CSC 200
Data Structures and Algorithms

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Fall 2024

Course Credits: Stanford-CS161

MULTIPLICATION!

What's the best way to multiply two numbers?

MULTIPLICATION: THE PROBLEM

Input: 2 non-negative numbers, x and y (n digits each)

Output: the product $x \cdot y$

$$
\begin{array}{r}
5678 \\
\times \;\; 1234 \\
\hline
\mathbf{7006652}
\end{array}
$$

GRADE-SCHOOL MULTIPLICATION

```
   45
 x 63
-----
  135
 2700
-----
 2835
```

GRADE-SCHOOL MULTIPLICATION

Algorithm description (informal*):
compute partial products (using multiplication & "carries" for digit overflows), and add all (properly shifted) partial products together

```
    45
x   63
------
   135
  2700
------
  2835
```

** This is not a good example of what your algorithm descriptions should look like on HW/quizzes*

GRADE-SCHOOL MULTIPLICATION

```
  45123456678093420581217332421
x 63782384198347750652091236423
-------------------------------
                             ):
```

GRADE-SCHOOL MULTIPLICATION

$$
\begin{array}{r}
\overbrace{45123456678093420581217332421}^{\boldsymbol{n}\ \textit{digits}} \\
\times\ 63782384198347750652091236423 \\
\hline
):
\end{array}
$$

How efficient is this algorithm?

(How many single-digit operations are required?)

GRADE-SCHOOL MULTIPLICATION

$$\overbrace{\begin{array}{r} 4512345667809342058121733242 1 \\ \times\ 6378238419834775065209123642 3 \\ \hline):\end{array}}^{\textbf{\textit{n}}\ \textit{digits}}$$

How efficient is this algorithm?

(How many single-digit operations *in the worst case*?)

n partial products: ~2n^2 ops (at most n multiplications & n additions per partial product)

adding n partial products: ~2n^2 ops (a bunch of additions & “carries”)

GRADE-SCHOOL MULTIPLICATION

$$\overbrace{4512345667809342058121733242\,1}^{n \text{ digits}}$$

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n partial products: ~$2n^2$ ops (at most n multiplications & n additions per partial product)

adding n partial products: ~$2n^2$ ops

(a bunch of additions & "carries")

~ $4n^2$ operations in the worst case

GRADE-SCHOOL MULTIPLICATION

n digits

4512345 ... 17332421

x 63782 ... 236423

):

How efficient is this algo

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in the worst case?)

oducts: ~$2n^2$ **ops** (at most n

& n additions per partial product)

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THE QUESTION IS...

CAN WE DO BETTER?

~ $4n^2$ operations in the worst case

WHAT EXACTLY DOES “BETTER” MEAN?

Is $1000000n$ operations better than $4n^2$?

Is $0.000001n^3$ operations better than $4n^2$?

Is $3n^2$ operations better than $4n^2$?

WHAT EXACTLY DOES "BETTER" MEAN?

Is 1000000n operations better than $4n^2$?

Is $0.000001n^3$ operations better than $4n^2$?

Is $3n^2$ operations better than $4n^2$?

- **The answers for the first two depend on what value n is…**
  - $1000000n < 4n^2$ only when n exceeds a certain value (in this case, 250000)
- **These constant multipliers are too environment-dependent…**
  - An operation could be faster/slower depending on the machine, so $3n^2$ ops on a slow machine might not be "better" than $4n^2$ ops on a faster machine

WHAT EXACTLY DOES "BETTER" MEAN?

INTRODUCING...

ASYMPTOTIC ANALYSIS

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- **Some guiding principles:** we care about how the running time/number of operations *scales* with the size of the input (i.e. the runtime's *rate of growth*), and we want some measure of runtime that's independent of hardware, programming language, memory layout, etc.

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    - Note: details like hardware/language/memory/compiler/etc. could totally be important to real world engineers, but in TheoryLand™, we want to reason about high-level algorithmic approaches rather than lower-level details

ASYMPTOTIC ANALYSIS (High Level Idea)

We'll express the asymptotic runtime of an algorithm using

BIG-O NOTATION

"big-oh of n squared"
or
"Oh of n squared"

