Average dwell-time minimization of switched systems via sequential convex programming

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Overview

Introduction

Background

- A switched system is defined by a collection of dynamical subsystems and a switching signal that governs the transitions between them.
- Switched systems are a class of hybrid systems which play an important role in modeling real-world processes [Sun and Ge, 2005]
- In general, switched systems do not inherit the stability properties of their subsystems under arbitrary switching, see e.g., [Liberzon, 2003].
- Moreover, the switching of a switched system may be graph-based (that is, the mode changing during a switch is restricted to be an edge of a graph).

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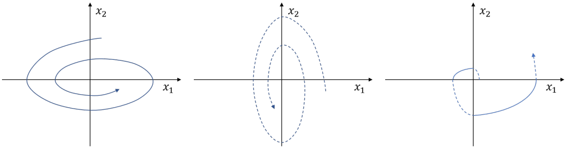

Mode 1 only | Mode 2 only | An unstable switched case

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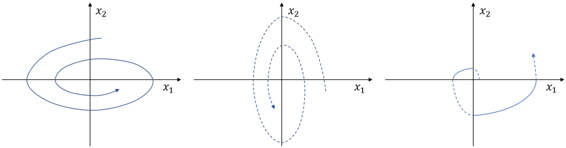

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Literature review

- To guarantee stability, switches should not occur too frequently.
- A bound on the dwell-time (DT) (resp. average dwell-time (ADT)) conditions, which bound the number (resp. the average number) of allowed switches over an arbitrary time interval can provide an important design criteria to prevent de-stabilization by the switching action of the signal.
  - Global asymptotic stability: [Morse, 1993, Hespanha and Morse, 1999]
  - Input-to-state stability: [Xie et al., 2001, Vu et al., 2007]
  - Integral input-to-state stability: [Russo et al., 2020, Liu et al., 2020]
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Literature review (contd.)

- For DT condition
  - Optimization: [Geromel and Colaneri, 2006, Briat and Seuret, 2013].
  - Using eigenspace: [Karabacak and Şengör, 2009].
- For ADT condition
  - Using cycle ratio: [Karabacak, 2013].
  - Verification: [Mitra et al., 2006].
- Research on discrete-time switched systems: [İlhan and Karabacak, 2016, Kundu and Chatterjee, 2017].

Switching signals and switched systems

- Consider a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V} = \{1, 2, \cdots, p\}$ and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. For each $i \in \mathcal{V}$ there is a locally Lipschitz vector field $f_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$.
- A switching signal $\sigma : [0, \infty) \mapsto \mathcal{V}$ is a right-continuous, piecewise constant function with locally finite number of discontinuities.
- A switching signal σ has an underlying switching graph $\mathcal{G}$ if $(\sigma(t^-), \sigma(t)) \in \mathcal{E}$ for all $t \in \mathcal{T}(\sigma) := \{t > 0 : \sigma(t) \neq \sigma(t^-)\}$.
- A switching signal σ has an average dwell-time (ADT) of τ_a if there exist $\tau_a > 0$ and $N_0 \geqslant 1$ such that

$$\forall t_2 \geqslant t_1 \geqslant 0 : \quad N_\sigma(t_1, t_2) \leqslant N_0 + \frac{t_2 - t_1}{\tau_a},$$

  where $N_\sigma(t_1, t_2) := |(t_1, t_2] \cap \mathcal{T}(\sigma)|$.
- The dynamics of the switched system is described by

$$\begin{aligned} \dot{x}(t) &= f_{\sigma(t)}(x(t), \omega(t)), && \text{if } t \notin \mathcal{T}(\sigma), \\ x(t) &= x(t^-), && \text{if } t \in \mathcal{T}(\sigma). \end{aligned}$$

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Switched linear systems and switched bilinear systems

- Switched unforced linear system

$$\dot{x} = A_\sigma x. \tag{1}$$

- Switched linear system with linear input

$$\dot{x} = A_\sigma x + B_\sigma \omega. \tag{2}$$

- Switched system with linear and bilinear inputs

$$\dot{x} = A_\sigma x + B_\sigma \omega + \sum_{j=1}^{m_c} C_{\sigma,j} x \omega_j. \tag{3}$$

- We aim to study global asymptotic stability (GAS), input-to-state stability (ISS) [Sontag, 1989] and integral input-to-state stability (iISS) [Angeli et al., 2000] for these systems.
- It is known that when not switched and A_σ's are all Hurwitz, (1) is GAS, (2) is ISS and (3) is iISS.

