PMATH 465

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Contents

Chapter 1

Smooth Manifolds

1.1 Introduction

- This note is a summary of *Introduction to Smooth Manifolds* by John M. Lee. Smooth manifolds are spaces that locally resemble Euclidean space $\mathbb{R}^n$, and on which one can perform calculus.
- We always write basis vectors (such as E_i) with lower indices, and components of a vector with respect to a basis (such as x^i) with upper indices.

1.2 Topological Manifolds

1.2.1 Definition (Topological Manifold)

Suppose M is a topological space. We say that M is a topological manifold of dimension n or topological n-manifold if it has the following properties:

i. M is a **Hausdorff space**: for every pair of distinct points $p, q \in M$, there are disjoint open subsets $U, V \subset M$ such that $p \in U$ and $q \in V$ (That is, any two distinct points can be separated by disjoint open sets.)

ii. M is **second-countable** : there exist a countable basis for the topology of M. (A topological space T is second-countable if there exists some countable collection $\mathcal{U} = \{U_i\}_{i=1}^{\infty}$ of open subsets of T such that any open subset of T can be written as a union of elements of some subfamily of U)

iii. M is **locally Euclidean of dimension n**: each point of M has a neighbourhood that is homeomorphic to an open subset of $\mathbb{R}^n$

The third property means, for each $p \in M$, we can find

1. an open subset $U \subset M$ containing p

2. an open subset $\hat{U} \subset \mathbb{R}^n$, and
3. a homeomorphism $\phi : U \to \hat{U}$

1.2.2 Definition (homeomorphism)

A function $f : X \to Y$ between two topological spaces is a homeomorphism if it has the following properties:

- f is a bijection
- f is continuous
- the inverse function f^{-1} is continuous

1.2.3 Examples

- $\mathbb{R}^n$ is an n-manifold
- Any countable discrete space is a 0-dimensional manifold (Notice that $\mathbb{R}^0$ is just collection of points, which is homeomorphic to the discrete space. Plus, it is a discrete space, so every subset is clopen.)
- A circle is a compact 1-dimensional manifold (Pick any point on the circle, we can always find an open arc of that point. This is homeomorphic to open interval $(-\epsilon, \epsilon)$ in $\mathbb{R}^1$)
- A torus and a Klein bottle are compact 2-manifolds (or surfaces)
- The n-dimensional sphere $\mathbb{S}^n$ or torus $\mathbb{T}^n$ is a compact n-manifold

1.2.4 Theorem

Every metric space is Hausdorff.

Proof 1:
Let $\varepsilon > 0$, and let $x, y \in A$ with $x \neq y$. Then there exist disjoint open balls $B_\varepsilon(x)$ and $B_\varepsilon(y)$. Hence, by definition, A is a Hausdorff space. □

Proof 2:
Let M be a metric space. Suppose, for the sake of contradiction, that M is not Hausdorff.

Then there exist $x, y \in A$ with $x \neq y$, and for all $\varepsilon > 0$, the open balls $B_\varepsilon(x)$ and $B_\varepsilon(y)$ are not disjoint. That is, there exists $z \in M$ such that

$$z \in B_\varepsilon(x) \cap B_\varepsilon(y).$$

Let $r = \frac{d(x,y)}{2}$. Then there exists $z \in B_r(x) \cap B_r(y)$. This means:

$$d(x, z) < r \quad \text{and} \quad d(y, z) < r.$$

Thus,

$$d(x,z) + d(z,y) < r + r = 2r = d(x,y),$$

This is a contradiction to the triangle inequality.

Therefore, the assumption that M is not Hausdorff must be false. Hence, M is a Hausdorff space. $\square$

1.2.5 Theorem (Topological Invariance of Dimension)

A non-empty n-dimensional topological manifold cannot be homeomorphic to an m-dimensional manifold unless $m = n$

1.3 Coordinates Charts

1.3.1 Definition (Coordinates Charts)

Let M be a topological n-manifold. A **coordinate chart** (or just a *chart*) on M is a pair (U, φ), where U is an open subset of M and $\varphi : U \to U'$ is a homeomorphism from U to an open subset $U' = \varphi(U) \subseteq \mathbb{R}^n$. In other words, φ maps a subset of M to $\mathbb{R}^n$.

By the definition of a topological manifold, each point $p \in M$ is contained in the domain of some chart (U, φ). If $\varphi(p) = 0$, we say that the chart is **centered at** p. Given any chart (U, φ) whose domain contains p, it is easy to obtain a new chart centred at p by subtracting the constant vector $\varphi(p)$.

Given a chart (U, φ), we call the set U a **coordinate domain**, or a **coordinate neighbourhood** of each of its points.

If, in addition, $\varphi(U)$ is an open ball in $\mathbb{R}^n$, then U is called a **coordinate ball**; if $\varphi(U)$ is an open cube, then U is called a **coordinate cube**.

The map φ is called a **(local) coordinate map**, and the component functions $(x^1, x^2, \ldots, x^n)$ of φ, defined by

$$\varphi(p) = (x^1(p), x^2(p), \ldots, x^n(p)),$$

are called **local coordinates** on U.

We sometimes denote the chart by $(U, (x^1, \ldots, x^n))$ or $(U, (x^i))$.

1.3.2 Example (Graphs of Continuous Functions).

Let $U \subseteq \mathbb{R}^n$ be an open subset, and let $f : U \to \mathbb{R}^k$ be a continuous function. The *graph of* f is the subset of $\mathbb{R}^n \times \mathbb{R}^k$ defined by

$$\Gamma(f) = \left\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^k : x \in U \text{ and } y = f(x)\right\},$$

with the subspace topology. Let $\pi_1 : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ denote the projection onto the first factor, and let $\varphi : \Gamma(f) \to U$ be the restriction of π_1 to $\Gamma(f)$:

$$\varphi(x,y) = x, \quad (x,y) \in \Gamma(f).$$

Because φ is the restriction of a continuous map, it is continuous; and it is a homeomorphism because it has a continuous inverse given by $\varphi^{-1}(x) = (x, f(x))$. Thus $\Gamma(f)$ is a topological manifold of dimension n. In fact, $\Gamma(f)$ is homeomorphic to U itself, and $(\Gamma(f), \varphi)$ is a global coordinate chart, called *graph coordinates*.

The same observation applies to any subset of $\mathbb{R}^{n+k}$ defined by setting any k of the coordinates (not necessarily the last k) equal to some continuous function of the other n, which are restricted to lie in an open subset of $\mathbb{R}^n$.

1.3.3 Example (Spheres).

For each integer $n \geq 0$, the unit n-sphere S^n is Hausdorff and second-countable because it is a topological subspace of $\mathbb{R}^{n+1}$. To show that it is locally Euclidean, for each index $i = 1, \dots, n+1$, let U_i^+ denote the subset of $\mathbb{R}^{n+1}$ where the ith coordinate is positive:

$$U_i^+ = \left\{(x^1, \dots, x^{n+1}) \in \mathbb{R}^{n+1} : x^i > 0\right\}.$$

(See 1.3.3) Similarly, U_i^- is the set where $x^i < 0$.

Let $f : \mathbb{B}^n \to \mathbb{R}$ be the continuous function

$$f(u) = \sqrt{1 - |u|^2}.$$

Then for each $i = 1, \dots, n+1$, it is easy to check that $U_i^+ \cap S^n$ is the graph of the function

$$x^i = f(x^1, \dots, \widehat{x^i}, \dots, x^{n+1}),$$

where the hat indicates that x^i is omitted. Indeed, the unit sphere satisfy :

$$(x^1)^2 + (x^2)^2 + (x^3)^2 + ... + (x^{n+1})^2 = 1$$

That is,

$$x^i \;=\; \sqrt{1 \;-\; \sum_{j \neq i} (x^j)^2}\,.$$

That's exactly the same as $f(u)$ based on the definition of $f(u)$. Similarly, $U_i^- \cap S^n$ is the graph of

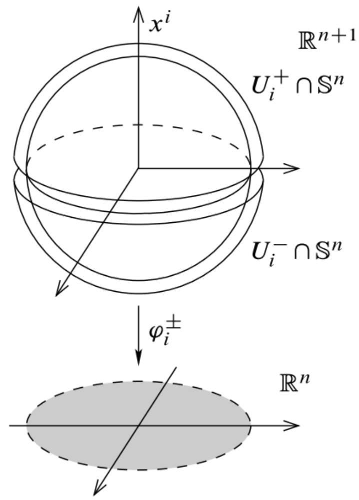
$$x^i = -f(x^1, \dots, \widehat{x^i}, \dots, x^{n+1}).$$

Thus, each subset $U_i^\pm \cap S^n$ is locally Euclidean of dimension n, and the maps

$$\varphi_i^\pm : U_i^\pm \cap S^n \to \mathbb{B}^n$$

are given by

$$\varphi_i^\pm(x^1, \dots, x^{n+1}) = (x^1, \dots, \widehat{x^i}, \dots, x^{n+1}).$$



are graph coordinates for S^n. (Think about φ as a projection from semi-sphere onto $\mathbb{R}^n$) Since each point of S^n is in the domain of at least one of these $2n+2$ charts, S^n is a topological n-manifold.

1.3.4 Example (Projective Spaces)

The ***n-dimensional real projective space***, denoted by $\mathbb{RP}^n$ (or sometimes just $\mathbb{P}^n$), is defined as the set of 1-dimensional linear subspaces of $\mathbb{R}^{n+1}$, with the quotient topology determined by the natural map

$$\pi : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$$

sending each point $x \in \mathbb{R}^{n+1} \setminus \{0\}$ to the subspace spanned by x. The 2-dimensional projective space $\mathbb{RP}^2$ is called the *projective plane*. For any point $x \in \mathbb{R}^{n+1} \setminus \{0\}$, let $[x] = \pi(x) \in \mathbb{RP}^n$ denote the line spanned by x.

For each $i = 1, \ldots, n+1$, let $\widetilde{U}_i \subseteq \mathbb{R}^{n+1} \setminus \{0\}$ be the set where $x^i \neq 0$, and let $U_i = \pi(\widetilde{U}_i) \subseteq \mathbb{RP}^n$. Since $\widetilde{U}_i$ is a saturated open subset, U_i is open and $\pi|_{\widetilde{U}_i} : \widetilde{U}_i \to U_i$ is a quotient map (see Theorem A.27). Define a map $\varphi_i : U_i \to \mathbb{R}^n$ by

$$\varphi_i\left([x^1, \ldots, x^{n+1}]\right) = \left(\frac{x^1}{x^i}, \ldots, \frac{x^{i-1}}{x^i}, \frac{x^{i+1}}{x^i}, \ldots, \frac{x^{n+1}}{x^i}\right).$$

This map is well defined because its value is unchanged by multiplying x by a nonzero constant. Because $\varphi_i \circ \pi$ is continuous, φ_i is continuous by the characteristic property of quotient maps (Theorem A.27). In fact, φ_i is a homeomorphism, because it has a continuous inverse given by

$$\varphi_i^{-1}(u^1, \ldots, u^n) = [u^1, \ldots, u^{i-1}, 1, u^i, \ldots, u^n],$$

as you can check. Geometrically, $\varphi_i([x]) = u$ means $(u, 1)$ is the point in $\mathbb{R}^{n+1}$ where the line $[x]$ intersects the affine hyperplane where $x^i = 1$ (Fig. 1.4). Because the sets $U_1, \dots, U_{n+1}$ cover $\mathbb{RP}^n$, this shows that $\mathbb{RP}^n$ is locally Euclidean of dimension n. The Hausdorff and second-countability properties are left as exercises.

1.4 Topological Properties of Manifolds

1.4.1 Definition (Precompact)

A topological space X is said to be **locally compact** if every point has a neighbourhood contained in a compact subset of X. A subset of X is said to be ***precompact*** in X if its closure in X is compact.

1.4.2 Proposition

For a Hausdorff space X, the following are equivalent:

(a) X is locally compact.

(b) Each point of X has a precompact neighbourhood.

(c) X has a basis of precompact open subsets.

1.4.3 Lemma

Every topological manifold has a countable basis of precompact coordinate balls.

Proof. Let M be a topological n-manifold. First we consider the special case in which M can be covered by a single chart. Suppose $\varphi : M \to \widehat{U} \subseteq \mathbb{R}^n$ is a global coordinate map, and let $\mathcal{B}$ be the collection of all open balls $B_r(x) \subseteq \mathbb{R}^n$ such that r is rational, x has rational coordinates, and $B_{r'}(x) \subseteq \widehat{U}$ for some $r' > r$. Each such ball is precompact in $\widehat{U}$, and it is easy to check that $\mathcal{B}$ is a countable basis for the topology of $\widehat{U}$. Because φ is a homeomorphism, it follows that the collection of sets of the form $\varphi^{-1}(B)$ for $B \in \mathcal{B}$ is a countable basis for the topology of M, consisting of precompact coordinate balls, with the restrictions of φ as coordinate maps.

Now let M be an arbitrary n-manifold. By definition, each point of M is in the domain of a chart. Because every open cover of a second-countable space has a countable subcover (Proposition A.16), M is covered by countably many charts $\{(U_i, \varphi_i)\}$. By the argument in the preceding paragraph, each coordinate domain U_i has a countable basis of coordinate balls that are precompact in U_i, and the union of all these countable bases is a countable basis for the topology of M. If $V \subseteq U_i$ is one of these balls, then the closure of V in U_i is compact, and because M is Hausdorff, it is closed in M. It follows that the closure of V in M is the same as its closure in U_i, so V is precompact in M as well. $\square$

1.5 Connectivity

1.5.1 Definition

A topological space X is

- **connected** if there do not exist two disjoint, nonempty, open subsets of X whose union is X.
- **path-connected** if every pair of X can be joint by a path in X; and
- **locally path-connected** if X has a basis of path-connected open subsets.

1.5.2 Proposition (Properties of Locally Path-Connected Spaces)

Let X be a locally path-connected topological space.

(a) The components of X are open in X.

(b) The path components of X are equal to its components.

(c) X is connected if and only if it is path-connected.

(d) Every open subset of X is locally path-connected.

1.5.3 Proposition

Let M be a topological manifold.

(a) M is locally path-connected.

(b) M is connected if and only if it is path-connected.

(c) The components of M are the same as its path components.

(d) M has countably many components, each of which is an open subset of M and a connected topological manifold.

Proof. Since each coordinate ball is path-connected, part (a) follows from the fact that M has a basis of coordinate balls. Parts (b) and (c) are immediate consequences of (a) and the proposition above.

To prove (d), note that each component is open in M by the proposition above, so the collection of components forms an open cover of M. Because M is second-countable, this cover must have a countable subcover. But since the components are all disjoint, the cover must have been countable to begin with, which is to say that M has only countably many components. Because the components are open, they are connected topological manifolds in the subspace topology. □

1.6 Local Compactness and Paracompactness

1.6.1 Proposition (Manifolds are Locally Compact)

Every topological manifold is locally compact.

Proof: The previous Lemma showed that every manifold has a basis of precompact open subsets. □

1.6.2 Definition

Let M be a topological space.

A collection $\mathcal{X}$ of subsets of M is said to be **locally finite** if each point of M has a neighborhood that intersects only finitely many sets in $\mathcal{X}$. (All intersections all finite)

Given a cover $\mathcal{U}$ of M, another cover $\mathcal{V}$ is called a **refinement** of $\mathcal{U}$ if for each $V \in \mathcal{V}$, there exists some $U \in \mathcal{U}$ such that $V \subseteq U$. (some V is contained in some U)

We say that M is **paracompact** if every open cover of M admits an open, locally finite refinement.

1.6.3 Lemma

Suppose $\mathcal{X}$ is a locally finite collection of subsets of a topological space M.

(a) The collection $\left\{\overline{X} : X \in \mathcal{X}\right\}$ is also locally finite.

(b) $\displaystyle\bigcup_{X \in \mathcal{X}} \overline{X} = \overline{\bigcup_{X \in \mathcal{X}} X}$.

