09-04-01-LinearAlgebra

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目录

Chapter 1　Overall

2 条主线：linear space, linear mapping

Chapter 2 Space

线性空间（加法、乘法）-> 线性映射 <-> 矩阵

现实几何空间还有距离和角度的概念，可用内积（双线性函数）来刻画。具有度量的线性空间。分类：欧几里得空间（实数域、有限维、线性空间、内积，内积有交换律）、酉空间（复数域、有限维、线性空间、内积，内积有共轭交换律）。

线性变换：空间 A 到空间 AZ 自身的线性映射。

2.0.1 Linear Space

向量加法、标量乘法构成的单位环。

2.0.2 Metric Space

Definition 2.1. *The set X with a distance function d, d satisfies*??. *Metric Space is noted as* (X, d).

Definition 2.2. 紧集：(X, d) 中的子集 A，A 中任意序列都存在一子列 x_n,x_n 收敛到 A 中某点。

Definition 2.3. 稠密集：(X, d) 中的子集 A，对于 X 中的任意点 x，A 中存在点 a，使得 $d(x, a) < \varepsilon$

Definition 2.4. X 可分：(X, d) 中存在一个可数稠密集。

2.0.2.1 Complete Metric Space

Definition 2.5. 收敛：*sequence* $\{x_n\}$ 收敛到 c，*means that* $\lim_{x\to\infty} d(x_n, c) = 0$, *noted as* $\lim_{x\to\infty} x_n = c$

Definition 2.6. *Cauchy* 基本列：$\lim_{m\to\infty,n\to\infty} d(x_m, x_n) = 0$

Definition 2.7. 完备距离空间：所有 *Cauchy* 基本列收敛于一点

Definition 2.8. 不完备：对于苹果空间，从宇宙开始到宇宙结束的所有苹果序列，收敛到我，则不完备。

2.0.3 Banach Space

完备、赋范、线性。

2.0.4 Inner Product Space

$$\begin{aligned}
&(\alpha x+\beta y)\cdot z=\alpha x\cdot z+\beta y\cdot z, && \text{线性}\\
&x\cdot(\alpha y+\beta z)=\bar{\alpha}x\cdot y+\bar{\beta}x\cdot z, && \text{共轭线性}\\
&x\cdot y=y\bar{\cdot}x, && \text{共轭对称}\\
&x\cdot x\geqslant 0, && \text{正定}\Rightarrow ||x||=\sqrt{x\cdot x}\\
&|x\cdot y|\leqslant ||x||\cdot||y||, && satisfiesCauchy-Schwarz
\end{aligned}\tag{2.1}$$

2.0.5 Hilbert Space

完备，内积。

内积 $\Rightarrow$ 范数 $\Rightarrow$ 完备

Proposition 2.1. $[0,1]$ 上的复连续函数空间 $C([0,1])$, 定义内积 $f\cdot g=\int_0^1 f(t)g(t)\,dt$, *proof that* $C([0,1])$ 不是 *Hilbert Space*

$$\begin{aligned}
&f_n(t)=\begin{cases}1, & 0\leqslant t\leqslant\frac{1}{2}\\ -2n(t-\frac{1}{2})+1, & \frac{1}{2}<t\leqslant\frac{1}{2n}+\frac{1}{2}\\ 0, & \frac{1}{2n}+\frac{1}{2}<t\leqslant 1\end{cases}\\
&||f_n-f_m||\leqslant(\frac{1}{n}+\frac{1}{m})^{\frac{1}{2}}\to 0, is\ Cauchy\ Sequence.\\
&\lim f_n=\begin{cases}1, 0\leqslant t\leqslant\frac{1}{2}\\ 0, \frac{1}{2}<t\leqslant 1\end{cases}\\
&\therefore \lim f_n\notin C([0,1])
\end{aligned}\tag{2.2}$$