- We would say Grade-school Multiplication **"runs in time $O(n^2)$"**
  - Informally, this means that the runtime "scales like" n^2
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suppress constant factors and lower-order terms

too system dependent

irrelevant for large inputs

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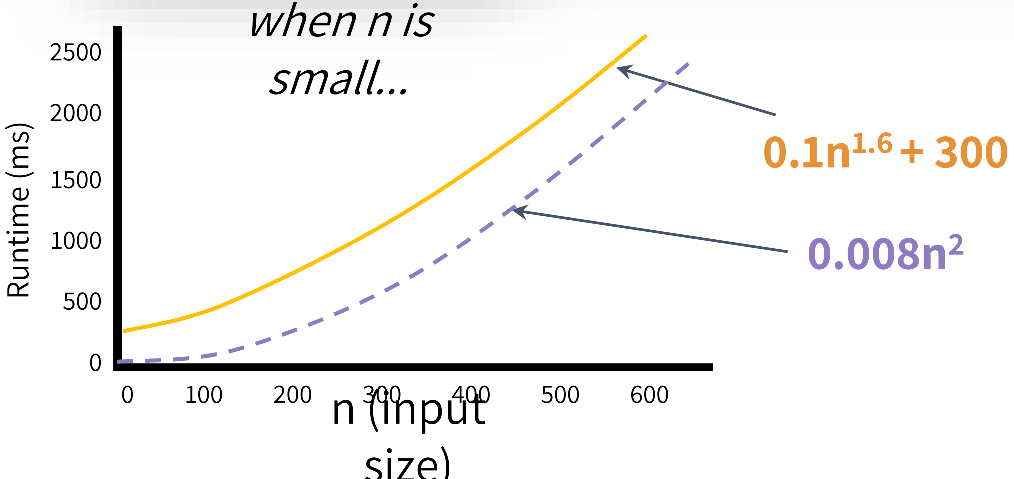

ASYMPTOTIC ANALYSIS (High Level Idea)

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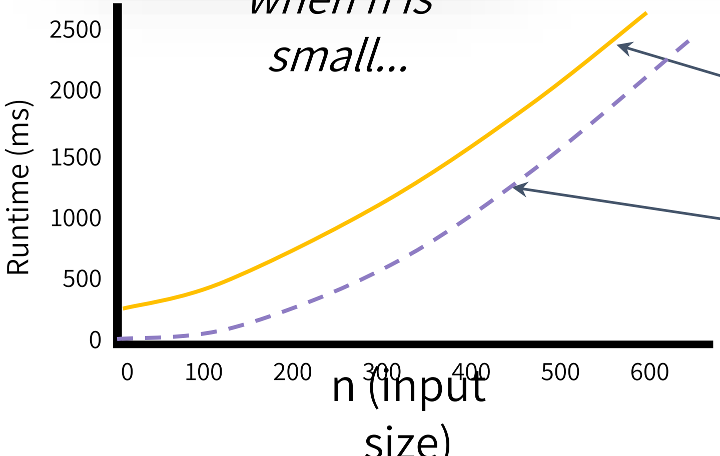
suppress constant factors and lower-order terms

too system dependent — *irrelevant for large inputs*

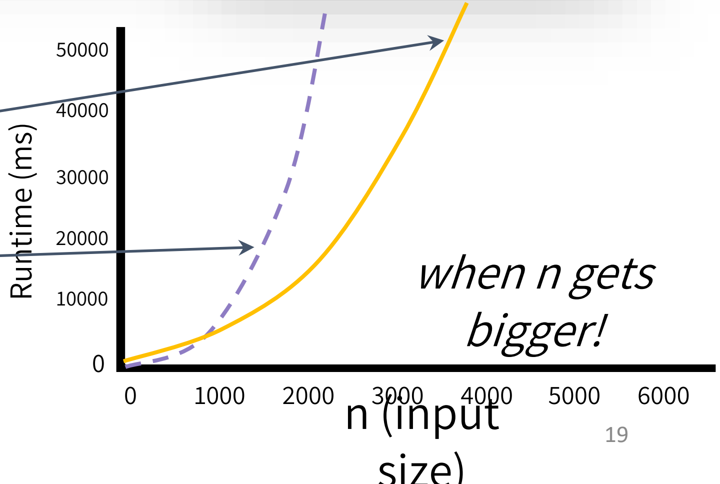
when n is small…



$0.1n^{1.6} + 300$

$0.008n^2$

when n gets bigger!



