# Interest Rate Models

## 7. LIBOR Market Model, I

Andrew Lesniewski

Baruch College
New York

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# Outline

# LIBOR market model

- The real challenge in modeling interest rates is the existence of a term structure of interest rates embodied in the shape of the forward curve. Fixed income instruments typically depend on a segment of the forward curve rather than a single point. Pricing such instruments requires thus a model describing a stochastic time evolution of the entire forward curve.
- There exists a large number of term structure models based on different choices of state variables parameterizing the curve, number of dynamic factors, volatility smile characteristics, etc. The industry standard for interest rates modeling that has emerged over the past few years is the LIBOR market model (LMM).
- Unlike the older approaches (short rate models which we discussed in the previous lecture), where the underlying state variable is an unobservable instantaneous rate, LMM captures the dynamics of the entire curve of interest rates by using the (market observable) LIBOR forwards as its state variables.
- The time evolution of the forwards is given by a set of intuitive stochastic differential equations in a way which guarantees arbitrage freeness of the process.

# LIBOR market model

- The model is intrinsically multi-factor, meaning that it captures accurately various aspects of the curve dynamics: parallel shifts, steepenings / flattenings, butterflies, etc. In this lecture we discuss two versions of the LMM methodology:
  (i) the classic LMM with a local volatility specification, and
  (ii) its stochastic volatility (SABR style) extension.
- One of the main difficulties experienced by the pre-LMM term structure models is the fact that they tend to produce unrealistic volatility structures of forward rates. The persistent “hump” occurring in the short end of the volatility curve leads to overvaluation of instruments depending on forward volatility.
- The LMM model offers a solution to this problem by allowing one to impose an approximately stationary volatility and correlation structure of LIBOR forwards. This reflects the view that the volatility structure of interest rates retains its shape over time, without distorting the valuation of instruments sensitive to forward volatility.
- On the downside, LMM is far less tractable than, for example, the Hull-White model. In addition, it is not Markovian in the sense short rate models are Markovian. As a consequence, all valuations based on LMM have to be done by means of Monte Carlo simulations.

# LIBOR market model

- We shall consider a sequence of approximately equally spaced dates $0 = T_0 < T_1 < \ldots < T_N$ which will be termed the *standard tenors*. A standard LIBOR forward rate $L_j$, $j = 0, 1, \ldots, N-1$ is associated with a FRA which starts on $T_j$ and matures on $T_{j+1}$.
- Usually, it is assumed that $N = 120$ and the $L_j$'s are 3 month LIBOR forward rates. Note that these dates refer to the actual start and end dates of the contracts rather than the LIBOR "fixing dates", i.e. the dates on which the LIBOR rates settle. To simplify the notation, we shall disregard the difference between the contract's start date and the corresponding forward rate's fixing date. Proper implementation, however, must take this distinction into account.
- Each LIBOR forward $L_j$ is modeled as a continuous time stochastic process $L_j(t)$. Clearly, this process has the property that it gets killed at $t = T_j$. The dynamics of the forward process is driven by an $N$-dimensional, correlated Wiener process $W_1(t), \ldots, W_N(t)$.
- We let $\rho_{jk}$ denote the correlation coefficient between $W_j(t)$ and $W_k(t)$:

$$\mathsf{E}\left[dW_j(t)\, dW_k(t)\right] = \rho_{jk}\, dt\,,$$

The period is fixed, e.g. 30Y, starts Lj(t), ends Tj
Each day, the model re-setup and re-calibrate
The fixed period shifts every day

where E denotes expected value.

# No arbitrage condition

- In order to motivate the form of the stochastic differential equations describing the dynamics of the LIBOR forwards, let us first consider the world in which there is no volatility of interest rates. The shape of the forward curve would be set once and for all by a higher authority, and each LIBOR forward would have a constant value $L_j(t) = L_{j0}$.
- In other words,

$$dL_j(t) = 0,$$

for all $j$'s.
- The fact that the rates are stochastic forces us to replace this simple dynamical system with a system of stochastic differential equations of the form:

$$dL_j(t) = \Delta_j(t)\,dt + C_j(t)\,dW_j(t)\,. \tag{1}$$

- Here

$$\begin{aligned}\Delta_j(t) &= \Delta_j(t, L(t)),\\ C_j(t) &= C_j(t, L(t)),\end{aligned}$$

are the drift and instantaneous volatility, respectively.

# No arbitrage condition

- As discussed in Lecture Notes 3, the *no arbitrage* requirement of asset pricing forces a relationship between the drift term and the diffusion term: the form of the drift term depends thus on the choice of numeraire.
- Recall that $L_k$ is a martingale under the $T_{k+1}$-forward measure $Q_k$, and so its dynamics reads:
  Starts Tk
  $$dL_k(t) = C_k(t)\, dW_k(t),$$
  where $C_k(t)$ is an instantaneous volatility function which will be defined later.
- For $j \neq k$,
  All under probability measure Qk
  $$dL_j(t) = \Delta_j(t)\, dt + C_j(t)\, dW_j(t).$$
  Since the $j$-th LIBOR forward settles at $T_j$, the process for $L_j(t)$ is killed at $t = T_j$. We shall determine the drifts $\Delta_j(t)$ by requiring lack of arbitrage.

