

Math 247: Applied Linear Algebra

Course Notes - Week 6

Main topics: Compositions of linear maps, matrix representations of isomorphisms, change of basis, similarity, normal form problem, equivalence relation.

1 From Last Time

Recall the example from last class:

$$\begin{aligned} L : \text{Mat}(2 \times 2, \mathbb{R}) &\rightarrow \text{Mat}(2 \times 2, \mathbb{R}) \\ A &\mapsto A - A^T \\ B &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

We can then obtain $[L]_{B,B}$ by unravelling the matrices from the basis and using them as the columns of $[L]_{B,B}$

$$[L]_{B,B} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Consider an alternative basis B' .

$$\begin{aligned} L : \text{Mat}(2 \times 2, \mathbb{R}) &\rightarrow \text{Mat}(2 \times 2, \mathbb{R}) \\ A &\mapsto A - A^T \\ B' &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \end{aligned}$$

Applying the transformation, we then obtain:

$$L(B') = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix}$$

We need to express the images as linear combinations of B' . Trivial for the first three:

$$= 0 \cdot \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + 2 \cdot \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

Collect coefficients into a 4×4 matrix:

$$\Rightarrow [L]_{B',B'} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$[L]_{B,B} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The normal form question: what basis can we choose to obtain the simplest matrix?

2 Compositions of linear maps

Let U , V , and W be finite-dimensional VS over k with bases B , B' , and B'' . Let's define two linear maps:

$$L_1 : U_B \rightarrow V_{B'}$$

$$L_2 : V_{B'} \rightarrow W_{B''}$$

Then, we consider the following transformation:

$$L_2 \circ L_1 : U_B \rightarrow W_{B''}$$

Question: how are $[L_1]_{B',B}$, $[L_2]_{B'',B'}$, and $[L_2 \circ L_1]_{B'',B}$ related?

Theorem 1. Let L_1 and L_2 be defined as above. Let $\dim(U) = m$, $\dim(V) = k$, and $\dim(W) = n$. Then:

$$[L_2 \circ L_1]_{B'',B} = [L_2]_{B'',B'} \cdot [L_1]_{B',B}$$

This theorem is basically saying that you can obtain a matrix composition by multiplying the matrix maps. This is why matrix multiplication is so complicated; it's defined to be that way such that we can use it for linear maps.

Proof. Let u be an arbitrary vector in U . Then, the idea is to simply compute what $[L_2 \circ L_1]_{B'',B}[u]_B = [(L_2 \circ L_1)(u)]_{B''}$ is. As was discussed last week, we know that:

$$[L_2(L_1(u))]_{B''} = [L_2]_{B'',B'}[L_1(u)]_{B'}$$

We are going to apply the same idea but backwards:

$$\begin{aligned} &\Rightarrow [L_2]_{B'',B'}[L_1]_{B',B}[u]_B \\ &\Rightarrow [L_2 \circ L_1]_{B'',B} = \text{product of } [L_2]_{B'',B'}[L_1]_{B',B} \\ &\Rightarrow [L_2 \circ L_1]_{B'',B} = [L_2]_{B'',B'} \cdot [L_1]_{B',B} \end{aligned}$$

□

3 Matrix Representations of Isomorphisms

Theorem 2. V, W finite-dimensional VS over k , $L : V \rightarrow W$ be linear and let B, B' be bases for V, W , respectively. Claim: the linear map is an isomorphism \iff the matrix representation of the linear map $[L]_{B',B}$ is invertible. This is the same thing as insisting that L is bijective.

Remark 3.1. Consider the identity map, $id : V \rightarrow V$, with $\dim(V) = n$, and a basis B . Then:

$$[id]_{B,B} = I_n \text{ (} n \times n \text{ identity matrix)}$$

Why? We can just use the definition of a matrix map.

