Assignment #5

1. The dynamics of $L_j(t)$ is

$$dL_j(t) = \Delta_j(t)\,dt + C_j(t)\,dW_j(t)$$

The numeraire for the forward T_{j+1} measure Q_j is

$$P(t, T_{j+1}) = P(t, T_{\gamma(t)}) \prod_{\gamma(t)\leq i\leq j} \frac{1}{1+\delta_j F_j(t)}$$

The numeraire for the spot measure Q_0 is

$$B(t) = \frac{P(t, T_{\gamma(t)})}{\prod_{1\leq i\leq \gamma(t)} P(T_{i-1}, T_i)}$$

$\because$ $\Delta_j(t)$ is zero under Q_j, when we switch from Q_j measure to spot measure, $\Delta_j(t)$ becomes

$$\Delta_j(t) = \frac{d}{dt}\left[L_j, \log\frac{B(t)}{P(t, T_{j+1})}\right](t)$$

$$= \frac{d}{dt}\left[L_j, \log\left(\frac{1}{\prod_{1\leq i\leq \gamma(t)} P(T_{i-1}, T_i)} \prod_{\gamma(t)\leq i\leq j}(1+\delta_i F_i(t))\right)\right](t)$$

$$= \frac{d}{dt}\left[L_j, \log \prod_{0\leq i\leq \gamma(t)-1}(1+\delta_i F_i(T_i)) \prod_{\gamma(t)\leq i\leq j}(1+\delta_i F_i(t))\right](t)$$

$\prod_{0\leq i\leq \gamma(t)-1}(1+\delta_i F_i(T_i))$ is constant since $T_i < t$ for $0\leq i\leq \gamma(t)-1$

$$\therefore \Delta_j(t) = \sum_{\gamma(t)\leq i\leq j} dL_j(t)\, \log(1+\delta_i F_i(t))$$

$$= \sum_{\gamma(t)\leq i\leq j} dL_j(t)\, \frac{\delta_i dF_i(t)}{1+\delta_i F_i(t)}$$

$$= C_j(t) \sum_{\gamma(t)\leq i\leq j} \frac{\rho_{ij}\,\delta_i\, C_i(t)}{1+\delta_i F_i(t)}$$

Since the change of measure doesn't affect the diffusion,

$$\therefore\ dL_j(t) = G_j(t)\Big(\sum_{\gamma(t)\leq i\leq j} \frac{\rho_{ij}\,\delta_i G_i(t)}{1+\delta_i F_i(t)}\,dt + dW_j(t)\Big)$$

under the spot measure

\# 2. $I_{(a,b)} + I_{(b,a)} = \int_t^{t+\delta}\int_t^s (dZ_a(u)\,dZ_b(s) + dZ_b(u)\,dZ_a(s))$

$$= \int_t^{t+\delta} (Z_a(s) - Z_a(t))\,dZ_b(s) + \int_t^{t+\delta} (Z_b(s) - Z_b(t))\,dZ_a(s)$$

$$= \int_t^{t+\delta} (Z_a(s)\,dZ_b(s) + Z_b(s)\,dZ_a(s)) - Z_a(t)\int_t^{t+\delta} dZ_b(s) - Z_b(t)\int_t^{t+\delta} dZ_a(s)$$

$$= \int_t^{t+\delta} (Z_a(s)\,dZ_b(s) + Z_b(s)\,dZ_a(s)) - 2\delta Z_a(t) Z_b(t)$$

Apply Ito's lemma to $Z_a(s) Z_b(s)$, we have

$$d(Z_a(s)Z_b(s)) = Z_a(s)\,dZ_b(s) + Z_b(s)\,dZ_a(s)$$

$$\Rightarrow\ Z_a(s)Z_b(s)\Big|_t^{t+\delta} = \int_t^{t+\delta} (Z_a(s)\,dZ_b(s) + Z_b(s)\,dZ_a(s))$$

$$\therefore\ I_{(a,b)} + I_{(b,a)} = Z_a(s)Z_b(s)\Big|_t^{t+\delta} - 2\delta Z_a(t) Z_b(t)$$

$$= (Z_a(t) + Z_a(t)\delta)(Z_b(t) + \delta Z_b(t)) - Z_a(t) Z_b(t) - 2\delta Z_a(t) Z_b(t)$$

$$= \delta Z_a\,\delta Z_b$$

3. After factor reduction, the dynamics of LMM is

$$dL_j(t) = \Delta_j(t)\,dt + \sum_{1\le a\le d} B_{ja}(t)\,dZ_a(t)$$

where $B_{ja}(t) = U_{ja} C_j(t)$ and $Z_a(t)$ are independent B.M.

$$\mathcal{L}^a B_{ib} = \sum_{1\le k\le n} B_{ka} \frac{\partial B_{ib}}{\partial X_k} = \sum_{1\le k\le n} U_{ka} C_k(t) \frac{\partial U_{ib} C_i(t)}{\partial L_k}$$

Since U_{ja} doesn't depend on L_j,

$$\mathcal{L}^a B_{ib} = \sum_{1\le k\le n} U_{ka} U_{ib} C_k(t) \frac{\partial C_i(t)}{\partial L_k}$$

Since $C_i(t, L_i(t))$ is a function of t and $L_i(t)$,

$$\frac{\partial C_i(t)}{\partial L_k} = 0 \quad \text{except for } k = i$$

$$\Rightarrow \mathcal{L}^a B_{ib} = U_{ia} U_{ib} C_i(t) \frac{\partial C_i(t)}{\partial L_i}$$

Using the same calculation, we can show that

$$\mathcal{L}^b B_{ia} = U_{ib} U_{ia} C_i(t) \frac{\partial C_i(t)}{\partial L_i}$$

$$\Rightarrow \mathcal{L}^a B_{ib} = \mathcal{L}^b B_{ia}$$

$\therefore$ The integrability condition is satisfied.