Path Planning and Obstacle Avoidance of Under-actuated System via Heat Flow Method

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Riemannian Geometry

Length functional & Geodesics

The length functional of a curve $x : [a, b] \to M$ is defined as:

$$L(v) := \int_a^b \sqrt{g_x(\dot{x}, \dot{x})} dt \tag{1}$$

where g_x is the Riemannian metric of M. For our problem, $M = \mathbb{R}^n$ and

$$g_x(\dot{x}, \dot{x}) := \dot{x}^T G(x) \dot{x} \tag{2}$$

where $G(x)$ is a symmetric, positive definite matrix.
The Riemannian distance stemming from the Riemannian metric is then:

$$d(x_1, x_2) := \inf_{C(x_1, x_2)} L(x) \tag{3}$$

If this infimum is achievable for some curve $C(x_1, x_2)$, then this path is called the *geodesic* between x_1 and x_2 on M.

$L(x)$ v.s. $E(x)$

Lemma 1

$L(x)$ is independent of parametrization.

The energy functional is

$$E(x) = \int_a^b g_x(\dot{x}, \dot{x}) ds \tag{4}$$

Lemma 2

If a simple curve $x \in C(x_1, x_2)$ is a minimizer of $E(x)$, it has constant velocity and it is also a minimizer of $L(x)$.

Hint of proof: Use Cauchy-Schwartz to show $L(x) \leq (b-a)E(x)$ and equality is only achieved when $|\dot{x}|$ is constant.

Geodesic Equation and Heat Flow Equation

Geodesic equation

If define the Lagrangian $l(x, \dot{x}) := g_x(\dot{x}, \dot{x}) = \sum_{i,j} G_{ij}\dot{x}_i\dot{x}_j$, then the minimum of $E(x)$ is achieved when the Euler-Lagrange equation is satisfied:

$$\frac{d}{dt}\frac{\partial l}{\partial \dot{x}} = \frac{\partial l}{\partial x} \tag{5}$$

Work out the details, it can be derived that

$$\ddot{x}_i + \sum_{j,k} \Gamma^i_{jk}\dot{x}_j\dot{x}_k = 0 \quad \forall i = 1, 2, \ldots, n \tag{6}$$

where

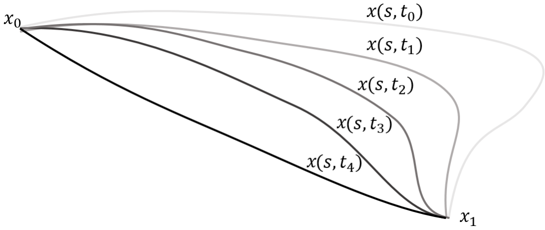
$$\Gamma^i_{jk} := \frac{1}{2}\sum_l (G^{-1})_{il}\left(\frac{\partial G_{lj}}{\partial x_k} + \frac{\partial G_{lk}}{\partial x_j} - \frac{\partial G_{jk}}{\partial x_l}\right) \tag{7}$$

Γ^i_{jk} are called the *Christoffel Symbols* of G. (6) is the *geodesic equation*.

Varying curve w.r.t. time

While directly solving (6) is usually difficult, an alternative way of finding a geodesic is to treat the energy functional (4) as a function of time and minimize it by varying the curve with respect to time.
Hence one more argument t is added to curve parameterization:

$$x(s,t) : [0,1] \times [0,\infty) \to M \tag{8}$$



Heat flow equation

The heat flow (HF) equation is defined to be

$$\frac{\partial}{\partial t}x_i(s,t) = \frac{\partial^2}{\partial s^2}x_i(s,t) + \sum_{j,k}\Gamma^i_{jk}\frac{\partial x_j}{\partial s}\frac{\partial x_k}{\partial s} \quad \forall i = 1,2,\dots,n \tag{9}$$

with initial condition:

