MATH 567: FUNCTIONAL ANALYSIS (FALL 2020 SEMESTER)

SHEREEN ELAIDI AS TAUGHT BY PROF. LIN; LAST UPDATED: DECEMBER 1, 2020

This class is about linear functional analysis. This has a lot in common with linear algebra in infinite-dimensional spaces. We can think of this as infinite-dimensional linear algebra. There are two main applications of this: (a) geometry and topology in infinite dimensions and (b) solving PDEs. Recall that in regular linear algebra, we used those tools to solve linear systems. The infinite-dimensional equivalent to this is a PDE. In this class, we'll focus on the second application of functional analysis.

1. Basic Functional Analysis

This corresponds to Chapters 4 and 5 of the textbook.

1.1. Banach Spaces and General Topology.

Let X be a vector space. Recall that this means that it is closed under addition and scalar multiplication.

Definition 1.1 (Norm)**.** A **norm** on a vector space X, $||\cdot|| : X \to [0,\infty[$, satisfies the following three properties:

(1) $||x|| = 0 \iff x = 0$.
(2) **(Homogeneity)**: $||\lambda x|| = |\lambda|||x||$ for each $x \in X$, $\lambda \in \mathbb{R}$.
(3) **(Triangle Inequality):** $||x + y|| \leq ||x|| + ||y||$.

Definition 1.2 (Completeness / Banach Space)**.** X is **complete** if every Cauchy sequence converges. $(X, ||\cdot||)$ is a **Banach space** if it is a complete normed vector space.

Definition 1.3 (Dense Subset)**.** $Y \subseteq X$ is **dense** if

(1) $\overline{Y} = X$ (one thing we need to note: $\overline{Y}$ is the closure, but we need to ask ourselves "in which topology"?).

This is equivalent to:

$$\forall \varepsilon > 0,\ \forall x \in X,\ \exists y \in Y \ \text{ s.t. } ||x - y|| < \varepsilon.$$

And also equivalent to,

$$\forall x \in X,\ \exists \{y_n\} \subseteq Y \ \text{ s.t. } y_n \to y \in X.$$

Definition 1.4 (Strong Topology)**.** The **strong topology** is the topology induced by the norm, $||\cdot||$ (the open sets are characterized by the balls, $B_r := \{x \mid ||x|| < r\}$). In this topology, the definitions of density given above are equivalent.

Definition 1.5 (Separable)**.** X is **separable** if $\exists$ a countable dense subset.

We have the following equivalent definitions of compactness.

Definition 1.6 (Compactness 1)**.** $E \subseteq X$ is **compact** if every open cover of E admits a finite subcover.

Definition 1.7 (Compactness 2)**.** Every sequence has a convergent sub-sequence.

Definition 1.8 (Compactness 3)**.** For any sequence $\{x_n\} \subseteq E$, there exists $\{x_{n_k}\}$ and $x^* \in E$ such that $x_{n_k} \to x^* \in E$.

Definition 1.9 (Pre-Compact)**.** $E \subseteq X$ is **pre-compact** if $\overline{E}$ is compact.

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1.2. **Euclidean Space $\mathbb{R}^n$.** Let $x \in \mathbb{R}^n$. This is denoted by $(x_1, ..., x_n)$. Then, recall,

$$||x|| = ||x||_{\ell^2} = \left(\sum_{j=1}^{n} x_j^2\right)^{1/2}.$$

We also have these other typical norms on Euclidean space:

$$\begin{aligned} ||x||_{\ell^1} &= \sum_{j=1}^{n} |x_j| \\ ||x||_{\ell^p} &= \left(\sum_{j=1}^{n} |x_j|^p\right)^{1/p} \\ ||x||_{\ell^\infty} &= \max_{1\leq j\leq n} |x_j|. \end{aligned}$$

Definition 1.10 (Equivalent Norms)**.** We say that two norms, $|\cdot|$ and $||\cdot||$, are equivalent if and only if there exist two constants a and b such that

$$\boxed{a||x|| \leq |x| \leq b||x|| \qquad \forall x \in X.} \tag{1.11}$$

In words, this is saying that you can't be big on one norm but small in another. These norms are comparable; they are bounded by constants on either side.

Theorem 1.12. *All norms on $\mathbb{R}^n$ are equivalent (all norms in finite dimensions are equivalent).*

Proof. Let $||\cdot||$ be the Euclidean norm, and let $|\cdot|$ be another norm. Let $\{e_i\}_{i=1}^n$ be the standard basis of $\mathbb{R}^n$; recall that this is $e_i = (0, ..., 1, ..., 0)$ where the 1 is in the ith slot. Since this is a basis, for $x \in X$:

$$x = \sum_{i=1}^{n} x_i e_i.$$

By the reverse triangle inequality,

$$\begin{aligned} ||x| - |y|| &\leq |x - y| \\ &= \left|\sum_{i=1}^{n} (x_i - y_i)e_i\right| \\ &\leq \sum_{i=1}^{n} |x_i - y_i||e_i| \\ &\leq \underbrace{\left(\sum_{i=1}^{n} |e_i|^2\right)^{1/2}}_{:=C} \left(\sum_{i=1}^{n} |x_i - y_i|^2\right)^{1/2} \qquad \text{(Cauchy-Schwarz)} \\ &\leq C||x - y|| \qquad (*), \end{aligned}$$

where C is some number. Norms are continuous; $x \mapsto ||x||$ is continuous $S = \{x \mid ||x|| = 1\}$ (the unit ball). By $(*)$, $x \mapsto |x|$ is continuous on S. S is closed and bounded on $\mathbb{R}^n$ which means that S is compact. By the extreme value theorem, this means that there exist two constants $a, b \in \mathbb{R}$ such that

$$a \leq |x| \leq b \qquad \forall x \in S. \tag{1.13}$$

Observe that $|x| = 0 \iff x = 0$, which implies that $a > 0$. For any $y \in \mathbb{R}^n$, let $x := \frac{y}{||y||} \in S$. Then,

$$a \leq \left|\frac{y}{||y||}\right| \leq b \iff a \leq \frac{1}{||y||}|y| \leq b \iff a||y|| \leq |y| \leq b||y|| \qquad \forall y \in \mathbb{R}^n \setminus \{0\}.$$

The case of $y = 0$ is straightforward. This proves that any norm in a finite-dimensional vector space are equivalent. Note that this proof rests on the fact that we have a basis. ☐

Remark 1.14. $\mathbb{R}^n$ is separable in any norm. The typical countable dense subset of $\mathbb{R}^n$ is $\mathbb{Q}^n$. We will see in infinite-dimensions that all norms are not equivalent.

1.3. **The Spaces of C^r, $C^{r,\gamma}$ of Continuous Functions.**

Definition 1.15 (C^0)**.** Let $\Omega \subseteq \mathbb{R}^n$ be open. Then,

$$C^0(\Omega) := \{f \mid \Omega \to \mathbb{R} \text{ s.t. } f \text{ is continuous on } \Omega\}$$
$$C^0(\overline{\Omega}) := \{f \mid \overline{\Omega} \to \mathbb{R} \text{ s.t. } f \text{ is continuous on } \overline{\Omega}\}.$$

This implies that $f \in C^0(\overline{\Omega})$ is bounded and uniformly continuous.

