INTEREST RATES AND FX MODELS

5. Short rate models

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1 Term structure modeling

The real challenge in modeling interest rates is the existence of a term structure of interest rates embodied in the shape of the forward curve. Fixed income instruments typically depend on a segment of the forward curve rather than a single point. Pricing such instruments requires thus a model describing a stochastic time evolution of the entire forward curve.

There exists a large number of term structure models based on different choices of state variables parameterizing the curve, number of dynamic factors, volatility smile characteristics, etc. Time permits us to discuss term structure modeling only in its crudest outline, and we focus on two approaches:

(a) *Short rate models*, in which the stochastic state variable is taken to be the instantaneous forward rate. Historically, these were the earliest successful term structure models. We shall focus on a tractable Gaussian model, namely Vasicek's model and its descendents.

(b) *LIBOR market model*, in which the stochastic state variable is the entire forward curve represented and as a collection of benchmark LIBOR forward rates. These, more recently developed, models are descendants of the HJM model and have been popular among practitioners.

Short rates models use the instantaneous spot rate $r(t)$ as the basic state variable. In the LIBOR / OIS framework, the short rate is defined as $r(t) = f(t,t)$, where $f(t,s)$ denotes the instantaneous discount (OIS) rate, as explained in Lecture 1. The stochastic differential equation describing the dynamics of $r(t)$ is usually stated under the spot measure, and has the form

$$dr(t) = A(t, r(t))dt + B(t, r(t))dW(t), \tag{1}$$

where A and B are suitably chosen drift and diffusion coefficients, and W is the Brownian motion driving the process. Models of this type are referred to as *one-factor models*, as there is only one stochastic drivers; models with multiple stochastic drivers are called *multi-factor models*.

Various choices of the coefficients A and B lead to different dynamics of the instantaneous rate. You should consult the literature cited at the end of these notes for a complete catalog of choices available in the repertoire. We shall focus on the Vasicek model and its descendent, the Hull-White model.

2 Vasicek's model and its descendants

2.1 Modeling mean reversion of rates

The simplest term structure model of any practical significance is *Vasicek's model*. Under the spot measure Q_0, its dynamics is given by:

$$dr(t) = \lambda\left(\mu - r(t)\right)dt + \sigma dW(t), \tag{2}$$

together with the initial condition:

$$r\left(0\right) = r_0. \tag{3}$$

Originally, this process has been studied in the physics literature, and is known as the *Ornstein - Uhlenbeck process*.

A special feature of Vasicek's model is that the stochastic differential equation (2) has a closed form solution. In order to find it we utilize the method of variations of constants. The homogeneous equation

$$dr(t) = -\lambda r(t)dt$$

has the obvious solution:

$$r(t) = Ce^{-\lambda t}, \tag{4}$$

with C an arbitrary constant. Seeking a particular solution to the inhomogeneous equation in the form of (4) with the constant C replaced by an unknown function $\psi(t)$,

$$r_1(t) = \psi(t)\, e^{-\lambda t},$$

we find readily that $\psi(t)$ has to satisfy the ordinary differential equation:

$$d\psi(t) = \lambda\mu e^{\lambda t}dt + \sigma e^{\lambda t}dW(t).$$

Consequently,

$$\psi(t) = \mu e^{\lambda t} + \sigma \int_0^t e^{\lambda s} dW(s).$$

The solution to our problem is the sum of the solution (4) with $C = r_0 - \mu$ (in order to enforce the initial condition) and the particular solution $r_1(t)$:

$$r(t) = r_0 e^{-\lambda t} + \mu \left(1 - e^{-\lambda t}\right) + \sigma \int_0^t e^{-\lambda(t-s)} dW(s). \tag{5}$$

To understand better the meaning of this solution, we note that the expected value of the instantaneous rate $r(t)$ is

$$\mathsf{E}^{\mathsf{Q}_0}\left[r(t)\right] = X_0 e^{-\lambda t} + \mu\left(1 - e^{-\lambda t}\right), \tag{6}$$

while its variance is

$$\mathsf{Var}\left[r(t)\right] = \frac{\sigma^2}{2\lambda}\left(1 - e^{-2\lambda t}\right). \tag{7}$$

This means that, on the average, as $t \to \infty$, $X(t)$ tends to μ, and this limit is approached exponentially fast. This property is referred to as *mean reversion* of the short rate. The rate of mean reversion is equal to λ, and the time scale τ on which it takes place is given by the inverse of λ, $\tau = 1/\lambda$. Random fluctuations interfering with the mean reversion are of the order of magnitude $\sigma/\sqrt{2\lambda}$. This tends to zero, as $\lambda \to \infty$, and thus strongly mean reverting processes are characterized by low volatility.

