end{aligned} \tag{14}$$

- Solution exists if and only if $H(x)F_d(x) \in \operatorname{span} H(x)F(x)$.
- When solution exists, the motion planning problem of system (14) is equivalent to the motion planning problem for the system

$$\dot{x} = \tilde{F}_d(x) + \tilde{F}(x)v \tag{15}$$

where $\tilde{F}_d(x) = \Big(I + (H(x)F(x))^{\dagger}H(x)\Big)F_d(x)$ and $\tilde{F}(x) = F(x)A(x)$, $A(x) = \{a_1(x), \cdots, a_l(x)\}$ is the basis of $\ker H(x)F(x)$.

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Indefinite boundary condition

- For physical systems with no external inputs, the conservation of momentum imposes constraints on the initial and finial configuration of the system.
- Arbitrary choice of the pair $x_i, x_f \in M$ may result in no solution; the relation between x_i, x_f is implicit.
- Let $S_{bc} \subset S := \{1, \cdots, d\} \times \{0, T\}$. For element-wise specified boundary states:

$$x_i(t, s) = x^{bc}(i, t) \quad \forall (i, t) \in S_{bc}, s \geq 0 \tag{16}$$

  For unspecified boundary states:

$$\frac{\partial L}{\partial (x_t)_i}(x(t, s), x_t(t, s)) = 0 \quad \forall (i, t) \in S \backslash S_{bc}, s \geq 0 \tag{17}$$

- (16) together with (17) give the correct number of boundary conditions when solving our AGHF (6).

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Free terminal time

- For a driftless affine system, If $u(\cdot) : [0,1] \to \mathbb{R}^m$ gives an admissible path satisfying the boundary condition $x(0) = x_i, x(1) = x_f$, then $\frac{1}{T} u\left(\frac{t}{T}\right)$ gives a time scaled admissible path with $x'(0) = x_i, x'(T) = x_f$.
- This is not true for systems with drift; minimization of the action functional $\mathcal{A}$ in (7) as studied in Lemma 1 should be with respect to T as well.
- In addition, unlike the driftless case, the reachable space of an affine system with drift or constrained inputs may be related to the terminal time T.
- To tackle free terminal time motion planning, we have introduced a true time state τ and under smoothness assumptions, let $\dot{\tau}(t) = a(t)^2$, $\dot{a}(t) = u_0(t)$.
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Summary

Motion Planning Problem	Appears in
Driftless	ACC '17
Driftless + Obstacle	
Driftless + Holonomic constraints	arXiv
Drift	NOLCOS '19
Drift + Obstacle	
Drift + Input constraints	
Drift + Holonomic constraints	Only in dissertation
Drift + Indefinite boundary condition	WROCO '19
Drift + Free terminal time	arXiv

Conclusion

- In our research we have proposed an innovative motion planning algorithm for dynamical system affine in controls and with or without drift.
- We have formulated the (A)GHF equation, obeying which the initial curve is deformed to a curve with locally minimal "length". Controls are extracted from this minimizer and the integrated path is derived by feeding the system with the extracted control, which gives us a solution to the relaxed motion planning problem.
- Variants of the basic motion planning problems are also studied, including obstacle avoidance, input constraints, holonomic constraints, indefinite boundary conditions and free terminal time.
- Our algorithm is demonstrated on many canonical examples and the simulations show great potential of our algorithm.

Stability analysis via non-monotonic Lyapunov function

1. Motion planning via geometric approach
   - Review
   - Motion planning for systems with drift
   - Extensions

2. Stability analysis via non-monotonic Lyapunov function
   - Review
   - Show ISS via almost Lyapunov functions
   - Show ISS via higher order derivatives of Lyapunov functions

Background

- Lyapunov's direct method tells that a time invariant, autonomous system

$$\dot{x} = f(x) \tag{18}$$

  has a globally asymptotically stable origin if there exists a positive definite, radially unbounded function $V(x)$ such that $\dot{V}(x) := \left\langle \frac{d}{dx} V(x), f(x) \right\rangle$ is negative definite [Khalil, 2002].
- Finding such V satisfying the opposite sign definite constraints is difficult.
- Non-monotonic Lyapunov functions are widely studied [Aeyels and Peuteman, 1998, Ahmadi and Parrilo, 2008, Karafyllis, 2011, Meigoli and Nikravesh, 2012]
- The time varying nature and presence of inputs complicate the analysis: $\dot{x} = f(t, x, u)$.

