Math 247: Applied Linear Algebra

Course Notes - Week 6

Main topics: Compositions of linear maps, matrix representations of isomorphisms, change of basis, similarity, normal form problem, equivalence relation.

1 From Last Time

Recall the example from last class:

$$\begin{aligned} L : &Mat(2\times 2,\mathbb{R}) \to Mat(2\times 2,\mathbb{R}) \\ &A \mapsto A - A^T \\ &B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

We can then obtain $[L]_{B,B}$ by unravelling the matrices from the basis and using them as the columns of $[L]_{B,B}$

$$[L]_{B,B} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Consider an alternative basis B'.

$$\begin{aligned} L : &Mat(2\times 2,\mathbb{R}) \to Mat(2\times 2,\mathbb{R}) \\ &A \mapsto A - A^T \\ &B' = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \end{aligned}$$

Applying the transformation, we then obtain:

$$L(B') = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix}$$

We need to express the images as linear combinations of B'. Trivial for the first three:

$$= 0 \cdot \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + 2 \cdot \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

Collect coefficients into a 4×4 matrix:

$$\Rightarrow [L]_{B',B'} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$[L]_{B,B} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The normal form question: what basis can we choose to obtain the simplest matrix?

2 Compositions of linear maps

Let U, V, and W be finite-dimensional VS over k with bases B, B', and B". Let's define two linear maps:

$$L_1 : U_B \to V_{B'}$$
$$L_2 : V_{B'} \to W_{B"}$$

Then, we consider the following transformation:

$$L_2 \circ L_1 = U_B \to W_{B"}$$

Question: how are $[L_1]_{B',B}$, $[L_2]_{B",B'}$, and $[L_2 \circ L_1]_{B",B}$ related?

Theorem 1. *Let L_1 and L_2 be defined as above. Let $dim(V) = m$, $dim(V) = k$, and $dim(W) = n$. Then:*

$$[L_2 \circ L_1]_{B",B} = [L_2]_{B",B'} \cdot [L_1]_{B',B}$$

This theorem is basically saying that you can obtain a matrix composition by multiplying the matrix maps. This is why matrix multiplication is so complicated; it's defined to be that way such that we can use it for linear maps.

Proof. Let u be an arbitrary vector in U. Then, the idea is to simply compute what $[L_2 \circ L_1]_{B",B}[u]_B = [(L_2 \circ L_1)(u)]_{B"}$ is. As was discussed last week, we know that:

$$[L_2(L_1(u))]_{B"} = [L_2]_{B",B'}[L_1(u)]_{B'}$$

We are going to apply the same idea but backwards:

$$\Rightarrow [L_2]_{B",B'}[L_1]_{B',B}[u]_B$$
$$\Rightarrow [L_2 \circ L_1]_{B",B} = \text{ product of } [L_2]_{B",B'}[L_1]_{B',B}$$
$$\Rightarrow [L_2 \circ L_1]_{B",B} = [L_2]_{B",B'} \cdot [L_1]_{B',B}$$

□

3 Matrix Representations of Isomorphisms

Theorem 2. *V, W finite-dimensional VS over k, $L : V \to W$ be linear and let B, B' be bases for V, W, respectively. Claim: the linear map is an isomorphism $\iff$ the matrix representation of the linear map $[L]_{B',B}$ is invertible. This is the same thing as insisting that L is bijective.*

Remark 3.1. Consider the identity map, $id : V \to V$, with $dim(V) = n$, and a basis B. Then:

$$[id]_{B,B} = I_n \text{ } (n \times n \text{ identity matrix})$$

Why? We can just use the definition of a matrix map.