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Motivation

Lyapunov approach

Theorem 1 ([Vu et al., 2007, Theorem 3.1])

Consider a switched system and suppose that there exist continuously differentiable functions $V_i : \mathbb{R}^n \mapsto [0, \infty), i \in \mathcal{V}$, class K_∞ functions $\alpha_1, \alpha_2, \gamma$, and numbers $\lambda > 0, \mu \geqslant 1$ such that $\forall x \in \mathbb{R}^n, u \in \mathbb{R}^m$, and $\forall i \in \mathcal{V}, (i,j) \in \mathcal{E}$, we have

$$\alpha_1(|x)| \leqslant V_i(x) \leqslant \alpha_2(|x|),$$
$$\frac{\partial V_i(x)}{\partial x} \cdot f_i(x,u) \leqslant -\lambda V_i(x) + \gamma(|\eta|),$$
$$V_j(x) \leqslant \mu V_i(x).$$

Let σ be a switching signal having average dwell-time τ_a. If $\tau_a > \frac{\ln \mu}{\lambda}$,then the switched system is ISS.

Question: What Lyapunov functions V_i should we choose?

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An intuitive choice of V_i

- Consider a switched unforced linear system (1) with $\mathcal{V} = \{1, 2\}$, $\mathcal{E} = \{(1, 2), (2, 1)\}$ and

$$A_1 = \begin{pmatrix} -15 & 9 & -12 & -1 \\ -2 & 2 & -5 & -7 \\ 13 & -5 & -17 & 23 \\ 2 & 2 & -15 & 10 \end{pmatrix}, \; A_2 = \begin{pmatrix} -14 & 11 & -19 & 6 \\ -10 & 7 & -15 & 5 \\ 3 & -1 & -7 & 9 \\ -6 & 5 & -15 & 8 \end{pmatrix}.$$

- Pick quadratic Lyapunov functions $V_i(x) := x^\top P_i x$ such that

$$A_i^\top P_i + P_i A_i + I = 0 \quad \forall i \in \mathcal{V}.$$

- The (flow) inequality holds for any $\lambda < \min_{i \in \mathcal{V}} \frac{1}{\lambda_{\max}(P_i)} \approx 0.205$
- The (jump) inequality holds for any $\mu < \max_{(i,j) \in \mathcal{E}} \lambda_{\max}(P_i P_j^{-1}) \approx 10.473$
- The lower-bound on τ_a is $\frac{\ln 10.473}{0.205} \approx 11.48$. How good is this lower-bound?

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An intuitive choice of V_i (contd.)

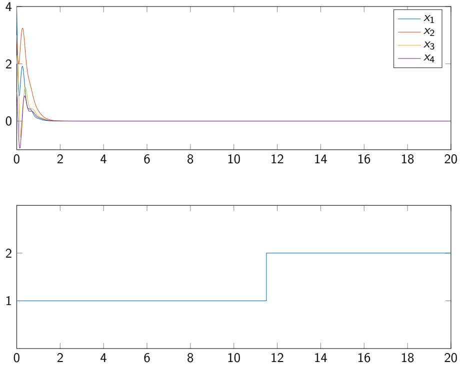

Figure: Top: state trajectory. Bottom: switching signal. $x_0 = (4, 3, 2, 1)^\top$ and periodic switching with period 11.5.

Too conservative!

