MTH 9878 IR Hw 6

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Q1:

① From formula (1) in Lecture 7, we have:

$$dL_j(t) = \Delta_j(t)dt + G(t)\,dW_j(t)$$

with

$$\Delta_j(t) = \Delta_j(t, L(t)) \quad \text{drift}$$

$$G(t) = G(t, L(t)) \quad \text{diffusion}$$

② Denote Q_j is the numeraire measure of zero coupon bonds expiring at T_{j+1}

F_i : OIS forward spanning the accrual period $[T_i, T_{i+1})$

$\gamma : [0, T_N] \to \mathbb{Z}$ defined as $\gamma(t) = m+1$, if $t \in [T_m, T_{m+1})$

Then we have:

$$P(t, T_{j+1}) = P(t, T_{\gamma(t)}) \cdot \prod_{\gamma(t) \le i \le j} \frac{1}{1 + \delta_i F_i(t)}$$

③ The spot measure:

$$B(t) = \frac{P(t, T_{\gamma(t)})}{\prod_{1 \le i \le \gamma(t)} P(T_{i-1}, T_i)}$$

④ Since drift of $L_j(t)$ under Q_j is zero. Then the formulas of Girsanov's theorem

$$E^P\left[\exp\left(\frac{1}{2}\int_0^t \theta(s)^T\theta(s)ds\right]<\infty$$

$$dD(t) = \theta(t)^T D(t)\,dW(t)$$

yields as:

$$\Delta_j(t) = \frac{d}{dt}\left[L_j,\ \log\frac{B(t)}{P(t,T_{j+1})}\right](t)$$

$$= \frac{d}{dt}\left[L_j,\ \log\frac{\dfrac{P(t,T_{r(t)})}{\prod_{1\le i\le r(t)} P(T_{i-1},T_i)}}{P(t,T_{r(t)})\prod_{r(t)\le i\le j}\dfrac{1}{1+\delta_i F_i(t)}}\right](t)$$

$$= \frac{d}{dt}\left[L_j,\ \log\frac{\prod_{r(t)\le i\le j}(1+\delta_i F_i(t))}{\prod_{1\le i\le r(t)} P(T_{i-1},T_i)}\right](t)$$

$$= \frac{d}{dt}\left[L_j,\ \log\frac{\prod_{r(t)\le i\le j}(1+\delta_i F_i(t))}{\prod_{0\le i\le r(t)-1}\dfrac{1}{1+\delta_i F_i(t)}}\right](t)$$

$$= \frac{d}{dt}\left[L_j,\ \log\left(\prod_{1\le i\le j}(1+\delta_i F_i(t))\right)\right](t)$$

As $T_i < t$, $\prod_{0\le i\le j}(1+\delta_i F_i(t))$ is constant for $\forall\ 0\le i\le r(t)-1$

$$\Rightarrow \Delta_j(t) = \sum_{r(t)\le i\le j} dL_j(t)\cdot\frac{\delta_i dF_i(t)}{1+\delta_i F_i(t)} = G(t)\cdot\sum_{r(t)\le i\le j}\cdot\frac{\rho_{ij}\delta_i G_i(t)}{1+\delta_i F_i(t)}$$

$\cancel{\sum_{r(t)\le i\le j} dL_j(t)}$).

⑤ Since change of numeraire measure by Girsanov's theorem only affect drift part (to be zero), we have:

$$dL_j(t) = G_j(t)\left[\sum_{\eta(t)\le i\le j} \frac{\rho_{ij}\,\delta_i\, G_i(t)}{1+\delta_i F_i(t)} + dW_j(t)\right].$$

i.e., formula 16) in lecture 7.