If $B = (v_1, \dots, v_n)$, then the matrix representation of $[id]_{B,B}$ would then be given by:

$$([id(v_1)]_B | \dots | [id(v_n)]_B)$$

From last week's lectures, it therefore follows:

$$= ([v_1]_B | \dots | [v_n]_B)$$

By expressing each v_i as a linear combination of v_1, \dots, v_n , we then obtain the following matrix:

$$= \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} = I_n$$

Now, to prove the theorem:

Proof. Proving both directions:

$[\Rightarrow]$ Let $L : V(B) \rightarrow W(B')$ be an isomorphism. This means that L is bijective \Rightarrow automatically has an inverse map that we automatically know is linear: L has an inverse L^{-1} which is $L^{-1} : W_{B'} \rightarrow V_B$. We now have: $L^{-1} \circ L : V_B \rightarrow V_B, L^{-1} \circ L = id_V$.

We now apply the remark and composition formula:

$$[L^{-1} \circ L]_{B,B} = I = [L^{-1}]_{B,B'} \cdot [L]_{B',B}$$

Matrices whose product is I implies that one of them is invertible. Similarly:

$$[L]_{B',B} \cdot [L^{-1}] = I \Rightarrow [L]_{B',B} \text{ is invertible.}$$

$[\Leftarrow]$ We know that our matrix is invertible. So, we need to prove that it is bijective. So, let $[L]_{B',B}$ be an invertible $n \times n$ matrix. We will prove that L is injective first. To prove this, we need to show that the kernel is trivial.

Let $v \in \text{Ker}(L) \Rightarrow L(v) = 0 \Rightarrow [L(v)]_{B'} = 0$. We then obtain:

$$[L]_{B',B} \cdot [v]_B = [L(v)]_{B'} = 0$$

We know that $[L]_{B',B}$ is an invertible matrix, so, it cannot be zero. This means that $[v]_B$ must be zero $\Rightarrow v = 0$. We chose an arbitrary vector in the kernel, showed that it was equal to zero. This means that $\text{Ker}(L) = \{0\}$, which means that L is injective.

Now, we need to show that L is **surjective**. Since $[L]_{B',B}$ is an $n \times n$ matrix, we have that:

$$\dim(V) = \dim(W) = n$$

By the dimension formula, we obtain:

$$\begin{aligned}\dim(\text{Ker}) + \dim(\text{Im}) &= \dim(V) \\ 0 + \dim(\text{Im}) &= n\end{aligned}$$

This implies that W is an n -dimensional subspace of an n -dimensional vector space. The only way this is possible is that if $\text{Im}(L) = W$, which is the very definition of L being surjective. So, since injectivity and surjectivity hold, we then obtain that L is bijective, and from this we obtain that L is an isomorphism. \square

4 Change of Basis

There are two questions that will motivate this section:

1. Let $v \in V$. Given the coordinate vector $[v]_B$ of v with respect to the basis B , how can we compute the coordinate vector $[v]_{B'}$ with respect to B' , another basis for the same vector space?
2. Let $L : V \rightarrow V$ be linear. Given the matrix representation $[L]_{B,B}$, how can we compute $[L]_{B',B'}$.

4.1 Question 1

Consider the linear map $\text{id} : V_B \rightarrow V_{B'}$. The matrix representation is given by $[\text{id}]_{B',B}$. By definition, we obtain:

$$[\text{id}]_{B',B}[v]_B = [\text{id}(v)]_{B'} = [v]_{B'}$$

The **transition matrix/change of basis matrix** from B to B' is given by $[\text{id}]_{B',B} \cdot [v]_B$. This formula is important:

$$[v]_{B'} = [\text{id}]_{B',B} \cdot [v]_B \tag{1}$$

4.2 Question 2

Consider the following sequence of maps:

$$V_{B'} \xrightarrow{\text{id}} V_B \xrightarrow{L} V_B \xrightarrow{\text{id}} V_{B'}$$

Then,

$$\begin{aligned}[L]_{B',B'} &= [\text{id} \circ L \circ \text{id}]_{B',B'} \\ &\Rightarrow [\text{id}]_{B',B} [L]_{B,B} [\text{id}]_{B,B'} \\ &\Rightarrow ([\text{id}]_{B,B'})^{-1} [L]_{B,B} [\text{id}]_{B,B'}\end{aligned}$$

Therefore, $[L]_{B,B}$ and $[L]_{B',B'}$ are similar matrices. This is **how we transform one matrix representation to another**. Representations of the same linear map are possible if there exists an invertible matrix P .