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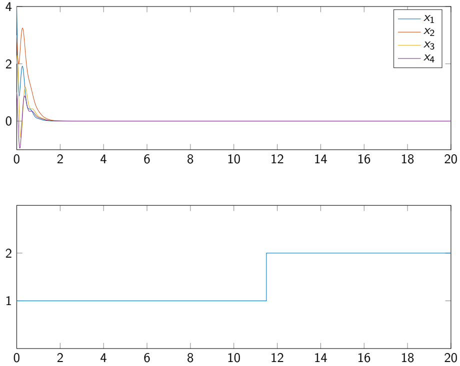

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The λ maximizing choice of V_i

- Alternatively, pick k_i close to $\mathrm{Re}(\lambda_{\max}(A_i))$ and solve

$$(A_i + k_i I)^\top P_i + P_i(A_i + k_i I) + I = 0 \quad \forall i \in \mathcal{V}.$$

- These P_i's give quadratic Lyapunov functions with maximal decay rates.
- In this case the (flow) inequality holds for any $\lambda < -2 \max_{i \in \mathcal{V}} \mathrm{Re}(\lambda_{\max}(A_i)) \approx 1.522$.
- The (jump) inequality holds for any $\mu < \max_{(i,j) \in \mathcal{E}} \lambda_{\max}(P_i P_j^{-1}) \approx 2245$.
- The lower-bound on τ_a is $\frac{\ln 2245}{1.522} \approx 5.08$. Better, but still conservative.

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- The (jump) inequality holds for any $\mu < \max_{(i,j)\in\mathcal{E}} \lambda_{\max}(P_i P_j^{-1}) \approx 2245$.
- The lower-bound on τ_a is $\frac{\ln 2245}{1.522} \approx 5.08$. Better, but still conservative.

The λ maximizing choice of V_i (contd.)

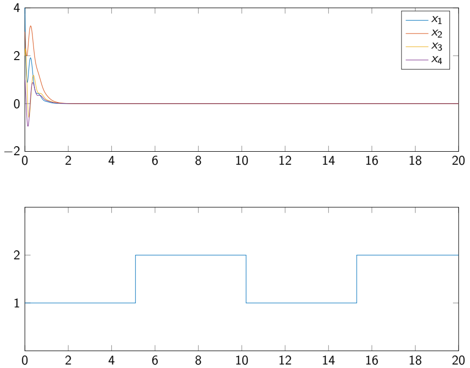
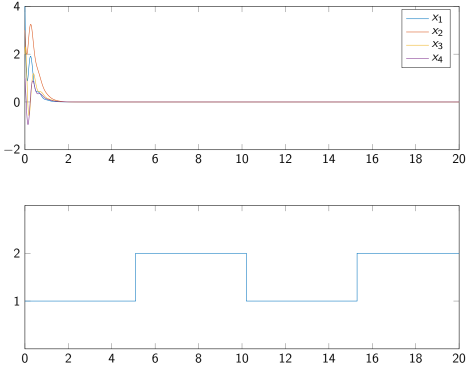


Figure: Top: state trajectory. Bottom: switching signal. $x_0 = (4, 3, 2, 1)^\top$ and periodic switching with period 5.1.

Question: How to choose V_i? (i.e., how to choose P_i?)

The λ maximizing choice of V_i (contd.)



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A uniform ADT lower bound for stable switched systems

Problem reformulation using matrix inequalities

Theorem 2

Given a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$*, let the matrix* $A_i \in \mathbb{R}^{n \times n}$ *be Hurwitz for all* $i \in \mathcal{V}$*. Consider the switched systems* (1), (2) *or* (3) *and assume the switching signal* σ *has an underlying switching graph* $\mathcal{G}$ *with ADT parameter* τ_a*. Let* P_i*'s be positive definite symmetric matrices and suppose that the inequalities*

$$\begin{aligned} A_i^\top P_i + P_i A_i + \lambda P_i \preceq 0, &\quad \forall i \in \mathcal{V}, \\ P_j - \mu P_i \preceq 0, &\quad \forall (i,j) \in \mathcal{E}, \end{aligned}$$

hold for some $\mu \geqslant 1, \lambda > 0$*. If* $\tau_a > \frac{\ln \mu}{\lambda}$*, then the systems* (1), (2) *and* (3) *are GAS, ISS and iISS, respectively.*