1.6.4 theorem [Manifolds Are Paracompact]

Every topological manifold is paracompact. In fact, given a topological manifold M, an open cover $\mathcal{X}$ of M, and any basis $\mathcal{B}$ for the topology of M, there exists a countable, locally finite open refinement of $\mathcal{X}$ consisting of elements of $\mathcal{B}$.

Proof. Given M, $\mathcal{X}$, and $\mathcal{B}$ as in the hypothesis of the theorem, let $(K_j)_{j=1}^{\infty}$ be an exhaustion of M by compact sets (Proposition A.60). For each j, let

$$V_j = \operatorname{Int} K_{j+1} \setminus K_{j-1} \quad \text{and} \quad W_j = \operatorname{Int} K_{j+2} \setminus K_{j-1}$$

(where we interpret $K_j = \varnothing$ if $j < 1$). Then V_j is a compact set contained in the open subset W_j. For each $x \in V_j$, there is some $X_x \in \mathcal{X}$ containing x, and because $\mathcal{B}$ is a basis, there exists $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq X_x \cap W_j$.

The collection of all such sets B_x as x ranges over V_j is an open cover of V_j, and thus has a finite subcover. The union of all such finite subcovers as j ranges over the positive integers is a countable open cover of M that refines $\mathcal{X}$. Because the finite subcover of V_j consists of sets contained in W_j, and $W_j \cap W_{j'} = \varnothing$ except when $j-2 \le j' \le j+2$, the resulting cover is locally finite. □

1.7 Fundamental Groups of Manifolds

Proposition *The fundamental group of a topological manifold is countable.*

Proof. Let M be a topological manifold. By Lemma 1.10, there is a countable collection $\mathcal{B}$ of coordinate balls covering M. For any pair of coordinate balls $B, B' \in \mathcal{B}$, the intersection $B \cap B'$ has at most countably many components, each of which is path-connected. Let $\mathcal{X}$ be a countable set containing a point from each component of $B \cap B'$ for each $B, B' \in \mathcal{B}$ (including $B = B'$). For each $B \in \mathcal{B}$ and each $x, x' \in \mathcal{X}$ such that $x, x' \in B$, let $h^B_{x,x'}$ be some path from x to x' in B.

Since the fundamental groups based at any two points in the same component of M are isomorphic, and $\mathcal{X}$ contains at least one point in each component of M, we may as well choose a point $p \in \mathcal{X}$ as base point. Define a *special loop* to be a loop based at p that is equal to a finite product of paths of the form $h^B_{x,x'}$. Clearly, the set of special loops is countable, and each special loop determines an element of $\pi_1(M, p)$. To show that $\pi_1(M, p)$ is countable, therefore, it suffices to show that each element of $\pi_1(M, p)$ is represented by a special loop.

Suppose $f : [0, 1] \to M$ is a loop based at p. The collection of components of sets of the form $f^{-1}(B)$ as B ranges over $\mathcal{B}$ is an open cover of $[0, 1]$, so by compactness it has a finite subcover. Thus, there are finitely many numbers

$$0 = a_0 < a_1 < \cdots < a_k = 1$$

such that $[a_{i-1}, a_i] \subseteq f^{-1}(B)$ for some $B \subseteq \mathcal{B}$. For each i, let f_i be the restriction of f to the interval $[a_{i-1}, a_i]$, reparametrized so that its domain is $[0, 1]$, and let $B_i \in \mathcal{B}$ be a coordinate ball containing the image of f_i. For each i, we have $f(a_i) \in B_i \cap B_{i+1}$, and there is some $x_i \in \mathcal{X}$ that lies in the same component of $B_i \cap B_{i+1}$ as $f(a_i)$. Let g_i be a path in $B_i \cap B_{i+1}$ from x_i to $f(a_i)$ (Fig. 1.5), with the understanding that $x_0 = x_k = p$, and g_0 and g_k are both equal to the constant path c_p based at p. Then, because $\overline{g}_i \cdot g_i$ is path-homotopic to a constant path (where $\overline{g}_i(t) = g_i(1 - t)$ is the reverse path of g_i),

$$\begin{aligned} f &\sim f_1 \cdot \cdots \cdot f_k \\ &\sim g_0 \cdot f_1 \cdot \overline{g}_1 \cdot g_1 \cdot f_2 \cdot \overline{g}_2 \cdots \overline{g}_{k-1} \cdot g_{k-1} \cdot f_k \cdot \overline{g}_k \\ &\sim \widetilde{f}_1 \cdot \widetilde{f}_2 \cdots \widetilde{f}_k, \end{aligned}$$

where $\widetilde{f}_i = g_{i-1} \cdot f_i \cdot \overline{g}_i$. For each i, $\widetilde{f}_i$ is a path in B_i from x_{i-1} to x_i. Since B_i is simply connected, $\widetilde{f}_i$ is path-homotopic to $h^{B_i}_{x_{i-1},x_i}$. It follows that f is path-homotopic to a special loop, as claimed. □

1.8 Smooth Structures

1.8.1 Definition (Smooth)

If U and V are open subsets of Euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, a function $F : U \to V$ is said to be **smooth** (or C^∞, or **infinitely differen-**

tiable) if each of its component functions has continuous partial derivatives of all orders. If in addition F is bijective and has a smooth inverse map, it is called a **diffeomorphism**. A diffeomorphism is, in particular, a homeomorphism.

1.8.2 Definition (Transition Map)

Let M be a topological n-manifold. If (U, φ), (V, ψ) are two charts such that $U \cap V \neq \emptyset$, the composite map $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is called the ***transition map from*** φ ***to*** ψ. It is a composition of homeomorphisms, and is therefore itself a homeomorphism. Two charts (U, φ) and (V, ψ) are said to be ***smoothly compatible*** if either $U \cap V = \emptyset$ or the transition map $\psi \circ \varphi^{-1}$ is a diffeomorphism. Since $\varphi(U \cap V)$ and $\psi(U \cap V)$ are open subsets of $\mathbb{R}^n$, smoothness of this map is to be interpreted in the ordinary sense of having continuous partial derivatives of all orders.

We define an ***atlas for*** M to be a collection of charts whose domains cover M. An atlas $\mathcal{A}$ is called a ***smooth atlas*** if any two charts in $\mathcal{A}$ are smoothly compatible with each other.

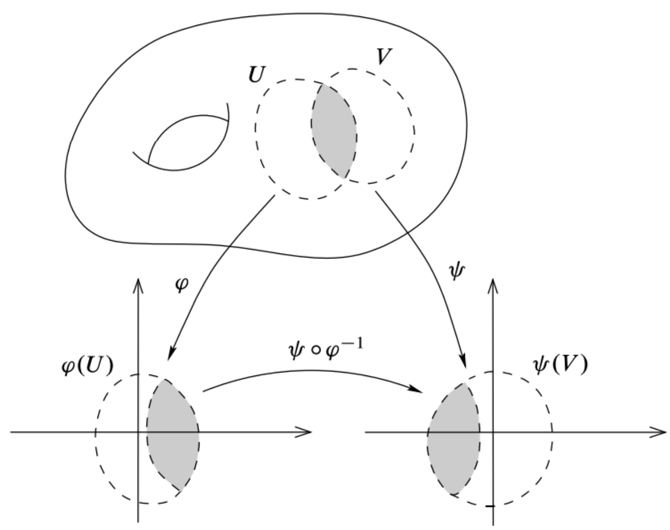


Fig. 1.6 A transition map

1.8.3 Definition (Smooth Structure)

A smooth atlas $\mathcal{A}$ on M is **maximal** if it is not properly contained in any larger smooth atlas. If M is a topological manifold, a **smooth structure on M** is a maximal smooth atlas. A **smooth manifold** is a pair $(M, \mathcal{A})$, where M is a topological manifold and $\mathcal{A}$ is a smooth structure on M. When the smooth structure is understood, we usually omit mention of it and just say "M is a smooth manifold." Smooth structures are also called **differentiable structures** or C^∞ **structures** by some authors. We also use the term **smooth manifold structure** to mean a manifold topology together with a smooth structure.

1.8.4 Proposition

Let M be a topological manifold.

(a) Every smooth atlas $\mathcal{A}$ for M is contained in a unique maximal smooth atlas, called the smooth structure determined by $\mathcal{A}$.

(b) Two smooth atlases for M determine the same smooth structure if and only if their union is a smooth atlas.

Proof. Let $\mathcal{A}$ be a smooth atlas for M, and let $\bar{\mathcal{A}}$ denote the set of all charts that are smoothly compatible with every chart in $\mathcal{A}$. To show that $\bar{\mathcal{A}}$ is a smooth atlas, we need to show that any two charts of $\bar{\mathcal{A}}$ are smoothly compatible with each other. That is, for any $(U,\varphi),(V,\psi) \in \bar{\mathcal{A}}$, the map

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$$

is smooth.

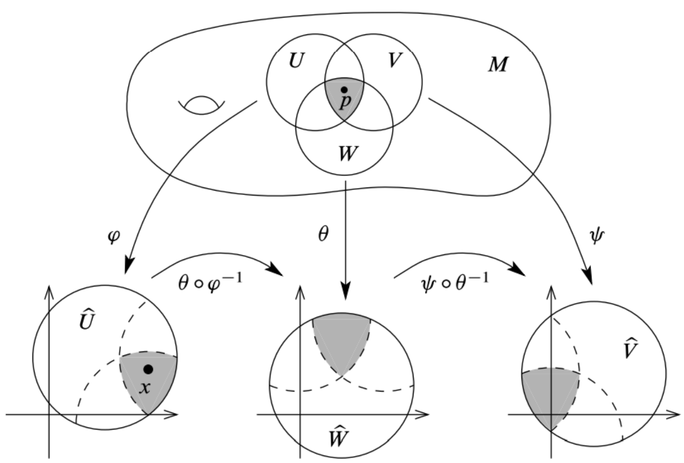


Figure 1.1: Proof of the proposition.

Let $x = \varphi(p) \in \varphi(U \cap V)$ be arbitrary. Because the domains of the charts in $\mathcal{A}$ cover M, there exists a chart $(W,\theta) \in \mathcal{A}$ such that $p \in W$. Since every chart in $\bar{\mathcal{A}}$ is smoothly compatible with (W,θ), both of the maps $\theta \circ \varphi^{-1}$ and $\psi \circ \theta^{-1}$ are smooth where defined. Since $p \in U \cap V \cap W$, it follows that

$$\psi \circ \varphi^{-1} = (\psi \circ \theta^{-1}) \circ (\theta \circ \varphi^{-1})$$

is smooth on a neighborhood of x. Thus, $\psi \circ \varphi^{-1}$ is smooth in a neighborhood of each point in $\varphi(U \cap V)$, so it is smooth on all of $\varphi(U \cap V)$. Therefore, $\bar{\mathcal{A}}$ is a smooth atlas.

To check that it is maximal, note that any chart that is smoothly compatible with every chart in $\bar{\mathcal{A}}$ is, in particular, smoothly compatible with every chart in $\mathcal{A}$. Hence, it is already in $\bar{\mathcal{A}}$ by definition. This proves the existence of a maximal smooth atlas containing $\mathcal{A}$.

If $\mathcal{B}$ is any other maximal smooth atlas containing $\mathcal{A}$, then each chart in $\mathcal{B}$ is smoothly compatible with each chart in $\mathcal{A}$, so $\mathcal{B} \subseteq \bar{\mathcal{A}}$. But by maximality of $\mathcal{B}$, we also have $\bar{\mathcal{A}} \subseteq \mathcal{B}$, and therefore $\mathcal{B} = \bar{\mathcal{A}}$.

The proof of (b) is left as exercise □

1.9 Local Coordinates Representation

1.9.1 Definition

If M is a *smooth manifold*, any chart (U, φ) contained in the given maximal smooth atlas is called a ***smooth chart***, and the corresponding coordinate map φ is called a ***smooth coordinate map***. It is useful also to introduce the terms ***smooth coordinate domain*** or ***smooth coordinate neighborhood*** for the domain of a smooth coordinate chart. A ***smooth coordinate ball*** means a smooth coordinate domain whose image under a smooth coordinate map is a ball in Euclidean space. A ***smooth coordinate cube*** is defined similarly.

It is often useful to restrict attention to coordinate balls whose closures sit nicely inside larger coordinate balls. We say a set $B \subseteq M$ is a ***regular coordinate ball*** if there is a smooth coordinate ball $B' \supseteq \overline{B}$ and a smooth coordinate map

$$\varphi \colon B' \to \mathbb{R}^n$$

such that for some positive real numbers $r < r'$,

$$\varphi(B) = B_r(0), \quad \varphi\big(\overline{B}\big) = \overline{B}_r(0), \quad \varphi(B') = B_{r'}(0).$$

Because $\overline{B}$ is homeomorphic to $\overline{B}_r(0)$, it is compact, and thus every regular coordinate ball is precompact in M.

1.9.2 Proposition

Every smooth manifold has a countable basis of regular coordinate balls.

1.9.3 Note

Here is how one usually thinks about coordinate charts on a smooth manifold. Once we choose a smooth chart (U, φ) on M, the coordinate map $\varphi \colon U \to \widehat{U} \subset \mathbb{R}^n$ can be thought of as giving a temporary ***identification*** between U and $\widehat{U}$. Using this identification, while we work in this chart, we can think of U simultaneously as an open subset of M and as an open subset of $\mathbb{R}^n$. You can visualize this identification by thinking of a "grid" drawn on U representing the preimages of the coordinate lines under φ (Fig. 1.2). Under this identification,

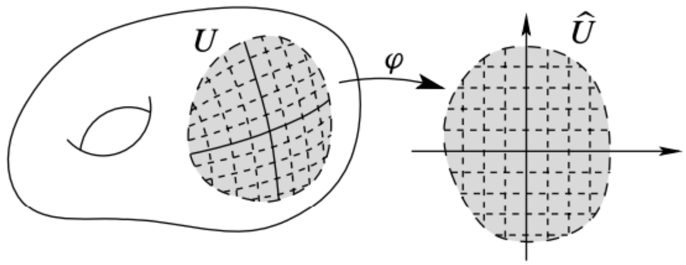


Fig. 1.8 A coordinate grid

Figure 1.2: Coordinate Grid

we can represent a point $p \in U$ by its coordinates

$$(x^1, \dots, x^n) = \varphi(p),$$

(for example, if a point $p \in U$ has a coordinate (2,1) inside U, after changing the system of coordinate, it may become (3,10) in $\widehat{U}$) and think of this n-tuple as *being* the point p. We typically express this by saying "$(x^1, \dots, x^n)$ is the (local) coordinate representation for p" or "$p = (x^1, \dots, x^n)$ in local coordinates."

1.10 Examples of Smooth Manifolds:

1.10.1 Example (0-Dimensional Manifolds)

A topological manifold M of dimension 0 is just a countable discrete space. For each point $p \in M$, the only neighborhood of p that is homeomorphic to an open subset of $\mathbb{R}^0$ is $\{p\}$ itself, and there is exactly one coordinate map $\varphi\colon \{p\} \to \mathbb{R}^0$. Thus, the set of all charts on M trivially satisfies the smooth compatibility condition, and each 0-dimensional manifold has a unique smooth structure.

1.10.2 Example (Euclidean Spaces)

For each nonnegative integer n, the Euclidean space $\mathbb{R}^n$ is a smooth n-manifold with the smooth structure determined by the atlas consisting of the single chart $(\mathbb{R}^n, \mathrm{Id}_{\mathbb{R}^n})$. We call this the ***standard smooth structure on*** $\mathbb{R}^n$ and the resulting coordinate map ***standard coordinates***. Unless we explicitly specify otherwise, we always use this smooth structure on $\mathbb{R}^n$. With respect to this smooth structure, the smooth coordinate charts for $\mathbb{R}^n$ are exactly those charts (U, φ) such that φ is a diffeomorphism (in the sense of ordinary calculus) from U to another open subset $\widehat{U} \subseteq \mathbb{R}^n$.

