Chapter 7: L^p Spaces: Completeness and Approximation

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Abstract

This document contains a summary of all the key definitions, results, and theorems from class. There are probably typos, and so I would be grateful if you brought those to my attention :-).

Syllabus: L^p space, duality, weak convergence, Young, Holder, and Minkowski inequalities, point-set topology, topological space, dense sets, completeness, compactness, connectedness, path-connectedness, separability, Tychnoff theorem, Stone-Weierstrass Theorem, Arzela-Ascoli, Baire category theorem, open mapping theorem, closed graph theorem, uniform boudnedness principle, Hahn Banch theorem.

7.1. Normed Linear Spaces

Definition 1 (ℓ^p space)**.** Let $(a_1, a_n, ...)$ be a sequence. Then, the ℓ^p-space is:

$$\ell^p := \left\{ (a_1, a_2, ...) \mid \sum_{n=1}^{\infty} |a_n|^p < +\infty \right\} \tag{7.1}$$

Theorem 1 (Riesz-Fisher)**.** $L^p(X)$ is complete.

Definition 2 (L^p space)**.** Let E be a measurable set and let $1 \leq p < \infty$. Then, $L^p(E)$ is the collection of measurable functions f for which $|f|^p$ is Lebesgue integrable over E.

Definition 3 (Equivalent Functions)**.** Let $\mathcal{F}$ be the collection of all measurable extended real-valued functions on E that are finite a.e. on E. Define two functions f and g to be equivalent, and write $f \sim g$ if $g(x) = f(x)$ a.e. on E.

Definition 4 (Essentially Bounded)**.** We call a function $f \in \mathcal{F}$ to be **essentially bounded** if there exists some $M \geq 0$, called the **essential upper bound** for f, for which

$$|f(x)| \leq M$$

for almost every $x \in E$. $L^\infty(E)$ is the collection of equivalence classes $[f]$ for which f is essentially bounded.

Definition 5 (Norm)**.** Let X be a linear space. A real-valued functional $||\cdot||$ on X is called a **norm** provided that for each f and g in X and each real number α,

(i) (The Triangle Inequality).

$$||f + g|| \leq ||f|| + ||g||$$

(ii) (Positive Homogeneity).

$$||\alpha f|| = |\alpha|||f||$$

(iii) (Non-Negativity).

$$||f|| \geq 0 \text{ and } ||f|| = 0 \text{ if and only if } f = 0$$

Definition 6 (Normed Linear Space)**.** X is said to be a **normed linear space** if X is equipped with a norm.

Definition 7 (Essential Supremum)**.** Let $f \in L^\infty(E)$. $||f||_\infty$ is called the **essential supremum** and is defined as:

$$||f||_\infty := \{M \mid M \text{ is an essential upper bound for } f\}$$

Theorem: $||\cdot||_\infty$ is a norm on $L^\infty(E)$.

7.2. The Inequalities of Young, Holder, and Minkowski

Definition 8 (p-norm)**.** Let E be a measurable set, $1 < p < \infty$, and let $f \in L^p(E)$. Then, define the **p-norm** to be:

$$||f||_p := \left[\int_E |f|^p\right]^{\frac{1}{p}} \tag{7.2}$$

Definition 9 (Conjugate)**.** The **conjugate** of a number $p \in]1, \infty[$ is the number $q = p/(p-1)$, which is the unique number $q \in]1, \infty[$ for which

$$\frac{1}{p} + \frac{1}{q} = 1 \tag{7.3}$$

The conjugate of 1 is defined to be ∞ and the conjugate of ∞ is defined to be 1.

Definition 10 (Young's Inequality)**.** For $1 < p < \infty$, q the conjugate of p, and any two positive numbers a and b, we have:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \tag{7.4}$$

Theorem 2 (Hölder's Inequality)**.** Let $E \subseteq \mathbb{R}$ be measurable, $1 \leq p < \infty$, and q the conjugate of p. If f belongs to $L^p(E)$, and g belongs to $L^q(E)$, then their product $f \cdot g$ is integrable over E and:

$$\int_E |f \cdot g| \leq ||f||_p \cdot ||g||_q. \tag{7.5}$$

Moreover, if $f \neq 0$, then the function defined as:

$$f^* := ||f||_p^{1-p} \cdot \operatorname{sgn}(f) \cdot |f|^{p-1} \tag{7.6}$$

belongs to $L^q(E)$,

$$\int_E f \cdot f^* = ||f||_p \text{ and } ||f^*||_q = 1$$

We call f^* defined as above to be called the **conjugate function** of f.

Theorem 3 (Minkowski's Inequality)**.** Let E be a measurable set and $1 \leq p \leq \infty$. If the functions f and g belong to $L^p(E)$, then so does their sum $f + g$. Moreover,

$$||f+g||_p \leq ||f||_p + ||g||_p \tag{7.7}$$

Theorem 4 (Cauchy-Schwarz Inequality)**.** Let E be a measurable set and let f and g be measurable functions over E for which f^2 and g^2 are integrable over E. Then, $f \cdot g$ is integrable over E and

$$\int_E |f \cdot g| \leq \sqrt{\int_E f^2} \cdot \sqrt{\int_E g^2} \tag{7.8}$$

Corollary 1. Let E be a measurable set and $1 < p < \infty$. Suppose $\mathcal{F}$ is a family of functions in $L^p(E)$ that is bounded in $L^p(E)$ in the sense that there is a constant M for which

$$||f||_p \leq M \text{ for all } f \in \mathcal{F}$$

Then, the family $\mathcal{F}$ is uniformly integrable over E.

