IR HW 8 Group 5

Hongchao Pan & Chendi Zhang

1.

(i) ① Since $\lambda(t) = \lambda\gamma(\lambda t)^{\gamma-1}$, λ, γ are constant, $\lambda(t)$ is only depend on time t.

② From Lecture 10, we know that prepayment modeling needs to capture both observable (cox model) and unobservable (fraity/threshold model) factors.

③ As hazard model only capture the factor of time t, which makes it is inadequate for prepayment modeling.

(ii)
$$S(t,T) = \exp\left(-\int_t^T \lambda(s)\,ds\right)$$
$$= \exp\left(-\int_t^T \lambda\gamma(\lambda s)^{\gamma-1}\,ds\right)$$
$$= \exp\left(-(\lambda s)^\gamma \Big|_t^T\right)$$
$$= e^{-(\lambda T)^\gamma + (\lambda t)^\gamma} = e^{-\lambda^\gamma[T^\gamma - t^\gamma]}$$

(iii) From Lecture 10, we have:

$$P(T) = E\left\{\sum_j Z^\lambda(T,T_j)\left(C_j + \bar\lambda(T_{j-1},T_j)B_j\right)\right\}$$

where $Z^\lambda(T,T_j) = Z(T,T_j)S(T,T_j)$, ~~$\bar\lambda(T_{j-1},T_j) \approx \int_{T_{j-1}}^{T_j} \lambda(t)dt$~~.

$$Z(T_j, T) = e^{-r(T-T_j)}$$

$$S(T_j, T) = e^{-\lambda^\gamma[T^\gamma - T_j^\gamma]}$$

$$\Rightarrow \quad Z^\lambda(T_j, T) = Z(T_j, T)S(T_{j-1}, T) = e^{-\lambda^\gamma[T^\gamma - T_j^\gamma] - r(T-T_j)}$$

Notes: $Z^\lambda(T, T_j) = Z(T, T_j)\, S(T, T_{j-1})$ in Lecture 10 should be

$$Z^\lambda(T_j, T) = Z(T_j, T)S(T_j, T).$$

$$\bar{\lambda}(T_{j-1}, T_j) = \frac{S(T_{j-1}, T) - S(T_j, T)}{S(T_{j-1}, T)} = 1 - \frac{S(T_j, T)}{S(T_{j-1}, T)}$$

$$= 1 - \frac{e^{-\lambda^\gamma[T^\gamma - T_j^\gamma]}}{e^{-\lambda^\gamma[T^\gamma - T_{j-1}^\gamma]}}$$

$$= 1 - e^{-\lambda^\gamma[T_{j-1}^\gamma - T_j^\gamma]}$$

$$\Rightarrow \quad p(T) = E\left\{ \sum_j \left[e^{-\lambda^\gamma(T^\gamma - T_j^\gamma) - r(T-T_j)} \cdot \left(G_j + (1 - e^{-\lambda^\gamma[T_{j-1}^\gamma - T_j^\gamma]})B_j\right)\right]\right\}$$

$$\overset{\text{deterministic}}{=} \sum_j \left[e^{-\lambda^\gamma(T^\gamma - T_j^\gamma) - r(T-T_j)} \cdot \left(G_j + (1 - e^{-\lambda^\gamma(T_{j-1}^\gamma - T_j^\gamma)}) \cdot B_j\right)\right].$$

where G_j and B_j are the notations used in Lecture 10.

2. (i) $\lambda(t)$ in model 2 is also only depend on time t. The reason of it is inadequate for prepayment modeling is same as $\lambda(t)$ in model 1 (Q1).

(ii) $$S(t,T) = \exp\left[-\int_t^T \lambda(s)\,ds\right]$$

$$= \exp\left(-\int_t^T \cdot \frac{\lambda\gamma(\lambda s)^{\gamma-1}}{1+(\lambda s)^{\gamma}}\,ds\right)$$

$$= \exp\left(-\int_t^T \cdot \frac{1}{1+(\lambda s)^{\gamma}} \cdot d((\lambda s)^{\gamma}+1)\right)$$

$$= \exp\left(- \ln(1+(\lambda s)^{\gamma})\Big|_t^T\right)$$

$$= \exp\left[\ln \frac{1+(\lambda t)^{\gamma}}{1+(\lambda T)^{\gamma}}\right] = \frac{1+(\lambda t)^{\gamma}}{1+(\lambda T)^{\gamma}}$$

(iii) $$Z(T_j,T) = e^{-r(T-T_j)}$$

$$S(T_j,T) = \frac{1+(\lambda T_j)^{\gamma}}{1+(\lambda T)^{\gamma}}$$

$$Z^{\pi}(T_j,T) = Z(T_j,T)\,S(T_j,T) = \frac{1+(\lambda T_j)^{\gamma}}{1+(\lambda T)^{\gamma}} \cdot e^{-r(T-T_j)}$$

$$\overline{\lambda}(T_{j-1},T_j) = 1-\frac{S(T_j,T)}{S(T_{j-1},T)} = 1-\frac{1+(\lambda T_j)^{\gamma}}{1+(\lambda T)^{\gamma}} \cdot \frac{1+(\lambda T)^{\gamma}}{1+(\lambda T_{j-1})^{\gamma}}$$

$$= 1-\frac{1+(\lambda T_j)^{\gamma}}{1+(\lambda T_{j-1})^{\gamma}}$$

$$p(T) = E\left\{ \sum_j \left[\frac{1+(\lambda T_j)^\gamma}{1+(\lambda T)^\gamma} \cdot e^{-\gamma(T-T_j)} \cdot \left(C_j + \left(1 - \frac{1+(\lambda T_j)^\gamma}{1+(\lambda T_{j-1})^\gamma}\right) \cdot B_j \right) \right] \right.$$

<u>deterministic</u> $\sum_j \left[\frac{1+(\lambda T_j)^\gamma}{1+(\lambda T)^\gamma} \cdot e^{-\gamma(T-T_j)} \cdot \left(C_j + \left(1 - \frac{1+(\lambda T_j)^\gamma}{1+(\lambda T_{j-1})^\gamma}\right) \cdot B_j \right) \right]$

with C_j and B_j as same notations in Letture10.

(iv) Model 1: $\lambda(t) = \lambda\gamma(\lambda t)^{\gamma-1}$

Model 2: $\lambda(t) = \dfrac{\lambda\gamma(\lambda t)^{\gamma-1}}{1+(\lambda t)^\gamma}$

Clearly, we can see $\lambda(t)$ in model 1 is monotone increasing as t increase. However, $\lambda(t)$ in model 2 has compensation term $\dfrac{1}{1+(\lambda t)^\gamma}$ to adjust the trend as t increasing.

Morever, $\lambda(t)$ in model 2 has a bell shape than increasing shape in model ~~a~~ 1.

Therefore, $\lambda(t)$ in model 2 can capture more factors than $\lambda(t)$ in model 1. ~~Also.~~ In other words, $\lambda(t)$ in model 2 is better for prepayment modeling.