Math 247: Applied Linear Algebra
Week 3 Course Notes

Main topics: Steinitz exchange lemma, bases, casting out algorithm, dimension

1 Theoretical applications of Steinitz Exchange

Theorem 1. *Every subspace of a finite-dimensional vector space is finite-dimensional itself.*

Proof. We use Steinitz Exchange Lemma to prove this theorem. Let V be a finite dimensional vector space and $U \leq V$. V is finite dimensional $\Rightarrow$ V has a finite spanning set $v_1, ..., v_n$. None of these vectors need to be in U. We can **recursively** show that U has a finite spanning set, which would imply that it is finite-dimensional.

Proof by contradiction: $\Rightarrow U \neq \{0\}$. So, let $u_1 \neq 0$, $u_1 \in U \Rightarrow u_1$ is linearly independent. By the hypothesis, it is not spanning, since it is a finite set $\Rightarrow$ we can find a vector $v_2 \in U$ with $u_2 \notin span\{u_1\}$. By the same argument as before, we can continue adding vectors since it is not a spanning set, by assumption.

After $n+1$ iterations, we've obtained a **linearly independent set** $\{u_1, ..., u_{n+1}\} \subseteq U \subseteq V$. Here is the problem: **We have a linearly independent set consisting of n+1 elements and a spanning set with n elements (for V), which is not allowed by the Steinitz Exchange Lemma.** This is a contradiction, since V has a spanning set containing n elements. The assumption was wrong, which forces U to be finite-dimensional. $\square$

Example 1.1. The vector space S of all sequences with real coefficients is infinite-dimensional. Why? We already know an infinite-dimensional sub-space: C_{00}.

- If S were finite-dimensional, any subspace of S would also be finite-dimensional.
- Since C_{00} is infinite-dimensional, S cannot be finite-dimensional, so, it must be infinite-dimensional.
- This is basically a contrapositive proof.
- From this, we also get the fact for free that C_0 is infinite-dimensional.

2 Basis of a Vector Space

Definition 2.1. Let V be a vector space over K. A subset B of V is called a **basis** if:

1. B is <u>linearly independent</u>.
2. B is <u>spanning</u>.

Note that these two factors tend to move in opposite directions. A more intuitive meaning of this: B is a subset that allows us to create anything that we want in V, but at the same time doesn't include anything extra. It is the optimal compromise.

Example 2.1. $V = P_n$, the vector space of all polynomials of degree less than or equal to n:

$$B = \{1, x, x^2, ..., x^n\}$$

This is in fact called the **standard basis for** P_n.

Example 2.2. V = mat($m \times n$, k). Define the following:

$$M_{ij} \mid (m_{ij}) \text{ where } m_{kl} \text{ is } \begin{cases} 0 \text{ if } kl \neq ij \\ 1 \text{ if } kl = ij \end{cases}$$

Then, $\{M_{ij}\}_{1 \leq i \leq m;\ 1 \leq j \leq n}$ is a basis for V.

Theorem 2. *Every finite-dimensional vector space has a basis.*

Proof. We prove this with the **casting-out algorithm**. Let V be a finite-dimensional vector space. Essentially, we will use the definition of a finite-dimensional vector space to obtain "for free" that we have a finite spanning set $\{v_1, ..., v_n\}$. There are two possibilities here:

1. The set could be linearly independent. If this is the case, then by definition of a basis we are done.
2. The set could be linearly dependent. Then, we know that at least one of the vectors could be written as a linear combination of the other vectors. So, we can then eliminate one vector and leave the span unchanged. We an recursively remove vectors until we obtain a finite, linearly independent set after a finite number of steps. As proven in the previous theorem, we would obtain:

   $$span\{v_1, ..., v_n\} = span\{v_1, ..., v_{n-1}\}$$

   Once we obtain the linearly independent set, we could have a linearly independent spanning set, and hence obtain a basis.

□

Question: What is a basis for { 0 }?
Solution: The only possibility is the empty set {}.

2.1 Constructing bases with a bottom-up algorithm

Let V be a finite-dimensional vector space over k. Then, V will have a spanning set consisting of n elements.

1. If V is the zero-space, then we are done.
2. Else, we can pick a non-zero vector, $v_1 \in V$ which is non-zero. Then, $\{v_1\}$ is linearly independent. From here, there are two possibilities:
   (a) It is a spanning set, so we are done.
   (b) Else: there is a vector $v_2 \in V$ s.t. $v_2 \notin span\{v_1\}$. By the previous theorem, this is a linearly independent set.
   (c) And so on...

