MTH 9878

Interest Rate Models HW 3

Group 5

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Q1

① Apply Itô's formula to $\log(\sigma_1 F_t + \sigma_0)$ we have:

$$d\left(\log(\sigma_1 F_t + \sigma_0)\right) = \log \frac{\sigma_1 F_T + \sigma_0}{\sigma_1 F_0 + \sigma_0} = \frac{\sigma_1}{\sigma_1 F_t + \sigma_0} dF_t - \frac{1}{2} \frac{\sigma_1^2}{(\sigma_1 F_t + \sigma_0)^2} dF_t \, dF_t$$

$$= \sigma_1 dW_t - \frac{1}{2}\sigma_1^2 dt$$

$$\Rightarrow \sigma_1 F_T + \sigma_0 = (\sigma_1 F_0 + \sigma_0) \cdot \exp\left(-\frac{1}{2}\sigma_1^2 dt + \sigma_1 dW_t\right)$$

$$\Rightarrow F_T = \frac{1}{\sigma_1}\left[(\sigma_1 F_0 + \sigma_0) \exp\left(-\frac{1}{2}\sigma_1^2 \overset{T}{\cancel{dt}} + \overset{\sigma_1 W_T}{\cancel{\sigma_1 dW_t}}\right) - \sigma_0 \right]$$. as $t=0$, $W_0 = 0$.

② for call option, $P_{sln}^{call} = E(F_T - k, 0)^{+} \cdot N(0)$

$$F_T > k \iff (\sigma_1 F_0 + \sigma_0) \cdot e^{-\frac{1}{2}\sigma_1^2 dt + \sigma_1 dW_t} > k\sigma_1 + \sigma_0$$

$$\iff -\frac{1}{2}\sigma_1^2 dt + \sigma_1 dW_t > \ln \frac{k\sigma_1 + \sigma_0}{\sigma_1 F_0 + \sigma_0}$$

$$\overset{t_0 = 0}{\underset{W_0 = 0}{\iff}} -\frac{1}{2}\sigma_1^2 T + \sigma_1 W_T > \ln \frac{k\sigma_1 + \sigma_0}{\sigma_1 F_0 + \sigma_0}$$

$$\iff W_T > \frac{\ln \frac{k\sigma_1 + \sigma_0}{\sigma_1 F_0 + \sigma_0} + \frac{1}{2}\sigma_1^2 T}{\sigma_1}$$

$$\overset{W_T = \sqrt{T} Z}{\underset{Z \sim N(0,1)}{\iff}} Z > \frac{\ln \frac{\sigma_1 k + \sigma_0}{\sigma_1 F_0 + \sigma_0} + \frac{1}{2}\sigma_1^2 T}{\sigma_1 \sqrt{T}} = -d_-$$

where $d_- = \dfrac{\ln \frac{\sigma_1 F_0 + \sigma_0}{\sigma_1 k + \sigma_0} - \frac{1}{2}\sigma_1^2 T}{\sigma_1 \sqrt{T}}$

③ pcall :

$$p^{call} = N(0) \cdot \int_{-\infty}^{\infty} (F_T - k, 0)^+ \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$= N(0) \cdot \int_{-d_-}^{\infty} (F_T - k) \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx .$$

$$= N(0) \cdot \int_{-d_-}^{\infty} \left[\frac{(\sigma_1 F_0 + \sigma_0) \cdot e^{-\frac{1}{2}\sigma_1^2 T + \sigma_1 W_T} - \sigma_0}{\sigma_1} - k \right] \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$= N(0) \cdot \int_{-d_-}^{\infty} \frac{(\sigma_1 F_0 + \sigma_0) e^{-\frac{1}{2}\sigma_1^2 T + \sigma_1 W_T} - (k\sigma_1 + \sigma_0)}{\sigma_1} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$\underset{x \sim N(0,1)}{\overset{W_T = \sqrt{T} x}{=}} N(0) \cdot \int_{-d_-}^{\infty} \frac{\sigma_1 F_0 + \sigma_0}{\sigma_1} \cdot e^{-\frac{1}{2}\sigma_1^2 T + \sigma_1 \sqrt{T} x} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$- N(0) \int_{-d_-}^{\infty} \frac{k\sigma_1 + \sigma_0}{\sigma_1} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$= N(0) \left[\int_{-d_-}^{\infty} \frac{\sigma_1 F_0 + \sigma_0}{\sigma_1} \cdot e^{-\frac{(x - \sigma_1\sqrt{T})^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} dx - \left(k + \frac{\sigma_0}{\sigma_1}\right) N(d_-) \right]$$

$$= N(0) \left[\int_{-d_- + \sigma_1\sqrt{T}}^{\infty} \left(F_0 + \frac{\sigma_0}{\sigma_1}\right) e^{-\frac{u^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} du - \left(k + \frac{\sigma_0}{\sigma_1}\right) N(d_-) \right]$$

$$= N(0) \left[\left(F_0 + \frac{\sigma_0}{\sigma_1}\right) N(d_+) - \left(k + \frac{\sigma_0}{\sigma_1}\right) N(d_-) \right]$$

where $\int_{-d_- + \sigma_1\sqrt{T}}^{\infty} \left(F_0 + \frac{\sigma_0}{\sigma_1}\right) e^{-\frac{u^2}{2}} \frac{1}{\sqrt{2\pi}} du = \int_{-d_+}^{\infty} \left(F_0 + \frac{\sigma_0}{\sigma_1}\right) e^{-\frac{u^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} du = \left(F_0 + \frac{\sigma_0}{\sigma_1}\right) N(d_+)$.