# No arbitrage condition

- Let us first assume that $j < k$. The numeraires for the measures $\mathrm{Q}_j$ and $\mathrm{Q}_k$ are the prices $P\left(t, T_{j+1}\right)$ and $P\left(t, T_{k+1}\right)$ of the zero coupon bonds expiring at $T_{j+1}$ and $T_{k+1}$, respectively.
- Explicitly,

$$P(t, T_{j+1}) = P(t, T_{\gamma(t)}) \prod_{\gamma(t) \leq i \leq j} \frac{1}{1 + \delta_i F_i\left(t\right)}\,, \tag{2}$$

  where $F_i$ denotes the OIS forward[1] spanning the accrual period $[T_i, T_{i+1})$, and where $\gamma : [0, \ T_N] \to \mathbb{Z}$ is defined by

$$\gamma\left(t\right) = m + 1, \quad \text{if } t \in \left[T_m, \ T_{m+1}\right)\,.$$

  Step function, can be replaced by exponential intended

- Notice that $P(t, T_{\gamma(t)})$ is the "stub" discount factor over the incomplete accrual period $[t, T_{\gamma(t)}]$.

[1] Recall that all discounting is done on OIS.

# No arbitrage condition

- Since the drift of $L_j(t)$ under $\mathsf{Q}_j$ is zero, formula (26) (or (27)) of Lecture 2 yields:

quadratic variation

$$
\begin{aligned}
\Delta_j(t) &= \frac{d}{dt}\left[L_j, \log\frac{P(\cdot, T_{j+1})}{P(\cdot, T_{k+1})}\right](t) \\
&= -\frac{d}{dt}\left[L_j, \log\prod\nolimits_{j+1\leq i\leq k}(1+\delta_i F_i)\right](t) \\
&= -\sum\nolimits_{j+1\leq i\leq k} dL_j(t)\,\frac{\delta_i dF_i(t)}{1+\delta_i F_i(t)} \\
&= -C_j(t)\sum\nolimits_{j+1\leq i\leq k}\frac{\rho_{ji}\delta_i C_i(t)}{1+\delta_i F_i(t)}\,,
\end{aligned}
$$

j<k

dF(t) formula(20) in lecture 3

+1/2….dt is zero

dWidWj=rhoij

where, in the third line, we have used the fact that the spread between $L_j$ and $F_j$ is deterministic, and thus its contribution to the quadratic variation is zero.

- Similarly, for $j > k$, we find that

$$
\Delta_j(t) = C_j(t)\sum\nolimits_{k+1\leq i\leq j}\frac{\rho_{ji}\delta_i C_i(t)}{1+\delta_i F_i(t)}\,.
$$

# No arbitrage condition

- We can thus summarize the above discussion as follows. In order to streamline the notation, we let $dW_j(t) = dW_j^{Q_k}(t)$ denote the Wiener process under the measure $Q_k$. Then the dynamics of the LMM model is given by the following system of stochastic differential equations. For $t < \min(T_k, T_j)$,

$$dL_j(t) = C_j(t) \times \begin{cases} -\sum_{j+1\leq i\leq k} \dfrac{\rho_{ji}\delta_i C_i(t)}{1+\delta_i F_i(t)}\, dt + dW_j(t), & \text{if } j < k, \\ dW_j(t), & \text{if } j = k, \\ \sum_{k+1\leq i\leq j} \dfrac{\rho_{ji}\delta_i C_i(t)}{1+\delta_i F_i(t)}\, dt + dW_j(t), & \text{if } j > k. \end{cases} \quad (3)$$

- These equations have to be supplied with initial values for the LIBOR forwards:

$$L_j(0) = L_{j0}, \quad (4)$$

where $L_{j0}$ is the current value of the forward which is implied by the current yield curve.

# No arbitrage condition

- In addition to the forward measures discussed above, it is convenient to use the spot measure. It is expressed in terms of the numeraire:

$$B(t) = \frac{P(t, T_{\gamma(t)})}{\prod_{1 \le i \le \gamma(t)} P(T_{i-1}, T_i)} . \tag{5}$$

- Under the spot measure, the LMM dynamics reads:

$$dL_j(t) = C_j(t) \Big( \sum_{\gamma(t) \le i \le j} \frac{\rho_{ji} \delta_i C_i(t)}{1 + \delta_i F_i(t)} dt + dW_j(t) \Big). \tag{6}$$

Great exam question for deriving formula(6)

# Structure of the instantaneous volatility

- So far we have been working with a general instantaneous volatility $C_j(t)$ for the forward $L_j(t)$. In practice, we assume $C_j(t)$ to be one of the following standard volatility specifications discussed in Lecture Notes 3:

$$C_j(t) = \begin{cases} \sigma_j(t) & \text{(normal model),} \\ \sigma_j(t) L_j(t)^{\beta_j} & \text{(CEV model),} \\ \sigma_j(t) L_j(t) & \text{(lognormal model),} \\ \sigma_j(t) L_j(t) + \vartheta_j(t) & \text{(shifted lognormal model),} \end{cases} \tag{7}$$

  where the functions $\sigma_j(t)$ and $\vartheta_j(t)$ are deterministic, and where $\beta_j \leq 1$.
- In the following, we will be assuming the CEV model specification, and thus the dynamics of the LIBOR forwards is given by the system:

$$\begin{aligned} dL_j(t) &= \sigma_j(t) L_j(t)^{\beta_j} \\ &\times \begin{cases} -\sum_{j+1\leq i\leq k} \dfrac{\rho_{ji}\delta_i\sigma_i(t) L_i(t)^{\beta_i}}{1+\delta_i F_i(t)}\, dt + dW_j(t), & \text{if } j < k, \\ dW_j(t), & \text{if } j = k, \\ \sum_{k+1\leq i\leq j} \dfrac{\rho_{ji}\delta_i\sigma_i(t) L_i(t)^{\beta_i}}{1+\delta_i F_i(t)}\, dt + dW_j(t), & \text{if } j > k, \end{cases} \end{aligned} \tag{8}$$

  under $\mathsf{Q}_k$.