Definition 1.16 ($||\cdot||_\infty$)**.** The standard norm on $C^0(\Omega)$ is

$$||u||_\infty := \sup_{x\in\Omega} |u(x)| \leftrightarrow \text{ uniform convergence.} \tag{1.17}$$

Proposition 1.18. *(1) $(C^0(\Omega), ||\cdot||_\infty)$ is a Banach space.*
(2) If $\Omega \subseteq \mathbb{R}^n$ is bounded, then $C^0(\overline{\Omega})$ is separable.

We will only give a sketch of the proof.

Proof. (1) The uniform limit of continuous functions is continuous.
(2) Follows from the Weierstrass approximation theorem: polynomials are dense in $C^0(\overline{\Omega})$; then, consider the polynomials with rational coefficients.

□

1.3.1. *Higher-Order Derivatives.* Recall some notation from advanced calculus:

$$\boxed{Du = \nabla u = \text{ gradient of } u = \begin{bmatrix} \partial_1 u \\ \vdots \\ \partial_n u \end{bmatrix}}. \tag{1.19}$$

We consider the **multi-index** $\alpha = (\alpha_1, ..., \alpha_n)$, where $|\alpha| := \alpha_1 + ... + \alpha_n$, and $\forall k \in \mathbb{R}^n$, define $k^\alpha := k_1^{\alpha_1} \cdots h_n^{\alpha_n}$. Then, in this notation,

$$\begin{aligned} D^\alpha u &= \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} u \\ &= \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \qquad \text{(partial derivative).} \end{aligned}$$

Definition 1.20 ($C^r(\Omega)$)**.**

$$C^r(\Omega) := \{f \mid D^\alpha f \in C^0(\Omega) \ \forall \ |\alpha| < r\} \tag{1.21}$$

In words, this means that all partial derivatives less than or equal to r are continuous. Then, we can define the following space:

$$C^\infty(\Omega) := \bigcap_{r=1}^{\infty} C^r(\Omega). \tag{1.22}$$

Definition 1.23 (Support of f)**.** The **support** of f is defined as the smallest closed set such that $f \equiv 0$ on $\mathbb{R}^n \setminus \text{supp}(f)$.

$$\text{supp}(f) := \overline{\{x \mid f(x) \neq 0\}}. \tag{1.24}$$

Definition 1.25 (Compactly Contained)**.** A set $K \subset\subset \Omega$ means that $K \subseteq \Omega$ is compact. We say that K is **compactly contained** in Ω if $K \subset\subset \Omega$, Ω is bounded, and that there exists an $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq \Omega$ for all $x \in K$. This is equivalent to for all $x \in K$,

$$\exists \ \varepsilon > 0 \text{ s.t. } d(x, \partial\Omega) := \inf_{y\in\partial\Omega} |x - y| > \varepsilon. \tag{1.26}$$

Definition 1.27 ($C_c^r(\Omega)$)**.**

$$C_c^r(\Omega) := \{f \mid f \in C^r(\Omega), \text{supp}(f) \subset\subset \Omega\}. \tag{1.28}$$

Definition 1.29 (Norm on $C^r(\overline{\Omega})$)**.** Let Ω be bounded. Then,

$$\boxed{||f||_{C^r} := \sum_{|\alpha|\leq r} \sup_{x\in\Omega} |D^\alpha f(x)|.} \tag{1.30}$$

Proposition 1.31. *Let $\Omega \subseteq \mathbb{R}^n$ be bounded. Then, $C^r(\Omega)$ is a separable Banach space (in fact, all you need for separable is that it is bounded) for all $r < \infty$.*

Remarks 1.32. $C_c^r(\Omega)$ is not complete. $C^\infty(\Omega)$ is not complete. However, subspaces of C^∞ is still complete with some norm.

We also introduce,

Definition 1.33 (Hölder Continuous $C^{0,\gamma}(\Omega)$)**.** $f : \Omega \to \mathbb{R}$ is **Hölder Continuous** with exponent $\gamma \in [0, 1[$ if there exists a C such that

$$\boxed{|f(x) - f(y)| \leq C||x - y||^\gamma.} \tag{1.34}$$

If $\gamma = 1 \Rightarrow f$ is **Lipschitz Continuous**.

Also,

$$\boxed{[f]_{C^{0,\gamma}(\Omega)} := \sup_{x,y\in\Omega} \frac{|f(x) - f(y)|}{||x - y||^\gamma}}$$

is called the **Hölder seminorm**. This is nor a norm, but we can make it a norm:

Definition 1.35 ($|| \cdot ||_{C^{0,\gamma}}$)**.**

$$\boxed{||f||_{C^{0,\gamma}(\Omega)} := ||f||_\infty + [f]_{C^{0,\gamma}(\Omega)}.} \tag{1.36}$$

On the homework, you'll show that

$$(C^{0,\gamma}(\Omega), ||f||_{C^{0,\gamma}(\Omega)})$$

is complete.

Definition 1.37 ($C^{r,\gamma}$)**.**

$$\boxed{C^{r,\gamma}(\Omega) := \{f \mid f \in C^r(\Omega) \text{ and } |D^\alpha f(x) - D^\alpha f(y)| \leq C||x - y||^\gamma \ \forall \ |\alpha| = r\}}. \tag{1.38}$$

The norm of this space is given by,

$$\boxed{||f||_{C^{r,\gamma}} := ||f||_{C^r} + \sup_{|\alpha|=r} [D^\alpha f]_{C^{0,r}}.} \tag{1.39}$$

Remark 1.40. If $f \in C^{0,\gamma}(\Omega)$, Ω bounded, then $f \in C^{0,\alpha}(\Omega)$ for all $0 < \alpha \leq \gamma$

Remark 1.41. **(Rademacher's Theorem)**. If $f \in C^{0,1}$, then f is differentiable a.e.

1.4. Integration Theorems.

Theorem 1.42 (Monotone Convergence Theorem)**.** *If $f_n \uparrow f$ pointwise for almost every x, then*

$$\boxed{\lim_{n\to\infty} \int_\Omega f_n(x)dx = \int_\Omega f(x)dx}. \tag{1.43}$$

Theorem 1.44 (Fatou's Lemma)**.** *Let $\{f_n\}$ be a sequence of measurable functions that are all positive. Then,*

$$\boxed{\int_\Omega \liminf_{n\to\infty} f_n(x) \leq \liminf_{n\to\infty} \int_\Omega f_n(x)dx} \tag{1.45}$$

Theorem 1.46 (Dominated Convergence Theorem)**.** *Assume that $\{f_n\}$ are measurable, $f_n \to f$ pointwise a.e. Then, if $|f_n(x)| \leq g(x)$ for all $n \in \mathbb{N}$ for almost every x, where $g \in L^1(\Omega)$, then*

$$\boxed{\lim_{n\to\infty} \int_\Omega f_n(x)dx = \int_\Omega f(x)dx} \tag{1.47}$$

This is the theorem that you use when you want to differentiate under integrals.

