1.

$$dF_t = (\sigma_1 F_t + \sigma_0)\, dW_t$$

$$d\ln(\sigma_1 F_t + \sigma_0) = \frac{\sigma_1 dF_t}{\sigma_1 F_t + \sigma_0} - \frac{1}{2}\frac{\sigma_1^2}{(\sigma_1 F_t + \sigma_0)^2}(dF_t)^2$$

$$= \sigma_1 dW_t - \frac{1}{2}\sigma_1^2 dt$$

$$\therefore \frac{\sigma_1 F_T + \sigma_0}{\sigma_1 F_0 + \sigma_0} = \exp\{\sigma_1 W_T - \frac{1}{2}\sigma_1^2 T\}$$

$$F_T = \frac{1}{\sigma_1}\left[(\sigma_1 F_0 + \sigma_0)\exp\{\sigma_1 W_T - \frac{1}{2}\sigma_1^2 T\} - \sigma_0\right]$$

For Call

$$P_{call} = E(F_T - K)^+$$

$$F_T > K \iff (\sigma_1 F_0 + \sigma_0)\exp\{\sigma_1 W_T - \frac{1}{2}\sigma_1^2 T\} > \sigma_1 K + \sigma_0$$

$$\iff \sigma_1 W_T - \frac{1}{2}\sigma_1^2 T > \ln\frac{\sigma_1 K + \sigma_0}{\sigma_1 F_0 + \sigma_0}$$

$$Z > \frac{\ln\frac{\sigma_1 F_0 + \sigma_0}{\sigma_1 K + \sigma_0} - \frac{1}{2}\sigma_1^2 T}{\sigma_1\sqrt{T}} = d_2$$

$$P_{call} = \int_{d_2}^{\infty}(F_T - K)\frac{1}{\sqrt{2\pi}}\exp\{-\frac{1}{2}Z^2\}dZ$$

$$= \int_{d_2}^{\infty}\frac{1}{\sigma_1}\left[(\sigma_1 F_0 + \sigma_0)\exp\{\sigma_1\sqrt{T}Z - \frac{1}{2}\sigma_1^2 T\} - \sigma_0 - \sigma_1 K\right]\frac{1}{\sqrt{2\pi}}\exp\{-\frac{1}{2}Z^2\}dZ$$

$$= \frac{1}{\sigma_1}\left[(\sigma_1 F_0 + \sigma_0)N(d_1) - (\sigma_1 K + \sigma_0)N(d_2)\right]$$

where $d_2 = \dfrac{\ln\frac{\sigma_1 F_0 + \sigma_0}{\sigma_1 K + \sigma_0} - \frac{1}{2}\sigma_1^2 T}{\sigma_1\sqrt{T}}$ $\quad d_1 = d_2 + \sigma_1\sqrt{T}$, N is CDF of Normal Distribution.

By Put-Call Parity.

$$= -\frac{1}{\sigma_1}(\sigma_1 F_0 + \sigma_0) + \frac{1}{\sigma_1}(\sigma_1 K + \sigma_0) + P_{call}$$

$$P_{put} = \frac{1}{\sigma_1}\left[(\sigma_1 K + \sigma_0)N(-d_2) - (\sigma_1 F_0 + \sigma_0)N(-d_1)\right]$$

2(a) Consider a Call

$$P = N\sigma_n\sqrt{T}(d_+ N(d_+) + N'(d_+)) \qquad d_+ = \frac{F_0 - K}{\sigma_n\sqrt{T}} = 0$$

$$= N\sigma_n\sqrt{T}\cdot\frac{1}{\sqrt{2\pi}}$$

$$P = N(F_0 N(d_+) - K N(d_-)) \qquad d_\pm = \frac{\log\frac{F}{K} \pm \frac{1}{2}\sigma_{ln}^2 T}{\sigma_{ln}\sqrt{T}} = \pm\frac{1}{2}\sigma_{ln}\sqrt{T}$$

$$= N F_0(2N(\tfrac{1}{2}\sigma_{ln}\sqrt{T}) - 1)$$

$$\therefore \sigma_n\sqrt{T}\frac{1}{\sqrt{2\pi}} = F_0(2N(\tfrac{1}{2}\sigma_{ln}\sqrt{T}) - 1)$$