1.10.3 Example (Another Smooth Structure on $\mathbb{R}$)

Consider the homeomorphism $\psi\colon \mathbb{R} \to \mathbb{R}$ given by

$$\psi(x) = x^3.$$

The atlas consisting of the single chart $(\mathbb{R}, \psi)$ defines a smooth structure on $\mathbb{R}$. This chart is not smoothly compatible with the standard smooth structure, because the transition map $\mathrm{Id}_{\mathbb{R}} \circ \psi^{-1}(y) = y^{1/3}$ is not smooth at the origin. Therefore, the smooth structure defined on $\mathbb{R}$ by ψ is not the same as the standard one. Using similar ideas, it is not hard to construct many distinct smooth structures on any given positive-dimensional topological manifold, as long as it has one smooth structure to begin with.

1.10.4 Example (Finite-Dimensional Vector Spaces)

Let V be a finite-dimensional real vector space. Any norm on V determines a topology, which is independent of the choice of norm (Exercise B.49). With this topology, V is a topological n-manifold, and has a natural smooth structure defined as follows. Each (ordered) basis $(E_1, \ldots, E_n)$ for V defines a basis isomorphism

$$E\colon \mathbb{R}^n \to V$$

by

$$E(x) = \sum_{i=1}^{n} x^i E_i.$$

(Notice each x^i is the basis for each direction and if one combine all components of a vector through every direction, one will get the resulting vector) This map is a homeomorphism, so (V, E^{-1}) is a chart. If $(\widetilde{E}_1, \ldots, \widetilde{E}_n)$ is any other basis and

$$\widetilde{E}(x) = \sum_{j} x^j \, \widetilde{E}_j$$

is the corresponding isomorphism, then there is some invertible matrix (A_i^j) such that

$$E_i = \sum_{j} A_i^j \, \widetilde{E}_j$$

for each i. The transition map between the two charts is then given by

$$\widetilde{E}^{-1} \circ E(x) = \widetilde{x},$$

where $\widetilde{x} = (\widetilde{x}^1, \ldots, \widetilde{x}^n)$ is determined by

$$\sum_{j=1}^{n} \widetilde{x}^j \, \widetilde{E}_j = \sum_{i=1}^{n} x^i \, E_i = \sum_{i,j=1}^{n} x^i \, A_i^j \, \widetilde{E}_j.$$

It follows that

$$\widetilde{x}^j = \sum_i A^j_i x^i.$$

Thus, the map sending x to $\widetilde{x}$ is an invertible linear map and hence a diffeomorphism, so any two such charts are smoothly compatible. The collection of all such charts thus defines a smooth structure, called the **standard smooth structure on V**.

1.10.5 Example (Spaces of Matrices)

Let $M(m \times n, \mathbb{R})$ denote the set of $m \times n$ matrices with real entries. Because it is a real vector space of dimension mn under matrix addition and scalar multiplication, $M(m \times n, \mathbb{R})$ is a smooth mn-dimensional manifold. (In fact, it is often useful to *identify* $M(m \times n, \mathbb{R})$ with $\mathbb{R}^{mn}$, just by stringing all the matrix entries out in a single row.) Similarly, the space $M(m \times n, \mathbb{C})$ of $m \times n$ complex matrices is a vector space of dimension $2mn$ over $\mathbb{R}$, and thus a smooth manifold of dimension $2mn$. (Here in this case, we put the matrix under $\mathbb{R}^n$, and notice $\mathbb{C} \cong \mathbb{R}^2$ since $a + ib \mapsto (a, b)$) In the special case in which $m = n$ (square matrices), we abbreviate $M(n \times n, \mathbb{R})$ and $M(n \times n, \mathbb{C})$ by $M(n, \mathbb{R})$ and $M(n, \mathbb{C})$, respectively.

1.10.6 Example (Open Submanifolds)

Let U be any open subset of $\mathbb{R}^n$. Then U is a topological n-manifold, and the single chart (U, Id_U) defines a smooth structure on U.

More generally, let M be a smooth n-manifold and let $U \subseteq M$ be any open subset. Define an atlas on U by

$$\mathcal{A}_U = \{\text{smooth charts } (V, \varphi) \text{ for } M \mid V \subseteq U\}.$$

Every point $p \in U$ is contained in the domain of some chart (W, φ) for M; if we set $V = W \cap U$, then $(V, \varphi|_V)$ is a chart in $\mathcal{A}_U$ whose domain contains p. Therefore, U is covered by the domains of charts in $\mathcal{A}_U$, and it is easy to verify that this is a smooth atlas for U. Thus any open subset of M is itself a smooth n-manifold in a natural way. Endowed with this smooth structure, we call any open subset an ***open submanifold* of M**.

1.10.7 Example (The General Linear Group)

The ***general linear group*** $\mathrm{GL}(n, \mathbb{R})$ is the set of invertible $n \times n$ matrices with real entries. It is a smooth n^2-dimensional manifold because it is an open subset of the n^2-dimensional vector space $M(n, \mathbb{R})$, namely the set where the (continuous) determinant function is nonzero.

1.10.8 Theorem C.40 (Implicit Function Theorem)

Let $U \subseteq \mathbb{R}^n \times \mathbb{R}^k$ be an open subset, and let $(x, y) = (x^1, \dots, x^n, y^1, \dots, y^k)$ denote the standard coordinates on U. Suppose $\Phi\colon U \to \mathbb{R}^k$ is a smooth function, $(a, b) \in U$, and $c = \Phi(a, b)$. If the $k \times k$ matrix

$$\Big(\frac{\partial \Phi^i}{\partial y^j}(a, b)\Big)$$

is nonsingular, then there exist neighborhoods $V_0 \subseteq \mathbb{R}^n$ of a and $W_0 \subseteq \mathbb{R}^k$ of b and a smooth function $F\colon V_0 \to W_0$ such that $\Phi^{-1}(c) \cap (V_0 \times W_0)$ is the graph of F; that is,

$$\Phi(x, y) = c \quad \text{for } (x, y) \in V_0 \times W_0 \quad \iff \quad y = F(x).$$

Proof. Consider the smooth function

$$\Psi\colon U \to \mathbb{R}^n \times \mathbb{R}^k, \qquad \Psi(x, y) = (x, \Phi(x, y)).$$

Its total derivative at (a, b) is

$$D\Psi(a, b) = \begin{pmatrix} I_n & 0 \\ \dfrac{\partial \Phi^i}{\partial x^j}(a, b) & \dfrac{\partial \Phi^i}{\partial y^j}(a, b) \end{pmatrix},$$

which is nonsingular because it is block lower-triangular with nonsingular diagonal blocks. Thus by the inverse function theorem there exist connected neighborhoods U_0 of (a, b) and Y_0 of (c, b) such that $\Psi\colon U_0 \to Y_0$ is a diffeomorphism. Shrinking U_0 and Y_0 if necessary, we may assume $U_0 = V \times W$ is a product neighborhood.

Writing

$$\Psi^{-1}(x, y) = (A(x, y), B(x, y))$$

for some smooth functions A and B, we compute

$$(x, y) = \Psi\big(\Psi^{-1}(x, y)\big) = \Psi\big(A(x, y), B(x, y)\big) = \big(A(x, y),\ \Phi\big(A(x, y), B(x, y)\big)\big).$$

Comparing the first components in this equation, we find that

$$A(x, y) = x,$$

so Ψ^{-1} has the form

$$\Psi^{-1}(x, y) = (x, B(x, y)).$$

Now let

$$V_0 = \{\, x \in V : (x, c) \in Y_0 \}, \quad W_0 = W,$$

and define $F\colon V_0 \to W_0$ by $F(x) = B(x, c)$. Comparing the second components in (C.21) yields

$$c = \Phi\big(x, B(x, c)\big) = \Phi\big(x, F(x)\big)$$

whenever $x \in V_0$, so the graph of F is contained in $\Phi^{-1}(c)$. Conversely, suppose $(x, y) \in V_0 \times W_0$ and $\Phi(x, y) = c$. Then

$$\Psi(x, y) = (x, \Phi(x, y)) = (x, c),$$

so

$$(x, y) = \Psi^{-1}(x, c) = (x, B(x, c)) = (x, F(x)),$$

which implies that $y = F(x)$. This completes the proof. □

1.10.9 Example (Level Sets)

Suppose $U \subseteq \mathbb{R}^n$ is an open subset and $\Phi\colon U \to \mathbb{R}$ is a smooth function. For any $c \in \mathbb{R}$, the set $\Phi^{-1}(c)$ is called a ***level set of*** Φ. Choose some $c \in \mathbb{R}$, let $M = \Phi^{-1}(c)$, and suppose it happens that the total derivative $D\Phi(a)$ is nonzero for each $a \in \Phi^{-1}(c)$. Because $D\Phi(a)$ is a row matrix whose entries are the partial derivatives

$$\big(\tfrac{\partial \Phi}{\partial x^1}(a), \ldots, \tfrac{\partial \Phi}{\partial x^n}(a)\big),$$

for each $a \in M$ there is some i such that $\frac{\partial \Phi}{\partial x^i}(a) \neq 0$. It follows from the implicit function theorem (Theorem C.40, with x^i playing the role of y) that there is a neighborhood U_0 of a such that $M \cap U_0$ can be expressed as the graph of an equation of the form

$$x^i \;=\; f\big(x^1, \ldots, \widehat{x^i}, \ldots, x^n\big),$$

for some smooth real-valued function f defined on an open subset of $\mathbb{R}^{n-1}$. Therefore, arguing just as in the case of the n-sphere, we see that M is a topological manifold of dimension (n - 1), and has a smooth structure such that each of the graph coordinate charts associated with a choice of f as above is a smooth chart.

1.10.10 Example (Projective Spaces)

The n-dimensional real projective space $\mathbb{RP}^n$ is a topological n-manifold by Example 1.5. Let us check that the coordinate charts (U_i, φ_i) constructed in that example are all smoothly compatible. Assuming for convenience that $i > j$, it is straightforward to compute that

$$\varphi_j \circ \varphi_i^{-1}(u^1, \ldots, u^n) = \left(\tfrac{u^1}{u^j}, \ldots, \tfrac{u^{j-1}}{u^j}, \tfrac{u^{j+1}}{u^j}, \ldots, \tfrac{u^{i-1}}{u^j}, \tfrac{1}{u^j}, \tfrac{u^i}{u^j}, \ldots, \tfrac{u^n}{u^j}\right),$$

which is a diffeomorphism from $\varphi_i(U_i \cap U_j)$ to $\varphi_j(U_i \cap U_j)$.

1.10.11 Example (Smooth Product Manifolds)

If $M_1, \ldots, M_k$ are smooth manifolds of dimensions $n_1, \ldots, n_k$, respectively, we showed in Example 1.8 that the product space

$$M_1 \times \cdots \times M_k$$

is a topological manifold of dimension $n_1 + \cdots + n_k$, with charts of the form

$$\big(U_1 \times \cdots \times U_k,\ \varphi_1 \times \cdots \times \varphi_k\big).$$

Any two such charts are smoothly compatible because, as is easily verified,

$$(\psi_1 \times \cdots \times \psi_k) \circ (\varphi_1 \times \cdots \times \varphi_k)^{-1} \ = \ (\psi_1 \circ \varphi_1^{-1}) \times \cdots \times (\psi_k \circ \varphi_k^{-1}),$$

which is a smooth map. This defines a natural smooth manifold structure on the product, called the *product smooth manifold structure*. For example, this yields a smooth manifold structure on the n-torus

$$\mathbb{T}^n \ = \ S^1 \times \cdots \times S^1.$$

1.10.12 Lemma (Smooth Manifold Chart)

Let M be a set, and suppose we are given a collection $\{U_\alpha\}$ of subsets of M together with maps $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$, such that the following properties are satisfied:

(i) For each α, φ_α is a bijection between U_α and an open subset $\varphi_\alpha(U_\alpha) \subseteq \mathbb{R}^n$.

(ii) For each α and β, the sets $\varphi_\alpha(U_\alpha \cap U_\beta)$ and $\varphi_\beta(U_\alpha \cap U_\beta)$ are open in $\mathbb{R}^n$.

(iii) Whenever $U_\alpha \cap U_\beta \neq \emptyset$, the map

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

is smooth.

(iv) Countably many of the sets U_α cover M.

(v) Whenever p, q are distinct points in M, either there exists some U_α containing both p and q, or there exist disjoint sets U_α, U_β with $p \in U_\alpha$ and $q \in U_\beta$.

Then M has a unique smooth manifold structure such that each $(U_\alpha, \varphi_\alpha)$ is a smooth chart.

Proof. We define the topology by taking all sets of the form $\varphi_\alpha^{-1}(V)$, with V an open subset of $\mathbb{R}^n$, as a basis. To prove that this is a basis for a topology, we need to show that for any point p in the intersection of two basis sets $\varphi_\alpha^{-1}(V)$ and $\varphi_\beta^{-1}(W)$, there is a third basis set containing p and contained in the intersection. It suffices to show that

$$\varphi_\alpha^{-1}(V) \cap \varphi_\beta^{-1}(W)$$

is itself a basis set. To see this, observe that (iii) implies that

$$(\varphi_\beta \circ \varphi_\alpha^{-1})^{-1}(W)$$

is an open subset of $\varphi_\alpha(U_\alpha \cap U_\beta)$, and (ii) implies that this set is also open in $\mathbb{R}^n$. It follows that

$$\varphi_\alpha^{-1}(V) \cap \varphi_\beta^{-1}(W) = \varphi_\alpha^{-1}\left(V \cap (\varphi_\beta \circ \varphi_\alpha^{-1})^{-1}(W)\right)$$

is also a basis set, as claimed.

Each map φ_α is then a homeomorphism onto its image (essentially by definition), so M is locally Euclidean of dimension n. The Hausdorff property follows easily from (v), and second-countability follows from (iv) and the result of Exercise A.22, because each U_α is second-countable. Finally, (iii) guarantees that the collection $\{(U_\alpha, \varphi_\alpha)\}$ is a smooth atlas. It is clear that this topology and smooth structure are the unique ones satisfying the conclusions of the lemma. □

1.11 Einstein Summation Convention

This is a good place to pause and introduce an **important notational convention** that is commonly used in the study of smooth manifolds. Because of the proliferation of summations such as

$$\sum_i x^i E_i$$

in this subject, we often abbreviate such a sum by omitting the summation sign, as in

$$E(x) = x^i E_i,$$

an abbreviation for

$$E(x) = \sum_{i=1}^n x^i E_i.$$

We interpret any such expression according to the following rule, called the **Einstein summation convention**: if the same index name (such as i in the expression above) appears exactly twice in any monomial term, once as an upper index and once as a lower index, that term is understood to be summed over all possible values of that index, generally from 1 to the dimension of the space in question. This simple idea was introduced by Einstein to reduce the complexity of expressions arising in the study of smooth manifolds by eliminating the necessity of explicitly writing summation signs.

Another important aspect of the summation convention is the positions of the indices. **We always write basis vectors (such as E_i) with lower indices, and components of a vector with respect to a basis (such as x^i) with upper indices.** These index conventions help to ensure that, in summations that make mathematical sense, each index to be summed over typically appears twice in any given term, once as a lower index and once as an upper index. Any index that is implicitly summed over is a dummy index, meaning that the value of such an expression is unchanged if a different name is substituted for each dummy index. For example, $x^i E_i$ and $x^j E_j$ mean exactly the same thing.

Since the coordinates of a point $(x^1, \ldots, x^n) \in \mathbb{R}^n$ are also its components with respect to the standard basis, in order to be consistent with our convention of writing components of vectors with upper indices, we need to use upper indices for these coordinates, and we do so throughout this book. Although this may seem awkward at first, in combination with the summation convention it offers enormous advantages when we work with complicated indexed sums, not the least of which is that expressions that are not mathematically meaningful often betray themselves quickly by violating the index convention. (The main exceptions are expressions involving the Euclidean dot product $x \cdot y = \sum_i x^i y^i$, in which the same index appears twice in the upper position, and the standard symplectic form on $\mathbb{R}^{2n}$, which we will define in Chapter 22. We always explicitly write summation signs in such expressions.)