Theorem 1.48. *The space $C_c^0(\mathbb{R}^n)$ is dense in $L^1(\mathbb{R}^n)$.*

Theorem 1.49 (Fubini-Tonelli)**.** *For all* $f : X \times Y \to \mathbb{R}^n$,

$$\boxed{\int_X \int_Y |f(x,y)|dydx = \int_Y \int_X |f(x,y)|dxdy = \int_{X\times Y} |f(x,y)|d(x,y).} \tag{1.50}$$

If, moreover, $f \in L^1(X \times Y)$,

$$\int_X \int_Y f(x,y)dydx = \int_Y \int_X f(x,y)dxdy = \int_{X\times y} f(x,y)d(x,y) \tag{1.51}$$

1.5. **Elementary L^p Spaces.**

Definition 1.52 (L^p)**.** Fix $1 \le p < \infty$, let $\Omega \subseteq \mathbb{R}^n$. Then, we define the L^p space to be

$$\boxed{L^p(\Omega) \coloneqq \{f : \Omega \to \mathbb{R} \mid f \text{ measurable } |f|^p \in L^1(\Omega)\}} \tag{1.53}$$

with the following norm,

$$||f||_{L^p} \coloneqq \left[\int_\Omega |f(x)|^p\right]^{1/p}. \tag{1.54}$$

Definition 1.55 (L^∞)**.** We define L^∞ to be:

$$L^\infty(\Omega) \coloneqq \{f : \Omega \to \mathbb{R} \mid f \text{ measurable }, \exists\, C \text{ s.t. } |f(x)| \le C \text{ a.e. }\}, \tag{1.56}$$

with the following norm,

$$||f||_{L^\infty} = ||f||_\infty = \inf\{c \mid |f(x)| \le c \textit{ a.e.}\} \tag{1.57}$$

This definition implies that $f(x) \le ||f||_\infty$ almost everywhere. Below are some fundamental tools that we'll be using

Theorem 1.58 (Hölder's Inequality)**.** *Let* $1 \le p, p' \le \infty$. *If* $f \in L^p(\Omega)$, $g \in L^{p'}(\Omega)$ *and* $1/p + 1/p' = 1$, *then* $fg \in L^1$ *and*

$$\boxed{\int |fg|dx \le ||f||_p ||g||_{p'}} \tag{1.59}$$

Theorem 1.60 (Minkowski's Inequality)**.** *For all* $p \in [1, \infty]$,

$$\boxed{||f + g||_p \le ||f||_p + ||g||_p.} \tag{1.61}$$

As a consequence of Minkowski's Inequality, L^p is a vector space.

Theorem 1.62 (Riesz-Fischer)**.** L^p *is a Banach space for all* $p \in [1, \infty]$.

Proof. **Case # 1:** $p = \infty$. Let $\{f_n\} \subseteq L^\infty$ be Cauchy. Hence, for all $k \in \mathbb{N}$, there exists an N_k such that for all $n, m \ge N_k$,

$$||f_n - f_m||_\infty < \frac{1}{k}. \tag{1.63}$$

Then, there exists a null set E_k such that $\forall n, m \ge N_k$.

$$|f_n(x) - f_m(x)| \le \frac{1}{k}.$$

for all $x \in \Omega \setminus E_k$, $\{f_n\} \subseteq \mathbb{R}$ is a Cauchy sequence. Since $\mathbb{R}$ is complete, there exists an $f(x) \in \mathbb{R}$ such that

$$f_n(x) \to f(x) \qquad x \in \Omega \setminus E.$$

So, in particular, $\forall m \ge N_k$,

$$|f_n(x) - f(x)| \le \frac{1}{k}\ \forall x \in \Omega \setminus E.$$

We can then take the supremum,

$$\sup_{x\in\Omega\setminus E} |f_m(x) - f(x)| \le \frac{1}{k}.$$

Extend f to be whatever on E:

$$\begin{aligned} &\Rightarrow ||f - f_m||_\infty \leq \frac{1}{k}, \qquad n \geq N_k, \\ &\Rightarrow f_n \to f \text{ in } L^\infty. \end{aligned}$$

□

Also, $f = (f - f_n) + f_n$. We have that $(f - f_n) \in L^\infty$ and $f_n \in L^\infty$. Hence, $f \in L^\infty$ since L^∞ i a vector space. Hence, we have proven that L^p is a Banach space for $p = \infty$.

Case # 2: $1 \leq p < \infty$. Similarly, $\{f_n\} \subseteq L^p$ be Cauchy. Choose a subsequence such that,

$$||f_{n_{k+1}} - f_{n_k}||_{L^p} < \frac{1}{2^k} \qquad \forall k \geq 1.$$

Then,

$$\left|\left|\sum_{k=1}^{N} |f_{n_{k+1}} - f_{n_k}|\right|\right|_{L^p} \leq \sum_{k=1}^{N} \left(\frac{1}{2^k}\right) < 1.$$

Define,

$$v(x) := \lim_{N \to \infty} \sum_{k=1}^{N} |f_{n_{k+1}}(x) - f_{n_k}(x)|.$$

(Possibly infinite, but always positive). By Fatou's Lemma,

$$\int_\Omega |v|^p dx \leq \liminf_{N \to \infty} \int_\Omega \left(\sum_{k=1}^{N} |f_{n_{k+1}}(x) - f_{n_k}(x)|\right)^p dx \leq 1.$$

Hence, $v \in L^p$ which implies that $|v(x)| < \infty$ a.e. and,

$$f_{n_{k+1}}(x) + \sum_{k=1}^{\infty} (f_{n_{k+1}}(x) - f_{n_k}(x)) \qquad (*)$$

converges almost everywhere for x. Observe that the partial sums of the above in $(*)$ are just $f_{n_{k+1}}(x)$ (telescoping series):

$$f(x) := \lim_{k \to \infty} f_{n_k}(x),$$

which we already knew converges for a.e. x and extend this to be whatever on a null set. Claim: $f \in L^p$ and $||f_n - f||_{L^p} \to 0$. By Fatou's Lemma, for k sufficiently large,

$$\int_\Omega |f - f_{n_k}|^p dx \leq \liminf_{j \to \infty} \int_\Omega |f_{n_j} - f_{n_k}|^p \leq \frac{\varepsilon}{2} \qquad \text{(since Cauchy)}$$

Which implies,

$$\begin{aligned} &\Rightarrow f - f_{n_k} \in L^p(\Omega) \\ &\Rightarrow f(x) = \underbrace{(f(x) - f_{n_k}(x))}_{\in L^p} + \underbrace{f_{n_k}}_{\in L^p}(x) \\ &\Rightarrow f \in L^p. \end{aligned}$$

Break at the subsequence, which means that the limiting guy is in L^p. Also, for all $n \geq N$, $n_k \geq N$,

$$\begin{aligned} ||f_n(x) - f(x)||_{L^p} &\leq ||f_n(x) - f_{n_k}(x)||_{L^p} + ||f_{n_k}(x) - f(x)||_{L^p} \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Hence, $f_n \to f$ in L^p.

