Higher Order Derivatives of Lyapunov Functions for Stability of Systems with Inputs

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Overview

Introduction

Background

- Lyapunov's direct method tells that a time invariant, autonomous system
$$\dot{x} = f(x) \tag{1}$$
has a globally asymptotically stable origin if there exists a positive definite, radially unbounded function $V(x)$ such that $\dot{V}(x) := \left\langle \frac{d}{dx} V(x), f(x) \right\rangle$ is negative definite [Khalil, 2002].
- Finding such V satisfying the opposite sign definite constraints is difficult.
- Non-monotonic Lyapunov functions are widely studied [Aeyels and Peuteman, 1998, Ahmadi and Parrilo, 2008, Karafyllis, 2011, Meigoli and Nikravesh, 2012]
- The time varying nature and presence of inputs complicate the analysis.

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Non-monotonic V

"$V(x(t))$ decreases more than it increases over any long enough time"
which can be guaranteed by

1. Considering the finite step difference of Lyapunov function, i.e., existence of $T > 0$ such that $V(x(T)) - V(x(0)) < 0$ for all $x(0) \neq 0$ [Aeyels and Peuteman, 1998]; or
2. Imposing some conditions on the higher order derivatives of V [Meigoli and Nikravesh, 2009]; or
3. The study of "almost" Lyapunov functions such that $\Omega := \{x \in \mathbb{R}^n : \dot{V}(x) > -aV(x)\}$ is small enough [Liu et al., 2016].

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Autonomous systems

Global asymptotic stability

$$\dot{x} = f(x) \tag{1}$$

- The system (1) is globally asymptotically stable (GAS) if there exists $\beta \in \mathcal{KL}$ such that
$$|x(t)| \leq \beta(|x_0|, t) \tag{2}$$
for all $x_0 \in \mathbb{R}^n$, $t \geq 0$.
- GAS can be shown via a Lyapunov function $V(x) : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that for some $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$
$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \tag{3}$$
and
$$\dot{V}(x) < 0 \tag{4}$$
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Higher order derivatives for autonomous systems

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Not necessarily $\dot{V}(x) < 0$ for all $x \neq 0$.

Theorem ([Butz, 1969])

Let $V(x) : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be a three times differentiable Lyapunov function and assume f is twice differentiable. If there exists $a_1, a_2 \geq 0$ such that

$$a_2 \dddot{V}(x) + a_1 \ddot{V}(x) + \dot{V}(x) < 0$$

for all $x \neq 0$, then (1) is GAS.

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Theorem ([Meigoli and Nikravesh, 2009])

Let $V(x) : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be a Lyapunov function. If there exists $m \in \mathbb{N}_{\geq 2}$ and $a_1, \cdots a_{m-1} \geq 0$ such that V is $m-1$ times differentiable and f is $m-2$ times differentiable and

$$V^{(m)}(x) + a_{m-1} V^{(m-1)}(x) + \cdots + a_1 \dot{V}(x) < 0$$

for all $x \neq 0$, then (1) is GAS.

Time varying systems with inputs

GUAS for time varying systems with inputs

$$\dot{x} = f(t, x, u) \tag{5}$$

- The system (5) is globally uniformly asymptotically stable (GUAS) if there exists $\beta \in \mathcal{KL}$ such that

$$|x(t; t_0, x_0, u)| \leq \beta(|x_0|, t - t_0) \tag{6}$$

for all $x_0 \in \mathbb{R}^n$, $t \geq t_0 \geq 0$ and $u \in \mathcal{U} \subset \mathbb{R}^m$.

- GUAS can be shown via a time varying Lyapunov function $V(t, x) : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that for some $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and a positive definite function $\psi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that

$$\alpha_1(|x|) \leq V(t, x) \leq \alpha_2(|x|), \tag{7}$$

$$\dot{V}(t, x, u) := \frac{\partial}{\partial t} V(t, x) + \left\langle \frac{\partial}{\partial x} V(t, x, u), f(t, x, u) \right\rangle \leq -\psi(|x|) \tag{8}$$

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Input-to-state Stability

$$\dot{x} = f(t, x, u) \tag{5}$$

- The system (5) is input-to-state stable (ISS) [Sontag, 1989] if there exists $\beta \in \mathcal{KL}, \gamma \in \mathcal{K}_\infty$ such that

$$|x(t; t_0, x_0, u)| \leq \beta(|x_0|, t - t_0) + \gamma(\operatorname*{ess\,sup}_{s \in [t_0, t]} |u(s)|) \tag{9}$$

for all $x_0 \in \mathbb{R}^n, t \geq t_0 \geq 0$ and $u \in \mathcal{U} \subseteq \mathbb{R}^m$.