2.0.6 Euclid Space

有序的 n 元组的全体称为 n 维 Euclid 空间，记为 $\mathbb{R}^n$，称 $\boldsymbol{p}=(p_i)_{i=1}^n\in\mathbb{R}^n$ 是 $\mathbb{R}^n$ 的一个点。为便于研究，本论文以 $\mathbb{R}^3$ 为背景空间，所涉及的函数默认为可微实值函数。如果实函数 f 的任意阶偏导数存在且连续，则称函数是可微的（或无限可微的，或光滑的，或 C^∞ 的）。

由于微分运算是函数的局部运算，限制所讨论函数的定义域在 $\mathbb{R}^3$ 中的任意开集，所讨论的结论仍然成立。

自然坐标函数：定义在 $\mathbb{R}^n$ 上的实值函数 $x_i : \mathbb{R}^n \to \mathbb{R}$，使得 $\boldsymbol{p} = (p_i)_{i=1}^n = (x_i(\boldsymbol{p}))_{i=1}^n$

切向量：由 $\mathbb{R}^n$ 中的二元组构成，$\boldsymbol{v_p} = (\boldsymbol{p}, \boldsymbol{v})$，其中 $\boldsymbol{p}$ 是作用点，$\boldsymbol{v}$ 是向量部分

切空间 $T_p\mathbb{R}^n$: 作用点 $\boldsymbol{p} \in \mathbb{R}^n$ 的所有切向量的集合。利用向量加法与数量乘法使某点的切空间称为向量空间，与背景空间存在非平凡同构。

向量场 $\boldsymbol{V}$: 作用于空间点的向量函数，$\boldsymbol{V}(\boldsymbol{p}) \in T_p\mathbb{R}^n$

逐点化原理：$(\boldsymbol{V} + \boldsymbol{W})(\boldsymbol{p}) = \boldsymbol{V}(\boldsymbol{p}) + \boldsymbol{W}(\boldsymbol{p}),\ (f\boldsymbol{V})(\boldsymbol{p}) = f(\boldsymbol{p})\boldsymbol{V}(\boldsymbol{p})$

自然标架场：定义 $\boldsymbol{U}_i = (\delta_j^i)_{j=1}^n$，按 Einstein 求和约定，有 $\boldsymbol{V}(\boldsymbol{p}) = v^i(\boldsymbol{p})\boldsymbol{U}_i(\boldsymbol{p})$, 称 v^i 为场的 Euclid 坐标函数，其中 Kronecker δ 函数定义为：

$$\delta_i^j = \begin{cases} 1,\ i = j \\ 0,\ i \neq j \end{cases} \tag{2.3}$$

与度量有关的线性变换：正交变换对称变换

2.0.7 Unitary Linear Space

酉空间

与度量有关的线性变换：酉变换 Hermite 变换

Chapter 3 linear mapping

Chapter 4 Vector

4.1 Basic Defination

we define the basic element as following, where $\boldsymbol{e}_i$ means $x_i = 1, x_j = 0$ for all $j \neq i$. When we say a vector, it means a column vector.

$$\vec{x} = \boldsymbol{x} = [x_1, x_2, \dots]^T = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \Sigma x_i \boldsymbol{e_i} \tag{4.1}$$

We define Kronecker sign to simply the description of $\boldsymbol{e}_i \cdot \boldsymbol{e}_j$.

$$\delta_{ij} := \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \tag{4.2}$$

The set of bases $\{\boldsymbol{e}_i\} \xrightarrow{apply} \boldsymbol{x} \longrightarrow \{x_i\}$.

4.2 Operation

4.2.1 Dot Product

We define in algebra, $\boldsymbol{x} \cdot \boldsymbol{y} := \sum x_i y_i \delta_{ij} = \boldsymbol{x}^T \cdot \boldsymbol{y}$.

Then the defination is restricted to the choose of the coordinate system.

