

Calculus: Single Variable, Multivariable, Differential Equations, and Vector Calculus Summary

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Abstract

The purpose of this document is to review Calculus. The content here should be equivalent to Math 140, Math 141, Math 222, and Math 248/358 at McGill.

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1 Single Variable Calculus

1.1 Limits and Derivatives

- **Precise Definition of a Limit:** Let f be a function defined on an open interval $]a, c[$ that contains the number a . Then, we say that the limit of $f(x)$ as x approaches a is L , and we write $\lim_{x \rightarrow a} f(x) = L$ if for every $\varepsilon > 0$, \exists a $\delta > 0$ such that $0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$.
 - Heuristically, this means that if any small interval $]L - \varepsilon, L + \varepsilon[$ is given around L , then we can find an interval $]a - \delta, a + \delta[$ around a such that f maps the points in $]a - \delta, a + \delta[$ (except possibly a) into the interval $]L - \varepsilon, L + \varepsilon[$.
- **Continuous:** A function f is said to be continuous at a number $a \in \mathbb{R}$ if $\lim_{x \rightarrow a} f(x) = f(a)$.
- **Intermediate Value Theorem:** Let f be continuous on the interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$ where $f(a) \neq f(b)$. Then, there exists a number $c \in]a, b[$ for which $f(c) = N$.
- **Tangent Line:** The tangent line to the curve $y = f(x)$ at the point $P = (a, f(a))$ is the line through P with the slope

$$m = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \iff m = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h} \quad (1)$$

- **Velocity / Instantaneous Velocity:** the instantaneous velocity $v(a)$ at the time $t = a$ is the limit of the average velocities:

$$v(a) := \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h} \quad (2)$$

- **Derivative:** The derivative of a function f at a number $a \in \mathbb{R}$, denoted by $f'(a)$, is

$$\lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h} = f'(a) \quad (3)$$

1.2 Differentiation Rules

- Derivative of a constant function: $\frac{d}{dx} [c] = 0$.
- **Power Rule:** if $n \in \mathbb{R}$, $\frac{d}{dx} [x^n] = nx^{n-1}$. One can prove this using geometric series.
- **Constant Multiple Rule:** if $c \in \mathbb{R}$ and f differentiable, then $\frac{d}{dx} [cf(x)] = cf'(x)$.
- **Constant Sum Rule:** if f, g are differentiable, then $\frac{d}{dx} [f(x) + g(x)] = \frac{d}{dx} [f(x)] + \frac{d}{dx} [g(x)]$.
- The rate of change of any exponential function is proportional to the function itself: for $f(x) := b^x$:

$$f'(x) = f'(0)b^x \quad (4)$$

- **Derivative of the Natural Exponential Function:**

$$\frac{d}{dx} [e^x] = e^x \quad (5)$$

- **Product Rule:** if f, g are differentiable, then:

$$\frac{d}{dx} [f(x)g(x)] = f(x)\frac{d}{dx} [g(x)] + g(x)\frac{d}{dx} [f(x)] \quad (6)$$

- **Quotient Rule:** If f, g are differentiable, then:

$$\frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] = \frac{g(x)\frac{d}{dx} [f(x)] - f(x)\frac{d}{dx} [g(x)]}{[g(x)]^2} \quad (7)$$

• **Derivatives of Trigonometric Functions:**

$$\begin{aligned} - \frac{d}{dx} [\sin(x)] &= \cos(x), \quad \frac{d}{dx} [\csc(x)] = -\csc(x) \cot(x) \\ - \frac{d}{dx} [\cos(x)] &= -\sin(x), \quad \frac{d}{dx} [\sec(x)] = \sec(x) \tan(x) \\ - \frac{d}{dx} [\tan(x)] &= \sec^2(x), \quad \frac{d}{dx} [\cot(x)] = -\csc^2(x). \end{aligned}$$

- **Chain Rule:** If g is differentiable at x and if f is differentiable at $g(x)$, then the composite function $F := f \circ g$ defined by $F(x) := f(g(x))$ is differentiable at x and F' is given by the product:

$$F'(x) = f'(g(x)) \cdot g'(x) \quad (8)$$

or, in Leibnitz notation,

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} \quad (9)$$

- **Method of Implicit Differentiation:** Differentiating both sides of the equation with respect to x , and then solving the resulting equation for y' .

– Application: finding the derivatives of inverse trigonometric functions:

$$\begin{aligned} * \frac{d}{dx} [\arcsin(x)] &= \frac{1}{\sqrt{1-x^2}}, \quad \frac{d}{dx} [\operatorname{arcsec}(x)] = \frac{-1}{x\sqrt{x^2-1}} \\ * \frac{d}{dx} [\arccos(x)] &= \frac{-1}{\sqrt{1-x^2}}, \quad \frac{d}{dx} [\operatorname{arcsec}(x)] = \frac{1}{x\sqrt{x^2-1}} \\ * \frac{d}{dx} [\arctan(x)] &= \frac{1}{1+x^2}, \quad \frac{d}{dx} [\operatorname{arccot}(x)] = \frac{-1}{x^2+1} \end{aligned}$$

– Application: derivatives of logarithmic functions, $y = \log_b(x)$ and $y = \ln(x)$.

$$\begin{aligned} * \frac{d}{dx} [\log_b(x)] &= \frac{1}{x \ln(b)} \\ * \frac{d}{dx} [\ln(x)] &= \frac{1}{x} \end{aligned}$$

– **Method of Logarithmic Differentiation:** the calculation of complex functions involving products, quotients, or powers can be simplified by taking logarithms.

- **Hyperbolic Trigonometric Functions:** hyperbolic functions \sim hyperbola like trigonometric functions \sim circle. They are defined as:

$$\begin{aligned} - \sinh(x) &:= \frac{e^x - e^{-x}}{2}, \quad \operatorname{csch}(x) := \frac{1}{\sinh(x)} \\ - \cosh(x) &:= \frac{e^x + e^{-x}}{2}, \quad \operatorname{sech}(x) := \frac{1}{\cosh(x)} \\ - \tanh(x) &:= \frac{\sinh(x)}{\cosh(x)}, \quad \operatorname{coth}(x) := \frac{\cosh(x)}{\sinh(x)} \end{aligned}$$

– Applications: whenever an entity such as light, velocity, electricity, or radioactivity is gradually absorbed or extinguished.

– Hyperbolic identities:

$$\begin{aligned} * \sinh(-x) &= -\sinh(x), \quad \cosh(-x) = \cosh(x) \\ * \cosh^2(x) - \sinh^2(x) &= 1, \quad 1 - \tanh^2(x) = \operatorname{sech}^2(x) \\ * \sinh(x+y) &= \sinh(x) \cosh(y) + \cosh(x) \sinh(y) \\ * \cosh(x+y) &= \cosh(x) \cosh(y) + \sinh(x) \sinh(y) \end{aligned}$$

– Derivatives of Hyperbolic Functions:

$$\begin{aligned} * \frac{d}{dx} [\sinh(x)] &= \cosh(x), \quad \frac{d}{dx} [\operatorname{csch}(x)] = -\operatorname{csch}(x) \coth(x), \\ * \frac{d}{dx} [\cosh(x)] &= \sinh(x), \quad \frac{d}{dx} [\operatorname{sech}(x)] = -\operatorname{sech}(x) \tanh(x), \\ * \frac{d}{dx} [\tanh(x)] &= \operatorname{sech}^2(x), \quad \frac{d}{dx} [\operatorname{coth}(x)] = -\operatorname{csch}^2(x). \end{aligned}$$