Elegant and simple as it is, the Vasicek model has a number of serious shortcomings:

?

(a) It is impossible to fit the entire forward curve as the initial condition.

(b) There is one volatility parameter only available for calibration (two, if you count the mean reversion rate). That makes fitting the volatility structure virtually impossible.

(c) The model is *one-factor*, meaning that there is only one stochastic driver of the process.

(d)) With non-zero probability, rates may become negative (typically, this probability is fairly low).

Some of these shortcomings can be easily overcome by means of a slight extension of the model.

2.2 One-factor Hull-White model

A suitable generalization of the Ornstein-Uhlenbeck process (2) is a process which mean reverts to a time dependent level $\mu(t)$ rather than a constant μ. Such a process is given by

$$dr(t) = \left(\frac{d\mu(t)}{dt} + \lambda \left(\mu(t) - r(t) \right) \right) dt + \sigma(t) dW(t), \tag{8}$$

where we have also allowed σ to be a function of time. The presence of the time derivative of $\mu(t)$ in the drift is somewhat surprising. However, solving (8) (using again the method of variation of constants) yields

$$r(t) = r_0 e^{-\lambda t} + \mu(t) - \mu(0) e^{-\lambda t} + \int_0^t e^{-\lambda(t-u)} \sigma(u) dW(u), \tag{9}$$

and thus

$$\begin{aligned} \mathsf{E}^{\mathsf{Q}_0} \left[r(t) \right] &= r_0 e^{-\lambda t} + \mu(t) - e^{-\lambda t} \mu(0), && (10) \\ \mathsf{Var} \left[r(t) \right] &= \int_0^t e^{-2\lambda(t-u)} \sigma(u)^2 du. && (11) \end{aligned}$$

That shows that $\mathsf{E} \left[r(t) \right] - \mu(t) \to 0$, as $t \to 0$.

Note that (9) implies that

$$r(t) = r(s) e^{-\lambda(t-s)} + \mu(t) - \mu(s) e^{-\lambda(t-s)} + \int_s^t e^{-\lambda(t-u)} \sigma(u) dW(u), \tag{12}$$

for any $s < t$. We shall use this fact in the following.

Let us now choose $\mu(t) = r_0(t)$, i.e.

$$dr(t) = \left(\frac{dr_0(t)}{dt} + \lambda \left(r_0(t) - r(t) \right) \right) dt + \sigma(t) dW(t), \tag{13}$$

where $r_0(0) = r(0) = r_0$. This process is called the *extended Vasicek* (or *Hull-White*) model. From (12),

$$r(t) = r_0(t) + e^{-\lambda(t-s)}\left(r(s) - r_0(s)\right) + \int_s^t e^{-\lambda(t-u)}\sigma(u)dW(u), \qquad (14)$$

and so the instantaneous rate is represented as a contribution from the current yield curve plus a random perturbation. This representation of $r(t)$ implies that

$$\mathsf{E}_s^{\mathsf{Q}_0}\left[r(t)\right] = r_0(t) + e^{-\lambda(t-s)}\left(r(s) - r_0(s)\right). \qquad (15)$$

In particular, choosing $s = 0$ in (14) we obtain

$$r(t) = r_0(t) + \int_0^t e^{-\lambda(t-u)}\sigma(u)dW(u). \qquad (16)$$

The instantaneous 3 month LIBOR rate $r_{\mathrm{3ML}}(t)$ is given by

$$r_{\mathrm{3ML}}(t) = r(t) + b(t), \qquad (17)$$

where $b(t)$ is the basis between the instantaneous LIBOR and OIS rates. For simplicity of exposition we assume that the basis curve is given by a deterministic function rather than a stochastic process.