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Non-monotonic V

"$V(x(t))$ decreases more than it increases over any long enough time"
which can be guaranteed by

1. Considering the finite step difference of Lyapunov function, i.e., existence of $T > 0$ such that $V(x(T)) - V(x(0)) < 0$ for all $x(0) \neq 0$ [Aeyels and Peuteman, 1998]; or
2. Imposing some conditions on the higher order derivatives of V [Meigoli and Nikravesh, 2009]; or
3. The study of almost Lyapunov functions such that $\Omega := \{x \in \mathbb{R}^n : \dot{V}(x) > -aV(x)\}$ is small enough [Liu et al., 2016, Liu et al., 2020].

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Global asymptotic stability

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- The system (18) is globally asymptotically stable (GAS) if there exists $\beta \in \mathcal{KL}$ such that

$$|x(t)| \leq \beta(|x_0|, t) \quad \forall x_0 \in \mathbb{R}^n, t \geq 0. \tag{19}$$

- GAS can be shown via a Lyapunov function $V(x) : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that for some $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $a > 0$ such that

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- The inequality (21) may not hold for all $x \in \mathbb{R}^n$, because of:
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    2. Numerical computation;
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    4. $\cdots$
- The inequality (21) may not be satisfied for $x \in \Omega \subset \mathbb{R}^n$.
- Our work [Liu et al., 2020] shows that as long as the volume of the connected components of Ω are small enough, the system (18) is GAS.

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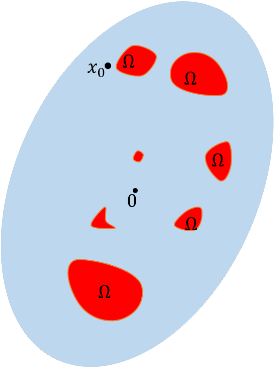

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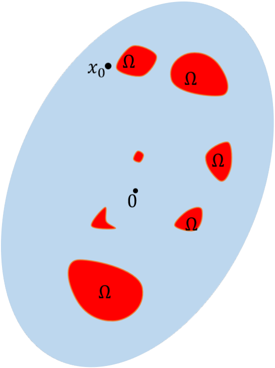

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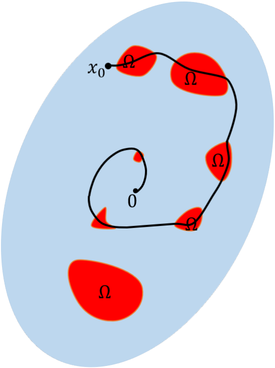
- Our work [Liu et al., 2020] shows that as long as the volume of the connected components of Ω are small enough, the system (18) is GAS.



Higher order derivatives of V for autonomous systems

$$\dot{x} = f(x) \tag{18}$$

Not necessarily $\dot{V}(x) < 0$ for all $x \neq 0$.

Theorem ([Meigoli and Nikravesh, 2009])

Let $V(x) : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be a Lyapunov function. If there exists $m \in \mathbb{N}_{\geq 2}$ and $a_1, \cdots a_{m-1} \geq 0$ such that V is $m-1$ times differentiable and f is $m-2$ times differentiable and

$$V^{(m)}(x) + a_{m-1}V^{(m-1)}(x) + \cdots + a_1\dot{V}(x) < 0$$

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Preliminaries in the presence of inputs

$$\dot{x} = f(t, x, u) \tag{22}$$

- The system (22) is globally uniformly asymptotically stable (GUAS) if there exists $\beta \in \mathcal{KL}$ such that

$$|x(t; t_0, x_0, u)| \le \beta(|x_0|, t - t_0) \tag{23}$$

for all $x_0 \in \mathbb{R}^n$, $t \ge t_0 \ge 0$ and $u \in \mathcal{U} \subset \mathbb{R}^m$.