– **Inverse Hyperbolic Functions:**

$$\begin{aligned} * \operatorname{arcsinh}(x) &:= \ln(x + \sqrt{x^2 + 1}) \text{ for } x \in \mathbb{R}. \\ * \operatorname{arccosh}(x) &:= \ln(x + \sqrt{x^2 - 1}) \text{ for } x \geq 1. \\ * \operatorname{arctanh}(x) &:= \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right) \text{ for } x \in [-1, 1] \end{aligned}$$

– Derivatives of Inverse Hyperbolic Functions:

$$\begin{aligned}
* \frac{d}{dx} [\operatorname{arcsinh}(x)] &= \frac{1}{\sqrt{1+x^2}}, \quad \frac{d}{dx} [\operatorname{arccsch}(x)] = \frac{-1}{|x|\sqrt{x^2+1}} \\
* \frac{d}{dx} [\operatorname{arccosh}(x)] &= \frac{1}{\sqrt{x^2-1}}, \quad \frac{d}{dx} [\operatorname{arcsech}(x)] = \frac{-1}{x\sqrt{1-x^2}} \\
* \frac{d}{dx} [\operatorname{arctanh}(x)] &= \frac{1}{1-x^2}, \quad \frac{d}{dx} [\operatorname{arccoth}(x)] = \frac{1}{1-x^2}
\end{aligned}$$

1.3 Applications of Differentiation

- **Extreme Value Theorem:** Let f be continuous on the closed and bounded interval $[a, b]$. Then, f attains an absolute maximum value $f(x)$ and an absolute minimum value $f(d)$ at some numbers $c, d \in [a, b]$.
- **Fermat's Theorem:** If f has a local maximum or minimum at c , and if $f'(c)$ exists, then $f'(c) = 0$.
- **Closed Interval Method:** To find the absolute maximum and minimum values of a continuous function f on a closed interval $[a, b]$,
 1. Find the values of f at the critical points of f in the open interval $]a, b[$.
 2. Compute $f(a)$ and $f(b)$.
 3. The max between (1) and (2) is the absolute max; the min between (1) and (2) is the absolute min.
- **Rolle's Theorem:** Let $f : [a, b] \rightarrow \mathbb{R}$ satisfy:
 1. f is continuous on $[a, b]$
 2. f is differentiable on $]a, b[$
 3. $f(a) = f(b)$.

Then, there exists a number $c \in]a, b[$ such that $f'(c) = 0$.

- **Mean Value Theorem:** Let $f : [a, b] \rightarrow \mathbb{R}$ satisfy:
 1. f is continuous on $[a, b]$
 2. f is differentiable on $]a, b[$

Then, there exists a number $c \in]a, b[$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \iff f(b) - f(a) = f'(c)[b - a] \quad (10)$$

- **Theorem** (Consequence of MVT): If $f'(x) = 0 \forall x \in]a, b[$, then f is constant on $]a, b[$.
 - **Corollary:** If $f'(x) = g'(x) \forall x \in]a, b[$, then $f - g$ is constant on $]a, b[$, i.e., $\exists c \in \mathbb{R}$ such that $f(x) = g(x) + c$.
- **L'Hopital's Rule:** Suppose f and g are differentiable and that $g(x) \neq 0$ on an open interval containing a . Suppose that

$$\lim_{x \rightarrow a} f(x) = 0 \text{ and } \lim_{x \rightarrow a} g(x) = 0 \quad (11)$$

or

$$\lim_{x \rightarrow a} f(x) = \pm\infty \text{ and } \lim_{x \rightarrow a} g(x) = \pm\infty \quad (12)$$

then

$$\lim_{x \rightarrow a} \left(\frac{f(x)}{g(x)} \right) = \lim_{x \rightarrow a} \left(\frac{f'(x)}{g'(x)} \right) \quad (13)$$

- **Antiderivative:** A function F is called an **anti-derivative** of f on an interval I if $F'(x) = f(x) \forall x \in I$.

1.4 Integrals

- **Area:** The area A of a region S that lies under the graph of a continuous function f is the limit of the sum of the approximating rectangles

$$A = \lim_{n \rightarrow \infty} R_n = \lim_{n \rightarrow \infty} [f(x_1)\Delta x + \dots + f(x_n)\Delta x] \quad (14)$$

- **Definite Integral:** Let $f : [a, b] \rightarrow \mathbb{R}$. Divide $[a, b]$ into n subintervals of equal width $\Delta x = \frac{b-a}{n}$. Let $a = x_0 < x_1 < \dots < x_n = b$ be the endpoints and let x_1^*, \dots, x_n^* be any sample points in these subintervals such that $x_i^* \in [x_{i-1}, x_i]$. Then, the definite integral of f from a to b is:

$$\int_a^b f(x)dx := \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*)\Delta x \quad (15)$$

provided that the limit exists and is the same for all possible choices of sample points. If it does exist, then we say that f is integrable on $[a, b]$.

- Formulae for the sums of positive integers:
 - $\sum_{i=1}^n i = \frac{n(n+1)}{2}$
 - $\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$
 - $\sum_{i=1}^n i^3 = \left[\frac{n(n+1)}{2} \right]^2$
- **Fundamental Theorem of Calculus** connects differential calculus and integral calculus. Deals with equations of the form

$$g(x) = \int_a^x f(t)dt \quad (16)$$

- **Fundamental Theorem of Calculus Part 1:** let $f : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$. Then, the function g defined by

$$g(x) := \int_a^x f(t)dt \quad (17)$$

is continuous on $[a, b]$ and differentiable on $]a, b[$. Moreover, $g'(x) = f(x)$.

- **Fundamental Theorem of Calculus Part 2:** If f is continuous on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a) \quad (18)$$

where F is any anti-derivative of f .

- Alternative expression for the FoC Part 1:

$$\frac{d}{dx} \left[\int_a^x f(t)dt \right] = f(x) \quad (19)$$

- **Table of Integration Formulae:**

– $\int x^n dx = \frac{x^{n+1}}{n+1}$ for $n \neq -1$.	– $\int \frac{1}{x} dx = \ln x $
– $\int e^x dx = e^x$	– $\int b^x dx = \frac{b^x}{\ln b}$
– $\int \sin(x) dx = -\cos(x)$	– $\int \cos(x) dx = \sin(x)$
– $\int \sec^2(x) dx = \tan(x)$	– $\int \csc^2(x) dx = -\cot(x)$
– $\int \sec(x) \tan(x) dx = \sec(x)$	– $\int \csc(x) \cot(x) dx = -\csc(x)$
– $\int \sec(x) dx = \ln \sec(x) + \tan(x) $	– $\int \csc(x) dx = \ln \csc(x) - \cot(x) $
– $\int \tan(x) dx = \ln \sec(x) $	– $\int \cot(x) dx = \ln \sin(x) $
– $\int \sinh(x) dx = \cosh(x)$	– $\int \cosh(x) dx = \sinh(x)$
– $\int \frac{1}{x^2+a^2} dx = \frac{1}{a} \arctan\left(\frac{x}{a}\right)$	– $\int \frac{1}{\sqrt{a^2-x^2}} dx = \arcsin\left(\frac{x}{a}\right)$ $a > 0$
– $\int \frac{1}{x^2-a^2} dx = \frac{1}{2a} \ln\left \frac{x-a}{x+a}\right $	– $\int \frac{1}{\sqrt{x^2 \pm a^2}} dx = \ln x + \sqrt{x^2 \pm a^2} $

1.5 Applications of Integration

- The average value of f on the interval $[a, b]$ is:

$$f_{avg} := \frac{1}{b-a} \int_a^b f(x) dx \quad (20)$$

- Mean Value Theorem for Integrals: If f is continuous on $[a, b]$ then there exists a $c \in [a, b]$ such that

$$f(x) = f_{avg} = \frac{1}{b-a} \int_a^b f(x) dx \iff \int_a^b f(x) dx = f(c)(b-a) \quad (21)$$

1.6 Integration Techniques

- Integration by Parts:

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx \quad (22)$$

- Trigonometric Integrals:

1. Strategy for evaluating $\int \sin^m(x) \cos^n(x) dx$:

- (a) If n is odd: save one cosine, use $\cos^2(x) = 1 - \sin^2(x)$ to express the remaining factors in terms of sine:

$$\int \sin^m(x) \cos^{2k+1}(x) dx = \int \sin^m(x) (1 - \sin^2(x))^k \cos(x) dx \quad (23)$$

and make the substitution $u = \sin(x)$.

