Math 247: Linear Algebra with Applications

Course Notes - Week 2

Main topics: subspaces, intersections, unions, linear combinations, span, linear independence, linear dependence, finite-dimensional, Steinitz exchange lemma.

1 Subspaces II

Theorem 1. *If we have a vector space V over K, then U (a subset $U \subseteq V$ of V), is called a* ***subspace*** *iff:*

1. *U contains the zero vector.*
2. *U is closed under the operations of a vector space $(+, \cdot)$.*

Proof. Need to prove in both directions

[$\Rightarrow$] was last class.
[$\Leftarrow$] we need to prove why U is a vector space in its own right. Most of the axioms hold for self-evident reasons. Since U is a subset of V, it inherits many properties from V. We will check the axioms:

- Addition: **associativity** and **commutativity** are inherited from V since V is a vector space. Additionally, the **neutral element** is contained in U by condition. Some more tricky aspects:
  - There exists an **additive inverse**: if we let u be an arbitrary vector. Since U is closed under scalar multiplication by condition 2, we have
  $$(-1)u = -u \in U$$
  - This proves that all the axioms of addition hold.
- Scalar multiplication: all axioms self-evidently hold since they hold $\forall v \in V$.

Hence, U is a vector space with respect to $+$ and $\cdot$, i.e., $U \leq V$. □

Some examples of sub-spaces

Example 1.1. All subspaces of $\mathbb{R}^2$: (1) $\{0\}$, (2) all lines through the origin, and (3) $\mathbb{R}^2$.

Example 1.2. Let $V := mat(n \times n, k)$. Define U to be the set of all upper-triangular matrices, i.e., $U := \{A \in Mat(n \times n, k) \mid a_{ij} = 0 \ \forall i > j, \ 0 \leq i, j, \leq n\}$ with coefficients in K. This is an important example. It is clearly a subset and a subspace.

Example 1.3. Here is a chain of subspaces: $C_{00} \leq C_0 \leq C \leq S$. In words: the set of all convergent sequences constantly equal to zero is a subspace of the set of all convergent sequences that converge to zero which is a subspace of all convergent sequences which is a subspace of all sequences.

Example 1.4. Let $P_n(k)$ be the set of all polynomials with degree at most n. Let $P(k)$ be the set of all polynomials following standard addition of polynomials. Then:

$$P_0(k) \leq P_1(k) \leq P_2(k) \leq ... \leq P_n(k) \leq ... \leq P$$

All polynomials are **infinite-dimensional vector spaces** in their own right. Additionally, the **dimension** of P_n is $n + 1$.

2 Intersections and unions of subspaces

Theorem 2. ***Intersections of subspaces***
(a) Let u, w be subspaces of V, which is a vector space over K. Then: $U \cap W \leq V$.
(b) Let $u_1, ..., u_n \leq V$. Then: $u_1 \cap ... \cap u_n$, which is equivalent to $\bigcap_{i=1}^{n} u_i$ is a subspace of V.
(c) Let I be an arbitrary index set, and $u_i \leq V \ \forall i \in I$. Then:

$$\bigcap_{i \in I} u_i \leq V$$

Proof. Proving each part:

1. Checking the axioms:
   (a) **Contains zero vector**: $0 \in u$ and $0 \in w$ iff $0 \in W \cap W$.
   (b) **Closure under addition**: Let $v_1, v_2 \in U \cap W$. Then, especially, we have that v_1 is in U and v_2 is in U, so $v_1 + v_2 \in U$ since U is closed under addition. Similarly, $v_1 + v_2 \in W$. Since the sum is in both subspaces, then it is also in the intersection. Hence, it is closed under addition.
   (c) **Closure under scalar multiplication**:

$$\begin{aligned} &u_1 \in U \Rightarrow ku_1 \in U \ \forall k \in K \text{ (by assumption)} \\ &u_1 \in W \Rightarrow ku_1 \in W \ \forall k \in K \text{ (by assumption)} \\ &\Rightarrow kv_1 \in W \cap W \Rightarrow \text{ the intersection is closed under multiplication.} \end{aligned}$$

$$\Rightarrow W \cap W \leq V$$

2. The proof is basically the same as (a).

□

Remark 2.1. In general, intersections of subspaces are not a problem. Unions, however, are more difficult. **Generally speaking, unions of subspaces are not subspaces themselves**. A union of two subspaces will always contain 0 and be closed under scalar multiplication; the problem arises with closure under addition.