For linear systems, GAS and ISS can be proven from [Vu et al., 2007, Theorem 3.1] by using quadratic Lyapunov functions $V_i(x) := x^\top P_i x$. For bilinear systems, iISS is proven in [Liu et al., 2021, Proposition 12] by using Lyapunov functions $V_i(x) := \ln(1 + x^\top P_i x)$.

Problem reformulation using matrix inequalities

Theorem 2

Given a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, let the matrix $A_i \in \mathbb{R}^{n\times n}$ be Hurwitz for all $i \in \mathcal{V}$. Consider the switched systems (1), (2) or (3) and assume the switching signal σ has an underlying switching graph $\mathcal{G}$ with ADT parameter τ_a. Let P_i's be positive definite symmetric matrices and suppose that the inequalities

$$\begin{aligned} A_i^\top P_i + P_i A_i + \lambda P_i \preceq 0, &\quad \forall i \in \mathcal{V}, \\ P_j - \mu P_i \preceq 0, &\quad \forall (i,j) \in \mathcal{E}, \end{aligned}$$

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Problem reformulation using matrix inequalities (contd.)

Corollary 3

Given a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, let the matrices $A_i \in \mathbb{R}^{n\times n}$ be Hurwitz for all $i \in \mathcal{V}$. Denote the optimal value of the optimization problem

$$
\begin{aligned}
(P1) \qquad & \underset{\{P_i\}_{i\in\mathcal{V}},\mu,\lambda}{\text{minimize}} \ \frac{\ln \mu}{\lambda} \\
& \textit{subject to} \quad \mu \geqslant 1, \\
& \qquad \lambda > 0, \\
& \qquad P_i \succ 0 \quad \forall i \in \mathcal{V}, \\
& \qquad A_i^\top P_i + P_i A_i + \lambda P_i \preceq 0, \quad \forall i \in \mathcal{V}, && \text{(flow)} \\
& \qquad P_j - \mu P_i \preceq 0, \quad \forall (i,j) \in \mathcal{E}. && \text{(jump)}
\end{aligned}
$$

Let $\tau^* = \frac{\ln \mu^*}{\lambda^*}$. If a switching signal σ has underlying switching graph $\mathcal{G}$ with ADT satisfying $\tau_a > \tau^*$, then the switched system (1) is GAS, (2) is ISS and (3) is iISS.

Some remarks

- The ADT lower bound τ^* in Corollary 3 only depends on A_i's.

(P1)

$$\begin{aligned}
\underset{\{P_i\}_{i\in\mathcal{V}},\mu,\lambda}{\text{minimize}} \quad & \frac{\ln\mu}{\lambda} \\
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& P_j - \mu P_i \preceq 0, \quad \forall (i,j) \in \mathcal{E}. && \text{(jump)}
\end{aligned}$$

- The problem (P1) is NP-hard in general because
  - The objective function is nonlinear and nonconvex.
  - the constraints (flow) and (jump) are bilinear matrix inequality (BMI) constraints, which are nonconvex.
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Solving the nonlinear optimization problem

Sketch of the algorithm

Algorithm Minimization of ADT lower bound

Input: $(\mathcal{V}, \mathcal{E}), \{A_i\}_{i\in\mathcal{V}}, \{P_i^{(0)}\}_{i\in\mathcal{V}}, \mu^{(0)}, \lambda^{(0)}$