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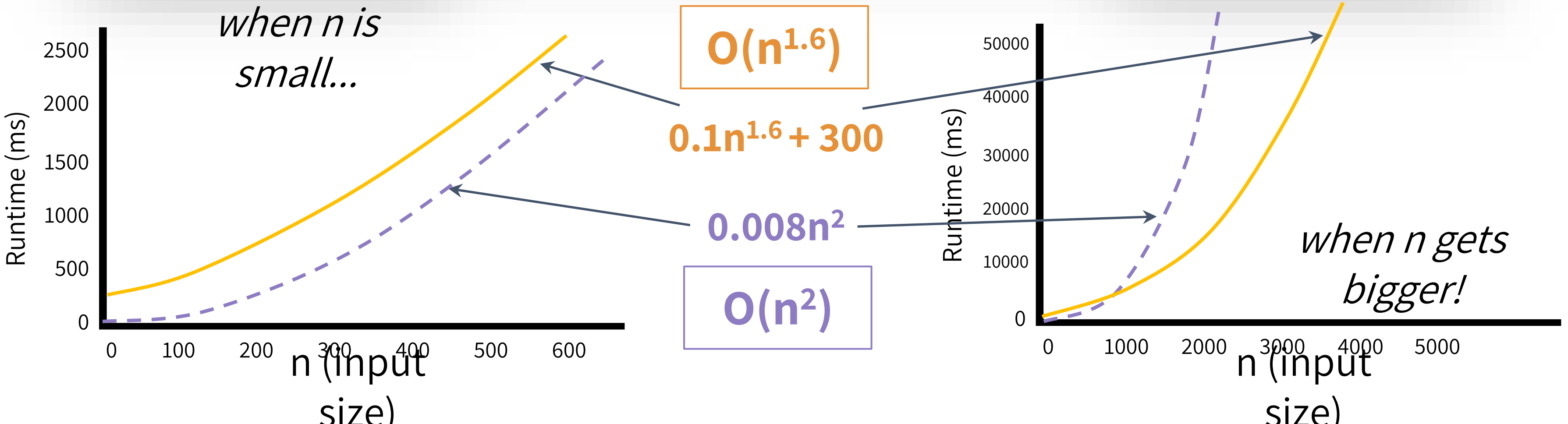

ASYMPTOTIC ANALYSIS (High Level Idea)

- To compare algorithm runtimes in this class, we compare their Big-O runtimes
  - Ex: a runtime of $O(n^2)$ is considered “better” than a runtime of $O(n^3)$
  - Ex: a runtime of $O(n^{1.6})$ is considered “better” than a runtime of $O(n^2)$
  - Ex: a runtime of $O(1/n)$ is considered “better” than $O(1)$

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  - Ex: a runtime of $O(1/n)$ is considered "better" than $O(1)$

So the question is:

Can we multiply n-digit integers faster than $O(n^2)$?

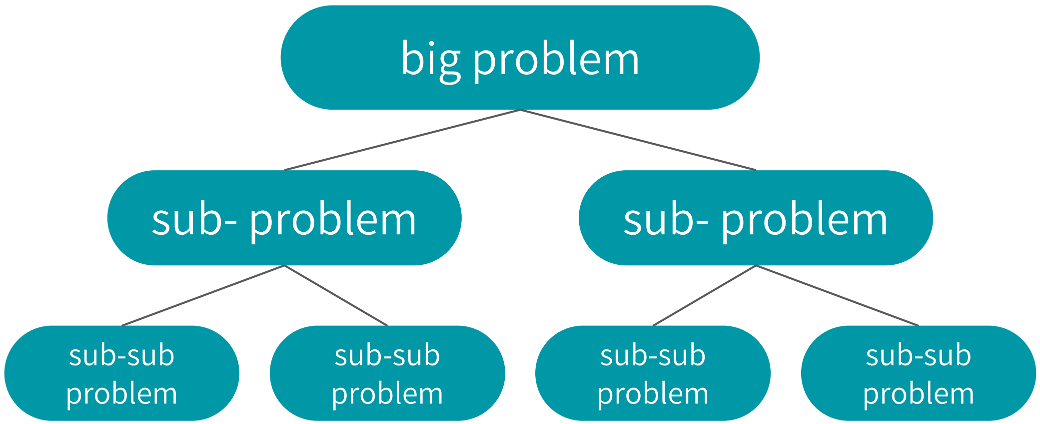
Don't worry, we'll revisit Asymptotic Analysis & Big-O stuff more formally in Lecture 2!

DIVIDE AND CONQUER

algorithm design paradigm

DIVIDE AND CONQUER

- **An algorithm design paradigm:**
  1. break up a problem into smaller subproblems
  2. solve those subproblems *recursively*
  3. combine the results of those subproblems to get the overall answer



MULTIPLICATION SUBPROBLEMS

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- **What are the subproblems?** Let's unravel some stuff...

