On Almost Lyapunov Functions for Systems with Inputs

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December 11th, 2019

Overview

Introduction

Background

- Lyapunov's direct method tells that a time invariant, autonomous system
$$\dot{x} = f(x) \tag{1}$$
has a globally asymptotically stable origin if there exists a positive definite, radially unbounded function $V(x)$ such that $\dot{V}(x) := \left\langle \frac{d}{dx} V(x), f(x) \right\rangle$ is negative definite [Khalil, 2002].
- Finding such V satisfying the opposite sign definite constraints is difficult.
- Non-monotonic Lyapunov functions are widely studied [Aeyels and Peuteman, 1998, Ahmadi and Parrilo, 2008, Karafyllis, 2011, Meigoli and Nikravesh, 2012]
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Non-monotonic V

"$V(x(t))$ decreases more than it increases over any long enough time"
which can be guaranteed by

1. Considering the finite step difference of Lyapunov function, i.e., existence of $T > 0$ such that $V(x(T)) - V(x(0)) < 0$ for all $x(0) \neq 0$ [Aeyels and Peuteman, 1998]; or
2. Imposing some conditions on the higher order derivatives of V [Meigoli and Nikravesh, 2009]; or
3. The study of almost Lyapunov functions such that $\Omega := \{x \in \mathbb{R}^n : \dot{V}(x) > -aV(x)\}$ is small enough [Liu et al., 2016], [Liu et al., 2019].

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Autonomous systems

Global asymptotic stability

$$\dot{x} = f(x) \tag{1}$$

- The system (1) is globally asymptotically stable (GAS) if there exists $\beta \in \mathcal{KL}$ such that

$$|x(t)| \leq \beta(|x_0|, t) \quad \forall x_0 \in \mathbb{R}^n, t \geq 0. \tag{2}$$

- GAS can be shown via a Lyapunov function $V(x) : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that for some $\alpha_1, \alpha_2 \in \mathcal{K}_\infty, a > 0$ such that

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \quad \forall x \in \mathbb{R}^n, \tag{3}$$

$$\dot{V}(x) = \left\langle \frac{d}{dx} V(x), f(x) \right\rangle \leq -aV(x) \quad \forall x \in \mathbb{R}^n. \tag{4}$$

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Almost Lyapunov functions for autonomous systems

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- The inequality (4) may not hold for all $x \in \mathbb{R}^n$, because of:
  1. Poor choice of V;
  2. Numerical computation;
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- The inequality (4) may not be satisfied for $x \in \Omega \subset \mathbb{R}^n$.
- Our previous work [Liu et al., 2019] shows that as long as the volume of the connected components of Ω are small enough, the system (1) is GAS.

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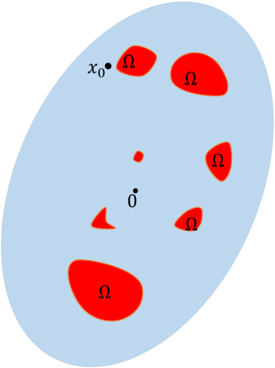

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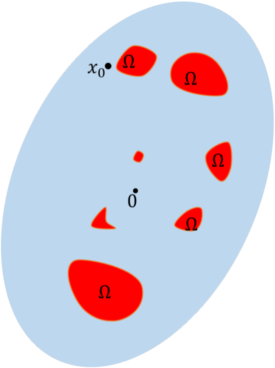

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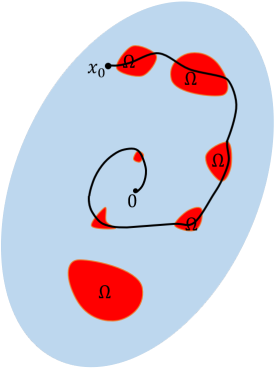

Systems with inputs

Input-to-state Stability

$$\dot{x} = f(x, u) \tag{5}$$

- The system (5) is input-to-state stable (ISS) [Sontag, 1989] if there exists $\beta \in \mathcal{KL}, \gamma \in \mathcal{K}_\infty$ such that

$$|x(t; x_0, u)| \leq \beta(|x_0|, t) + \gamma(\operatorname{ess\,sup}_{s \in [0,t]} |u(s)|) \tag{6}$$

for all $x_0 \in \mathbb{R}^n, t \geq 0$ and $u \in \mathcal{U} \subseteq \mathbb{R}^m$.