and $$d_+ = \frac{\ln \frac{F_0\sigma_1+\sigma_0}{k\sigma_1+\sigma_0} + \frac{1}{2}\sigma_1^2 T}{\sigma_1\sqrt{T}}, \quad d_- = \frac{\ln \frac{F_0\sigma_1+\sigma_0}{k\sigma_1+\sigma_0} - \frac{1}{2}\sigma_1^2 T}{\sigma_1\sqrt{T}}. \qquad (29)$$

Therefore

$$P_{call} = N(0)\left[(F_0+\frac{\sigma_0}{\sigma_1})N(d_+) - (k+\frac{\sigma_0}{\sigma_1})N(d_-)\right]$$

and $$B_{sln}^{call} = (F_0+\frac{\sigma_0}{\sigma_1})N(d_+) - (k+\frac{\sigma_0}{\sigma_1})N(d_-) \qquad (28)$$

$$P^{call} = N(0)\cdot B_{sln}^{call} \qquad (27).$$

From put-call parity, we know

$$B_{sln}^{call} - B_{sln}^{put} = F_0 - k.$$

$$\Rightarrow B_{sln}^{put} = B_{sln}^{call} - F_0 + k.$$

$$= (F_0+\frac{\sigma_0}{\sigma_1})N(d_+) - F_0 - \left[(k+\frac{\sigma_0}{\sigma_1})N(d_-) - k\right]$$

$$= F_0 - F_0 - \frac{\sigma_0}{\sigma_1} - (F_0+\frac{\sigma_0}{\sigma_1})N(-d_+) - \left[k-k-\frac{\sigma_0}{\sigma_1} - (k+\frac{\sigma_0}{\sigma_1})N(-d_-)\right]$$

$$= -(F_0+\frac{\sigma_0}{\sigma_1})N(-d_+) + (k+\frac{\sigma_0}{\sigma_1})N(-d_-) \qquad (30)$$

$$P^{put} = N(0)\cdot B_{sln}^{put} \qquad (27)$$

Therefore, all formulas have been approved.

Q2

(a) Let's consider call option:

for Normal model:

$$P_n^{call} = N(0)\,\sigma_n\sqrt{T}\,\left(d_+ N(d_+) + N'(d_+)\right), \quad d_\pm = \pm\frac{F_0-K}{\sigma_n\sqrt{T}}$$

for lognormal model:

$$P_{ln}^{call} = N(0)\left(F_0 N(d_1) - K N(d_2)\right), \quad d_{1,2} = \frac{\log\frac{F_0}{K} \pm \frac{1}{2}\sigma_{ln}^2 T}{\sigma_{ln}\sqrt{T}}$$

for ATM call option: $F_0 = K, \Rightarrow d_\pm = 0, \quad d_{1,2} = \pm\frac{1}{2}\sigma_{ln}\sqrt{T}$

Then $$P_n^{call} = N_0\,\sigma_n\sqrt{T}\cdot\left(0 + \frac{1}{\sqrt{2\pi}}e^{-\frac{(F_0-K)^2}{2\sigma_n^2 T}}\right) = N_0\cdot\frac{\sigma_n\sqrt{T}}{\sqrt{2\pi}}$$

$$P_{ln}^{call} = N(0)\,F_0\left(N(d_1) - N(d_2)\right)$$

Let $P_n^{call} = P_{ln}^{call}$.

$$\sigma_n = F_0\cdot\sqrt{\frac{2\pi}{T}}\cdot\left(N(d_1) - N(d_2)\right)$$

$$N(d_1) - N(d_2) = \int_{-\frac{1}{2}\sigma_{ln}\sqrt{T}}^{\frac{1}{2}\sigma_{ln}\sqrt{T}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx$$

$$\xrightarrow[u=\frac{\sqrt{2}}{2}x]{x=\sqrt{2}u} \int_{-\frac{1}{2\sqrt{2}}\sigma_{ln}\sqrt{T}}^{\frac{1}{2\sqrt{2}}\sigma_{ln}\sqrt{T}} \frac{1}{\sqrt{2\pi}} e^{-u^2}\cdot\sqrt{2}\,du = \frac{2}{\sqrt{\pi}}\int_0^{\frac{1}{2\sqrt{2}}\sigma_{ln}\sqrt{T}} e^{-u^2}\,du$$

Therefore, $\sigma_n = F_0\frac{\sqrt{2\pi}}{\sqrt{T}}\,\mathrm{erf}\left(\frac{\sqrt{T}}{2\sqrt{2}}\sigma_{ln}\right)$, where $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}\,du$. 4