Theorem 3. ***Unions of subspaces***
$U \cup W \leq V \iff U \subseteq W$ or $W \subseteq U$. These are trivial cases; in other cases, the union is not a subspace.

3 Sums of subspaces

Definition 3.1. If we let V be a vector space over K and $u, w \leq V$. Then:

$$u + w := \{u + w \mid u \in U, w \in W\} \tag{1}$$

is the **sum** of u and w. It is defined, in words, as follows: **all the possible sums with one vector rom U and one vectors from W**. Why this is not useless:

1. It is also a subspace of V
2. It is the smallest subspace of V that contains the union of U and W.

Theorem 4. *Let V be a vector space over K and $u, w \leq V$. Then:*

1. *The sum is a subspace of V: $u + w \leq V$.*
2. *The sum $u + w$ is the **smallest** subspace of V that contains BOTH U and W and hence their **union**.*

Aside: we defined the **smallest subspace** in this context as such: if $\widetilde{V}$ is any subspace of V with $V \cup W \subseteq \widetilde{V}$, then the sum $u + w \subseteq \widetilde{V}$. In a similar way, we can define $u_1 + ... + u_n$ where $u_1, ..., u_n$ are all subspaces of V as such:

$$\{u_1 + ... + u_n \mid u_k \in U_k,\ \forall 1 \leq k \leq n\}$$

If this holds, then $u_1 + ... + u_n \leq V$.

4 Linear Combinations and Span

Definition 4.1. Let V be a vector space over k, $v_1, .., v_n \in V$, and $k_1, ..., k_n \in K$. Then, an expression of the form

$$k_1 v_1 + ... + k_n v_n$$

is called a **linear combination** of $v_1, ..., v_n$.

Remark 4.1. It is possible to extend the idea of a linear combination to infinitely many vectors:

$$k_1 v_1 + k_2 v_2 + ...$$

this, however, is **completely meaningless.** We do not know what it means to add infinitely many vectors. There is no notion of convergence here, since there is no notion of distance in a vector space like there is, for say, sequences.

What we can do is define $k_1 v_1 + k_2 v_2 + ...$ where all but **finitely many** of the $k_i = 0$.

Definition 4.2. If we let V be a vector space over K and $v_1, ..., v_n \in V$. Then, **span** is defined as:

$$span\{v_1, ..., v_n\} = \{k_1 v_1 + ... + k_n v_n \mid k_i \in K\, 1 \leq i \leq n\} \tag{2}$$

in other words, it is the set of all linear combinations.

Remark 4.2. Note that:

$$\begin{aligned} span\{v_1, ..., v_n\} &= span\{v_1\} + ... + span\{v_n\} \\ &\Rightarrow \{k_1 v_1 + k_n v_n \mid k_i \in K\} \end{aligned}$$

Since sums of subspaces are also subspaces, then the entire thing is also a subspace. From this, we also get something for free: the $span\{v_1, ..., v_n\}$ is the smallest subspace of V containing all the vectors.

Example 4.1. Our vector space is $V = \mathbb{R}^n$. Then, the following are the standard basis vectors of $\mathbb{R}^n$:

$$\text{Standard basis vectors of } \mathbb{R}^n = \begin{cases} e_1 =< 1, 0, ... > \\ e_2 =< 0, 1, ... > \\ \vdots \\ e_n =< 0, ..., 1 > \end{cases}$$

It's clear that $\mathbb{R}^n = span\{e_1, ..., e_n\}$

Example 4.2. A more abstract vector space: consider the vector space $P_n(k)$, defined as

$$P_n(k) = \{1, x, x^2, x^3, ..., x^n\}$$

You can think of these as the "building blocks" of the vector space. Consider the linear combinations of this. You'd obtain:

$$span\{1, x, x^2, ..., x^n\} = \{k_0 1 + k_1 x + ...k_n x^n \mid k_i \in K, 0 \leq i \leq n\} = P_n(k)$$

Example 4.3. Consider the vector space S of all sequences with coefficients in $\mathbb{R}$. We'd define the following:

$$\begin{aligned} S_1 &= (1, 0, 0, ...) \\ S_2 &= (0, 1, 0, ...) \\ S_3 &= (0, 0, 1, ...) \\ &\vdots \end{aligned}$$

These are analogous to the standard basis vectors of $\mathbb{R}^n$. **Question**: What is the span? **Answer**: it is NOT a spanning set, since linear combinations only involve finitely many terms.