4.2.2 Cross Product

$a, b \in \mathbb{F}^m, a \wedge b = c \in \mathbb{F}^n$, if m = n, we have m = 0, 1, 3, 7. Therefore, we define cross product in 3d.

$$u \times v = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \tag{4.3}$$

Proposition 4.1. 外积对于 u、v 双线性。从定义易知。

Proposition 4.2. $(a \times b) \times c = (c \cdot a)b - (b \cdot c)a$

证明:

$$\begin{vmatrix} i & j & k \\ 23 & 31 & 12 \\ 1 & 2 & 3 \end{vmatrix} \tag{4.4}$$

例如对 i 分量，有 $31 \cdot 3 - 12 \cdot 2$，形式上 ijk 一样，因而证明 i 即可。展开后，按正负号分类，we have $(313 + 212) - (133 + 122)$, 两部分都加上 111 即得。b 和-a 的线性组合。

Proposition 4.3. 混合积 $(u, v, w) = (u \times v) \cdot w$，其具有轮换对称性。

证明: *for* $\cdot w$, *we have* $23 \cdot 1 + 31 \cdot 2 + 12 \cdot 3$

$231 - 321, 312 - 132, 123 - 213$

对 $\cdot v$，即中间元素按 *1,2,3* 顺序组合，易有 wu；同样对 $\cdot u$，易有 vw。即证。

另外，从展开后的分量对应上，易有

$$(u, v, w) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \tag{4.5}$$

4.2.3 Add

$$\begin{aligned} \boldsymbol{x} + \boldsymbol{y} &:= \sum (x_i + y_i)\boldsymbol{e}_i \\ k \cdot \boldsymbol{x} &:= \sum k x_i \boldsymbol{e}_i \end{aligned} \tag{4.6}$$

Law $\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{y} + \boldsymbol{x}$, law $(\boldsymbol{x} + \boldsymbol{y}) + \boldsymbol{z} = \boldsymbol{x} + (\boldsymbol{y} + \boldsymbol{z})$ is not obvious in the view of Set Theory.

4.2.4 geometry Properties

4.2.4.1 Length and Angle

$$
\begin{aligned}
&\parallel \boldsymbol{x} \parallel := \sqrt{\boldsymbol{x} \cdot \boldsymbol{x}} \\
&\cos \theta_{x,y} := \frac{\boldsymbol{x} \cdot \boldsymbol{x}}{\parallel \boldsymbol{x} \parallel \cdot \parallel \boldsymbol{y} \parallel}
\end{aligned} \tag{4.7}
$$

4.2.4.2 Distance

Distance function satisfies the following:

$$
\begin{aligned}
&d(\boldsymbol{x}, \boldsymbol{y}) \geqslant 0 \\
&d(\boldsymbol{x}, \boldsymbol{y}) = d(\boldsymbol{y}, \boldsymbol{x}) \\
&d(\boldsymbol{x}, \boldsymbol{y}) \leqslant d(\boldsymbol{x}, \boldsymbol{z}) + d(\boldsymbol{z}, \boldsymbol{y})
\end{aligned} \tag{4.8}
$$

$$
\begin{aligned}
&d_p = [\sum |x_i - y_i|^p]^{\frac{1}{p}}, 1 \leqslant p < \infty \\
&d_\infty = \max_i |x_i - y_i|
\end{aligned} \tag{4.9}
$$

Chapter 5 Matrix Theory

5.1 Basic Notation

Normally, we consider vector space over the fields of real or complex numbers.

5.1.1 linear equation $\mathbf{Ax} = \mathbf{B}$

5.1.1.1 Defination

linear equation in n variables.$\sum_{i=1}^{i=n} a_i x^i = b$, which can be written as $\boldsymbol{a}^T\boldsymbol{x} = b$. We collect m equations and write like this:

$$\begin{bmatrix} \boldsymbol{a}_1^T \\ \vdots \\ \boldsymbol{a}_m^T \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{x} \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} \tag{5.1}$$

Noticed that x_1 is only applied toe the first column of the left matrix, we can say that $\boldsymbol{x}$ is one point, or a specific composition, of the space spanned by the column vector of the matrix. Then it is easy to see that this equation has the solution, only if the vector $\boldsymbol{b}$ is in the space spanned by the column vector of the matrix.

