09-04-05-LinearAlgebra

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Chapter 1 Overall

2 条主线：linear space, linear mapping

Chapter 2 Vector

2.1 Basic Defination

we define the basic element as following, where $\boldsymbol{e}_i$ means $x_i = 1, x_j = 0$ for all $j \neq i$. When we say a vector, it means a column vector.

$$\vec{x} = \boldsymbol{x} = [x_1, x_2, \dots]^T = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \Sigma x_i \boldsymbol{e_i} \tag{2.1}$$

We define Kronecker sign to simply the description of $\boldsymbol{e}_i \cdot \boldsymbol{e}_j$.

$$\delta_{ij} := \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \tag{2.2}$$

The set of bases $\{\boldsymbol{e}_i\} \xrightarrow{apply} \boldsymbol{x} \longrightarrow \{x_i\}$.

2.2 Operation

2.2.1 Dot Product

We define in algebra, $\boldsymbol{x} \cdot \boldsymbol{y} := \sum x_i y_i \delta_{ij} = \boldsymbol{x}^T \cdot \boldsymbol{y}$.

Then the defination is restricted to the choose of the coordinate system.

2.2.2 Cross Product

$a, b \in \mathbb{F}^m, a \wedge b = c \in \mathbb{F}^n$, if m = n, we have m = 0, 1, 3, 7. Therefore, we define cross product in 3d.

$$u \times v = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \tag{2.3}$$

Proposition 2.1. 外积对于 u、v 双线性。从定义易知。

Proposition 2.2. $(a \times b) \times c = (c \cdot a)b - (b \cdot c)a$

证明：

$$\begin{vmatrix} i & j & k \\ 23 & 31 & 12 \\ 1 & 2 & 3 \end{vmatrix} \tag{2.4}$$

例如对 i 分量，有 $31 \cdot 3 - 12 \cdot 2$，形式上 ijk 一样，因而证明 i 即可。展开后，按正负号分类，we have $(313 + 212) - (133 + 122)$, 两部分都加上 111 即得。b 和-a 的线性组合。

Proposition 2.3. 混合积 $(u, v, w) = (u \times v) \cdot w$，其具有轮换对称性。

证明：*for* $\cdot w$*, we have* $23 \cdot 1 + 31 \cdot 2 + 12 \cdot 3$

$231 - 321, 312 - 132, 123 - 213$

对 $\cdot v$，即中间元素按 *1,2,3* 顺序组合，易有 wu；同样对 $\cdot u$，易有 vw。即证。

另外，从展开后的分量对应上，易有

$$(u, v, w) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \tag{2.5}$$

2.2.3 Add

$$\begin{aligned} \boldsymbol{x} + \boldsymbol{y} &:= \sum (x_i + y_i)\boldsymbol{e}_i \\ k \cdot \boldsymbol{x} &:= \sum k x_i \boldsymbol{e}_i \end{aligned} \tag{2.6}$$

Law $\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{y} + \boldsymbol{x}$, law $(\boldsymbol{x} + \boldsymbol{y}) + \boldsymbol{z} = \boldsymbol{x} + (\boldsymbol{y} + \boldsymbol{z})$ is not obvious in the view of Set Theory.

2.2.4 geometry Properties

2.2.4.1 Length and Angle

$$
\begin{aligned}
&\parallel \boldsymbol{x} \parallel := \sqrt{\boldsymbol{x} \cdot \boldsymbol{x}} \\
&\cos\theta_{x,y} := \frac{\boldsymbol{x} \cdot \boldsymbol{x}}{\parallel \boldsymbol{x} \parallel \cdot \parallel \boldsymbol{y} \parallel}
\end{aligned} \tag{2.7}
$$

2.2.4.2 Distance

Distance function satisfies the following:

$$
\begin{aligned}
&d(\boldsymbol{x}, \boldsymbol{y}) \geqslant 0 \\
&d(\boldsymbol{x}, \boldsymbol{y}) = d(\boldsymbol{y}, \boldsymbol{x}) \\
&d(\boldsymbol{x}, \boldsymbol{y}) \leqslant d(\boldsymbol{x}, \boldsymbol{z}) + d(\boldsymbol{z}, \boldsymbol{y})
\end{aligned} \tag{2.8}
$$

$$
\begin{aligned}
&d_p = [\sum |x_i - y_i|^p]^{\frac{1}{p}}, 1 \leqslant p < \infty \\
&d_\infty = \max_i |x_i - y_i|
\end{aligned} \tag{2.9}
$$

