Homework 1

Thursday, February 19, 2015 2:29 PM

1. (i) Couple rate is C, in total $2T$ payment, since $\alpha = 0.5$

$$PV(\text{fixed Leg}) = \sum_{j=1}^{2T} \alpha C \cdot P_{0,T_j} = \sum_{j=1}^{2T} \frac{1}{2} C P_{0,\frac{j}{2}}$$

$\delta = \frac{1}{4}$, in total $4T$ payment

$$PV(\text{float leg}) = \sum_{j=1}^{4T} \delta L_j P_{0,T_j} = \sum_{j=1}^{n} \frac{1}{4} L_j P_{0,\frac{j}{4}}$$

$$F(t,S,T) = \frac{1}{\delta}\left(e^{\int_S^T f_0 \, ds} - 1\right)$$

$$= \frac{1}{\delta}\left(e^{f_0(T-S)} - 1\right)$$

$$P_{0,T_j} = \frac{1}{1-\delta F(0,T_j)} = e^{\frac{j}{2} f_0}$$

$$L_j = \frac{1}{\delta}\left(e^{\int_{\frac{j}{4}}^{\frac{j+1}{4}} f_0 \, ds} - 1\right)$$

$$= 4\left(e^{\frac{1}{4} f_0} - 1\right)$$

$$\therefore PV(\text{Fixe Leg}) = \sum_{j=1}^{2T} \frac{1}{2} C e^{-f_0 \frac{1}{2} j} = \frac{c\left(e^{-\frac{1}{2} f_0} - e^{-\frac{1}{2} f_0 (2T+1)}\right)}{2\left(1-e^{-\frac{1}{2} f_0}\right)} = Ac.$$

$$PV(\text{Float leg}) = \sum_{i=1}^{4T} \left(e^{\frac{1}{4} f_0 \frac{i}{4}} - 1\right) e^{-\frac{i}{4} f_0} = \frac{e^{\frac{1}{4}(f_0 - f_0)} - e^{\frac{1}{4}(f_0 - f_0)(4T+1)}}{1-e^{\frac{1}{4}(f_0 - f_0)} - 1} + \frac{e^{-\frac{1}{4} f_0} - e^{-\frac{1}{4} f_0 (4T+1)}}{1-e^{-\frac{1}{4} f_0}} = L.$$

(2) if $PV(\text{Fixed Leg}) = PV(\text{Float leg})$

$$C^* = \frac{L}{A}. \quad \text{which is the par coupon}$$

$$= \frac{2\left(e^{-\frac{1}{4} f_0} - e^{-\frac{1}{4} f_0 (4T+1)}\right)\left(e^{\frac{1}{4} f_0} - 1\right)\left(1 - e^{-\frac{1}{2} f_0}\right)}{\left(1-e^{-\frac{1}{4} f_0}\right)\left(e^{-\frac{1}{2} f_0} - e^{-\frac{1}{2} f_0 (2T+1)}\right)}$$

3. Since $S_1(t) \ldots S_N(t)$ has a self-financing Portfolio that weight is $W_1(t) \cdots W_N(t)$, and it is frictionless

$$V_t = \sum_{i=1}^{N} w_i(t) S_i(t)$$

$$dV(t) = \sum_{i=1}^{N} w_i(t) \, dS_i(t)$$

$$V(t) = V_0 + \int_0^t \sum_{i=1}^{N} w_i(s) \, dS_i(s)$$

Given Numeraire $N(t)$

$$S_i^N(t) = \tilde{E}\left[S_i^N(T)/N(T) \mid \mathcal{F}_t\right]$$

$$= \frac{S_i(t)}{N(t)}$$

$$d\frac{V_t}{N_t} = \frac{1}{N_t} dV_t - \frac{V_t}{N_t^2} dN_t + \frac{V_t}{N_t^3}(dN_t)^2 - \frac{1}{N_t^2} dV_t dN_t$$

$$= \sum_{i=1}^{N} w_i(t) \frac{1}{N_t} dS_i(t) - \sum_{i=1}^{N} w_i(t) \frac{S_{i,t}}{N_t^2} dN_t + \sum_{i=1}^{N} w_i(t) \frac{S_{i,t}}{N_t^3}(dN_t)^2 - \sum_{i=1}^{N} w_i(t) \frac{1}{N_t^2} dS_{i,t} dN_t$$

$$= \sum_{i=1}^{N} w_i(t) \, dS_i^N(t)$$

$\therefore$ It is also frictionless

2. We'll induction to prove.

We first prove the case for $d=1$, s.t.

