Math 455: Analysis IV Summary
Midterm Date: 27 April 2020 14.00 - 17.00
Key Results, Theorems, Definitions, etc.

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Abstract

This document contains a summary of all the key definitions, results, and theorems from class. There are probably typos, and so I would be grateful if you brought those to my attention :-).

Syllabus: L^p space, duality, weak convergence, Young, Holder, and Minkowski inequalities, point-set topology, topological space, dense sets, completeness, compactness, connectedness, path-connectedness, separability, Tychnoff theorem, Stone-Weierstrass Theorem, Arzela-Ascoli, Baire category theorem, open mapping theorem, closed graph theorem, uniform boudnedness principle, Hahn Banch theorem.

Contents

1 L^p Spaces: Completeness and Approximation 1
- 1.1 Normed Vector Spaces . . . 1
- 1.2 The Inequalities of Young, Hölder, and Minkowski . . . 2
- 1.3 L^p is complete: the Reisz-Fischer Theorem . . . 3
- 1.4 Approximation and Separability . . . 4
- 1.5 Results from the Homework . . . 4

2 L^p Spaces: Duality and Weak Convergence 5
- 2.1 Riesz Representation Theorem for the Dual of L^p, $1 \leq p < \infty$ . . . 5
- 2.2 Weak Sequential Convergence in L^p . . . 6
- 2.3 Weak Sequential Compactness ("Compactness Found!") . . . 7
- 2.4 Results from the Homework . . . 7

3 Metric Spaces 8
- 3.1 Examples of Metric Spaces . . . 8
- 3.2 Open Sets, Closed Sets, and Convergent Sequences . . . 9
- 3.3 Continuous Mappings Between Metric Spaces . . . 10
- 3.4 Complete Metric Spaces . . . 10
- 3.5 Compact Metric Spaces . . . 11
- 3.6 Separable Metric Spaces . . . 12
- 3.7 Results from the Homework . . . 13

4 Topological Spaces 13
- 4.1 Open Sets, Closed Sets, Bases, and Sub-bases . . . 13
- 4.2 Separation Properties . . . 15
- 4.3 Countability and Separability . . . 15
- 4.4 Continuous Mappings between Topological Spaces . . . 16
- 4.5 Compact Topological Spaces . . . 16
- 4.6 Connected Topological Space . . . 17
- 4.7 Results from Homework . . . 17

5 Metric Spaces: Three Fundamental Theorems 18
- 5.1 The Arzela-Ascoli Theorem . . . 18
- 5.2 Baire Category Theorem . . . 18
- 5.3 The Banach Contraction Principle . . . 20

6 Topological Spaces: Three Fundamental Properties 20

1. L^p Spaces: Completeness and Approximation

1.1. Normed Vector Spaces

Definition 1 (ℓ^p space)**.** Let $(a_1, a_n, ...)$ be a sequence. Then, the ℓ^p-space is:

$$\ell^p := \left\{ (a_1, a_2, ...) \mid \sum_{n=1}^{\infty} |a_n|^p < +\infty \right\} \tag{1}$$

Theorem 1 (Riesz-Fisher)**.** $L^p(X)$ is complete.

Definition 2 (L^p space)**.** Let E be a measurable set and let $1 \leq p < \infty$. Then, $L^p(E)$ is the collection of measurable functions f for which $|f|^p$ is Lebesgue integrable over E.

Definition 3 (Equivalent Functions)**.** Let $\mathcal{F}$ be the collection of all measurable extended real-valued functions on E that are finite a.e. on E. Define two functions f and g to be equivalent, and write $f \sim g$ if $g(x) = f(x)$ a.e. on E.

Definition 4 (Essentially Bounded)**.** We call a function $f \in \mathcal{F}$ to be **essentially bounded** if there exists some $M \geq 0$, called the **essential upper bound** for f, for which

$$|f(x)| \leq M$$

for almost every $x \in E$. $L^\infty(E)$ is the collection of equivalence classes $[f]$ for which f is essentially bounded.

Definition 5 (Norm)**.** Let X be a linear space. A real-valued functional $||\cdot||$ on X is called a **norm** provided that for each f and g in X and each real number α,

(1) (The Triangle Inequality).

$$||f + g|| \leq ||f|| + ||g||$$

(2) (Positive Homogeneity).

$$||\alpha f|| = |\alpha| ||f||$$

(3) (Non-Negativity).

$$||f|| \geq 0 \text{ and } ||f|| = 0 \text{ if and only if } f = 0$$

Definition 6 (Normed Linear Space)**.** X is said to be a **normed linear space** if X is equipped with a norm.

Definition 7 (Essential Supremum)**.** Let $f \in L^\infty(E)$. $||f||_\infty$ is called the **essential supremum** and is defined as:

$$||f||_\infty := \{M \mid M \text{ is an essential upper bound for } f\}$$

Theorem: $||\cdot||_\infty$ is a norm on $L^\infty(E)$.

1.2. The Inequalities of Young, Hölder, and Minkowski

Definition 8 (p-norm)**.** Let E be a measurable set, $1 < p < \infty$, and let $f \in L^p(E)$. Then, define the **p-norm** to be:

$$||f||_p := \left[\int_E |f|^p\right]^{\frac{1}{p}} \tag{2}$$

Definition 9 (Conjugate)**.** The **conjugate** of a number $p \in]1, \infty[$ is the number $q = p/(p-1)$, which is the unique number $q \in]1, \infty[$ for which

$$\frac{1}{p} + \frac{1}{q} = 1 \tag{3}$$

The conjugate of 1 is defined to be ∞ and the conjugate of ∞ is defined to be 1.

Definition 10 (Young's Inequality)**.** For $1 < p < \infty$, q the conjugate of p, and any two positive numbers a and b, we have:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \tag{4}$$

Theorem 2 (Hölder's Inequality)**.** Let $E \subseteq \mathbb{R}$ be measurable, $1 \leq p < \infty$, and q the conjugate of p. If f belongs to $L^p(E)$, and g belongs to $L^q(E)$, then their product $f \cdot g$ is integrable over E and:

$$\int_E |f \cdot g| \leq ||f||_p \cdot ||g||_q. \tag{5}$$

Moreover, if $f \neq 0$, then the function defined as:

$$f^* := ||f||_p^{1-p} \cdot \operatorname{sgn}(f) \cdot |f|^{p-1} \tag{6}$$

belongs to $L^q(E)$,

$$\int_E f \cdot f^* = ||f||_p \text{ and } ||f^*||_q = 1$$

We call f^* defined as above to be called the **conjugate function** of f.

Theorem 3 (Minkowski's Inequality)**.** Let E be a measurable set and $1 \leq p \leq \infty$. If the functions f and g belong to $L^p(E)$, then so does their sum $f + g$. Moreover,

$$||f + g||_p \leq ||f||_p + ||g||_p \tag{7}$$

Theorem 4 (Cauchy-Schwarz Inequality)**.** Let E be a measurable set and let f and g be measurable functions over E for which f^2 and g^2 are integrable over E. Then, $f \cdot g$ is integrable over E and

$$\int_E |f \cdot g| \leq \sqrt{\int_E f^2} \cdot \sqrt{\int_E g^2} \tag{8}$$

Corollary 1. Let E be a measurable set and $1 < p < \infty$. Suppose $\mathcal{F}$ is a family of functions in $L^p(E)$ that is bounded in $L^p(E)$ in the sense that there is a constant M for which

$$||f||_p \leq M \text{ for all } f \in \mathcal{F}$$

Then, the family $\mathcal{F}$ is uniformly integrable over E.

Corollary 2. Let E be a measurable set of finite measure and $1 \leq p_1 < p_2 \leq \infty$. Then, $L^{p_2}(E) \subseteq L^{p_1}(E)$. Furthermore,

$$||f||_{p_1} \leq c||f||_{p_2}$$

for all f in $L^{p_2}(E)$, where $c = [m(E)]^{\frac{p_2 - p_1}{q_1 p_2}}$ if $p_2 < \infty$ and $c = [m(E)]^{\frac{1}{p_1}}$ if $p_2 = \infty$.