1.12 Manifolds with Boundary

1.12.1 Definition

Points in smooth manifolds will have neighborhoods modeled either on open subsets of $\mathbb{R}^n$ or on open subsets of the *closed n-dimensional upper half-space* $\mathbb{H}^n \subset \mathbb{R}^n$, defined as

$$\mathbb{H}^n = \left\{(x^1, \ldots, x^n) \in \mathbb{R}^n : x^n \geq 0\right\}.$$

We will use the notations $\operatorname{Int} \mathbb{H}^n$ and $\partial\mathbb{H}^n$ to denote the interior (largest open subset) and boundary of $\mathbb{H}^n$, respectively, as a subset of $\mathbb{R}^n$. When $n > 0$, this means

$$\operatorname{Int} \mathbb{H}^n = \left\{(x^1, \ldots, x^n) \in \mathbb{R}^n : x^n > 0\right\},$$

$$\partial\mathbb{H}^n = \left\{(x^1, \ldots, x^n) \in \mathbb{R}^n : x^n = 0\right\}.$$

In the $n = 0$ case, $\mathbb{H}^0 = \mathbb{R}^0 = \{0\}$, so $\operatorname{Int} \mathbb{H}^0 = \mathbb{R}^0$ and $\partial\mathbb{H}^0 = \varnothing$.

An n-dimensional topological manifold with boundary is a second-countable Hausdorff space M in which every point has a neighborhood homeomorphic either to an open subset of $\mathbb{R}^n$ or to a (relatively) open subset of $\mathbb{H}^n$.

An open subset $U \subseteq M$ together with a map $\varphi : U \to \mathbb{R}^n$ that is a homeomorphism onto an open subset of $\mathbb{R}^n$ or $\mathbb{H}^n$ will be called a ***chart for*** M, just as in the case of manifolds.

When it is necessary to make the distinction, we will call (U, φ) an *interior chart* if $\varphi(U)$ is an open subset of $\mathbb{R}^n$ (which includes the case of an open subset of $\mathbb{H}^n$ that does not intersect $\partial\mathbb{H}^n$), and a *boundary chart* if $\varphi(U)$ is an open subset of $\mathbb{H}^n$ such that $\varphi(U) \cap \partial\mathbb{H}^n \neq \varnothing$.

A boundary chart whose image is a set of the form $B_r(x) \cap \mathbb{H}^n$ for some $x \in \partial\mathbb{H}^n$ and $r > 0$ is called a *coordinate half-ball.*

A point $p \in M$ is called an *interior point of* M if it is in the domain of some interior chart. It is a *boundary point of* M if it is in the domain of a boundary chart that sends p to $\partial\mathbb{H}^n$.

The *boundary of* M (the set of all its boundary points) is denoted by ∂M; similarly, its *interior*, the set of all its interior points, is denoted by $\operatorname{Int} M$.

It follows from the definition that each point $p \in M$ is either an interior point or a boundary point: if p is not a boundary point, then either it is in the domain of an interior chart or it is in the domain of a boundary chart (U, φ) such that $\varphi(p) \notin \partial\mathbb{H}^n$, in which case the restriction of φ to $U \cap \varphi^{-1}(Int\mathbb{H}^n)$ is an interior chart whose domain contains p. However, it is not obvious that a given point cannot be simultaneously an interior point with respect to another. In fact, this cannot happen, but the proof requires more machinery than we have available at this point. For convenience, we state the theorem here.

1.12.2 Theorem (Topological Invariance of the Boundary)

If M is a topological manifold with boundary, then each point of M is either a boundary point or an interior point (x is an interior point of S if there exists an open ball centered at x which is completely contained in S), but not both. Thus ∂M and Int M are disjoint sets whose union is M .

1.12.3 Note

Only part of Manifold Boundary and Smooth Structures on Manifolds with Boundary are included from the textbook is included in this note.

Chapter 2

Smooth Maps

2.2 Smooth Functions and Smooth Maps

2.2.1 Smooth Functions on Manifolds

Definition 2.2.1 [smooth function]. Suppose M is a smooth n-manifold, k is a nonnegative integer, and

$$f \colon M \to \mathbb{R}^k$$

is any function. We say that f is a ***smooth function*** if for every $p \in M$, there exists a smooth chart (U, φ) for M whose domain contains p and such that the composite

$$f \circ \varphi^{-1}$$

is smooth on the open subset

$$\widehat{U} = \varphi(U) \subseteq \mathbb{R}^n$$

See figure 2.1 If M is a smooth manifold with boundary, the definition is exactly the same, except that $\varphi(U)$ is now an open subset of either $\mathbb{R}^n$ or $\mathbb{H}^n$, and in the latter case we interpret smoothness of

$$f \circ \varphi^{-1}$$

to mean that each point of $\varphi(U)$ has a neighborhood (in $\mathbb{R}^n$) on which $f \circ \varphi^{-1}$ extends to a smooth function in the ordinary sense.

The most important special case is that of smooth real-valued functions

$$f \colon M \to \mathbb{R};$$

the set of all such functions is denoted by $C^\infty(M)$. Because sums and constant multiples of smooth functions are smooth, $C^\infty(M)$ is a vector space over $\mathbb{R}$.

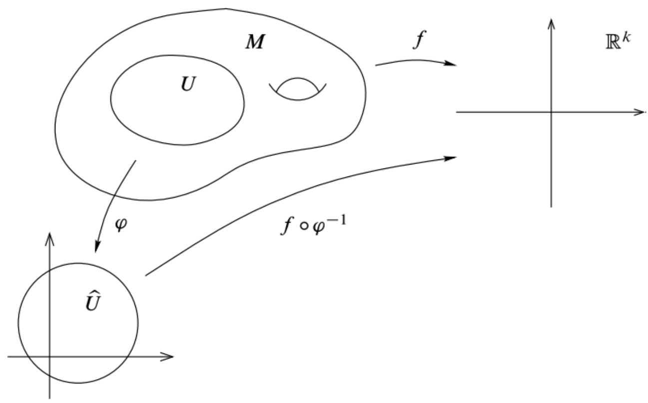


Figure 2.1: Definition of Smooth Functions

Definition 2.2.2 [coordinate representation]. Given a function $f\colon M \to \mathbb{R}^k$ and a chart (U, φ) for M, the function

$$\widehat{f}\colon \varphi(U) \to \mathbb{R}^k,$$

defined by

$$\widehat{f}(x) = f \circ \varphi^{-1}(x)$$

is called the ***coordinate representation*** of f. By definition, f is smooth if and only if its coordinate representation is smooth in some smooth chart around each point. By the preceding exercise, smooth functions have smooth coordinate representations in every smooth chart.

For example, consider the real-valued function

$$f(x, y) = x^2 + y^2$$

defined on the plane. In polar coordinates on, say, the set

$$U = \{(x, y)\colon x > 0\},$$

it has the coordinate representation

$$\widehat{f}(r, \theta) = r^2.$$

(Indeed,

$$x^2 + y^2 = (rcos\theta)^2 + (rsin\theta)^2 = r^2)$$

In keeping with our practice of using local coordinates to identify an open subset of a manifold with an open subset of Euclidean space, in cases where it causes no confusion we often do not even observe the distinction between $\widehat{f}$ and f itself, and instead say something like "f is smooth on U because its ***coordinate representation*** $f(r, \theta) = r^2$ is smooth."

2.2.2 Smooth Maps Between Manifolds

Definition 2.2.3 [Smooth map]. The definition of ***smooth functions*** generalizes easily to maps between manifolds. Let M, N be smooth manifolds, and let

$$F\colon M \to N$$

be any map. We say that F is a ***smooth map*** if for every $p \in M$, there exist smooth charts (U, φ) containing p and (V, ψ) containing $F(p)$ such that $F(U) \subseteq V$ and the composite map

$$\psi \circ F \circ \varphi^{-1}$$

is smooth from $\varphi(U)$ to $\psi(V)$ (Fig 2.2).

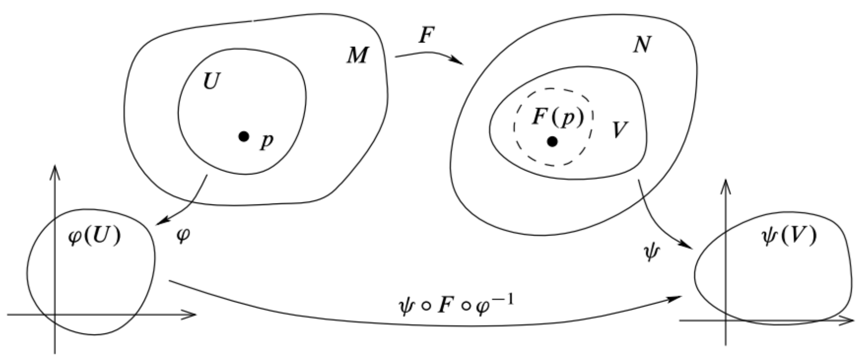


Fig. 2.2 Definition of smooth maps

Figure 2.2: Definition of smooth maps

If M and N are smooth manifolds with boundary, smoothness of F is defined in exactly the same way, with the usual understanding that a map whose domain is a subset of $\mathbb{H}^n$ is smooth if it admits an extension to a smooth map in a neighborhood of each point, and a map whose codomain is a subset of $\mathbb{H}^n$ is smooth if it is smooth as a map into $\mathbb{R}^n$. Note that our previous definition of smoothness of real-valued or vector-valued functions can be viewed as a special case of this one, by taking $N = V = \mathbb{R}^k$ and $\psi = \mathrm{Id}\colon \mathbb{R}^k \to \mathbb{R}^k$.

The first important observation about our definition of smooth maps is that, as one might expect, smoothness implies continuity.

Proposition 2.2.4. *Every smooth map is continuous*

Proof. Suppose M and N are smooth manifolds with or without boundary, and

$$F\colon M \to N$$

is smooth. Given $p \in M$, smoothness of F means there are smooth charts (U, φ) containing p and (V, ψ) containing $F(p)$ such that $F(U) \subseteq V$ and

$$\psi \circ F \circ \varphi^{-1} \colon \varphi(U) \to \psi(V)$$

is smooth, hence continuous. Since $\varphi \colon U \to \varphi(U)$ and $\psi \colon V \to \psi(V)$ are homeomorphisms, this implies in turn that

$$F\big|_U = \psi^{-1} \circ \left(\psi \circ F \circ \varphi^{-1}\right) \circ \varphi \colon U \to V.$$

which is a composition of continuous maps. Since F is continuous in a neighborhood of each point, it is continuous on M.

$\square$

Proposition 2.2.5 (***Equivalent Characterizations of Smoothness***)**.** *Suppose M and N are smooth manifolds with or without boundary, and*

$$F \colon M \to N$$

is a map. Then F is smooth if and only if either of the following conditions is satisfied:

(a) For every $p \in M$, there exist smooth charts (U, φ) containing p and (V, ψ) containing $F(p)$ such that $U \cap F^{-1}(V)$ is open in M and the composite map

$$\psi \circ F \circ \varphi^{-1} \colon \varphi\big(U \cap F^{-1}(V)\big) \to \psi(V)$$

is smooth.

(b) F is continuous and there exist smooth atlases $\{(U_\alpha, \varphi_\alpha)\}$ and $\{(V_\beta, \psi_\beta)\}$ for M and N, respectively, such that for each α, β, the map

$$\psi_\beta \circ F \circ \varphi_\alpha^{-1} \ : \ \varphi_\alpha\big(U_\alpha \cap F^{-1}(V_\beta)\big) \to \psi_\beta(V_\beta)$$

is smooth.

Proposition 2.2.6 (***Smoothness Is Local***)**.** *Let M and N be smooth manifolds with or without boundary, and let*

$$F \colon M \to N$$

be a map.

(a) If every point $p \in M$ has a neighborhood U such that the restriction $F\big|_U$ is smooth, then F is smooth.

(b) Conversely, if F is smooth, then its restriction to every open subset is smooth.

Corollary 2.2.7 (***Gluing Lemma for Smooth Maps***)**.** *Let M and N be smooth manifolds with or without boundary, and let $\{U_\alpha\}_{\alpha\in A}$ be an open cover of M. Suppose that for each $\alpha \in A$, we are given a smooth map*

$$F_\alpha\colon U_\alpha \to N$$

such that the maps agree on overlaps:

$$F_\alpha\big|_{U_\alpha\cap U_\beta} = F_\beta\big|_{U_\alpha\cap U_\beta} \quad \text{for all } \alpha,\beta.$$

Then there exists a unique smooth map

$$F\colon M \to N$$

such that

$$F\big|_{U_\alpha} = F_\alpha \quad \text{for each } \alpha \in A.$$

□

Definition 2.2.8 [coordinate representation of F]. If $F\colon M \to N$ is a smooth map, and (U,φ) and (V,ψ) are any smooth charts for M and N, respectively, we call

$$\widehat{F} = \psi \circ F \circ \varphi^{-1}$$

the ***coordinate representation of*** F with respect to the given coordinates. It maps the set

$$\varphi\big(U \cap F^{-1}(V)\big)$$

to $\psi(V)$. As with real-valued or vector-valued functions, once we have chosen specific local coordinates in both the domain and codomain, we can often ignore the distinction between F and Fy. Next we examine some simple classes of maps that are automatically smooth

Proposition 2.2.9. Let M, N, and P be smooth manifolds with or without boundary.

(a) Every constant map $c\colon M \to N$ is smooth.

(b) The identity map $\mathrm{id}_M\colon M \to M$ is smooth.

(c) If $U \subseteq M$ is an open submanifold (with or without boundary), then the inclusion $\iota\colon U \hookrightarrow M$ is smooth.

(d) If $F\colon M \to N$ and $G\colon N \to P$ are smooth, then so is $G \circ F\colon M \to P$.

Proof. We prove (d) and leave the rest as exercises. Let

$$F\colon M \to N, \quad G\colon N \to P$$

be smooth maps, and let $p \in M$. By definition of smoothness of G, there exist smooth charts (V,θ) containing $F(p)$ and (W,ψ) containing $G(F(p))$ such that $G(V) \subseteq W$ and

$$\psi \circ G \circ \theta^{-1}\colon \theta(V) \;\to\; \psi(W)$$

is smooth. Since F is continuous, $F^{-1}(V)$ is a neighborhood of p in M, so there is a smooth chart (U, φ) for M with

$$p \in U \subseteq F^{-1}(V).$$

By Exercise 2.9,

$$\theta \circ F \circ \varphi^{-1} \colon \varphi(U) \to \theta(V)$$

is smooth. Hence

$$\psi \circ (G \circ F) \circ \varphi^{-1} = \left(\psi \circ G \circ \theta^{-1}\right) \circ \left(\theta \circ F \circ \varphi^{-1}\right) \colon \varphi(U) \to \psi(W)$$

is smooth, being a composition of smooth maps between subsets of Euclidean space. □

Proposition 2.2.10. *Suppose $M_1, \ldots, M_k$ and N are smooth manifolds with or without boundary, such that at most one of $M_1, \ldots, M_k$ has nonempty boundary. For each i, let*

$$\pi_i \colon M_1 \times \cdots \times M_k \to M_i$$

denote the projection onto the ith factor. A map

$$F \colon N \to M_1 \times \cdots \times M_k$$

is smooth if and only if each of the component maps

$$F_i = \pi_i \circ F : N \to M_i$$

is smooth.

Note

On the other hand, because the definition of a smooth map requires smooth structures in the domain and codomain, if we say "$F : M \to N$ is a smooth map" without specifying what M and N are, it should always be understood that they are smooth manifolds with or without boundaries.