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$$= (12 \times 100 + 34) \times (56 \times 100 + 78)$$

$$= (12 \times 56)100^2 + (12 \times 78 + 34 \times 56)100 + (34 \times 78)$$

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- **What are the subproblems?** Let's unravel some stuff...

$$1234 \times 5678$$

$$= (\ 12 \times 100 + 34\ ) \times (\ 56 \times 100 + 78\ )$$

$$= (\ \underset{1}{12 \times 56}\ )100^2 + (\ \underset{2}{12 \times 78} + \underset{3}{34 \times 56}\ )100 + (\ \underset{4}{34 \times 78}\ )$$

One 4-digit problem → *Four 2-digit subproblems*

MULTIPLICATION SUBPROBLEMS

- **Original large problem:** multiply 2 n-digit numbers
- **What are the subproblems?** More generally:

$$[x_1 x_2 \cdots x_{n-1} x_n] \times [y_1 y_2 \cdots y_{n-1} y_n]$$

$$= (a \times 10^{n/2} + b) \times (c \times 10^{n/2} + d)$$

$$= (\underset{1}{a \times c})10^n + (\underset{2}{a \times d} + \underset{3}{b \times c})10^{n/2} + (\underset{4}{b \times d})$$

One n-digit problem → *Four (n/2)-digit subproblems*

LET'S SEE SOME PSEUDOCODE

```
MULTIPLY( x, y ):
```

x & y are n-digit numbers

Note: *we're making a big assumption that n is a power of 2 just to make the pseudocode simpler*

LET'S SEE SOME PSEUDOCODE

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Base case: we can just reference some memorized 1-digit multiplication tables

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a, b, c, & d are (n/2)-digit numbers