If $B = (v_1, ..., v_n)$, then the matrix representation of $[id]_{B,B}$ would then be given by:

$$([id(v_1)]_B| \dots |[id(v_n)]_B)$$

From last week's lectures, it therefore follows:

$$= ([v_1]_B| \dots |[v_n]_B)$$

By expressing each v_i as a linear combination of $v_1, ..., v_n$, we then obtain the following matrix:

$$= \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} = I_n$$

Now, to prove the theorem:

Proof. Proving both directions:
[$\Rightarrow$] Let L: V (B) $\to$ W (B') be an isomorphism. This means that L is bijective $\Rightarrow$ automatically has an inverse map that we automatically know is linear: L has an inverse L^{-1} which is $L^{-1} : W_{B'} \to V_B$ We now have: $L^{-1} \circ L : V_B \to V_B, L^{-1} \circ L = id_V$.

We now apply the remark and composition formula:

$$[L^{-1} \circ L]_{B,B} = I = [L^{-1}]_{B,B'} \cdot [L]_{B',B}$$

Matrices whose product is I implies that one of them is invertible. Similarly:

$$[L]_{B',B} \cdot [L^{-1}] = I \Rightarrow [L]_{B',B} \text{ is invertible.}$$

[$\Leftarrow$] We know that our matrix is invertible. So, we need to prove that it is bijective. So, let $[L]_{B',B}$ be an invertible $n \times n$ matrix. We will prove that L is injective first. To prove this, we need to show that the kernel is trivial.

Let $v \in Ker(L) \Rightarrow L(v) = 0 \Rightarrow [L(v)]_{B'} = 0$. We then obtain:

$$[L]_{B',B} \cdot [v]_B = [L(v)]_{B'} = 0$$

We know that $[L]_{B',B}$ is an invertible matrix, so, it cannot be zero. This means that $[v]_B$ must be zero $\Rightarrow v = 0$. We chose an arbitrary vector in the kernel, showed that it was equal to zero. This means that $Ker(L) = \{0\}$, which means that L is injective.

Now, we need to show that L is **surjective**. Since $[L]_{B',B}$ is an $n \times n$ matrix, we have that:

$$dim(V) = dim(W) = n$$

By the dimension formula, we obtain:

$$\begin{aligned} &dim(Ker) + dim(Im) = dim(V) \\ &0 + dim(Im) = n \end{aligned}$$

This implies that W is an n-dimensional subspace of an n-dimensional vector space. The only way this is possible is that if $Im(L) = W$, which is the very definition of L being surjective. So, since injectivity and surjectivity hold, we then obtain that L is bijective, and from this we obtain that L is an isomorphism. □

4 Change of Basis

There are two questions that will motivate this section:

1. Let $v \in V$. Given the coordinate vector $[v]_B$ of v with respect to the basis B, how can we compute the coordinate vector $[v]_{B'}$ with respect to B', another basis for the same vector space?
2. Let $L : V \to V$ be linear. Given the matrix representation $[L]_{B,B}$, how can we compute $[L]_{B',B'}$.

4.1 Question 1

Consider the linear map $id : V_B \to V_{B'}$. The matrix representation is given by $[id]_{B',B}$. By definition, we obtain:

$$[id]_{B',B}[v]_B = [id(v)]_{B'} = [v]_{B'}$$

The **transition matrix/change of basis matrix** from B to B' is given by $[id]_{B',B} \cdot [v]_B$. This formula is important:

$$[v]_{B'} = [id]_{B',B} \cdot [v]_B \tag{1}$$

4.2 Question 2

Consider the following sequence of maps:

$$V_{B'} \xrightarrow{id} V_B \xrightarrow{L} V_B \xrightarrow{id} V_{B'}$$

Then,

$$\begin{aligned} &[L]_{B',B'} = [id \circ L \circ id]_{B',B'} \\ &\Rightarrow [id]_{B',B}[L]_{B,B}[id]_{B,B'} \\ &\Rightarrow ([id]_{B,B'})^{-1}[L]_{B,B}[id]_{B,B'} \end{aligned}$$

Therefore, $[L]_{B,B}$ and $[L]_{B',B'}$ are similar matrices. This is **how we transform one matrix representation to another**. Representations of the same linear map are possible if there exists an invertible matrix P.

Definition 4.1. Two n by n matrices A,B with coefficients in K are called **similar** if there exists an invertible $n \times n$ matrix P with:

$$B = P^{-1}AP$$

Thus, any two matrix representations of the same linear map $L : V \to V$ with respect to bases B, B' are similar.