1.3. L^p is complete: the Reisz-Fischer Theorem

Definition 11 (Converge)**.** A sequence $\{f_n\}$ in a linear space X normed by $||\cdot||$ is said to **converge to f in X** provided:

$$\lim_{n\to\infty} ||f - f_n|| = 0$$

Definition 12 (Cauchy)**.** A sequence $\{f_n\}$ in a linear space X that is normed by $||\cdot||$ is said to be **Cauchy** in X provided for each $\varepsilon > 0$, there exists a $N \in \mathbb{N}$ such that

$$||f_n - f_m|| < \varepsilon \ \forall\ m, n \geq N \tag{9}$$

Definition 13 (Complete)**.** A normed linear space X is called **complete** if every Cauchy sequence in X converges to a function in X. A complete normed linear space is called a **Banach space**.

Proposition 1. Let X be a normed linear space. Then, every convergent sequence in X is Cauchy. Moreover, a Cauchy sequence in X converges if it has a convergent subsequence.

Definition 14. Let X be a linear space normed by $||\cdot||$. A sequence $\{f_n\}$ in X is said to be **rapidly Cauchy** if there is a convergent series of positive numbers $\sum_{k=1}^{\infty} \varepsilon_k$ for which

$$||f_{k+1} - f_k|| \leq \varepsilon_k^2 \text{ for all } k$$

Proposition 2. Let X be a normed linear space. Then, every rapidly Cauchy sequence in X is Cauchy. Furthermore, every Cauchy sequence has a rapidly Cauchy subsequence.

Proposition 3. Let E be a measurable set and $1 \leq p \leq \infty$. Then, every rapidly Cauchy sequence in $L^p(E)$ converges with respect to the $L^p(E)$ norm and pointwise a.e. on E to a function in $L^p(E)$.

Theorem 5 (Riesz-Fischer Theorem)**.** Let E be a measurable set and $1 \leq p \leq \infty$. Then $L^p(E)$ is a Banach space. Moreover, if $\{f_n\} \to f$ in $L^p(E)$, a subsequence of $\{f_n\}$ converges pointwise a.e. on E to f.

Theorem 6. Let E be a measurable set and $1 \leq p < \infty$. Suppose $\{f_n\}$ is a sequence in $L^p(E)$ that converges pointwise a.e. on E to the function f which belongs to $L^p(E)$. Then:

$$\{f_n\} \to f \text{ in } L^p(E) \iff \lim_{n\to\infty} \int_E |f_n|^p = \int_E |f|^p$$

Definition 15 (Tight)**.** A family $\mathcal{F}$ of measurable functions on E is said to be **tight** over E provided that for each $\varepsilon > 0$, there exists a subset E_0 of E of finite measure for which

$$\int_{E \setminus E_0} |f| < \varepsilon \text{ for all } f \in \mathcal{F}$$

Theorem 7. Let E be a measurable set and let $1 \leq p < \infty$. Suppose $\{f_n\}$ is a sequence in $L^p(E)$ that converges pointwise a.e. on E to the function f which belongs to $L^p(E)$. Then, $\{f_n\} \to f$ in $L^p(E) \iff \{|f_n|^p\}$ is uniformly integrable and tight over E.

1.4. Approximation and Separability

Definition 16 (Dense)**.** Let X be a normed linear space with norm $||\cdot||$. Given two subsets $\mathcal{F}$ and $\mathcal{G}$ of X with $\mathcal{F} \subseteq \mathcal{G}$, we say that $\mathcal{F}$ is **dense** in $\mathcal{G}$ provided for each function g in $\mathcal{G}$ and $\varepsilon > 0$, there is a function $f \in \mathcal{F}$ for which $||f - g|| < \varepsilon$.

Proposition 4. Let E be a measurable set and let $1 \leq p \leq \infty$. Then, the subspace of simple functions in $L^p(E)$ is dense in $L^p(E)$.

Proposition 5. Let $[a, b]$ be a closed, bounded interval and $1 \leq p < \infty$. Then, the subspace of step functions on $[a, b]$ is dense in $L^p[a, b]$.

Definition 17 (Separable)**.** A normed linear space X is said to be **separable** provided there is a countable subset that is dense in X.

Theorem 8. Let E be a measurable set and $1 \leq p < \infty$. Then, the normed linear space $L^p(E)$ is separable.

Theorem 9. Suppose E is measurable and let $1 \leq p < \infty$. Then, $C_c(E)$ (the set of all continuous functions with compact support on E) is dense in $L^p(E)$.

1.5. Results from the Homework

(1) (When Hölder's inequality $\rightarrow$ equality): There is equality in Hölder's Inequality $\iff$ there exists constants α, β, both of which non-zero, for which:

$$\alpha|f|^p = \beta|g|^q$$

a.e. on E.

(2) (Extension of Hölder's Inequality for 3 functions): Let $E \subseteq \mathbb{R}$ be measurable, let $1 \leq p < \infty$, $1 \leq q < \infty$, $1 \leq r < \infty$ such that:

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$$

If $f \in L^p(E)$, $q \in L^q(E)$, and $h \in L^r(E)$, then $fgh \in L^(E)$ and:

$$\int_E |fgh| \leq ||f||_p ||g||_q ||h||_r$$

(3) For $1 \leq p \leq \infty$, q conjugate of p, $f \in L^p(E)$:

$$||f||_p = \max_{g \in L^q(E), ||g||_q \leq 1} \int_E fg$$

(4) (L^p dominated convergence theorem): Let $\{f_n\}$ be a sequence of measurable functions that converge pointwise a.e. on E to f. For $1 \leq p < \infty$, suppose $\exists$ a function $g \in L^p(E)$ such that $\forall$ $n \in \mathbb{N}$, $|f_n| \leq g$ a.e. on E. Then, $\{f_n\} \rightarrow f$ in $L^p(E)$.

(5) Assume $1 \leq p < \infty$, if $E \subseteq \mathbb{R}$ has finite measure, $1 \leq p < \infty$, and $\{f_n\}$ is a sequence of measurable functions which converge pointwise a.e. on E to f, then $\{f_n\} \rightarrow f$ in $L^p(E)$ if $\exists$ a $\theta > 0$ such that $\{f_n\}$ belongs to and is <u>bounded</u> as a subset of $L^{p+\theta}(E)$.

(6) The space c of all convergent sequences of real numbers and the space c_0 of all sequences which converge to zero are Banach spaces with respect to the ℓ^∞ norm.

(7) Let $E \subseteq \mathbb{R}$ be measurable, $1 \leq p \leq \infty$, q the conjugate of p, and $\mathcal{S}$ a dense subset of $L^q(E)$. If $g \in L^p(E)$ and $\int_E g \cdot g = 0$ for all $g \in \mathcal{S}$, then $g = 0$.

(8) (Separability of ℓ^p): For $1 \leq p < \infty$, ℓ^p is separable. ℓ^∞ is <u>not</u> separable.

2. L^p Spaces: Duality and Weak Convergence

2.1. Riesz Representation Theorem for the Dual of L^p, $1 \le p < \infty$

Definition 18 (Linear Functional)**.** A **linear functional** on a linear space X is a real-valued function T on X such that for f and g in X and α and β real numbers,

$$T(\alpha \cdot g + \beta \cdot h) = \alpha \cdot T(g) + \beta \cdot T(h) \tag{10}$$

Definition 19 (Bounded)**.** For a normed linear space X, a linear functional T on X is said to be **bounded** provided there is an $M \ge 0$ for which

$$|T(f)| \le M \cdot ||f|| \text{ for all } f \in X \tag{11}$$

The infimum of all such M is called the **norm** of T and is denoted by $||T||_*$.

Proposition 6 (Continuity Property of a Bounded Linear Functional)**.** Let T be a bounded linear functional on the normed space X. Then, if $\{f_n\} \to f$ in X, then $\{T(f_n)\} \to \{T(f)\}$.

Proposition 7. Let X be a normed vector space. Then, the collection of bounded linear functionals on X is a linear space which is normed by $|| \cdot ||_*$. This normed vector space is called the **dual space** of X, and is denoted by X^*.

Proposition 8. Let $E \subseteq \mathbb{R}$ be measurable, $1 \le p < \infty$, q the conjugate of p, $g \in L^q(E)$. Define the functional T on $L^p(E)$ by:

$$T(f) := \int_E g \cdot f \ \forall f \in L^p(E) \tag{12}$$

Then, T is a bounded linear functional on $L^p(E)$ and $||T||_* = ||g||_q$.

Proposition 9. Let T, S be bounded linear functionals on the normed vector space X. If $T = S$ on a dense subset X_0 of X, then $T = S$.