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    return ac·10^n + (ad + bc)·10^(n/2) + bd
```

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a, b, c, & d are (n/2)-digit numbers

These are **recursive** calls that provide subproblem answers

Add them up to get our overall answer!

***Note**: we're making a big assumption that n is a power of 2 just to make the pseudocode simpler*

HOW EFFICIENT IS THIS ALGORITHM?

- **Let's start small:** if we're multiplying two 4-digit numbers, how many 1-digit multiplications does the algorithm perform?
  - In other words, how many times do we reach the base case where we actually perform a "multiplication" (a.k.a. a table lookup)?
  - This at least lower bounds the number of operations needed overall

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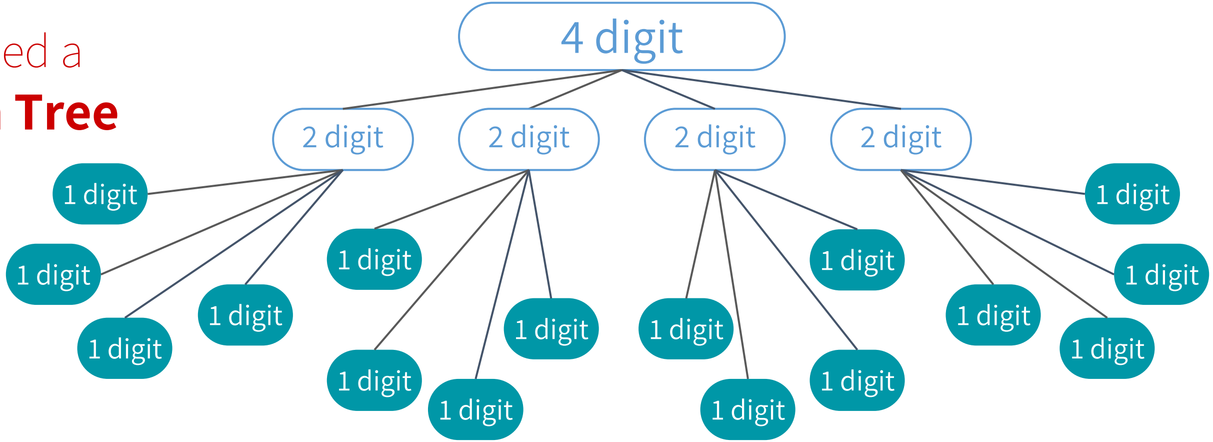
This is called a

Recursion Tree

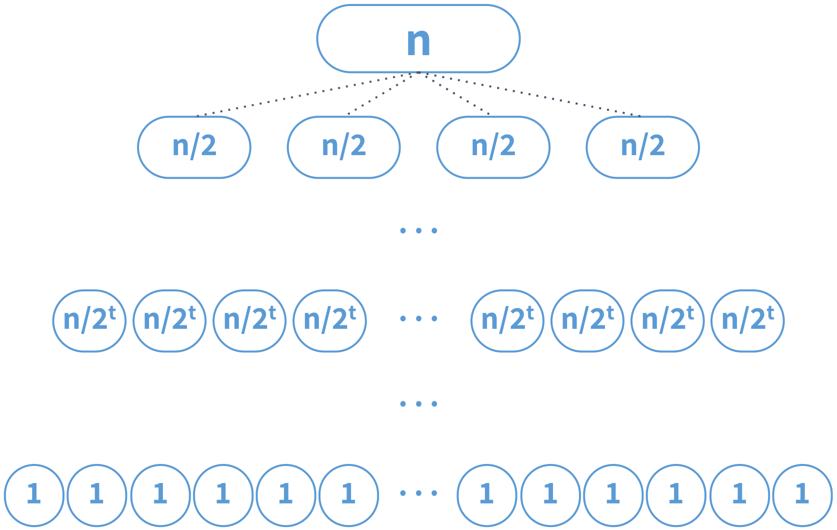


Sixteen 1-digit multiplications!

HOW EFFICIENT IS THIS ALGORITHM?

- **Now let's generalize:** if we're multiplying two n-digit numbers, how many 1-digit multiplications does the algorithm perform?

Recursion Tree



Level 0: 1 problem of size n

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Level $\log_2 n$: _____ problems of size 1

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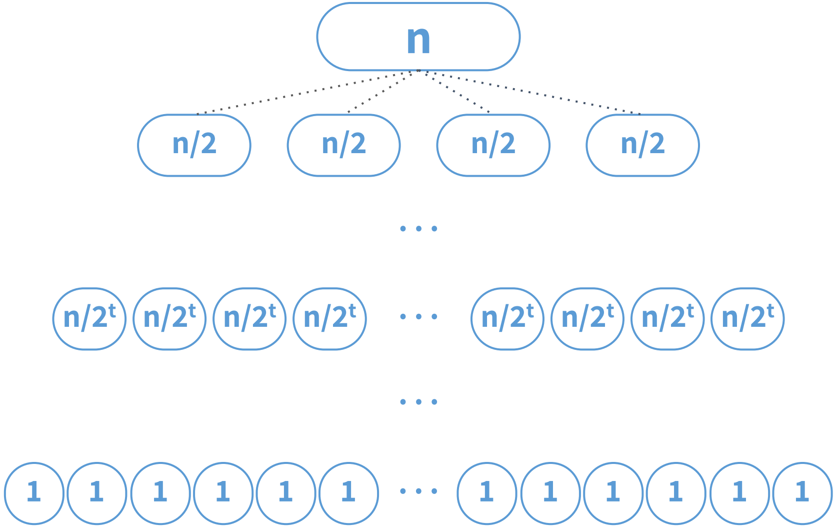