Note (Methods to prove smooth maps)

There are three common ways to do prove that a particular map is smooth:

- Write the map in smooth local coordinates and recognize its component functions as compositions of smooth elementary functions.
- Exhibit the map as a composition of maps that are known to be smooth.
- Use some special-purpose theorem that applies to the particular case under consideration.

Example 2.2.11 [*Smooth Maps*].

(a) Any map from a zero-dimensional manifold into a smooth manifold with or without boundary is automatically smooth, because each coordinate representation is constant.

(b) If the circle $\mathbb{S}^1$ is given its standard smooth structure, the map

$$\varepsilon\colon \mathbb{R} \to \mathbb{S}^1, \qquad \varepsilon(t) = e^{2\pi i t}$$

is smooth, because with respect to any angle coordinate θ for $\mathbb{S}^1$ (see Problem 1-8) it has a coordinate representation of the form

$$\widehat{\varepsilon}(t) = 2\pi t + c$$

for some constant c, as you can check.

(c) The map

$$\varepsilon^n\colon \mathbb{R}^n \to \mathbb{T}^n, \qquad \varepsilon^n(x^1, \ldots, x^n) = \left(e^{2\pi i x^1}, \ldots, e^{2\pi i x^n}\right)$$

is smooth by Proposition 2.2.10.

2.2.3 Diffeomorphism

Definition 2.2.12. If M and N are smooth manifolds with or without boundary, a ***diffeomorphism from M to N*** is a smooth bijective map $F : M \to N$ that has a smooth inverse. We say that ***M and N are diffeomorphic*** if there exists a diffeomorphism between them. Sometimes this is symbolized by $M \approx N$.

Example 2.2.13 [*Diffeomorphism*].

(a) Consider the maps $F : \mathbb{B}^n \to \mathbb{R}^n$ and $G : \mathbb{R}^n \to \mathbb{B}^n$ given by

$$F(x) = \frac{x}{\sqrt{1 - |x|^2}}, \qquad G(y) = \frac{y}{\sqrt{1 + |y|^2}}.$$

These maps are smooth, and it is straightforward to compute that they are inverses of each other. Thus they are both diffeomorphisms, and therefore $\mathbb{B}^n$ is diffeomorphic to $\mathbb{R}^n$.

(b) If M is any smooth manifold and (U, φ) is a smooth coordinate chart on M, then $\varphi : U \to \varphi(U) \subseteq \mathbb{R}^n$ is a diffeomorphism. (In fact, it has an identity map as a coordinate representation.)

Proposition 2.2.14 (Properties of Diffeomorphism)**.**

- *Every composition of diffeomorphisms is a diffeomorphism.*
- *Every finite product of diffeomorphisms between smooth manifolds is a diffeomorphism.*

- *Every diffeomorphism is a homeomorphism and an open map.*
- *The restriction of a diffeomorphism to an open submanifold with or without boundary is a diffeomorphism onto its image.*
- *"Diffeomorphic" is an equivalence relation on the class of all smooth manifolds with or without boundary.*

Proposition 2.2.15. *Suppose $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ are open subsets and $F : U \to V$ is a diffeomorphism. Then $m = n$, and for each $a \in U$, the total derivative $DF(a)$ is invertible, with*

$$DF(a)^{-1} = D(F^{-1})(F(a)).$$

Proof. Because $F^{-1} \circ F = \mathrm{Id}_U$, the chain rule implies that for each $a \in U$,

$$\mathrm{Id}_{\mathbb{R}^n} = D(\mathrm{Id}_U)(a) = D(F^{-1} \circ F)(a) = D(F^{-1})(F(a)) \circ DF(a). \tag{2.1}$$

Similarly, $F \circ F^{-1} = \mathrm{Id}_V$ implies that $DF(a) \circ D(F^{-1})(F(a))$ is the identity on $\mathbb{R}^m$. This implies that $DF(a)$ is invertible with inverse $D(F^{-1})(F(a))$, and therefore $m = n$. □

Theorem 2.2.16 (***Diffeomorphism Invariance of Dimension***)**.** *A nonempty smooth manifold of dimension m cannot be diffeomorphic to an n-dimensional smooth manifold unless $m = n$.*

Proof. Suppose M is a nonempty smooth m-manifold, N is a nonempty smooth n-manifold, and $F : M \to N$ is a diffeomorphism. Choose any point $p \in M$, and let (U, φ) and (V, ψ) be smooth coordinate charts containing p and $F(p)$, respectively. Then (the restriction of) $\widehat{F} = \psi \circ F \circ \varphi^{-1}$ is a diffeomorphism from an open subset of $\mathbb{R}^m$ to an open subset of $\mathbb{R}^n$, so it follows from Proposition 2.15 that $m = n$. □

There is a similar invariance statement for boundaries.

Theorem 2.2.17 (***Diffeomorphism Invariance of the Boundary***)**.** *Suppose M and N are smooth manifolds with boundary and $F : M \to N$ is a diffeomorphism. Then $F(\partial M) = \partial N$, and F restricts to a diffeomorphism from* $\mathrm{Int}\, M$ *to* $\mathrm{Int}\, N$.

2.2.4 Partitions of Unity

Lemma 2.2.18. *The function $f : \mathbb{R} \to \mathbb{R}$ defined by*

$$f(x) = \begin{cases} e^{-1/t}, & t > 0 \\ 0, & t \leq 0 \end{cases}$$

is smooth.

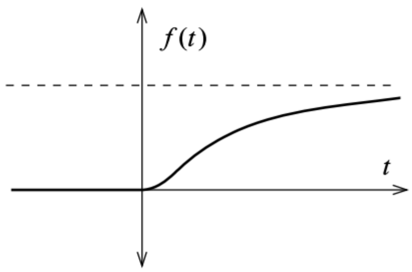


Fig. 2.4 $f(t) = e^{-1/t}$

Figure 2.3: $f(t) = e^{-1/t}$

Proof. The function in question is pictured in Fig 2.3. It is smooth on $\mathbb{R}\setminus\{0\}$ by composition, so we need only show that f has continuous derivatives of all orders at the origin. Because existence of the $(k+1)$st derivative implies continuity of the kth, it suffices to show that each such derivative exists. We begin by noting that f is continuous at 0 because

$$\lim_{t\to 0} e^{-1/t} = 0.$$

In fact, a standard application of l'Hôpital's rule and induction shows that for any integer $k \geq 0$,

$$\lim_{t\to 0} \frac{e^{-1/t}}{t^k} \;=\; \lim_{t\to 0} \frac{t^{-k}}{e^{1/t}} \;=\; 0. \tag{2.2}$$

We show by induction that for $t > 0$, the kth derivative of f is of the form

$$f^{(k)}(t) \;=\; \frac{p_k(t)\, e^{-1/t}}{t^{2k}}, \tag{2.3}$$

for some polynomial p_k of degree at most k. This is clearly true (with $p_0(t) = 1$) for $k = 0$, so suppose it is true for some $k \geq 0$. By the product rule,

$$\begin{aligned} f^{(k+1)}(t) &= p_k'(t)\frac{e^{-1/t}}{t^{2k}} + p_k(t)\Big(\frac{t^{-2}e^{-1/t}}{t^{2k}} - 2k\frac{p_k(t)\, e^{-1/t}}{t^{2k+1}}\Big) \\ &= \frac{\big(t^2 p_k'(t) + p_k(t) - 2kp_k(t)\, t\big)e^{-1/t}}{t^{2(k+1)}}, \end{aligned}$$

which is of the required form.

Finally, we prove by induction that $f^{(k)}(0) = 0$ for each integer $k \geq 0$. For $k = 0$ this is true by definition, so assume it is true for some $k \geq 0$. To prove that $f^{(k+1)}(0)$ exists, it suffices to show that $f^{(k)}$ has one-sided derivatives from both sides at $t = 0$ and that they are equal. Clearly, the derivative from the left is zero. Using (2.3) and (2.2) again, we find that the right-hand derivative of $f^{(k)}$ at $t = 0$ is

$$\lim_{t\to 0} \frac{\dfrac{p_k(t)\, e^{-1/t}}{t^{2k}} - 0}{t} \;=\; \lim_{t\to 0} p_k(t)\, \frac{e^{-1/t}}{t^{2k+1}} \;=\; p_k(0) \lim_{t\to 0} \frac{e^{-1/t}}{t^{2k+1}} \;=\; 0.$$

Thus $f^{(k+1)}(0) = 0$. □

Lemma 2.2.19. *Given any real numbers r_1 and r_2 with $r_1 < r_2$, there exists a smooth function*

$$h\colon \mathbb{R} \to \mathbb{R}$$

such that

$$\begin{aligned} H(x) &\equiv 1, && \textit{on } \overline{B}_{r_1}(0),\\ 0 < H(x) &< 1, && \textit{for all } x \in B_{r_2}(0) \setminus \overline{B}_{r_1}(0),\\ H(x) &\equiv 0, && \textit{on } \mathbb{R}^n \setminus B_{r_2}(0). \end{aligned}$$

Proof. Let f be the function of the previous lemma, and set

$$h(t) \;=\; \frac{f(r_2 - t)}{f(r_2 - t) + f(t - r_1)}.$$

Note that the denominator is positive for all t, since at least one of $r_2 - t$ and $t - r_1$ is positive. The desired properties of h follow easily from those of f. □

A function with the properties of h in the preceding lemma is usually called a ***cutoff function***.

Lemma 2.2.20. *Given any positive real numbers $r_1 < r_2$, there is a smooth function*

$$H\colon \mathbb{R}^n \to \mathbb{R}$$

such that

$$\begin{aligned} H &\equiv 1 && \textit{on } \overline{B}_{r_1}(0),\\ 0 < H(x) &< 1 && \textit{for all } x \in B_{r_2}(0) \setminus \overline{B}_{r_1}(0),\\ H &\equiv 0 && \textit{on } \mathbb{R}^n \setminus B_{r_2}(0). \end{aligned}$$

Proof. Set

$$H(x) = h\big(|x|\big),$$

where h is the cutoff function from Lemma 2.2.19. Clearly, H is smooth on $\mathbb{R}^n \setminus \{0\}$ as a composition of smooth functions, and since $H \equiv 1$ on $B_{r_1}(0)$ it is smooth there as well. □

Definition 2.2.21 [*Support of a Function*]. If f is any real-valued or vector-valued function on a topological space M, the ***support of*** f, denoted by $\operatorname{supp} f$, is the closure of the set of points where f is nonzero:

$$\operatorname{supp} f \;=\; \overline{\{\, p \in M : f(p) \neq 0\}}.$$

Definition 2.2.22 [***Partition of Unity Subordinate to*** $\mathcal{X}$]**.** Let M be a topological space, and let $\mathcal{X} = \{X_\alpha\}_{\alpha\in A}$ be an open cover of M. A ***partition of unity subordinate to*** $\mathcal{X}$ is an indexed family of continuous functions

$$\psi_\alpha \colon M \to \mathbb{R} \quad (\alpha \in A)$$

with the following properties:

(i) $0 \le \psi_\alpha(x) \le 1$ for all $\alpha \in A$ and all $x \in M$.

(ii) $\operatorname{supp}\psi_\alpha \subseteq X_\alpha$ for each $\alpha \in A$.

(iii) The family of supports $\{\operatorname{supp}\psi_\alpha\}_{\alpha\in A}$ is locally finite, meaning every point of M has a neighborhood intersecting only finitely many of the supports.

(iv) $\displaystyle\sum_{\alpha\in A}\psi_\alpha(x) = 1$ for all $x \in M$.

Because of the local finiteness condition (iii), the sum in (iv) actually has only finitely many nonzero terms in a neighborhood of each point, so there is no issue of convergence. If M is a smooth manifold with or without boundary, a ***smooth partition of unity*** is one for which each of the functions ψ_α is smooth.

Theorem 2.2.23 (***Existence of Partitions of Unity***)**.** *Suppose M is a smooth manifold with or without boundary, and*

$$\mathcal{X} = (X_\alpha)_{\alpha\in A}$$

is any indexed open cover of M. Then there exists a smooth partition of unity subordinate to $\mathcal{X}$.

Proof. For simplicity, suppose M is a smooth manifold without boundary; the general case is left as an exercise. Each X_α is itself a smooth manifold, so by Proposition 1.19 it admits a basis $\mathcal{B}_\alpha$ of regular coordinate balls. Let

$$\mathcal{B} = \bigcup_{\alpha\in A}\mathcal{B}_\alpha,$$

which is then a basis for M. By Theorem 1.15 the cover $\mathcal{X}$ has a countable, locally finite refinement $\{B_i\}$ drawn from $\mathcal{B}$, and by Lemma 1.13(a) the family $\{\overline{B}_i\}$ remains locally finite.

For each i, the fact that B_i is a regular coordinate ball in some X_α guarantees that there is a coordinate ball $B_i' \subseteq X_\alpha$ such that $B_i' \supseteq \overline{B}_i$, and a smooth coordinate map

$$\varphi_i \colon B_i' \to \mathbb{R}^n$$

such that $\varphi_i(\overline{B}_i) = \overline{B}_{r_i}(0)$ and $\varphi_i(B_i') = B_{r_i'}(0)$ for some $r_i < r_i'$. For each i, define a function $f_i \colon M \to \mathbb{R}$ by

$$f_i = \begin{cases} H_i \circ \varphi_i, & \text{on } B_i', \\ 0, & \text{on } M\backslash\overline{B}_i, \end{cases}$$

where $H_i\colon \mathbb{R}^n \to \mathbb{R}$ is a smooth function that is positive in $B_{r_i}(0)$ and zero elsewhere, as in Lemma 2.2.20. On the set $B_i' \setminus \overline{B}_i$ where the two definitions overlap, both definitions yield the zero function, so f_i is well defined and smooth, and supp $f_i = \overline{B}_i$.

Finally, we need to reindex our functions so that they are indexed by the same set A as our open cover. Because the cover $\{B_i'\}$ is a refinement of $\mathcal{X}$, for each i we can choose some index $a(i) \in A$ such that $B_i' \subseteq X_{a(i)}$. For each $\alpha \in A$, define

$$\psi_\alpha\colon M \to \mathbb{R} \quad \text{by} \quad \psi_\alpha = \sum_{i:a(i)=\alpha} g_i.$$

If there are no indices i for which $a(i) = \alpha$, then this sum should be interpreted as the zero function. It follows from Lemma 1.13(b) that

$$\operatorname{supp}\psi_\alpha = \overline{\bigcup_{i:a(i)=\alpha} B_i} = \bigcup_{i:a(i)=\alpha} \overline{B_i} \subseteq X_\alpha.$$

Each ψ_α is a smooth function that satisfies $0 \le \psi_\alpha \le 1$. Moreover, the family of supports $\{\operatorname{supp}\psi_\alpha\}_{\alpha\in A}$ is still locally finite, and $\sum_{\alpha\in A}\psi_\alpha \equiv \sum_i g_i \equiv 1$, so this is the desired partition of unity. $\square$

2.2.5 Applications of Partitions of Unity

Definition 2.2.24 [Bump Function]. If M is a topological space, $A \subseteq M$ is a closed subset, and $U \subseteq M$ is an open subset containing A, a continuous function

$$\psi\colon M \to \mathbb{R}$$

is called a ***bump function for*** A ***supported in*** U if

$$0 \le \psi \le 1 \quad \text{on } M, \qquad \psi \equiv 1 \quad \text{on } A, \qquad \operatorname{supp}\psi \subseteq U.$$

(See fig 2.4 for illustrate.)