- (b) If m is odd: save one sine, use $\sin^2(x) = 1 - \cos^2(x)$ to express the remaining factors in terms of cosine:

$$\int \sin^{2k+1}(x) \cos^n(x) dx = \int (1 - \cos^2(x))^k \cos^n(x) \sin(x) dx \quad (24)$$

and make the substitution $u = \cos(x)$.

- (c) If sine and cosine are even, then use the half-angle identities:

$$\sin^2(x) = \frac{1}{2}(1 - \cos(2x)) \text{ and } \cos^2(x) = \frac{1}{2}(1 + \cos(2x)) \quad (25)$$

A helpful identity is $\sin(x) \cos(x) = \frac{1}{2} \sin(2x)$.

2. Strategy for evaluating $\int \tan^m(x) \sec^n(x) dx$:

- (a) If n is even: save one secant squared, use the identity $\sec^2(x) = 1 + \tan^2(x)$ to express the remaining factors in terms of $\tan(x)$:

$$\int \tan^m(x) \sec^{2k}(x) dx = \int \tan^m(x) (1 + \tan^2(x))^{k-1} \sec^2(x) dx \quad (26)$$

and make the substitution $u = \tan(x)$.

- (b) If m is odd: save one $\sec(x) \tan(x)$, use $\tan^2(x) = \sec^2(x) - 1$ to express the remaining factors in terms of $\sec(x)$:

$$\int \tan^{2k+1}(x) \sec^n(x) dx = \int (\sec^2(x) - 1)^k \sec^{n-1}(x) \sec(x) \tan(x) dx \quad (27)$$

substitute $u = \sec(x)$.

3. Important product identities to remember:

- (a) $\sin(A) \cos(B) = \frac{1}{2}[\sin(A - B) + \sin(A + B)]$
- (b) $\sin(A) \sin(B) = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$
- (c) $\cos(A) \cos(B) = \frac{1}{2}[\cos(A - B) + \cos(A + B)]$

• **Trigonometric Substitution**

- $\sqrt{a^2 - x^2} \rightarrow x = a \sin(\theta), \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \rightarrow 1 - \sin^2(\theta) = \cos^2(\theta).$
- $\sqrt{a^2 + x^2} \rightarrow x = a \tan(\theta), \theta \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\rightarrow 1 + \tan^2(\theta) = \sec^2(\theta).$
- $\sqrt{x^2 - a^2} \rightarrow x = a \sec(\theta), \theta \in \left[0, \frac{\pi}{2}\right[\cup \left[\pi, \frac{3\pi}{2}\right[\rightarrow \sec^2(\theta) - 1 = \tan^2(\theta).$

• **Partial Fractions:**

1. Case I: Denominator $Q(x)$ is a product of distinct linear factors:

$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \dots + \frac{A_k}{a_kx + b_k} \quad (28)$$

2. Case II: Denominator $Q(x)$ is a product of linear factors, some of which are repeated r times:

$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \dots + \frac{A_k}{(a_1x + b_1)^r} \quad (29)$$

3. Case III: $Q(x)$ contains irreducible quadratic factors, none of which is repeated. Then, expression will have a term of the form

$$\frac{Ax + B}{ax^2 + bx + c} \quad (30)$$

which can be integrated by completing the square and using the formula:

$$\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C \quad (31)$$

4. Case IV: $Q(x)$ contains a repeated irreducible factor. Then, the expression will have a term of the form:

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \dots + \frac{A_rx + B_r}{(ax^2 + bx + c)^r} \quad (32)$$

• **General Strategy for Integrating:**

1. Simplify the integrand if possible using algebraic manipulation and trigonometric identities.
2. Look for obvious substitutions.
3. Classify integrand according to its form
 - (a) Trigonometric functions
 - (b) Rational functions (\rightarrow partial fractions)
 - (c) Integration by parts
 - (d) Radicals
 - i. $\sqrt{\pm x^2 \pm a^2} \rightarrow$ trigonometric substitution
 - ii. $(ax + b)^{1/n} \rightarrow$ rationalising substitution $u = (ax + b)^{1/n}$

• **Improper Integral:** if in the definite integral, $\int_a^n f(x)dx$, either $[a, b]$ is an unbounded interval or $f(x)$ has an infinite discontinuity in $[a, b]$

1.7 Further Applications of Integration

- **Arc-length formula:** If f' is continuous on $[a, b]$, then the length of the curve $y = f(x)$, $a \leq x \leq b$ is:

$$L = \int_a^b \sqrt{1 + [f'(x)]^2} \quad (33)$$

- **Arc-Length Function:** If a smooth curve C has the equation $y = f(x)$, $a \leq x \leq b$, let $s(x)$ be the distance along C from the initial point $P_0(a, f(a))$ to the point $Q(x, f(x))$. Then, s is a function given by:

$$s(x) = \int_a^x \sqrt{1 + [f'(t)]^2} dt \quad (34)$$

1.8 Parameter Equations and Polar Coordinates

Motivation: some curves are best handled when both x and y are given as a function of a third variable t : $x = f(t)$, $y = g(t)$.

- Suppose f, g are differentiable functions and suppose we want to find the tangent line at a point on the parametric curve $x = f(t)$, $y = g(t)$, where y is also a differentiable function of x . If $\frac{dx}{dt} \neq 0$, then the slope of the parametric curve is given by:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad (35)$$

- We can consider $\frac{d^2y}{dx^2}$:

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{\frac{d}{dt} \left(\frac{dy}{dx} \right)}{\frac{dx}{dt}} \quad (36)$$

- **Areas:** if a curve is traced out by the parametric equations $x = f(t)$ and $y = g(t)$ for $t \in [\alpha, \beta]$, then using the substitution rule for integrals one has the following formula:

$$\mathcal{A} = \int_a^b y dx = \int_\alpha^\beta g(t) f'(t) dt \quad (37)$$

- **Arc Length** if a curve C is described by parametric equations $x = f(t)$, $y = g(t)$, $\alpha \leq t \leq \beta$, where f' and g' are continuous on $[a, b]$ and C is traversed exactly once as t travels from α to β , then the length of C is:

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2} dt \quad (38)$$

- **Surface Area:** similar to the conditions in the previous theorem, the surface area of a curve obtained by rotating it about the x -axis is given by:

$$S = \int_\alpha^\beta 2\pi y \sqrt{\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2} dt \quad (39)$$

- Equations to convert between cartesian and polar coordinates:

$$x = r \cos(\theta) \quad y = r \sin(\theta) \quad (40)$$

$$r^2 = x^2 + y^2 \quad \tan(\theta) = \frac{y}{x} \quad (41)$$

- If $r = f(\theta)$ is a polar curve, then we can find the tangent line to a polar curve by regarding θ as a parameter:

$$\begin{aligned}x &= f(\theta) \cos(\theta) \\y &= f(\theta) \sin(\theta)\end{aligned}$$

and the tangent is given by:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta} \sin(\theta) + r \cos(\theta)}{\frac{dr}{d\theta} \cos(\theta) - r \sin(\theta)} \quad (42)$$

- Areas and lengths in polar coordinates

– The area of a sector of a circle: $A = \frac{1}{2}r^2\theta$. The area of a polar region \mathcal{R} :

$$A(\mathcal{R}) = \int_a^b \frac{1}{2} [f(\theta)]^2 d\theta \quad (43)$$

– The arc-length of a polar curve with the equation $r = f(\theta)$, $a \leq \theta \leq b$ is:

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta \quad (44)$$

1.9 Infinite Sequences and Series

- A **Sequence** is a list of numbers written in a definite order

$$a_1, a_2, a_3, \dots \quad (45)$$

- A sequence has a **limit** L and we write

$$\lim_{n \rightarrow \infty} a_n = L$$

if we can make the terms a_n as close to L as we'd like by taking n sufficiently large. if $\lim_{n \rightarrow \infty} a_n$ exists, then we say that $\{a_n\}$ is **convergent**. Else, it is **divergent**.