Or we can write like this:

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ & \ddots & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} \tag{5.2}$$

The equation $\boldsymbol{Ax} = \boldsymbol{b}$ has solution, means y 可由 A 的列向量线性表出。

If $\boldsymbol{b} = \mathbf{0}$, called homogeneous linear equations, homogeneous because 所有非 0 项是 1 次的。if $\boldsymbol{b} \neq \mathbf{0}$, it is inhomogeneous. 显然 0 向量 (zero solution, or trivial soltution) 是一个解. A 的列向量正交，只有零解；若 A 的列向量线性相关，有多解，即可按多种方式回到原点。

5.1.1.2 Complex Matrices

5.1.1.2.1 Conjugate Transposition for matrix $\boldsymbol{C} \in \mathbb{C}^{m\times n}$, we mark the Conjugate Transposition as C^H, where $c_{ji}^T = \overline{c}_{ij}$

in representation, $\boldsymbol{C} = \boldsymbol{A} + i\boldsymbol{B}$. Usually 4 real matrix multiplications are needed to calculate $(C+iD)(E+iF)$, actually 3 multiplications are enough. $(C+iD)(E+iF) = (C+D)(E-F) + CF - DE + i(DE+CF)$

5.1.1.3 Number of solution

5.1.1.3.1 非齐次线性 n 元线性方程组解的个数等解集结构的研究，期待在不求解的情况下有所了解，就需要研究系数矩阵表示的 n 维向量空间的性质。

构造增广矩阵 $[\boldsymbol{A}, \boldsymbol{b}]$ 后，初等行变换化为阶梯型，如??所示，解的个数讨论。

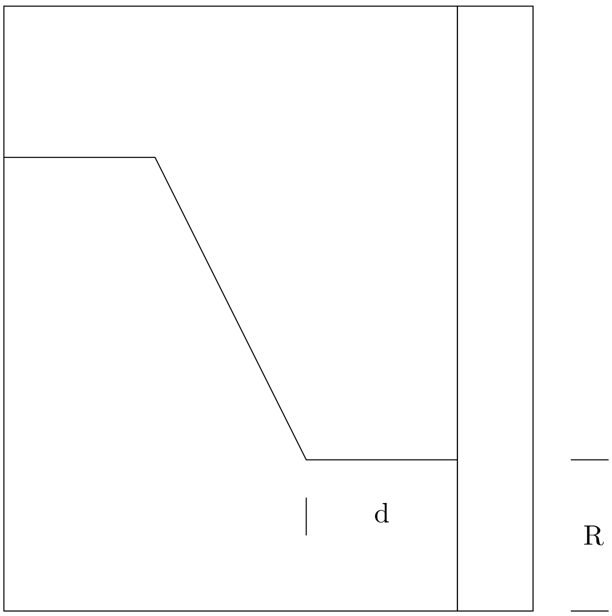

Figure 5.1: 【number of solution】

总共有 n+1 列，下面 r 行都是 0.

(1)$d=0$，即最后一个个主元在第 n+1 列，即存在方程 0=1，无解, no solution。

(2)$d=1$，即最后一个个主元在第 n 列，唯一解, one solution, $tr\boldsymbol{A}_{mn} = m$。

(3)$d>1$，即最后一个个主元在第 t 列，$t<n$。高度 R 所在的行号记为 r。有无穷个解。解可以这样写出，共 R 行，即 R 个主元，每个主元都用所在行的常数项 d 和 n-r 个自由元表示出来。

根据主元的构造过程，t 的列号一定大于等于 r。

当 A_{ii} 都是主元的时候,$d \neq 1$, $tr\boldsymbol{A}_{mn} < m$,inifinity solution，最后一行是解的超平面方程, 图中 d 是解的维度，$d = n - tr\boldsymbol{A}_{mn}$, 如 d 为 3，有 3 列独立的，即解空间是三维的。齐次方程组的未知数个数大于方程个数，有无数解。

det $\boldsymbol{A} = 0$, no solution, or infinite solution. det $\boldsymbol{A} \neq 0$, one solution.