Q2: $I(a,b) + I(b,a) = \int_t^{t+\delta}\int_t^{s} (dZ_a(u)\,dZ_b(s) + dZ_b(u)\,dZ_a(s))$.

$$= \int_t^{t+\delta}\int_t^{s} (dZ_a(u)\,dZ_b(s)) + \int_t^{t+\delta}\int_t^{s} dZ_b(u)\,dZ_a(s)$$

$$= \int_t^{t+\delta} (Z_a(s)-Z_a(t))\,dZ_b(s) + \int_t^{t+\delta} (Z_b(s)-Z_b(t))\,dZ_a(s).$$

$$= \int_t^{t+\delta} Z_a(s)\,dZ_b(s) + \int_t^{t+\delta} Z_b(s)\,dZ_a(s) - \int_t^{t+\delta} Z_a(t)\,dZ_b(s) - \int_t^{t+\delta} Z_b(t)\,dZ_a(s)$$

$$= \int_t^{t+\delta} [Z_a(s)\,dZ_b(s) + Z_b(s) + dZ_a(s)] - Z_a(t)\Delta Z_b(t) - Z_b(t)\Delta Z_a(t)$$

$$= \int_t^{t+\delta} [Z_a(s)\,dZ_b(s) + Z_b(s)\,dZ_a(s)] - 2\Delta Z_a(t)\,Z_b(t)$$

Apply Itô's formula to $Z_a(s)\,Z_b(s)$, we have:

$$d(Z_a(s)\,Z_b(s)) = Z_a(s)\,dZ_b(s) + Z_b(s)\,dZ_a(s)$$

$$Z_a(t+\delta)\,Z_b(t+\delta) - Z_a(t)\,Z_b(t) = \int_t^{t+\delta} [Z_a(s)\,dZ_b(s) + Z_b(s)\,dZ_a(s)]$$

As $Z_a(t+\delta) Z_b(t+\delta) = (Z_a(t) + \delta Z_a(t))(Z_b(t) + \delta Z_b(t))$

$$= Z_a(t) Z_b(t) + 2\delta Z_a(t) Z_b(t) + \delta Z_a(t)\, \delta Z_b(t)$$

$$\Rightarrow \int_t^{t+\delta} [Z_a(s) dZ_b(s) + Z_b(s) dZ_a(s)] = 2\delta Z_a(t) Z_b(t) + \delta Z_a(t) \delta Z_b(t)$$

$$\Rightarrow I_{(a,b)} + I_{(b,a)} = \delta Z_a \delta Z_b$$

Q3:

① From Lecture 7, we can write the dynamics of LMM of the independent Brownian motions:

$$dL_j(t) = \Delta_j(t) dt + \sum_{1 \le a \le d} B_{ja}(t) dZ_a(t)$$

with $$B_{ja}(t) = U_{ja} G_j(t)$$

② $$\sum_{1 \le k \le n} B_{ka} \cdot \frac{\partial B_{ib}}{\partial X_k} = \sum_{1 \le k \le n} U_{ka} G_k(t) \cdot \frac{\partial U_{ib} G_i(t)}{\partial X_k}$$

$$= \sum_{1 \le k \le n} U_{ka} U_{ib} G_k(t) \cdot \frac{\partial G_i(t)}{\partial L_k}$$ U_{ib} is not depend on L_j

$$= U_{ia} U_{ib} G_i(t) \cdot \frac{\partial G_i(t)}{\partial L_i}$$ since $G_i(t) = G_i(t, L_j(t))$

$$\Rightarrow \frac{\partial G_i(t)}{\partial L_k(t)} = \begin{cases} 0, & k \ne i \\ \frac{\partial G_i(t)}{\partial L_i}, & k = i \end{cases}$$

Similarly :

$$\sum_{1\le k\le n} B_{kb} \cdot \frac{\partial B_{ia}}{\partial X_k} = \sum_{1\le k\le n} U_{kb} C_k(t) \cdot \frac{\partial\, U_{ia} C_i(t)}{\partial X_k}$$

$$= \sum_{1\le k\le n} U_{kb} U_{ia} C_k(t) \cdot \frac{\partial\, C_i(t)}{\partial X_k}$$

$$= U_{ia} U_{ib} C_i(t) \cdot \frac{\partial\, C_i(t)}{\partial L_i}$$

$$\Rightarrow \sum_{1\le k\le n} B_{ka} \cdot \frac{\partial B_{ib}}{\partial X_k} = \sum_{1\le k\le n} B_{kb} \cdot \frac{\partial B_{ia}}{\partial X_k}$$

i.e., formula (31) in Lecture 8

dynamics of LMM satisfies integrability condition (31) in Lecture 8.