- GUAS can be shown via a time varying Lyapunov function $V(t, x) : \mathbb{R}_{\ge 0} \times \mathbb{R}^n \to \mathbb{R}_{\ge 0}$ such that for some $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and a positive definite function $\psi : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ such that

$$\alpha_1(|x|) \le V(t, x) \le \alpha_2(|x|), \tag{24}$$

$$\dot{V}(t, x, u) := \frac{\partial}{\partial t} V(t, x) + \left\langle \frac{\partial}{\partial x} V(t, x, u), f(t, x, u) \right\rangle \le -\psi(|x|) \tag{25}$$

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Input-to-state Stability

$$\dot{x} = f(t, x, u) \tag{22}$$

- The system (22) is input-to-state stable (ISS) [Sontag, 1989] if there exists $\beta \in \mathcal{KL}, \gamma \in \mathcal{K}_\infty$ such that

$$|x(t; t_0, x_0, u)| \leq \beta(|x_0|, t - t_0) + \gamma(\operatorname*{ess\,sup}_{s \in [t_0, t]} |u(s)|) \tag{26}$$

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Auxiliary system; GUAS and ISS

Lemma ([Sontag and Wang, 1996])

The system

$$\dot{x} = f(t, x, u) \tag{22}$$

is ISS if and only if its auxiliary system

$$\dot{x} = f_\rho(t, x, d) := f(t, x, \rho(|x|)d) \tag{27}$$

is GUAS with $d \in \mathcal{U} = \mathbb{B}_1^m$ for some $\rho \in \mathcal{K}_\infty$.

Problem set up

Consider a time invariant system with input

$$\dot{x} = f(x, u) \tag{28}$$

- If there exist a Lyapunov function satisfying (20) and some $\rho \in \mathcal{K}_\infty, a > 0$ such that $\dot{V}(x) = \langle \frac{d}{dx} V(x), f_\rho(x, d) \rangle \leq -aV(x)$ for all $|d| \leq 1, x \in \mathbb{R}^n$, then (28) is ISS. Equivalently, we need

$$V'(x) := \sup_{|d| \leq 1} \left\langle \frac{d}{dx} V(x), f_\rho(x, d) \right\rangle \leq -aV(x). \tag{29}$$

- Appealing to our almost Lyapunov function framework, we allow (29) to be violated for $x \in \Omega \subset \mathbb{R}^n$.
- Can we still show ISS if we only have (29) for all $x \in \mathbb{R}^n \backslash \Omega$, while Ω is small enough?

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How to quantify the size of Ω?

- Although volume of Ω in Euclidean space is used in our previous work, it is conservative.
- Small Ω does not necessarily imply $x_\rho(t; x_0, d)$ will stay inside Ω for finite time.
- Instead, we directly impose assumptions on the Ω dwell time:

$$T := \sup_{x_0 \in \Omega, \|d\| \le 1} \inf_{t \ge 0} \{t : x_\rho(t; x_0, d) \notin \Omega\} \tag{30}$$

- Only upper bound of T is needed; depending on the size and shape of Ω and the vector field $f(x, u)$, it can be estimated without computing the solutions of the system.

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Let $V \in C^1(\mathbb{R}^n \to \mathbb{R}_{\ge 0})$ be a positive definite function and assume the system (22) has an equilibrium at 0. Then V' defined via (31) exists for all $x \in \mathbb{R}^n$ and is Lipschitz when both $f_\rho(x,d)$, $\frac{d}{dx}V(x)$ are Lipschitz in x.