Chapter 3 Matrix Theory

Chapter 4 determinant

行列式论行列式，定义、性质、展开、Gramer 法则等

Chapter 5 Matrix Analysis

《矩阵理论-陈大新》

5.1 矩阵序列

5.2 矩阵幂级数

5.3 矩阵函数

5.3.1 定义

5.3.2 e^At

5.3.3 计算

Chapter 6 Generalized Inverse

6.1 单边逆

6.2 Moore-Penrose Pseudoniverse

$$\boldsymbol{AGA} = \boldsymbol{A}$$
$$\boldsymbol{GAG} = \boldsymbol{G}$$
$$(\boldsymbol{GA})^T = \boldsymbol{GA}$$
$$(\boldsymbol{AG})^T = \boldsymbol{AG}$$

Lemma 6.1. $\boldsymbol{G}$ *is unique.*

Prove: suppose $\boldsymbol{G}_1 \neq \boldsymbol{G}_2$,

$$\begin{aligned}
\boldsymbol{G}_2 &= \boldsymbol{G}_2\underline{\boldsymbol{A}}\boldsymbol{G}_2 \\
&= \boldsymbol{G}_2\underline{\boldsymbol{AG}_1} \cdot \underline{\boldsymbol{AG}_2} = \boldsymbol{G}_2[(\boldsymbol{AG}_1)^T \cdot (\boldsymbol{AG}_2)^T] = \boldsymbol{G}_2(\boldsymbol{AG}_2\boldsymbol{AG}_1)^T = \boldsymbol{G}_2(\boldsymbol{AG}_1)^T \\
&= \boldsymbol{G}_2\underline{\boldsymbol{A}}\boldsymbol{G}_1 \\
&= \underline{\boldsymbol{G}_2\boldsymbol{A}} \cdot \underline{\boldsymbol{G}_1\boldsymbol{A}}\boldsymbol{G}_1 = \boldsymbol{G}_1\boldsymbol{A} \cdot \boldsymbol{G}_2\boldsymbol{A} \cdot \boldsymbol{G}_1 = \boldsymbol{G}_1\boldsymbol{AG}_1 \\
&= \boldsymbol{G}_1 \quad \square
\end{aligned}$$

6.2.1 Properties

6.2.1.1 solution of Ax=y

$\boldsymbol{Ax} = \boldsymbol{y}$, where$\boldsymbol{A}_{mn}, m > n$

Algorithm 1: Algorithm LinearEquation:MP

Input: $\boldsymbol{Ax} = \boldsymbol{y}$, where $\boldsymbol{A}_{mn}, m > n$, and $\boldsymbol{y}$ are unknown.

Output: $\boldsymbol{x}$

$\boldsymbol{A}(\boldsymbol{A}^T\boldsymbol{A}^{-T})\boldsymbol{x} = \boldsymbol{y}$;

let $\boldsymbol{C} = \boldsymbol{AA}^T$, $\boldsymbol{I} = \boldsymbol{A}^{-T}\boldsymbol{x}$, $\boldsymbol{CI} = \boldsymbol{y}$;

solve $\boldsymbol{I}$;

$\boldsymbol{x} =$;

return $\boldsymbol{x} = \boldsymbol{A}^T\boldsymbol{I}$;

The algorithm meas $\boldsymbol{x} = (\boldsymbol{AA}^T\boldsymbol{A}^{-T})^{-1}\boldsymbol{y} = [(\boldsymbol{AA}^T)\boldsymbol{A}^{-T}]^{-1}\boldsymbol{y} = \boldsymbol{A}^T(\boldsymbol{AA}^T)^{-1}\boldsymbol{y}$.

$\boldsymbol{A}^T(\boldsymbol{AA}^T)^{-1}$, and $(\boldsymbol{A}^T\boldsymbol{A})^{-1}\boldsymbol{A}^T$ are Moore-Penrose Pseudoniverse

6.2.2 相容方程的解

6.2.3 反射广义逆

6.2.4 最小范数解

6.2.5 最小二乘解

Chapter 7　多线性代数

Chapter 8　向量代数、因子代数、代数不变量论

Chapter 9　线性不等式

Chapter 10 线性代数的应用

Chapter 11 参考文献说明

《矩阵理论-陈大新》[?]：好的观点的来源。

参考文献