1: $\tau^{(0)} \leftarrow \frac{\ln \mu^{(0)}}{\lambda^{(0)}}$
2: **for** $k = 1, 2, \cdots$ **do**
3: Convexify (P1) around $\{P_i^{(k-1)}\}_{i\in\mathcal{V}}$, $\mu^{(k-1)}$, $\lambda^{(k-1)}$
4: Solve the convexified problem, set $\{P_i^{(k)}\}_{i\in\mathcal{V}}$, $\mu^{(k)}$, $\lambda^{(k)}$ equal to the obtained minimizer
5: $\tau^{(k)} \leftarrow \frac{\ln \mu^{(k)}}{\lambda^{(k)}}$

Approximation of the objective function

We approximate the objective function $f(\mu, \lambda)$ in (P1) linearly around $(\mu^\dagger, \lambda^\dagger)$ by

$$\mathbf{L}\, f_{\mu^\dagger,\lambda^\dagger}(\mu, \lambda) := \frac{\ln \mu^\dagger}{\lambda^\dagger} + \begin{pmatrix} \frac{1}{\mu^\dagger \lambda^\dagger} & -\frac{\ln \mu^\dagger}{(\lambda^\dagger)^2} \end{pmatrix} \begin{pmatrix} \mu - \mu^\dagger \\ \lambda - \lambda^\dagger \end{pmatrix}.$$

Approximation of the BMI constraints

$$A_i^\top P_i + P_i A_i + \lambda P_i \preceq 0, \quad \forall i \in \mathcal{V}, \tag{flow}$$

- For each $i \in \mathcal{V}$, (flow) can be rewritten in quadratic form as

$$\begin{pmatrix} A_i \\ P_i \\ \frac{\lambda}{2} I \end{pmatrix}^\top \Sigma \begin{pmatrix} A_i \\ P_i \\ \frac{\lambda}{2} I \end{pmatrix} \preceq 0, \quad \text{with } \Sigma := \begin{pmatrix} 0 & I & 0 \\ I & 0 & I \\ 0 & I & 0 \end{pmatrix}.$$

- Equivalently, $\hat{R}(P_i, \lambda) - \check{R}(P_i, \lambda) \preceq 0$ where

$$\hat{R}(P_i, \lambda) := \begin{pmatrix} A_i \\ P_i \\ \frac{\lambda}{2} I \end{pmatrix}^\top V \begin{pmatrix} 0 \\ 0 \\ I \end{pmatrix} \begin{pmatrix} 0 & 0 & I \end{pmatrix} V^\top \begin{pmatrix} A_i \\ P_i \\ \frac{\lambda}{2} I \end{pmatrix},$$

$$\check{R}(P_i, \lambda) := \begin{pmatrix} A_i \\ P_i \\ \frac{\lambda}{2} I \end{pmatrix}^\top V \begin{pmatrix} I \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} I & 0 & 0 \end{pmatrix} V^\top \begin{pmatrix} A_i \\ P_i \\ \frac{\lambda}{2} I \end{pmatrix},$$

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Approximation of the BMI constraints (contd.)

- By definition, both $\hat{R}$ and $\check{R}$ are positive semidefinite and convex. Hence

$$\begin{aligned}\check{R}(P_i,\lambda) &\succeq \mathbf{L}\,\check{R}_{P_i^\dagger,\lambda^\dagger}(P_i,\lambda)\\ &:= \check{R}(P_i^\dagger,\lambda^\dagger) + D\check{R}(P_i^\dagger,\lambda^\dagger)(P_i - P_i^\dagger, \lambda - \lambda^\dagger),\end{aligned}$$

  and $\hat{R}(P_i,\lambda) - \mathbf{L}\,\check{R}_{P_i^\dagger,\lambda^\dagger}(P_i,\lambda) \preceq 0$.