5.1.1.3.2 齐次线性 一定有 0 解，因而当有非 0 解时，有无穷个解。n 列时，系数矩阵的秩 $r < n$。

方程个数 $s < n$ 时，由于 $r \leqslant s < n$，易知有无穷个解。

5.1.2 Matrix

5.1.2.1 Defination

If we have a serious of $\boldsymbol{x}$, we have a serious of b, like this:

$$A_{mn} \cdot X_{nt} = B_{mt} \Longrightarrow \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ & \ddots & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \cdot \left[\begin{array}{c:c:c} x_{11} & \cdots & x_{1t} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nt} \end{array}\right] = \left[\begin{array}{c:c:c} b_{11} & \cdots & b_{1t} \\ \vdots & \cdots & \vdots \\ b_{m1} & \cdots & b_{mt} \end{array}\right] \tag{5.3}$$

The normal definition of the product of two matrix is as above.

5.1.2.2 Multiplication

There are 6 views sorting with the loop order, we fully understand that. for example, we can think the order jki(j is the outer, i is the inner) as follows

$$\begin{aligned} &i : | \ \ \cdot = | \\ &k : [\,|\,|\,] \ \ | = \sum | \\ &j : [\,|\,|\,] \ \ [\,|\,|\,] = [\,|\,|\,] \end{aligned} \tag{5.4}$$

We collect the 6 vews into one table as fallows.

Table 5.1: $\boldsymbol{A}_{ik}\boldsymbol{X}_{kj} = \boldsymbol{B}_{ij}$

Order	innerLoop	MiddleLoop	dataAccess	view	comment
ijk	S-S:dot	rowV-M	$\boldsymbol{A}_{\alpha:}, [\boldsymbol{X}_{:\beta}], \boldsymbol{B}_{\alpha:}$	$[-] \cdot [\vert\vert\vert\vert] = [-]$	dot view $\rightharpoonup\downharpoonleft$
jik	S-S:dot	M-columnV	$[\boldsymbol{A}_{\alpha:}], \boldsymbol{X}_{:\beta}, \boldsymbol{B}_{:\beta}$	$[\equiv] \cdot [\vert\vert] = [\vert\vert]$	dot view $\downharpoonleft\rightharpoonup$
ikj	S-rowV:saxpy	rowV-M:gaxpy	$\boldsymbol{A}_{\alpha:}, [\boldsymbol{X}_{\beta:}], \boldsymbol{B}_{\alpha:}$	$[-]gaxpy[\equiv] = [-]$	useOfA $\rightharpoonup\downharpoonleft$
jki	colV-S:saxpy	M-colV:gaxpy	$[\boldsymbol{A}_{:\alpha}], \boldsymbol{X}_{:\beta}, \boldsymbol{B}_{:\alpha}$	$[\vert\vert\vert\vert]gaxpy[\vert\vert] = [\vert\vert]$	useOfB $\downharpoonleft\rightharpoonup$
kij	S-rowV:saxpy	colV-rowV:outP	$\boldsymbol{A}_{:\alpha}, \boldsymbol{X}_{\beta:}, \sum \boldsymbol{B}_{row}$	$\sum[\vert\vert]outProd[-] = \sum[\equiv]$	$on\ A \downharpoonleft outProd \rightharpoonup$
kji	colV-S:saxpy	colV-rowV:outP	$\boldsymbol{A}_{:\alpha}, \boldsymbol{X}_{\beta:}, \sum \boldsymbol{B}_{col}$	$\sum[\vert\vert]outProd[-] = \sum[\vert\vert\vert\vert]$	$on\ X \downharpoonleft outProd \rightharpoonup$

