MATH 567: FUNCTIONAL ANALYSIS (FALL 2020 SEMESTER)

SHEREEN ELAIDI AS TAUGHT BY PROF. LIN; LAST UPDATED: SEPTEMBER 15, 2020

This class is about linear functional analysis. This has a lot in common with linear algebra in infinite-dimensional spaces. We can think of this as infinite-dimensional linear algebra. There are two main applications of this: (a) geometry and topology in infinite dimensions and (b) solving PDEs. Recall that in regular linear algebra, we used those tools to solve linear systems. The infinite-dimensional equivalent to this is a PDE. In this class, we'll focus on the second application of functional analysis.

1. Basic Functional Analysis

This corresponds to Chapters 4 and 5 of the textbook.

1.1. Banach Spaces and General Topology.

Let X be a vector space. Recall that this means that it is closed under addition and scalar multiplication.

Definition 1.1 (Norm)**.** A **norm** on a vector space X, $||\cdot|| : X \to [0,\infty[$, satisfies the following three properties:

(1) $||x|| = 0 \iff x = 0$.
(2) **(Homogeneity)**: $||\lambda x|| = |\lambda| ||x||$ for each $x \in X$, $\lambda \in \mathbb{R}$.
(3) **(Triangle Inequality):** $||x + y|| \leq ||x|| + ||y||$.

Definition 1.2 (Completeness / Banach Space)**.** X is **complete** if every Cauchy sequence converges. $(X, ||\cdot||)$ is a **Banach space** if it is a complete normed vector space.

Definition 1.3 (Dense Subset)**.** $Y \subseteq X$ is **dense** if

(1) $\overline{Y} = X$ (one thing we need to note: $\overline{Y}$ is the closure, but we need to ask ourselves "in which topology"?).

This is equivalent to:

$$\forall \varepsilon > 0,\ \forall x \in X,\ \exists y \in Y \ \text{ s.t. } ||x - y|| < \varepsilon.$$

And also equivalent to,

$$\forall x \in X,\ \exists \{y_n\} \subseteq Y \ \text{ s.t. } y_n \to y \in X.$$

Definition 1.4 (Strong Topology)**.** The **strong topology** is the topology induced by the norm, $||\cdot||$ (the open sets are characterized by the balls, $B_r := \{x \mid ||x|| < r\}$). In this topology, the definitions of density given above are equivalent.

Definition 1.5 (Separable)**.** X is **separable** if $\exists$ a countable dense subset.

We have the following equivalent definitions of compactness.

Definition 1.6 (Compactness 1)**.** $E \subseteq X$ is **compact** if every open cover of E admits a finite subcover.

Definition 1.7 (Compactness 2)**.** Every sequence has a convergent sub-sequence.

Definition 1.8 (Compactness 3)**.** For any sequence $\{x_n\} \subseteq E$, there exists $\{x_{n_k}\}$ and $x^* \in E$ such that $x_{n_k} \to x^* \in E$.

Definition 1.9 (Pre-Compact)**.** $E \subseteq X$ is **pre-compact** if $\overline{E}$ is compact.

Date: Fall 2020 Semester.

1.2. **Euclidean Space $\mathbb{R}^n$.** Let $x \in \mathbb{R}^n$. This is denoted by $(x_1, ..., x_n)$. Then, recall,

$$||x|| = ||x||_{\ell^2} = \left(\sum_{j=1}^{n} x_j^2 \right)^{1/2}.$$

We also have these other typical norms on Euclidean space:

$$\begin{aligned} ||x||_{\ell^1} &= \sum_{j=1}^{n} |x_j| \\ ||x||_{\ell^p} &= \left(\sum_{j=1}^{n} |x_j|^p \right)^{1/p} \\ ||x||_{\ell^\infty} &= \max_{1 \le j \le n} |x_j|. \end{aligned}$$

Definition 1.10 (Equivalent Norms)**.** We say that two norms, $|\cdot|$ and $||\cdot||$, are equivalent if and only if there exist two constants a and b such that

$$\boxed{a||x|| \le |x| \le b||x|| \qquad \forall x \in X.} \tag{1.11}$$

In words, this is saying that you can't be big on one norm but small in another. These norms are comparable; they are bounded by constants on either side.

