HW2

2015年2月19日 19:41

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1.

$$dF_t = (\sigma_1 F_t + \sigma_0)\,dW_t$$

$$d\ln(\sigma_1 F_t + \sigma_0) = \frac{\sigma_1\,dF_t}{\sigma_1 F_t + \sigma_0} - \frac{1}{2}\frac{\sigma_1^2}{(\sigma_1 F_t + \sigma_0)^2}(dF_t)^2$$

$$= \sigma_1\,dW_t - \frac{1}{2}\sigma_1^2\,dt$$

$$\therefore \frac{\sigma_1 F_T + \sigma_0}{\sigma_1 F_0 + \sigma_0} = \exp\{\sigma_1 W_T - \tfrac{1}{2}\sigma_1^2 T\}$$

$$F_T = \frac{1}{\sigma_1}\left[(\sigma_1 F_0 + \sigma_0)\exp\{\sigma_1 W_T - \tfrac{1}{2}\sigma_1^2 T\} - \sigma_0\right]$$

For Call

$$P_{call} = E(F_T - K)^+$$

$$F_T > K \iff (\sigma_1 F_0 + \sigma_0)\exp\{\sigma_1 W_T - \tfrac{1}{2}\sigma_1^2 T\} > \sigma_1 K + \sigma_0$$

$$\iff \sigma_1 W_T - \tfrac{1}{2}\sigma_1^2 T > \ln\frac{\sigma_1 K + \sigma_0}{\sigma_1 F_0 + \sigma_0}$$

$$Z > \frac{\ln\frac{\sigma_1 F_0 + \sigma_0}{\sigma_1 K + \sigma_0} - \frac{1}{2}\sigma_1^2 T}{\sigma_1\sqrt{T}} = d_2$$

$$P_{call} = \int_{d_2}^{\infty}(F_T - K)\frac{1}{\sqrt{2\pi}}\exp\{-\tfrac{1}{2}Z^2\}\,dZ$$

$$= \int_{d_2}^{\infty}\frac{1}{\sigma_1}\left[(\sigma_1 F_0 + \sigma_0)\exp\{\sigma_1\sqrt{T}Z - \tfrac{1}{2}\sigma_1^2 T\} - \sigma_0 - \sigma_1 K\right]\frac{1}{\sqrt{2\pi}}\exp\{-\tfrac{1}{2}Z^2\}\,dZ$$

$$= \frac{1}{\sigma_1}\left[(\sigma_1 F_0 + \sigma_0)N(d_1) - (\sigma_1 K + \sigma_0)N(d_2)\right]$$

where $d_2 = \dfrac{\ln\frac{\sigma_1 F_0 + \sigma_0}{\sigma_1 K + \sigma_0} - \frac{1}{2}\sigma_1^2 T}{\sigma_1\sqrt{T}}$ $\quad d_1 = d_2 + \sigma_1\sqrt{T}$, N is CDF of Normal Distribution.

By Put-Call Parity.

$$P_{put} = -\frac{1}{\sigma_1}(\sigma_1 F_0 + \sigma_0) + \frac{1}{\sigma_1}(\sigma_1 K + \sigma_0) + P_{call}$$

$$= \frac{1}{\sigma_1}\left[(\sigma_1 K + \sigma_0)N(-d_2) - (\sigma_1 F_0 + \sigma_0)N(-d_1)\right]$$

2(a) Consider a Call

$$P = N\sigma_n\sqrt{T}\,(d_+ N(d_+) + N'(d_+)) \qquad d_+ = \frac{F_0 - K}{\sigma_n\sqrt{T}} = 0$$

$$= N\sigma_n\sqrt{T}\cdot\frac{1}{\sqrt{2\pi}}$$

$$P = N(F_0 N(d_+) - K N(d_-)) \qquad d_\pm = \frac{\ln\frac{F_0}{K} \pm \frac{1}{2}\sigma_{ln}^2 T}{\sigma_{ln}\sqrt{T}} = \pm\frac{1}{2}\sigma_{ln}\sqrt{T}$$

$$= N F_0(2N(\tfrac{1}{2}\sigma_{ln}\sqrt{T}) - 1)$$

$$\therefore \sigma_n\sqrt{T}\frac{1}{\sqrt{2\pi}} = F_0(2N(\tfrac{1}{2}\sigma_{ln}\sqrt{T}) - 1)$$

