Max Weight Matching in bipartite graphs

Input: Bipartite graph $G=(V,E)$, $w: E \rightarrow \mathbb{Z}^+$

Output: Matching M with maximum $\sum_{e \in M} w(e)$

Simplification:

Assume: $|\text{Outer}| = |\text{Inner}|$, G = complete bipartite graph.

(by adding dummy nodes, dummy edges of weight 0)

Optimal solutions are perfect matchings.

Define: Labeling $L: V \rightarrow \mathbb{Z}^+$ is feasible

if for all $(u,v) \in E$, $L(u) + L(v) \geq w(u,v)$.

Theorem: Let M be a matching of G, let L be a feasible labeling of G. Then

$$w(M) \leq \sum_{u \in V} L(u)$$

Proof:

For any edge $e = (u,v) \in M$: $L(u) + L(v) \geq w(e)$

$$W(M) = \sum_{e \in M} w(e) \leq \sum_{u = \text{matched node}} L(u) \leq \sum_{u \in V} L(u)$$

$\blacksquare$

Corollary: If M is a matching, and L is a valid labeling, and if

$$\sum_{e \in M} w(e) = \sum_{u \in V} L(u),$$

then M is a maximum weight matching.