MATH 314: Advanced Calculus
Meeting 1 - Suggested Problems

Problem 1 (Practice with Conservative Vector Fields)**.** For each of the following vector fields, determine if it's conservative or not. If it's conservative, find the potential function for the vector field.

1. $\mathbf{F} = (x^3 - 4xy^2 + 2)\mathbf{i} + (6x - 7y + x^3y^3)\mathbf{j}$
2. $\mathbf{F} = (6x^2 - 2xy^2 + \frac{y}{2\sqrt{x}})\mathbf{i} - (2x^2y - 4 - \sqrt{x})\mathbf{j}$
3. $\mathbf{F} = (2x\sin(2y) - 3y^2)\mathbf{i} + (2 - 6xy + 2x^2\cos(2y))\mathbf{j}$
4. $\mathbf{F} = y^2(1 + \cos(x + y))\mathbf{i} + (2xy - 2y + y^2\cos(x + y) + 2y\sin(x + y))\mathbf{j}$
5. $\mathbf{F} = (6 - 2xy + y^3)\mathbf{i} + (x^2 - 8y + 3xy^2)\mathbf{j}$
6. $\mathbf{F} = (2z^4 - 2y - y^3)\mathbf{i} + (z - 2x - 3xy^2)\mathbf{j} + (6 + y + 8xz^3)\mathbf{k}$

Problem 2. Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F}(x, y, z) = xy\mathbf{i}+yz\mathbf{j}+zx\mathbf{k}$ and C is the twisted cube parameterized by:

$$x = t,\ y = t^2,\ z = t^3,\ 0 \le t \le 1.$$

Problem 3. Compute the following line integral:

$$\int_C xy^4 ds,$$

where C is the right half of the circle $x^2 + y^2 = 16$.

Problem 4. Evaluate the following line integral: $\int_C(x^2y + \sin(x))dy$, where C is the arc of the parabola $y = x^2$ from $(0, 0)$ to (π, π^2).

Problem 5. (Exam Question)

1. Show that $\mathbf{F}$ is a conservative vector field.
2. Consider the vector field $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ defined by:

$$\mathbf{F}(x, y) := (y^5 + 2x, 5xy^4 - 2). \tag{1}$$

   Let C be the semi-circle $x^2 + y^2 = 1$, $x \ge 0$, oriented clock-wise. Evaluate the line-integral of $\mathbf{F}$ along the curve C, that is, the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$.

Problem 6. (Similar in spirit to the previous problem)

1. Say we have the following 3 pieces of information about some parametric curve:
   (a) $\mathbf{r}''(t) = \langle 6, 0, 0\rangle$ for all t.
   (b) $\mathbf{r}(0) = \langle 0, 3, 4\rangle$.
   (c) $\mathbf{r}'(0) = \langle 0, 0, 1\rangle$.

   Find an expression for $\mathbf{r}(t)$ for all t. Use this to find $\mathbf{r}(1)$.
2. Using this same curve that you found in (a), use an integral theorem to compute the line integral of the following vector field:

$$\mathbf{F}(x, y, z) = \pi\cos(\pi x)\mathbf{i} + (3y^2 + z)\mathbf{j} + (4z^3 + y)\mathbf{k}, \tag{2}$$

   along the path $\mathbf{r}(t)$ from $t = 0$ to $t = 1$.

Problem 7. Compute the line integral,

$$\oint_C \frac{-ydx}{x^2+y^2} + \frac{xdy}{x^2+y^2},$$

where C is the circle $x^2+y^2=a^2$ with $a \neq 0$ in the counter-clockwise direction.

Problem 8. Find the work done by the force,

$$\mathbf{F} = \left(2e^{2x}\cos(\pi y)\right)\mathbf{i} - \left(\pi e^{2x}\sin(\pi y)\right)\mathbf{j} \tag{3}$$

in moving

1. from $(0,0)$ to $(1,1)$ on the curve

$$x^3(y-1)^2 = 4y^4(x-1)^3. \tag{4}$$

2. once around the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$.

Problem 9. Show that the vector field,

$$\mathbf{F} = \frac{2x}{\pi}\sin(\pi y)\mathbf{i} + (x^2\cos(\pi y) - 2ye^{-z})\mathbf{j} + y^2e^{-z}\mathbf{k}, \tag{5}$$

is conservative. Hence, evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where C is a straight line from $(0,0,0)$ to $(1,1,-1)$.

Problem 10 (Careful – True or False?)**.** For any vector field $\mathbf{F}$ and for any parameterized curve $\mathbf{r}(t)$ with $t \in [a,b]$, we have

$$\int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)dt = \mathbf{F}(\mathbf{r}(b)) - \mathbf{F}(\mathbf{r}(a)). \tag{6}$$

Problem 11. Find the work done by the force field,

$$\mathbf{F}(x,y) = x\mathbf{i} + (y+2)\mathbf{j}, \tag{7}$$

in moving an object along an arch of the cycloid,

$$\mathbf{r}(t) = (t-\sin(t))\mathbf{i} + (1-\cos(t))\mathbf{j},\ 0 \le t \le 2\pi. \tag{8}$$

Problem 12.

1. Show that a constant force field does zero work on a particle that moves once uniformly around a circle $x^2+y^2=1$.
2. Is it also true for a force field $\mathbf{F}(\mathbf{x}) = k\mathbf{x}$, where k is constant and $\mathbf{x} = \langle x, y\rangle$?

Problem 13. Show that if the vector field $\mathbf{F} = P\mathbf{i}+Q\mathbf{j}+R\mathbf{k}$ is conservative and P,Q,R have continuous first-order partial derivatives, then:

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x},\ \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x},\ \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$$

Problem 14. Let

$$\mathbf{F}(x,y) = \frac{-y\mathbf{i}+x\mathbf{j}}{x^2+y^2}.$$

1. Show that

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

2. Show that $\int_C \mathbf{F} \cdot d\mathbf{r}$ is not path-independent. Is there a contradiction?
   (Hint: Compute $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$ and $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$, where C_1 and C_2 are the upper and lower halves of the circle $x^2 + y^2 = 1$ from $(1, 0)$ to $(-1, 0)$).

Problem 15. Let $\mathbf{F} = \nabla f$, where $f(x, y) = \sin(x - 2y)$. Find curves C_1 and C_2 that are not closed and satisfy the equation.

1. $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = 0$.
2. $\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = 1$.

Problem 16. Compute the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where

$$\mathbf{F}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + (xy + 2z)\mathbf{k}, \tag{9}$$

along C, where C is the line segment from $(1, 0, -2)$ to $(4, 6, 3)$.