– A sequence $\{a_n\}$ has a limit L if $\forall \varepsilon > 0$, \exists an $N \in \mathbb{N}$ such that $\forall n \geq N$, one has that $|a_n - L| < \varepsilon$.

- **Squeeze Theorem:** if $a_n \leq b_n \leq c_n \quad \forall n \geq n_0$, and if $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n = L$, then $\lim_{n \rightarrow \infty} b_n = L$.

– If $\lim_{n \rightarrow \infty} |a_n| = 0$, then $\lim_{n \rightarrow \infty} a_n = 0$.

– If $\lim_{n \rightarrow \infty} a_n = L$ and if f is a continuous function at L , then

$$\lim_{n \rightarrow \infty} f(a_n) = f(L)$$

- The sequence $\{r^n\}$ is convergent if $r \in]-1, 1[$ and divergent for all other values of r . If $r \in]-1, 1[$:

$$\lim_{n \rightarrow \infty} r^n = \begin{cases} 0 & \text{if } r \in]-1, 1[\\ 1 & \text{if } r = 1 \end{cases}$$

- **Monotonic Sequence Theorem:** every bounded, monotonic sequence converges.

- **Series:** Given a series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \dots$$

let s_n denote the n th partial sum:

$$s_n := \sum_{i=1}^n a_i = a_1 + \dots + a_n$$

if the *sequence* $\{s_n\}$ is convergent and $\lim_{n \rightarrow \infty} s_n = s \in \mathbb{R}$, then the series $\sum_{n=1}^{\infty} a_n$ is convergent.

- **Geometric Series:** is an important example of an infinite series.

$$a + ar + ar^2 + \dots + ar^{n-1} + \dots = \sum_{n=1}^{\infty} ar^{n-1} \quad (a \neq 0) \quad (46)$$

- If $r = 1$, then $s_n = na \rightarrow \infty$.
- If $r \neq 1$, then $s_n = \frac{a(1-r^n)}{1-r}$. If $r \in]-1, 1[$, then $r^n \rightarrow 0$ as $n \rightarrow \infty$, and so

$$\lim_{n \rightarrow \infty} s_n = \frac{a}{1-r} \quad (47)$$

otherwise the geometric series diverges.

- **Harmonic Series** is defined as $\sum_{n=1}^{\infty} \frac{1}{n}$. It's divergent.
- If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n \rightarrow \infty} a_n = 0$.
 - **Test for Divergence:** if $\lim_{n \rightarrow \infty} a_n$ does not exist or if $\lim_{n \rightarrow \infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
- **Integral Test:** Suppose f is a continuous, positive, decreasing function on $[1, \infty[$ and let $a_n = f(n)$. Then, the series $\sum_{n=1}^{\infty} a_n$ is convergent \iff the improper integral $\int_1^{\infty} f(x)dx$ is convergent.
 - **p-series:** the p -series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges $\iff p > 1$.
- **Remainder Estimate for the Integral Test:** Suppose that $f(k) = a_k$, where f is a continuous, decreasing, positive function for $x \geq n$ and suppose that $\sum a_n$ is convergent. If $R_n := S - S_n$, then

$$\int_{n+1}^{\infty} f(x)dx \leq R_n \leq \int_n^{\infty} f(x)dx \quad (48)$$

- **Comparison Test:** Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.
 1. If $\sum b_n$ is convergent and $a_n \leq b_n \forall n$, then $\sum a_n$ converges.
 2. If $\sum b_n$ is divergent and $b_n \leq a_n \forall n$, then $\sum a_n$ diverges.
- **Limit Comparison Test:** Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c$$

where $c \in]0, \infty[$, then both series have the same behaviour; i.e, either both series converge or both diverge.

- **Alternating Series Test:** If the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ for $b_n > 0$ satisfies:
 1. $b_{n+1} \leq b_n \forall n \in \mathbb{N}$
 2. $\lim_{n \rightarrow \infty} b_n = 0$

then, the series converges.

- **Absolutely Convergent:** A series $\sum a_n$ is called **absolutely convergent** if the series of absolute values $\sum |a_n|$ is convergent.
- **Conditionally Convergent:** A series $\sum a_n$ is called **conditionally convergent** if its convergent but not absolutely convergent.
- If $\sum a_n$ is absolutely convergent then it is convergent.
- **Ratio Test** Let $\{a_n\}$ be a sequence.
 1. If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$, then $\sum_{n=1}^{\infty} a_n$ absolutely converges (and thus converges).
 2. If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$ or $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| - L = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.
 3. If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L = 1$ then the Ratio test is inconclusive.
- **Root Test** Let $\{a_n\}$ be a sequence.
 1. If $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = L < 1$, then the series absolutely converges (and therefore converges).
 2. If $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = L > 1$ or if $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \infty$ then the series diverges.
 3. If $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = L = 1$, then the root test will be inconclusive.

1.10 Strategy for Testing Series

1. If the series is of the form $\sum \frac{1}{n^p}$, then apply the p-series rule.
2. If the series is of the form $\sum ar^{n-1}$ or $\sum ar^n$, then apply the geometric series rule.
3. If the series is similar to a p-series or a geometric series, then use a comparison test.
4. If $\lim_{n \rightarrow \infty} a_n \neq 0$, use the divergence test to conclude that the series diverges.
5. If the series is of the form $\sum (-1)^{n-1} b_n$ or $\sum (-1)^n b_n$, then use the alternating series test.
6. If the series has factorials in it, consider applying a ratio test.
7. If the series is of the form $(b_n)^n$, then consider the root test.
8. If $a_n = f(n)$ where $\int_1^{\infty} f(x)dx$ is easily evaluated, then consider the integral test.

1.11 Power Series

- **Power Series:** a power series is of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \dots \quad (49)$$

A series of the form

$$\sum_{n=0}^{\infty} c_n (x - a)^n = c_0 + c_1 (x - a) + c_2 (x - a)^2 \quad (50)$$

is called a **power series in $(x - a)$**

- **Theorem:** for a given power series $\sum_{n=0}^{\infty} c_n (x - a)^n$, there are only three possibilities:
 1. The series converges only when $x = a$.
 2. The series converges for all x .
 3. \exists an $R > 0$ such that the series converges if $|x - a| < R$ and diverges if $|x - a| > R$.
- **Theorem (Term-by-term Differentiation and Integration):** If the power series $\sum c_n (x - a)^n$ has a radius of convergence $R > 0$, then the function defined by:

$$f(x) := c_0 + c_1 (x - a) + c_2 (x - a)^2 + \dots = \sum_{n=0}^{\infty} c_n (x - a)^n \quad (51)$$

is differentiable (and thus continuous) on the interval $]a - R, a + R[$ and:

1. $f'(x) = c_0 + 2c_1(x-a) + 3c_2(x-a)^2 + \dots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$.
2. $\int f(x)dx = c + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \dots$

$$\int f(x)dx = c + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} \quad (52)$$

The radius of convergence of both (1) and (2) remain R .