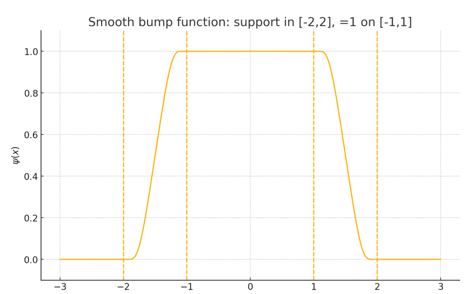


Figure 2.4: Bump function

Proposition 2.2.25 (Existence of Smooth Bump Functions)**.** *Let M be a smooth manifold with or without boundary. For any closed subset $A \subseteq M$ and any open subset U containing A, there exists a smooth bump function for A supported in U.*

Proof. Let $U_0 = U$ and $U_1 = M \setminus A$, (notice since A is closed, $U \setminus A$ is open) and let $\{\psi_0, \psi_1\}$ be a smooth partition of unity subordinate to the open cover $\{U_0, U_1\}$. Since

$$\psi_1 \equiv 0 \text{ on } A \quad \implies \quad \psi_0 = \sum_i \psi_i = 1 \text{ on } A,$$

the function ψ_0 has the required properties. □

Definition 2.2.26 [Smooth on a Subset]. Suppose M and N are smooth manifolds with or without boundary, and $A \subseteq M$ is an arbitrary subset. We say that a map

$$F \colon A \to N$$

is ***smooth on*** A if it admits a smooth extension in a neighborhood of each point; that is, for every $p \in A$ there exists an open subset $W \subseteq M$ with $p \in W$ and a smooth map

$$\widetilde{F} \colon W \to N$$

whose restriction to $W \cap A$ agrees with F.

Lemma 2.2.27 (Extension Lemma for Smooth Functions)**.** *Suppose M is a smooth manifold with or without boundary, $A \subseteq M$ is a closed subset, and*

$$f \colon A \to \mathbb{R}^k$$

is a smooth function. For any open subset U containing A, there exists a smooth function

$$\widetilde{f} \colon M \to \mathbb{R}^k$$

such that $\widetilde{f}|_A = f$ and $\operatorname{supp} \widetilde{f} \subseteq U$.

(The assumption in the extension lemma that the codomain of f is $\mathbb{R}^k$, and not some other smooth manifold, is needed: for other codomains, extensions can fail to exist for topological reasons.)

Proof. For each $p \in A$, choose a neighborhood W_p of p and a smooth function

$$\widetilde{f}_p \colon W_p \to \mathbb{R}^k$$

that agrees with f on $W_p \cap A$. Replacing W_p by $W_p \cap U$, we may assume $W_p \subseteq U$. The family

$$\{W_p : p \in A\} \,\cup\, \{M \setminus A\}$$

is an open cover of M. Let

$$\{\psi_p : p \in A\} \,\cup\, \{\psi_0\}$$

be a smooth partition of unity subordinate to this cover, with $\operatorname{supp}\psi_p \subseteq W_p$ and $\operatorname{supp}\psi_0 \subseteq M \setminus A$.

On each W_p, the product $\psi_p \widetilde{f}_p$ is smooth and extends by zero smoothly to all of M (since on $W_p \setminus \operatorname{supp}\psi_p$ the two definitions agree). We then define

$$\widetilde{f}(x) \;=\; \sum_{p\in A} \psi_p(x)\, \widetilde{f}_p(x).$$

Because the supports $\{\operatorname{supp}\psi_p\}_{p\in A}$ are locally finite, this sum has only finitely many nonzero terms near any $x \in M$, so $\widetilde{f}\colon M \to \mathbb{R}^k$ is smooth.

If $x \in A$, then $\psi_0(x) = 0$ and $\widetilde{f}_p(x) = f(x)$ whenever $\psi_p(x) \neq 0$. Hence

$$\widetilde{f}(x) = \sum_{p\in A} \psi_p(x)\, f(x) = \Big(\psi_0(x) + \sum_{p\in A} \psi_p(x)\Big) f(x) = f(x),$$

so $\widetilde{f}$ indeed extends f. Finally, by Lemma 1.13(b),

$$\operatorname{supp}\widetilde{f} = \bigcup_{p\in A} \operatorname{supp}\psi_p \subseteq U,$$

as required. □

Definition 2.2.28 [Exhaustion Function for M]. Suppose M is a topological space. An **exhaustion function for** M is a continuous function

$$f\colon M \to \mathbb{R}$$

with the property that, for each $c \in \mathbb{R}$, the set

$$f^{-1}\big((-\infty, c]\big)$$

(called a **sublevel set of** f) is compact.

The name comes from the fact that as n ranges over the positive integers, the sublevel sets

$$f^{-1}\big((-\infty, n]\big)$$

form an exhaustion of M by compact sets; thus an exhaustion function provides a sort of continuous version of an exhaustion by compact sets. For example, the functions

$$f\colon \mathbb{R}^n \to \mathbb{R}, \quad f(x) = |x|^2, \qquad g\colon \mathbb{B}^n \to \mathbb{R}, \quad g(x) = \frac{1}{1-|x|^2}$$

are smooth exhaustion functions. Of course, if M is compact, any continuous real-valued function on M is an exhaustion function, so such functions are interesting only for noncompact manifolds.

Proposition 2.2.29 (Existence of Smooth Exhaustion Functions)**.** *Every smooth manifold with or without boundary admits a smooth positive exhaustion function.*

Proof. Let M be a smooth manifold with or without boundary, let $\{V_j\}_{j=1}^\infty$ be any countable open cover of M by precompact open subsets, and let $\{\psi_j\}_{j=1}^\infty$ be a smooth partition of unity subordinate to this cover. Define $f \in C^\infty(M)$ by

$$f(p) \ = \ \sum_{j=1}^\infty j\, \psi_j(p).$$

Then f is smooth because in a neighborhood of any point only finitely many ψ_j are nonzero, and f is strictly positive since

$$f(p) \ = \ \sum_{j=1}^\infty j\, \psi_j(p) \ \geq \ \sum_{j=1}^\infty \psi_j(p) = 1.$$

To see that f is an exhaustion function, fix $c \in \mathbb{R}$ and choose $N \in \mathbb{N}$ with $N > c$. If

$$p \notin \bigcup_{j=1}^N \overline{V_j},$$

then $\psi_j(p) = 0$ for $1 \leq j \leq N$, so

$$f(p) = \sum_{j=N+1}^\infty j\, \psi_j(p) \ \geq \ \sum_{j=N+1}^\infty N\, \psi_j(p) = N \sum_{j=1}^\infty \psi_j(p) = N > c.$$

Equivalently, if $f(p) \leq c$ then $p \in \bigcup_{j=1}^N \overline{V_j}$. Hence

$$f^{-1}\big((-\infty, c]\big)$$

is a closed subset of the compact set $\bigcup_{j=1}^N \overline{V_j}$, and is therefore compact. □

Theorem 2.2.30 (Level Sets of Smooth Functions)**.** *Let M be a smooth manifold. If K is any closed subset of M, there is a smooth nonnegative function*

$$f\colon M \to \mathbb{R}$$

such that

$$f^{-1}(0) = K.$$

Proof. We begin with the special case in which $M = \mathbb{R}^n$ and $K \subseteq \mathbb{R}^n$ is closed. For each $x \in \mathbb{R}^n \setminus K$, choose a positive number $r \leq 1$ such that

$$B_r(x) \subseteq \mathbb{R}^n \setminus K.$$

By Proposition A.16, $\mathbb{R}^n \setminus K$ is the union of countably many such balls $\{B_{r_i}(x_i)\}_{i=1}^\infty$.

Let

$$h\colon \mathbb{R}^n \to \mathbb{R}$$

be a smooth bump function with $h \equiv 1$ on $\overline{B_{1/2}(0)}$ and $\operatorname{supp} h \subseteq B_1(0)$. For each $i \in \mathbb{N}$, pick $C_i \geq 1$ so that every partial derivative of h up to order i has absolute value $\leq C_i$. Define

$$f\colon \mathbb{R}^n \to \mathbb{R}, \qquad f(x) \;=\; \sum_{i=1}^{\infty} \frac{r_i^i}{2^i\, C_i}\; h\Big(\frac{x - x_i}{r_i}\Big).$$

Since $\left|\frac{r_i^i}{2^i C_i}\, h(\frac{x-x_i}{r_i})\right| \;\leq\; \frac{1}{2^i}$, the series converges uniformly by the Weierstrass M–test to a continuous—and in fact smooth—function. Moreover, the ith term is positive exactly when $x \in B_{r_i}(x_i)$, so f vanishes on K and is strictly positive on $\mathbb{R}^n \setminus K$.

It remains only to show that f is smooth. We have already shown that it is continuous, so suppose $k \geq 1$ and assume by induction that all partial derivatives of f of order less than k exist and are continuous. By the chain rule and induction, every kth partial derivative of the ith term in the series can be written in the form

$$\frac{(r_i)^{\,i-k}}{2^i\, C_i}\; D^k h\big(\tfrac{x-x_i}{r_i}\big),$$

where $D^k h$ is some kth partial derivative of h. By our choices of r_i and C_i, as soon as $i \geq k$, each of these terms is bounded in absolute value by $1/2^i$, so the differentiated series also converges uniformly to a continuous function. It then follows from Theorem C.31 that the kth partial derivatives of f exist and are continuous. This completes the induction, and shows that f is smooth.

Now let M be an arbitrary smooth manifold, and let $K \subseteq M$ be any closed subset. Let $\{B_\alpha\}$ be an open cover of M by smooth coordinate balls, and let $\{\psi_\alpha\}$ be a subordinate partition of unity. Since each B_α is diffeomorphic to $\mathbb{R}^n$, the preceding argument shows that for each α there is a smooth nonnegative function

$$f_\alpha\colon B_\alpha \to \mathbb{R}$$

such that $f_\alpha^{-1}(0) = B_\alpha \cap K$. The function

$$f \;=\; \sum_\alpha \psi_\alpha\, f_\alpha$$

does the trick. $\square$

Chapter 3

Tangent Vectors

3.3 Tangent Vectors

3.3.1 Geometric Tangent Vectors

Definition 3.3.1 [Geometric Tangent Vector]. Given a point $a \in \mathbb{R}^n$, let us define the ***geometric tangent space* to** $\mathbb{R}^n$ ***at*** a, denoted by $\mathbb{R}^n_a$, to be the set $\{a\} \times \mathbb{R}^n = \{(a, v) : v \in \mathbb{R}^n\}$. A ***geometric tangent vector*** in $\mathbb{R}^n$ is an element of $\mathbb{R}^n_a$ for some $a \in \mathbb{R}^n$. As a matter of notation, we abbreviate (a, v) as v_a (or sometimes $v|_a$ if it is clearer, for example if v itself has a subscript). We think of v_a as the vector v with its initial point at a (Fig. 3.1). The set $\mathbb{R}^n_a$ is a real vector space under the natural operations

$$v_a + w_a = (v + w)_a, \quad c(v_a) = (cv)_a.$$

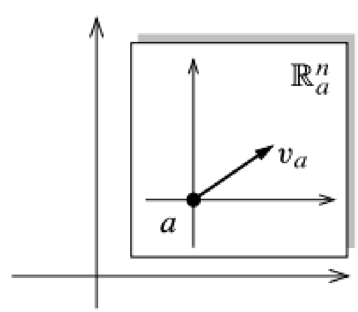


Figure 3.1: Geometric Tangent Space

The vectors $e_i|_a$, $i = 1, \ldots, n$, are a basis for $\mathbb{R}^n_a$. In fact, as a vector space, $\mathbb{R}^n_a$ is essentially the same as $\mathbb{R}^n$ itself; the only reason we add the index a is so that the geometric tangent spaces $\mathbb{R}^n_a$ and $\mathbb{R}^n_b$ at distinct points a and b will be disjoint sets.

With this definition we could think of the tangent space to S^{n-1} at a point $a \in S^{n-1}$ as a certain subspace of $\mathbb{R}^n_a$ (Fig. 3.2), namely the space of vectors that are orthogonal to the radial unit vector through a, using the inner product that $\mathbb{R}^n_a$ inherits from $\mathbb{R}^n$ via the natural isomorphism $\mathbb{R}^n \cong \mathbb{R}^n_a$. The problem with this definition, however, is that it gives us no clue as to how we might define tangent vectors on an arbitrary smooth manifold, where there is no ambient Euclidean space. So we need to look need to look for another characterization of tangent vectors that might make sense on a manifold.

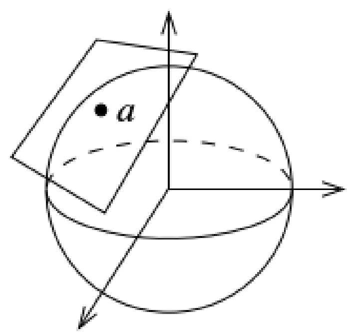


Figure 3.2: Tangent Space to $\mathbb{S}^{n-1}$

The only things we have to work with on smooth manifolds so far are smooth functions, smooth maps, and smooth coordinate charts. One thing that a geometric tangent vector provides is a means of taking directional derivatives of functions. For example, any geometric tangent vector $v_a \in \mathbb{R}^n_a$ yields a map

$$D_v\big|_a : C^\infty(\mathbb{R}^n) \longrightarrow \mathbb{R},$$

which takes the directional derivative in the direction v at a:

$$D_v\big|_a f \;=\; D_v f(a) \;=\; \frac{d}{dt}\bigg|_{t=0} f(a + t\,v). \tag{3.1}$$

This operation is linear over $\mathbb{R}$ and satisfies the product rule:

$$D_v\big|_a (fg) \;=\; f(a)\, D_v\big|_a g \;+\; g(a)\, D_v\big|_a f. \tag{3.2}$$

If $v_a = v^i e_i\big|_a$ in terms of the standard basis, then by the chain rule $D_v\big|_a f$ can be written more concretely as

$$D_v\big|_a f \;=\; v^i \frac{\partial f}{\partial x^i}(a).$$

(Here we are using the summation convention as usual, so the expression on the right-hand side is understood to be summed over $i = 1, \ldots, n$. This sum is consistent with our index convention if we stipulate that an upper index "in the denominator" is to be regarded as a lower index.) For example, if $v_a = e_j\big|_a$, then

$$D_v\big|_a f \;=\; \frac{\partial f}{\partial x^j}(a).$$

Definition 3.3.2. With this construction in mind, we make the following definition. If a is a point of $\mathbb{R}^n$, a map

$$\omega : C^\infty(\mathbb{R}^n) \;\longrightarrow\; \mathbb{R}$$

is called a ***derivation at*** a if it is linear over $\mathbb{R}$ and satisfies the following product rule:

$$\omega(fg) \;=\; f(a)\,\omega g \;+\; g(a)\,\omega f. \tag{3.3}$$

Let $T_a\mathbb{R}^n$ denote the set of all derivations of $C^\infty(\mathbb{R}^n)$ at a. Clearly, $T_a\mathbb{R}^n$ is a vector space under the operations

$$(w_1 + w_2)f = w_1 f + w_2 f, \qquad (c\,w)f = c\,(wf).$$

The most important (and perhaps somewhat surprising) fact about $T_a\mathbb{R}^n$ n is that it is finite-dimensional, and in fact is naturally isomorphic to the geometric tangent space $\mathbb{R}^n_a$ that we defined above.