- Using Schur complement, it can be converted to a linear matrix inequality (LMI):

$$\begin{pmatrix} I & \begin{pmatrix}0 & 0 & I\end{pmatrix} V^\top \begin{pmatrix} A_i \\ P_i \\ \frac{\lambda}{2} I \end{pmatrix} \\ \begin{pmatrix} A_i^\top & P_i & \frac{\lambda}{2} I \end{pmatrix} V \begin{pmatrix} 0 \\ 0 \\ I \end{pmatrix} & \mathbf{L}\,\check{R}_{P_i^\dagger,\lambda^\dagger}(P_i,\lambda) \end{pmatrix} \succeq 0. \qquad \text{(LMI1)}$$

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Approximation of the BMI constraints (contd.)

$$P_j - \mu P_i \preceq 0, \quad \forall (i,j) \in \mathcal{E}, \qquad \text{(jump)}$$

- Using similar convex-concave decomposition and linearization, the constraints (jump) can also be approximated by LMI constraints

$$\begin{pmatrix} I & \begin{pmatrix} I & 0 \end{pmatrix} U^\top \begin{pmatrix} P_i \\ \frac{\mu}{2} I \end{pmatrix} \\ \begin{pmatrix} P_i & \frac{\mu}{2} I \end{pmatrix} U \begin{pmatrix} I \\ 0 \end{pmatrix} & \mathbf{L}\, \hat{S}_{P_i^\dagger, \mu^\dagger}(P_i, \mu) - P_j \end{pmatrix} \succeq 0, \qquad \text{(LMI2)}$$

where the columns of U are the eigenvectors of $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ and

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The convex subproblem

- Define the regularization function

$$r_{P_i^\dagger,\mu^\dagger,\lambda^\dagger}(\{P_i\}_{i\in\mathcal{V}},\mu,\lambda) := \quad c_P \sum_{i\in\mathcal{V}} \|P_i - P_i^\dagger\|_F^2 + c_\mu(\mu-\mu^\dagger)^2 + c_\lambda(\lambda-\lambda^\dagger)^2.$$

- Consider the following problem

$$\text{(P2)} \qquad \begin{aligned} \underset{\{P_i\}_{i\in\mathcal{V}},\mu,\lambda}{\text{minimize}}\ & \mathbf{L}\, f_{\mu^\dagger,\lambda^\dagger}(\mu,\lambda) + r_{P_i^\dagger,\mu^\dagger,\lambda^\dagger}(\{P_i\}_{i\in\mathcal{V}},\mu,\lambda), \\ \text{subject to}\ & \mu \geqslant 1, \\ & \lambda > 0, \\ & P_i \succ 0 \quad \forall i \in \mathcal{V}, \\ & \text{and (LMI 1)}\ \forall i \in \mathcal{V}, \text{(LMI 2)}\ \forall (i,j) \in \mathcal{E}, \end{aligned}$$

- (P2) is the convexified problem of (P1), solved repeatedly in Algorithm 1.
- If the point $(\{P_i^\dagger\}_{i\in\mathcal{V}},\mu^\dagger,\lambda^\dagger)$ is itself a solution to (P2), then we call it a fixed point.

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Convergence of Algorithm 1

Proposition 1

Suppose that there is a compact subset D of the feasible set of (P1) such that $c_\lambda \geqslant \frac{1+2\ln\mu}{\lambda^3}$ for all $(\{P_i\}_{i\in\mathcal{V}}, \mu, \lambda) \in D$. Let $(\{P^*\}_{i\in\mathcal{V}}, \mu^*, \lambda^*) \in D$ be a solution of (P1), then it is a fixed point. In addition, if Algorithm 1 generates the sequence $(P_i^{(k)}, \mu^{(k)}, \lambda^{(k)}) \in D$ for all $k \in \mathbb{N}$, then the associated $\tau^{(k)}$ monotonically decrease and the sequence $(\{P_i^{(k)}\}_{i\in\mathcal{V}}, \mu^{(k)}, \lambda^{(k)})$ converges to a fixed point when k approaches infinity.

- Only local convergence is guaranteed for Algorithm 1.
- In practice, we use a sufficiently large parameter c_λ for the assumptions in Proposition 1 to hold.
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- The initial guesses $P_i^{(0)}, \mu^{(0)}, \lambda^{(0)}$ affect the output of Algorithm 1. Nevertheless, monotonicity implies that the ADT lower bound improves by applying Algorithm 1.