Lemma 10. Let $E \subseteq \mathbb{R}$ be measurable, $1 \le p < \infty$. Suppose that g is integrable over E and there exists a $M \ge 0$ for which

$$\left| \int_E g \cdot f \right| \le M||f||_p \ \forall f \in L^p(E),\ f \text{ simple}$$

Then, $g \in L^q(E)$, where q is the conjugate of p. Moreover, $||g||_q \le M$.

Theorem 11. Let $[a, b]$ be a closed, bounded interval, and $1 \le p < \infty$. Suppose that T is a bounded linear functional on $L^p[a, b]$. Then, there is a functional $g \in L^q[a, b]$, where q is the conjugate of p, for which:

$$T(f) = \int_a^b g \cdot f \ \forall f \in L^p[a, b] \tag{13}$$

Theorem 12 (Riesz-Representation Theorem for the Dual of $L^p(E)$)**.** Let $E \subseteq \mathbb{R}$ be measurable, $1 \le p < \infty$, and q the conjugate of p. For all $g \in L^q(E)$, define the bounded linear functional $\mathcal{R}_g$ on $L^p(E)$ by:

$$\mathcal{R}_g := \int_E g \cdot f \ \forall f \in L^p(E) \tag{14}$$

Then, for each bounded linear functional T on $L^p(E)$, there exists a unique $g \in L^q(E)$ for which

(1) $\mathcal{R}_g = T$ and
(2) $||T||_* = ||g||_q$

2.2. Weak Sequential Convergence in L^p

Definition 20 (Converge Weakly)**.** Let X be a normed vector space. A sequence $\{f_n\}$ in X is said to **converge weakly** in X to f provided that

$$\lim_{n\to\infty} T(f_n) = T(f)\ \forall T \in X^* \tag{15}$$

we write

$$\{f_n\} \rightharpoonup f$$

to mean that f and each f_n belong to X and $\{f_n\}$ converges weakly in X to f.

Definition 21. Let $E \subseteq \mathbb{R}$ be measurable, $1 \leq p < \infty$, q the conjugate of p. Then, $\{f_n\} \rightharpoonup f$ in $L^p(E)$ $\iff$

$$\lim_{n\to\infty} \int_E g \cdot f_n = \int_E g \cdot f\ \forall g \in L^q(E) \tag{16}$$

Theorem 13. Let $E \subseteq \mathbb{R}$ be measurable, $1 \leq p < \infty$. Suppose that $\{f_n\} \rightharpoonup f$ in $L^p(E)$. Then:

$$\{f_n\} \text{ is bounded and } ||f||_p \leq \liminf ||f_n||_p$$

Corollary 3. Let $E \subseteq \mathbb{R}$ be measurable, $1 \leq p < \infty$, q the conjugate of p. Suppose $\{f_n\}$ converges weakly to f in $L^p(E)$ and $\{g_n\}$ converges strongly to $g \in L^q(E)$. Then:

$$\lim_{n\to\infty} \int_E g_n \cdot f_n = \int_E g \cdot f \tag{17}$$

Definition 22 (Linear Span)**.** Let X be a normed vector space, and let $S \subseteq X$. Then, the **linear span of** S is the vector space consisting of all linear functionals of the form:

$$f = \sum_{k=1}^{n} \alpha_k \cdot f_k \tag{18}$$

where each $\alpha_k \in \mathbb{R}$ and $f_k \in S$. It is the set of all *finite linear combinations of elements in* S. We care about this since L^p is an infinite dimensional space, so we want to find a way to approximate it with finitely many elements.

Proposition 10 (Characterisation of Weak Convergence in $L^p(E)$)**.** Let $E \subseteq \mathbb{R}$ be measurable, $1 \leq p < \infty$, q the conjugate of P. Assume that $\mathcal{F} \subseteq L^q(E)$ whose linear span is dense in $L^q(E)$. Let $\{f_n\}$ be a bounded sequence in $L^p(E)$, and let $f \in L^p(E)$. Then, $\{f_n\} \rightharpoonup f$ in $L^p(E)$ $\iff$

$$\lim_{n\to\infty} \int_E f_n \cdot g = \int_E f \cdot g\ \forall g \in \mathcal{F} \tag{19}$$

Theorem 14. Let $E \subseteq \mathbb{R}$ be measurable, $1 \leq p < \infty$. Suppose that $\{f_n\}$ is a bounded sequence in $L^p(E)$ and f belongs to $L^p(E)$. Then, $\{f_n\} \rightharpoonup f$ in $L^p(E)$ $\iff$ $\forall$ measurable sets $A \subseteq E$:

$$\lim_{n\to\infty} \int_A f_n = \int_A f \tag{20}$$

if $p > 1$, then it is sufficient to consider sets A of finite measure.

Theorem 15. Let $[a, b]$ be a closed and bounded interval, $1 < p < \infty$. Suppose that $\{f_n\}$ is a bounded sequence in $L^p[a, b]$ and $f \in L^p[a, b]$. Then, $\{f_n\} \rightharpoonup f$ in $L^p(E)$ in $L^p[a, b]$ $\iff$

$$\lim_{n\to\infty} \left[\int_a^x f_n\right] = \int_a^x f\ \forall x \in [a, b] \tag{21}$$

Lemma 16 (Riemann-Lebesgue Lemma; used in Fourier Series :-))**.** Let $I = [-\pi, \pi]$, $1 \leq p < \infty$. $\forall n \in \mathbb{N}$, define $f_n(x) := \sin(nx)$ for $x \in I$. Then, $\{f_n\}$ converges weakly in $L^p(I)$ to $f \equiv 0$.

Theorem 17. Let $E \subseteq \mathbb{R}$ be measurable, $1 < p < \infty$. Suppose that $\{f_n\}$ is a bounded sequence in $L^p(E)$ that converges pointwise a.e. on E to f. Then, $\{f_n\} \rightharpoonup f$ in $L^p(E)$.

This theorem was used in the proof but was not covered in Analysis 3:

> **Theorem 18** (Vitali Convergence Theorem)**.** Let $E \subseteq \mathbb{R}$ be measurable and of finite measure. Suppose that the sequence of functions $\{f_n\}$ is uniformly integrable over E. Then, if $\{f_n\} \to f$ pointwise a.e. on E, then f is integrable over E and $\lim_{n\to\infty} \int_E f_n = f$.

Theorem 19 (Radon-Riesz Theorem)**.** Let $E \subseteq \mathbb{R}$ be measurable, $1 < p < \infty$. Suppose that $\{f_n\} \rightharpoonup f$ in $L^p(E)$. Then:

$$\{f_n\} \to f \text{ in } L^p(E) \iff \lim_{n\to\infty} ||f_n||_p = ||f||_p \tag{22}$$

Corollary 4. (Not Covered in Class): Let $E \subseteq \mathbb{R}$ be measurableand $1 < p < \infty$. Suppose that $\{f_n\} \rightharpoonup f$ in $L^p(E)$. Then, a subsequence of $\{f_n\}$ converges strongly to $f \iff ||f||_p = \liminf ||f_n||_p$.

2.3. Weak Sequential Compactness ("Compactness Found!")

Theorem 20. Let $E \subseteq \mathbb{R}$ be measurable, $1 < p < \infty$. Then, every bounded sequence in $L^p(E)$ has a subsequence that converges weakly in $L^p(E)$ to a function in $L^p(E)$.

Theorem 21 (Helly's Theorem)**.** Let X be a *SEPARABLE* normed vector space and $\{T_n\}$ a sequence in its dual space X^* that is bounded; that is, $\exists$ a $M > 0$ for which

$$|T_n(f)| \leq M \cdot ||f|| \; \forall f \in X, \; \forall n \in \mathbb{N}$$

Then, there is a subsequence $\{T_{n_k}\}$ of $\{T_n\}$ and $T \in X^*$ for which

$$\lim_{k\to\infty} T_{n_k}(f) = T(f) \; \forall f \in X \tag{23}$$

Definition 23 (Weakly Sequentially Compact (Compact in the "weak topology")**.** Let X be a normed vector space. Then, a subset $K \subseteq X$ is **weakly sequentially compact** in X provided that every sequence $\{f_n\}$ in K has a subsequence that converges weakly to $f \in K$.

Theorem 22 (The Unit Ball is Weakly Sequentially Compact)**.** Let $E \subseteq \mathbb{R}$ be measurable, $1 < p < \infty$. Define:

$$B_1 := \{f \in L^p(E) \mid ||f||_p \leq 1\}. \tag{24}$$

B_1 is weakly sequentially compact in $L^p(E)$.