2.3 Two-factor Hull-White model

In the two-factor Hull-White model, the instantaneous rate is represented as the sum of

(a) the current rate $r_0(t)$, and

(b) two stochastic state variables $r_1(t)$ and $r_2(t)$.

In other words, $r(t) = r_0(t) + r_1(t) + r_2(t)$. A natural interpretation of these variables is that $r_1(t)$ controls the levels of the rates, while $r_2(t)$ controls the steepness of the forward curve.

We assume the stochastic dynamics:

$$\begin{aligned} dr_1(t) &= -\lambda_1 r_1(t)dt + \sigma_1(t)dW_1(t), \\ dr_2(t) &= -\lambda_2 r_2(t)dt + \sigma_2(t)dW_2(t), \end{aligned} \qquad (18)$$

where $\sigma_1(t)$ and $\sigma_2(t)$ are the instantaneous volatilities of the state variables $r_1(t)$ and $r_2(t)$, respectively. The two Brownian motions are correlated,

$$\mathsf{E}\left[dW_1(t)dW_2(t)\right] = \rho\, dt. \tag{19}$$

The correlation coefficient ρ is typically a large negative number ($\rho \sim -0.9$) reflecting the fact that steepening curve moves tend to correlate negatively with parallel moves.

3 Zero coupon bonds

The key to all pricing is the ability to compute the forward price of a zero coupon bond $P(t,T)$. It is given by the expected value of the stochastic discount factor,

$$P(t,T) = \mathsf{E}_t^{\mathsf{Q}_0}\left[e^{-\int_t^T r(u)du}\right], \tag{20}$$

where the subscript t indicates conditioning on $\mathcal{F}_t$. Within the Hull-White model (and thus, as a special case, in the Vasicek model), this expected value can be computed in closed form!

Let us consider the one-factor case. We proceed as follows:

$$\begin{aligned}
&\mathsf{E}_t^{\mathsf{Q}_0}\left[e^{-\int_t^T r(u)du}\right] \\
&\quad = \mathsf{E}_t^{\mathsf{Q}_0}\left[e^{-\int_t^T \left(r_0(u)+e^{-\lambda(u-t)}(r(t)-r_0(t))+\int_t^u e^{-\lambda(u-s)}\sigma(s)dW(s)\right)du}\right] \\
&\quad = e^{-\int_t^T r_0(u)du - h_\lambda(T-t)(r(t)-r_0(t))}\ \mathsf{E}_t^{\mathsf{Q}_0}\left[e^{-\int_t^T\int_t^u e^{-\lambda(u-s)}\sigma(s)dW(s)du}\right],
\end{aligned}$$

where

$$h_\lambda(t) = \frac{1-e^{-\lambda t}}{\lambda}\ .$$

Integrating by parts we can transform the double integral in the exponent into a single integral

$$\int_t^T\int_t^u e^{-\lambda(u-s)}\sigma(s)\,dW(s)\,du = \int_t^T h_\lambda\,(T-s)\,\sigma(s)dW(s).$$

Finally, using the fact that

$$\mathsf{E}_t\left[e^{\int_t^T\varphi(s)dW(s)}\right] = e^{\frac{1}{2}\int_t^T\varphi(s)^2ds}, \tag{21}$$

we obtain the following expression for the price of a zero coupon bond:

$$P\left(t,T\right)=A\left(t,T\right)e^{-h_{\lambda}(T-t)r(t)}, \tag{22}$$

where

$$A\left(t,T\right)=e^{-\int_{t}^{T}r_{0}(u)du+h_{\lambda}(T-t)r_{0}(t)+\frac{1}{2}\int_{t}^{T}h_{\lambda}(T-s)^{2}\sigma(s)^{2}ds}. \tag{23}$$

Note in particular that the discount factor $P_0\left(0,T\right)$ has the form

$$\begin{aligned}P_{0}\left(0,T\right)&=P\left(0,T\right)\\&=e^{-\int_{0}^{T}r_{0}(s)ds+\frac{1}{2}\int_{0}^{T}h_{\lambda}(T-s)^{2}\sigma(s)^{2}ds}.\end{aligned} \tag{24}$$