Definition 4.1. Two n by n matrices A, B with coefficients in K are called **similar** if there exists an invertible $n \times n$ matrix P with:

$$B = P^{-1}AP$$

Thus, any two matrix representations of the same linear map $L : V \rightarrow V$ with respect to bases B, B' are similar.

Special Case (Most important)

$V = K^n$ with two bases: the standard basis S and an arbitrary basis B . **First Question:** if we consider the vectors in K^n as row vectors, what is the coordinate vector of $[v]_s$? The solution:

$$[v]_s = v^t$$

How can we compute the coordinate vector $[v]_B$?

$$[v]_B = [id]_{B,S}[v]_S$$

Where $[id]_{S,B}$ is the transition matrix B to S . Now, we can **feed in arbitrary members of B** by expressing them in terms of the standard basis. Let $B = (v_1, \dots, v_n)$. Then: $[id]_{S,B} = (v_1^t | \dots | v_n^t)$ and $[id]_{B,S} = ([id]_{S,B})^{-1}$. So, we can basically find the vector in terms of the original basis by “inverting” the transformation:

$$[v]_B = P^{-1}[v]_S$$

Where $P = (v_1^t | \dots | v_n^t)$.

Example 4.1. $V = \mathbb{R}^2$, S standard basis, and $B = ((1, 1), (3, 2))$. From this, we can obtain two change of basis matrixes:

$[id]_{S,B}$: from B to S . Relatively straightforward; just take the rows of B and use them as the columns of the change of basis matrix:

$$[id]_{S,B} = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}$$

$[id]_{B,S}$ this is more complicated; we need to compute $([id]_{S,B})^{-1}$. We then obtain:

$$[id]_{B,S} = \begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix}$$

Putting it all together, we then obtain:

$$[v]_B = [id]_{B,S}[v]_S$$

Example 4.2. $v = (2, 3)$. So, $[v]_S = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$. We can obtain $[v]_B$ by:

$$[v]_B = \begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

Example 4.3. $L : \mathbb{R}^2 \rightarrow \mathbb{R}^2$. Remark that this is an endomorphism. Let this linear map define the projection onto the x-axis; i.e.,:

$$\begin{aligned} L((1, 0)) &= (1, 0) \\ L((0, 1)) &= (0, 0) \\ \Rightarrow [L]_{S,S} &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

Using the formula, we then obtain:

$$\begin{aligned} [L]_{B,B} &= [id]_{B,S} \cdot [L]_{S,S} \cdot [id]_{S,B} \\ &= ([id]_{S,B})^{-1} [L]_{S,S} [id]_{S,B} \\ &= \begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & -6 \\ 1 & 3 \end{bmatrix} \end{aligned}$$

To check if your solution is correct, you feed in a coordinate vector, compute the output, and verify that it is correct.

Example 4.4. $V = \text{Mat}(2 \times 2, \mathbb{R})$, $L : A \mapsto A - A^t$. Define the following bases:

$$\begin{aligned} B &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\ B' &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \end{aligned}$$

Last class, we computed the following matrices:

$$\begin{aligned} [L]_{B,B} &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, [L]_{B',B'} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \\ [id]_{B,B'} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, [id]_{B',B} = ([id]_{B,B'})^{-1} = \frac{1}{2} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & -1 & 0 \end{bmatrix} \end{aligned}$$

5 The Normal Form Problem

Similar matrices represent the same linear map with respect to different bases. So, let $A \in \text{MAT}(n \times n, k)$. Let $L_A : K^n \rightarrow K^n$, $v \mapsto Av$. Let S be a standard basis for K^n be an arbitrary basis. Then, $[L_A]_{S,S} = A$, $B = \{v_1, \dots, v_n\}$. Then:

$$\begin{aligned} [L_A]_{S,S} &= [id]_{B,S} [L_A]_{S,S} [id]_{S,B} \\ &= P^{-1}AP \end{aligned}$$

Where the columns of P consist of the vectors in B written as column vectors: $P = (v_1^t | \dots | v_n^t)$. P is invertible \Rightarrow we have n linearly independent columns in $K^n \Rightarrow$ they span $K^n \Rightarrow$ they form a basis for K^n .