Theorem 1.12. *All norms on $\mathbb{R}^n$ are equivalent (all norms in finite dimensions are equivalent).*

Proof. Let $||\cdot||$ be the Euclidean norm, and let $|\cdot|$ be another norm. Let $\{e_i\}_{i=1}^n$ be the standard basis of $\mathbb{R}^n$; recall that this is $e_i = (0, ..., 1, ..., 0)$ where the 1 is in the ith slot. Since this is a basis, for $x \in X$:

$$x = \sum_{i=1}^{n} x_i e_i.$$

By the reverse triangle inequality,

$$\begin{aligned} ||x| - |y|| &\le |x - y| \\ &= \left| \sum_{i=1}^{n} (x_i - y_i) e_i \right| \\ &\le \sum_{i=1}^{n} |x_i - y_i| |e_i| \\ &\le \underbrace{\left(\sum_{i=1}^{n} |e_i|^2 \right)^{1/2}}_{:=C} \left(\sum_{i=1}^{n} |x_i - y_i|^2 \right)^{1/2} \qquad \text{(Cauchy-Schwarz)} \\ &\le C||x - y|| \qquad (*), \end{aligned}$$

where C is some number. Norms are continuous; $x \mapsto ||x||$ is continuous $S = \{x \mid ||x|| = 1\}$ (the unit ball). By $(*)$, $x \mapsto |x|$ is continuous on S. S is closed and bounded on $\mathbb{R}^n$ which means that S is compact. By the extreme value theorem, this means that there exist two constants $a, b \in \mathbb{R}$ such that

$$a \le |x| \le b \qquad \forall x \in S. \tag{1.13}$$

Observe that $|x| = 0 \iff x = 0$, which implies that $a > 0$. For any $y \in \mathbb{R}^n$, let $x := \frac{y}{||y||} \in S$. Then,

$$a \le \left| \frac{y}{||y||} \right| \le b \iff a \le \frac{1}{||y||} |y| \le b \iff a||y|| \le |y| \le b||y|| \qquad \forall y \in \mathbb{R}^n \setminus \{0\}.$$

The case of $y = 0$ is straightforward. This proves that any norm in a finite-dimensional vector space are equivalent. Note that this proof rests on the fact that we have a basis. ☐

Remark 1.14. $\mathbb{R}^n$ is separable in any norm. The typical countable dense subset of $\mathbb{R}^n$ is $\mathbb{Q}^n$. We will see in infinite-dimensions that all norms are not equivalent.

1.3. **The Spaces of C^r, $C^{r,\gamma}$ of Continuous Functions.**

Definition 1.15 (C^0)**.** Let $\Omega \subseteq \mathbb{R}^n$ be open. Then,

$$C^0(\Omega) \coloneqq \{f \mid \Omega \to \mathbb{R} \text{ s.t. } f \text{ is continuous on } \Omega\}$$
$$C^0(\overline{\Omega}) \coloneqq \{f \mid \overline{\Omega} \to \mathbb{R} \text{ s.t. } f \text{ is continuous on } \overline{\Omega}\}.$$

This implies that $f \in C^0(\overline{\Omega})$ is bounded and uniformly continuous.

Definition 1.16 ($||\cdot||_\infty$)**.** The standard norm on $C^0(\Omega)$ is

$$||u||_\infty \coloneqq \sup_{x\in\Omega} |u(x)| \leftrightarrow \text{ uniform convergence.} \tag{1.17}$$

Proposition 1.18. *(1) $(C^0(\Omega), ||\cdot||_\infty)$ is a Banach space.*
(2) If $\Omega \subseteq \mathbb{R}^n$ is bounded, then $C^0(\overline{\Omega})$ is separable.

We will only give a sketch of the proof.

Proof. (1) The uniform limit of continuous functions is continuous.
(2) Follows from the Weierstrass approximation theorem: polynomials are dense in $C^0(\overline{\Omega})$; then, consider the polynomials with rational coefficients.