$$x(s,0) = v(s) \quad s \in [0,1] \tag{10}$$

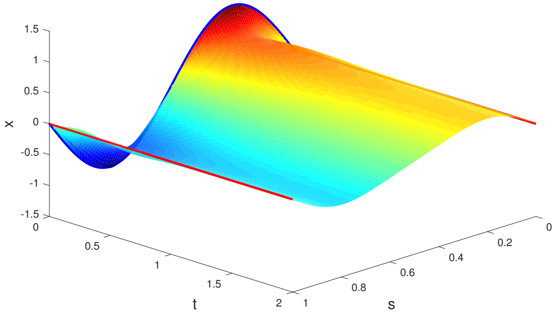
and boundary conditions:

$$x(0,t) = x_0, x(1,t) = x_1 \quad \forall t \geq 0 \tag{11}$$

Solution illustration

$$\frac{\partial}{\partial t}x_i(s,t) = \frac{\partial^2}{\partial s^2}x_i(s,t) + \sum_{j,k}\Gamma^i_{jk}\frac{\partial x_j}{\partial s}\frac{\partial x_k}{\partial s} \quad \forall i = 1,2,\ldots,n$$

$$x(s,0) = v(s), \quad x(0,t) = x_0, x(1,t) = x_1$$



Convergence of solution

Lemma 3

The solution $x(s,t)$ of heat flow equation (9) converges to a geodesic on M, i.e., converges to a solution $x^*(s)$ of geodesic equation (6) as $t \to \infty$. In addition, $E(x(\cdot,t)) \searrow E(x^*)$ as $t \to \infty$.

The idea of its proof is showing that $E(x(\cdot,t))$ is non-increasing along heat flow.

In addition to show that the energy converges to minimum with $\frac{\partial}{\partial t}x(s,t) \to 0$, LaSalle-like argument is need.

Instead, $\frac{d^2}{dt^2}E(x) \geq 0$ suggests energy indeed drops to minimum.

$S^1 \times S^1$ torus

Points on a torus of major radius c and minor radius a is usually parameterised by (θ, ϕ) as

$$\begin{aligned} x &= (c + a\cos\phi)\cos\theta \\ y &= (c + a\cos\phi)\sin\theta \\ z &= a\sin\phi \end{aligned} \tag{12}$$

Christoffel symbols are

$$\begin{aligned} &\Gamma^{\phi}_{\phi\phi} = \Gamma^{\phi}_{\phi\theta} = \Gamma^{\phi}_{\theta\phi} = \Gamma^{\theta}_{\phi\phi} = \Gamma^{\theta}_{\theta\theta} = 0; \\ &\Gamma^{\phi}_{\theta\theta} = \frac{1}{a}\sin\phi(c + a\cos\phi) \\ &\Gamma^{\theta}_{\theta\phi} = \Gamma^{\theta}_{\phi\theta} = -\frac{a\sin\phi}{c + a\cos\phi} \end{aligned} \tag{13}$$

Torus cont'd.

The geodesic equation on torus is

$$\begin{aligned} \ddot{\phi} + \frac{1}{a}\sin\phi(c + a\cos\phi)\dot{\theta}^2 = 0 \\ \ddot{\theta} - \frac{2a\sin\phi}{c + a\cos\phi}\dot{\theta}\dot{\phi} = 0 \end{aligned} \tag{14}$$

The corresponding heat flow equation are:

$$\begin{aligned} \frac{\partial}{\partial t}\theta(s,t) &= \frac{\partial^2}{\partial s^2}\theta(s,t) + \frac{1}{a}\sin\phi(c + a\cos\phi)\left(\frac{\partial\theta}{\partial s}\right)^2 \\ \frac{\partial}{\partial t}\phi(s,t) &= \frac{\partial^2}{\partial s^2}\phi(s,t) - \frac{2a\sin\phi}{c + a\cos\phi}\frac{\partial\theta}{\partial s}\frac{\partial\phi}{\partial s} \end{aligned} \tag{15}$$