$$\therefore \sigma_n = \sqrt{\frac{2\pi}{T}}F_0\left(2\int_{-\infty}^{\frac{1}{2}\sigma_{ln}\sqrt{T}} e^{-\frac{1}{2}x^2}\frac{1}{\sqrt{2\pi}}dx - 1\right)$$

$$= \sqrt{\frac{2\pi}{T}}F_0\frac{2}{\sqrt{2\pi}}\int_0^{\frac{1}{2}\sigma_{ln}\sqrt{T}} e^{-\frac{1}{2}x^2}dx.$$

$$= \sqrt{\frac{2\pi}{T}}F_0\frac{2}{\sqrt{\pi}}\int_0^{\frac{\sqrt{T}}{2\sqrt{2}}\sigma_{ln}} e^{-u^2}du \qquad u = \frac{x}{\sqrt{2}}.$$

$$= \sqrt{\frac{2\pi}{T}}F_0\,\mathrm{erf}(\frac{\sqrt{T}}{2\sqrt{2}}\sigma_{ln})$$

(b) By Taylor Expansion.

$$f(x) = \frac{\sqrt{\pi}}{2}\mathrm{erf}(x) = \int_0^x e^{-u^2}du \qquad \text{where } x = \frac{\sqrt{T}\,\sigma_{ln}}{2\sqrt{2}}.$$

$$f(x) = f(0) + x f'(0) + \frac{1}{2}x^2 f''(0) + \frac{1}{6}x^3 f^{(3)}(0) + \frac{1}{24}x^4 f^{(4)}(0) + \cdots$$

$$f'(0) = 1,\quad f''(0) = 0,\quad f^{(3)}(0) = -2,\ f^{(4)} = 0,\ f^{(5)} = 12$$

$$\therefore \sigma_n = 2\sqrt{2}\frac{1}{\sqrt{T}}F_0 f(x)$$

$$= 2\sqrt{2}\frac{F_0}{\sqrt{T}}\left(\frac{\sigma_{ln}\sqrt{T}}{2\sqrt{2}}f'(0) + \frac{1}{6}\left(\frac{\sigma_{ln}\sqrt{T}}{2\sqrt{2}}\right)^3 f^{(3)}(0) + \frac{1}{120}\left(\frac{\sigma_{ln}\sqrt{T}}{2\sqrt{2}}\right)^5 f^{(5)}(0) + O((\sigma_{ln}\sqrt{T})^7)\right)$$

$$= F_0 \sigma_n \left(1 - \frac{1}{24}\sigma_{ln}^2 T + \frac{1}{640}(\sigma_{ln}^2 T)^2 + \cdots\right)$$

3. in normal SABR model

$$\sigma_n = \alpha \frac{F_0 - K}{D(\zeta)} = \frac{\zeta}{D(\zeta)}$$

$$D(\zeta) = \ln\left(\frac{\sqrt{\zeta^2 - 2\rho\zeta + 1} + \zeta - \rho}{1-\rho}\right)$$

By l'Hopital's rule. $\sigma_n = \lim_{\zeta \to 0} \frac{\alpha}{\ln\left(\frac{\sqrt{\zeta^2-2\rho\zeta+1}+\zeta-\rho}{1-\rho}\right)'} = \frac{\cancel{\frac{1}{1-\rho}}\left[(\zeta-\rho)(\zeta^2-2\rho\zeta+1)^{-\frac{1}{2}}+1\right]\alpha}{\cancel{\frac{1}{1-\rho}}\left(\sqrt{\zeta^2-2\rho\zeta+1}+\zeta-\rho\right)}$

$= \alpha.$

$$\sigma_{ln} = \lim_{\zeta \to 0} \alpha \frac{\log(F_0/K)}{D(\zeta)} = \lim_{\zeta \to 0} \frac{\log\left(\frac{F_0 - K}{K} + 1\right)}{D(\zeta)} = \frac{\log\frac{\zeta+1}{K}\cdot\alpha}{D(\zeta)} = \lim_{\zeta\to 0}\frac{K}{\zeta+1}\alpha = K\alpha$$