Math 247: Applied Linear Algebra
Week 5 Course Notes

Main topics: Endomorphisms, surjectivity, injectivity, coordinate vectors, bijective, matrices, transformations

1 Endomorphisms

Definition 1.1. A linear map $L : V \to V$, V VS over K, is called an **endomorphism**. In other words, this is a linear from a vector space to itself.

Theorem 1. *Let V be a finite-dimensional VS over K, $L : V \to V$ an endomorphism. Then, we have:*

$$L \text{ is bijective} \iff L \text{ is injective} \iff L \text{ is a surjection}$$

This theorem does not hold for infinite-dimensional VS. Examples were already discussed...

Before the proof, here is a useful lemma

Lemma 2. *Let V be a finite-dimensional VS over k and let $U \geq V$ s.t. dim(U) = dim(V). Then, we have that* $\boldsymbol{U} = \boldsymbol{V}$.

Proof. Let $\{v_1, ..., v_n\}$ be a basis for U. This especially means that the vectors are linearly independent. This means we the basis of V will also contain n elements, since their dimensions are equal.

$$\Rightarrow \{u_1, ..., u_n\} \text{ is linearly independent} \Rightarrow n = dim(U) = dim(V).$$
$$\Rightarrow \{u_1, ..., u_n\} \text{ is a basis or V, and it especially spans V} \Rightarrow U = V$$

Trick: vector spaces are equal if they contain bases that span the same area. □

Now, we need to prove the theorem.

Proof. We only need to prove the non-trivial implications.
[$\Rightarrow$] Injectivity $\Rightarrow$ surjectivity
The proof of this direction is a consequence of the dimension formula. Recall:

$$dim(Ker) + dim(Im) = dim(V)$$

By assumption, we are given that L is injective. So: $ker(K) = \{0\} \Rightarrow dim(Ker) = 0$. Plugging this into our formula, we then obtain:

$$dim(Im) = dim(V)$$

Now, we are in the exact situation of the lemma: we have an endomorphism with $im(L) \leq V$ of the same dimension. By the lemma, we then obtain that $im(L) = V \Rightarrow L$ is surjective.

[$\Rightarrow$] Surjective $\Rightarrow$ injective
We assume that L is surjective. It then follows that $im(L) = V$. We then obtain the following:

$$\Rightarrow dim(Im(L)) = dim(V)$$
$$\Rightarrow dim(Ker(L)) + dim(Im) = dim(V)$$
$$\Rightarrow dim(Ker(L)) = 0 \Rightarrow Ker = \{0\}$$

If the kernel is 0, then L is injective and we are done. □

Remark 1.1. Comparable results are wrong in almost any other context. The result is also in general wrong for endomorphisms between infinite-dimensional vector spaces.

2 Coordinate Vectors

All bases in this section are considered to be **ordered**, and are therefore written as a tuple to maintain order:

$$B = (v_1, v_2, ..., v_n)$$

In this section, we will consider vectors in $\mathbb{R}^n$ or $\mathbb{C}^n$ as **column-vectors** since we will be introducing matrices soon.

Let V be a finite-dimensional VS over k. Considered the ordered basis below:

$$B = (v_1, ..., v_n)$$

Let $v \in V$. Then, there exists uniquely determined $a_1, .., a_n \in K$ such that:

$$v = a_1 v_1 + ... + a_n v_n$$

This uniqueness implies that we can find any vector just with its coefficients. If we collect all these coefficients into a single object of size $n \times 1$, we obtain a **column vector**:

$$\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{K}^n$$

This object uniquely represents v and is called the **coordinate vector of v with respect to our choice of basis**. It is symbolized as: $[\ v\ ]_B$. Now, we have an object representing the coefficients. It induces a map from $V \to K^n$, and, it turns out that this map is both bijective and linear.

Now, consider the following coordinate map:

$$[\ ]_B : V \to K^n;\ v \mapsto [\ v\ ]_b$$

Here, v maps to its coordinate vector with respect to B, which lives in K^n. We obtain the following useful theorem:

Theorem 3. *Let V be finite-dimensional, B an ordered basis for V. Then, $[\ ]_B : V \to K^n;\ v \mapsto [\ v\ ]_b$ is linear and bijective, and it is called an **isomorphism**.*

Proof. We need to prove two things: (1) linearity and (2) bijectivity.