[1] S for scalar, V for vector, M for matrix；colV for column vector；outP for out product.
[2] $[-]gaxpy[\equiv] = [-]$ is $\sum[\cdot]gaxpy[-] = \sum[-]$.
[3] $[\vert\vert\vert\vert]gaxpy[\vert\vert] = [\vert\vert]$ is $\sum[\vert\vert]gaxpy[\cdot] = \sum[\vert\vert]$.

Table 5.2: $\boldsymbol{A}_{ik}\boldsymbol{X}_{kj} = \boldsymbol{B}_{ij}$

Order	InnerLoop	MiddleLoop	OuterLoop
ijk	$(rowV, colV) = S$	$(rowV, [colV]) = rowV$	collection
jik	$(rowV, colV) = S$	$([rowV], colV) = rowV$	collection
ikj	$(S, colV) = colV$	$(rowV, [colV]) = \sum colV$	collection
jki	$(colV, S) = colV$	$([colV], colV) = \sum colV$	collection
kij	$(S, rowV) = rowV$	$(colV, rowV) = [rowV]$	collection and $\sum[rowV]$
kji	$(colV, S) = colV$	$(colV, rowV) = [colV]$	collection and $\sum[colV]$

[1] $\sum$ comes with k.

5.1.2.3 Transposition

Defination: $a_{ij}^T = a_{ji}$

Proposition 5.1. $(\boldsymbol{AB})^T = \boldsymbol{B}^T\boldsymbol{A}^T$

Proof: $L = (a_{ik}b_{kj})^T = c_{ij}^T = c_{ji} = b_{jk}a_{ki} = R$ □

Proposition 5.2. *We take a look a the product with reflect* $T : \boldsymbol{x} \to \boldsymbol{T} \cdot \boldsymbol{x}$. $(\boldsymbol{Tx})^T\boldsymbol{Ty} = \boldsymbol{x}^T(\boldsymbol{T}^T\boldsymbol{T})\boldsymbol{y} = [(\boldsymbol{TT}^T)\boldsymbol{x}]^T\boldsymbol{y}$. $0 \leqslant \|\boldsymbol{TT}^T\| < 1$, $\boldsymbol{T}$ *is a contractive mapping.*

5.1.3 operation

5.1.3.1 dot procuct AX = B

Focus on each element of B.

5.1.3.1.1 vector vector for vector, $\boldsymbol{x}.*\boldsymbol{y} = \boldsymbol{x}^T\boldsymbol{y}$,

5.1.3.1.2 matrix matrix for matrix, this is the definition of the multiplication of the matrix, $\boldsymbol{A}_{mn}.*\boldsymbol{B}_{mn} = [a_{ij} \cdot b_{ij}]_{mn}$

5.1.3.2 outer procuct AX = B

Focus on each element of X, with X is seperated as row by row.

5.1.3.2.1 vector vector $\boldsymbol{xy}^T := [x_i]_{m1} \cdot [y_j]_{1n} = [x_iy_j]_{mn}$

In row view, we have $i \to$: $\boldsymbol{A}_{i:} = x_i \cdot \boldsymbol{y}^T$, this notation means that for each i, we do the follows. And $\boldsymbol{A}_{i:}$ means the ith row of the row seperation of $\boldsymbol{A}$

In column view, we have $j \to$: $\boldsymbol{A}_{:j} = \boldsymbol{x} \cdot y_j$

5.1.3.2.2 matrix matrix $[||||] outerProduct[—] = [\quad]$, we just sum each matrix $\boldsymbol{M}$, where $\boldsymbol{M} = [||] outerProduct[—]$.