# Structure of the instantaneous volatility

- Under the spot measure:

$$dL_j(t) = \sigma_j(t) L_j(t)^{\beta_j} \Big( \sum_{\gamma(t)\leq i\leq j} \frac{\rho_{ji}\delta_i\sigma_i(t) L_i(t)^{\beta_i}}{1+\delta_i F_i(t)}\, dt + dW_j(t) \Big). \tag{9}$$

- We recall from the discussion in Lecture Notes 3 that the CEV model needs care at zero forward.
- Experience shows that the Dirichlet (absorbing) boundary condition at zero works better than the Neumann (reflecting) condition, and we will assume that the Dirichlet condition is imposed. What it means is that if a path realizing the process for $L_j$ hits zero, it gets killed and stays zero forever.

# Factor reduction

- In a market where the forward curve spans 30 years, there are 120 quarterly LIBOR forwards and thus 120 stochastic factors. So far we have not imposed any restrictions on the number of these factors, and thus the number of Brownian motions driving the LIBOR forward dynamics is equal to the number of forwards.
- Having a large number of factors poses severe problems with the model's implementation. On the numerical side, the "curse of dimensionality" kicks in, leading to unacceptably slow performance. On the financial side, the parameters of the model are severely underdetermined and the calibration of the model becomes unstable.
- We are thus led to the idea that only a small number $d$ of independent Brownian motions $Z_a(t)$, $a = 1, \ldots, d$, with

do PCA reduce
variables

$$\mathrm{E}\left[dZ_a(t)\, dZ_b(t)\right] = \delta_{ab} dt\,, \tag{10}$$

should drive the process. Typically, $d = 1, 2, 3$, or $4$.

# Factor reduction

- We set
$$dW_j(t) = \sum_{1 \le a \le d} U_{ja}\, dZ_a(t)\,, \tag{11}$$
where $U$ is an $N \times d$ matrix with the property that $UU'$ is close to the correlation matrix.
- Of course, it is in general impossible to have $UU' = \rho$. We can easily rewrite the dynamics of the model in terms of the independent Brownian motions:
$$dL_j(t) = \Delta_j(t)\, dt + \sum_{1 \le a \le d} B_{ja}(t)\, dZ_a(t)\,, \tag{12}$$
where
$$B_{ja}(t) = U_{ja}\, C_j(t)\,. \tag{13}$$
- We shall call this system the *factor reduced* LMM dynamics. It is the factor reduced form of LMM that is used in practice.

# Calibration of the LMM model

Calibrations for caps or floors are easy

- Calibration (to a selected collection of benchmark instruments) is a choice of the model parameters so that the model reprices the benchmark instruments to a desired accuracy. The choice of the calibrating instruments is dictated by the characteristics of the portfolio to be managed by the model.
- An special feature of LMM is that it leads to pricing formulas for caps and floors which are consistent with the market practice of quoting the prices of these products in terms of Black's model. This makes the calibration of LMM to caps and floors very easy.
- On the other hand, from the point of view of the LMM model, swaptions are exotic structures whose fast pricing poses serious challenges. In this section we describe our strategy of dealing with these issues.
- A key ingredient of any efficient calibration methodology for LMM is rapid and accurate swaption valuation.

# Approximate valuation of swaptions

- A swap rate is a non-linear function of the underlying LIBOR forward rates. The stochastic differential equation for the swap rate implied by the LMM model cannot be solved in closed form, and thus pricing swaptions within LMM requires Monte Carlo simulations. This poses a serious issue for efficient model calibration, as such simulations are very time consuming.
- Let us describe a closed form approximation which can be used to calibrate the model. We consider a standard forward starting swap, whose start and end dates are denoted by $T_m$ and $T_n$, respectively. Recall from Lecture 1 that the level function of the swap is defined by:

Amn: benchmark swaption

$$A_{mn}(t) = \sum_{m \le j \le n-1} \alpha_j P(t, T_{j+1}), \tag{14}$$

where $\alpha_j$ are the day count fractions for fixed rate payments, and where $P(t, T_j)$ is the time $t$ value of \$1 paid at time $T_j$.
- Typically, the payment frequency on the fixed leg is not the same as that on the floating leg[2] (which we continue to denote by $\delta_j$). This fact causes a bit of a notational nuisance but needs to be taken properly into account for accurate pricing. We let $S_{mn}(t)$ denote the corresponding forward swap rate. In order to lighten up the notation, we will suppress the subscripts $mn$ throughout the remainder of this lecture.

[2] Remember, the default convention on US dollar swaps is a semiannual 30/360 fixed leg versus a quarterly

# Approximate valuation of swaptions

- A straightforward calculation shows that, under the forward measure $Q_k$, the dynamics of the swap rate process can be written in the form:

S: breakeven forward swap rate

$$dS(t) = \Omega(t, L)dt + \sum_{m \leq j \leq n-1} \Lambda_j(t, L)dW_j(t), \tag{15}$$

where

$$\Omega = \sum_{m \leq j \leq n-1} \frac{\partial S}{\partial F_j} \Delta_j + \frac{1}{2} \sum_{m \leq i,j, \leq n-1} \rho_{ij} \frac{\partial^2 S}{\partial F_i \partial F_j} C_i C_j, \tag{16}$$

and

$$\Lambda_j = \frac{\partial S}{\partial F_j} C_j. \tag{17}$$

- Not surprisingly, the stochastic differential equation for $S$ has a drift term: the forward swap rate is not a martingale under a forward measure.