Linearity: most of the work for this was done at the beginning of class. Let $B = (v_1, ..., v_n)$. We then obtain:

$$[\ v_1\ ]_B = 1 \cdot v_1 + 0 \cdot v_2 + ... + 0 \cdot v_n = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = e_1$$

$$[\ v_2\ ]_B = 0 \cdot v_1 + 1 \cdot v_2 + ... + 0 \cdot v_n = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} = e_2$$

$$\vdots$$

$$[\ v_n\ ]_B = 0 \cdot v_1 + 0 \cdot v_2 + ... + 1 \cdot v_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} = e_n$$

We then use the existence theorem (proven last class): there exists a uniquely determined linear map L , $V \to k^n$ with $L(v_1) = e_1, ..., L(v_n) = e_n$. We have a formula for this:

$$L(a_1 v_1, ..., a_n v_n) = a_1 e_1 + ... + a_n e_n$$
$$= \begin{pmatrix} a_1 \\ a_2 \\ ... \\ a_n \end{pmatrix}$$

This is precisely what the coordinate map $L = [\ ]_B$ is $\Rightarrow [\ ]_B$ is linear (we obtain that for free from Theorem 3). **Proving bijectivity**: To prove bijectivity, we will prove injectivity and surjectivity.

Injective: We know that $[\ v\ ]_B = \begin{pmatrix} 0 \\ 0 \\ ... \\ 0 \end{pmatrix} \iff v = 0 \cdot v_1 + ... + 0 \cdot v_n$. The matrix is the image. This means that $Ker([\ v\ ]_B$ is trivial. A trivial kernel means injectivity.

Surectivity: If we let $\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in K^n$ be arbitrary. We need to then find a vector in V whose coordinate map is that object. But, we can express any vector in v as a linear combination of the basis vectors, i.e.,:

$$v := a_1 v_1 + ... + a_n v_n$$

Then: $[\ v\ ]_B = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$. Therefore, we have proven that it is surjective.

So, we have a bijective map. Combined with linearity, we have that this is an isomorphic map. This is useful because in algebra, isomorphic objects have the same properties. The "different objects" are simply re-labeling. □

Theorem 4. *Crux: if you have bijectivity, then you have an inverse.*
More formally: let $L : V \to W$ be an isomorphism. Since L is bijective, it is invertible $\to$ it has an inverse map L^{-1}. So, the claim we need to prove that

$$L^{-1} : W \to V$$

is linear.

Proof. Let $w_1, w_2 \in W$. Then, there exists a uniquely determined $v_1, v_2 \in V$ with $L(v_1) = w_1, L(v_2)\ w_2$. Then, by the linearity of L, we obtain:

$$L^{-1}(w_1 + w_2) = L^{-1}(L(v_1) + L(v_2))$$

Since we know that **L is linear**, we can then simplify:

$$\begin{aligned} &\Rightarrow L^{-1}(L(v_1 + v_2)) \\ &\Rightarrow v_1 + v_2 \text{ (since inverses cancel out)} \\ &\Rightarrow L^{-1}(w_1) + L^{-1}(w_2) \text{ (by construction)} \end{aligned}$$

We have proven that L^{-1} respects addition. Similarly, we can prove that it also respects scalar multiplication. Let $k \in K$. We then obtain:

$$L^{-1}(kw_1) = L^{-1}(k(L(v_1)) = kv_1 = KL^{-1}(w_1)$$
$$\Rightarrow L^{-1}(kw_1) = kL^{-1}(w_1)$$

Therefore, L^{-1} is linear and we are done; we have no additional condition. This case holds in finite and infinite-dimensional cases. □

Theorem 5. *Let V be a VS over k of dim n. Then, V is isomorphic to K^n. This means that there is only ONE vector space over K per dimension. This theorem makes linear algebra very applicable to other fields.*

Proof. Let B be an ordered basis for V. Then, $[\ ]_B : V \to K^n$ is an isomorphism by Theorem 3. □

All of linear algebra is over K^n. Matrices are the equivalent of linear maps. So, the key question that remains is: **how can we interpret linear maps as matrices?**.

3 Representations of Linear Maps as Matrices

Motivation: Let V, W be finite-dimensional VS over k. Let L be a linear map, $V \to W$. Let B be a set of ordered bases for V, and B' be an ordered basis for W.

Consider the coordinate vector:

$$[\ v\ ]_B;\ dim(V) = m, dim(W) = n$$

If we let $v \in V$ and $w \in W$. Then, consider the following:

$$[\ v\ ]_B \in K^m;\ [\ w\ ]_B \in K^w$$

If w is the image under L(v), then L would map m-tuple to n-tuple by an $n \times m$ matrix. **Does there exist such a matrix that mimics what the linear map L does? In other words, does there exist an $n \times m$ matrix A with coefficients in K that mimics the action of L?**

$$A \cdot [\ v\ ] = [\ w\ ]$$

If this happens, then we say that **A is the matrix representation of L with respect to B and B'**.