Lemma

Assume Ω is bounded and all the assumptions in the above lemma hold. Then $\frac{d}{dx}V'$ exists almost everywhere in Ω. In addition, there exists $c > 0$ such that for all $x \in \Omega$ where $\frac{d}{dx}V'(x)$ exists,

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Recap

- $f_\rho(x,d) := f(x, \rho(|x|)d)$ is Lipschitz in x;
- $V \in C^1(\mathbb{R}^n \to \mathbb{R}_{\geq 0})$ has Lipschitz gradient;
- $\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \quad \forall x \in \mathbb{R}^n$
- $V'(x) := \sup_{|d|\leq 1} \left\langle \frac{d}{dx} V(x), f_\rho(x,d) \right\rangle$;
- $V'(x) \leq -aV(x) \quad \forall x \in \mathbb{R}^n \backslash \Omega$;
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Main result

Theorem

Consider a system with input (28) with the assumptions listed in the previous slide satisfied with some $a, c > 0$. There exists an increasing function $\alpha : [0,1) \to [0,\infty)$ with $\alpha(0) = 0, \lim_{t\to 1^-} \alpha(t) = \infty$ such that as long as the Ω dwell time T satisfies

$$T < \frac{1}{\sqrt{c}} \min\left\{\frac{\pi}{2}, \alpha\left(\frac{a}{\sqrt{c}}\right)\right\}, \tag{33}$$

the system (22) is ISS.

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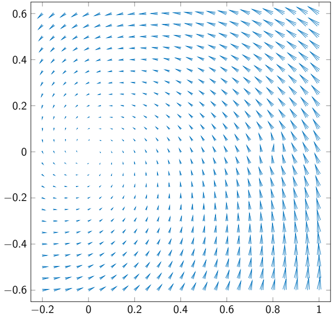
Example

$$\dot{x} = \begin{pmatrix} -\lambda(x) & -\mu \\ \mu & -\lambda(x) \end{pmatrix} x + u, \tag{35}$$

where $\lambda(x) = \left(\frac{a+b}{2}\right) \min\left\{\frac{|x-x_c|}{r}, 1\right\} - \frac{b}{2} + k.$



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- For numerical values $a = 1, b = 0.5, k = 0.1, r = 0.1, \mu = 2$ and $x_c = (0.8, 0)^\top$, it is computed $c \approx 30.54$ and $T < 0.125$.
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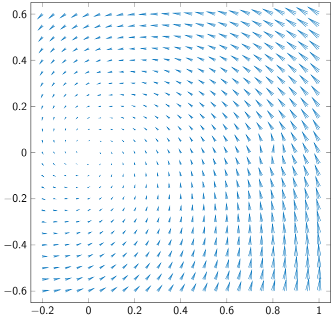

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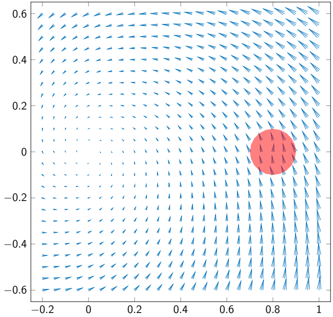

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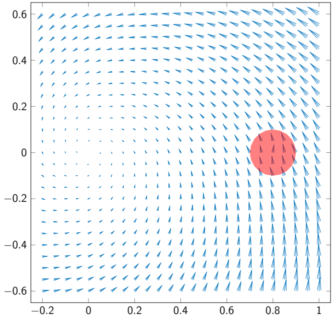

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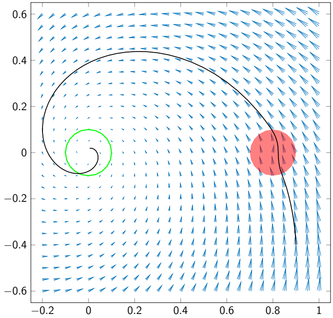

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- Because different assumption on the size of Ω is used in this work compared with [Liu et al., 2020], The system is shown to be ISS with even much larger Ω.
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Sketch of proof