Special Case (Most important)
V $= K^n$ with two bases: the standard basis S and an arbitrary basis B. **First Question:** if we consider the vectors in K^n as row vectors, what is the coordinate vector of $[v]_s$? The solution:

$$[v]_s = v^t$$

How can we compute the coordinate vector $[v]_B$?

$$[v]_B = [id]_{B,S}[v]_S$$

Where $[id]_{S,B}$ is the transition matrix B to S. Now, we can **feed in arbitrary members of B** by expressing them in terms of the standard basis. Let $B = (v_1, ..., v_n)$. Then: $[id]_{S,B} = (v_1^t | \cdots | v_n^t)$ and $[id]_{B,S} = ([id]_{S,B})^{-1}$. So, we can basically find the vector in terms of the original basis by "inverting" the transformation:

$$[v]_B = P^{-1}[v_s]$$

Where $P = (v_1^t | \cdots | v_n^t)$.

Example 4.1. V $= \mathbb{R}^2$, S standard basis, and $B = ((1, 1), (3, 2))$. From this, we can obtain two change of basis matrixes:

$[id]_{S,B}$: from B to S. Relatively straightforward; just take the rows of B and use them as the columns of the change of basis matrix:

$$[id]_{S,B} = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}$$

$[id]_{B,S}$ this is more complicated; we need to compute $([id]_{S,B})^{-1}$. We then obtain:

$$[id]_{B,S} = \begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix}$$

Putting it all together, we then obtain:

$$[v]_B = [id]_{B,S} = [v]_S$$

Example 4.2. $v = (2, 3)$. So, $[v]_S = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$. We can obtain $[v]_B$ by:

$$[v]_B = \begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

Example 4.3. $L : \mathbb{R}^2 \to \mathbb{R}^2$. Remark that this is an endomorphism. Let this linear map define the projection onto the x-axis; i.e,:

$$\begin{aligned} L((1,0)) &= (1,0) \\ L((0,1,)) &= (0,0) \\ \Rightarrow [L]_{S,S} &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

Using the formula, we then obtain:

$$\begin{aligned} [L]_{B,B} &= [id]_{B,S} \cdot [L]_{S,S} \cdot [id]_{S,B} \\ &= ([id]_{S,B})^{-1}[L]_{S,S}[id]_{S,B} \\ &= \begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & -6 \\ 1 & 3 \end{bmatrix} \end{aligned}$$

To check if your solution is correct, you feed in a coordinate vector, compute the output, and verify that it is correct.

Example 4.4. V = Mat$(2 \times 2, \mathbb{R})$, $L : A \mapsto A - A^t$. Define the following bases:

$$\begin{aligned} B &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\ B' &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \end{aligned}$$

Last class, we computed the following matrices:

$$[L]_{B,B} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, [L]_{B',B'} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$[id]_{B,B'} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, [id]_{B',B} = ([id]_{B,B'})^{-1} = \frac{1}{2} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

5 The Normal Form Problem

Similar matrices represent the same linear map with respect to different bases. So, let $A \in MAT(n \times n, k)$. Let $L_A : K^n \to k^n$, $v \mapsto Av$. Let S be a standard basis for K^n be an arbitrary basis. Then, $[L_A]_{S,S} = A, B = \{v_1, ..., v_n\}$. Then:

$$\begin{aligned} [L_A]_{S,S} &= [id]_{B,S}[L_A]_{S,S}[id]_{S,B} \\ &= P^{-1}AP \end{aligned}$$

Where the columns of P consist of the vectors in B written as column vectors: $P = (v_1^t|\cdots|v_n^t)$. P is invertible $\Rightarrow$ we have n linearly independent columns in K^n $\Rightarrow$ they span K^n $\Rightarrow$ they form a basis for K^n.

Algebra Crash Course

General linear group: denoted $P \in GL(n,k)$, a general linear group is the set of all invertible $n \times n$ matrices with coefficients in k. It's a group.