The generalization of formula of (22) to the case of the two-factor Hull-White model reads:

$$P\left(t,T\right)=A\left(t,T\right)e^{-h_{\lambda_{1}}(T-t)r_{1}(t)-h_{\lambda_{2}}(T-t)r_{2}(t)}, \tag{25}$$

where now

$$\begin{aligned}A\left(t,T\right)&=e^{-\int_{t}^{T}r_{0}(u)du}\\&\times e^{\frac{1}{2}\int_{t}^{T}\left(h_{\lambda_{1}}(T-s)^{2}\sigma_{1}(s)^{2}+2\rho h_{\lambda_{1}}(T-s)h_{\lambda_{2}}(T-s)\sigma_{1}(s)\sigma_{2}(s)+h_{\lambda_{2}}(T-s)^{2}\sigma_{2}(s)^{2}\right)ds}.\end{aligned} \tag{26}$$

4 Options on a zero coupon bond

Using the above expressions for the zero coupon bond, it is possible to derive explicit, closed form expressions for valuation of European options on such bonds. The calculations are elementary, if a bit tedious, and we shall defer them to the next homework assignment. We focus on the one factor Hull-White model; it is straightforward to extend these calculations to the two factor model.

Consider a call option struck at K and expiring at T on a zero coupon bond maturing at $T_{\text{mat}} > T$. Then, its price is equal to

$$\mathsf{PV}_{\text{call}}=P_{0}\left(0,T_{\text{mat}}\right)N\left(d_{+}\right)-KP_{0}\left(0,T\right)N\left(d_{-}\right), \tag{27}$$

where

$$d_{\pm}=\frac{1}{\overline{\sigma}}\,\log\frac{P_{0}\left(0,T_{\text{mat}}\right)}{P_{0}\left(0,T\right)K}\pm\frac{\overline{\sigma}}{2}\,, \tag{28}$$

with

$$\overline{\sigma}=\left(\int_{0}^{T}e^{-2\lambda(T-s)}\sigma(s)^{2}ds\right)^{1/2}h_{\lambda}\left(T_{\text{mat}}-T\right). \tag{29}$$

Similarly, the price of a put struck at K is given by

$$\mathsf{PV}_{\text{put}} = KP_0\left(0, T\right) N\left(-d_-\right) - P_0\left(0, T_{\text{mat}}\right) N\left(-d_+\right). \tag{30}$$

Since floorlets and caplets can be thought of as calls and puts on FRAs, these formulas can be used as building blocks for valuation of caps and floors in the Hull-White model.

5 Pricing under the Hull-White model

A term structure model has to be *calibrated* to the market before it can be used for valuation purposes. All the free parameters of the model have to be assigned values, so that the model reprices exactly (or close enough) the prices of a selected set of liquid vanilla instruments.

In the case of the Hull-White model, this amounts to

(a) Matching the current forward curve, which is accomplished by choosing $r_0(t)$ to match the current instantaneous OIS curve.

(a) Matching the volatilities of selected options. This is a bit more difficult, and we proceed as follows. We choose the instantaneous volatility function $\sigma(t)$ to be locally constant. That means that we divide up the time axis into, say, 3 month period $[T_j, T_{j+1})$ and set $\sigma(t) = \sigma_j$, for $t \in [T_j, T_{j+1})$. Now, we select sufficiently many calibrating instruments, so that their number exceeds the number of the σ's. Next, we optimize the choice of σ's and λ, requiring that the suitable sum of pricing errors is minimal.

Despite the simple structure of the Hull-White model, most instruments cannot be priced by means of closed form expressions such as those for caps and floors of the previous section. One has to resort numerical techniques. Among them, two are particularly important:

(a) Tree methods.

(a) Monte Carlo methods.

Time does not permit us to discuss these numerical techniques, and I defer you to literature cited at the end of these notes.

6 Application: Eurodollar / FRA convexity correction

As a simple application of the Hull-White model, we shall now derive a closed form expression for the Eurodollar / FRA convexity correction discussed in Lecture 4.