- **Theorem (Taylor Series Representation):** If f has a power series representation (expansion) at a , that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n \quad (53)$$

for $|x-a| < R$, then the coefficients are given by the formula:

$$c_n = \frac{f^{(n)}(a)}{n!} \quad (54)$$

Then, the **Taylor Series** for f about a is given by:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n \quad (55)$$

For the special case of $a = 0$, then the above becomes:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \dots \quad (56)$$

- **Theorem (Remainder):** Define the remainder of the Taylor series by $R_n := f(x) - T_n(x)$. If $f(x) = T_n(x) + R_n(x)$, where T_n is the n th degree Taylor polynomial of f at a and

$$\lim_{n \rightarrow \infty} R_n(x) = 0 \quad (57)$$

for $|x-a| < R$, then f is equal to the sum of its Taylor series on the interval $|x-a| < R$.

- The following theorem is often used when trying to show that $\lim_{n \rightarrow \infty} R_n = 0$ for a specific function f :

- **Taylor's Inequality:** If $|f^{(n+1)}(x)| \leq M \forall |x-a| \leq d$, then the remainder $R_n(x)$ of the Taylor series satisfies the following inequality:

$$|R_n(x)| \leq \frac{M}{(n+1)!} |x-a|^{n+1} \quad (58)$$

for $|x-a| \leq d$.

• **Important MacLaurin Series and their Radii of Convergence:**

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots \quad (R=1) \quad (59)$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \quad (R=\infty) \quad (60)$$

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \dots \quad (R=\infty) \quad (61)$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \quad (R=\infty) \quad (62)$$

$$\arctan(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)} = x - \frac{x^3}{3} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \quad (R=1) \quad (63)$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \quad (R=1) \quad (64)$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!} x^2 + \frac{k(k-1)(k-2)}{3!} x^3 + \dots \quad (R=1) \quad (65)$$

2 Multi Variable Calculus

2.1 Vectors and Geometry of Space

- If θ is the angle between vectors \mathbf{a} and \mathbf{b} , then $\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos(\theta)$
 - Two vectors are orthogonal $\iff \mathbf{a} \cdot \mathbf{b} = 0$.
 - **Scalar Projection of \mathbf{b} onto \mathbf{a} :**

$$\text{comp}_{\mathbf{a}}(\mathbf{b}) := \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|} \quad (66)$$

- **Vector projection of \mathbf{b} onto \mathbf{a} :**

$$\text{proj}_{\mathbf{a}}(\mathbf{b}) := \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|} \right) \frac{\mathbf{a}}{\|\mathbf{a}\|} \quad (67)$$

- **Cross Product:** if $\mathbf{a} := \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} := \langle b_1, b_2, b_3 \rangle$, then their **cross product** is:

$$\mathbf{a} \times \mathbf{b} := \det \begin{bmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} \quad (68)$$

- If θ is the angle between \mathbf{a} and \mathbf{b} , then

$$\|\mathbf{a} \times \mathbf{b}\| := \|\mathbf{a}\| \|\mathbf{b}\| \sin(\theta) \quad (69)$$

- Two non-zero vectors \mathbf{a} and \mathbf{b} are parallel $\iff \mathbf{a} \times \mathbf{b} = 0$.
- The volume of the parallelepiped spanned by the vectors \mathbf{a} , \mathbf{b} , \mathbf{c} is $|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$
- A parametric equation for a line going through the point (x_0, y_0, z_0) parallel to the direction vector $\langle a, b, c \rangle$ are:

$$x = x_0 + at$$

$$y = y_0 + bt$$

$$z = z_0 + ct$$

- The scalar equation of the plane through the point $P_0(x_0, y_0, z_0)$ with the normal vector $\mathbf{n} := \langle a, b, c \rangle$ is:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \quad (70)$$

- The distance D from a point $P_1(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$ is:

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}} \quad (71)$$

- **Vector-Valued Function:** a function whose domain is the set of real numbers and whose range is a set of vectors.

- The **limit** of a vector-valued function \mathbf{r} is defined by taking the limits of the component functions as follows:

$$\lim_{t \rightarrow a} \mathbf{r}(t) := \langle \lim_{t \rightarrow a} f(t), \lim_{t \rightarrow a} g(t), \lim_{t \rightarrow a} h(t) \rangle \quad (72)$$

- **Space-curve:** the set C of all points (x, y, z) in space where $x = f(t)$, $y = g(t)$, and $z = h(t)$, where $t \in I$, is called a **space-curve**.

- The **derivative** \mathbf{r}' of a vector-valued function \mathbf{r} is defined as:

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h \rightarrow 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} \quad (73)$$

- The **unit tangent** vector $\mathbf{T}(t)$ is defined as:

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \quad (74)$$

- The **definite integral** of a vector-valued function is exactly what one would expect:

$$\int_a^b \mathbf{r}(t) dt = \left(\int_a^b f(t) dt \right) \hat{\mathbf{i}} + \left(\int_a^b g(t) dt \right) \hat{\mathbf{j}} + \left(\int_a^b h(t) dt \right) \hat{\mathbf{k}} \quad (75)$$

- The length L of a space-curve between the points a and b is parameterisation-independent and is given by:

$$L = \int_a^b \|\mathbf{r}'(t)\| dt \quad (76)$$

- The **arc-length function** of a curve, s , is defined as:

$$s(t) := \int_a^t \|\mathbf{r}'(u)\| du \quad (77)$$

We can use the above equation to parameterise a curve with respect to arc-length by differentiating both sides of the equation above with respect to t and applying the fundamental theorem of calculus:

$$\frac{ds}{dt} = \|\mathbf{r}'(t)\| \quad (78)$$

Advantages of an arc-length parametrisation include: it arises naturally from the shape of the curve and it's coordinate-system independent.

- **Smooth:** a parameterisation $\mathbf{r}(t)$ is called **smooth** on an interval I if \mathbf{r}' is continuous and $\mathbf{r}'(t) \neq 0$ on I .
- **Curvature:** the **curvature** of a curve is given by:

$$\kappa := \left\| \frac{d\mathbf{T}}{ds} \right\| \quad (79)$$

where \mathbf{T} is the unit tangent vector. We have three other formulae for curvature:

1.

$$\kappa(t) = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} \quad (80)$$

2.

$$\kappa(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} \quad (81)$$

3. For plane curves, write $\mathbf{r}(x) = x\hat{\mathbf{i}} + f(x)\hat{\mathbf{j}}$

$$\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}} \quad (82)$$

- When $\kappa(t) \neq 0$, one can define the **principle unit normal** $\mathbf{N}(t)$:

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} \quad (83)$$

- **Normal plane:** the plane determined by the normal and binormal vectors. Consists of all lines orthogonal to the unit tangent vector \mathbf{T} .
- **Osculating circle:** the plane determined by the vectors \mathbf{T} and \mathbf{N} .
 - * Closest plane to containing the part of the curve near P .
 - * **Osculating circle:** the circle that lies on the osculating plane of C at P , has the same tangent as C at P , and lies on the concave side of C (towards where \mathbf{N} is pointing). This best describes the behaviour of C near P .