Corollary 1.64. *Let $\{f_n\} \subseteq L^p$ and let $f \in L^p$ such that $||f_n - f||_{L^p} \to 0$, then there exists a subsequence such that,*

$$f_{n_k}(x) \to f(x) \qquad on \Omega.$$

Proof. Hidden in Reisz-Fischer. □

Theorem 1.65. *$C_c(\mathbb{R}^N)$ is dense in $L^p(\mathbb{R}^N)$ for all $p \in [1, \infty[$.*

Proof. We'll work with the truncation operator. It's a function $T_n : \mathbb{R} \to R$ defined by,

$$T_n r := \begin{cases} r & \text{if } |r| \leq n, \\ \frac{nr}{|r|} & \text{if } |r| \geq n. \end{cases}$$

Claim: for all $f \in L^p(\mathbb{R}^N)$ and for all $\varepsilon > 0$ there exist a $g \in L^\infty(\mathbb{R}^N)$ and a compact set $K \subseteq \mathbb{R}^N$ such that,

$$\text{supp}(g) \subseteq K \text{ and } ||f - g||_{L^p} < \varepsilon.$$

Let $f_n := T_n(f)\chi_{B(0,n)}$. Note that $f_n - f \to 0$ a.e. Then,

$$|f_n - f| \leq 2|f| \in L^p.$$

By the DCT, $||f_n - f||_{L^p} \to 0$. Thus, let $g(x) = f_n(x)$ for n large. So, $g \in L^p(\mathbb{R}^N)$ and is compactly supported. Hence, by inclusions in L^p and Hölder's inequality, we obtain:

$$g \in L^1(\mathbb{R}^N).$$

Thus, for all $\delta > 0$, by the density in L^p, there exists a $g \in C_c^0$ such that

$$||g - g_1||_{L^1} < \delta.$$

WLOG, we may assume that $||g_1||_\infty \leq ||g||_\infty$ (by replacing g_1 for $T_n g_1$ for n large). Since $p \in]1, \infty[$,

$$\begin{aligned} ||g - g_1||_{L^p} &= \left(\int |g - g_1|^p\right)^{1/p} \\ &= \left(\int |g - g_1||g - g_1|^{p-1}\right)^{1/p} \\ &= ||g - g_1||_\infty^{(p-1)/p} ||g - g_1||_{L^1}^{1/p} \\ &= \delta^{1/p} ||g - g_1||_\infty^{1-1/p} \\ &= 2||g||_{L^\infty}^{1-1/p} \delta^{1/p}. \end{aligned}$$

Choosing δ sufficiently small,

$$\leq \varepsilon.$$

By Minkowski, $g \in C_c^0(\mathbb{R}^N)$,

$$||f - g_1||_{L^p} \leq ||f - g||_{L^p} + ||g - g_1||_{L^p} \leq 2\varepsilon,$$

as desired. Hence, $C_c(\mathbb{R}^N)$ is dense in $L^p(\mathbb{R}^N)$. ☐

Theorem 1.66. *The vector space $L^p(\mathbb{R}^N)$ is separable.*

Proof. Define the following,

$$\mathcal{R} := \left\{ \prod_{k=1}^N]a_k, b_k[, \ a_k, b_k \in \mathbb{Q} \text{ rational rectangles.} \right\}.$$

And let,

$$\mathcal{E} := \{\text{finite linear combination of elements of } \chi_{\mathcal{R}}, R \in \mathcal{R}, \text{ with rational coefficients.}\}$$

(You can think of this as a vector space over the rationals $\mathbb{Q}$). **Claim:** $\mathcal{E}$ is dense in Ł^p. Given an $f \in L^p(\mathbb{R}^N)$, $\varepsilon > 0$ we know that there is a $f_1 \in C_c^0(\mathbb{R}^N)$ such that,

$$||f - f_1|| < \varepsilon.$$

Let $\text{supp}(f_1) \subseteq R \subseteq \mathcal{R}$. Now, for all $\delta > 0$, build an $f_2 \in \mathcal{E}$ such that $||f_1 - f_2||_\infty < \delta$. Indeed, re-write,

$$R := \bigcup_{i=1}^N R_i \text{ where } R_i \in \mathcal{R} \text{ and } \forall i, \text{osc}_{R_i} f_1 = \sup_{R_i} f_1 - \inf_{R_i} f_1 < \delta.$$

Hence,

$$f_2 = \sum_{i=1}^N q_i \chi_{R_i} \text{ with } q_i \in \mathbb{Q}, q \approx f_1|_{R_i}.$$

Which implies,

$$||f_1 - f_2||_{L^\infty} \leq \delta.$$

So,

$$\begin{aligned} ||f_1 - f_2||_{L^p} &\leq ||f_1 - f_2||_\infty |R|^{1/p} \text{ (compact support)} \\ &\leq \delta |R|^{1/p} \\ &< \delta \text{ for } \delta \text{ chosen.} \end{aligned}$$

Which implies that,

$$\begin{aligned} ||f - f_2||_{L^p} &\leq ||f - f_1||_{L^p} + ||f - f_2||_{L^p} \\ &\leq \varepsilon + \varepsilon = 2\varepsilon, \end{aligned}$$

as asserted. □

Remark 1.67. These results are more general. In particular, if Ω is separable, then $L^p(\Omega)$ is separable.

1.6. **Convolutions and Mollifers.**

Definition 1.68 (Convolution)**.** Let f and g be functions. Their **convolution** is defined as:

$$(f * g)(x) := \int_{\mathbb{R}^N} f(x-y)g(y)dy = \int_{\mathbb{R}^N} g(x-y)f(y)dy = (g * f)(x).$$

Theorem 1.69 (Young's Inequality)**.** *If $f \in L^1$, $g \in L^p$, then,*

$$\boxed{||f * g||_{L^p} \leq ||f||_{L^1} ||g||_{L^p}}$$

Hence, $f * g \in L^p$ and hence $f * g$ is defined almost everywhere.

Proposition 1.70. *Let f, g be functions. Then,*

$$supp(f * g) \subseteq \overline{sup(f) + sup(g)}$$

Remark 1.71. If f, g are both compactly supported, then $f * g$ is compactly supported. If only one is compactly supported, you cannot say anything.

Definition 1.72 (L^p_{loc})**.** Let f be a function. $f \in L^p_{\text{loc}}$ if $f\chi_K \in L^p$ for all K compact, $K \subseteq \Omega$.

Remark 1.73. By using some sort of a Hölder-estimate, we can show that $f \in L^p_{\text{loc}} \Rightarrow f \in L^1_{\text{loc}}$.