Lemma 3.3.3 (Properties of Derivations)**.** *Suppose $a \in \mathbb{R}^n$, $w \in T_a\mathbb{R}^n$, and $f, g \in C^\infty(\mathbb{R}^n)$.*

1. *If f is a constant function, then $wf = 0$.*
2. *If $f(a) = g(a) = 0$, then $w(fg) = 0$.*

Proof. It suffices to prove (a) for the constant function $f_1(x) \equiv 1$. Indeed, if $f(x) \equiv c$ then

$$wf \;=\; w(c\,f_1) \;=\; c\,wf_1 \;=\; 0$$

by linearity. For f_1 the product rule gives

$$wf_1 \;=\; w(f_1 f_1) \;=\; f_1(a)\,wf_1 \;+\; f_1(a)\,wf_1 \;=\; 2\,wf_1,$$

which forces $wf_1 = 0$. Similarly, (b) follows immediately from the product rule:

$$w(fg) \;=\; f(a)\,wg \;+\; g(a)\,wf \;=\; 0 + 0 \;=\; 0.$$

□

Theorem 3.3.4 (Taylor's Theorem)**.** *Let $U \subseteq \mathbb{R}^n$ be an open subset, and let $a \in U$ be fixed. Suppose $f \in C^{k+1}(U)$ for some $k \geq 0$. If W is any convex subset of U containing a, then for all $x \in W$,*

$$f(x) = P_k(x) + R_k(x), \tag{3.4}$$

where P_k is the k^{th}-order Taylor polynomial of f at a, defined by

$$P_k(x) = f(a) + \sum_{m=1}^{k} \frac{1}{m!} \sum_{I:|I|=m} \partial_I f(a)\,(x-a)^I, \tag{3.5}$$

and R_k is the k^{th} remainder term, given by

$$R_k(x) = \frac{1}{k!} \sum_{I:|I|=k+1} (x-a)^I \int_0^1 (1-t)^k\, \partial_I f\big(a+t(x-a)\big)\, dt. \tag{3.6}$$

Proof. For $k=0$ (where we interpret P_0 to mean $f(a)$), the result is just the fundamental theorem of calculus applied to the function

$$u(t) = f\big(a+t(x-a)\big),$$

together with the chain rule. Now assume the theorem holds for some k, and consider the remainder term

$$\int_0^1 (1-t)^k\, \partial_I f\big(a+t(x-a)\big)\, dt.$$

Integrating by parts gives

$$\int_0^1 (1-t)^k\, \partial_I f\big(a+t(x-a)\big)\, dt = \left[-\frac{(1-t)^{k+1}}{k+1}\, \partial_I f\big(a+t(x-a)\big)\right]_{t=0}^{t=1} + \int_0^1 \frac{(1-t)^{k+1}}{k+1} \frac{\partial}{\partial t}\big(\partial_I f(a+t(x-a))\big)\, dt$$

$$= \frac{1}{k+1}\, \partial_I f(a) + \frac{1}{k+1} \sum_{j=1}^{n} (x^j - a^j) \int_0^1 (1-t)^{k+1} \frac{\partial}{\partial x^j}\big(\partial_I f(a+t(x-a))\big)\, dt.$$

Inserting this into (3.4) yields the same formula with k replaced by $k+1$, completing the induction. □

Proposition 3.3.5. *Let $a \in \mathbb{R}^n$.*

1. *For each geometric tangent vector $v_a \in \mathbb{R}^n_a$, the map*

   $$D_v\big|_a : C^\infty(\mathbb{R}^n) \longrightarrow \mathbb{R}$$

   defined by (3.1) *is a derivation at a.*

2. *The map*

   $$\mathbb{R}^n_a \longrightarrow T_a\mathbb{R}^n, \quad v_a \mapsto D_v\big|_a$$

   is an isomorphism.

Proof. The fact that $D_v\big|_a$ is a derivation at a is an immediate consequence of the product rule (3.2).

To show the map $v_a \mapsto D_v\big|_a$ is an isomorphism, first note it is linear. For injectivity, suppose $v_a \in \mathbb{R}^n_a$ and $D_v\big|_a \equiv 0$. Writing

$$v_a = v^i\, e_i\big|_a$$

and taking f to be the jth coordinate function $x^j : \mathbb{R}^n \to \mathbb{R}$, we get

$$0 = D_v\big|_a(x^j) = v^i\, \frac{\partial}{\partial x^i}(x^j)\Big|_{x=a} = v^j,$$

since $\partial x^j/\partial x^i = 0$ for $i \neq j$ and $= 1$ for $i = j$. Hence all $v^j = 0$ and $v_a = 0$.

For surjectivity, let $w \in T_a\mathbb{R}^n$. Define

$$v = v^i\, e_i, \qquad v^i = w(x^i).$$

We claim $w = D_v\big|_a$. Indeed, for any $f \in C^\infty(\mathbb{R}^n)$, Taylor's theorem around a (Theorem 3.3.4) gives

$$f(x) = f(a) + \sum_{i=1}^n \frac{\partial f}{\partial x^i}(a)\,(x^i - a^i) + \sum_{i,j=1}^n (x^i - a^i)(x^j - a^j) \int_0^1 (1-t)\, \frac{\partial^2 f}{\partial x^i \partial x^j}(a + t(x-a))\, dt.$$

Applying w, the constant term vanishes by Lemma 3.3.3(a), and the remainder term vanishes by Lemma 3.3.3(b) since it carries at least two factors of $(x^k - a^k)$. Thus

$$w(f) = \sum_{i=1}^n \frac{\partial f}{\partial x^i}(a)\, w(x^i) = \sum_{i=1}^n v^i\, \frac{\partial f}{\partial x^i}(a) = D_v\big|_a f,$$

as claimed. □

Corollary 3.3.6. *For any $a \in \mathbb{R}^n$, the n derivations*

$$\frac{\partial}{\partial x^1}\Big|_a, \; \ldots, \; \frac{\partial}{\partial x^n}\Big|_a$$

defined by

$$\frac{\partial}{\partial x^i}\Big|_a f = \frac{\partial f}{\partial x^i}(a)$$

form a basis for $T_a\mathbb{R}^n$, which therefore has dimension n.

Proof. Apply Proposition 3.3.5 and note that

$$\frac{\partial}{\partial x^i}\Big|_a = D_{e_i}\big|_a.$$

□

3.3.2 Tangent Vectors on Manifolds

Definition 3.3.7. Let M be a smooth manifold with or without boundary, and let p be a point of M. A linear map

$$v : C^\infty(M) \longrightarrow \mathbb{R}$$

is called a ***derivation at*** p if it satisfies

$$v(fg) = f(p)\, vg + g(p)\, vf \quad \text{for all } f, g \in C^\infty(M). \tag{3.7}$$

The set of all derivations of $C^\infty(M)$ at p, denoted by T_pM, is a vector space called the ***tangent space to*** M ***at*** p. An element of T_pM is called a ***tangent vector at*** p. The following lemma is the analogue of Lemma 3.3.3 for manifolds.

Lemma 3.3.8 (Properties of Tangent Vectors on Manifolds)**.** *Suppose M is a smooth manifold with or without boundary, $p \in M$, $v \in T_pM$, and $f, g \in C^\infty(M)$.*

(a) If f is a constant function, then $vf = 0$.

(b) If $f(p) = g(p) = 0$, then $v(fg) = 0$.

Note

With the motivation of geometric tangent vectors in $\mathbb{R}^n$ in mind, you should visualize tangent vectors to M as "arrows" that are tangent to M and whose base points are attached to M at the given point. Proofs of theorems about tangent vectors must, of course, be based on the abstract definition in terms of derivations, but your intuition should be guided as much as possible by the geometric picture

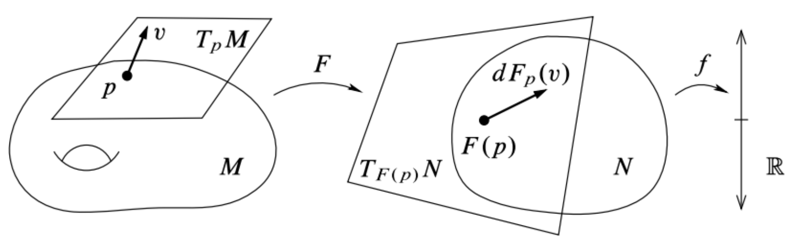


Fig. 3.3 The differential

Figure 3.3: The differential

3.4 The Differential of a Smooth Map

If M and N are smooth manifolds with or without boundary and $F\colon M \to N$ is a smooth map, then for each $p \in M$ we define a map

$$dF_p\colon T_pM \longrightarrow T_{F(p)}N,$$

called the **differential of** F **at** p (Fig. 3.3), as follows. Given $v \in T_pM$, we let $dF_p(v)$ be the derivation at $F(p)$ that acts on $f \in C^\infty(N)$ by the rule

$$dF_p(v)(f) \;=\; v\big(f \circ F\big).$$

Note that if $f \in C^\infty(N)$, then $f \circ F \in C^\infty(M)$, so $v(f \circ F)$ makes sense. The operator

$$dF_p(v)\colon C^\infty(N) \longrightarrow \mathbb{R}$$

is linear because v is, and is a derivation at $F(p)$ because for any $f, g \in C^\infty(N)$ we have

$$\begin{aligned} dF_p(v)(fg) &= v\big((fg) \circ F\big) = v\big((f \circ F)(g \circ F)\big) \\ &= f\big(F(p)\big)\, v(g \circ F) + g\big(F(p)\big)\, v(f \circ F) \\ &= f\big(F(p)\big)\, dF_p(v)(g) + g\big(F(p)\big)\, dF_p(v)(f). \end{aligned}$$

Proposition 3.4.1 (Properties of Differentials)**.** *Let M, N, and P be smooth manifolds with or without boundary, let*

$$F\colon M \longrightarrow N, \quad G\colon N \longrightarrow P$$

be smooth maps, and let $p \in M$.

(a) $dF_p\colon T_pM \to T_{F(p)}N$ is linear.

(b) $d(G \circ F)_p = dG_{F(p)} \circ dF_p\colon T_pM \to T_{G\circ F(p)}P$.

(c) $d(\mathrm{Id}_M)_p = \mathrm{Id}_{T_pM}\colon T_pM \to T_pM$.

(d) If F is a diffeomorphism, then $dF_p\colon T_pM \to T_{F(p)}N$ is an isomorphism, and

$$(dF_p)^{-1} = d\big(F^{-1}\big)_{F(p)}.$$

Proposition 3.4.2. *Let M be a smooth manifold with or without boundary, $p \in M$, and $v \in T_pM$. If $f, g \in C^\infty(M)$ agree on some neighborhood of p, then*

$$vf \;=\; vg.$$

Proof. Let $h = f - g$, so that h is a smooth function vanishing on a neighborhood of p. Choose $\psi \in C^\infty(M)$ a bump function with $\psi \equiv 1$ on $\operatorname{supp} h$ and $\operatorname{supp} \psi \subset M \setminus \{p\}$. Since $\psi = 1$ wherever h is nonzero, we have $\psi\, h \equiv h$. But $h(p) = \psi(p) = 0$, so by Lemma 3.3.8

$$vh \;=\; v(\psi\, h) \;=\; 0.$$

By linearity of v, it follows that

$$vf - vg \;=\; v(h) \;=\; 0,$$

hence $vf = vg$. ☐

Proposition 3.4.3 (The Tangent Space to an Open Submanifold)**.** *Let M be a smooth manifold with or without boundary, let $U \subseteq M$ be an open subset, and let $\iota\colon U \hookrightarrow M$ be the inclusion map. ($\iota : A \to B$, $\iota(x) = x$, where $A \subset B$) For every $p \in U$, the differential*

$$d\iota_p\colon T_pU \longrightarrow T_pM$$

is an isomorphism.

Proof. To prove injectivity, suppose $v \in T_pU$ and $d\iota_p(v) = 0 \in T_pM$. Let B be a neighborhood of p such that $\overline{B} \subseteq U$. If $f \in C^\infty(U)$ is arbitrary, the extension lemma for smooth functions guarantees that there exists $\tilde{f} \in C^\infty(M)$ with $\tilde{f} \equiv f$ on $\overline{B}$. Since f and $\tilde{f}|_U$ agree on a neighborhood of p, Proposition 3.4.2 implies

$$v\,f = v(\tilde{f}|_U) = v(\tilde{f} \circ \iota) = d\iota_p(v)\,\tilde{f} = 0.$$

As this holds for every $f \in C^\infty(U)$, we conclude $v = 0$, so $d\iota_p$ is injective.

For surjectivity, let $w \in T_pM$. Define an operator $v\colon C^\infty(U) \to \mathbb{R}$ by

$$v\,f \;=\; w\,\tilde{f},$$

where $\tilde{f} \in C^\infty(M)$ satisfies $\tilde{f} \equiv f$ on $\overline{B}$. By Proposition 3.4.2, this value is independent of the choice of $\tilde{f}$, so v is well-defined and one checks easily that $v \in T_pU$. Then for any $g \in C^\infty(M)$,

$$d\iota_p(v)\,g = v(g \circ \iota) = w(\tilde{g} \circ \iota) = w\,g,$$

where the last equalities use that $g \circ \iota$, $\tilde{g} \circ \iota$, and g agree on B. Hence $d\iota_p$ is surjective as well. ☐

Proposition 3.4.4 (Dimension of the Tangent Space)**.** *If M is an n-dimensional smooth manifold, then for each $p \in M$, the tangent space T_pM is an n-dimensional vector space.*

Proof. Given $p \in M$, let (U, φ) be a smooth coordinate chart containing p. Because φ is a diffeomorphism from U onto an open subset $\widehat{U} \subseteq \mathbb{R}^n$, it follows from Proposition 3.4.1 that

$$d\varphi_p : T_pU \;\longrightarrow\; T_{\varphi(p)}\widehat{U}$$

is an isomorphism. Since Proposition 3.4.3 guarantees that $T_pM \cong T_pU$ and

$$T_{\varphi(p)}\widehat{U} \;\cong\; T_{\varphi(p)}\mathbb{R}^n,$$

we conclude

$$\dim T_pM \;=\; \dim T_{\varphi(p)}\mathbb{R}^n \;=\; n.$$

☐

Lemma 3.4.5. *Let $\iota : \mathbb{H}^n \hookrightarrow \mathbb{R}^n$ denote the inclusion map. For any $a \in \partial\mathbb{H}^n$, the differential*

$$d\iota_a : T_a\mathbb{H}^n \longrightarrow T_a\mathbb{R}^n$$

is an isomorphism.

Proof. Suppose $a \in \partial\mathbb{H}^n$. To show that $d\iota_a$ is injective, assume

$$d\iota_a(v) = 0.$$

Let $f : \mathbb{H}^n \to \mathbb{R}$ be any smooth function, and choose an extension $\widetilde{f}$ of f to a smooth function on all of $\mathbb{R}^n$. (Such an extension exists by the extension lemma for smooth functions.) Then $\widetilde{f} \circ \iota = f$, so

$$vf \;=\; v\big(\widetilde{f} \circ \iota\big) \;=\; d\iota_a(v)\,\widetilde{f} \;=\; 0,$$

(Recall that

$$v\big(\widetilde{f} \circ \iota\big) = \big(d\big(\widetilde{f} \circ \iota\big)_a\big)(v)$$

which implies that $d\iota_a$ is injective. (Recall injective means $L(p) = 0 \implies p = 0$ for a linear map, which is stated in proposition 3.4.1)

To show surjectivity, let $w \in T_a\mathbb{R}^n$ be arbitrary. Define $v \in T_a\mathbb{H}^n$ by

$$vf \;=\; w\,\widetilde{f},$$

where $\widetilde{f}$ is any smooth extension of f. Writing

$$w \;=\; w^i \,\frac{\partial}{\partial x^i}\Big|_a$$

in terms of the standard basis for $T_a\mathbb{R}^n$, this means that

$$vf \;=\; w^i\,\frac{\partial \widetilde{f}}{\partial x^i}(a).$$

This is independent of the choice of $\widetilde{f}$, because by continuity the derivatives of $\widetilde{f}$ at a are determined by those of f in $\mathbb{H}^n$. It is easy to check that v is a derivation at a and that $w = d\iota_a(v)$, so $d\iota_a$ is surjective. $\square$

Proposition 3.4.6 (Dimension of Tangent Spaces on a Manifold with Boundary)**.** *Suppose M is an n-dimensional smooth manifold with boundary. For each $p \in M$, the tangent space T_pM is an n-dimensional vector space.*

Proof. Let $p \in M$ be arbitrary. If p is an interior point, then because $\operatorname{Int} M$ is an open submanifold of M, Proposition 3.4.3 implies

$$T_p(\operatorname{Int} M) \;\cong\; T_pM.$$

Since $\operatorname{Int} M$ is a smooth n-manifold without boundary, its tangent spaces all have dimension n.