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$\log_2 n$ levels
(you need to cut n in half $\log_2 n$ times to get to size 1)

# of problems on last level (size 1)
$= 4^{\log_2 n} = n^{\log_2 4}$
$= \mathbf{n^2}$

HOW EFFICIENT IS THIS ALGORITHM?

The running time of this Divide-and-Conquer multiplication algorithm is **at least $O(n^2)$**!

We know there are already n^2 multiplications happening at the bottom level of the recursion tree, so that's why we say "at least" $O(n^2)$

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Is Divide-and-Conquer really that useless?

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no!!!

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Is Divide-and-Conquer really that useless?

Karatsuba says no!!!

KARATSUBA INTEGER MULTIPLICATION

Three subproblems instead of four!

CHOOSING SUBPROBLEMS WISELY

$$[x_1 x_2 \ldots x_{n-1} x_n] \times [y_1 y_2 \ldots y_{n-1} y_n]$$

$$= (a \times 10^{n/2} + b) \times (c \times 10^{n/2} + d)$$

$$= (a \times c)10^n + (a \times d + b \times c)10^{n/2} + (b \times d)$$

The subproblems we choose to solve just need to provide these quantities:

ac   ad + bc   bd

Originally, we assembled these quantities by computing FOUR things: ac, ad, bc, and bd.

KARATSUBA'S TRICK

end result = (ac)10^n + (ad + bc)$10^{n/2}$ + (bd)

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ac & bd can be recursively computed as usual

