(1)

$T = 10 + 1 = 11$, $f = 2$, $f_{CMS} = 4$, $n = 20$

10 year CMS settling in 1 year and paying 3 months later, $T_0 = 1$

Using the data from the previous assignment,

$S_0(1, 11) = 3.872426347\%$ and $\sigma = 30.220506\%$

We need to compute the convexity correction,

$$\begin{aligned}
\Delta_{corr} &\simeq S_0(T_0, T)\theta_c\,(e^{\sigma T^2} - 1)\\
&= S_0(T_0, T)(1 - \frac{1}{1 + S_0/f}\frac{n\,S_0/f}{(1 + S_0/f)^n - 1})\,(e^{\sigma T^2} - 1)\\
&= 0.012728583\\
\Delta_{delay} &\simeq -\,S_0(T_0, T)\theta_d(e^{\sigma T^2} - 1)\\
&= -\,S_0(T_0, T)\frac{S_0/f_{CMS}}{1 + S_0/f}\,(e^{\sigma T^2} - 1)\\
&= 0.0645207\%
\end{aligned}$$

Thus, $C(T_0, T|T_p) = S_0(1, 10) + \Delta_{corr} + \Delta_{delay} = 0.050805776$

The fraction contributed by the payment delay in CMS convexity is

$$\frac{|\Delta_{corr}|}{\Delta_{corr} + \Delta_{delay}} = 0.053396 = 5.34\%$$

10 year CMS settling in 5 year and paying 3 months later, $T_0 = 5$, and

$T = 5 + 10 = 15$

Using the data from the previous assignment,

$S_0(5, 15) = 5.03030935\%$ and $\sigma = 22.15447\%$

The convexity correction is,

$$
\begin{aligned}
\Delta_{corr} &\simeq S_0(T_0,T)\theta_c\,(e^{\sigma T^2}-1)\\
&= S_0(T_0,T)(1-\frac{1}{1+S_0/f}\frac{n\,S_0/f}{(1+S_0/f)^n-1})\,(e^{\sigma T^2}-1)\\
&= 0.012995136
\end{aligned}
$$

$$
\begin{aligned}
\Delta_{delay} &\simeq -\,S_0(T_0,T)\theta_d(e^{\sigma T^2}-1)\\
&= -\,S_0(T_0,T)\frac{S_0/f_{CMS}}{1+S_0/f}\,(e^{\sigma T^2}-1)\\
&== 0.0671414\%
\end{aligned}
$$

Thus, $C(T_0,T|T_p) = S_0(1,10)+\Delta_{corr}+\Delta_{delay} = 0.062626812$

The fraction contributed by the payment delay in CMS convexity is

$$
\frac{|\Delta_{corr}|}{\Delta_{corr}+\Delta_{delay}} = 0.0544814 = 5.45\%
$$

2. $P(T, T_{mat}) = A(T, T_{mat}) e^{-h_\lambda(T_{mat}-T) r(T)}$. (22)

where $r(T) = r_0(T) + \int_0^T e^{-\lambda(T-s)} \sigma(s) dW(s)$ (16)

$E(r(T)) = r_0(T) \quad Var(r(T)) = \int_0^T e^{-2\lambda(T-s)} \sigma^2(s) ds$

$\therefore \; r(T) \sim N(r_0(T), \int_0^T e^{-2\lambda(T-s)} \sigma^2(s) ds)$

$h_\lambda(T_{mat}-T) r(T) \sim N(h_\lambda(T_{mat}-T) r_0(T), \; h_\lambda(T_{mat}-T)^2 \int_0^T e^{-2\lambda(T-s)} \sigma^2 ds)$

Let $\bar{\mu} = h_\lambda(T_{mat}-T) r_0(T)$

$\Rightarrow h_\lambda(T_{mat}-T) r(T) \sim N(\bar{\mu}, \bar{\sigma}^2)$

$\therefore \; h_\lambda(T_{mat}-T) r(T) = \bar{\mu} + \bar{\sigma} Z$

$\therefore \; P(T, T_{mat}) = A(T, T_{mat}) e^{-\bar{\mu} - \bar{\sigma} Z}$

$$A(T, T_{mat}) = e^{-\int_0^T r_0(u) du + \int_0^t r_0(u) du + h_\lambda(T-t) r_0(t) + \frac{1}{2}\int_t^T h_\lambda(T-s)^2 \sigma(s)^2 ds}$$

$$= P(0, T_{mat}) e^{h_\lambda(T-t) r_0(t) + \int_0^t r_0(u) du - \frac{1}{2}\int_0^t h_\lambda(t-s)^2 \sigma(s)^2 ds + \frac{1}{2}\int_0^t h_\lambda(t-s)^2 \sigma(s)^2 ds - \frac{1}{2}\int_0^t h_\lambda(T-s)^2 \sigma^2 ds}$$

$$= \frac{P(0, T_{mat})}{P(0, T)} e^{h_\lambda(T-t) r_0(t) + \frac{1}{2}\int_0^t (h_\lambda(t-s)^2 - h_\lambda(T-s)^2) \sigma(s)^2 ds}$$

$$\Rightarrow P(T, T_{mat}) = \frac{P(0, T_{mat})}{P(0, T)} e^{\frac{1}{2}\int_0^t (h_\lambda(t-s)^2 - h_\lambda(T-s)^2) \sigma(s)^2 ds - \bar{\sigma} Z}$$

In order to compute the option price, we need to change from the current spot measure into forward-T measure

From the previous Lecture, we know that under the forward-T measure, Use $P(t,T)$ as the numeraire, the process of $P(t, T_{mat})$ becomes,

$$F_t = \frac{P(t, T_{mat})}{P(t, T)}$$

F_t is the forward price of $P(t, T_{mat})$ under $P(t,T)$.