- The **velocity vector** $\mathbf{v}(t)$ at time t :

$$\mathbf{v}(t) := \lim_{h \rightarrow 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} = \mathbf{r}'(t) \quad (84)$$

- **Speed:** the magnitude of the velocity vector $\|\mathbf{v}(t)\|$.

$$\|\mathbf{v}(t)\| = \|\mathbf{r}'(t)\| = \frac{ds}{dt} \quad (85)$$

- **Acceleration:** the derivative of the velocity

$$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t) \quad (86)$$

- Often times it can be useful to resolve the acceleration of a particle into its tangential and normal components:

$$\mathbf{a} = \underbrace{v'}_{:=\mathbf{a}_T} \mathbf{T} + \underbrace{\kappa v^2}_{:=\mathbf{a}_N} \mathbf{N} \quad (87)$$

where $v := \|\mathbf{r}'(t)\|$. One can re-write Equation (87) so that it only depends on \mathbf{r} , \mathbf{r}' , and \mathbf{r}'' :

$$\mathbf{a}_T = \frac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{\|\mathbf{r}'(t)\|} \quad (88)$$

$$\mathbf{a}_N = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|} \quad (89)$$

2.2 Partial Derivatives

- **Graph:** let f be a function in two variables with domain Ω . Then, the **graph** of f is the set of all points $(x, y, z) \in \mathbb{R}^3$ such that $z = f(x, y)$ for $(x, y) \in \Omega$.
- **Level Curves:** The **level curves** of a function f in two variables are the curves with equations $f(x, y) = k$, where $k \in \mathbb{R}$ is a constant.
- **Limit:** Let f be a function of two variables whose domain Ω includes points arbitrarily close to (a, b) . Then, we say that the **limit of $f(x, y)$ as $(x, y) \rightarrow (a, b)$** is L and we write:

$$\lim_{(x,y) \rightarrow (a,b)} f(x, y) = L \quad (90)$$

if $\forall \varepsilon > 0, \exists \delta > 0$ such that if $(x, y) \in \Omega, 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$, then $|f(x, y) - L| < \varepsilon$.

- **Partial Derivative:** the partial derivative of f with respect to x at (a, b) is:

$$f_x(a, b) := \lim_{h \rightarrow 0} \frac{f(a+h) - f(a, b)}{h} \quad (91)$$

- **Claircut's Theorem:** suppose f is defined on a disk D that contains the point (a, b) . If the functions f_{xx} and f_{yy} are both continuous on D , then,

$$f_{xy}(a, b) = f_{yx}(a, b) \quad (92)$$

- **Tangent Plane:** Suppose f has continuous partial derivatives. An equation for the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is given by:

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \quad (93)$$

- **Linearisation:** an equation for the tangent plane to the graph f at the point $(a, b, f(a, b))$ is given by:

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \quad (94)$$

the graph of this tangent plane is called the **linearisation** of f at (a, b) :

$$L(x, y) := f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \quad (95)$$

- **Differentiable:** If $z = f(x, y)$, then f is **differentiable** at (a, b) if Δz can be expressed in the form:

$$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y \quad (96)$$

where $\varepsilon_1, \varepsilon_2 \rightarrow 0$ as $(\Delta x, \Delta y) \rightarrow (0, 0)$.

- If the partial derivatives f_x and f_y exist near (a, b) , and are continuous at (a, b) , then f is differentiable at (a, b) .
- **Total Differential:** for a differentiable function of two variables $z = f(x, y)$, then the **total differential** is defined as:

$$\begin{aligned} dz &:= f_x(x, y)dx + f_y(x, y)dy \\ &= \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \end{aligned}$$

and so in the language of differentials, we write:

$$f(x, y) \approx f(a, b) + dz$$

- **Chain Rule (1. Case):** Suppose that $z = f(x, y)$ is a differentiable function of x and y , and suppose that both x and y are differentiable functions of t (i.e, $x = x(t)$, $y = y(t)$) so that $z = (f(x(t), y(t)))$. Then, z is a differentiable function of t and:

$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} \quad (97)$$

- **Chain Rule (General Version):** Suppose that u is a differentiable function of n variables (x_1, \dots, x_n) and each x_j is a differentiable function of m variables t_1, \dots, t_m . Then, u is a function of t_1, \dots, t_m and one has:

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1} \frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2} \frac{\partial x_2}{\partial t_i} + \dots + \frac{\partial u}{\partial x_n} \frac{\partial x_n}{\partial t_i} \quad (98)$$

$\forall i = 1, 2, \dots, m$.

- **Implicit Function Theorem:** Let $y = f(x)$. If F is defined on a disc containing (a, b) , where $F(a, b) = 0$, $F_y(a, b) \neq 0$, and F_y and F_x are continuous on the disc, then the equation $F(x, y) = 0$ defines y as a function of x near the point (a, b) whose derivative is given by:

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y} \quad (99)$$

Now, for $z = f(x, y)$, if z is implicitly given as a function by an equation of the form $F(x, y, z) = 0$, then the derivative of the implicitly defined function is:

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \quad (100)$$

$$\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} \quad (101)$$

- **Directional Derivative:** the directional derivative of f at (x_0, y_0) in the direction of the unit vector $\mathbf{u} = \langle a, b \rangle$ is:

$$D_{\mathbf{u}}f(x_0, y_0) := \lim_{h \rightarrow 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h} \quad (102)$$

- If f is a differentiable function of x and y , then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b \rangle$ and

$$D_{\mathbf{u}}f(x, y) = f_x(x, y)a + f_y(x, y)b \quad (103)$$

- **Gradient:** Let $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be a function of two variables. Then, the gradient of f is the vector function ∇f defined by:

$$\begin{aligned} \nabla f(x, y) &:= \langle f_x(x, y), f_y(x, y) \rangle \\ &= \frac{\partial f}{\partial x} \hat{\mathbf{i}} + \frac{\partial f}{\partial y} \hat{\mathbf{j}} \end{aligned}$$

Using this notation, we can re-write the directional derivative as a dot product:

$$D_{\mathbf{u}}f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$$

Thus, we can interpret the directional derivative as the scalar projection of the gradient function onto \mathbf{u} .

- Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a differentiable function of n variables. The maximum value of the directional derivative $D_{\mathbf{u}}f(x)$ is $\|\nabla f(x)\|$; this occurs when \mathbf{u} is parallel to $\nabla f(x)$.
- The tangent plane to the level surface $F(x, y, z) = k$ at the point $P(x_0, y_0, z_0)$ is the plane that passes through P with the normal vector $\nabla F(x_0, y_0, z_0)$. This is written as:

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0 \quad (104)$$

2.3 Multiple Integrals

- **Double Integral:** the double integral of f over the rectangle \mathcal{R} is:

$$\iint_{\mathcal{R}} f(x, y) dA := \lim_{m, n \rightarrow \infty} \sum_{i=1}^m \sum_{j=1}^n f(x_{i,j}^*, y_{i,j}^*) \Delta A \quad (105)$$

if the limit exists.