Examples

Motivating example revisited

- Consider a switched unforced linear system (1) with $\mathcal{V} = \{1, 2\}$, $\mathcal{E} = \{(1, 2), (2, 1)\}$ and

$$A_1 = \begin{pmatrix} -15 & 9 & -12 & -1 \\ -2 & 2 & -5 & -7 \\ 13 & -5 & -17 & 23 \\ 2 & 2 & -15 & 10 \end{pmatrix}, \; A_2 = \begin{pmatrix} -14 & 11 & -19 & 6 \\ -10 & 7 & -15 & 5 \\ 3 & -1 & -7 & 9 \\ -6 & 5 & -15 & 8 \end{pmatrix}.$$

- By applying Algorithm 1, we get an ADT bound of 0.2844

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Motivating example revisited (contd.)

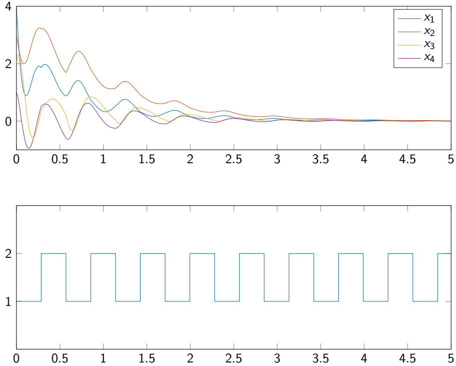

Figure: Top: state trajectory. Bottom: switching signal. $x_0 = (4, 3, 2, 1)^\top$ and periodic switching with period 0.285.

Comparison

	Naive	Max. λ	Min. μ	Alg. 1	[Karabacak, 2013]
μ	10.47	2245	1.056	1.171	-
λ	0.2046	1.522	0.0053	0.5568	-
ADT lb.	11.48	5.08	10.39	0.2844	2.899

Table: ADT lower bounds computed using different approaches.

Five-mode, 3-dimensional switched system

- Consider a five-mode, 3-dimensional switched system of form (1) with matrices given by

$$A_1 = \begin{pmatrix} -5 & 1 & 2 \\ 0 & -5 & 1 \\ 0 & 1 & -2 \end{pmatrix},\ A_2 = \begin{pmatrix} -1 & 3 & 1 \\ 0 & -2 & 0 \\ 0 & 1 & -1 \end{pmatrix},\ A_3 = \begin{pmatrix} 0 & 0 & 3 \\ -2 & -1 & -3 \\ -1 & 0 & -2 \end{pmatrix},$$

$$A_4 = \begin{pmatrix} -4 & 0 & -3 \\ 2 & -2 & 4 \\ 1 & 0 & -1 \end{pmatrix},\ A_5 = \begin{pmatrix} -1 & 0 & 0 \\ -1 & -1 & -1 \\ -3 & 0 & -4 \end{pmatrix}.$$

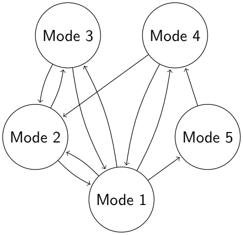

Figure: Switching graph $\mathcal{G}$.

Comparison

	Naive	Max. λ	Min. μ	Alg. 1	[Karabacak, 2013]
μ	20.17	3964900	1.443	3.071	-
λ	0.286	1.959	0.0011	0.9178	-
ADT lb.	10.5	7.757	334.6	1.222	-

Table: ADT lower bounds computed using different approaches.

Conclusion

Conclusion

- The problem of finding ADT lower bounds for switching signals that can guarantee GAS, ISS or iISS of continuous-time, graph-based switched systems was studied.
- The problem was formulated as an optimization problem over the parameters given by different choices of quadratic Lyapunov functions.
- This optimization problem was then solved via an iterative algorithm with local convergence guarantees.
- Numerical examples and the comparison with previous results showed that the ADT lower bounds produced by our algorithm are relatively small and, hence, favorable for practical switching-signal design purposes.

Thank you!

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