2.4. Results from the Homework

(1) (Reisz-Representation Theorem for the Dual of ℓ^p): Let $1 \leq p < \infty$, q the conjugate of p. Then for all $\{g_n\} \in \ell^q$, define the bounded linear functional $\mathcal{R}_g$ on ℓ^p by:

$$\mathcal{R}_g := T(\{f_n\}) = \sum_{n=1}^{\infty} g_n f_n \tag{25}$$

$\forall \{f_n\} \in \ell^p$. Then, for each bounded linear functional T on ℓ^p, there exists a unique $\{g_n\} \in \ell^q$ for which:

(1) $\mathcal{R}_g = T$
(2) $||T||_* = ||\{g_n\}||_q$

(2) Let c be the vector space of all real sequences that converge to a real number and let c_0 be the subspace of c comprising of all sequences that converge to zero. Norm each vector space with the ℓ^∞ norm. Then, $c^* = \ell^1$ and $c_0^* = \ell^1$.

(3) Assume that h is a continuous function defined on all of $\mathbb{R}$ that is periodic with period T and $\int_0^T h = 0$. Let $[a, b]$ be a closed + bounded interval. For each $n \in \mathbb{N}$, define $f_n(x) := h(nx)$. Define $f \equiv 0$ on $[a, b]$. Then, $\{f_n\}$ converges weakly to f in $L^p[a, b]$.

(4) Let $1 < p < \infty$, assume $f_0 \in L^p(\mathbb{R})$. For each $n \in \mathbb{N}$, define $f_n(x) := f_0(x - n)$. Define $f \equiv 0$ on $\mathbb{R}$. Then, $\{f_n\}$ converges weakly to f in $L^p(\mathbb{R})$. Not true for $p = 1$!

(5) For $1 \leq p < \infty$, for each $n \in \mathbb{N}$, let $e_n \in \ell^p$ be the standard basis sequence. If $p > 1$, then $\{e_n\}$ converges weakly to zero in ℓ^p, but no subsequence converges strongly to zero. $\{e_n\}$ does not converge at all in ℓ^1.

(6) (Uniform Boundedness Principle): Let $E \subseteq \mathbb{R}$ be measurable, $1 \leq p < \infty$, and q the conjugate of p. Suppose that $\{f_n\}$ is a sequence in $L^p(E)$ for which for each $g \in L^q(E)$, the sequence $\{\int_E g f_n\}$ is bounded. Show that $\{f_n\}$ is bounded in $L^p(E)$.

(7) $\{x^n\}$ in $C[0, 1]$ fails to have a strongly convergent subsequence. Suitably modify this to work in any $C[a, b]$ by:

$$f_n := \left(\frac{x - a}{b - a}\right)^n$$

(8) In ℓ^p, $1 < p < \infty$, every bounded sequence in ℓ^p has a weakly convergent subsequence.

(9) Let X be a normed vector space, and let $\{T_n\}$ be a sequence in X^* for which there exists an $M \geq 0$ such that $||T_n||_* \leq M$ for all $n \in \mathbb{N}$. Let $\mathcal{S} \subseteq X$ be a dense subset such that $\{T_n(g)\}$ is Cauchy for all $g \in \mathcal{S}$. Then:

(1) $\{T_n(g)\}$ is Cauchy for all $g \in X$.
(2) The limiting functional is linear and bounded.

(10) Helly's theorem fails when $X = L^\infty[0, 1]$. To see why, consider a sequence of linear functionals induced by the Rademacher functions.

3. Metric Spaces

This section was not covered in class, but since we have homework on this chapter I figured having this as a review from analysis 2 might be helpful. Also, there are a few terms/results that I don't think we covered in analysis 2.

3.1. Examples of Metric Spaces

Definition 24 (Metric Space)**.** Let X be a non-empty set. A function $\rho : X \times X \to \mathbb{R}$ is called a **metric** if $\forall\ x, y \in X$:

(1) $\rho(x, y) \geq 0$
(2) $\rho(x, y) = 0 \iff x = y$
(3) $\rho(x, y) = \rho(y, x)$

(4) $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$ (**Triangle Inequality**).

A non-empty set together with a metric, denoted (X,ρ) is called a **metric space**.

Definition 25 (Discrete Metric)**.** For any non-empty set X, the **discrete metric** ρ is defined by setting $\rho(x,y) = 0$ if $x = y$ and $\rho(x,y) = 1$ if $x \neq y$.

Definition 26 (Metric Subspace)**.** For any metric space (X,ρ), let $Y \subseteq X$ be non-empty. Then, the restriction of ρ to $Y \times Y$ defines a metric on Y. We define this induced metric space as a **metric subspace**.

Example 3.1 (Examples of metric spaces)**.** The following are examples of metric spaces:

(1) Every non-empty subset of a Euclidean space.
(2) $L^p(E)$, where $E \subseteq \mathbb{R}$ is a measurable set.
(3) $C[a,b]$.

Definition 27 (Product Metric)**.** For metric spaces (X_1,ρ_1) and (X_2,ρ_2), we define the **product metric** τ on the cartesian product $X_1 \times X_2$ by setting, for (x_1,x_2) and (y_1,y_2) in $X_1 \times X_2$:

$$\tau((x_1,x_2),(y_1,y_2)) := \{[\rho_1(x_1,x_2)]^2 + [\rho_2(y_1,y_2)]^2\}^{1/2} \tag{26}$$

Definition 28. Two metrics ρ and σ on a set X are said to be **equivalent** if there are positive numbers c_1 and c_2 such that $\forall\ x_1, x_2 \in X$,

$$c_1\sigma(x_1,x_2) \leq \rho(x_1,x_2) \leq c_2\sigma(x_1,x_2)$$

Definition 29 (Isometry)**.** A mapping $f : (X,\rho) \to (Y,\sigma)$ between two metric spaces is called an **isometry** provided that f is surjective and $\forall x_1, x_2 \in X$:

$$\sigma(f(x_1), f(x_2)) = \rho(x_1,x_2) \tag{27}$$

We say that two metric spaces are **isometric** if there is an isometry from one to another.

3.2. Open Sets, Closed Sets, and Convergent Sequences

Definition 30 (Open Ball)**.** Let (X,ρ) be a metric space. For a point $x \in X$ and $r > 0$, the set:

$$B(x,r) := \left\{x' \in X \mid \rho(x',x) < r\right\} \tag{28}$$

is called the **open ball** centred at x of radius r. A subset $\mathcal{O} \subseteq X$ is said to be **open** if $\forall x \in \mathcal{O}$, there exists an open ball centred at x and contained in $\mathcal{O}$. For a point $x \in X$, an open set containing x is called a **neighbourhood** of x.

Proposition 11. Let X be a metric space. The whole set X and the empty set $\emptyset$ are open. The intersection of any two open sets is open. The union of any collection of open sets is open.

Proposition 12. Let X be a subspace of a metric space Y and $E \subseteq X$. Then, E is **open in** $X \iff E = X \cap \mathcal{O}$, where $\mathcal{O}$ is open in Y.

Definition 31 (Closure)**.** For a subset $E \subseteq X$, a point $x \in X$ is called a **point of closure** of E provided that every neighbourhood of x contains a point in E. The collection of the points of closure of E is called the **closure** of E and is denoted by $\overline{E}$.

Proposition 13. For $E \subseteq X$, where X is a metric space, its closure $\overline{E}$ is closed. Moreover, $\overline{E}$ is the smallest closed subset of X containing E in the sense that if F is closed and if $E \subseteq F$, then $\overline{E} \subseteq F$.

Definition 32 (Converge)**.** A sequence $\{x_n\}$ in a metric space (X, ρ) is said to **converge** to the point $x \in x$ provided that:

$$\lim_{n\to\infty} \rho(x_n, x) = 0$$

that is, $\forall\ \varepsilon > 0$, $\exists$ an index N such that $\forall n \geq N$, $\rho(x_n, x) < \varepsilon$.

Proposition 14. Let ρ and σ be equivalent metrics on a non-empty set X. Then, a subset X is open in a metric space $(X, \rho) \iff$ it is open in (X, σ).

3.3. Continuous Mappings Between Metric Spaces

Definition 33 (Continuous)**.** A mapping f from a metric space X to a metric space Y is continuous at the point $x \in X$ if $\forall\ \{x_n\} \in X$, if $\{x_n\} \to x$, then $\{f(x_n)\} \to f(x)$. f is said to be **continuous** if it is continuous at every point in X.