- Let t_1, t_2 be the time the solution trajectory $x_\rho(t; x_0, d)$ enters and leaves Ω. We want to ultimately show $V(x_\rho(t_2; x_0, d)) \leq \eta V(x_\rho(t_1; x_0, d))$ for some $\eta < 1$.
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Sketch of proof

- For any $t \in [t_1, t_2]$, It is proven that

$$\frac{V(x_\rho(t; x_0, d))}{V(x_\rho(t_1; x_0, d))} \leq R_1(t - t_1),$$
$$\frac{V(x_\rho(t; x_0, d))}{V(x_\rho(t_2; x_0, d))} \geq R_2(t - t_2)$$

where

$$R_1(s) := \left(\cosh\sqrt{c}s - \frac{a}{\sqrt{c}}\sinh\sqrt{c}s\right),$$
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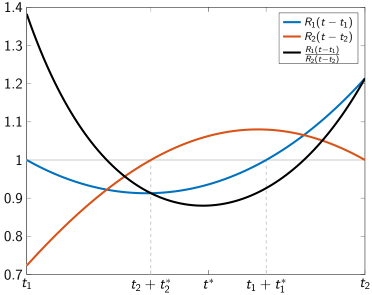

- It is then shown that when (33) holds, there always exists $t^* \in [t_1, t_2]$ such that $R_1(t^* - t_1) \leq \eta_1 < 1$, $R_2(t^* - t_2) \geq \eta_2 > 1$ and thus

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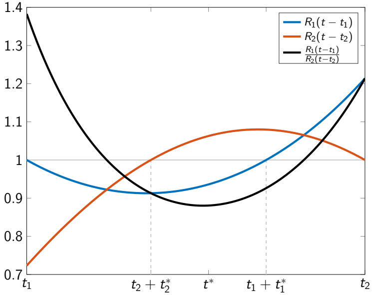

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1. Motion planning via geometric approach
   - Review
   - Motion planning for systems with drift
   - Extensions

2. Stability analysis via non-monotonic Lyapunov function
   - Review
   - Show ISS via almost Lyapunov functions
   - Show ISS via higher order derivatives of Lyapunov functions

Problem set up

$$\dot{x} = f(t, x, u) \tag{22}$$

- Known some $V(t, x)$ satisfying the "sandwich condition" (24) and $\rho \in \mathcal{K}_\infty$; but not necessarily

$$\dot{V}(t, x, d) := \frac{\partial}{\partial t} V(t, x) + \left\langle \frac{\partial}{\partial x} V(t, x, d), f_\rho(t, x, d) \right\rangle \leq -\psi(|x|) \tag{25}$$

  for all $x \in \mathbb{R}^n, t \geq t_0$ and $|d| \leq 1$.
- Like the autonomous case, we would like to study the "higher order derivatives" of V to show stability.
- Difficulty: $\dot{V}(t, x, d)$ depends on d, which may not be differentiable w.r.t. time t, $\ddot{V}(t, x, d) = \frac{d}{dt}\dot{V}(t, x, d)$ does not exist in general.

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Defining higher order derivatives for systems with inputs

- The higher order derivatives of V are defined and constructed iteratively:
  1. $v_0(t,x) := V(t,x)$;
  2. $V_i(t,x) := \frac{\partial}{\partial t} v_{i-1}(t,x) + \sup_{|d|\le 1} \langle \frac{\partial}{\partial x} v_{i-1}(t,x), f_\rho(t,x,d) \rangle$;
  3. Construct $v_i(t,x) \in C^1(\mathbb{R}_{\ge 0} \times \mathbb{R}^n)$ such that $v_i(t,x) \ge V_i(t,x)$ for all $x \in \mathbb{R}^n, t \ge t_0$.
- $v_i(t,x)$ is sign indefinite for $i > 0$;
- $\frac{d}{dt} v_{i-1}(t,x) \le v_i(t,x)$.
- The higher order derivatives $v_i(t,x)$ are globally decrescent up to order $m \in \mathbb{N}$ if there exists $\phi \in \mathcal{K}_\infty$ such that

$$v_i(t,x) \le \phi(|x|) \quad \forall x \in \mathbb{R}^n, t \ge t_0, i = 0, \cdots m.$$