# Approximate valuation of swaptions

- We shift to the martingale measure $\mathsf{Q}_{mn}$ (the swap measure),

$$dW(t) = \frac{\sum_{m \le j \le n-1} \Lambda_j(t, F) dW_j(t) + \Omega(t, F) dt}{\nu_{mn}(t)}, \tag{18}$$

where

$$\nu_{mn}(t)^2 = \sum_{m \le i, j \le n-1} \rho_{ij} \Lambda_i(t, L) \Lambda_j(t, L). \tag{19}$$

- Then the SDE for the swap rate reads

$$dS(t) = \nu(t) dW(t). \tag{20}$$

# Approximate valuation of swaptions

- In order to be able to use this dynamics effectively, we have to approximate it by quantities with tractable analytic forms.
- The simplest approximation consists in replacing the values of the stochastic forwards $L_j(t)$ by their initial values $L_{j0}$. This amounts to "freezing" the curve at its current shape.
- Within this approximation, the coefficients in the diffusion process (15) for the swap rate are deterministic:

$$\Lambda_j(t, L) \approx \Lambda_j(t, L_0), \tag{21}$$

and

$$\Omega(t, L) \approx \Omega(t, L_0). \tag{22}$$

# Approximate valuation of swaptions

- Let $\nu_0(t)$ denote the value of $\nu(t)$ in this approximation, i.e. $\nu_0(t)$ is given by (19) with all $\Lambda_j(t, L)$ replaced by $\Lambda_j(t, L_0)$. The stochastic differential equation (20) can then be solved in closed form,

$$S(t) = S_0 + \int_0^t \nu_0(s)\, dW(s)\,. \tag{23}$$

- This is a normal model with deterministic time dependent volatility and thus the swaption implied normal volatility $\zeta_{mn}$ is approximately given by

$$\begin{aligned}\zeta_{mn}^2 &\approx \frac{1}{T_m}\int_0^t \nu_0(s)^2\, ds\\ &= \frac{1}{T_m}\sum_{m\le j,l\le n-1}\rho_{jl}\int_0^t \Lambda_j(s, F_0)\Lambda_l(s, F_0) ds\,.\end{aligned} \tag{24}$$

  This formula is easy to implement in code, and leads to reasonably accurate results.
- The frozen curve approximation can be regarded as the lowest order term in the "small noise expansion". With a bit of extra work, one can compute higher order terms in that expansion.

# Parametrization of the volatility surface

- For the purpose of calibration we require that the deterministic instantaneous CEV volatilities $\sigma_j(t)$ in (8) are piecewise constant. In order to help the intuition, we organize constant components as a lower triangular matrix in Table 1.
- Clearly, the problem of determining all the $\sigma_{j,i}$'s is vastly overparametrized. Table 1 contains 7140 parameters (assuming $N = 120$)!

| | $t \in [T_0,\, T_1)$ | $t \in [T_1,\, T_2)$ | $\dots$ | $t \in [T_{N-1},\, T_N)$ |
|---|---|---|---|---|
| $\sigma_0(t)$ Has to be 0 | $0$ | $0$ | $\dots$ | $0$ |
| $\sigma_1(t)$ | $\sigma_{1,0}$ Only exists 1 period (3M here) | $0$ | $\dots$ | $0$ |
| $\sigma_2(t)$ | $\sigma_{2,0}$ | $\sigma_{2,1}$ | $\dots$ | $0$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $\sigma_{N-1}(t)$ | $\sigma_{N-1,0}$ | $\sigma_{N-1,1}$ | $\dots$ | $0$ Only this is 0 |

Table: 1. General volatility structure

# Parametrization of the volatility surface

- A natural remedy to the overparametrization problem is assuming that the instantaneous volatility is *stationary*, i.e.,

$$\begin{aligned}\sigma_{j,i} &= \sigma_{j-i,0}\\ &\equiv \sigma_{j-i}\,,\end{aligned} \tag{25}$$

for all $i < j$.

- This assumption appears natural and intuitive, as it implies that the structure of cap volatility will look in the future exactly the same way as it does currently. Consequently, the "forward volatility problem" plaguing the traditional terms structure models would disappear. Under the stationary volatility assumption, the instantaneous volatility has the structure summarized in Table 2.

| | $t \in [T_0, T_1)$ | $t \in [T_1, T_2)$ | $\ldots$ | $t \in [T_{N-1}, T_N)$ |
|---|---|---|---|---|
| $\sigma_0(t)$ | 0 | 0 | $\ldots$ | 0 |
| $\sigma_1(t)$ | $\sigma_1$ | 0 | $\ldots$ | 0 |
| $\sigma_2(t)$ | $\sigma_2$ | $\sigma_1$ | $\ldots$ | 0 |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $\sigma_{N-1}(t)$ | $\sigma_{N-1}$ | $\sigma_{N-2}$ | $\ldots$ | 0 |

Table: 2. Stationary volatility structure

# Parametrization of the volatility surface

- It is a good idea to reduce the number of parameters even further, and try to find a parametric fit $\sigma_i = h(T_i)$, $i = 1, \ldots, N-1$.
- A popular (but, by no means the only) choice is the hump function

$$h(t) = (at + b)e^{-\lambda t} + \mu. \tag{26}$$

Starts linearly, ends asymptotically approaches to constant mu as t->infinity

- Despite its intuitive appeal, the stationarity assumption is not sufficient for accurate calibration of the model. The financial reason behind this fact appears to be the phenomenon of mean reversion of long term rates. Unlike the Vasicek style models, it is impossible to take this phenomenon into account by adding an Ornstein - Uhlenbeck style drift term to the LMM dynamics as this would violate the arbitrage freeness of the model.
- On the other hand, one can achieve a similar effect by suitably specifying the instantaneous volatility function.

