Q3:

(i) Since $dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))dW_t$

If $P(t,T)$ satisfied the PDE and terminal value $P(T,T)=1$

$$\Rightarrow P(t,T) = E^Q[e^{-\int_t^T r(s)ds} P(T,T)] = E^Q[e^{-\int_t^T r(s)ds}]$$

Apply Ito's formula to $p e^{-\int_0^t r(s)ds}$, we have:

$$d[pe^{-\int_0^t r(s)ds}] = e^{-\int_0^t r(s)ds}[P_t dt - rPdt + P_r dr + \frac{1}{2} P_{rr} d[r]_t]$$

$$= e^{-\int_0^t r(s)ds}[(P_t + \frac{1}{2}\sigma^2 P_{rr} + \mu p - rp) dt + \sigma P_r dW_t]$$

$$= e^{-\int_0^t r(s)ds} \sigma P_r dW_t \qquad (\text{since } P_t + \frac{1}{2}\sigma^2 P_{rr} + \mu p = rp).$$

$$\Rightarrow P(T,T) e^{-\int_0^T r(s)ds} - p(t,T) e^{-\int_0^t r(s)ds} = e^{-\int_0^t r(s)ds} \sigma P_r dW_t$$

$$\Rightarrow P(T,T) e^{-\int_t^T r(s)ds} - p(t,T) = \int_t^T e^{-\int_0^\tau r(s)ds} \sigma P_r dW_\tau$$

Take conditional expectation on both side, we have:

$$p(t,T) = E^Q[e^{-\int_t^T r(s)ds} P(T,T)] = E^Q[e^{-\int_t^T r(s)ds}]$$

with $P(T,T) = 1$

Formula is proofed.

(ii) In Affine term structure model, we have:

① $$P(t,T) = A(t,T)\, e^{-B(t,T)r(t)}$$

$$\frac{\partial P}{\partial t} = A_t \cdot e^{-Br} + A \cdot [-rB_t - Br_t] e^{-Br} = e^{-Br}[A_t - ArB_t - ABr_t]$$

$$\frac{\partial P}{\partial r} = -AB e^{-Br}, \quad \frac{\partial^2 P}{\partial r^2} = AB^2 e^{-Br}$$

plug in PDE ~~of~~ part (i), we have

$$e^{-Br}[A_t - ArB_t - \cancel{ABr_t}] + \frac{1}{2}\sigma^2 AB^2 e^{-Br} - \mu AB e^{-Br} = rAe^{-Br}$$

$$\Rightarrow \frac{1}{A}\frac{\partial A}{\partial t} - rB_t - Br_t + \frac{1}{2}\sigma^2 B^2 - \mu B = r$$

Since from part (i), we know $r(t) = r = \text{const}$. $\Rightarrow r_t = 0$

We have:

$$\frac{\partial \log A}{\partial t} - r \cdot \frac{\partial B}{\partial t} + \frac{1}{2}\sigma(t,r)^2 B^2 - \mu(t,r) B = r. \qquad (*)$$

② As $P(T,T) = A(T,T)\, e^{-B(T,T)\cdot r} = 1$

$$\Rightarrow \begin{cases} A(T,T) = 1 \\ B(T,T) = 0 \end{cases}$$

Formulas are ~~approv~~ proofed.

(iii)

Let $\mu(t,r) = a(t)r + b(t)$

$\sigma(t,r)^2 = c(t)r + d(t)$

plug into formula (*) in part (ii), we have:

$$\frac{\partial \log A}{\partial t} - r \cdot \frac{\partial B}{\partial t} + \frac{1}{2}B^2[c(t)r + d(t)] = r + B[a(t)r + b(t)]$$

$$\Leftrightarrow \frac{\partial \log A}{\partial t} - b(t)B + \frac{1}{2}d(t)B^2 = r\left[1 + a(t)B + \frac{\partial B}{\partial t} - \frac{1}{2}c(t)B^2\right] \quad (**)$$

Since $r=0$, left side is a function of t, right side of (**) is a function of ~~r and~~ t with coefficient r. In order to left both sides ~~to~~ be equal, we have:

$$\begin{cases} \frac{\partial \log A}{\partial t} - b(t)B + \frac{1}{2}d(t)B^2 = 0 \\ \frac{\partial B}{\partial t} + a(t)B - \frac{1}{2}c(t)B^2 + 1 = 0 \end{cases}$$

with terminal value

$$\begin{cases} B(T,T) = 0 \\ A(T,T) = 1 \end{cases}$$

formulas are proofed.