Proposition 1.74. *Let f, g be functions. If $f \in C^0_c(\mathbb{R}^N)$, $g \in L^1_{loc}(\mathbb{R}^N)$. Then, $(f * g)(x)$ is defined for every x and $(f * g) \in C(\mathbb{R}^N)$.*

Proof. We have that for every $x \in \mathbb{R}^N$,

$$\begin{aligned} \left| \int f(x-y)g(y)dy \right| &= \left| \int g(x-y)f(y)dy \right| \\ &\leq ||f||_\infty \int_K |g(x-y)|dy \text{ (by compactly supported)} \\ &\leq ||f||_\infty ||g||^1_L(\tilde{K}) \text{ (since } g \in L^1_{\text{loc}}) \\ &< \infty \end{aligned}$$

Since $\tilde{K}$ is compact. Now suppose that $x_n \to x$ (which means that $|x_n - x| \leq B_1$ for all $n \geq N$). Then, since supp(f) is compact, there exists a compact set such that,

$$|f(x_n - y) - f(x-y)| \leq \varepsilon_n \chi_K(y)$$

(by the uniform continuity and by taking $\varepsilon_n \to 0$). So, now it's obvious,

$$|(f * g)(x_n) - (f * g)(x)| \leq \varepsilon_n \int_K |g(y)|dy \to 0,$$

where the last limit follows from the fact that $g \in L^1_{\text{loc}}$. □

Mollification is approximating a function by a smooth function. We define a **mollifier** by:

$$\rho(x) := \begin{cases} C\exp\left(-\frac{1}{1-|x|^2}\right) & \text{if } |x| \leq 1, \\ 0 & \text{if } |x| > 1 \end{cases}$$

This is continuous and smooth, with C chosen based on the dimensions such that,

$$\int_{\mathbb{R}^N} \rho(x)dx = 1.$$

Note that $\rho \in C_c^\infty(\mathbb{R}^N)$. We define:

$$\rho_h(x) := \frac{\rho(x/h)}{h^N} \qquad u_h(x) := (\rho_h * u)(x) = \frac{1}{h^N}\int_\Omega \rho\left(\frac{x-y}{h}\right)u(y)dy.$$

Proposition 1.75. *Let $u \in C_c^0(\Omega)$. Then, $u_h \in C_c^\infty(\Omega)$ and if $u < \operatorname{dist}(\operatorname{supp}, \partial\Omega)$, then $u_h \to u$ uniformly on Ω as $h \to 0$.*

Proof. Let $h < \operatorname{dist}(\operatorname{supp}, \partial\Omega)$. Observe that by the dominated convergence theorem,

$$\partial_i u_h(x) = \int_\Omega \partial_i\rho\left(\frac{x-y}{h}\right)u(y)dy,$$

where $\partial_i\rho\left(\frac{x-y}{h}\right)$ is smooth with compact support, and hence the integral is finite. This implies that $\rho \in C_c^\infty(\Omega)$ and hence $u_h \in C_c^\infty(\Omega)$. Observe,

$$\begin{aligned} &\frac{1}{h^N}\int_{\mathbb{R}^N} \rho\left(\frac{y}{h}\right)dy = 1 \\ \Rightarrow &\frac{1}{h^N}\int_{B(0,h)} \rho\left(\frac{y}{h}\right)dy = 1 \\ \Rightarrow &\frac{1}{h^N}\int_{B(0,h)} \rho\left(\frac{x-y}{h}\right)dy = 1\ \forall x. \end{aligned}$$

Hence,

$$u_h(x) - u(x) = \frac{1}{h^N}\int_\Omega \rho\left(\frac{x-y}{h}\right)[u(y)-u(x)]dy \text{ (multiplying by 1 in a smart way)}$$

Hence,

$$\begin{aligned} |u_h(x) - u(x)| &= \left|\frac{1}{h^N}\int_{|x-y|\leq h} \rho\left(\frac{x-y}{h}\right)|u(y)-u(x)|dy\right| \\ &\leq \sup_{|x-y|\leq h}|u(y)-u(x)|\frac{1}{h^N}\int_{|x-y|\leq h}\rho\left(\frac{x-y}{h}\right)dy \\ &= \sup_{|x-y|\leq h}|u(y)-u(x)|. \end{aligned}$$

Invoking the uniform continuity of $u \in C_c^0$, we can bound $\sup_{|x-y|\leq h}|u(y)-u(x)| \leq u(h)$. Hence,

$$\sup_{x\in\Omega}|u_h(x)-u(x)| \to 0 \text{ as } h \to 0.$$

□

Theorem 1.76. *Assume that $f \in L^p(\mathbb{R}^N)$ with $1 \leq p < \infty$. Then,*

$$(\rho_h * f) \to f \text{ as } h \to 0 \text{ in } L^p.$$

Proof. Fix an $\varepsilon > 0$. We know that there exists an $f_1 \in C_c^0(\mathbb{R}^N)$, $||f - f_1||_L^P < \varepsilon$. Also, since $f_1 \in C_c^0(\mathbb{R}^N)$, we know that

$$(\rho_h * f_1) \to f_1 \text{ uniformly.}$$

We also have that

$$\begin{aligned} \operatorname{supp}(\rho_h * f_1) &\subseteq \overline{B(0,h) + \operatorname{supp}(f_1)} \\ &\subseteq \underbrace{\overline{B(0,1) + \operatorname{supp}(f_1)}}_{\text{compact}} \end{aligned}$$

Hence,

$$||(\rho_h * f_1) - f_1||_{L^p} \to 0.$$

Thus,

$$(\rho_h * f) - f = (\rho_h * (f - f_1)) + [(\rho_h * f_1) - f_1] + f_1 - f.$$

By the triangle inequality and Young's inequality,

$$||(\rho_h * f) - f||_{L^p} \leq 2\underbrace{||f - f_1||_{L^p}}_{:=(1)} + \underbrace{||(\rho_h * f_1) - f_1||_{L^p}}_{:=(2)}$$

Where (1) is small by density and (2) is small because we just did it. Hence,

$$\limsup_{h\to 0} ||(\rho_h * f) - f||_{L^p} \leq \varepsilon$$
$$\Rightarrow \lim_{h\to 0} ||(\rho_h * f) - f||_{L^p} = 0$$

□

Corollary 1.77. *Let $\Omega \subseteq \mathbb{R}^N$ (possibly all of $\mathbb{R}^N$) with $1 \leq p < \infty$. Then, $(\rho_h * f) \to f$ in $L^P(\mathbb{R}^N)$ as $h \to 0$.*

Proof. Given an $f \in L^p(\Omega)$, we extend to $\overline{f} \in L^p(\mathbb{R}^n)$ by:

$$\overline{f}(x) := \begin{cases} f(x) & x \in \Omega, \\ 0 & x \in \mathbb{R}^n \setminus \Omega. \end{cases}$$

Let $\{K_N\} \subseteq \mathbb{R}^N$ be a compact set such that $\bigcup_{n=1}^\infty K_N = \Omega$. (Remark: if $\Omega \subseteq \mathbb{R}^N$ is bounded, then $\text{dist}(K_n \cap \Omega^c) > 2/n$). Now let $g_n := \overline{f}\chi_{K_n}$. This is compactly supported. Also, $f_n := \rho_{1/n} * g_n$ is compactly supported. Hence,

$$\text{supp}(f_n) \subseteq \overline{B(0, 1/n) + K_n} \subseteq \Omega,$$

and $f_n \in C_c^\infty(\Omega)$ for all $n \in \mathbb{N}$. Also,

$$\begin{aligned} ||f_n - f||_{L^p(\Omega)} &= ||f_n - \overline{f}||_{L^p(\mathbb{R}^N)} \\ &\leq ||(\rho_{1/n} * g_n) - (\rho_{1/n} * \overline{f})||_{L^p(\mathbb{R}^N)} + ||(\rho_{1/n} * \overline{f}) - \overline{f}||_{L^p(\mathbb{R}^N)} \text{ (Minkowski and triangle inequality)} \end{aligned}$$