F_t is a martingale under forward-T measure

$$F_T = \frac{P(T, T_{mat})}{P(T, T)} = P(T, T_{mat})$$

$\Rightarrow$ $P(T, T_{mat})$ is a martingale under forward-T measure with zero drift and same volatility as in the spot measure.

$$\Rightarrow P(T, T_{mat}) = P_0(T, T_{mat})\, e^{-\frac{\bar{\sigma}^2}{2} - \bar{\sigma} Z} \quad \left(\begin{array}{l}\text{under T-forward}\\ \text{measure}\end{array}\right)$$

Notice that this expression is identical to the price of equity under risk-neutral measure

$$\Rightarrow PV_{call} = P(0,T)\, E^{Q^T}\left[(P(T, T_{mat}) - K)^+\right] \quad (P(T,T) = 1)$$

Apply the same logic we use in B-S model for equity option,

$$E^{Q^T}\left[(P(T, T_{mat}) - K)^+\right] = P_0(T, T_{mat}) N(d_+) - K N(d_-)$$

where $d_+ = \dfrac{\log(P_0(T, T_{mat})) - \log K + \frac{1}{2}\bar{\sigma}^2}{\bar{\sigma}} = \dfrac{1}{\bar{\sigma}} \log \dfrac{P(0, T_{mat})}{P(0,T) K} + \dfrac{1}{2}\bar{\sigma}$

$$d_- = d_+ - \bar{\sigma} = \frac{1}{\bar{\sigma}} \log \frac{P(0, T_{mat})}{P(0,T)K} - \frac{1}{2}\bar{\sigma}.$$

$$\Rightarrow PV_{call} = P_0(0, T_{mat}) N(d_+) - K P_0(0, T) N(d_-)$$

Using the same approach for put option

$$PV_{put} = P(0,T)\, E^{Q_T}\left[(K - P(T, T_{mat}))^+ \right]$$

$$= P(0,T)\left(K N(-d_-) - P_0(T, T_{mat}) N(-d_+) \right)$$

$$= K P(0,T) N(-d_-) - P_0(0, T_{mat}) N(-d_+)$$

3.(1). $P(t,T) = E_t^{Q_0}[e^{-\int_t^T r(u)\,du}]$. $\quad P(T,T) = 1$

Let $dr(t) = \mu(r(t),t)\,dt + \sigma(r(t),t)\,dt$

From the Feynman-Kac formula, if $P = P(r(t),t)$, it is the solution to the terminal value problem of

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma(t,r)^2\frac{\partial^2 P}{\partial r^2} + \mu(t,r)\frac{\partial P}{\partial r} = rP$$

$$P(T,T) = 1.$$

Then $P(r,t)$ has the following representation

$$P(r,t) = E_t[e^{-\int_t^T r(u)du}P(T,T)]$$
$$= E_t[e^{-\int_t^T r(u)du}] = P(t,T).$$

$\therefore$ $P(t,T)$ satisfies the partial differential equation.

(2). Since $r(t)$ is affine, $\Rightarrow P(t,T) = A(t,T)\exp(-B(t,T)r(t))$

From the above PDE, we have.

$$\frac{1}{P}\frac{\partial P}{\partial t} + \frac{1}{2P}\sigma(r,t)^2\frac{\partial^2 P}{\partial r^2} + \frac{\mu(r,t)}{P}\frac{\partial P}{\partial r} = r$$

$$\frac{1}{P}\frac{\partial P}{\partial t} = \frac{\partial \log P}{\partial t} = \frac{\partial(A(t,T)\exp(-B(t,T)r(t)))}{\partial t} = \frac{\partial \log A}{\partial t} - r\frac{\partial B}{\partial t}$$

$$\frac{1}{2P}\sigma(r,t)^2\frac{\partial^2 P}{\partial r^2} = \frac{\sigma(r,t)^2}{2P}B^2(t,T)A(t,T)\exp(-B(t,T)r(t))$$
$$= \frac{1}{2}\sigma(t,r)^2B^2$$

$$\mu(t,r)\frac{1}{P}\frac{\partial P}{\partial r} = \mu(t,r)\frac{\partial \log P}{\partial r} = -\mu(t,r)B$$

$\Rightarrow$ The PDE becomes, $\dfrac{\partial \log A}{\partial t} - r\dfrac{\partial B}{\partial t} + \dfrac{1}{2}\sigma(t,r)^2B^2 - \mu(t,r)B = r$

$\because P(T,T)=1 \Rightarrow A(T,T)\exp(-B(T,T)r(t))=1$

At maturity, $\exp(-B(T,T)r(T))=1$

$\Rightarrow B(T,T)=0 \qquad \Rightarrow A(T,T)=1$

$\therefore A(T,T)=1 \quad , \quad B(T,T)=0$

(iii) $\mu(t,r)=a(t)r+b(t)$

$\sigma(t,r)^2=c(t)r+d(t)$

Plug into the PDE from (ii)

$$\frac{\partial \log A}{\partial t}-r\frac{\partial B}{\partial t}+\frac{1}{2}(c(t)r+d(t))B^2-(a r(t,r)+b(t))B=r$$

$$\Rightarrow \frac{\partial \log A}{\partial t}-b(t)B+\frac{1}{2}d(t)B^2=r\left[\frac{\partial B}{\partial t}+aB-\frac{1}{2}c(t)B^2\right]$$

The LHS is the function of t, RHS is the function of t and r,

For both sides to be equal, we have

$$\frac{\partial \log A}{\partial t}-b(t)B+\frac{1}{2}d(t)B^2=0$$

$$\frac{\partial B}{\partial t}+aB-\frac{1}{2}c(t)B^2+1=0$$

with $B(T,T)=0 \qquad A(T,T)=1$