$\boldsymbol{X}_{mk} \cdot \boldsymbol{Y}_{kn} = k \rightarrow$: $outerProduct\ of(\boldsymbol{X}_{:k}, \boldsymbol{Y}_{k:})$

We carefully focus on the use of each element of the matrix $\boldsymbol{Y}$, like $A_{11}, A_{12}, A_{13}, \cdots$, we can see it is true.

5.1.3.2.3 question

Question 5.1. *power function 001 solve* $(\boldsymbol{x}\boldsymbol{y}^T)^k$. *If k=1, easy. if k>1, ans =* $(\boldsymbol{y}^T\boldsymbol{x})^{k-1}\boldsymbol{x}\boldsymbol{y}^T$

Question 5.2. *power function 002*

solve $(\boldsymbol{X}\boldsymbol{Y}^T)^k, X, Y \in \mathbb{R}^{n\times 2}$. *Same trick like power function 001.*

5.1.3.3 saxpi

5.1.3.3.1 scalar scalar $y = ax + y$

5.1.3.3.2 scalar vector $\boldsymbol{y} = a \cdot \boldsymbol{x} + \boldsymbol{y}$

5.1.3.3.3 matrix vector $\boldsymbol{y} = \boldsymbol{A} \cdot \boldsymbol{x} + \boldsymbol{y}$

5.1.3.3.3.1 view row: $[—] \cdot | = [—]$ *This is the basic view of the dot product of the matrix. in view row first, we have:*

Algorithm 1: saxpyMatrixVectorRowAlgo1

Input: $\boldsymbol{A}_{mn}, \boldsymbol{x}, \boldsymbol{y}$

Output: $\boldsymbol{y}$

```
Initialization: i = 0, j = 0;
for i ← 0 to m − 1 do
    for j ← 0 to n − 1 do
        y_i ← A_ij x_j + y_i
    end
end
return y;
```

We seperate $\boldsymbol{A}$ *as row,* $\boldsymbol{A}_{mn} = [\boldsymbol{r}_i^T, ...]^T$, *the j range can be shinked, the algorithm is as follows. This means that, we operate each row at a time, and think each row is one whole object.*

Algorithm 2: saxpyMatrixVectorRowAlgo2

Input: $\boldsymbol{A}_{mn} = [\boldsymbol{r}_i^T, ...]^T, \boldsymbol{x}, \boldsymbol{y}$

Output: $\boldsymbol{y}$

Initialization:$i = 0, j = 0$;

for $i \leftarrow 0$ **to** $m - 1$ **do**

  $y_i \leftarrow \boldsymbol{r}_i^T \cdot \boldsymbol{x} + y_i$

end

return $\boldsymbol{y}$;

5.1.3.3.3.2 view column: $[||||] outerProduct[—] = [\ \ ]$ $\boldsymbol{A}_{mn}\boldsymbol{x} = \boldsymbol{y}$, *we seperate A column by column, x row by row, use outer product , focus on the use of x.*

in column view, we add each column of $\boldsymbol{A}$ *to the same output column to get the new* $\boldsymbol{y}$*, and the weight of each column comes from each row of* $\boldsymbol{x}$

Algorithm 3: saxpyMatrixVectorColumnAlgo1

Input: $\boldsymbol{A}_{mn}, \boldsymbol{x}, \boldsymbol{y}$

Output: $\boldsymbol{y}$

Initialization:$i = 0, j = 0$;

for $j \leftarrow 0$ **to** $n - 1$ **do**

  for $i \leftarrow 0$ **to** $m - 1$ **do**

    $y_i \leftarrow A_{ij}x_j + y_i$

  end

end

return $\boldsymbol{y}$;

Also with column seperation of $\boldsymbol{A}_{mn} = [\boldsymbol{c}_i, ...]$, *we have the vector view algorithm:*