Algebra Crash Course

General linear group: denoted $P \in GL(n, k)$, a general linear group is the set of all invertible $n \times n$ matrices with coefficients in k . It's a group.

Question: can we interpret $P^{-1}AP$ as the matrix representation of our linear map L with respect to some basis B of K^N ? Yes! We can just take $P = (v_1^t | \dots | v_n^t)$. Here's the chain of reasoning: P is invertible $\Rightarrow (v_1, \dots, v_n)$ are linearly independent; $\dim(K^n) = n \Rightarrow \{v_1, \dots, v_n\}$ also span $K^n \Rightarrow$ basis for K^n . So, we can write:

$$[L_A] = P^{-1}AP$$

We will see that there is a one-to-one correspondence between matrix representations of our linear map L and matrices similar to A . In other words: **matrices similar to A can be interpreted as representing the same linear map, just with respect to a different basis**. So, in the Linear Algebra Universe, they're practically equivalent. This brings us to the final section.

6 Similarity Problem

Motivation: we are interested in knowing the set of all matrices $\in MAT(n \times n, k)$ similar to A . This is the **similarity problem**. This introduces a relation: a set of any two matrices can be similar or not. We obtain two mathematical relations from this:

1. Order
2. Class of equivalence

Definition 6.1. Let $A, B \in MAT(n \times m, k)$. B is said to be **similar** to A , in symbols, $B \sim A$, if there exists a $P \in GL(n, k)$ such that

$$B = P^{-1}AP$$

Before this, we need more quick algebra.

6.1 Quick Algebra

Definition 6.2. Let S be a set. An **equivalence relation** is a subset of the cartesian product, $S \times S$. In maths, it's denoted by a \sim and has the following three properties:

1. **Identity:** every element of S is in relation with itself:

$$\forall x \in S, x \sim x$$

2. **Commutativity:**

$$\forall x, y \in S, \text{ if } x \sim y, \text{ then } y \sim x$$

Alternatively: $x \sim y \iff y \sim x$

3. **Transitivity:**

$$\forall x, y, z \in S, \text{ if } x \sim y \wedge y \sim z \Rightarrow x \sim z$$

Example 6.1. $S = \mathbb{Z}, n \sim k \iff n - k$ is even. **Claim:** this is an equivalence relation. We can check this with the axioms.

1. $n \sim n$ since $n - n = 0$ which is even.
2. $n \sim k \Rightarrow n - k$ is even $\iff k - n$ is even $\Rightarrow k \sim n$.
3. $n \sim k \wedge k \sim m \Rightarrow$ show that $n - m$ is also even: $(n - k) + (k - m) \Rightarrow$ we have the sum of two even things, so the sum will also be even and we have $n \sim m$, as desired.

Main idea: equivalence relationships partition things into classes. In this case, all even numbers are similar/equivalent to zero and all odd numbers are equivalent to 1. So, we have the set of all integers partitioned into two, disjoint groups: (1) odd numbers, and (2) even numbers.

The set of all $\{n \in \mathbb{Z} \mid n \sim 0\} = \{\dots, -4, -2, 0, 2, 4, \dots\}$ (the set of all even numbers)

The set of all $\{n \in \mathbb{Z} \mid n \sim 1\} = \{\dots, -3, -1, 1, 3, \dots\}$ (the set of all odd numbers)

Definition 6.3. Let S be a set, $s \in S$, \sim an equivalence relation on S . Then, we can define the **equivalence class** of S as follows:

$$[s]_{\sim} := \{t \in S \mid t \sim s\}$$

Remark that $s \in [s]_{\sim}$ by axiom 1 of equivalence relations. Back to our example:

Example 6.2. $S = \mathbb{Z}, \sim, n - k$ even. Then:

1. $[0]$ is the set of all even numbers.
2. $[1]$ is the set of all odd numbers.

These are equivalence classes that **partition** \mathbb{Z} (the set containing all integers). This, in fact, is true in general:

Theorem 3. Let S be a set, \sim be an equivalence relation. The equivalence classes of \sim **partition** S , i.e., every $s \in S$ is contained in an equivalence class and any two equivalence classes are either **identical** or **disjoint**.

Proof is next class...