Corollary 2. Let E be a measurable set of finite measure and $1 \leq p_1 < p_2 \leq \infty$. Then, $L^{p_2}(E) \subseteq L^{p_1}(E)$. Furthermore,

$$||f||_{p_1} \leq c||f||_{p_2}$$

for all f in $L^{p_2}(E)$, where $c = [m(E)]^{\frac{p_2 - p_1}{q_1 p_2}}$ if $p_2 < \infty$ and $c = [m(E)]^{\frac{1}{p_1}}$ if $p_2 = \infty$.

7.3. L^p is complete: the Reisz-Fischer Theorem

Definition 11 (Converge)**.** A sequence $\{f_n\}$ in a linear space X normed by $||\cdot||$ is said to **converge to f in X** provided:

$$\lim_{n\to\infty} ||f - f_n|| = 0$$

Definition 12 (Cauchy)**.** A sequence $\{f_n\}$ in a linear space X that is normed by $||\cdot||$ is said to be **Cauchy** in X provided for each $\varepsilon > 0$, there exists a $N \in \mathbb{N}$ such that

$$||f_n - f_m|| < \varepsilon \;\forall\; m, n \geq N \tag{7.9}$$

Definition 13 (Complete)**.** A normed linear space X is called **complete** if every Cauchy sequence in X converges to a function in X. A complete normed linear space is called a **Banach space**.

Proposition 1. Let X be a normed linear space. Then, every convergent sequence in X is Cauchy. Moreover, a Cauchy sequence in X converges if it has a convergent subsequence.

Definition 14. Let X be a linear space normed by $||\cdot||$. A sequence $\{f_n\}$ in X is said to be **rapidly Cauchy** if there is a convergent series of positive numbers $\sum_{k=1}^{\infty} \varepsilon_k$ for which

$$||f_{k+1} - f_k|| \leq \varepsilon_k^2 \text{ for all } k$$

Proposition 2. Let X be a normed linear space. Then, every rapidly Cauchy sequence in X is Cauchy. Furthermore, every Cauchy sequence has a rapidly Cauchy subsequence.

Proposition 3. Let E be a measurable set and $1 \leq p \leq \infty$. Then, every rapidly Cauchy sequence in $L^p(E)$ converges with respect to the $L^p(E)$ norm and pointwise a.e. on E to a function in $L^p(E)$.

Theorem 5 (Riesz-Fischer Theorem)**.** Let E be a measurable set and $1 \leq p \leq \infty$. Then $L^p(E)$ is a Banach space. Moreover, if $\{f_n\} \to f$ in $L^p(E)$, a subsequence of $\{f_n\}$ converges pointwise a.e. on E to f.

Theorem 6. Let E be a measurable set and $1 \leq p < \infty$. Suppose $\{f_n\}$ is a sequence in $L^p(E)$ that converges pointwise a.e. on E to the function f which belongs to $L^p(E)$. Then:

$$\{f_n\} \to f \text{ in } L^p(E) \iff \lim_{n\to\infty} \int_E |f_n|^p = \int_E |f|^p$$

Definition 15 (Tight)**.** A family $\mathcal{F}$ of measurable functions on E is said to be **tight** over E provided that for each $\varepsilon > 0$, there exists a subset E_0 of E of finite measure for which

$$\int_{E\setminus E_0} |f| < \varepsilon \text{ for all } f \in \mathcal{F}$$

Theorem 7. Let E be a measurable set and let $1 \leq p < \infty$. Suppose $\{f_n\}$ is a sequence in $L^p(E)$ that converges pointwise a.e. on E to the function f which belongs to $L^p(E)$. Then, $\{f_n\} \to f$ in $L^p(E) \iff \{|f_n|^p\}$ is uniformly integrable and tight over E.

7.4. Approximation and Separability

Definition 16 (Dense)**.** Let X be a normed linear space with norm $||\cdot||$. Given two subsets $\mathcal{F}$ and $\mathcal{G}$ of X with $\mathcal{F} \subseteq \mathcal{G}$, we say that $\mathcal{F}$ is **dense** in $\mathcal{G}$ provided for each function g in $\mathcal{G}$ and $\varepsilon > 0$, there is a function $f \in \mathcal{F}$ for which $||f - g|| < \varepsilon$.

Proposition 4. Let E be a measurable set and let $1 \leq p \leq \infty$. Then, the subspace of simple functions in $L^p(E)$ is dense in $L^p(E)$.

Proposition 5. Let $[a, b]$ be a closed, bounded interval and $1 \leq p < \infty$. Then, the subspace of step functions on $[a, b]$ is dense in $L^p[a, b]$.

Definition 17 (Separable)**.** A normed linear space X is said to be **separable** provided there is a countable subset that is dense in X.

Theorem 8. Let E be a measurable set and $1 \leq p < \infty$. Then, the normed linear space $L^p(E)$ is separable.

Theorem 9. Suppose E is measurable and let $1 \leq p < \infty$. Then, $C_c(E)$ (the set of all continuous functions with compact support on E) is dense in $L^p(E)$.