System and Cost Definitions

System

The actual system (Σ) has state x and its dynamics is

$$\dot{x} = \sum_{i=1}^{m} f_i(x) u_i \tag{16}$$

where $m < n$ and hence it is an under-actuated system. Let $F(x) := \begin{pmatrix} f_1(x) & f_2(x) & \cdots & f_m(x) \end{pmatrix}$ and $u := \begin{pmatrix} u_1 & u_2 & \cdots & u_m \end{pmatrix}^T$, then (16) has the compact form $\dot{x} = F(x)u$. In addition we assume $\text{Rank}(F(x)) = m$ for all $x \in \mathbb{R}^n$. (Σ) is associated with a cost functional

$$E = \int_0^1 |u(t)|^2 dt \tag{17}$$

with fixed boundary conditions:

$$x(0) = x_0, x(1) = x_1 \tag{18}$$

Augmented system

The augmented system $(\bar{\Sigma})$ is

$$\dot{x} = \sum_{i=1}^{m} f_i(x)u_i + \sum_{i=m+1}^{n} \bar{f}(x)_i \bar{u}_i \tag{19}$$

where f_i's are the same as in (16) and $\bar{f}_i(x)^T \bar{f}_j(x) = \delta_{ij}$ for all $x \in \mathbb{R}^n, i, j \in \{m+1, m+2, \cdots, n\}$. Similarly we can define $\bar{F}(x) = \begin{pmatrix} \bar{f}_{m+1}(x) & \bar{f}_{m+2}(x) & \cdots & \bar{f}_n(x) \end{pmatrix}$ and $u = \begin{pmatrix} \bar{u}_{m+1}(x) & \bar{u}_{m+2}(x) & \cdots & \bar{u}_n(x) \end{pmatrix}$ then (19) is equivalent to $\dot{x} = F(x)u + \bar{F}(x)\bar{u}$.

$(\bar{\Sigma})$ is associated with a cost functional

$$E_k(u, \bar{u}) = \int_0^1 |u(t)|^2 + k|\bar{u}(t)|^2 dt \tag{20}$$

where k is a constant and same boundary conditions (18).

Path Planning

Controllability

A key difference between linear and nonlinear systems is that the direction generated by Lie bracket of two admissible directions $f_i(x), f_j(x)$:

$$[f_i, f_j](x) := \frac{df_j}{dx}(x) f_i(x) - \frac{df_i}{dx}(x) f_j(x)$$

is also an admissible direction for system (Σ).
When at state x, the space of available directions of motion for system (Σ) is given by the distribution

$$\begin{aligned} \Delta_1(x) &= \text{span}\{f_1, \dots, f_p, [f_1, f_2], \dots, [f_{p-1}, f_p]\} \\ &= \Delta_0 \oplus \text{span}\{[f_i, f_j] \mid f_i, f_j \in \Delta_0(x)\}. \end{aligned}$$

Controllability contd.

Using this construction iteratively, we see that the distributions

$$\Delta_i(x) := \Delta_{i-1}(x) \oplus \mathrm{span}\{[f_1, f_2](x) \mid f_1, f_2 \in \Delta_{i-1}(x)\} \tag{21}$$

are key to understanding the reachable space of Σ. The precise relationship is given by the following theorem:

Theorem 1 (Chow)

Consider the control system (16) *and the associated distribution* $\Delta_0(x) := \mathrm{span}\{f_1(x), \ldots, f_p(x)\}$. *If* $\lim_{i\to\infty} \Delta_i(x) = \mathbb{R}^n$ *for all* $x \in \mathbb{R}^n$, *then the system is controllable.*

Useful lemma

How to construct the control?

Lemma 4

For system $(\bar{\Sigma})$ *as in* (19) *and cost* E_k *as in* (20), *the optimal trajectory* x_k^* *is smooth and regular, i.e.,* $\frac{dx_k^*}{ds}(s)$ *exists and does not equal to* $\mathbf{0}$ *for all* $s \in [0, 1]$.