Defining higher order derivatives for systems with inputs

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  2. $V_i(t,x) := \frac{\partial}{\partial t} v_{i-1}(t,x) + \sup_{|d|\leq 1} \left\langle \frac{\partial}{\partial x} v_{i-1}(t,x), f_\rho(t,x,d) \right\rangle$;
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Defining higher order derivatives for systems with inputs

- The higher order derivatives of V are defined and constructed iteratively:
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  3. Construct $v_i(t,x) \in C^1(\mathbb{R}_{\geq 0} \times \mathbb{R}^n)$ such that $v_i(t,x) \geq V_i(t,x)$ for all $x \in \mathbb{R}^n, t \geq t_0$.
- $v_i(t,x)$ is sign indefinite for $i > 0$;
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Defining higher order derivatives for systems with inputs

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  3. Construct $v_i(t,x) \in C^1(\mathbb{R}_{\geq 0} \times \mathbb{R}^n)$ such that $v_i(t,x) \geq V_i(t,x)$ for all $x \in \mathbb{R}^n, t \geq t_0$.
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$$v_i(t,x) \le \phi(|x|) \quad \forall x \in \mathbb{R}^n, t \ge t_0, i = 0, \cdots m.$$

Main result

Theorem

Given a system (22), a C^1 positive definite function $V(t,x)$ satisfying (24) and some $\rho \in \mathcal{K}_\infty$, generate the higher order derivatives v_i by f_ρ and V. If v_i's are globally decrescent up to order $m \in \mathbb{N}$ and there exist

$$a_0 > 0,\ a_i \geq 0 \quad \forall i = 1, \cdots, m \tag{37}$$

such that

$$\sum_{i=0}^{m} a_i v_i(t,x) \leq 0 \quad \forall x \in \mathbb{R}^n, t \geq t_0, \tag{38}$$

then the system (22) is ISS.

Sketch of proof

The following lemma can be proven using induction, similar to what is done in [Meigoli and Nikravesh, 2009]:

Lemma

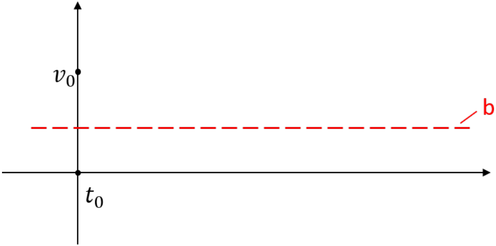
Let $x(t; t_0, x_0, d)$ be a solution of system (27). When (38) holds with some a_i's satisfying (37) and $a_m = 1$, for any $b > 0$ if $v_0(t, x(t; t_0, x_0, d)) \geq b$ for all $t \in [t_0, t_0 + T]$ for some $T \geq 0$, then

$$v_0(t, x(t; t_0, x_0, d)) \leq -b \sum_{j=1}^{m} a_{m-j} \frac{(t-t_0)^j}{j!} + \sum_{j=0}^{m-1} \sum_{i=0}^{j} \frac{(t-t_0)^j}{j!} a_{m+i-j} v_i(t_0, x_0) \tag{39}$$

for all $t \in [t_0, t_0 + T], \|d\| \leq 1$.