By the linearity of convolution,

$$\leq ||\rho_{1/n} * (g_n - \overline{f})||_{L^p(\mathbb{R}^N)} + ||(\rho_{1/n} * \overline{f}) - \overline{f}||_{L^p(\mathbb{R}^N)}.$$

Apply Young's Inequality to the first term,

$$\leq ||g_n - \overline{f}||_{L^p(\mathbb{R}^N)} + ||(\rho_{1/n} * \overline{f}) - \overline{f}||_{L^p(\mathbb{R}^N)}.$$

Note that $g_n := \overline{f}\chi_{k_n}$, and hence by the dominated convergence theorem,

$$||g_n - \overline{f}||_{L^p(\mathbb{R}^N)} \to 0,$$

and by the last theorem,

$$||(\rho_{1/n} * \overline{f}) - \overline{f}||_{L^p(\mathbb{R}^N)} \to 0.$$

Combining everything together, we get

$$||f_n - f||_{L^p(\Omega)} \to 0.$$

Which proves that smooth functions with compact support are dense in L^p. □

1.7. **Hilbert Spaces.**

Definition 1.78 (Inner Product)**.** An **inner product** over a vector space X is a map $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{R}$ such that:

(1) $\langle \lambda x + \mu y, z\rangle = \lambda\langle x, z\rangle + \mu\langle y, z\rangle$ for all $\lambda, \mu \in \mathbb{R}$.
(2) $\langle y, x\rangle = \langle x, y\rangle$ for all $x, y \in X$.
(3) $\langle x, x\rangle \geq 0$ for all $x \in X$ and $\langle x, x\rangle = 0 \iff x = 0$.

An inner product generates a norm:

$$||x|| := \sqrt{\langle x, x, \rangle}.$$

And we have Cauchy-Schwarz:

$$\boxed{|\langle x, y\rangle| \leq ||x|| ||y||}$$

Proof of Cauchy-Schwarz:

Proof. Let $z := x - \frac{\langle x,y\rangle}{\langle y,y\rangle}y$. We have that,

$$\begin{aligned}
0 \leq ||z||^2 &= \left\langle x - \frac{\langle x, y\rangle}{\langle x, x\rangle}y, x - \frac{\langle x, y\rangle}{\langle y, y\rangle} \right\rangle \\
&= \langle x, x\rangle - \frac{\langle x, y\rangle^2}{\langle y, y\rangle} - \frac{\langle x, y\rangle^2}{\langle y, y\rangle} + \frac{\langle x, y\rangle^2 \langle y, y\rangle}{\langle y, y\rangle^2} \\
&= ||x||^2 - \frac{\langle x, y\rangle^2}{||y||^2}
\end{aligned}$$

And hence,

$$\langle x, y\rangle^2 \leq ||x||^2 ||y||^2 \iff \boxed{.||\langle x, y\rangle|| \leq ||x|| ||y||}$$

□

Definition 1.79 (Hilbert Space)**.** A **Hilbert Space** $\mathcal{H}$ is a complete inner product space (with respect to the norm induced by the inner product). This satisfies the parallelogram law,

$$\boxed{||u + v||^2 + ||u - v||^2 = 2||u||^2 + 2||v||^2}$$

Examples of Hilbert Spaces you've encountered:

(1) ℓ^2 with the inner product $\langle x, y\rangle = \sum_{i=1}^{\infty} x_i y_i$.
(2) L^2 with inner product $\langle f, g\rangle := \int_\Omega f(x)g(x)dx$.

1.7.1. *Orthogonal Projections.* If $M \subseteq H$ is a subset, then the **orthogonal complement** of H is defined as:

$$\boxed{M^\perp := \{u \in H \mid \langle u, v\rangle = 0\} \ \forall v \in M}$$

Proposition 1.80. *If M is a closed linear subspace of H, then every $x \in H$ has the following unique decomposition:*

$$x = u + v, \ u \in M, \ v \in M^\perp$$

Proof. Fix an $x \in H$. If $x \in M$, then $u = x$ and $v = 0$ (straightforward). Now assume that $x \notin H$; then, we claim that there exists a $u \in M$ such that

$$||x - u|| = \inf_{y \in M} ||x - y|| = \delta > 0.$$

By the definition of infimum, there exists a sequence $\{u_n\} \subseteq M$ such that

$$||x - u_n|| \leq \delta^2 + \frac{1}{n} \qquad (*)$$

We'll show that this sequence $\{u_n\}$ is Cauchy. To that end, let

$$\begin{aligned}
v &:= u_m - x, \\
w &:= u_n - x
\end{aligned}$$

Using the Parallelogram law,

$$||u_m - u_n||^2 + ||u_m + u_n + 2x||^2 = 2||u_m - x||^2 + 2||u_n - x||^2.$$

Note. $||u_m + u_n + 2x||^2 = 4||1/2(u_m + u_n) - x||^2$ and therefore,

$$||u_m - u_n||^2 = 2||u_m - x||^2 + 2||u_n - x||^2 - 4||(1/2)(u_m + u_n) - x||^2.$$

However, M is a linear subspace, and therefore

$$1/2(u_n + u_m) \in M \Rightarrow ||x - (1/2)(u_m + u_n)|| \geq \delta.$$

By $(*)$ and the above,

$$\begin{aligned}||u_m - u_n||^2 &\le 2\left(\delta^2 + \frac{1}{m}\right) + 2\left(\delta^2 + \frac{1}{n}\right) - 4\delta^2 \\ &= \frac{1}{m} - \frac{1}{n}.\end{aligned}$$

Hence, $\{u_n\} \subseteq M$ is Cauchy and since H is complete, M is closed. Hence, $u_n \to u \in M$. Using the triangle inequality,

$$\begin{aligned}||x-u|| &\le ||x-u_n|| + ||u_n - u|| \\ &\le \left(\delta^2 + \frac{1}{n}\right)^{1/2} + ||u_n - u||\end{aligned}$$

Letting $n \to \infty$, we get that $||x-u|| \le \delta$. Hence,

$$\delta \le ||x-u|| \le \Delta \iff ||x-u|| = \delta,$$

which proves that the infimum was attained and hence proves the claim. Next claim – **Claim:** if $v = x - u$, then $v \in M^\perp$. To show this, consider $y \in M$. Then,

$$||x-(u-ty)||^2 = ||v+ty|| = ||v||^2 + 2t\langle v, y\rangle + t^2||y||^2.$$

Note that the map $t \mapsto ||v+ty||$ is minimized when $t = 0$. So,

$$\begin{aligned}\frac{\partial}{\partial t}\left[||v||^2 + 2t\langle v,y\rangle + t^2||y||^2\right]_{t=0} &= [2\langle v,y\rangle + 2t||y||^2]_{t=0} \\ &= 2\langle v,y\rangle = 0 \ \forall y \in M, \\ &\Rightarrow v \in M^\perp\end{aligned}$$

Hence, $x = u + v$ where $u \in M$ and $v \in M^\perp$. Now, to show uniqueness, assume, not:

$$x = u_1 + v_1 = u_2 + v_2.$$

Then,

$$u_1 - u_2 = v_2 - v_1 \iff ||v_1 - v_2||^2 = \langle v_1 - v_2, v_1 - v_2\rangle = \langle v_1 - v_2, u_2 - u_2\rangle = 0$$

where $v_1 - v_2 \in M^\perp$ and $u_2 - u_1 \in M$. Hence, $v_1 \equiv v_2$ and $u_2 \equiv u_2$. □

We can now define orthogonal projections onto M: $P_M(x) = u$. Observe that the following hold:

- $P_M^2 = P_M \circ P_M = P_M$.
- By orthogonality, one has

$$||x||^2 = \underbrace{||u||^2}_{=P_M(x)} + ||x-u||^2$$

  which implies that

$$||P_M(x)|| \le ||x||,$$

  that is, projecting can only shorten vectors.