ad + bc is equivalent to **(a+b)(c+d) - ac - bd**

= (ac + ad + bc + bd) - ac - bd

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= ad + bc

So, instead of computing ad & bc as two separate subproblems, let's just compute (a+b)(c+d) instead!

OUR THREE SUBPROBLEMS

These *three* subproblems give us everything we need to compute our desired quantities:

1 ac

2 bd

3 (a+b)(c+d)

Assemble our overall product by combining these three subproblems:

$$(\overset{1}{ac})10^n + (\overset{3 - 1 - 2}{ad + bc})10^{n/2} + (\overset{2}{bd})$$

OUR THREE SUBPROBLEMS

These *three* subproblems give us everything we need to compute our desired quantities:

1 **ac**

2 **bd**

3 **(a+b)(c+d)**

(a+b) and (c+d) are both going to be n/2-digit numbers!

⇩

This means we still have half-sized subproblems!

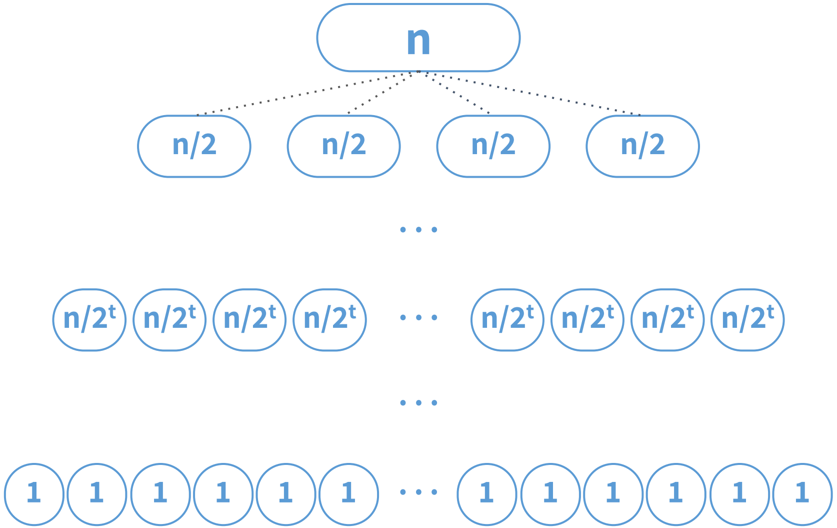
Assemble our overall product by combining these three subproblems:

\- \- 1 3 1 2 2

$$(ac)10^{n} + (ad + bc)10^{n/2} + (bd)$$

WHAT'S THE RUNTIME?

This was the Recursion Tree + Analysis from Divide-and-Conquer Attempt 1:



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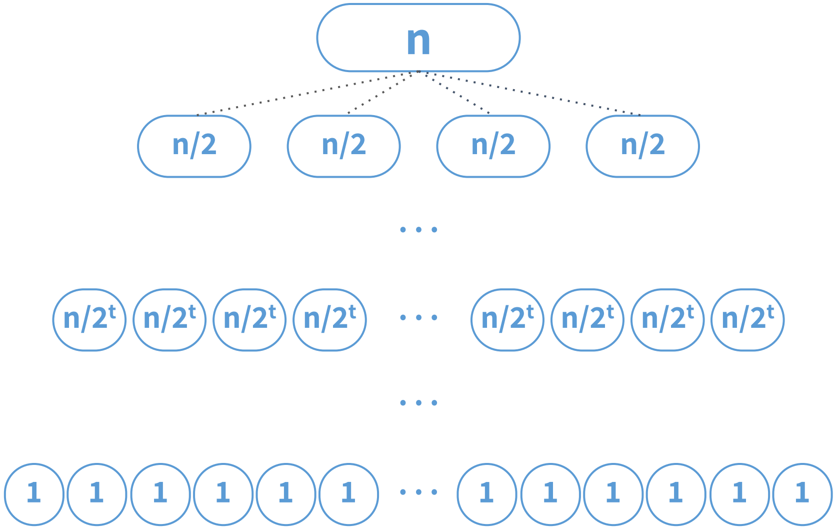

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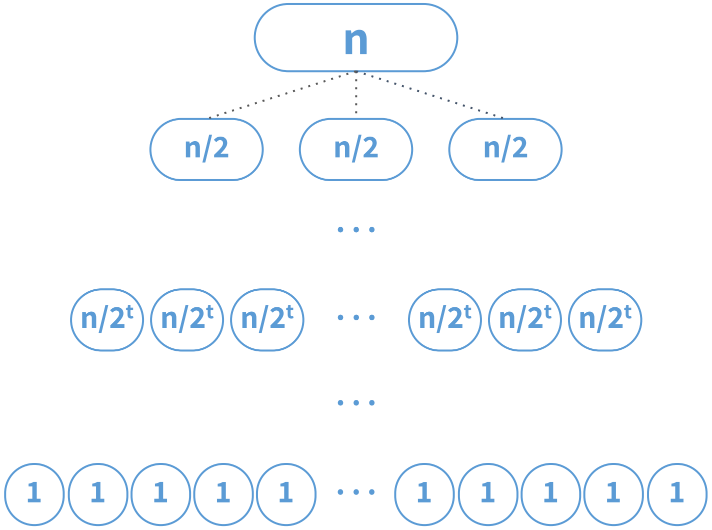
For Karatsuba's, we'll replace the branching factor of 4 with a 3! ⇒

WHAT'S THE RUNTIME?

Karatsuba Multiplication Recursion Tree