- ISS can be shown via a Lyapunov function $V(x) : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that for some $\alpha_1, \alpha_2, \rho \in \mathcal{K}_\infty, a > 0$, (3) holds and

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Problem set up

$$\left\langle \frac{d}{dx}V(x), f(x,u) \right\rangle \leq -aV(x) \tag{7}$$

- Denote the solution of the auxiliary system

$$\dot{x} = f_\rho(x,d) := f(x, \rho(|x|)d) \tag{8}$$

by $x_\rho(t; x_0, d)$, where $\|d\| \leq 1$.

- (7) holds for all $|u| \leq \rho(|x|)$ is equivalent to

$$V'(x) := \sup_{|d|\leq 1} \left\langle \frac{d}{dx}V(x), f_\rho(x,d) \right\rangle \leq -aV(x). \tag{9}$$

- Appealing to our almost Lyapunov function framework, we allow (9) to be violated for $x \in \Omega \subset \mathbb{R}^n$.
- Can we still show ISS if we only have (9) for all $x \in \mathbb{R}^n \backslash \Omega$, while Ω is "small" enough?

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How to quantify the size of Ω?

- Although volume of Ω in Euclidean space is used in our previous work, it is conservative.
- Small Ω does not necessarily imply $x_\rho(t; x_0, d)$ will stay inside Ω for finite time.
- Instead, we directly impose assumptions on the Ω dwell time:

$$T := \sup_{x_0 \in \Omega, \|d\| \leq 1} \inf_{t \geq 0} \{t : x_\rho(t; x_0, d) \notin \Omega\} \tag{10}$$

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The upper bound of time derivative of V

$$V'(x) := \sup_{|d|\leq 1} \left\langle \frac{d}{dx} V(x), f_\rho(x,d) \right\rangle, \tag{11}$$

Lemma

Let $V \in C^1(\mathbb{R}^n \to \mathbb{R}_{\geq 0})$ be a positive definite function and assume the system (5) has an equilibrium at 0. Then V' defined via (11) exists for all $x \in \mathbb{R}^n$ and is Lipschitz when both $f_\rho(x,d)$, $\frac{d}{dx}V(x)$ are Lipschitz in x.

Lemma

Assume Ω is bounded and all the assumptions in the above lemma hold. Then $\frac{d}{dx}V'$ exists almost everywhere in Ω. In addition, there exists $c > 0$ such that for all $x \in \Omega$ where $\frac{d}{dx}V'(x)$ exists,

$$\left| \left\langle \frac{d}{dx} V'(x), f_\rho(x,d) \right\rangle \right| \leq cV(x) \quad \forall |d| \leq 1 \tag{12}$$

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Main result

Recap

- $f_\rho(x,d) := f(x,\rho(|x|)d)$ is Lipschitz in x;
- $V \in C^1(\mathbb{R}^n \to \mathbb{R}_{\geq 0})$ has Lipschitz gradient;
- $\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \quad \forall x \in \mathbb{R}^n$
- $V'(x) := \sup_{|d|\leq 1} \left\langle \frac{d}{dx} V(x), f_\rho(x,d) \right\rangle$;
- $V'(x) \leq -aV(x) \quad \forall x \in \mathbb{R}^n \backslash \Omega$;
- $\left| \left\langle \frac{d}{dx} V'(x), f_\rho(x,d) \right\rangle \right| \leq cV(x) \quad \forall |d| \leq 1, a.a.\ x \in \Omega$;
- $T := \sup_{x_0 \in \Omega, \|d\| \leq 1} \inf_{t\geq 0} \{t : x_\rho(t; x_0, d) \notin \Omega\}$.

Main result

Theorem

Consider a system with input (5) *with the assumptions listed in the previous slide satisfied with some* $a, c > 0$. *There exists an increasing function* $\alpha : [0, 1) \to [0, \infty)$ *with* $\alpha(0) = 0, \lim_{t \to 1^-} \alpha(t) = \infty$ *such that as long as the* Ω *dwell time* T *satisfies*

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the system (5) *is ISS.*

$$\alpha(t) = \ln\left(\frac{1+t}{1-t}\right) + 2\arccos\left(\frac{1}{\sqrt{t^2+1}}\right) \tag{14}$$

Main result

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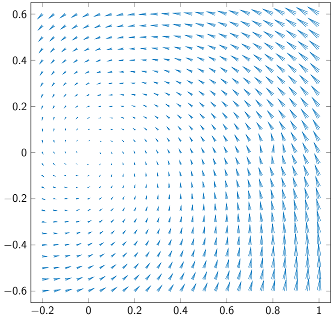
where $\lambda(x) = \left(\frac{a+b}{2}\right) \min\left\{\frac{|x-x_c|}{r}, 1\right\} - \frac{b}{2} + k.$