$$\frac{d}{dt}B_k^{(1)}(t) = \frac{1}{t_{k+1}-t_k}B_k^{(0)}(t) - \frac{1}{t_{k+2}-t_{k+1}}B_{k+1}^{(0)}(t)$$

By definition, we have

$$B_k^{(1)}(t) = \frac{t-t_k}{t_{k+1}-t_k}B_k^{(0)}(t) + \frac{t_{k+2}-t}{t_{k+2}-t_{k+1}}B_{k+1}^{(0)}(t)$$

$$B_k^{(0)} = \begin{cases} 1 & t \in [t_k, t_{k+1}] \\ 0 & \text{o.w.} \end{cases}$$

When $t \notin [t_k, t_{k+1}] \Rightarrow B_k^{(0)} = 0$

$\Rightarrow B_k^{(1)}(t) = 0 \Rightarrow$ The derivative formula holds true.

when $t \in [t_k, t_{k+1}]$

$$B_k^{(1)} = \frac{t-t_k}{t_{k+1}-t_k}B + \frac{t_{k+2}-t}{t_{k+2}-t_{k+1}}$$

$$\frac{d}{dt}B_k^{(1)} = \frac{1}{t_{k+1}-t_k} - \frac{1}{t_{k+2}-t_{k+1}}$$

$\therefore$ The formula holds true for $d=1$.

Next, we assume $\frac{d}{dt}B_k^{(d)}(t) = \frac{d}{t_{k+1}-t_k}B_k^{(d-1)}(t) - \frac{d}{t_{k+2}-t_{k+1}}B_{k+1}^{(d-1)}(t)$

We need to show

$$\frac{d}{dt}B_k^{(d+1)}(t) = \frac{d+1}{t_{k+1}-t_k}B_k^{(d)}(t) - \frac{d+1}{t_{k+2}-t_{k+1}}B_{k+1}^{(d)}(t)$$

Apply the definition, we have

$$\frac{d}{dt}B_k^{(d+1)}(t) = \frac{d}{dt}\left(\frac{t-t_k}{t_{k+d+1}-t_k}B_k^{(d)}(t) + \frac{t_{k+d+2}-t}{t_{k+d+2}-t_{k+1}}B_{k+1}^{(d)}(t)\right)$$

Take the derivative,

$$\Rightarrow \frac{B_k^{(d)}(t)}{t_{k+d+1}-t_k} + \frac{t-t_k}{t_{k+d+1}-t_k}\, dB_k^{(d)}(t) + \frac{t_{k+d+2}-t}{t_{k+d+2}-t_{k+1}}\, dB_{k+1}^{d}(t)$$

$$- \frac{B_{k+1}^{(d)}(t)}{t_{k+d+2}-t_{k+1}}$$

We focus on $\frac{t-t_k}{t_{k+d+1}-t_k}\, dB_k^{(d)}(t) + \frac{t_{k+d+2}-t}{t_{k+d+2}-t_{k+1}}\, dB_{k+1}^{(d)}(t)$

Using our assumption, we have

$$\frac{t-t_k}{t_{k+d+1}-t_k}\, dB_k^{(d)}(t) = \frac{t-t_k}{t_{k+d+1}-t_k}\left(\frac{d}{t_{k+d}-t_k}B_k^{(d-1)}(t) - \frac{d}{t_{k+d+2}-t_{k+1}}B_{k+1}^{(d-1)}(t)\right)$$

$$\frac{t_{k+d+2}-t}{t_{k+d+2}-t_{k+1}}\, dB_{k+1}^{(d)}(t) = \frac{t_{k+d+2}-t}{t_{k+d+2}-t_{k+1}}\left(\frac{d}{t_{k+d+1}-t_{k+1}}B_{k+1}^{(d-1)}(t) - \frac{d}{t_{k+d+2}-t_{k+2}}B_{k+2}^{(d-1)}(t\right.$$