Sketch of proof

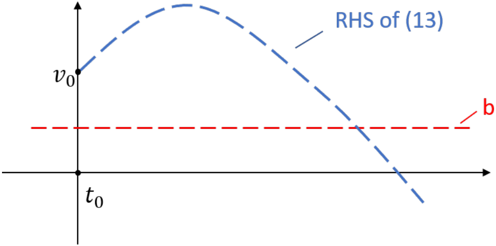
$$v_0(t, x(t; t_0, x_0, d)) \leq -b \sum_{j=1}^{m} a_{m-j} \frac{(t-t_0)^j}{j!} + \sum_{j=0}^{m-1} \sum_{i=0}^{j} \frac{(t-t_0)^j}{j!} a_{m+i-j} v_i(t_0, x_0) \tag{39}$$



The origin can be subsequently shown globally stable and uniformly attractive for the auxiliary system and hence it is GUAS; the original system (22) is therefore ISS.

Sketch of proof

$$v_0(t, x(t; t_0, x_0, d)) \leq -b \sum_{j=1}^{m} a_{m-j} \frac{(t-t_0)^j}{j!} + \sum_{j=0}^{m-1} \sum_{i=0}^{j} \frac{(t-t_0)^j}{j!} a_{m+i-j} v_i(t_0, x_0) \tag{39}$$



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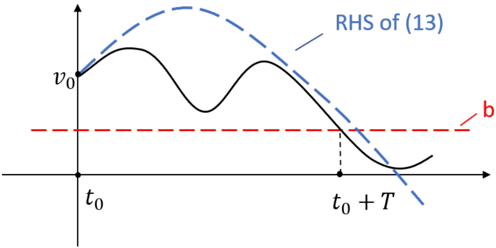

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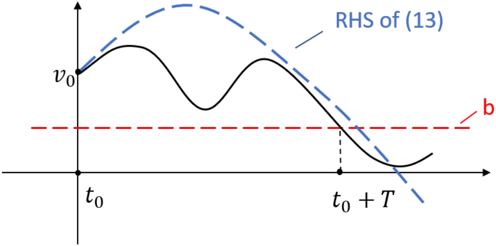

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Example: linear system with unaligned V

$$\dot{x} = f(x,u) = Ax + u, \qquad A = \begin{pmatrix} -0.1 & -1 \\ 2 & -0.1 \end{pmatrix} \tag{40}$$

- Choosing $V = |x|^2$ gives $\dot{V} = -0.2(x_1 - 5x_2)^2 + 4.8x_2^2$ even if $u \equiv 0$; stability is inconclusive.
- Pick $\rho(s) = 0.05s$, Higher order derivatives are constructed:

$$v_1 = -0.1x_1^2 + 2x_1x_2 - 0.1x_2^2,$$
$$v_2 = 4.13x_1^2 - 0.6x_1x_2 - 1.87x_2^2,$$
$$v_3 = -1.5907x_1^2 - 15.62x_1x_2 + 1.4093x_2^2.$$

- Let $a_0 = 0.1, a_1 = 8, a_2 = 0.5, a_3 = 1,$

$$\sum_{i=0}^{3} a_i v_i = -0.2257x_1^2 + 0.08x_1x_2 - 0.2257x_2^2 = -x^\top \begin{pmatrix} 0.2257 & -0.04 \\ -0.04 & 0.2257 \end{pmatrix} x$$

So (40) is ISS.

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So (40) is ISS.

Example: slowly varying between two stable linear modes

$$\dot{x} = f(t,x,u) = \sin^2(kt)A_1x + \cos^2(kt)A_2x + u =: A(k,t)x + u \qquad (41)$$

where $A_1 = \begin{pmatrix} -0.1 & -1 \\ 2 & -0.1 \end{pmatrix}, A_2 = \begin{pmatrix} -0.1 & -2 \\ 1 & -0.1 \end{pmatrix}$

- No common Lyapunov function for A_1, A_2.
- Again choose $V(t,x) = |x|^2, \rho(s) = 0.05s$. It can be shown that $v_i(t,x) = x^\top(P_i(k,t) + kp(k)Q_i(k,t))x$ where Q_i are uniformly bounded, $p(k)$ is a polynomial in k and

$$P_1 = \begin{pmatrix} -0.1 & -C \\ -C & -0.1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} C^2 - 3C + 0.13 & 0.3C \\ 0.3C & C^2 + 3C + 0.13 \end{pmatrix},$$

$$P_3 = \begin{pmatrix} -0.5C^2 + 1.5C + 0.224 & -C^3 + 8.81C \\ -C^3 + 8.81C & -0.5C^2 - 1.5C + 0.224 \end{pmatrix}$$