# Parametrization of the volatility surface

- In order to implement this idea, we assume that the long term volatility structure is given by $\overline{\sigma}_i = \overline{h}(T_i)$, $i = 1, \ldots, N-1$, where $\overline{h}(t)$ is another hump shaped function.
- We then set

$$\sigma_{j,i} = p_i \sigma_{j-i} + q_i \overline{\sigma}_{j-i}, \tag{27}$$

  i.e. the $\sigma$'s are mixtures of the short term $\sigma$'s and the equilibrium $\overline{\sigma}$'s. The weights $p_i$ and $q_i$ are parametrized so that $p_i, q_i \geq 0$, $p_i + q_i = 1$, and $p_i \to 0$, as $i \to \infty$.
- In other words, as we move forward in time, the volatility structure looks more and more like the long term limit.

# Parametrization of the volatility surface

- This specification is summarized in Table 3.
- The lower triangular matrix in Table 3, LMM's internal representation of volatility, is referred to as the *LMM volatility surface*. We leave out the details of this methodology, as that would make the presentation a bit tedious. In the final result, we have a parametrization of the volatility surface by a manageable number of parameters $\theta = (\theta_1, \ldots, \theta_d)$ (such as the parameters of the hump functions $h(t)$ and $\overline{h}(t)$, and of the weights $p_i$), such that $\sigma_{j,i} = \sigma_{j,i}(\theta)$ can be calibrated to the market and has an intuitive shape.

| | $t \in [T_0, T_1)$ | $t \in [T_1, T_2)$ | $\ldots$ | $t \in [T_{N-1}, T_N)$ |
|---|---|---|---|---|
| $\sigma_0(t)$ | $0$ | $0$ | $\ldots$ | $0$ |
| $\sigma_1(t)$ | $p_1\sigma_1 + q_1\overline{\sigma}_1$ | $0$ | $\ldots$ | $0$ |
| $\sigma_2(t)$ | $p_1\sigma_2 + q_1\overline{\sigma}_2$ | $p_2\sigma_1 + q_2\overline{\sigma}_1$ | $\ldots$ | $0$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $\sigma_{N-1}(t)$ | $p_1\sigma_{N-1} + q_1\overline{\sigma}_{N-1}$ | $p_2\sigma_{N-2} + q_2\overline{\sigma}_{N-2}$ | $\ldots$ | $0$ |

Table: 3. Approximately stationary volatility structure

# Parametrization of the correlation matrix

- The central issue is to calibrate the model, at the same time, to the cap / floor and swaption markets in a stable and consistent manner. An important part of this process is determining the correlation matrix $\rho = \{\rho_{jk}\}_{0\leq j,k\leq N-1}$. The dimensionality of $\rho$ is $N(N+1)/2$, clearly far too high to assure a stable calibration procedure.
- A convenient approach to correlation modeling is to use a parameterized form of $\rho_{ij}$. An intuitive and flexible parametrization is given by the formula:

$$\rho_{ij} = \overline{\rho}_{\min(i,j)} + \left(1 - \overline{\rho}_{\min(i,j)}\right) \exp\left(-\beta_{\min(i,j)} \left|T_i - T_j\right|\right), \tag{28}$$

min: minimum value of the whole curve

where

$$\overline{\rho}_k = \rho \tanh\left(\alpha T_k\right), \tag{29}$$

and

$$\beta_k = \beta T_k^{-\kappa}. \tag{30}$$

Corr(T1,T2) is much less than Corr(T118,T119)
Corr(T118,T119) all most 1

# Parametrization of the correlation matrix

- The meaning of the parameters is as follows: $\rho$ is the asymptotic level of correlations, $\alpha$ is a measure of speed at which $\rho$ is approached, $\beta$ is a the decay rate of correlations, and $\kappa$ is an asymmetry parameter.
- Intuitively, positive $\kappa$ means that two consecutive forwards with short maturities are less correlated than two such forwards with long maturities. The parameters in this formula can be calibrated by using, for example, historical data.
- A word of caution is in order: this parametrization produces a matrix that is only approximately positive definite. If negative value should be very small

# Optimization

- In order to calibrate the model we seek instantaneous volatility parameters $\sigma_i$ so that to fit the at the money caplet and swaption volatilities. These can be expressed in terms of the instantaneous volatilities are as follows.
- Let $\zeta_m$ denote the at the money implied normal volatility of the caplet expiring at $T_m$. Then, within the frozen curve approximation,

$$\zeta_m(\theta)^2 = \frac{1}{T_m} L_{m0}^{2\beta_m} \sum_{0 \leq i \leq m-1} \sigma_{m,i}(\theta)^2 \, \delta_i, \tag{31}$$

where $\delta_i = T_{i+1} - T_i$, and $\theta$ denotes the set of parameters of the LMM volatility surface.
- This relationship is reasonably accurate, and can be used for calibration. However, in practice, one needs to improve on this formula by going beyond the frozen curve approximation.