Hence can use Euler-Lagrange (so as geodesic equation, heat flow equation)

Lemma 5

The cost functional E_k *along system* $(\bar{\Sigma})$ *and boundary condition* (18) *is equivalent to some length functional with same boundary condition.*

In fact $G(x) := \left((F|\bar{F})^{-1}(x)\right)^T K \left((F|\bar{F})^{-1}(x)\right)$

Intuition

$$
\begin{aligned}
(\Sigma) : \dot{x} = F(x)u, \qquad & E = \int_0^1 |u(s)|^2 ds \\
(\bar{\Sigma}) : \dot{x} = F(x)u + \bar{F}(x)\bar{u}, \quad & E_k = \int_0^1 |u(s)|^2 + k|\bar{u}(s)|^2 ds
\end{aligned}
$$

E_k includes infeasible control efforts $\bar{u}$ which is augmented to system (Σ). This additional cost is penalized by the parameter k. In order to achieve a minimal cost, $\bar{u}$ is forced to be small. Hence the trajectory will gets closer to a horizontal path of (Σ) if k gets larger.

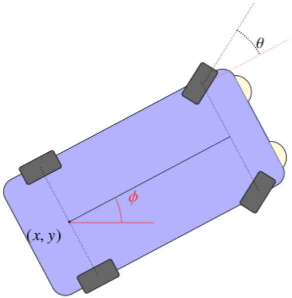
Unicycle example

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ \theta \end{pmatrix} = u_1 \underbrace{\begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}}_{f_1} + u_2 \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{f_2} + \bar{u}_3 \underbrace{\begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}}_{\bar{f}_3}. \tag{22}$$

And use the formula in lemma 5 we computed the Riemannian tensor G:

$$G = \begin{pmatrix} k\sin^2\theta + \cos^2\theta & (1-k)\sin\theta\cos\theta & 0 \\ (1-k)\sin\theta\cos\theta & \sin^2\theta + k\cos^2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Boxcar example



$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ \theta \\ \phi \end{pmatrix} = u_1 \underbrace{\begin{pmatrix} \cos\phi \\ \sin\phi \\ 0 \\ \frac{1}{d}\sin\theta \end{pmatrix}}_{f_1} + u_2 \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}}_{f_2}. \tag{23}$$

The augmented two control directions are

$$\bar{f}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \bar{f}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

And G has entries

$$G_{11} = \frac{k+1}{d^2} \sin^2\theta \cos^2\phi + (k-1)\sin^2\phi + 1$$

$$G_{22} = \frac{k+1}{d^2} \sin^2\theta \sin^2\phi - (k-1)\sin^2\phi + k$$

$$G_{33} = 1, G_{44} = k$$

$$G_{12} = G_{21} = \frac{k+1}{d^2} \sin^2\theta \sin\phi \cos\phi - (k-1)\sin\phi\cos\phi$$

$$G_{14} = G_{41} = -\frac{k}{d} \sin\theta\cos\phi, G_{24} = G_{42} = -\frac{k}{d}\sin\theta\sin\phi$$

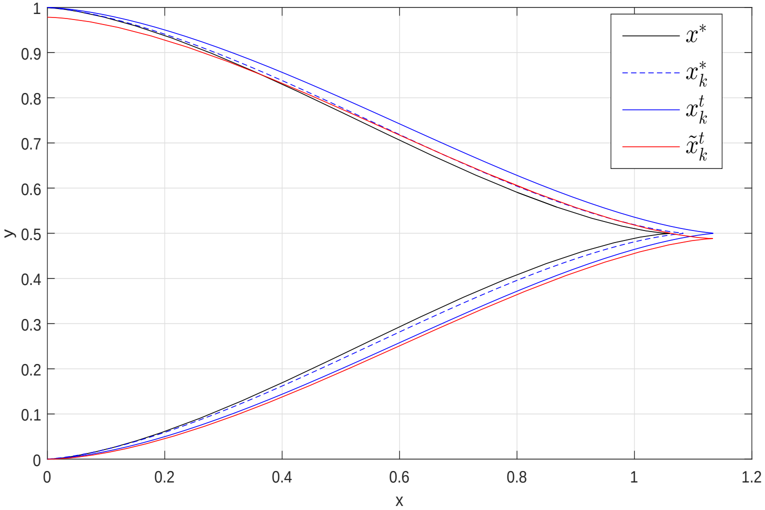
$$G_{13} = G_{31} = G_{23} = G_{32} = G_{34} = G_{43} = 0$$