Proposition 15 (ε-δ criteria for continuity)**.** A mapping from a metric space (X, ρ) to a metric (Y, σ) is continuous at the point $x \in X \iff \forall\ \varepsilon > 0$, $\exists\ \delta > 0$ such that if $\rho(x, x') < \delta$, then $\sigma(f(x), f(x')) < \varepsilon$. That is:

$$f(B(x, \delta)) \subseteq B(f(x), \varepsilon) \tag{29}$$

Proposition 16. A mapping f from a metric space X to a metric space Y is continuous $\iff \forall$ open subsets $\mathcal{O} \subseteq Y$, the inverse image under f of $\mathcal{O}$, $f^{-1}(\mathcal{O})$, is an open subset of X.

Proposition 17. The composition of continuous mappings between metric spaces, when defined, is continuous.

Definition 34 (Uniformly Continuous)**.** A mapping from a metric space (X, ρ) to a metric space (Y, σ) is said to be **uniformly continuous** if $\forall\ \varepsilon > 0$, $\exists\ \delta > 0$ such that $\forall u, v \in X$, if $\rho(u, v) < \delta$, $\sigma(f(u), f(v)) < \varepsilon$.

Definition 35 (Lipschitz)**.** A mapping $f : (X, \rho) \to (Y, \sigma)$ is said to be **Lipschitz** if $\exists$ a $c \geq 0$ such that $\forall\ u, v \in X$:

$$\sigma(f(u), f(v)) \leq c\rho(u, v)$$

3.4. Complete Metric Spaces

Definition 36 (Cauchy)**.** A sequence $\{x_n\}$ in a metric space (X, ρ) is said to be a **Cauchy sequence** if $\forall \varepsilon > 0$, there exists a $N \in \mathbb{N}$ such that if $m, n \geq N$, then $\rho(x_n, x_m) < \varepsilon$.

Definition 37 (Complete)**.** A metric space X is said to be **complete** if every Cauchy sequence in X converges to a point in X.

Proposition 18. Let $[a, b]$ be a closed and bounded interval of real numbers. Then, $C[a, b]$ with the metric induced by the max norm is complete.

Proposition 19 (Characterisation of Complete Subspaces of Metric Spaces)**.** Let $E \subseteq X$, where X is a complete metric space. Then, the metric subspace E is complete $\iff E$ is a closed subset of X.

Theorem 23. The following are complete metric spaces:

(1) Every non-empty closed subset of $\mathbb{R}^n$.
(2) $E \subseteq \mathbb{R}$ measurable, $1 \leq p \leq \infty$, each non-empty closed subset of $L^p(E)$.
(3) Each non-empty closed subset of $C[a, b]$.

Definition 38 (Diameter)**.** Let E be a non-empty subset of a metric space (X, ρ). We define the **diameter** of E, denoted by $\text{diam}(E)$, by:

$$\text{diam}(E) := \sup\{\rho(x, y) \mid x, y \in E\} \tag{30}$$

We say that E is **bounded** if it has finite diameter.

Definition 39 (Contracting Sequence)**.** A decreasing sequence $\{E_n\}$ of non-empty subsets of X is called a **contracting sequence** if:

$$\lim_{n\to\infty} \text{diam}(E_n) = 0 \tag{31}$$

Theorem 24 (Cantor Intersection Theorem)**.** Let X be a metric space. Then, X is complete $\iff$ whenever $\{F_n\}$ is a contracting sequence of non-empty closed subsets of X, there is a point $x \in X$ for which:

$$\bigcap_{n=1}^{\infty} F_n = \{x\} \tag{32}$$

Theorem 25. Let (X, ρ) be a metric space. Then, there is a complete metric space $(\widetilde{X}, \tilde{\rho})$ for which X is a dense subset of $\widetilde{X}$ and

$$\rho(u, v) = \tilde{\rho}(u, v)\ \forall\ u, v \in X \tag{33}$$

we call such a space the **completion** of (X, ρ).

3.5. Compact Metric Spaces

Definition 40 (Compact Metric Space)**.** A metric space X is called **compact** if every open cover of X has a finite sub-cover. A subset $K \subseteq X$ is compact if K, considered as a metric subspace of X, is compact.

<u>**Formulation of compactness in terms of closed sets:**</u> Let $\mathcal{T}$ be a collection of open subsets of a metric space X. Define $\mathcal{F}$ to be the collection of the complements of elements in $\mathcal{T}$. Since the elements of $\mathcal{T}$ are open, the elements of $\mathcal{F}$ are closed. Thus, $\mathcal{T}$ is a cover $\iff$ the elements of $\mathcal{F}$ have *empty intersection*. By deMorgan's law, we can formulate compactness in terms of closed sets as:

> A metric space X is compact $\iff$ every collection of closed sets with empty intersection has a finite sub-collection whose intersection is non-empty.

This property is called the **finite intersection property**.

Definition 41 (Finite Intersection Property)**.** A collection of sets $\mathcal{F}$ is said to have the **finite intersection property** if any finite sub-collection of $\mathcal{F}$ has a non-empty intersection.

Proposition 20 (Compactness in terms of closed sets)**.** A metric space X is compact $\iff$ every collection $\mathcal{F}$ of closed subsets of X with the finite intersection property has a non-empty intersection.

Definition 42 (Totally Bounded)**.** A metric space X is **totally bounded** if $\forall\ \varepsilon > 0$, the space X can be covered by a finite number of open balls of radius ε. A subset $E \subseteq X$ is said to be **totally bounded** if E, a s a subspace of the metric space X, is totally bounded.

Definition 43 (ε-net)**.** Let E be a subset of a metric space X. A **ε-net** for R is a finite collection of open balls $\{B(x_k, \varepsilon)\}_{k=1}^{n}$ with centres $x_k \in X$ whose union covers E.

Proposition 21. A metric space E is totally bounded $\iff$ $\forall\ \varepsilon > 0$, there is a finite ε-net for E.

Proposition 22. A subset of Euclidean space $\mathbb{R}^n$ is bounded $\iff$ it is totally bounded.

Definition 44 (Sequentially Compact)**.** A metric space X is **sequentially compact** if every sequence in X has a subsequence that converges to a point in X.

Theorem 26 (Characterisation of Compactness for a metric space)**.** . Let X be a metric space. Then, TFAE:

(1) X is complete and totally bounded.
(2) X is compact.
(3) X is sequentially compact.

The following three propositions of this chapter are just breaking down these equivalences, so I will not write them.

Theorem 27. Let $K \subseteq \mathbb{R}^n$. Then, TFAE:

(1) K is closed and bounded.
(2) K is compact.
(3) K is sequentially compact.

Observe: The equivalence (1) $\iff$ (2) is the Heine-Borel theorem. The equivalence (2) $\iff$ (3) is the Bolzano-Weierstrass theorem.

Proposition 23. Let f be a continuous mapping from a compact metric space X to a compact metric space Y. Then, its image $f(X)$ is compact.

Theorem 28 (Extreme Value Theorem)**.** Let X be a metric space. Then, X is compact $\iff$ every continuous real-valued function on X attains a minimum and maximum value.

Definition 45 (Lebesgue Number)**.** Let X be a metric space, and let $\{\mathcal{O}_\lambda\}_{\lambda\in\Lambda}$ be an open cover of X. Thus, each $x \in X$ is contained in a member of the cover, $\mathcal{O}_\lambda$. Since $\mathcal{O}_\lambda$ is open, $\exists\ \varepsilon > 0$ such that:

$$B(x,\varepsilon) \subseteq \mathcal{O}_\lambda$$

In general, ε on X, but for compact metric spaces we can get *uniform control*. This ε that uniformly works is called the **Lebesgue number** for the cover $\{\mathcal{O}_\lambda\}_{\lambda\in\Lambda}$.

Lemma 29. Let $\{\mathcal{O}_\lambda\}_{\lambda\in\Lambda}$ be an open cover of a compact metric space X. Then, there is a number $\varepsilon > 0$ such that for each $x \in X$, the open ball $B(x,\varepsilon)$ is contained in some member of the cover.

Proposition 24. A continuous mapping from a compact space (X,ρ) to a metric space (Y,σ) is uniformly continuous.

3.6. Separable Metric Spaces

Definition 46 (Dense & Separable)**.** A subset D of a metric space X is **dense** in X if every non-empty subset of X contains a point of D. A metric space is **separable** if there is a countable subset of X that is dense in X.