- If $f \geq 0$, then the volume V of the solid above the rectangle \mathcal{R} and below the surface $z = f(x, y)$ is:

$$V = \iint_{\mathcal{R}} f(x, y) dA \quad (106)$$

- **Fubini's Theorem:** If f is continuous on the rectangle $\mathcal{R} := \{(x, y) \mid a \leq x \leq b, c \leq y \leq d\}$ then:

$$\iint_{\mathcal{R}} f(x, y) dA = \int_a^b \int_c^d f(x, y) dy dx = \int_c^d \int_a^b f(x, y) dx dy \quad (107)$$

- **Average Value:** the average value of a function f of two variables defined on the rectangle \mathcal{R} is defined to be:

$$f_{\text{avg}} := \frac{1}{A(\mathcal{R})} \iint_{\mathcal{R}} f(x, y) dA \quad (108)$$

- **Double Integral of f over D :** let D be a bounded region in \mathbb{R}^n . Define the following function F ; let \mathcal{R} be a rectangle such that $D \subseteq \mathcal{R}$ and

$$F(x, y) := \begin{cases} f(x, y) & (x, y) \in D \\ 0 & (x, y) \notin D \end{cases} \quad (109)$$

then, the double integral of f over D is given by:

$$\iint_D f(x, y) dA := \iint_{\mathcal{R}} F(x, y) dA \quad (110)$$

- We have various “types” of domains/regions:
 - **Type I:** A plane region D is Type I if it lies between the graphs of two continuous functions of x :

$$D := \{(x, y) \in \mathbb{R} \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\} \quad (111)$$

in this case, the double integral is given by:

$$\iint_D f(x, y) dA := \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx \quad (112)$$

- **Type II:** a plane region of Type II is:

$$D := \{(x, y) \in \mathbb{R}^2 \mid c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\} \quad (113)$$

where $h_1, h_2 \in C(\mathbb{R})$. In this case, the double integral is given by:

$$\iint_D f(x, y) dA := \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) dx dy$$

- If $m \leq f(x, y) \leq M \forall (x, y) \in D$, then:

$$mA(D) \leq \iint_D f(x, y) dA \leq MA(D) \quad (114)$$

- **Polar Rectangle:** $\mathcal{R} := \{(r, \theta) \mid a \leq r \leq b, \alpha \leq \theta \leq \beta\}$

– **Change to Polar Coordinates in a Double Integral:** Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be continuous, \mathcal{R} a polar rectangle, f continuous. If \mathcal{R} is given by $0 \leq a \leq r \leq b$, $\alpha \leq \theta \leq \beta$, $0 \leq \beta - \alpha \leq 2\pi$. Then:

$$\iint_{\mathcal{R}} f(x, y) dA := \int_{\alpha}^{\beta} \int_a^b f(r \cos(\theta), r \sin(\theta)) r dr d\theta \quad (115)$$

if f is continuous on a polar region of the form $D := \{(r, \theta) \mid \alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta)\}$, then:

$$\iint_D f(x, y) dA = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} f(r \cos(\theta), r \sin(\theta)) r dr d\theta \quad (116)$$

- **Surface Area:** the area of the surface $z = f(x, y)$ for $(x, y) \in D$ where $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are continuous is given by:

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \quad (117)$$

2.4 Triple Integrals

- **Triple Integrals over a Box:** The **triple integral** of f over the box B is:

$$\iiint_B f(x, y, z) dV := \lim_{\ell, m, n \rightarrow \infty} \sum_{i=1}^{\ell} \sum_{j=1}^m \sum_{k=1}^n f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*) \Delta V \quad (118)$$

- **Cylindrical Coordinate System:** a point $P \in \mathbb{R}^3$ is represented by the ordered triple (r, θ, z) ; the equations to convert are given by:

$$\begin{aligned} x &= r \cos(\theta) & y &= r \sin(\theta) & z &= z \\ r^2 &= x^2 + y^2 + z^2 & \tan(\theta) &= y/x & z &= z \end{aligned}$$

Often useful in problems involving symmetry about an axis.

- **Triple Integration in Polar Coordinates:** Suppose E is a Type I region whose projection D onto the xy -plane is described in polar coordinates:

$$\begin{aligned} E &= \{(x, y, z) \mid (x, y) \in D \mid u_1(x, y) \leq z \leq u_2(x, y)\} \\ D &= \{(r, \theta) \mid \alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta)\} \end{aligned}$$

Then,

$$\iiint_E f(x, y, z) dV = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r \cos(\theta), r \sin(\theta))}^{u_2(r \cos(\theta), r \sin(\theta))} f(r \cos(\theta), r \sin(\theta)) r dz dr d\theta \quad (119)$$

- **Spherical Coordinates:** the spherical coordinates (ρ, θ, φ) of a point $p \in \mathbb{R}^3$ are given by:

$$\begin{aligned} x &= \rho \sin(\varphi) \cos(\theta), \quad y = \rho \sin(\theta) \sin(\theta), \quad z = \rho \cos(\varphi) \\ \rho^2 &= x^2 + y^2 + z^2 \end{aligned}$$

- The formula for a triple integral in spherical coordinates is given by:

$$\iiint_E f(x, y, z) dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho \sin(\varphi) \cos(\theta), \rho \sin(\theta) \sin(\theta), \rho \cos(\varphi)) \rho^2 \sin(\varphi) d\rho d\theta d\varphi \quad (120)$$

- **Jacobian:** the **Jacobian** of a transformation T given by $x = g(u, v)$ and $y = h(u, v)$ is given by:

$$\frac{\partial(x, y)}{\partial(u, v)} = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} \quad (121)$$

- Suppose T is a C^1 -transformation whose Jacobian is non-zero and that T maps S in the uv -plane to region \mathcal{R} in the xy -plane. Suppose that f is continuous on \mathcal{R} and that both \mathcal{R} and S are Type I or Type II regions. Moreover, assume that T is bijective, except for potentially on ∂S . Then:

$$\iint_{\mathcal{R}} f(x, y) dA = \iint_S f(x(u, v), y(u, v)) |\det(\mathbf{Jac})| du dv \quad (122)$$

3 Vector Calculus

3.1 Vector Fields

- **Vector Field on \mathbb{R}^n :** Let $D \subseteq \mathbb{R}^n$. A **vector field** on \mathbb{R}^n is a function \mathbf{F} that assigns to every point $(x_1, \dots, x_n) \in D$ an n -dimensional vector $\mathbf{F}(x_1, \dots, x_n)$. One can write this in terms of component functions, e.g. in \mathbb{R}^2 :

$$\mathbf{F}(x, y) = P(x, y)\hat{\mathbf{i}} + Q(x, y)\hat{\mathbf{j}} + Q(x, y)\hat{\mathbf{k}} = \langle P(x, y), Q(x, y), Q(x, y) \rangle \quad (123)$$

- **Gradient Vector Field:** if $f : \mathbb{R}^2 \rightarrow \mathbb{R}$, recall that ∇f is:

$$\nabla f(x, y) = f_x(x, y)\hat{\mathbf{i}} + f_y(x, y)\hat{\mathbf{j}} \quad (124)$$

which means that ∇f is a vector field on \mathbb{R}^2 (we call this vector field a **gradient vector field**).

- **Conservative Vector Field:** A vector field \mathbf{F} is a **conservative vector field** if there exists a scalar function f such that $\nabla f = \mathbf{F}$.

3.2 Line Integrals

- **Line Integral:** parameterise a smooth curve C by

$$x = x(t), \quad y = y(t) \quad t \in [a, b]$$

or, equivalently,

$$\mathbf{r}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}}$$

then, the line integral of f along C is:

$$\int_C f(x, y) ds = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*, y_i^*) \Delta s_i \quad (125)$$

the line integral is given by:

$$\int_C f(x, y) ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \quad (126)$$

– A more compact notation for line integrals can be given by:

$$\int_a^b f(\mathbf{r}(t)) \|\mathbf{r}'(t)\| dt \quad (127)$$

- **Line Integrals over Vector Fields:** Let \mathbf{F} be a continuous vector field defined on a smooth curve C be given by a vector function $\mathbf{r}(t)$, $t \in [a, b]$. Then, the line integral of \mathbf{F} along C is:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b F(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt = \int_C \mathbf{F} \cdot \mathbf{T} ds \quad (128)$$

Suppose a vector field \mathbf{F} on \mathbb{R}^3 is given in compact-form by the equation $\mathbf{F} = P\hat{\mathbf{i}} + Q\hat{\mathbf{j}} + R\hat{\mathbf{k}}$. Then,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P dx + Q dy + R dz \quad (129)$$

- **Fundamental Theorem for Line Integrals:** Let C be a smooth curve given by the vector function $\mathbf{r}(t)$ for $t \in [a, b]$. Let f be a differentiable function f of two or three variables whose gradient vector ∇f is continuous on C . Then:

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)) \quad (130)$$

- Path Independence

- $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in $D \iff \int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path C in D .
- Let D be an open, connected domain. Suppose that F is a vector field on D . If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path on D , then \mathbf{F} is a conservative vector field on D , that is, \exists an f such that $\nabla f = \mathbf{F}$.
- Let $F(x, y) = P(x, y)\hat{\mathbf{i}} + Q(x, y)\hat{\mathbf{j}}$ be a conservative vector field, where P and Q have continuous first-order partial derivatives on a domain D . Then, throughout D , we have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad (131)$$

- **Simple Curve:** a curve that doesn't intersect itself anywhere between its endpoints.
- **Simply-Connected Region:** a connected region D such that every simple closed curve in D encloses points that are only in D .
- Let $\mathbf{F} = P\hat{\mathbf{i}} + Q\hat{\mathbf{j}}$ be a vector field of an open, simply-connected region D . Suppose that P and Q have continuous first-order partial derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad (132)$$

throughout D . Then, \mathbf{F} is conservative.