Algorithm 4: saxpyMatri

Input: $\boldsymbol{A}_{mn} = [\boldsymbol{c}_i, ...],$

Output: $\boldsymbol{y}$

Initialization:$i = 0, j = 0$

for $j \leftarrow 0$ **to** $n - 1$ **do**

  $\boldsymbol{y} \leftarrow \boldsymbol{c}_i \cdot x_j + \boldsymbol{y}$

end

return $\boldsymbol{y}$;

5.1.4 solve equation

$\boldsymbol{A}_{mn}\boldsymbol{x} = \boldsymbol{y}$ 求解方法，如消元法、迭代法等。

5.1.4.1 elimination 消元法

5.1.4.1.1 Gaussian Elimination 基础步骤的 $O(n)$ 的，但是最终组合起来就是 $O(n^3)$ 的。

利用初等变换化（同解变换）为“阶梯形（或称上三角形）”，从下往上回代。

阶梯型：*1*）*0* 行在下方；*2*）每行首个非 *0* 元的列号随行号增大而严格增大。

简化阶梯型：*1*）阶梯型；*2*）主元是 *1*；*3*）主元所在列其他元素是 *0.*

简化阶梯型后，可直接写出一般解，如下方程，其中主变量是 x_1, x_3，其余是自由未知量。

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \tag{5.5}$$

$$Ans : x_1 = x_2 + 2; x_3 = -1$$

Augmented matrix

5.2 determinant

行列式，定义、性质、展开、*Gramer* 法则等

5.2.1 排列

5.2.1.0.1 偶排列 如 *2431*,顺序对有 $24, 23$, 逆序对有 $21, 43, 41, 31$,逆序数是 *4*,记为 $\tau(2431) = 4$, 是偶数则为偶排列。

Lemma 5.1. 对换改变奇偶性，如 *2431* 是偶排列，对换 *4* 和 *1* 后得到的 *2134* 是奇排列。

证明：对换 ab,

若 ab 相邻：偏序函数原来查询 (ab), 记为 $P(a, b)$, 对换后改为 $P(b, a)$，反号，而 b 更后面的元素相关的查询不受影响，因而改变符号；

若 ab 不相邻：记为 $ax_1 \cdots x_t b$, 经过 t 次对换变为 $x_1 \cdots x_t ab$, 经过 *t+1* 次对换变为 $bx_1 \cdots x_t a$, 即改变符号。若 ab 不相邻，还可以这样考虑：对换前后，与 a 和 b 有关的查询为 $(a, [x_i, b]), (x_i, b)$, 对换后即将其中 a 和 b 互换，影响的查询共有 *2t+1* 个，即改变符号。即证。

5.3 polynomial

因式分解定理，多项式的根，多元多项式。

5.4 operation

初等变换、代数运算、分块运算、乘法、秩

5.5 Transformation

线性变换、坐标变换、像与核、特征向量、特征子空间、商空间

正交变换规范变换

酉相似

5.5.1 Elementary Transformation

初等变换。

1) 交换两行*:* $\boldsymbol{A} \xrightarrow{(i,j)} \boldsymbol{B}$

2) 某行乘以不为 *0* 的数*:* $\boldsymbol{A} \xrightarrow{\lambda(i)} \boldsymbol{B}$

3) 某行乘以不为 *0* 的数加到另一行上*:* $\boldsymbol{A} \xrightarrow{\lambda(i)+(j)} \boldsymbol{B}$

初等矩阵*:* 单位矩阵执行一系列初等变换得到的矩阵*.*

初等变换作用于矩阵 $\boldsymbol{A}$*,* 等于初等变换作用于单位阵之后得到的初等矩阵 $\boldsymbol{E}$ 再作用于 $\boldsymbol{A}$*.*

5.6 Form

5.6.1 Jordan

Jordan 型、根子空间分解、循环子空间、多项式矩阵相抵不变量、特征方阵与相似标准型

5.6.2 二次

配方法构造、对称方阵的相合、相合不变量