On the other hand, if $p \in \partial M$, let (U, φ) be a smooth boundary chart containing p, and set

$$\widehat{U} = \varphi(U) \subseteq \mathbb{H}^n.$$

There are isomorphisms

$$T_pM \cong T_pU \quad \text{(by Proposition 3.4.3)}; \quad T_pU \cong T_{\varphi(p)}\widehat{U}$$

(by Proposition 3.4.1, since φ is a diffeomorphism);

$$T_{\varphi(p)}\widehat{U} \cong T_{\varphi(p)}\mathbb{H}^n \quad \text{(by Proposition 3.4.3 again)}; \quad T_{\varphi(p)}\mathbb{H}^n \cong T_{\varphi(p)}\mathbb{R}^n \quad \text{(by Lemma 3.4.5)}.$$

Chaining these together shows $\dim T_pM = n$ in the boundary case as well, and the proof is complete. □

Proposition 3.4.7 (The Tangent Space to a Vector Space)**.** *Suppose V is a finite-dimensional vector space and $a \in V$. Just as we did earlier in the case of $\mathbb{R}^n$, for any vector $v \in V$, we define a map*

$$D_v \big|_a \colon C^\infty(V) \longrightarrow \mathbb{R}$$

by

$$D_v \big|_a f = \left.\frac{d}{dt}\right|_{t=0} f\big(a + tv\big). \tag{3.8}$$

Suppose V is a finite-dimensional vector space with its standard smooth manifold structure. For each point $a \in V$, the map

$$v \longmapsto D_v\big|_a,$$

defined by 3.8*, is a canonical isomorphism from V to T_aV, such that for any linear map $L \colon V \to W$, the following diagram commutes:*

$$\begin{array}{ccc} V & \xrightarrow{\cong} & T_aV \\ {\scriptstyle L}\downarrow & & \downarrow{\scriptstyle dL_a} \\ W & \xrightarrow[\cong]{} & T_{L(a)}W \end{array}$$

Proof. Once we choose a basis for V, we can use the same argument as in the proof of Proposition 3.2 to show that $D_v|_a$ is indeed a derivation at a, and that the map $v \mapsto D_v|_a$ is an isomorphism.

Now suppose $L \colon V \to W$ is a linear map. Because its components with respect to any choices of bases for V and W are linear functions of the coordinates, L is smooth. Unwinding the definitions and using the linearity of L, we compute

$$\begin{aligned} dL_a\big(D_v\big|_a\big)f &= D_v\big|_a\big(f \circ L\big) \\ &= \left.\frac{d}{dt}\right|_{t=0} f\big(L(a + tv)\big) = \left.\frac{d}{dt}\right|_{t=0} f\big(L(a) + t\,L(v)\big) \\ &= D_{L(v)}\big|_{L(a)} f. \end{aligned}$$

□

3.4.1 Note

It is important to understand that each isomorphism

$$V \;\cong\; T_a V$$

is canonically defined, independently of any choice of basis (notwithstanding the fact that we used a choice of basis to prove that it is an isomorphism). Because of this result, we can routinely *identify* tangent vectors to a finite-dimensional vector space with elements of the space itself. More generally, if M is an open submanifold of a vector space V, we can combine our identifications

$$T_p M \;\longleftrightarrow\; T_p V \;\longleftrightarrow\; V$$

to obtain a canonical identification of each tangent space to M with V. For example, since $GL(n,\mathbb{R})$ is an open submanifold of the vector space $M(n,\mathbb{R})$, we can identify its tangent space at each point $X \in GL(n,\mathbb{R})$ with the full space of matrices $M(n,\mathbb{R})$.

Proposition 3.4.8 (The Tangent Space to a Product Manifold)**.** *Let $M_1, \ldots, M_k$ be smooth manifolds, and for each j, let*

$$\pi_j : M_1 \times \cdots \times M_k \;\longrightarrow\; M_j$$

denote the projection onto the jth factor. For any point

$$p = (p_1, \ldots, p_k) \in M_1 \times \cdots \times M_k,$$

the map

$$\alpha\colon T_p\big(M_1 \times \cdots \times M_k\big) \;\longrightarrow\; T_{p_1}M_1 \oplus \cdots \oplus T_{p_k}M_k$$

defined by

$$\alpha(v) \;=\; \big(d(\pi_1)_p(v),\, \ldots,\, d(\pi_k)_p(v)\big) \tag{3.9}$$

is an isomorphism. The same statement holds if one of the M_i is a smooth manifold with boundary.

3.5 Computations in Coordinates

First, suppose M is a smooth manifold (without boundary), and let (U,φ) be a smooth coordinate chart on M. Then φ is, in particular, a diffeomorphism from U to an open subset $\widehat{U} \subseteq \mathbb{R}^n$. Combining Propositions 3.4.3 and 3.4.1, we see that

$$d\varphi_p\colon T_pM \;\longrightarrow\; T_{\varphi(p)}\mathbb{R}^n$$

is an isomorphism.

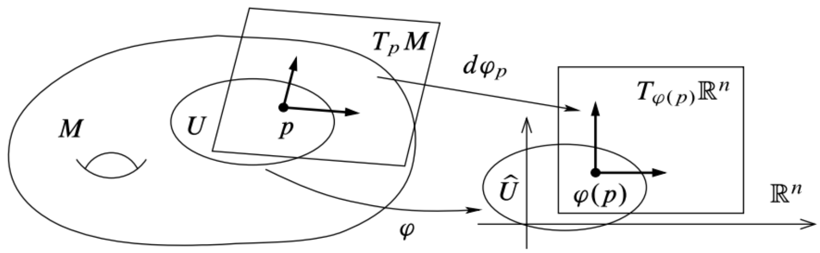


Figure 3.4: Tangent vectors in coordinates

By Corollary 3.3.6, the derivations

$$\left.\frac{\partial}{\partial x^1}\right|_{\varphi(p)}, \ldots, \left.\frac{\partial}{\partial x^n}\right|_{\varphi(p)}$$

form a basis for $T_{\varphi(p)}\mathbb{R}^n$.

Therefore, the preimages of these vectors under the isomorphism $d\varphi_p$ form a basis for T_pM (Fig. 3.4). In keeping with our standard practice of treating coordinate maps as identifications whenever possible, we use the notation

$$\left.\frac{\partial}{\partial x^i}\right|_p$$

for these vectors, characterized by either of the following expressions:

$$\begin{aligned}\left.\frac{\partial}{\partial x^i}\right|_p &= (d\varphi_p)^{-1}\Big(\left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)}\Big)\\ &= d(\varphi^{-1})_{\varphi(p)}\Big(\left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)}\Big).\end{aligned} \tag{3.10}$$

Unwinding the definitions, we see that $\left.\frac{\partial}{\partial x^i}\right|_p$ acts on a function $f \in C^\infty(U)$ by

$$\left.\frac{\partial}{\partial x^i}\right|_p f = \left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)}\big(f \circ \varphi^{-1}\big) = \frac{\partial \widehat{f}}{\partial x^i}(\widehat{p}).$$

where $\widehat{f} = f \circ \varphi^{-1}$ is the coordinate representation of f, and $\widehat{p} = (p^1, \ldots, p^n) = \varphi(p)$ is the coordinate representation of p. In other words,

$$\left.\frac{\partial}{\partial x^i}\right|_p$$

is just the derivation that takes the ith partial derivative of (the coordinate representation of) f at (the coordinate representation of) p. The vectors

$$\left.\frac{\partial}{\partial x^i}\right|_p$$

are called the ***coordinate vectors at*** p associated with the given coordinate system. In the special case of standard coordinates on $\mathbb{R}^n$, the vectors $\frac{\partial}{\partial x^i}\big|_p$ are literally the partial-derivative operators.

When M is a smooth manifold with boundary and p is an interior point, the discussion above applies verbatim. For $p \in \partial M$, the only change that needs to be made is to substitute $\mathbb{H}^n$ for $\mathbb{R}^n$, with the understanding that the notation

$$\frac{\partial}{\partial x^i}\bigg|_{\varphi(p)}$$

can be used interchangeably to denote either an element of $T_{\varphi(p)}\mathbb{R}^n$ or an element of $T_{\varphi(p)}\mathbb{H}^n$, in keeping with our convention of considering the isomorphism

$$d\iota_{\varphi(p)} : T_{\varphi(p)}\mathbb{H}^n \longrightarrow T_{\varphi(p)}\mathbb{R}^n$$

as an identification. The nth coordinate vector

$$\frac{\partial}{\partial x^n}\bigg|_p$$

should be interpreted as a one-sided derivative in this case.

The following proposition summarizes the discussion so far.

Proposition 3.5.1. *Let M be a smooth n-manifold with or without boundary, and let $p \in M$. Then T_pM is an n-dimensional vector space, and for any smooth chart $(U, (x^i))$ containing p, the coordinate vectors*

$$\frac{\partial}{\partial x^1}\bigg|_p, \ \ldots, \ \frac{\partial}{\partial x^n}\bigg|_p$$

form a basis for T_pM.

Thus, a tangent vector $v \in T_pM$ can be written uniquely as a linear combination

$$v \ = \ v^i \, \frac{\partial}{\partial x^i}\bigg|_p,$$

where we use the summation convention as usual, with an upper index in the denominator being considered as a lower index. The ordered basis $\left(\frac{\partial}{\partial x^i}\big|_p\right)$ is called a ***coordinate basis*** for T_pM, and the numbers $(v^1, \ldots, v^n)$ are called the ***components*** of v with respect to the coordinate basis. If v is known, its components can be computed easily from its action on the coordinate functions. For each j, the components of v are given by

$$v^j = v(x^j),$$

where we think of x^j as a smooth real-valued function on U, because

$$v(x^j) \ = \ \left(v^i \, \frac{\partial}{\partial x^i}\bigg|_p\right)(x^j) \ = \ v^i \, \frac{\partial x^j}{\partial x^i}(p) \ = \ v^j.$$

3.5.1 The Differential in Coordinates

Next we explore how differentials look in coordinates. We begin by considering the special case of a smooth map $F : U \to V$, where $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ are open subsets of Euclidean spaces. For any $p \in U$, we will determine the matrix of $dF_p : T_p\mathbb{R}^n \to T_{F(p)}\mathbb{R}^m$ in terms of the standard coordinate bases. Using $(x^1, \ldots, x^n)$ to denote the coordinates in the domain and $(y^1, \ldots, y^m)$ to denote those in the codomain, we use the chain rule to compute the action of dF_p on a typical basis vector as follows:

$$
\begin{aligned}
dF_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) f &= \left.\frac{\partial}{\partial x^i}\right|_p (f \circ F) = \frac{\partial f}{\partial y^j}(F(p))\frac{\partial F^j}{\partial x^i}(p) \\
&= \left(\frac{\partial F^j}{\partial x^i}(p)\left.\frac{\partial}{\partial y^j}\right|_{F(p)}\right) f.
\end{aligned}
$$

Thus

$$
dF_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) = \frac{\partial F^j}{\partial x^i}(p)\left.\frac{\partial}{\partial y^j}\right|_{F(p)}. \tag{3.9}
$$

In other words, the matrix of dF_p in terms of the coordinate bases is

$$
\begin{pmatrix}
\frac{\partial F^1}{\partial x^1}(p) & \cdots & \frac{\partial F^1}{\partial x^n}(p) \\
\vdots & \ddots & \vdots \\
\frac{\partial F^m}{\partial x^1}(p) & \cdots & \frac{\partial F^m}{\partial x^n}(p)
\end{pmatrix}.
$$

Now consider the more general case of a smooth map $F : M \to N$ between smooth manifolds with or without boundary. Choosing smooth coordinate charts (U, φ) for M containing p and (V, ψ) for N containing $F(p)$, we obtain the coordinate representation

$$
\widehat{F} = \psi \circ F \circ \varphi^{-1} : \varphi(U \cap F^{-1}(V)) \to \psi(V).
$$

(See figure 3.5)

Let $\hat{p} = \varphi(p)$ denote the coordinate representation of p. By the computation above, $d\widehat{F}_{\hat{p}}$ is represented with respect to the standard coordinate bases by the Jacobian matrix of $\widehat{F}$ at $\hat{p}$. Using the fact that $F \circ \varphi^{-1} = \psi^{-1} \circ \widehat{F}$, we compute

$$
\begin{aligned}
dF_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) &= dF_p\left(d(\varphi^{-1})_{\hat{p}}\left(\left.\frac{\partial}{\partial x^i}\right|_{\hat{p}}\right)\right) = d(\psi^{-1})_{\widehat{F}(\hat{p})}\left(d\widehat{F}_{\hat{p}}\left(\left.\frac{\partial}{\partial x^i}\right|_{\hat{p}}\right)\right). \\
&= d(\psi^{-1})_{\widehat{F}(\hat{p})}\left(\frac{\partial \widehat{F}^j}{\partial x^i}(\hat{p})\left.\frac{\partial}{\partial y^j}\right|_{\widehat{F}(\hat{p})}\right) = \frac{\partial \widehat{F}^j}{\partial x^i}(\hat{p})\left.\frac{\partial}{\partial y^j}\right|_{F(p)}.
\end{aligned} \tag{3.10}
$$

Thus, dF_p is represented in coordinate bases by the Jacobian matrix of (the coordinate representative of) F. In fact, the definition of the differential was

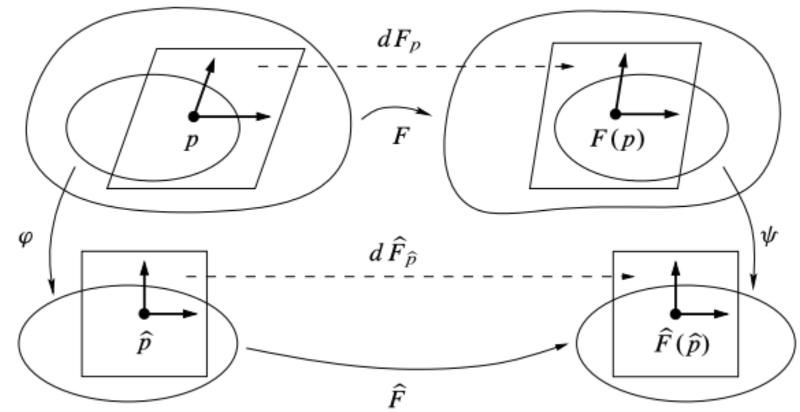


Fig. 3.6 The differential in coordinates

Figure 3.5: The differential in coordinates

cooked up precisely to give a coordinate-independent meaning to the Jacobian matrix.

$$= d(\psi^{-1})_{\widehat{F}(\hat{p})} \left(\frac{\partial \widehat{F}^j}{\partial x^i}(\hat{p}) \left. \frac{\partial}{\partial y^j} \right|_{\widehat{F}(\hat{p})} \right) = \frac{\partial \widehat{F}^j}{\partial x^i}(\hat{p}) \left. \frac{\partial}{\partial y^j} \right|_{F(p)}. \tag{3.10}$$

Thus, dF_p is represented in coordinate bases by the Jacobian matrix of (the coordinate representative of) F. In fact, the definition of the differential was cooked up precisely to give a coordinate-independent meaning to the Jacobian matrix.

In the differential geometry literature, the differential is sometimes called the ***tangent map***, the ***total derivative***, or simply the ***derivative of*** F. Because it "pushes" tangent vectors forward from the domain manifold to the codomain, it is also called the *(pointwise) pushforward*. Different authors denote it by symbols such as

$$F'(p), \quad DF, \quad DF(p), \quad F_*, \quad TF, \quad T_pF.$$

We will stick with the notation dF_p for the differential of a smooth map between manifolds, and reserve $DF(p)$ for the total derivative of a map between finite-dimensional vector spaces, which in the case of Euclidean spaces we identify with the Jacobian matrix of F.

3.5.2 Change of Coordinates