4 Curves

1. x^*: minimizer of E for system (Σ). (Horizontal geodesic)
2. x_k^*: minimizer of E_k for system $(\bar{\Sigma})$.
3. x_k^t: the solution $x(s,t)$ of heat flow equation (9) with fixed t.
4. $\tilde{x}_k^t$: the trajectory of (Σ) starting from $x(0) = x_0$ under the control extracted from $x_k^t(s)$, i.e., if $\dot{x}_k^t(s) = F(x_k^t(s))u_k^t(s) + \bar{F}(x_k^t(s))\bar{u}_k^t(s)$, then $\dot{\tilde{x}}_k^t(s) = F(\tilde{x}_k^t(s))u_k^t(s)$.

Conjecture: $x_k^t(s) \to x^*(s)$ when both $t, k \to \infty$?

4 Curves for parallel parking



Theoretical result

Theorem 2

For system Σ with F globally Lipschitz of L, let $k > 0$ be arbitrary. Define $x^*, x_k^*, x_k^t, \tilde{x}_k^t$ as earlier. In addition assume that x_k^* is a global minimizer of cost functional E_k defined in (20) for system $(\bar{\Sigma})$. Denote $U := E(x^*)$. Then for any $\epsilon > 0$, there exists $T = T(\epsilon, k)$ such that for all $t \geq T$,

1. $E(\tilde{x}_k^t) \leq U + \epsilon$
2. $|\tilde{x}_k^t(s) - x_k^t(s)| \leq \left(\sqrt{\frac{2s}{k}(U+\epsilon)}\right) e^{sL^2(U+\epsilon)}$ *for all* $k \in [0,1]$. *In particular,* $|\tilde{x}_k^t(1) - x_1| \leq \left(\sqrt{\frac{2}{k}(U+\epsilon)}\right) e^{L^2(U+\epsilon)}$.

Obstacle Avoidance

Modified Riemannian metric

Obstacles are defined to be a set $\Omega \in \mathbb{R}^n$. A path with obstacle avoidance is a system trajectory $x(s)$ such that $x(s) \notin \Omega$ for all $s \in [0, 1]$.
Define the new Riemannian metric tensor as

$$\tilde{g}_x(\dot{x}, \dot{x}) = \dot{x}^T \left(\lambda(x) G(x)\right) \dot{x}$$

where $G(x)$ is define in the remark of lemma 5 and $\lambda(\cdot) : \mathbb{R}^n \backslash D \to \mathbb{R}_+$ is a function with the following properties:

1. $\lambda(x) \in \mathcal{C}^1$,
2. $\lambda(x) \to \infty$ as $x \to \partial\Omega$,
3. $\lambda(x) = 1$ when x is far away from Ω.

Candidate λ function

In the case when obstacles are balls, i.e., $\Omega = \cup_{i=1}^{l} B_{r_i}(p_i)$, one candidate of such $\lambda(x)$ function will be a modification of penalty function from avoidance control:

$$\lambda(x) = 1 + \sum_{i=1}^{l} \left(\min\{0, \frac{|x - p_i|^2 - R_i^2}{|x - p_i|^2 - r_i^2} \right)^2 \tag{24}$$

where $R_i > r_i, i = 1, 2, \cdots, l$ such that $B_{R_i}(p_i)$ are the buffer region.

Advantage

It is observed that an advantage of path planning via HFE is that by customizing initial curve, we are able to configure how these obstacles are by-passed. On the contrary, directly solving the geodesic equation will not give the customized shortest path.

Conclusion

Conclusion

1. Studied geodesics on Riemannian manifold and finding geodesics via heat flow method.
2. Proposed a way to find optimal control for under-actuated nonlinear affine system by applying heat flow method to an augmented system with a penalty in the infeasible control direction.
3. Proven that the actual cost can be made arbitrarily closer to optimal cost and the error in trajectory is bounded and can also be made arbitrarily small by setting the penalty large enough.
4. Studied the application of our heat flow method in path planning with obstacle avoidance. Pointed out its advantage in path shape configuration.

The End