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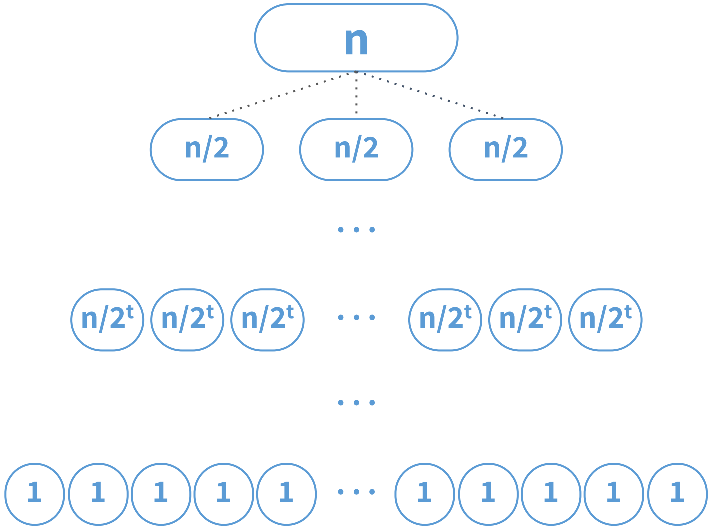

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$= 3^{\log_2 n} = n^{\log_2 3}$

$\approx \mathbf{n^{1.6}}$

Thus, the runtime is $O(n^{1.6})$!

WHAT'S THE RUNTIME?

NOTE: I know it looks like we didn't account for the work done on higher levels in the recursion tree, but as we'll learn later, the work on the last level actually dominates *in this particular recursion tree*!

Multiplication Recursion Tree

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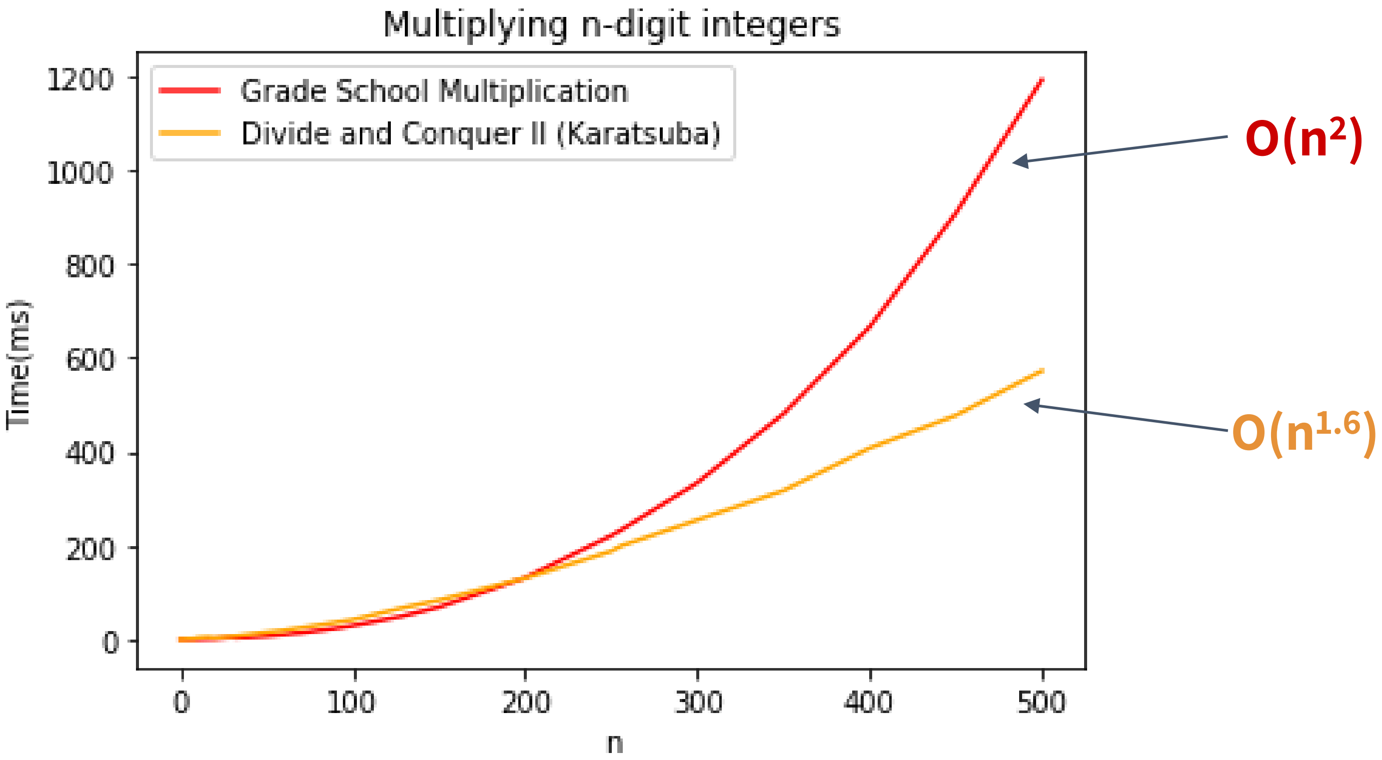

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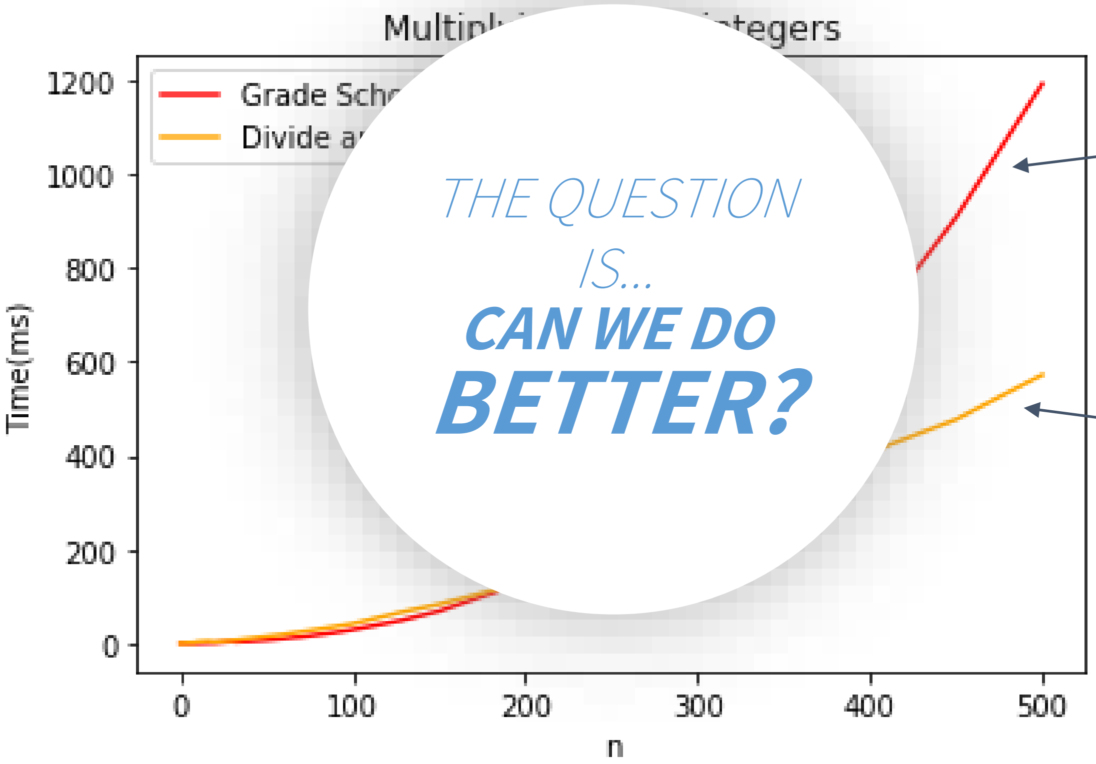

$O(n^2)$

$O(n^{1.6})$

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We won't expect you to know any of these algorithms by the way!

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- **Fürer (2007):** uses Fourier Transforms over complex numbers
    - Runtime: $O(n \log(n)\, 2^{O(\log^*(n))})$

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- **Schönhage–Strassen (1971):** uses fast polynomial multiplications
  - Runtime: $O(n \log n \log \log n)$
- **Fürer (2007):** uses Fourier Transforms over complex numbers
  - Runtime: $O(n \log(n)\, 2^{O(\log^*(n))})$
- **Harvey and van der Hoeven (2019!):** wild stuff
  - Runtime: $O(n \log(n))$

We won't expect you to know any of these algorithms by the way!