Since V is finite-dimensional, it cannot: it will stop after at most n steps. So, we have the very important conclusion:

$$\boxed{\textbf{Size of any linearly independent set} \leq \textbf{size of any spanning set}}$$

By Steinitz, this algorithm must terminate after at most n steps. We could then end up with a linearly independent and spanning set, which is by definition a basis.

Example 2.3. P_2:= all polynomials of degree at most 2 with coefficients in k. We already know that the set $\{1, x, x^2\}$ is a basis and therefore spanning. We can run the algorithm to construct a new basis:

$$\{2, 3 + 5x, 7 - 5x + 11x^2\}$$

If we apply Steinitz, we have a linearly independent set of 3 vectors. So, it must be spanning. (Because if we added a fourth vector and still had a linearly independent set, it would be a contradiction to Steinitz).

Fact: every VS except the trivial VS has infinitely many choices of bases.

Theorem 3 (Very important application of Steinitz)**.** *Let V be a* ***finite-dimensional*** *VS over k. Let $\{u_1, ..., u_m\}$ and $\{v_1, ..., v_n\}$ be a basis for V. Then,* ***m=n****. In other words:* ***any two bases of a finite-dimensional vector space must contain the same number of elements***

Proof. "Very nice proof. This is a super arrogant and short proof." We pick two completely arbitrary bases:

$$B_1 = \{u_1, ..., u_m\}$$
$$B_2 = \{v_1, ..., v_n\}$$

[$\Rightarrow$] Especially, B_1 is linearly independent, and especially B_2 is spanning. So, by Steinitz, $m \leq n$. Now, we do the exact opposite:
[$\Leftarrow$] Especially, B_2 is linearly independent and especially B_1 is spanning. Hence, $n \leq m$. Therefore, the only possibility is that $n = m$. □

Definition 2.2. Let V be a finite-dimensional VS over k. Let $\{v_1, ..., v_n\}$ be any basis for V. Then, n is called the **dimension of V**.

- **Dimension** is the number of elements in a basis for V.

Remark 2.1. Only by Theorem 3 is this definition well-defined.

Example 2.4. $\mathbb{R}^n$ as a VS over $\mathbb{R}$ has dim(n) since $\{e_1, ..., e_n\}$.

Example 2.5. $P_n(k)$ a VS over k has dim(n+1) since $\{1, x, x^2, ..., x^n\}$ is a basis with n+1 elements.

Example 2.6. Mat($n \times m$) VS over k has dimension $m \cdot n$ since its basis has $m \cdot n$ elements.

3 Theorems on Bases and Dimension

Theorem 4 (Very important)**.** *Let V be a finite-dimensional VS over K. Say $V = \{B_1, ..., B)n\}$ is a basis for V. Let $v \in V$. Then, there exists a* ***uniquely determined*** *$k_1, ..., k_n \in K$ such that:*

$$v = k_1 v_1 + ... + k_n v_n$$

Proof. V is spanning $\Rightarrow v$ can be written as a **linear combination** of $v_1, ..., v_n$. Assume that we have two such representations: Let

$$v = k_1 v_1 + ... + k_n v_n \text{ and}$$
$$v = l_1 v_1 + ... + l_n v_n$$

Subtract the equations and use the axioms of a vector space to simplify the resulting expressions:

$$0 = (k_1 v_1 + ... + k_n v_n) - (l_1 v_1 + ... + l_n v_n)$$
$$0 = (K_1 - l_1)v_1 + ... + (k_n - l_n)v_n$$

The heart of this proof is the **clever definition of linear independence**. We have that 0 is expressed as a linear combination of linearly independent vectors. Therefore, all coefficients MUST be zero. This forces all the coefficients to be equal, which means that v **can be expressed in exactly one way as a linear combination**. □

Theorem 5. *Let V be a finite-dimensional VS over k. Let $U \leq V$. Then,* $\mathbf{dim(U)} \leq \mathbf{dim(V)}$*. This implicitly uses the result from last class (that U is finite-dimensional, which is a subspace of a finite-dimensional space).*

Proof. The proof is "very straightforward if you do it right."

Let $\{u_1, ..., u_m\}$ be a basis for U. Especially, this is **linearly independent**.
Let $\{v_1, ..., v_n\}$ be a basis for V. Especially, this is a spanning set.