The **Weierstrass Approximation Theorem** states that polynomials are dense in $C[a,b]$. So, $C[a,b]$ is separable, with the countable dense set being the set of polynomials with rational coefficients.

Proposition 25. A compact metric space is separable.

Proposition 26. A metric space X is separable $\iff$ there is a countable collection of $\{\mathcal{O}_n\}$ of open subsets of X such that any open subset of X is the union of a sub-collection of $\{\mathcal{O}_n\}$.

Proposition 27. Every subspace of a separable metric space is separable.

Theorem 30. Each of the following are separable metric spaces:

(1) Every non-empty subset of Euclidean space $\mathbb{R}^n$.
(2) $1 \leq p < \infty$, $L^p(E)$ and all non-empty subsets of $L^p(E)$.
(3) Each non-empty subset of $C[a, b]$.

3.7. Results from the Homework

(1) $\{(X_n, \rho_n)\}_{n=1}^{\infty}$ a countable collection of metric spaces. Then, the following is a metric on the Cartesian product:

$$\rho_*(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{\rho_n(x_n, y_n)}{1 + \rho_n(x_n + y_n)}$$

(2) A continuous mapping between metric spaces remains continuous if an equivalent metric is imposed on the domain and an equivalent metric is imposed on the domain.
(3) The distance function (from a point to a set) is continuous.
(4) $\{x \in X \mid \text{dist}(x, E) = 0\} = \overline{E}$.
(5) (Sequential Definition of Uniform Continuity): For a mapping f of a metric space (X, ρ) to the metric space (Y, σ), f is uniformly continuous $\iff$ for all sequences $\{u_n\}$ and $\{v_n\}$ in X:

$$\text{if } \lim_{n\to\infty} \rho(u_n, v_n) = 0 \text{ then } \lim_{n\to\infty} \sigma(f(u_n), f(v_n)) = 0$$

(6) If X and Y are metric spaces, with Y complete, and f a uniformly continuous mapping from $E \subseteq X \to Y$, then f has a uniquely uniformly continuous extension mapping $\overline{f}$ of $\overline{E}$ to Y.
(7) Let $E \subseteq X$, X a compact metric space. Then, the metric subspace E is compact $\iff$ E is a closed subset of X.
(8) $E \subseteq X$, X complete. Then, E is totally bounded $\iff$ $\overline{E}$ is totally bounded.
(9) The closed unit ball in ℓ^2 is not compact.

4. Topological Spaces

4.1. Open Sets, Closed Sets, Bases, and Sub-bases

Definition 47 (Open Sets)**.** Let X be a non-empty set. A **topology** $\mathcal{T}$ for X is a collection of subsets of X, called **open sets**, posessing the following properties:

(1) The entire set X and the empty set $\emptyset$ are open.
(2) The finite intersection of open sets are open.
(3) The union of any collection of open sets is open.

A non-empty set X, together with a topology on X, is called a **topological space**. For a point $x \in X$, an open set that contains x is called a **neighbourhood** of x.

Proposition 28. A subset $E \subseteq X$ is open $\iff$ for each $x \in E$, there exists a neighbourhood of x that is contained in E.

Example 1 (Metric Topology)**.** Let (X, ρ) be a metric space. Let $\mathcal{O} \subseteq X$ be open if for all $x \in \mathcal{O}$, $\exists$ an open ball at x that is contained in $\mathcal{O}$. This collection of open sets forms a topology; we call this the **metric topology** induced by ρ.

Example 2 (Discrete Topology)**.** This topology is "too much." Let X be a non-empty subset. Let $\mathcal{T} := \mathcal{P}(X)$. Then, every set containing a point is a neighbourhood of that point. This is induced by the discrete metric.

Example 3 (Trivial Topology)**.** Let X be non-empty. Define $\mathcal{T} := \{X, \emptyset\}$. The only neighbourhood of any point is the whole set X.

Definition 48 (Topological Subspaces)**.** Let $(X, \mathcal{T})$ be a topological space and let E be a non-empty subset of X. The inherited topology $\mathcal{S}$ for E is the set of all sets of the form $E \cap \mathcal{T}$, where $\mathcal{O} \in \mathcal{T}$. The topological space $(E, \mathcal{S})$ is called a **subspace** of $(X, \mathcal{T})$.

Definition 49 (Base for the Topology)**.** The building blocks of a topology is called a **base**. Let $(X, \mathcal{T})$ be a topological space. For a point $x \in X$, a collection of neighbourhoods of x, B_x, is called a **base for the topology at** X if $\forall$ neighbourhoods $\mathcal{U}$ of x, there exists a set B in the collection B_x for which $B \subseteq \mathcal{U}$.

A collection of open sets $\mathcal{B}$ is called a **base for the topology** $\mathcal{T}$ provided it contains a base for the topology at each point.

A base for a topology completely determines a topology, alongside $\emptyset$ and X.

Proposition 29. For a non-empty set X, let $\mathcal{B}$ be a collection of subsets of X. Then, $\mathcal{B}$ is a base for a topology for $X \iff$:

(1) $\mathcal{B}$ covers X. That is:

$$X = \bigcup_{B \in \mathcal{B}} B \tag{34}$$

(2) If $B_1, B_2 \in \mathcal{B}$, and $x \in B_1 \cap B_2$, then there is a set $B_3 \in \mathcal{B}$ for which $x \in B_3 \subseteq B_1 \cap B_2$.

The unique topology that has $\mathcal{B}$ as its base consists of $\emptyset$ and unions of sub-collections of $\mathcal{B}$.

Definition 50 (Product Topology)**.** Let $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ be two topological spaces. In the cartesian product $X \times Y$, consider the collection of sets $\mathcal{B}$ containing the products $\mathcal{O}_1 \times \mathcal{O}_2$, where $\mathcal{O}_1$ is open in X and $\mathcal{O}_2$ is open in Y. Then, $\mathcal{B}$ is a base for a topology on $X \times Y$, which we call the **product topology.**

Definition 51 (Sub-base)**.** Let $(X, \mathcal{T})$ be a topological space. The collection of $\mathcal{S}$ of $\mathcal{T}$ that covers X is called a **sub-base** for the topology $\mathcal{T}$ provided intersections of finite collections of $\mathcal{S}$ are a base for $\mathcal{T}$.

Definition 52 (Closure)**.** Let $E \subseteq X$ be a subset of a topological space. A point $x \in E$ is called a **point of closure** of E if every neighbourhood of x contains a point in E. The collection of the points of closure of E is called the **closure** of E, denoted $\overline{E}$.

Proposition 30. Let X be a topological space, $E \subseteq X$. Then, $\overline{E}$ is closed. Moreover, $\overline{E}$ is the smallest closed subset of X containing E in the sense that if F is closed and $E \subseteq F$, then $\overline{E} \subseteq F$.

Proposition 31. A subset of a topological space X is open $\iff$ its complement is closed.

Proposition 32. Let X be a topological space. Then, (a) $\emptyset$ and X are closed, (b) the union of a finite collection of closed sets is closed, (c) the intersection of any collection of closed sets in X is closed.

4.2. Separation Properties

Motivation: Separation properties for a topology allow us to discriminate between which topologies discriminate between certain disjoint pairs of sets, which will then allow us to study a robust collection of cts real-valued functions on X.

Definition 53 (Neighbourhood)**.** A **neighbourhood** of K for a subset $K \subseteq X$ is an open set that contains K.

Definition 54 (Separated by Neighbourhoods)**.** We say that two disjoint sets A and B in X can be separated by disjoint neighbourhoods provided that there exists neighbourhoods of A and B, respectively, that are disjoint.

Definition 55 (Separation Properties of Topological Spaces)**.** . In the order of most general to least general, they are:

(1) **Tychonoff Separation Property**: For each two points $u, v \in X$, there exists a neighbourhood of u that does not contain v and a neighbourhood of v that does not contain u.
(2) **Hausdorff Separation Property**: Each two points in X can be separated by disjoint neighbourhoods.
(3) **Regular Separation Property**: Tychonoff + each closed set and a point not in the set can be separated by disjoint neighbourhoods.
(4) **Normal Separation Property**: Tychonoff + each two disjoint closed sets can be separated by disjoint neighbourhoods.

Proposition 33. A topological space is Tychonoff $\iff$ every set containing a single point, $\{x\}$, is closed.

Proposition 34. Every metric space is normal.

Lemma 31. F is closed $\iff$ $\operatorname{dist}(x, F) > 0 \ \forall \ x \notin F$.