$$\therefore \sigma_n = \sqrt{\frac{2\pi}{T}}F_0\left(2\int_{-\infty}^{\frac{1}{2}\sigma_{ln}\sqrt{T}} e^{-\frac{1}{2}x^2}\frac{1}{\sqrt{2\pi}}\,dx - 1\right)$$

$$= \sqrt{\frac{2\pi}{T}}F_0\frac{2}{\sqrt{2\pi}}\int_0^{\frac{1}{2}\sigma_{ln}\sqrt{T}} e^{-\frac{1}{2}x^2}\,dx$$

$$= \sqrt{\frac{2\pi}{T}}F_0\frac{2}{\sqrt{\pi}}\int_0^{\frac{\sqrt{T}}{2\sqrt{2}}\sigma_{ln}} e^{-u^2}\,du \qquad u = \frac{x}{\sqrt{2}}$$

$$= \sqrt{\frac{2\pi}{T}}F_0\,\mathrm{erf}\left(\frac{\sqrt{T}}{2\sqrt{2}}\sigma_{ln}\right)$$

(b) By Taylor Expansion

$$f(x) = \frac{\sqrt{\pi}}{2}\mathrm{erf}(x) = \int_0^x e^{-u^2}\,du \qquad \text{where } x = \frac{\sqrt{T}\,\sigma_{ln}}{2\sqrt{2}}$$

$$f(x) = f(0) + x f'(0) + \frac{1}{2}x^2 f''(0) + \frac{1}{6}x^3 f^{(3)}(0) + \frac{1}{24}x^4 f^{(4)}(0) + \cdots$$

$$f'(0) = 1,\ f''(0) = 0,\ f^{(3)}(0) = -2,\ f^{(4)} = 0,\ f^{(5)} = 12$$

$$\therefore \sigma_n = 2\sqrt{2}\frac{1}{\sqrt{T}}F_0 f(x)$$

$$= 2\sqrt{2}\frac{F_0}{\sqrt{T}}\left(\frac{\sigma_{ln}\sqrt{T}}{2\sqrt{2}}f'(0) + \frac{1}{6}\left(\frac{\sigma_{ln}\sqrt{T}}{2\sqrt{2}}\right)^3 f^{(3)}(0) + \frac{1}{120}\left(\frac{\sigma_{ln}\sqrt{T}}{2\sqrt{2}}\right)^5 f^{(5)}(0) + O((\sigma_{ln}\sqrt{T})^7)\right)$$

$$= F_0 \sigma_n \left(1 - \frac{1}{24}\sigma_{ln}^2 T + \frac{1}{640}(\sigma_{ln}^2 T)^2 + \cdots\right)$$

3. in normal SABR model.

$$\sigma_n = \alpha \frac{F_0 - K}{D(\zeta)}(1+O(\zeta)) = \frac{\zeta\sigma_0}{D(\zeta)}(1+O(\zeta))$$

$$D(\zeta) = \ln\left(\frac{\sqrt{\zeta^2 - 2\rho\zeta + 1} + \zeta - \rho}{1-\rho}\right)$$ if at the money, $\zeta \to 0$

By l'Hopital's rule. $$\sigma_n = \lim_{\zeta \to 0} \frac{\sigma_0(1+O(\zeta))}{\ln\left(\frac{\sqrt{\zeta^2-2\rho\zeta+1}+\zeta-\rho}{1-\rho}\right)'} = \frac{\frac{1}{1-\rho}\left[(\zeta-\rho)(\zeta^2-2\rho\zeta+1)^{-\frac{1}{2}}+1\right]\sigma_0}{\frac{1}{1-\rho}\left(\sqrt{\zeta^2-2\rho\zeta+1}+\zeta-\rho\right)}(1+O(\zeta))$$

$$= \sigma_0(1+O(\zeta))$$

$$\sigma_{ln} = \lim_{\zeta\to 0}\alpha\frac{\log(F_0/K)}{D(\zeta)}(1+O(\zeta)) = \lim_{\zeta\to 0}\frac{\log\left(\frac{F_0-K}{\alpha K}+1\right)(1+O(\zeta))}{D(\zeta)} = \frac{\log\frac{\zeta+1}{\alpha K}\alpha\sigma_0}{D(\zeta)}(1+O(\zeta)) = \lim_{\zeta\to 0}\frac{\alpha K}{\zeta+1}\alpha\sigma_0(1+O(\zeta)) = K\sigma_0(1+O(\zeta))$$

where adding the error term.

$$\sigma_n = 1 + O(\zeta)$$

$$\sigma_{ln} = K\sigma_0(1+O(\zeta))$$