Auxiliary system; GUAS and ISS

Lemma ([Sontag and Wang, 1996])

The system

$$\dot{x} = f(t, x, u) \tag{5}$$

is ISS if and only if its auxiliary system

$$\dot{x} = f_\rho(t, x, d) := f(t, x, \rho(|x|)d) \tag{10}$$

is GUAS with $d \in \mathcal{U} = \mathbb{B}_1^m$ *for some* $\rho \in \mathcal{K}_\infty$.

Problem set up

$$\dot{x} = f(t, x, u) \tag{5}$$

- Known some $V(t, x)$ satisfying the "sandwich condition" (7) and $\rho \in \mathcal{K}_\infty$; but not necessarily $\dot{V}(t, x, d) \leq -\psi(|x|)$ for all $x \in \mathbb{R}^n$, $t \geq t_0$ and $|d| \leq 1$.
- Like the autonomous case, we would like to study the "higher order derivatives" of V to show stability.
- Difficulty: $\dot{V}(t, x, d)$ depends on d, which may not be differentiable w.r.t. time t, $\ddot{V}(t, x, d) = \frac{d}{dt}\dot{V}(t, x, d)$ does not exist in general.

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Defining higher order derivatives for systems with inputs

- The higher order derivatives of V are defined and constructed iteratively:
  1. $v_0(t,x) := V(t,x)$;
  2. $V_i(t,x) := \frac{\partial}{\partial t} v_{i-1}(t,x) + \sup_{|d|\leq 1} \langle \frac{\partial}{\partial x} v_{i-1}(t,x), f_\rho(t,x,d) \rangle$;
  3. Construct $v_i(t,x) \in C^1(\mathbb{R}_{\geq 0} \times \mathbb{R}^n)$ such that $v_i(t,x) \geq V_i(t,x)$ for all $x \in \mathbb{R}^n, t \geq t_0$.
- $v_i(t,x)$ is sign indefinite for $i > 0$;
- $\frac{d}{dt} v_{i-1}(t,x) \leq v_i(t,x)$.
- The higher order derivatives $v_i(t,x)$ are globally decrescent up to order $m \in \mathbb{N}$ if there exists $\phi \in \mathcal{K}_\infty$ such that

$$v_i(t,x) \leq \phi(|x|) \quad \forall x \in \mathbb{R}^n, t \geq t_0, i = 0, \cdots m.$$

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Main result

Theorem

Given a system (5), a C^1 positive definite function $V(t,x)$ satisfying (7) and some $\rho \in \mathcal{K}_\infty$, generate the higher order derivatives v_i by f_ρ and V. If v_i's are globally decrescent up to order $m \in \mathbb{N}$ and there exist

$$a_0 > 0,\ a_i \geq 0 \quad \forall i = 1, \cdots, m \tag{11}$$

such that

$$\sum_{i=0}^{m} a_i v_i(t,x) \leq 0 \quad \forall x \in \mathbb{R}^n, t \geq t_0, \tag{12}$$

then the system (5) is ISS.

Sketch of proof

The following lemma can be proven using induction, similar to what is done in [Meigoli and Nikravesh, 2009]:

Lemma

Let $x(t; t_0, x_0, d)$ be a solution of system (10). When (12) holds with some a_i's satisfying (11) and $a_m = 1$, for any $b > 0$ if $v_0(t, x(t; t_0, x_0, d)) \geq b$ for all $t \in [t_0, t_0 + T]$ for some $T \geq 0$, then

$$v_0(t, x(t; t_0, x_0, d)) \leq -b \sum_{j=1}^{m} a_{m-j} \frac{(t-t_0)^j}{j!} + \sum_{j=0}^{m-1} \sum_{i=0}^{j} \frac{(t-t_0)^j}{j!} a_{m+i-j} v_i(t_0, x_0) \tag{13}$$

for all $t \in [t_0, t_0 + T], \|d\| \leq 1$.

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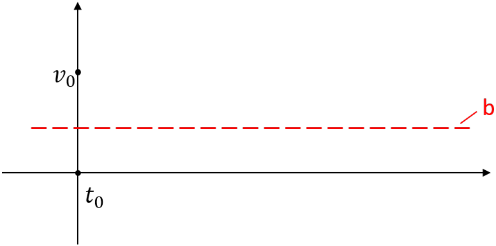

The origin can be subsequently shown globally stable and uniformly attractive for the auxiliary system and hence it is GUAS; the original system (5) is therefore ISS.