Proposition 35. Let X be a Tychonoff topological space. Then, X is normal $\iff$ whenever $\mathcal{U}$ is a neighbourhood of a closed subset of F of X, there is another neighbourhood of F whose closure is contained in $\mathcal{U}$. that is, there is an open set $\mathcal{O}$ for which:

$$F \subseteq \mathcal{O} \subseteq \overline{\mathcal{O}} \subseteq \mathcal{U} \tag{35}$$

4.3. Countability and Separability

Definition 56 (Converge, Limit)**.** A sequence $\{x_n\}$ in a topological space X is said to **converge** to the point $x \in X$ if for each neighbourhood $\mathcal{U}$ of x, there exists an index $N \in \mathbb{N}$ such that if $n \geq N$, then x_n belongs to $\mathcal{U}$. This point is called a **limit** of the sequence.

Definition 57 (First and Second Countable)**.** A topological space X is **first countable** if there is a countable base at each point. A space X is said to be **second countable** if there is a countable base for the topology.

Example 4. Every metric space is first countable.

Proposition 36. Let X be a first countable topological space. For a subset $E \subseteq X$, a point $x \in X$ is called a point of closure of E $\iff$ it is a limit of a sequence in E. Thus, a subset E of X is closed $\iff$ whenever a sequence in E converges to $x \in X$, we have that $x \in E$.

Definition 58 (Dense/Separable)**.** A subset $E \subseteq X$ is **dense** in X if every open set in X contains a point of E. We call X **separable** if it has a countable dense subset.

Definition 59 (Metrisable)**.** A topological space X is said to be **metrisable** if the topology is induced by the metric.

Theorem 32. Let X be a second countable topological space. Then, X is metrisable $\iff$ it is normal.

4.4. Continuous Mappings between Topological Spaces

Definition 60 (Continuous)**.** For topological spaces $(X, \mathcal{T})$, $(Y, \mathcal{S})$, a mapping $f : X \to Y$ is said to be **continuous** at the point x_0 in X if, for every neighbourhood $\mathcal{O}$ if $f(x_0)$, there is a neighbourhood $\mathcal{U}$ of x_0 for which $f(\mathcal{U}) \subseteq \mathcal{O}$. We say that f is continuous provided it is continuous at each point in X.

Proposition 37. A mapping $f : X \to Y$ between topological spaces X and Y is continuous $\iff$ for any open subset $\mathcal{O}$ in Y, its inverse image under f, $f^{-1}(\mathcal{O})$, is an open subset of X.

Proposition 38. The composition of continuous mappings between topological spaces, when defined, is continuous.

Definition 61 (Stronger)**.** Given two topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ for a set X, if $\mathcal{T}_2 \subseteq \mathcal{T}_1$, then we say that $\mathcal{T}_2$ is **weaker** than $\mathcal{T}_1$, and that $\mathcal{T}_1$ is **stronger** than $\mathcal{T}_2$.

Proposition 39. Let X be a non-empty set and let $\mathcal{S}$ be a collection of subsets of X that covers X. The collection of subsets of X consisting of intersections of finite collections of $\mathcal{S}$ is a base for a topology $\mathcal{T}$ of X. It is the weakest topology containing $\mathcal{S}$ in the sense that if $\mathcal{T}'$ is any other topology for X containing $\mathcal{S}$, then $\mathcal{T} \subseteq \mathcal{T}'$.

Definition 62 (Weak Topology)**.** Let X be a non-empty set and $\mathcal{F} := \{f_\alpha \mid X \to X_\alpha\}_{\alpha \in \Lambda}$ a collection of mappings, where each X_α is a topological space. The weakest topology for X that contains the collection of sets

$$\{f_\alpha^{-1}(\mathcal{O}_\alpha) \mid f_\alpha \in \mathcal{F},\ \mathcal{O}_\alpha \text{ open in } X_\alpha\} \tag{36}$$

is called the **weak topology for X induced by $\mathcal{F}$.**

Proposition 40. Let X be a non-empty set, $\mathcal{F} := \{f_\lambda \mid X \to X_\lambda\}_{\lambda \in \Lambda}$ a collection of mappings where each X_λ is a topological space. The weak topology for X induced by $\mathcal{F}$ is the topology on X that has the fewest number of sets covering the topologies on X for which each mapping $f_\lambda : X \to X_\lambda$ is continuous.

Definition 63 (Homeomorphism)**.** A mapping from a topological space $X \to Y$ is said to be a **homeomorphism** if it is bijective and has a continuous inverse $f^{-1} : Y \to X$. Two topological spaces are said to be **homeomorphic** if there exists a homeomorphism between them. The notion of homeomorphism induces a notion of an equivalence relation between spaces.

4.5. Compact Topological Spaces

Definition 64 (Cover)**.** A collection of sets $\{E_\lambda\}_{\lambda \in \Lambda}$ is said to be a **cover** of a set E if $E \subseteq \bigcup_{\lambda \in \Lambda} E_\lambda$.

Definition 65 (Compact)**.** A topological space X is said to be **compact** if every open cover of X has a finite sub-cover. A subset $K \subseteq X$ is compact if K, considered as a topological space with the subspace topology inherited from X, is compact.

Proposition 41. A topological space X is compact $\iff$ every collection of closed subsets of X that posesses the finite intersection property has non-empty intersection.

Proposition 42. A closed subset K of a compact topological space is compact.

Proposition 43. A compact subspace K of a Hausdorff topological space is a closed subset of X.

Definition 66 (Sequentially Compact)**.** A topological space X is said to be **sequentially compact** if every sequence in X has a subsequence that converges to a point in X.

Proposition 44. Let X be a second countable topological space. Then, X is compact $\iff$ it is sequentially compact.

Theorem 33. A compact Hausdorff space is normal.

Proposition 45. A continuous one-to-one mapping f of a compact space X onto a Hausdorff space Y is a homeomorphism.

Proposition 46. The continuous image of a compact topological space is compact.

Corollary 5. A continuous real-valued function on a compact topological space takes on a minimum and maximum functional value.

Definition 67 (Countably Compact)**.** A topological space is **countably compact** if every countable open cover has a finite subcover.

4.6. Connected Topological Space

Definition 68 (Separated)**.** Two non-empty subsets of a topological space **separate** X if they are disjoint and their union is X.

Definition 69 (Connected)**.** A topological space which cannot be separated by open sets is said to be **connected**. A subset $E \subseteq X$ is **connected** if there do NOT exist open subsets $\mathcal{O}_1$, $\mathcal{O}_2$ of X for which:

$$\begin{aligned}
&\mathcal{O}_1 \cap E \neq \emptyset \\
&\mathcal{O}_2 \cap E \neq \emptyset \\
&E \subseteq \mathcal{O}_1 \cup \mathcal{O}_2, \\
&E \cap \mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset
\end{aligned}$$

Proposition 47. Let f be a continuous mapping of a connected space X to a topological space Y. Then, its image $f(X)$ is connected.

Proposition 48. For A set $C \in \mathbb{R}$, the following are equivalent.

(1) C is an interval.
(2) C is convex.
(3) C is connected.

Definition 70 (Intermediate Value Property)**.** A topological space X has the **intermediate value property** if the image of any continuous real-valued function on X is an interval.

Proposition 49. A topological space has the intermediate value property $\iff$ it is connected.

Definition 71 (Arcwise connected)**.** A topological space X is **arcwise connected** if, for each pair $u, v \in X$, there exists a continuous map $f : [0, 1] \to X$ for which $f(0) = u$ and $f(1) = v$. Note:

(1) Connected $\iff$ arcwise connected in $\mathbb{R}^n$.
(2) Arcwise connected $\Rightarrow$ connected (in general)
(3) There exist connected but non-arcwise connected spaces (in general).

4.7. Results from Homework

(1) Let X be a topological space. Then, X is Hausdorff $\iff$ the diagonal $D := \{(x_1, x_2) \in X \times X \mid x_1 = x_2\}$ is closed as a subset of $X \times X$.
(2) The Moore plane is separable. The subspace $\mathbb{R} \times \{0\}$ is not separable. Thus, the Moore plane is not metrisable and not second countable.
(3) Let X and Y be topological spaces. Then, you can construct a continuous map from a Hausdorff space to a non-Hausdorff space, and you can do the same for a normal space to a non-normal space.