# Optimization

- Similarly, for the at the money implied normal volatility $\zeta_{mn}$ of the swaption expiring at $T_m$ into a swap maturing at $T_n$ we have an approximate expression:

$$\zeta_{mn}(\theta)^2 = \frac{1}{T_m} \sum_{0 \le i \le m-1} \sum_{m \le j, l \le n-1} \rho_{jl} \Lambda_{j,i} \Lambda_{l,i} \delta_i, \tag{32}$$

where $\Lambda_{j,i}$ is the (constant) value of $\Lambda_j(s, L_0)$, the frozen curve approximation to (17), for $s \in [T_i, T_{i+1})$. Note that the coefficients $\Lambda_{j,i}$ depend on the parameters of the LMM volatility surface.

- The objective function for optimization is given by:

$$\mathcal{L}(\theta) = \sum_m w_m \left(\zeta_m(\theta) - \overline{\zeta}_m\right)^2 + \sum_{m,n} w_{mn} \left(\zeta_{mn}(\theta) - \overline{\zeta}_{mn}\right)^2, \tag{33}$$

where $\overline{\zeta}_m$ and $\overline{\zeta}_{mn}$ are the market observed caplet and swaption implied normal volatilities.

# Optimization

- The coefficients $w_m$ and $w_{mn}$ are weights which allow the user select the calibration instruments and their relative importance.
- Finally, it is a good idea to add a Tikhonov style regularization in order to maintain stability of the calibration. A convenient and computationally efficient choice of the Tikhonov penalty term is the integral of the square of the mean curvature:
$$\frac{1}{2}\lambda \iint_{\text{LMM vol surface}} R(u,v)^2 du dv \tag{34}$$
(of elementary differential geometry of surfaces) of the parameterized LMM volatility surface[3].
- The impact of this penalty term is to discourage regions of extreme curvature (such as a sharp ridge along the diagonal) at the expense of slightly less accurate fit.

[3] For computational efficiency, this integral has to be approximated by a discrete sum.

# LMM and smile dynamics

- The classic LMM model has a severe drawback: while it is possible to calibrate it so that it matches at the money option prices, it generally misprices out of the money options.
- The main reason for this is its specification. While the market uses stochastic volatility models in order to price out of the money vanilla options, LMM is incompatible with such models.
- In order to remedy the problem, we describe a model that combines the key features of the LMM and SABR models.
- To this end, we assume that the instantaneous volatilities $C_j(t)$ of the forward rates $L_j$ are of the form

$$C_j(t) = \sigma_j(t) L_j(t)^{\beta_j}, \tag{35}$$

with stochastic volatility parameters $\sigma_j(t)$.

# LMM and smile dynamics

- Furthermore, we assume that, under the $T_{k+1}$-forward measure $\mathsf{Q}_k$, the full dynamics of the forward is given by the stochastic system:

$$\begin{aligned} dL_k(t) &= C_k(t)\, dW_k(t)\,, \\ d\sigma_k(t) &= D_k(t)\, dZ_k(t)\,, \end{aligned} \tag{36}$$

  where the diffusion coefficient of the process $\sigma_k(t)$ is of the form

$$D_k(t) = \alpha_k(t)\,\sigma_k(t)\,. \tag{37}$$

- Note that $\alpha_k(t)$ is assumed here to be a (deterministic) function of $t$ rather than a constant. This extra flexibility is added in order to make sure that the model can be calibrated to market data.

# LMM and smile dynamics

- In addition, we impose the following instantaneous volatility structure:

$$\mathsf{E}\left[dW_j\left(t\right)dZ_k\left(t\right)\right]=r_{jk}\,dt, \tag{38}$$

  and

$$\mathsf{E}\left[dZ_j\left(t\right)dZ_k\left(t\right)\right]=\eta_{jk}dt. \tag{39}$$

- The block correlation matrix

$$\Pi=\begin{bmatrix}\rho & r\\ r' & \eta\end{bmatrix} \tag{40}$$

  is assumed to be positive definite.

# No arbitrage condition for SABR / LMM

- Let us now derive the dynamics of such an extended LIBOR market model under the common forward measure $Q_k$. According to the arbitrage pricing theory, the form of the stochastic differential equations defining the dynamics of the LIBOR forward rates depends on the choice of numeraire.
- Under the $T_{k+1}$-forward measure $Q_k$, the dynamics of the forward rate $L_j(t)$, $j \neq k$ reads:
$$dL_j(t) = \Delta_j(t)\,dt + C_j(t)\,dW_j(t)\,.$$
- We determine the drifts $\Delta_j(t) = \Delta_j(t, L(t), \sigma(t))$ by requiring absence of arbitrage. This is essentially the same calculation as in the derivation of the drift terms for the classic LMM, and we can thus summarize the result as follows.
- In order to streamline the notation, we let $dW(t) = dW^{Q_k}(t)$ denote the Wiener process under the measure $Q_k$. Then, as expected,

$$dL_j(t) = C_j(t) \times \begin{cases} -\sum_{j+1 \le i \le k} \dfrac{\rho_{ji}\delta_i C_i(t)}{1+\delta_i F_i(t)}\,dt + dW_j(t)\,, & \text{if } j < k, \\ dW_j(t)\,, & \text{if } j = k, \\ \sum_{k+1 \le i \le j} \dfrac{\rho_{ji}\delta_i C_i(t)}{1+\delta_i F_i(t)}\,dt + dW_j(t)\,, & \text{if } j > k\,. \end{cases} \tag{41}$$