We know from Lectures 2 and 3 that the (currently observed) LIBOR forward rate $L_0\left(T_1, T_2\right)$ is the expected value of

$$\frac{1}{\delta}\left(\frac{1}{P(t, T_1, T_2)} - 1\right) + B_0(T_1, T_2) \tag{31}$$

under the T_2-forward measure Q_{T_2}. Here $B_0(T_1, T_2)$ denotes the credit spread between LIBOR and OIS. This is, indeed, almost the definition of the T_2-forward measure! Consequently, $L_0\left(T_1, T_2\right)$ is given by:

$$\begin{aligned} L_0\left(T_1, T_2\right) &= \frac{1}{\delta}\left(\frac{1}{P_0\left(T_1, T_2\right)} - 1\right) + B_0(T_1, T_2) \\ &= \frac{1}{\delta}\left(\frac{P_0\left(0, T_1\right)}{P_0\left(0, T_2\right)} - 1\right) + B_0(T_1, T_2). \end{aligned} \tag{32}$$

It is easy to calculate this rate within the Hull-White model. Let us first consider the one-factor case. Using (24), we find that

$$\begin{aligned} &L_0\left(T_1, T_2\right) \\ &\quad= \frac{1}{\delta}\left(e^{\int_{T_1}^{T_2} r_0(s)ds - \frac{1}{2}\left(\int_0^{T_2} h_\lambda(T_2-s)^2\sigma(s)^2 ds - \int_0^{T_1} h_\lambda(T_1-s)^2\sigma(s)^2 ds\right)} - 1\right) \\ &\quad+ B_0(T_1, T_2). \end{aligned} \tag{33}$$

On the other hand, the rate $L_0^{\text{fut}}\left(T_1, T_2\right)$ implied from the Eurodollar futures contract is given by the expected value of (31) under the spot measure Q_0, namely

$$L_0^{\text{fut}}(T_1, T_2) = \frac{1}{\delta}\Big(\mathsf{E}^{\mathsf{Q}_0}\left[e^{\int_{T_1}^{T_2} r(t)dt}\right] - 1\Big) + B_0(T_1, T_2).$$

We have explained this fact in Lecture 4, attributing it to the practice of daily[1] margin account adjustments by the Exchange. In order to calculate this expected

[1] which we model as continuous

value we proceed as in the calculation leading to the explicit formula for $P(t,T)$:

$$
\begin{aligned}
&\mathsf{E}^{\mathsf{Q}_0}\Big[e^{\int_{T_1}^{T_2} r(t)dt}\Big] \\
&\quad = \mathsf{E}^{\mathsf{Q}_0}\Big[e^{\int_{T_1}^{T_2}(r_0(t)+\int_0^t e^{-\lambda(t-s)}\sigma(s)dW(s))dt}\Big] \\
&\quad = e^{\int_{T_1}^{T_2} r_0(t)dt}\ \mathsf{E}^{\mathsf{Q}_0}\Big[e^{\int_0^{T_2} h_\lambda(T_2-s)\sigma(s)dW(s)-\int_0^{T_1} h_\lambda(T_1-s)\sigma(s)dW(s)}\Big] \\
&\quad = e^{\int_{T_1}^{T_2} r_0(t)dt}\ \mathsf{E}^{\mathsf{Q}_0}\Big[e^{\int_0^{T_1}(h_\lambda(T_2-s)-h_\lambda(T_1-s))\sigma(s)dW(s)+\int_{T_1}^{T_2} h_\lambda(T_2-s)\sigma(s)dW(s)}\Big] \\
&\quad = e^{\int_{T_1}^{T_2} r_0(t)dt+\frac{1}{2}\big(\int_0^{T_1}(h_\lambda(T_2-s)-h_\lambda(T_1-s))^2\sigma(s)^2ds+\int_{T_1}^{T_2} h_\lambda(T_2-s)^2\sigma(s)^2ds\big)},
\end{aligned}
$$

