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5) Show that $(C^{0,\gamma}(\Omega), \|\cdot\|)$ is a Banach Space.
The $C^{0,\gamma}(\Omega)$ norm is given by:

$$\boxed{\|u\| := \sup_{x\in\Omega}|u(x)| + \sup_{\substack{x,y\in\Omega\\ x\neq y}} \frac{|u(x)-u(y)|}{\|x-y\|^{\gamma}}}$$

~~1) $\|\cdot\|_{C^{0,\gamma}(\Omega)}$ is a Norm.~~

~~(N1): $\|x\|=0 \Leftrightarrow x=0$:~~
~~"$\Rightarrow$": Clear.~~
~~"$\Leftarrow$": Assume that $\|u\|_{C^{0,\gamma}(\Omega)}=0$. then, $\sup_{x\in\Omega}|u|=0$ since $\|\cdot\|_{C^0(\Omega)}$ is a norm.~~

We need to show that it is complete. Let $\{f_n\}\subseteq C^{0,\gamma}(\Omega)$ be a Cauchy Sequence. We need to show that $f_n \to f\in C^{0,\gamma}(\Omega)$ wrt. $\|\cdot\|_{C^{0,\gamma}(\Omega)}$. First, we need a candidate limit function; we'll get this from $C^0(\Omega)$; $\{f_n\}$ is a Cauchy Sequence in $C^0(\Omega)$, and $C^0(\Omega)$ is complete (Proposition 1.8) and hence $f_n\to f\in C^0(\Omega)$. Hence, $\exists N_1\in\mathbb{N}$ s.t.
$\|f_n - f\|_{C^0(\Omega)} < \varepsilon/2$.

Now, for the $\sup_{\substack{x,y\in\Omega\\ x\neq y}}\frac{|f(x)-f(y)|}{\|x-y\|^{\gamma}}$ part of the norm First, we'll show that our candidate limiting function has a finite $[f]_{C^{0,\gamma}(\Omega)}$ component (which would prove that $\|f\|_{C^{0,\gamma}(\Omega)}<\infty$). However, for some $x,y\in\Omega$, $x\neq y$ arbitrary, $\forall n\in\mathbb{N}$,

$$\frac{|f_n(x)-f_n(y)|}{\|x-y\|^{\gamma}} \leq C < \infty$$

$$\Leftrightarrow \lim_{n\to\infty}\frac{|f_n(x)-f_n(y)|}{\|x-y\|^{\gamma}} \leq \lim_{n\to\infty} C$$

$$\Leftrightarrow \frac{\left|\lim_{n\to\infty}(f_n(x)-f_n(y))\right|}{\|x-y\|^{\gamma}} \leq C \qquad (x\mapsto |x| \text{ is cts})$$

$$\Leftrightarrow \frac{|f(x)-f(y)|}{\|x-y\|^{\gamma}} \leq C \quad \Rightarrow \quad \boxed{[f]_{C^{0,\gamma}(\Omega)} < \infty}$$

(4D)

$\Rightarrow f \in C^{0,\gamma}(\Omega)$.

All that's left to show is convergence in the $\|\cdot\|_{C^{0,\gamma}(\Omega)}$ norm.
We already bounded $\|f - f_n\|_{C^0(\Omega)}$ by $\varepsilon/2$ $\forall n \geq N_1$.
For the $[f - f_n]_{C^0(\Omega)}$, we have

$$[f - f_n]_{C^0(\Omega)} = \sup_{\substack{x,y\in\Omega \\ x \neq y}} \frac{|(f_n - f)(x) - (f_n - f)(y)|}{\|x-y\|^\gamma}$$

$$= \lim_{m\to\infty} \frac{|(f_n - f_m(x)) - (f_n - f_m)(y)|}{\|x-y\|^\gamma}$$

$$= \lim_{m\to\infty} [f_m - f_n] \to 0$$

So, we can choose an $N_2 \in \mathbb{N}$ s.t. $\forall n \geq N_2$, $[f_m - f_n] < \varepsilon/2$
For all $n \geq \max\{N_1, N_2\}$,

$$\|f - f_n\|_{C^{0,\gamma}(\Omega)} = [f - f_n]_{0(\Omega)} + \|f - f_n\|_\infty$$
$$\leq \varepsilon/2 + \varepsilon/2$$
$$= \varepsilon$$

Which proves that $f_n \to f$ wrt $\|\cdot\|_{C^{0,\gamma}(\Omega)}$
$\Rightarrow (C^{0,\gamma}(\Omega), \|\cdot\|_{C^{0,\gamma}(\Omega)})$ is complete
$\Rightarrow (C^{0,\gamma}(\Omega), \|\cdot\|_{C^{0,\gamma}(\Omega)})$ is a Banach Space.