# No arbitrage condition for SABR / LMM

- Similarly, under the spot measure, the SABR / LMM dynamics reads:

$$dL_j(t) = C_j(t)\left(\sum_{\gamma(t)\le i\le j}\frac{\rho_{ji}\delta_i C_i(t)}{1+\delta_i F_i(t)}\,dt + dW_j(t)\right). \tag{42}$$

- Let us now compute the drift term $\Gamma_j(t) = \Gamma_j(t, L(t), \sigma(t))$ for the dynamics of $\sigma_j(t)$, $j \neq k$, under $\mathsf{Q}_k$,

$$d\sigma_j(t) = \Gamma_j(t)\,dt + D_j(t)\,dZ_j(t).$$

# No arbitrage condition for SABR / LMM

- Let us first assume that $j < k$. The numeraires for the measures $Q_j$ and $Q_k$ are the prices $P\left(t, T_{j+1}\right)$ and $P\left(t, T_{k+1}\right)$ of the zero coupon bonds maturing at $T_{j+1}$ and $T_{k+1}$, respectively. Since the drift of $L_j\left(t\right)$ under $Q_j$ is zero, formula (26) of Lecture Notes 2 yields:

$$
\begin{aligned}
\Gamma_j\left(t\right) &= \frac{d}{dt}\left[\sigma_j, \log \frac{P(\,\cdot\,, T_{j+1})}{P(\,\cdot\,, T_{k+1})}\right]\left(t\right) \\
&= -\frac{d}{dt}\left[\sigma_j, \log \prod\nolimits_{j+1\le i\le k}\left(1+\delta_i F_i\right)\right]\left(t\right) \\
&= -\sum\nolimits_{j+1\le i\le k} d\sigma_j\left(t\right)\frac{\delta_i dF_i\left(t\right)}{1+\delta_i F_i\left(t\right)} \\
&= -D_j\left(t\right)\sum_{j+1\le i\le k}\frac{r_{ji}\delta_i C_i\left(t\right)}{1+\delta_i F_i\left(t\right)}\,dt.
\end{aligned}
$$

- Similarly, for $j > k$, we find that

$$
\Gamma_j\left(t\right) = D_j\left(t\right)\sum_{k+1\le i\le j}\frac{r_{ji}\delta_i C_i\left(t\right)}{1+\delta_i F_i\left(t\right)}\,.
$$

# No arbitrage condition for SABR / LMM

- This leads to the following system:

$$
\begin{aligned}
d\sigma_j(t) &= D_j(t) \\
&\times \begin{cases} -\sum_{j+1\leq i\leq k} \dfrac{r_{ji}\delta_i C_i(t)}{1+\delta_i F_i(t)}\, dt + dZ_j(t)\,, & \text{if } j<k, \\ dZ_j(t)\,, & \text{if } j=k, \\ \sum_{k+1\leq i\leq j} \dfrac{r_{ji}\delta_i C_i(t)}{1+\delta_i F_i(t)}\, dt + dZ_j(t)\,, & \text{if } j>k\,, \end{cases}
\end{aligned}
\tag{43}
$$

  under $\mathsf{Q}_k$.
- Similarly,

$$
d\sigma_j(t) = D_j(t)\Big( \sum_{\gamma(t)\leq i\leq j} \frac{r_{ji}\delta_i D_i(t)}{1+\delta_i F_i(t)}\, dt + dZ_j(t) \Big), \tag{44}
$$

  under the spot measure.

# No arbitrage condition for SABR / LMM

- We now plug in the explicit choices made in (35) and (37). Under the $T_{k+1}$-forward measure $Q_k$, the dynamics of the full model reads:

$$dF_j(t) = \sigma_j(t)\, L_j(t)^{\beta_j} \times \begin{cases} -\sum_{j+1\le i\le k} \dfrac{\rho_{ji}\delta_i\sigma_i(t)\, L_i(t)^{\beta_i}}{1+\delta_i F_i(t)}\, dt + dW_j(t)\,, & \text{if } j<k, \\ dW_j(t)\,, & \text{if } j=k, \\ \sum_{k+1\le i\le j} \dfrac{\rho_{ji}\delta_i\sigma_i(t)\, L_i(t)^{\beta_i}}{1+\delta_i F_i(t)}\, dt + dW_j(t)\,, & \text{if } j>k\,, \end{cases} \tag{45}$$

and

$$d\sigma_j(t) = \alpha_j(t)\,\sigma_j(t) \times \begin{cases} -\sum_{j+1\le i\le k} \dfrac{r_{ji}\delta_i\sigma_i(t)\, L_i(t)^{\beta_i}}{1+\delta_i F_i(t)}\, dt + dZ_j(t)\,, & \text{if } j<k, \\ dZ_j(t)\,, & \text{if } j=k, \\ \sum_{k+1\le i\le j} \dfrac{r_{ji}\delta_i\sigma_i(t)\, L_i(t)^{\beta_i}}{1+\delta_i F_i(t)}\, dt + dZ_j(t)\,, & \text{if } j>k\,, \end{cases} \tag{46}$$