Question: how do we construct this map?
Idea: before we construct A in the general case, we will re-visit the more concrete or classical case of $V = K^m$ and $w = K^n$, with B being the standard basis of K^m and B' the standard basis of K^n. Then, by what we know from Math 133, the linear map $L : K^m \to K^n$ is of the form $L(x) = Ax$, where A is an $n \times n$ matrix. Once we understand the concrete case, we will copy-and-paste it to the abstract case.

$$A = \begin{pmatrix} a_{11} & \dots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nm} \end{pmatrix}$$

We know that $Ae_1 = L(e_1) = \begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \end{pmatrix}$, i.e., the first column of A. Continuing this, we then obtain that: $Ae_m = L(e_m) = \begin{pmatrix} a_{nm} \\ \vdots \\ a_{nn} \end{pmatrix}$,i.e., the mth coolumn of A.

The following construction is CRITICAL:

$$A = (L(e_1)\ \mid\ L(e_2)\ \mid\ \cdots \mid\ L(e_m)\ )$$

This actually makes perfect sense with our basis being $(b_1, b_2, ..., b_m)$ From this, we can obtain any $n \times m$ matrix. This will motivate our matrix representation of a map.
Generalizing...
Let $L : V \to W$ be linear, V,W, VS over K, dimension of V = m and dimension of W = n. $B = \{v_1, ..., v_m\}$ is a basis for V, $B' = \{w_1, ..., w_n\}$ is a basis for W. We could map all the members of our basis under L.

We will define the following: let $[\ L\ ]_{B',B}$ be the matrix representation of L with respect to the two bases B and B' by:

$$[\ L\ ]_{B',B} = (L(v_1)_{B'} \mid \cdots \mid L(v_m)_{B'}\ )$$

Each of the components are $n \times 1$ vectors, and we have m columns in total. Hence, w obtain a $n \times m$ matrix.

Theorem 6. *$L : V \to W$ be linear, dim(V) = m, dim(W) = n, $\{v_1, ..., v_n\} = B$ basis for V, $B' = \{w_1, ..., w_n\}$ basis for W. Then:*

$$[(v)\ ]_{B'} = [\ L\ ]_{B',B}[\ v\ ]_B\ \forall v \in V$$

Breaking this down, we obtain that....

$$\begin{aligned} &[L]_{B'B} \text{ is an } n \text{ by } m \text{ matrix} \\ &[v]_B = \text{ is an } m \text{ by 1 column vector} \\ &[L(v)]_{B'} = \text{ is an } n \text{ by 1 column vector} \end{aligned}$$

The idea here is that we can **mimic the action of L by matrix multiplication**.

Proof. Let $v \in V$ be arbitrary. Then, we can write v as a linear combination of some sort of basis vector:

$$v = a_1v_1 + ... + a_mv_m$$

First, we need to obtain the coordinate vector of v. That is given by:

$$[\ v\ ]_B = \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix}$$

Using the formula from above, we calculate the expression on the RHS:

$$[\ L\ ]_{B'B}[\ v\ ]_B = ([L(v_1)_{B'}] \mid \ldots \mid [L(v_m)]_{B'}) \cdot \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

□

Recall from Math 133 that we can take a linear combination of the columns and obtain:

$$= a_1[L(v_1)]_{B'} + ... + a_m[L(v_m)]_{B'}$$

Because the coordinate map $[\ ]_{B'} : W \to K^n$ is linear, we can move in the constants:

$$= [a_1L(v_1)]_{B'} + ... + [a_mL(v_m)]_{B'}$$

Applying linearity once more, we then obtain:

$$= [a_1L(v_1) + ... + a_mL(v_m)]_{B'}$$

And, we can play the same game once more:

$$\begin{aligned} &= [\ L(a_1v_1 + ... + a_mv_m)\ ]_{B'} \\ &= [L(v)]_{B'} \end{aligned}$$

Theorem 7. *Let $L : V \to W$, V, W, finite-dimensional, L linear, B a basis for V, and B' a basis for W, $A := [\ L\ ]_{B',B}$. Then:*

- *Then, Ker(L) isomorphic to nullspace(A).*

- *Then, Im(L) isomorphic to colspace(A).*

The isomorphism is given by:

- $Ker(L) \to nullspace(A)$

$$v \mapsto [\ v\ ]_B$$

- $im(L) \to colspace(A)$

$$w \mapsto [\ w\ ]_{B'}$$

Proof. $v \in Ker(L) \iff L(v) = 0 \iff [L(v)]_{B'} = 0$
Using the formula, we then obtain that

$$0 = A[\ v\ ]_B$$

This happens iff $[v]_B \in$ nullspace(A). Now, for image: Let $w \in Im(L)$ This happens $\iff$ $\exists v \in V \mid L(v) = w$ which happens $\iff$ $[L(v)]_{B'} = [w]_{B/}$. We know that the LHS is equal to: $[L]_{B',B} \cdot [v]$. But, this is precisely Ax, so we then obtain the following: $\iff$ $[\ w\ ]_B \in colspace(A)$. □

Example 3.1. Consider $D : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ (this is an endomorphism, so it is customary to choose the same basis).

$$B = B' = (1, x, x^2);\ D(p) := p'$$
$$[D]_B = ?\ \text{How can we calculate this?}$$

We will work through the machinery:

$$D(1) = 0$$

We will then express the result as a linear combination of the basis members and collect coefficients:

$$D(1) = 0 = 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2$$
$$D(x) = 1 = 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2$$
$$D(x^2) = 2x = 0 \cdot 1 + 2 \cdot x + 0 \cdot x^2$$

We then take each derivative entry and put the as the columns of the matrix, as we did with the formula:

$$\Rightarrow [D]_{BB} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

This matrix represents the transformation of the derivative. The theory basically tells us how we can differentiate a polynomial via matrix multiplication. **Obtaining the kernel and the image**:

- Ker(D) = set of all constant polynomials, $P_0(\mathbb{R})$.
- Im(D) = $p_1(\mathbb{R})$.

Example 3.2. $L : Mat(2 \times 2, \mathbb{R}) \to Mat(2 \times 2, \mathbb{R})$, $A \mapsto A - A^T$. The fact that L is linear is trivial, just simplify using the properties of transposes from Math 133. Now, let $B = B'$. We again have an endomorphism. Standard basis:

$$B = B' = \{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\}$$

We then obtain the matrix representation of our linear transformation:

$$\begin{aligned} L(M_{11}) &= 0 \cdot M_{11} + 0 \cdot M_{12} + 0 \cdot M_{21} + 0 \cdot M_{22} \\ L(M_{12}) &= 0 \cdot M_{11} + 1 \cdot M_{12} + (-1) \cdot M_{21} + 0 \cdot M_{22} \\ L(M_{21}) &= 0 \cdot M_{11} + (-1) \cdot M_{12} + 1 \cdot M_{21} + 0 \cdot M_{22} \\ L(M_{22}) &= 0 \cdot M_{11} + 0 \cdot M_{12} + 0 \cdot M_{21} + 0 \cdot M_{22} \end{aligned}$$

$$\Rightarrow [L]_{BB} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

For example, if we wanted to differentiate $A := \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}$, we could carry out the following matrix multiplication:

$$\begin{aligned} &\Rightarrow [A]_B \cdot [L]_{BB} = [L(A)]_B \\ &\Rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 2 \\ 0 \end{bmatrix} = [L(A)]_B \end{aligned}$$

Tobtain the **kernel**, we need to carry the matrix to reduced row-echelon form. This is given by:

$$\begin{bmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We will call our variables $a_{11}, a_{12}, a_{21}, a_{22}$. From the reduced matrix, we know that a_{11} and a_{22} are free. We additionally obtain:

$$\begin{aligned} &\Rightarrow a_{12} - a_{21} = 0 \\ &\Rightarrow a_{21} = a_{12} \end{aligned}$$

Since we have a_{11}, a_{21}, and a_{22} as free parameters, we have a three-dimensional nulls-space.

Symbolically, this means:

$$
\begin{aligned}
nullspace = span &\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\} \\
&= \left\{ \begin{pmatrix} a_{11} \\ a_{12} \\ a_{12} \\ a_{22} \end{pmatrix} \middle| \; a_{11}, a_{12}, a_{22} \in \mathbb{R} \right\} \\
&= \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} \middle| \; a_{11}, a_{12}, a_{22} \in \mathbb{R} \right\}
\end{aligned}
$$

We can obtain the image since it is the column space:

$$
Im(L) = span \left\{ \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} \right\}
$$

Dim(Im) $= 1$. We can interpret the result as a matrix: the set of all skew symmetric matrices:

$$
im(L) = \left\{ \begin{pmatrix} 0 & a_{12} \\ -a_{12} & 0 \end{pmatrix} \middle| \; a_{12} \in \mathbb{R} \right\}
$$