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4) Use Hölder's Inequality to obtain the following L^p interpolation inequality: for $q<p<r$,

$$\|u\|_p \leq \|u\|_q^{\frac{q(r-p)}{p(r-q)}} \|u\|_r^{\frac{r(p-q)}{p(r-q)}}$$

In particular, this implies that if $u \in L^q \cap L^r$, then $u \in L^p \ \forall p \in]q,r[$.

For notational simplicity, set $\lambda := \frac{q(r-p)}{p(r-q)}$ and then observe that

$$(1-\lambda) = \frac{p(r-q)}{p(r-q)} - \frac{q(r-p)}{p(r-q)} = \frac{pr-pq-qr+qp}{p(r-q)} = \frac{r(p-q)}{p(r-q)}$$

(i.e. the other exponent). Moreover,

$$\lambda + (1-\lambda) = 1$$

$$\Leftrightarrow \quad 1 = \frac{q(r-p)}{p(r-q)} + \frac{r(p-q)}{p(r-q)}$$

$$\Leftrightarrow \quad p = \frac{q(r-p)}{(r-q)} + \frac{r(p-q)}{(r-q)}$$

$$= p\lambda + p(1-\lambda) \qquad (*)$$

We have the following pair of conjugate exponents: $q/\lambda p$ and $r/(1-\lambda)p$. Hence, we can apply Hölder's Inequality:

$$\int |f|^p \underset{(*)}{\leq} \int |f|^{p\lambda} |f|^{p(1-\lambda)} \leq \left(\int |f|^{\cancel{p\lambda} \frac{q}{\cancel{\lambda p}}} \right)^{\frac{\lambda p}{q}} \left(\int |f|^{\cancel{p(1-\lambda)} \frac{r}{\cancel{(1-\lambda)p}}} \right)^{\frac{p(1-\lambda)}{r}}$$

$$= \left[\left(\int |f|^q \right)^{1/q} \right]^{\lambda p} \left[\left(\int |f|^r \right)^{1/r} \right]^{p(1-\lambda)}$$

Taking $1/p$ of both sides, we get

$$\boxed{\|f\|_p \leq \|f\|_q^{\lambda} \|f\|_r^{(1-\lambda)} = \|f\|_q^{\frac{q(r-p)}{p(r-q)}} \|f\|_r^{\frac{r(p-q)}{p(r-q)}}}$$

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