# No arbitrage condition for SABR / LMM

- These equations are supplemented by the initial conditions:

$$\begin{aligned} L_j(0) &= L_{j0}, \\ \sigma_j(0) &= \sigma_{j0}, \end{aligned} \tag{47}$$

  where $L_{j0}$'s and $\sigma_{j0}$'s are the currently observed values.
- Similarly, under the spot measure $\mathsf{Q}_0$, the dynamics is given by the stochastic system:

$$\begin{aligned} dL_j(t) &= \sigma_j(t)\, L_j(t)^{\beta_j} \Big( \sum_{\gamma(t) \leq i \leq j} \frac{\rho_{ji}\delta_i\sigma_i(t)\, L_i(t)^{\beta_i}}{1+\delta_i F_i(t)}\, dt + dW_j(t) \Big), \\ d\sigma_j(t) &= \alpha_j(t)\, \sigma_j(t) \Big( \sum_{\gamma(t) \leq i \leq j} \frac{r_{ji}\delta_i\sigma_i(t)\, L_i(t)^{\beta_i}}{1+\delta_i F_i(t)}\, dt + dZ_j(t) \Big). \end{aligned} \tag{48}$$

# Practicalities of the SABR / LMM model

- Time does not permit us to get into any detailed discussion of the practical aspects of SABR / LMM, and we will just highlight a number of issues. To a large degree, in order to implement SABR / LMM one follows the steps described above in the case of the classic LMM.
- The model has to be factor reduced in order to make it practical, sensible parametrizations for the volatilities and correlations have to be found, and a good deal of analytic work needs to be done to prepare ground for calibration. Let us note a number of new features of the SABR / LMM model as compare to the original models.
- Not surprisingly, unlike the classic LMM model, exact closed form valuation of caps and floors is not possible in SABR / LMM. This is simply a reflection of the fact that SABR itself does not have have closed form solutions, and one either relies on sensible approximations or Monte Carlo simulations.
- However, the reassuring fact is that SABR / LMM allows for pricing of caps / floors which is in principle consistent with market practice.

# Practicalities of the SABR / LMM model

- This can be seen as follows. Assume that we have chosen the $T_{k+1}$-forward measure $\mathsf{Q}_k$ for pricing. A cap is a basket of caplets spanning a number of consecutive accrual periods. Consider the caplet spanning the period $\left[T_j, T_{j+1}\right]$. Shifting from $\mathsf{Q}_k$ to the $T_{j+1}$-forward measure $\mathsf{Q}_j$, we note that its dynamics is that of the classic SABR model. Since instrument valuation is invariant under change of numeraire, this shows that the price of the caplet is consistent with its SABR price.
- The correlation structure of SABR / LMM is very rich: in addition to the block of correlations between the forwards, we have the blocks of correlations between the volatilities, and the block of correlations between the forwards and volatilites. Together, these correlations determine the shape of volatility smile.

# Practicalities of the SABR / LMM model

- SABR / LMM specifies the values of the CEV exponents $\beta_j$ for each benchmark forward $L_j$ but it does not use explicit CEV exponents $\beta_{mn}$ for the benchmark forward swap rates $S_{mn}$. These are internally implied by the model.
- There is no simple relation between the caplet $\beta$'s and the swaption $\beta$'s. An approximation which works well in practice is given by the following formulas:

$$\beta_{mn} = \sum_{m \le j \le n-1} a_{mn,j} \beta_j + b_{mn}, \tag{49}$$

where

$$\begin{aligned} a_{mn,k} &= \frac{2 \log L_{k0}}{(n-m)^2} \sum_{m \le j \le n-1} \frac{1}{\log L_{j0} + \log L_{k0}}, \\ b_{mn} &= \frac{1}{(n-m)^2} \sum_{m \le j,k \le n-1} \frac{\log \rho_{jk}}{\log L_{j0} + \log L_{k0}}. \end{aligned} \tag{50}$$

- Note that

$$\sum_{m \le j \le n-1} a_{mn,j} = 1. \tag{51}$$

# Practicalities of the SABR / LMM model

- Consequently, the CEV power of a swaption is a weighted average of the CEV powers of the spanning forwards plus a convexity correction. Under a perfectly flat forward curve $a_{mn,j} = 1/(n-m)$, for all $j$. The convexity correction $b$ is rather small. On a typical market snapshot it is of the order of magnitude $10^{-3}$, and thus for all practical purposes it can be assumed zero.
- Swaptions are the most liquid volatility instruments in the interest rates markets, and a term structure model should be calibrated to a suitable set of swaptions. Calibration of SABR / LMM to swaptions requires understanding the relationships between swaption SABR parameters (as discussed in Lecture 3), and the caplet parameters appearing in the SABR / LMM model. An example of such a relationship is the approximate equality (49).
- Other relations of this type are: relations between caplet $\beta_j$-volatility processes $\sigma_j(t)$ and the swaption $\beta_{mn}$-volatility processes, relations between the corresponding “volvols”, and between the swaption SABR correlation coefficient and the correlation structure of SABR / LMM. Such relationships are fairly easy to derive within the crude “frozen curve” approximation discussed above but, even then, they take some space to write down, and a good deal of coding effort to make them work.

# References


- Brigo, D., and Mercurio, F.: *Interest Rate Models - Theory and Practice*, Springer Verlag (2006).
- Hagan, P., and Lesniewski, A.: LIBOR market model with SABR style stochastic volatility, working paper (2006).
- Mercurio, F.: Joining the SABR and Libor models together, *Risk*, 2009.
- Rebonato, R., McKay, K., and White, R.: *The SABR / LIBOR Market Model: Pricing, Calibration and Hedging for Complex Interest-Rate Derivatives*, Wiley (2009).