and so

$$
\begin{aligned}
&L_0^{\text{fut}}(T_1,T_2) \\
&\quad = \frac{1}{\delta}\Big(e^{\int_{T_1}^{T_2} r_0(t)dt+\frac{1}{2}\big(\int_0^{T_1}(h_\lambda(T_2-s)-h_\lambda(T_1-s))^2\sigma(s)^2ds+\int_{T_1}^{T_2} h_\lambda(T_2-s)^2\sigma(s)^2ds\big)}-1\Big) \\
&\quad = L_0(T_1,T_2)+\frac{1}{\delta}\big(1+\delta F_0(T_1,T_2)\big) \\
&\qquad\qquad \times\Big(e^{\int_0^{T_2} h_\lambda(T_2-s)^2\sigma(s)^2ds-\int_0^{T_1} h_\lambda(T_2-s)h_\lambda(T_1-s)\sigma(s)^2ds}-1\Big),
\end{aligned}
$$

where $F_0(T_1,T_2)=L_0(T_1,T_2)-B_0(T_1,T_2)$ is the forward rate calculated off the OIS curve.

Consequently, the Eurodollar / FRA convexity adjustment is given by

$$
\begin{aligned}
\Delta_{\text{ED / FRA}}(T_1,T_2) &= \frac{1}{\delta}\big(1+\delta F_0(T_1,T_2)\big) \\
&\quad \times\Big(e^{\int_0^{T_2} h_\lambda(T_2-s)^2\sigma(s)^2ds-\int_0^{T_1} h_\lambda(T_2-s)h_\lambda(T_1-s)\sigma(s)^2ds}-1\Big).
\end{aligned}
\tag{34}
$$

This expression can be approximated by a much simpler expression, if we expand the exponential function to the first order and neglect all higher order terms. We also neglect the terms proportional to $F_0(T_1,T_2))$, as well as the integral $\int_{T_1}^{T_2} h_\lambda(T_2-s)^2\sigma(s)^2ds$. A moment of reflection shows that all these terms do not, indeed, contribute significantly to $\Delta_{\text{ED / FRA}}(T_1,T_2)$. As a result we find the following expression for the convexity adjustment:

$$
\Delta_{\text{ED / FRA}}(T_1,T_2) \simeq \frac{1}{\delta}\int_0^{T_1} h_\lambda(T_2-s)\big(h_\lambda(T_2-s)-h_\lambda(T_1-s)\big)\sigma(s)^2ds. \tag{35}
$$

In the case of a constant instantaneous volatility, $\sigma(t) = \sigma$, the last integral can be evaluated in closed form, and the result is:

$$\begin{aligned}\Delta_{\text{ED / FRA}} \simeq \frac{\sigma^2}{2\lambda^3\delta} \Big(& \big(1 - e^{-2\lambda T_1}\big)\big(1 - e^{-\lambda(T_2-T_1)}\big)^2 \\ & + \big(1 - e^{-\lambda(T_2-T_1)}\big)\big(1 - e^{-\lambda T_1}\big)^2\Big) .\end{aligned} \tag{36}$$

This formula is very easy to implement in computer code.

The calculations in the case of the two-factor Hull-White model are similar, if a bit more tedious. The corresponding formula reads:

$$\begin{aligned}\Delta_{\text{ED / FRA}}(T_1, T_2) \simeq \frac{1}{\delta} \sum_{1\leq j,k\leq 2} \rho_{jk} \int_0^{T_1} & h_{\lambda_j}(T_2 - s) \\ & \times \big(h_{\lambda_k}(T_2 - s) - h_{\lambda_k}(T_1 - s)\big)\sigma_j(s)\sigma_k(s)\, ds,\end{aligned} \tag{37}$$

where $\rho_{11} = \rho_{22} = 1$, $\rho_{12} = \rho_{21} = \rho$.

Note that for any real value λ, $h_\lambda(s)$ is non-negative and monotone increasing. Therefore, the convexity adjustments implied by the Hull-White model are always positive (as they should be!).

References

[1] Andersen, L., and Piterbarg, V.: *Interest Rate Modeling*, Vol. 2, Atlantic Financial Press (2010).

[2] Brigo, D., and Mercurio, F.: *Interest Rate Models - Theory and Practice*, Springer Verlag (2006).

[3] Hull, J.: *Options, Futures and Other Derivatives* Prentice Hall (2005).