Now, we apply Steinitz and the trick from last class. The first set is linearly independent and the second is spanning, so by Steinitz, we immediately have that $m \leq n \Rightarrow dim(U) \leq dim(V)$. □

The next theorem answers the question: when can we stop constructing bases?

Theorem 6. *Let V be a finite-dimensional V over K of dim(V) = n. Once we obtain n linearly independent vectors, then we are done. In more mathematical language: Let $B = \{v_1, ..., v_n\}$ be linearly independent. Then, B is a basis for V.*

Proof. We have for free that B is linearly independent. Now we need to show that it is a spanning set. **Proof by contradiction**: assume it is not.

- Apply the bottom-up algorithm. We would then obtain a linearly independent set with n+1 elements.

$$\Rightarrow \exists v_{n+1} \in V \mid v_{n+1} \notin span\{v_1, ..., v_n\}$$
$$\Rightarrow \{v_1, ..., v_n, v_{n+1}\} \text{ is linearly independent.}$$

- **Contradiction:** since dim(V) = n, V has a spanning set of length n, so a linearly independent set of size n+1 is a contradiction to Steinitz. Thus, V must also be spanning $\Rightarrow$ it's a basis.

□

Theorem 7 (Kind of converse of previous theorem)**.** *Let V be a finite-dimensional VS over K of dim(V) = n. Let $B = \{v_1, ..., v_n\}$ be spanning. Then: B is a basis for V.*

Proof. We get for free that B is spanning. So, we need to show that it is linearly independent. Again, we will prove this by contradiction

- Apply the top-down algorithm. We can cast out at least one member by retaining spanning. We will obtain a spanning set that's too small for a VS of dimension n.
- Assume: not linearly independent, then:

$$\exists j, 1 \leq j \leq n \mid v_j \text{ is a linear combination of } v_1, .., \hat{v}_j, ..., v_n$$
$$\Rightarrow span\{v_1, ..., v_n\} = V = span\{v_1, ..., \hat{v}_j, ..., v_n\}$$

- **Contradiction**: The new spanning set only has n-1 members. But, since dim(V) = n, V has a spanning set of length n. This is a contradiction to Steinitz. This, B is also linearly independent and thus a basis.

□

Example 3.1. $V = P_2(k)$. We have that dim(V) = 3. Define the following basis:

$$B = \{3, 5 - 9x, 7 - 3x + 13x^2\}$$

B is linearly independent for degree reasons. Thus, B is spanning.

Theorem 8. *Let V be a finite-dimensional VS over k, $U \leq K$. Let $B_u = \{u_1, ..., u_m\}$ be a basis for U. Then, B_u can be* ***extended*** *to a basis B_v. In mathematical language:*

$$\exists u_{m+1}, ..., u_n \in V \;\; s.t.$$
$$B_v := \{u_1, ..., u_m, u_{m+1}, ..., u_n\}$$

is a basis for V. This is essentially the bottom-up approach, but not beginning with the empty set.

The proof is essentially the same argument as the one used for the existence of a basis by the bottom-up algorithm, except we start with B_u as the original step.

Theorem 9. *Let V be a finite-dimensional VS over K, $U \leq V$ and $W \leq V$. (In other words, we are given two subspaces of V). There are various ways to combine these to form new subspaces. In particular:*

$$\boxed{dim(U + W) = dim(U) + dim(W) - dim(U \cap W)}$$

Example 3.2. Any two planes through the origin in $\mathbb{R}^3$ have a non-trivial intersection; i.e., they intersect in **at least** a line through the origin. Using the result from the previous theorem, we can give a formal proof of this.

Planes through the origin are exactly the two-dimensional sub-spaces of $\mathbb{R}^3$. Let U and W be two such planes. Then, we obtain from the dimension formula:

$$\begin{aligned} dim(U + W) &= dim(U) + dim(W) - dim(U \cap W) \\ dim(U \cap W) &= dim(U) + dim(W) - dim(U + W) \\ &= 2 + 2 - dim(U + W) \end{aligned}$$

The dimension of U + W in $\mathbb{R}^3$ is at most 3. So we then obtain the following inequality:

$$\Rightarrow 4 - dim(U + W) \geq 4 - 3$$
$$\Rightarrow 4 - dim(U + W) \geq 1$$

Hence, the intersection is of at least dimension 1, which implies that it is at least a line. So, we have a subspace of $\mathbb{R}^3$ of dim ≥ 1 and is thus at least a line.