Remark 4.3. $\{s_1, s_2, ...\}$ does NOT span S, since there are **finitely** many rows. So, this will be constantly equal to zero; you'd obtain C_{00}. This is a very small subspace. In other words: **every linear combination of S results in a sequence that is eventually constantly equal to 0**. In fact, $span\{s_1, s_2, ...\}$ is C_{00}.

5 Linear Independence and Dependence

Definition 5.1. Let V be a vector space over K, $v_1, ..., v_n \in V$. We say that $v_1, ..., v_n$ are **linearly independent** if

$$k_1 v_1 + ... + k_n v_n = 0 \Rightarrow k_1 = k_2 = ... = k_n = 0 \tag{3}$$

This combination is known as the **trivial linear combination**. $v_1, ..., v_n$ are called **linearly dependent** if they are not linearly independent.

Theorem 5. *Let V be a vector space over K. Then:*

1. *Any (finite) subset of V that contains the zero vector is linearly dependent.*
2. *If $v_1, ..., v_n$ are linearly dependent (say, v_j, then at least one of them can be written as a **linear combination** of the remaining vectors. In that case:*

   Consider $span\{v_1, ..., v_n\}$. Then, we can kick v_j out and obtain the same span:
   $span\{v_1, ..., v_n\} = span\{v_1, ..., \hat{v_j}, ..., v_n\}$

Theorem 6. *Let $v_1, ..., v_n \in V$, $v_1, ..., v_n$ be linearly independent. Let $v_{n+1} \notin span\{v_1, ..., v_n\}$. Then, $v_1, ..., v_n, v_{n+1}$ are **linearly independent**.*

This is a useful theorem; we will use this in the future to construct bases.

Definition 5.2. Let V be a vector space over K. V is called **finite-dimensional** if V has a finite spanning set; i.e., there are finitely many $v_1, ..., v_n \in V$ s.t. they all span V.

Remark 5.1. Let V be a vector space over K, $W \leq V$. Assume that V is finite dimensional. It is tempting to assume that W is finite dimensional. While it is true, we cannot currently prove it. We will currently conclude that W is finite dimensional, but we will prove it later.

Definition 5.3. V is called **infinite-dimensional** if it is not finite dimensional. Methods of showing that a vector space is infinite-dimensional are very different from finite-dimensional. To prove something is finite-dimensional, it is sufficient to write one example of a finite spanning set. However, to prove something is infinite-dimensional, you need to show that NO finite spanning set can possibly exist, which is much trickier.

Example 5.1. $\mathbb{R}^n$ is finite dimensional. This is trivial: just write out the following finite spanning set:

$$\mathbb{R}^n = span\{e_1, ..., e_n\}$$

Example 5.2. Prove that C_{00} is infinite-dimensional. We do a proof by contradiction.

Assume that C_{00} is finite-dimensional. We will let $S_1, ..., S_n$ be sequences in C_{00}, which, recall, is the vector space of all sequences that are eventually constantly equal to zero, such that $C_{00} = span\{s_1, ..., s_n\}$.
Written proof:
Here, only finitely many terms do not equal zero. This means that every sequence has finitely many non-zero terms, so we can obtain an index where all sequences are zero after the index.

Theorem 7. ***Very important: Steinitz Exchange Lemma***
Let V be a finite-dimensional vector space over K. The idea is to compare the size of linear independent sets to the size of spanning sets. We let $u_1, ..., u_m \in V$ be linearly independent. Let $v_1, ..., v_n$ be spanning. ***then,*** $\mathbf{m} \leq \mathbf{n}$ *and there exists finitely many indices $K_1, ..., k_{n-m}$ such that:*

$$u_1, ..., u_m, v_k, ..., v_{k_{n-1}} \text{ is } \textbf{spanning}.$$

Basically, we can ***exchange m of the vectors*** $v_1, ..., v_n$ ***by*** $u_1, ..., u_m$ ***without changing the span.***