(b) As we know, Taylor expansion of e^{-u^2} is following

$$e^{-u^2} \approx 1 - u^2 + \frac{(-u^2)^2}{2!} + \frac{(-u^2)^3}{3!} + \frac{(-u^2)^4}{4!} + \cdots \qquad u \to 0.$$

$$= 1 - u^2 + \frac{u^4}{2!} - \frac{u^6}{3!} + \frac{u^8}{4!} + \cdots$$

Then: when $\sigma_{ln}\sqrt{T}$ is small.

$$\int_0^{\frac{1}{2\sqrt{2}}\sigma_{ln}\sqrt{T}} e^{-u^2}\,du \approx \int_0^{\frac{1}{2\sqrt{2}}\sigma_{ln}\sqrt{T}} \left(1 - u^2 + \frac{u^4}{2!} - \frac{u^6}{3!} + \frac{u^8}{4!} + \cdots\right) du$$

$$= \left(u - \frac{u^3}{3} + \frac{u^5}{2! \times 5} - \frac{u^7}{3! \times 7} + \frac{u^9}{4! \times 5} + \cdots \right)\Bigg|_0^{\frac{1}{2\sqrt{2}}\sigma_{ln}\sqrt{T}}$$

$$= \frac{1}{2\sqrt{2}}\sigma_{ln}\sqrt{T}\left[1 - \frac{u^2}{3} + \frac{u^4}{10} - \frac{u^6}{42} + \frac{u^8}{120} + \cdots\right]\Bigg|_0^{\frac{1}{2\sqrt{2}}\sigma_{ln}\sqrt{T}}$$

Therefore

$$\sigma_n = F_0 \cdot \sqrt{\frac{2\pi}{T}} \cdot \frac{2}{\sqrt{\pi}} \cdot \frac{1}{2\sqrt{2}}\sigma_{ln}\sqrt{T}\left[1 - \frac{\left(\frac{1}{2\sqrt{2}}\sigma_{ln}\sqrt{T}\right)^2}{3} + \frac{\left(\frac{1}{2\sqrt{2}}\sigma_{ln}\sqrt{T}\right)^4}{10} + \cdots\right]$$

$$= F_0 \cdot \sigma_{ln}\left(1 - \frac{1}{24}\sigma_{ln}^2 T + \frac{1}{640}\left(\sigma_{ln}^2 T\right)^2 + \cdots\right)$$

Q3:

(1) Normal SABR model

The asymptotic expression for implied normal volatility is:

$$\sigma_n = \frac{\alpha(F_0-k)}{D(\xi)}\left(1+O(\varepsilon)\right)$$

where $D(\xi) = \log\left(\frac{\sqrt{\xi^2-2\rho\xi+1}+\xi-\rho}{1-\rho}\right)$

$$\xi = \frac{\alpha(F_0-k)}{\sigma_0}$$

Therefore $\sigma_n = \dfrac{\sigma_0\cdot\xi\cdot(1+O(\varepsilon))}{D(\xi)}$

for $\xi\to 0$, we have

$$\lim_{\xi\to 0}\sigma_n = \lim_{\xi\to 0}\frac{\sigma_0\cdot\xi\,(1+O(\varepsilon))}{D(\xi)} = \lim_{\xi\to 0}\frac{\sigma_0\,(1+O(\varepsilon))}{D'(\xi)}$$

As $D'(\xi) = \dfrac{1-\rho}{\sqrt{\xi^2-2\rho\xi+1}+\xi-\rho}\cdot\dfrac{1}{1-\rho}\cdot\left(1+\dfrac{2\xi-2\rho}{2\sqrt{\xi^2-2\rho\xi+1}}\right)$

$$= \frac{1}{\sqrt{\xi^2-2\rho\xi+1}}$$

Then $\sigma_n = \lim_{\xi\to 0}\dfrac{\sigma_0\,(1+O(\varepsilon))}{\frac{1}{\sqrt{\xi^2-2\rho\xi+1}}} = \sigma_0\,(1+O(\varepsilon))$

(2) lognormal SABR model

$$\sigma_{ln} = \frac{\alpha \log(F_0/K)}{D(\xi)}(1+O(\varepsilon))$$

where $$D(\xi) = \log\left(\frac{\sqrt{\xi^2-2\rho\xi+1}+\xi-\rho}{1-\rho}\right)$$

$$\xi = \frac{\alpha(F_0-K)}{\sigma_0}$$

As $\log(F_0/K) = \log\left(\frac{F_0-K}{K}+1\right) = \log\left(\frac{\xi\cdot\sigma_0}{\alpha K}+1\right)$

Then we have

$$\sigma_{ln} = \lim_{\xi\to 0} \frac{\alpha \log\left(\frac{\xi\cdot\sigma_0}{\alpha K}+1\right)(1+O(\varepsilon))}{D(\xi)}$$

$$= \lim_{\xi\to 0}\frac{\alpha(1+O(\varepsilon))\cdot\left(\frac{\alpha K}{\xi\cdot\sigma_0+\alpha K}\cdot\frac{1}{\alpha K}\cdot\sigma_0\right)}{\frac{1}{\sqrt{\xi^2-2\rho\xi+1}}}$$

$$= \frac{\sigma_0}{K}(1+O(\varepsilon))$$