3.3 Green's Theorem

Counterpart of the Fundamental Theorem of Calculus for double integrals.

- **Green's Theorem:** Let C be a positively-oriented, piece-wise smooth simple closed curve in the plane and let D be a region bounded by C . If P and Q have continuous partial derivatives on an open region containing D , then:

$$\oint_C Pdx + Qdy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \quad (133)$$

- Can be used to calculate areas:

$$A(D) = \oint_C xdy = - \oint_C ydx = \frac{1}{2} \oint_C xdy - ydx \quad (134)$$

3.4 Curl and Divergence

Each of the following operations resemble differentiation, but one produces a vector field and the other produces a scalar field.

- **Curl:** Let $\mathbf{F} := P\hat{\mathbf{i}} + Q\hat{\mathbf{j}} + R\hat{\mathbf{k}}$ be a vector field on \mathbb{R}^3 . Assume that the partial derivatives P , Q , and R all exist. Then, the **curl** of \mathbf{F} is the vector field on \mathbb{R}^3 defined by:

$$\begin{aligned} \text{curl}(\mathbf{F}) &:= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \hat{\mathbf{i}} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \hat{\mathbf{j}} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \hat{\mathbf{k}} \\ &= \nabla \times \mathbf{F} \\ &= \det \begin{bmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{bmatrix} \end{aligned}$$

- If f is a function of three variables and has continuous, second-order partial derivatives, then:

$$\text{curl}(\nabla f) = 0 \quad (135)$$

- Conservative vector fields have $\mathbf{F} = \nabla f$, and so $\text{curl}(\mathbf{F}) = 0$ for conservative vector fields.
- Let \mathbf{F} be a vector field defined on all of \mathbb{R}^3 whose component functions have continuous partial derivatives and $\text{curl}(\mathbf{F}) = 0$. Then, \mathbf{F} is a conservative vector field.
- We say that a vector field \mathbf{F} is irrotational if $\text{curl}(\mathbf{F}) = 0$.

- **Divergence:** Let $\mathbf{F} := P\hat{\mathbf{i}} + Q\hat{\mathbf{j}} + R\hat{\mathbf{k}}$ be a vector field on \mathbb{R}^3 and assume that $\frac{\partial P}{\partial x}$, $\frac{\partial Q}{\partial y}$, and $\frac{\partial R}{\partial z}$ all exist. Then, the **divergence** of \mathbf{F} is defined by:

$$\text{div}(\mathbf{F}) := \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \quad (136)$$

$$= \nabla \cdot \mathbf{F} \quad (137)$$

- If $\mathbf{F} := P\hat{\mathbf{i}} + Q\hat{\mathbf{j}} + R\hat{\mathbf{k}}$ be a vector field on \mathbb{R}^3 and if P , Q , and R have continuous second-order partial derivatives, then

$$\text{div}(\text{curl}(\mathbf{F})) = 0 \quad (138)$$

- A vector field \mathbf{F} is called **incompressible** if $\text{div}(\mathbf{F}) = 0$.

- We can use what we've built up here to formulate Green's Theorem in terms of vector forms.

- The first vector form of Green's theorem is:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\text{curl}(\mathbf{F})) \cdot \hat{\mathbf{k}} dA \quad (139)$$

- The second vector form of Green's Theorem is:

$$\oint_C \mathbf{F} \cdot \hat{\mathbf{n}} ds = \iint_D \text{div}(\mathbf{F}) dA \quad (140)$$

3.5 Parametric Surfaces and their Areas

4 Ordinary Differential Equations

4.1 Basic Concepts

- A **differential equation** is an equation that involves an unknown function and its derivatives. We say that a function is an **ordinary differential equation** if the unknown function depends only on one independent variable.
- The **order** of a differential equation is the order of the highest derivative appearing in the equation.
- A **solution** of an ODE in an unknown function y and an independent variable x on an interval I is a function $y(x)$ satisfying the ODE for all $x \in I$.

4.2 Classification of First Order Differential Equations

First of all, the two forms that we can see ODEs come in:

- **Standard Form:** $y' = f(x, y)$.
- **Differential Form:** $M(x, y)dx + N(x, y)dy = 0$.

And now, some types of first order differential equations that we will encounter:

- **Linear Equation:** If $f(x, y)$ can be written as:

$$f(x, y) = -p(x)y + q(x) \quad (141)$$

then the ODE is **linear** and it can always be expressed as:

$$y' + p(x)y = q(x) \quad (142)$$

- **Bernoulli Equation:** A **Bernoulli differential equation** is an equation of the form:

$$y' + p(x)y = q(x)y^n \quad (143)$$

- **Homogeneous Equation:** A differential equation in standard form is said to be **homogeneous** if:

$$f(tx, ty) = f(x, y) \quad (144)$$

- **Separable Equation:** A differential equation given in differential form

$$M(x, y)dx + N(x, y)dy = 0$$

is said to be **separable** if $M(x, y) = A(x)$ and $B(x, y) = B(y)$.

- **Exact Equations:** A differential equation in the differential form is called **exact** if:

$$\frac{\partial M(x, y)}{\partial y} = \frac{\partial N(x, y)}{\partial x} \quad (145)$$

- Separable ODE \Rightarrow exact ODE.

4.3 Separable First Order Differential Equations

Suppose our ODE is of the form:

$$A(x)dx + B(y)dy = 0 \quad (146)$$

i.e., we have a separable first-order differential equation. Then, the solution is obtained by integrating:

$$\int A(x)dx + \int B(y)dy = c \quad (147)$$

where $c \in \mathbb{R}$ is our constant of integration. This is the **general solution**; if we have an initial value problem:

$$\begin{aligned} A(x)dx + B(y)dy &= 0 \\ y(x_0) &= y_0 \end{aligned}$$

then we can obtain the particular solution by first obtaining the general solution, and then applying the initial condition to solve for $c \in \mathbb{R}$.

4.3.1 Reduction of a Homogeneous Equation

Suppose we have a homogeneous equation (i.e, $f(xt, yt) = f(x, t) \forall t$). Then, there are two transformations which we can apply to obtain a separable equation:

1. Make the following substitution: $y = xv$. This has the corresponding derivative:

$$\frac{dy}{dx} = v + x \frac{dv}{dx} \quad (148)$$

Then, the ODE is separable in v and x .

2. Re-write the differential equation as:

$$\frac{dx}{dy} = \frac{1}{f(x, y)} \quad (149)$$

and make the following substitution for x $x := yu$, which gives the corresponding derivative:

$$\frac{dx}{dy} = u + y \frac{du}{dy}$$

and substitute back into the ODE.