Now look at the $B_{k+1}^{(d-1)}(t)$ terms,

$$-\frac{(t-t_k)}{t_{k+d+1}-t_k}\,\frac{d}{t_{k+d+2}-t_{k+1}}B_{k+1}^{(d-1)}(t) + \frac{t_{k+d+2}-t}{t_{k+d+2}-t_{k+1}}\,\frac{d}{t_{k+d+1}-t_{k+1}}B_{k+1}^{(d-1)}(t).$$

We add and subtract the following.

$$\frac{d}{t_{k+d+1}-t_k}\,\frac{t_{k+d+1}-t}{t_{k+d+1}-t_{k+1}}B_{k+1}^{(d-1)}(t) - \frac{d}{t_{k+d+2}-t_{k+1}}\left(\frac{t-t_{k+1}}{t_{k+d+1}-t_{k+1}}\right)B_{k+1}^{(d-1)}(t)$$

$$\Rightarrow dB_{k+1}^{(d-1)}(t)\Big[\frac{t_{k+d+2}-t}{t_{k+d+2}-t_{k+1}}\,\frac{1}{t_{k+d+1}-t_{k+1}} - \frac{t_{k+d+1}-t}{(t_{k+d+1}-t_k)(t_{k+d+1}-t_{k+1})}$$

$$- \frac{t-t_k}{(t_{k+d+1}-t_k)(t_{k+d+1}-t_{k+1})} + \frac{t-t_{k+1}}{(t_{k+d+2}-t_{k+1})(t_{k+d+1}-t_{k+1})}\Big]$$

$$= dB_{k+1}^{(d-1)}(t)\left[\frac{t_{k+d+2}-t_{k+1}}{(t_{k+d+2}-t_{k+1})(t_{k+d+1}-t_{k+1})} - \frac{t_{k+d+1}-t_k}{(t_{k+d+1}-t_k)(t_{k+d+1}-t_{k+1})}\right]$$

The last term $= 0$

So the $B_{k+1}^{(d+1)}(t)$ terms become

$$d B_{k+1}^{(d+1)}(t)\left[\frac{t_{k+d+1}-t}{(t_{k+d+1}-t_k)(t_{k+d+1}-t_{k+1})}-\frac{t-t_{k+1}}{(t_{k+d+2}-t_{k+1})(t_{k+d+1}-t_{k+1})}\right]$$

$$\Rightarrow \frac{t-t_k}{t_{k+d+1}-t_k} d B_k^{(d)}(t)+\frac{t_{k+d+2}-t}{t_{k+d+2}-t_{k+1}} d B_{k+1}^{(d)}(t)$$

$$=\frac{d}{t_{k+d+1}-t_k}\frac{t-t_k}{t_{k+d}-t_k}B_k^{(d-1)}(t)+\frac{d}{t_{k+d+1}-t_k}\frac{t_{k+d+1}-t}{t_{k+d+1}-t_{k+1}}B_{k+1}^{(d-1)}(t)$$

$$-\frac{d}{t_{k+d+2}-t_{k+1}}\frac{t_{k+d+2}-t}{t_{k+d+2}-t_{k+2}}B_{k+2}^{(d-1)}(t)-\frac{d}{t_{k+d+2}-t_{k+1}}\frac{t-t_{k+1}}{t_{k+d+1}-t_{k+1}}B_{k+1}^{(d-1)}(t)$$

$$=\frac{d}{t_{k+d+1}-t_k}B_k^{(d)}(t)-\frac{d}{t_{k+d+2}-t_{k+1}}B_{k+1}^{(d)}(t)$$

$$\Rightarrow \frac{d}{dt}B_k^{(d+1)}(t)=\frac{B_k^{(d)}(t)}{t_{k+d+1}-t_k}+\frac{d}{t_{k+d+1}-t_k}B_k^{(d)}(t)$$

$$-\frac{d}{t_{k+d+2}-t_{k+1}}B_{k+1}^{(d)}(t)-\frac{B_{k+1}^{(d)}(t)}{t_{k+d+2}-t_{k+1}}$$

$$=\frac{d+1}{t_{k+d+1}-t_k}B_k^{(d)}(t)-\frac{d+1}{t_{k+d+2}-t_{k+1}}B_{k+1}^{(d)}(t)$$

$\therefore$ The formula also holds true for $d+1$

$\because$ This formula holds true for $d=1$, $d=d$, $d+1$

It is true for any positive integer