(4) If ρ_1 and ρ_2 are metrics on a set X that induce topologies $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively, then if they generate the same topology $\mathcal{T}_1 = \mathcal{T}_2$, then they are NOT necessarily equivalent. A counter example would be:

$$\rho_1 := |x-y|$$
$$\rho_2 := \frac{|x-y|}{1+|x-y|}$$

5. Metric Spaces: Three Fundamental Theorems

5.1. The Arzela-Ascoli Theorem

Proposition 50. Let X be a compact metric space. Then $C(X)$ is complete.

Definition 72 (Epicontinuous)**.** A collection $\mathcal{F}$ of real-valued functions on a metric space X is said to be **epicontinuous** at the point $x \in X$ provided that $\forall \varepsilon > 0$, $\exists\, \delta > 0$ such that $\forall f \in \mathcal{F}$, $x' \in X$:

$$\text{if } \rho(x', x) < \delta \text{ then } |f(x') - f(x)| < \varepsilon$$

The collection $\mathcal{F}$ is said to be epicontinuous on X if it is epicontinuous at every point in X.

- Any finite collection of continuous functions will be epicontinuous.
- In general, infinite collections of epicontinuous functions are not epicontinuous. For example, consider $f_n := x^n$ where $x \in [0,1]$. This collection fails to be epicontinuous at $x = 1$.

Definition 73 (Pointwise bounded)**.** A sequence $\{f_n\}$ of real-valued functions on a set X are said to be **pointwise bounded** if $\forall x \in X$, the sequence $\{f_n\}$ is bounded. A sequence is **uniformly bounded** if $\exists M \geq 0$ for which

$$|f_n| \leq M \text{ on } X \text{ for all } n \in \mathbb{N}$$

Lemma 34 (The Arzela-Ascoli Lemma)**.** Let X be a separable metric space. Let $\{f_n\}$ be an equicontinuous sequence in X that is pointwise bounded. Then a subsequence $\{f_n\}$ converges pointwise on all of X to a real-valued function f on X.

Definition 74 (Uniformly Epicontinuous)**.** Let X be a compact metric space, $\mathcal{F}$ an epicontinuous collection of real-valued functions on X. Then, $\mathcal{F}$ is **uniformly equicontinuous** in the sense that $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ such that $\forall\, u, v \in X$, $\forall f \in \mathcal{F}$ if

$$\rho(u, v) < \delta \;\Rightarrow |f(u) - f(v)| < \varepsilon$$

Theorem 35 (Arzela-Ascoli Theorem)**.** Let X be a compact metric space, $\{f_n\}$ a uniformly bounded, equicontinuous sequence of real-valued functions on X. Then $\{f_n\}$ has a subsequence that converges uniformly on X to a continuous function f on X.

Theorem 36. Let X be a compact metric space and $\mathcal{F} \subseteq C(X)$. Then, $\mathcal{F}$ is a compact subspace of $C(X)$ $\iff$ $\mathcal{F}$ is closed, uniformly bounded, and epicontinuous.

5.2. Baire Category Theorem

Definition 75 (Hallow)**.** A subset of a metric space X is **hallow** if it has an empty interior.

- For $E \subseteq X$, E is **hallow** in X $\iff$ its complement is dense in X.

- Let X be a metric space. Let $0 < r_1 < r_2$. Bu the continuity of the metric, $\overline{B(x,r_1)} \subseteq B(x,r_2)$. Thus, $\overline{B(x,r_1)}$ is a closed set for which the following holds:

$$B(x,r_1) \subseteq \overline{B(x,r_1)} \subseteq B(x,r_2) \tag{37}$$

Theorem 37 (Baire Category Theorem)**.** Let X be a complete metric space.

(1) Let $\{\mathcal{O}_n\}_{n=1}^{\infty}$ be a countable collection of open, dense subsets of X. Then, the intersection $\bigcap_{n=1}^{\infty} \mathcal{O}_n$ is also dense.
(2) Let $\{F_n\}_{n=1}^{\infty}$ be a countable collection of closed, hallow subsets of X. Then, their union $\bigcup_{n=1}^{\infty} F_n$ is also hallow.

Equivalent formulation:

- In a complete metric space, the union of a countable collection of nowhere dense sets is hallow.

Definition 76 (Nowhere Dense)**.** A subset $E \subseteq X$, X a metric space, is called **nowhere dense** provided that its closure $\overline{E}$ is hallow. We have the following equivalence:

- A subset $E \subseteq X$ is nowhere dense $\iff$ for each open subset $\mathcal{O}$ of X, $E \cap \mathcal{O}$ is not dense in X.

Corollary 6. Let X be a complete metric space and $\{F_n\}_{n=1}^{\infty}$ a countable collection of closed subsets of X. If $\cup_{n=1}^{\infty} F_n$ has a non-empty interior, then at least one of the F_n's has a non-empty interior. In particular, if $X = \bigcup_{n=1}^{\infty} F_n$, then at least one of the $F_n's$ has empty interior.

Corollary 7. Let X be a complete metric space and $\{F_n\}_{n=1}^{\infty}$ a countable collection of closed subsets of X. Then $\bigcup_{n=1}^{\infty} \partial F_n$ is hallow.

Theorem 38. Let $\mathcal{F}$ be. family of continuous real-valued functions on a complete metric space X that is pointwise bounded in the sense that $\forall x \in X$, $\exists$ a constant M_x for which

$$|f(x)| \leq M_x \ \forall f \in \mathcal{F} \tag{38}$$

Then, there is a non-empty open subset $\mathcal{O}$ of X on which $\mathcal{F}$ is uniformly bounded in the sense that $\exists$ a constant M such that

$$|f| \leq M \text{ on } \mathcal{O} \ \forall f \in \mathcal{F} \tag{39}$$

Theorem 39. Let X be a complete metric space and let $\{f_n\}$ be a sequence of continuous real-valued functions on X that converges pointwise on X to the real-valued function f. Then, there is a dense set $D \subseteq X$ for which $\{f_n\}$ is epicontinuous and f is continuous at each point in D.

Some standard terminology:

- **First Category/Meger**: a subset $E \subseteq X$ is of the **first category** if E is the union of a countable collection of nowhere dense subsets of X.
- **Second Category/Non-Meger** a set that is not of the second category.
- **Residual**: the complement of a set of the first category.

Equivalent formulation of the Baire Category Theorem: *A non-empty set of a complete metric space is of the second category*

5.3. The Banach Contraction Principle

Definition 77 (Fixed Point). A point $x \in X$ is called a **fixed point** of the mapping $T : X \to X$ provided that $T(x) = x$.

Definition 78 (Convex). A subset $K \subseteq \mathbb{R}^n$ is said to be **convex** provided that whenever $u, v \in K$, the segment $\{tu + (1-t)v \mid 0 \leq t \leq 1\} \subseteq K$.

Theorem 40 (Brouwer's Fixed Point Theorem). If $K \subseteq \mathbb{R}^n$ is a compact, convex subset of $\mathbb{R}^n$, and if the mapping $T : K \to K$ is continuous, then T has a fixed point.

Definition 79 (Lipschitz). A mapping T from a metric space (X, ρ) to itself is said to be **Lipschitz** provided that there is a number $c \geq 0$, called a Lipschitz constant for the mapping, for which

$$\rho(T(u), T(v)) \leq c\rho(u, v) \ \forall u, v \in X \tag{40}$$

If $c < 1$, then the Lipschitz mapping is called a **contraction**

Theorem 41 (Banach Contraction Principle). Let X be a complete metric space and the mapping $T : X \to X$ a contraction. Then, $T : X \to X$ has exactly one fixed point.

Theorem 42 (Picard Local Existence Theorem). Let $\mathcal{O} \subseteq \mathbb{R}^n$ be open, $(x_0, y_0) \in \mathcal{O}$. Let $g : \mathcal{O} \to \mathbb{R}$ be a function. Suppose we want to find an open interval of real numbers I containing the point x_0 and a differentiable function $f : I \to \mathbb{R}$ such that

$$\begin{cases} f'(x) = g(x, f(x)) \ \forall x \in I \\ f(x_0) = y_0 \end{cases} \tag{41}$$

Suppose the function $g : \mathcal{O} \to \mathbb{R}$ is continuous and there is a positive number M for which the following Lipschitz property holds:

$$|g(x, y_1) - g(x, y_2)| \leq M|y_1 - y_2| \ \forall (x, y_1), (x, y_2) \in \mathcal{O}$$

Then, $\exists$ an open interval I containing x_0 on which the ODE above has a unique solution.

6. Topological Spaces: Three Fundamental Properties