- Take $V = \frac{1}{2}|x|^2, \rho(s) = ks$. $V'(x) = 2(-\lambda(x) + k)V(x)$.
- $\Omega = \{x \in \mathbb{R}^2 : |x - x_c| < r\}$.
- For numerical values $a = 1, b = 0.5, k = 0.1, r = 0.1, \mu = 2$ and $x_c = (0.8, 0)^\top$, it is computed $c \approx 30.54$ and $T < 0.125$.
- (13),(14) give an upper bound of Ω dwell time of 0.131. Hence the system (15) is ISS.

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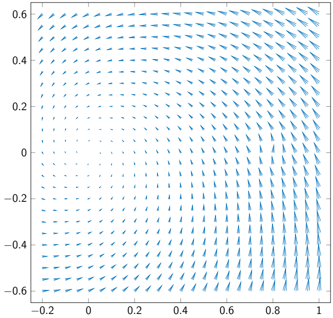

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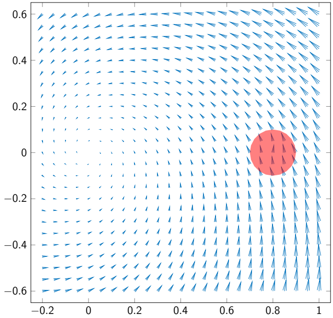

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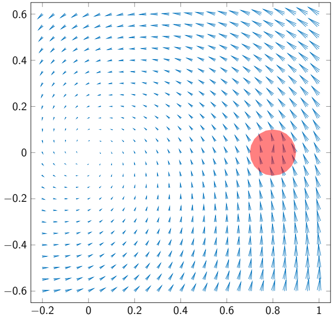

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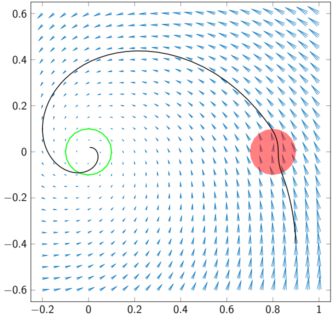

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Comments

- Our analysis via almost Lyapunov function also applies when there are multiple "bad" regions Ω.
- Because different assumption on the size of Ω is used in this work compared with our previous work [Liu et al., 2019], The system is shown to be ISS with even much larger Ω.
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Sketch of proof

- Let t_1, t_2 be the time the solution trajectory $x_\rho(t; x_0, d)$ enters and leaves Ω. We want to ultimately show $V(x_\rho(t_2; x_0, d)) \leq \eta V(x_\rho(t_1; x_0, d))$ for some $\eta < 1$.
- From the definition of V' in (11) and the assumption (12) on V', we have

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$$\frac{V(x_\rho(t; x_0, d))}{V(x_\rho(t_1; x_0, d))} \leq R_1(t - t_1),$$
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where

$$R_1(s) := \left(\cosh \sqrt{c s} - \frac{a}{\sqrt{c}} \sinh \sqrt{c s}\right),$$
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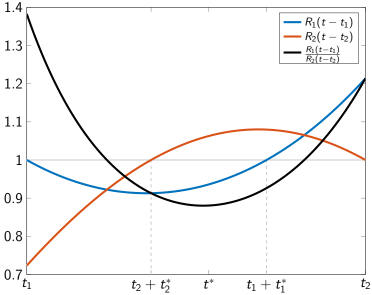

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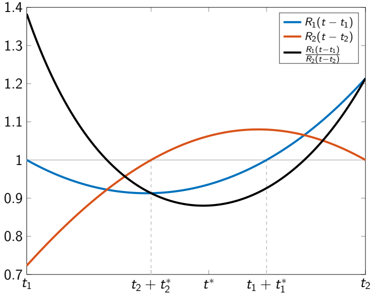

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Conclusion

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- The study of almost Lyapunov functions from our previous work was generalized to systems with inputs.
- When there are "bad" regions Ω in the state space where V does not decrease fast enough, an upper bound of the Ω dwell time was found to guarantee that the system with inputs is still ISS.
- Example showed that our method of showing stability via almost Lypapunov function is able to handle non-trivial perturbations and it is believed to be applicable to a broader class of systems.

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