Question: can we interpret $P^{-1}AP$ as the matrix representation of our linear map L with respect to some basis B of K^N? Yes! We can just take $P = (v_1^t|\cdots|v_n^t)$. Here's the chain of reasoning: P is invertible $\Rightarrow (v_1, ..., v_n)$ are linearly independent; $dim(K^n) = n \Rightarrow \{v_1, ..., v_n\}$ also span $K^n \Rightarrow$ basis for K^n. So, we can write:

$$[L_A] = P^{-1}AP$$

We will see that there is a one-to-one correspondence between matrix representations of our linear map L and matrices similar to A. In other words: **matrices similar to A can be interpreted as representing the same linear map, just with respect to a different basis**. So, in the Linear Algebra Universe, they're practically equivalent. This brings us to the final section.

6 Similarity Problem

Motivation: we are interested in knowing the set of all matrices $\in$ MAT $(n \times n, k)$ similar to A. This is the **similarity problem**. This introduces a relation: a set of any two matrices can be similar or not. We obtain two mathematical relations from this:

1. Order
2. Class of equivalence

Definition 6.1. Let $A, B \in MAT(n \times m, k)$. B is said to be **similar** to A, in symbols, $B \sim A$, if there exits a $P \in GL(n,k)$ such that

$$B = P^{-1}AP$$

Before this, we need more quick algebra.

6.1 Quick Algebra

Definition 6.2. Let S be a set. An **equivalence relation** is a subset of the cartesian product, $S \times S$. In maths, it's denoted by a $\sim$ and has the following three properties:

1. **Identity:** every element of S is in relation with itself:

$$\forall x \in S, x \sim s$$

2. **Commutativity**:

$$\forall x, y \in S, \text{ if } x \sim y, \text{ then } y \sim x$$
$$\text{Alternatively: } x \sim y \iff y \sim x$$

3. **Transitivity:**

$$\forall x, y, z \in S, \text{ if } x \sim y \wedge y \sim z \Rightarrow x \sim z$$

Example 6.1. $S = \mathbb{Z}, n \sim k \iff n - k$ is even. **Claim:** this is an equivalence relation. We can check this with the axioms.

1. $n \sim n$ since $n - n = 0$ which is even.
2. $n \sim k \Rightarrow n - k$ is even $\iff k - n$ is even $\Rightarrow k \sim n$.
3. $n \sim k \wedge k \sim m \Rightarrow$ show that $n - m$ is also even: $(n - k) + (k - m) \Rightarrow$ we have the sum of two even things, so the sum will also be even and we have $n \sim m$, as desired.

<u>Main idea:</u> equivalence relationships partition things into classes. In this case, all even numbers are similar/equivalent to zero and all odd numbers are equivalent to 1. So, we have the set of all integers partitioned into two, disjoint groups: (1) odd numbers, and (2) even numbers.

$$\text{The set of all } \{n \in Z \ n \sim 0\} = \{..., -4, -2, 0, 2, 4, ...\} \text{ (the set of all even numbers)}$$
$$\text{The set of all } \{n \in Z \ n \sim 1\} = \{..., -3, -1, 1, 3, ...\} \text{ (the set of all odd numbers)}$$

Definition 6.3. Let S be a set, $s \in S, \sim$ an equivalence relation on S. Then, we can define the **equivalence class** of S as follows:

$$[s]_\sim := \{t \in S \mid t \sim S\}$$

Remark that $s \in [S]_\sim$ by axiom 1 of equivalence relations. Back to our example:

Example 6.2. $S \in \mathbb{Z}, \sim, n - k$ even. Then:

1. $[0]$ is the set of all even numbers.
2. $[1]$ is the set of all odd numbers.

These are equivalence classes that **partition** $\mathbb{Z}$ (the set containing all integers). This, in fact, is true in general:

Theorem 3. *Let S be a set, $\sim$ be an equivalence relation. The equivalence classes of $\sim$* ***partition*** *S, i.e., every $s \in S$ is contained in an equivalence class and any two equivalence classes are either* ***identical*** *or* ***disjoint****.*

Proof is next class...