- Pick $a_0 = 0.1, a_1 = 8, a_2 = 0.5, a_3 = 1$, $\sum_{i=0}^{3} a_i v_i \approx \sum_{i=0}^{3} a_i x^\top P_i x \leq 0$ for sufficiently small k and hence (41) is ISS.

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- No common Lyapunov function for A_1, A_2.
- Again choose $V(t,x) = |x|^2, \rho(s) = 0.05s$. It can be shown that $v_i(t,x) = x^\top(P_i(k,t) + kp(k)Q_i(k,t))x$ where Q_i are uniformly bounded, $p(k)$ is a polynomial in k and

$$P_1 = \begin{pmatrix} -0.1 & -C \\ -C & -0.1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} C^2 - 3C + 0.13 & 0.3C \\ 0.3C & C^2 + 3C + 0.13 \end{pmatrix},$$

$$P_3 = \begin{pmatrix} -0.5C^2 + 1.5C + 0.224 & -C^3 + 8.81C \\ -C^3 + 8.81C & -0.5C^2 - 1.5C + 0.224 \end{pmatrix}$$

- Pick $a_0 = 0.1, a_1 = 8, a_2 = 0.5, a_3 = 1$, $\sum_{i=0}^3 a_i v_i \approx \sum_{i=0}^3 a_i x^\top P_i x \leq 0$ for sufficiently small k and hence (41) is ISS.

Example: slowly varying between two stable linear modes

$$\dot{x} = f(t,x,u) = \sin^2(kt)A_1x + \cos^2(kt)A_2x + u =: A(k,t)x + u \quad (41)$$

where $A_1 = \begin{pmatrix} -0.1 & -1 \\ 2 & -0.1 \end{pmatrix}, A_2 = \begin{pmatrix} -0.1 & -2 \\ 1 & -0.1 \end{pmatrix}$

- No common Lyapunov function for A_1, A_2.
- Again choose $V(t,x) = |x|^2, \rho(s) = 0.05s$. It can be shown that $v_i(t,x) = x^\top(P_i(k,t) + kp(k)Q_i(k,t))x$ where Q_i are uniformly bounded, $p(k)$ is a polynomial in k and

$$P_1 = \begin{pmatrix} -0.1 & -C \\ -C & -0.1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} C^2 - 3C + 0.13 & 0.3C \\ 0.3C & C^2 + 3C + 0.13 \end{pmatrix},$$
$$P_3 = \begin{pmatrix} -0.5C^2 + 1.5C + 0.224 & -C^3 + 8.81C \\ -C^3 + 8.81C & -0.5C^2 - 1.5C + 0.224 \end{pmatrix}$$

- Pick $a_0 = 0.1, a_1 = 8, a_2 = 0.5, a_3 = 1$, $\sum_{i=0}^{3} a_i v_i \approx \sum_{i=0}^{3} a_i x^\top P_i x \leq 0$ for sufficiently small k and hence (41) is ISS.

Conclusion

- We have proposed two alternative methods for determining ISS for systems with inputs: via almost Lyapunov functions or via higher order derivatives of Lyapunov functions.
- In the first method, the study of almost Lyapunov functions from our previous work was generalized to systems with inputs. When there are "bad" regions Ω in the state space where V does not decrease fast enough, an upper bound of the Ω dwell time was found to guarantee that the system with inputs is still ISS.
- In the second method, it is claimed and proven that if there exists a linear combination of those higher order derivatives with non-negative coefficients (except that the coefficient of the 0-th order term needs to be positive) which is negative semi-definite, then the system is ISS.

The End

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