1.7.2. *Orthonormal Bases.*

Definition 1.81 (Orthonormal Set)**.** Let $\{e_j\} \subseteq H$. Then, $\{e_j\}$ is orthonormal $\iff$ $\langle e_i, e_j\rangle = \delta_{ij}$.

Definition 1.82 (Orthonormal Basis)**.** We say that $\{e_j\} \subseteq H$ is an orthonormal basis for H if $\{e_j\}$ is orthonormal and,

$$x = \sum_{j=1}^{\infty} \langle x, e_j\rangle e_j \qquad \forall x \in H.$$

This happens $\iff$:

$$\left|\left| x - \sum_{j=1}^{N}\langle x, e_j\rangle e_j \right|\right| \to 0 \text{ as } N \to \infty$$

Proposition 1.83 (Parseval's Equality)**.** *Let $\{e_j\} \subseteq H$ be orthonormal. Then, its a basis $\iff$*

$$||x||^2 = \sum_{j=1}^{\infty} \langle x_j, e_j \rangle^2 \qquad \forall x \in H. \tag{1.84}$$

Proof. "$\Rightarrow$": Suppose that we have a basis. Then,

$$\left|\left|\sum_{j=1}^{n} \langle x, e_j \rangle e_j\right|\right| = \sum_{j=1}^{n} \langle x_i, e_j \rangle^2.$$

Taking the limit, we obtain:

$$||x||^2 = \sum_{j=1}^{\infty} \langle x, e_j \rangle^2.$$

"$\Leftarrow$": Suppose that Parseval's holds for all $x \in H$. Set:

$$Y := \{x \mid x = \sum_{j=1}^{\infty} \langle x, e_j \rangle e_j\}.$$

Claim: $Y = H$. We'll show that Y is closed and dense in H. This will be sufficient to prove the claim.

(1) Y is closed in H: Let $\{Y_n\} \subseteq Y$ be a Cauchy sequence. Note that by Parseval's,

$$Y \simeq \ell^2 \text{ (square-summable sequence)}.$$

We have that ℓ^2 is complete, which means that Cauchy sequences converge to points in the set ℓ^2:

$$\Rightarrow \{y_n\} \to y^*,\ y^* \in Y,$$
$$\Rightarrow Y \text{ is closed.}$$

(2) Y is dense in H: Suppose that Y is not dense. Then, $H \setminus Y$ has a non-zero element. Hence, $Y^\perp$ has a non-zero element. But, if $x \in Y^\perp$, $\langle x, e_j \rangle = 0 \forall j$, and hence by Parseval, $x \equiv 0$ which is a contradiction.

Hence, $Y = H$. $\square$

Proposition 1.85. *H is separable $\iff$ H has a countable basis.*

Proof. If H has a countable basis , then,

$$\text{span}_{\mathbb{Q}}\{\text{basis}\} \text{ is a countable dense set.}$$

If H is separable, let $\{x_n\}$ be the countable dense subset. Use Gram Schmidt: for $\{x_n\}$, remove a linear combination of the proceeding elements as follows:

$$e_i = \frac{x_1}{||x_1||}$$
$$y_n = x_n - \sum_{i=1}^{n-1} \langle x_n, e_i \rangle e_i,$$

and let $e_n := \frac{y_n}{||y_n||}$. Then, $\text{span}\{e_n\} = \text{span}\{x_n\}$ and e_n are orthonormal and dense (the linear combination is dense). $\square$

All of this allows us to see the following proposition:

Proposition 1.86. *The nit ball in any infinite-dimensional Hilbert Space is not compact.*

Proof. Let $\{w_n\}_{n=1}^{\infty}$ be orthonormal with $||w_n|| = 1$ for all $n \in \mathbb{N}$. Then,

$$\begin{aligned} ||w_n - w_m||^2 &= \langle w_n - w_m, w_n - w_m \rangle, \\ &= ||w_n||^2 + ||w_m||^2 \\ &= 2 \text{ (if } m \neq n \text{).} \end{aligned}$$

Hence, no subsequence can converge, and hence the unit ball is not compact. $\square$

2. Hahn-Banach Theorem and Convex Conjugate Functions

Let E be a vector space over $\mathbb{R}$. Recall that $f : E \to R$ is a functional. **Goal:** given $G \subseteq E$ a linear space and a linear functional g on G, when can we extend this to a functional defined on all of E? Before stating the main theorem, we will first provide two definitions and a lemma.

Definition 2.1 (Partially-Ordered)**.** Let P be a set. We say that a **partial order**, denoted $\leq$, on P satisfies the following:

(1) $a \leq a$ for all $a \in P$.
(2) $a \leq b$ and $b \leq c$ implies that $a \leq c$.
(3) If $a \leq b$ and $b \leq a$, then $a = b$.

Definition 2.2 (Totally Ordered Set/Upper Bound/Maximal Element)**.** (1) We say that $Q \subseteq P$ is **totally-ordered** if for all $a, b \in Q$, either $a \leq b$ or $b \leq a$ (or both).
(2) We say that $c \in P$ is an upper bound of Q if $\forall a \in Q$, $a \leq c$.
(3) We say that $m \in P$ is a **maximal element** if $m \leq x \Rightarrow m = x$.
(4) We say that P is **inductive** if every totally-ordered subset $Q \subseteq P$ has an upper-bound.

Lemma 2.3. *Every non-empty partially-ordered set that is inductive has a maximal element.*

Theorem 2.4 (Hahn-Banach)**.** *Let E be a vector space on $\mathbb{R}$. Let $p : E \to \mathbb{R}$ be a function satisfying:*

(1) $p(\lambda x) = \lambda p(x) \forall \lambda \geq 0$
(2) $p(x + y) \leq p(x) + p(y)$.

Let $G \subseteq E$ be a linear subspace and let $g : G \to \mathbb{R}$ be a linear functional such that $g(x) \leq p(x)$ for all $x \in G$. Then, there exists an $f : E \to \mathbb{R}$ linear functional such that

(1) $f(x) = g(x)$ for all $x \in G$ ("f extends g") and
(2) $f(x) < p(x)$ for all $x \in E$.

Proof. To do... □

2.1. Application of Hahn-Banach and Duals.

Let E be a normed vector space.