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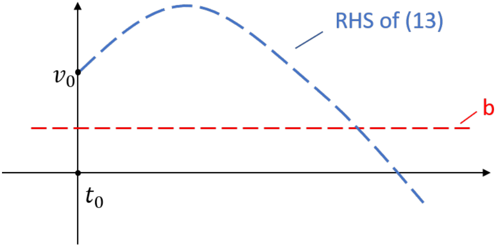

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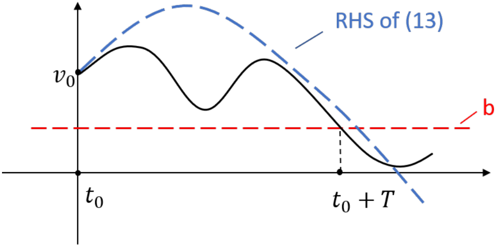

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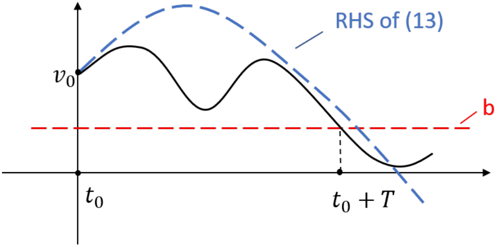

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Examples

Linear system with unaligned V

$$\dot{x} = f(x,u) = Ax + u, \qquad A = \begin{pmatrix} -0.1 & -1 \\ 2 & -0.1 \end{pmatrix} \tag{14}$$

- Choosing $V = |x|^2$ gives $\dot{V} = -0.2(x_1 - 5x_2)^2 + 4.8x_2^2$ even if $u \equiv 0$; stability is inconclusive.
- Pick $\rho(s) = 0.05s$, Higher order derivatives are constructed:

$$\begin{aligned} v_1 &= -0.1x_1^2 + 2x_1x_2 - 0.1x_2^2, \\ v_2 &= 4.13x_1^2 - 0.6x_1x_2 - 1.87x_2^2, \\ v_3 &= -1.5907x_1^2 - 15.62x_1x_2 + 1.4093x_2^2. \end{aligned}$$

- Let $a_0 = 0.1, a_1 = 8, a_2 = 0.5, a_3 = 1,$

$$\sum_{i=0}^{3} a_i v_i = -0.2257x_1^2 + 0.08x_1x_2 - 0.2257x_2^2 = -x^\top \begin{pmatrix} 0.2257 & -0.04 \\ -0.04 & 0.2257 \end{pmatrix} x$$

So (14) is ISS.

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Slowly varying between two stable linear modes

$$\dot{x} = f(t,x,u) = \sin^2(kt)A_1x + \cos^2(kt)A_2x + u =: A(k,t)x + u \quad (15)$$

where $A_1 = \begin{pmatrix} -0.1 & -1 \\ 2 & -0.1 \end{pmatrix}, A_2 = \begin{pmatrix} -0.1 & -2 \\ 1 & -0.1 \end{pmatrix}$

- No common Lyapunov function for A_1, A_2.
- Again choose $V(t,x) = |x|^2, \rho(s) = 0.05s$. It can be shown that $v_i(t,x) = x^\top(P_i(k,t) + kp(k)Q_i(k,t))x$ where Q_i are uniformly bounded, $p(k)$ is a polynomial in k and

$$P_1 = \begin{pmatrix} -0.1 & -C \\ -C & -0.1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} C^2 - 3C + 0.13 & 0.3C \\ 0.3C & C^2 + 3C + 0.13 \end{pmatrix},$$

$$P_3 = \begin{pmatrix} -0.5C^2 + 1.5C + 0.224 & -C^3 + 8.81C \\ -C^3 + 8.81C & -0.5C^2 - 1.5C + 0.224 \end{pmatrix}$$

- Pick $a_0 = 0.1, a_1 = 8, a_2 = 0.5, a_3 = 1$, $\sum_{i=0}^3 a_i v_i \approx \sum_{i=0}^3 a_i x^\top P_i x \leq 0$ for sufficiently small k and hence (15) is ISS.

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Conclusion

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- An alternative method for determining ISS for time varying systems with inputs was studied.
- Higher order derivatives of Lyapunov functions were defined.
- It is claimed and proven that if there exists a linear combination of those higher order derivatives with non-negative coefficients (except that the coefficient of the 0-th order term needs to be positive) which is negative semi-definite, then the system is GUAS. Consequently if a system whose auxiliary system admits a positive definite function which satisfies the aforementioned conditions, this system is ISS.

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