(11)

6) Let $\{f_n\} \subseteq L^p(\Omega)$ with $1 \le p < \infty$ and let $\{g_n\}$ is a bounded sequence in $L^\infty(\Omega)$. Assume $f_n \to f$ in $L^p(\Omega)$ and $g_n \to g$ a.e. Prove that $f_n g_n \to fg$ in $L^p(\Omega)$.

$$\begin{aligned} \|f_n g_n - fg\|_p &= \|f_n g_n + fg_n - fg_n - fg\|_p \\ &= \|(f_n g_n - fg_n) + (fg_n - fg)\|_p \\ &\le \underbrace{\|f_n g_n - fg_n\|_p}_{:=(1)} + \underbrace{\|fg_n - fg\|_p}_{:=(2)} \qquad \text{(Triangle Inequality)} \end{aligned}$$

1. Term:

$$\|f_n g_n - fg_n\|_p = \|g_n(f_n - f)\|_p$$

$\{g_n\}$ bounded in $L^\infty(\Omega) \Rightarrow \exists M \in \mathbb{R}$ s.t. $g_n \le M$ $\forall n \in \mathbb{N}$.
Hence,

$$\begin{aligned} &\le \|M(f_n - f)\|_p \\ &= |M| \, \|f_n - f\|_p \qquad \text{(Positive homogenity of } \|\cdot\|_p\text{)} \end{aligned}$$

Choose $N_1 \in \mathbb{N}$ s.t. $\forall n \geq N_1$,

$$\|f_n - f\|_p \le \varepsilon/2,$$

which we can do since $f_n \to f$ in $L^p(\Omega)$.

2. Term:

$$\begin{aligned} \|fg_n - fg\|_p &= \int_\Omega |fg_n - fg|^p \, dx \\ &= \int_\Omega |f|^p \underbrace{|g_n - g|}\, dx \end{aligned}$$

$\longrightarrow$ $g_n \to g$ in $L^\infty(E)$ a.e $\Rightarrow$ we can roughly bound the difference by $M+1$ for large enough $n \in \mathbb{N}$
↳ then, get a uniform bound with $B := \max\{|g_1|, \dots, |g_n|, (M+1)\}$

$$\begin{aligned} &\le \int_\Omega |f|^p (B)^p \, dx \\ &= \underbrace{(B)^p \underbrace{\int_\Omega |f|^p \, dx}_{< \infty \text{ since } f \in L^p(\Omega)}} \end{aligned}$$

$|fg_n - fg|$ dominated by an integrable fun $\Rightarrow$ we can use the Dominated Cv Thm to exchange limit + integral

(12)

$\Rightarrow \lim_{n\to\infty} \|fg_n - fg\| = \lim_{n\to\infty} \int_\Omega |f|^p \, |g_n - g| \, dx$

Let $J := \int_\Omega |f|^p dx < \infty$. Choose $N_2 \in \mathbb{N}$ s.t. $|g_n - g| < \frac{\varepsilon}{2J}$, which we can do since $g_n \to g$. Hence,

$$\lim_{n\to\infty} \|f_n g_n - fg\| = \lim_{n\to\infty} \left(\|g_n f_n - g_n f\| + \|fg_n - fg\| \right)$$

$$= \lim_{n\to\infty} \|g_n f_n - g_n f\| + \lim_{n\to\infty} \|fg_n - fg\|$$

Take $N := \max\{N_1, N_2\}$. Then, $\forall n \geq N$,

$$\leq \frac{\varepsilon}{2} + J \frac{\varepsilon}{2J} = \varepsilon \checkmark$$