Definition 2.5 (Dual Space)**.** We denote $E^* = E =$ dual space of E .

$$E^* := \{f : E \to \mathbb{R}, \text{ctslinear functionals.}\},$$

with the following norm,

$$||f||_{E^*} = ||f|| = \sup_{||x|| \leq 1} |f(x)| = \sup_{x \in E} = \frac{|f(x)|}{||x||}$$

We will write $f(x) = \langle f, x \rangle$, where $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{E^*, E}$ (scalar product, not to be confused with inner product).

3. Compact Operators, Spectral Decomposition, Self-Adjoint Compact Operators

3.1. Definitions, Elementary Properties.

Definition 3.1 (Compact Operator)**.** A linear operator $T : E \to F$ is **compact** if $T(B_E)$ is pre-compact (compact closure).

This is true if $\overline{T(W)}$ is compact for any $W \subseteq E$ bounded. This is true if and only if the following hold:

- $T(B_E)$ can be covered by finitely many balls in the strong topology of F.
- $\{x_n\} \subseteq E$ bounded and T compact means that $\{Tx_n\} \subseteq$ compact subset of F. Hence, there exists a strongly convergent subsequence in F.

Example 3.2. An example of a non-compact operator is the identity operator:

$$T : H \to H,$$
$$Tx = x$$

We saw that $T(B_H)$ is not compact if H is an infinite-dimensional vector space. Hence, the identity operator is a non-compact operator.

Lemma 3.3. *Any compact operator is bounded.*

Proof. Since B_E is bounded and T is compact, $\overline{T(B_E)} \subseteq B_F(0, R)$ for some $R > 0$. Hence, it's compact in the strong topology. Hence,

$$\begin{aligned} &\Rightarrow \sup_{||x|| \le 1} ||Tx||_F \le R \\ &\Rightarrow ||T|| \le R \text{ (operator norm)} \\ &\Rightarrow T \text{ is bounded.} \end{aligned}$$

So, WLOG, we could define a compact operator as belonging to $\mathcal{L}(E, F)$ (bounded linear operators between E and F). We'll call $\mathcal{K}(E, F) \subseteq \mathcal{L}(E, F)$ the set of **compact operators** from E to F. ☐

Theorem 3.4. *$\mathcal{K}(E, F)$ forms a Banach space with the $\mathcal{L}_{(E,F)}$ norm.*

$$||T||_{\mathcal{L}(E,F)} = \sup_{x \in E, ||x|| \le 1} ||Tx||_F$$

Proof. Clearly, $\mathcal{K}(E, F)$ is closed under linear combinations. We claim now that $\mathcal{K}(E, F)$ is closed under limits. This will require some work. Let $\{T_n\} \subseteq \mathcal{K}(E, F)$ be a convergence sequence such that:

$$||T_n - T||_{\mathcal{L}(\mathcal{E}, \mathcal{F})} \to 0.$$

Note that $\mathcal{L}(E, F)$ is a Banach space, and so $T \in \mathcal{L}(E, F)$. We need to now check the compactness of T. Let $\{x_n\} \subseteq E$ be a bounded sequence. We need to extract from this a convergent subsequence. We will construct this using diagonalization.

T_1 is compact. Hence, $\{T_1(x_{n_{1_j}})\}_n$ has a convergent subsequence in F, since T_1 is compact. Call it $T_1(x_{1_j})$. Similarly, $T_2(x_{n_{2_j}})$ has a convergent sequence. (to do: finish the proof) ☐

Definition 3.5 (Finite-Rank)**.** $T \in \mathcal{L}(E, F)$ has **finite rank** if $R(T)$ has finite dimensions.

Lemma 3.6. *If $T \in \mathcal{L}(E, F)$ and is finite-rank, then T is a compact operator.*

Proof. Let $\{x_n\} \subseteq E$ be bounded. Then,

$$||Tx_n|| \le ||T|| \underbrace{||x_n||}_{:=C} < \infty$$

which implies that $\{Tx_n\}$ is bounded in F. But, $R(T)$ is finite-dimensional. However, finite-dimensional Banach spaces are isomorphic to $\mathbb{R}^n$. By the Bolzano-Weierstrass theorem plus the fact that all norms on a finite-dimensional space are equivalent, we have that $\{Tx_n\}$ has a strongly convergent subsequence. Hence, T is compact. ☐

Corollary 3.7. *If $\{T_n\} \subseteq \mathcal{L}(E, F)$ is finite-rank, $T_n \to T$ in $\mathcal{L}(E, F)$. Then, T is compact.*

Example 3.8. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain. The **kernel**, denoted by $K(x, y) \in L^2(\Omega \times \Omega)$, is defined as:

$$T_u(x) := \int_\Omega K(x, y) u(y) dy,$$

where $T : L^2(\Omega) \to L^2(\Omega)$ and $u \in L^2(\Omega)$.

Claim: the T_u operator is a compact operator. To see why, recall that L^2 is a Hilbert space. Hence, it has an orthonormal basis; let $\{\phi_j\}$ be an orthonormal basis of L^2. Then, we claim that $\{\phi_i(x)\phi_j(y)\}$ forms an orthonormal basis for $L^2(\Omega \times \Omega)$. To see why, we may apply Fubini's theorem:

$$\iint (\phi_i(x)\phi_j(y))\phi_m(x)\phi_n(y) dx dy = \int \phi_i(x)\phi_m(x) dx \int \phi_j(y)\phi_n(y) dy$$

where $\int \phi_i(x)\phi_m(x)dx = 1$ if and only if $i = m$ and 0 otherwise, and similarly for $\int \phi_j(y)\phi_n(y)dy$.

Now, $K(x, y) \in L^2(\Omega \times \Omega)$ implies that $K^2(x, y) \in L^1(\Omega \times \Omega)$. Hence, we can re-arrange the integrals as we'd like, and we obtain that $K(x, \cdot) \in L^2(\Omega)$. We can write out $K(x, y)$ in the orthonormal basis as:

$$K(x, y) = \sum k_{ij} \phi_i(x) \phi_j(y)$$

Let $K_n(x, y)$ be given by:

$$K_n(x, y) := \sum_{i,j=1}^{n} k_{ij} \phi_j(x) \phi_j(y),$$

which gives us a new set of operators:

$$T_n(u) := \int_\Omega K_n(x,y)u(y)dy.$$

Since $u \in L^2(\Omega)$, we can write u out in terms of the basis of L^2:

$$u = \sum_{\ell=1}^{\infty} c_\ell \phi_\ell$$

By orthogonality:

$$\begin{aligned} T_n(u) &= \int_\Omega \sum_{i,j=1}^{n} K_{ij}\phi_i(x)\phi_j(y)\left(\sum_{\ell=1}^{\infty} c_\ell \phi_\ell(y)\right) dy \\ &= \sum_{i,j=1}^{n} K_{ij} c_j \phi_i(x) \end{aligned}$$

which implies that T_n has rank n. By the Lemma, this implies that T_n is compact for all n. We now claim that $||T_n - T|| \to 0$ as $n \to \infty$. We have:

$$||T - T_n||^2 = \sup_{||u|| \leq 1} \left|\left| \int_\Omega \int_\Omega [K(x,y) - K_n(x,y)]u(y)dydx \right|\right|^2$$