□

1.3.1. *Higher-Order Derivatives.* Recall some notation from advanced calculus:

$$\boxed{Du = \nabla u = \text{ gradient of } u = \begin{bmatrix} \partial_1 u \\ \vdots \\ \partial_n u \end{bmatrix}}. \tag{1.19}$$

We consider the **multi-index** $\alpha = (\alpha_1, ..., \alpha_n)$, where $|\alpha| \coloneqq \alpha_1 + ... + \alpha_n$, and $\forall k \in \mathbb{R}^n$, define $k^\alpha \coloneqq k_1^{\alpha_1} \cdots h_n^{\alpha_n}$. Then, in this notation,

$$\begin{aligned} D^\alpha u &= \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} u \\ &= \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \qquad \text{(partial derivative).} \end{aligned}$$

Definition 1.20 ($C^r(\Omega)$)**.**

$$C^r(\Omega) \coloneqq \{f \mid D^\alpha f \in C^0(\Omega) \ \forall\ |\alpha| < r\} \tag{1.21}$$

In words, this means that all partial derivatives less than or equal to r are continuous. Then, we can define the following space:

$$C^\infty(\Omega) \coloneqq \bigcap_{r=1}^{\infty} C^r(\Omega). \tag{1.22}$$

Definition 1.23 (Support of f)**.** The **support** of f is defined as the smallest closed set such that $f \equiv 0$ on $\mathbb{R}^n \setminus \operatorname{supp}(f)$.

$$\operatorname{supp}(f) \coloneqq \overline{\{x \mid f(x) \neq 0\}}. \tag{1.24}$$

Definition 1.25 (Compactly Contained)**.** A set $K \subset\subset \Omega$ means that $K \subseteq \Omega$ is compact. We say that K is **compactly contained** in Ω if $K \subset\subset \Omega$, Ω is bounded, and that there exists an $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq \Omega$ for all $x \in K$. This is equivalent to for all $x \in K$,

$$\exists\ \varepsilon > 0 \text{ s.t. } d(x, \partial\Omega) \coloneqq \inf_{y\in\partial\Omega} |x - y| > \varepsilon. \tag{1.26}$$

Definition 1.27 ($C_c^r(\Omega)$)**.**

$$C_c^r(\Omega) \coloneqq \{f \mid f \in C^r(\Omega), \operatorname{supp}(f) \subset\subset \Omega\}. \tag{1.28}$$

Definition 1.29 (Norm on $C^r(\overline{\Omega})$)**.** Let Ω be bounded. Then,

$$\boxed{||f||_{C^r} \coloneqq \sum_{|\alpha|\leq r} \sup_{x\in\Omega} |D^\alpha f(x)|.} \tag{1.30}$$

Proposition 1.31. *Let $\Omega \subseteq \mathbb{R}^n$ be bounded. Then, $C^r(\Omega)$ is a separable Banach space (in fact, all you need for separable is that it is bounded) for all $r < \infty$.*

Remarks 1.32. $C_c^r(\Omega)$ is not complete. $C^\infty(\Omega)$ is not complete. However, subspaces of C^∞ is still complete with some norm.

We also introduce,

Definition 1.33 (Hölder Continuous $C^{0,\gamma}(\Omega)$)**.** $f : \Omega \to \mathbb{R}$ is **Hölder Continuous** with exponent $\gamma \in [0, 1[$ if there exists a C such that

$$\boxed{|f(x) - f(y)| \leq C||x - y||^\gamma.} \tag{1.34}$$

If $\gamma = 1 \Rightarrow f$ is **Lipschitz Continuous**.

Also,

$$\boxed{[f]_{C^{0,\gamma}(\Omega)} \coloneqq \sup_{x,y\in\Omega} \frac{|f(x) - f(y)|}{||x - y||^\gamma}}$$

is called the **Hölder seminorm**. This is nor a norm, but we can make it a norm:

Definition 1.35 ($|| \cdot ||_{C^{0,\gamma}}$)**.**

$$\boxed{||f||_{C^{0,\gamma}(\Omega)} \coloneqq ||f||_\infty + [f]_{C^{0,\gamma}(\Omega)}.} \tag{1.36}$$

On the homework, you'll show that

$$(C^{0,\gamma}(\Omega), ||f||_{C^{0,\gamma}(\Omega)})$$

is complete.

Definition 1.37 ($C^{r,\gamma}$)**.**

$$\boxed{C^{r,\gamma}(\Omega) \coloneqq \{f \mid f \in C^r(\Omega) \text{ and } |D^\alpha f(x) - D^\alpha f(y)| \leq C||x - y||^\gamma \ \forall \ |\alpha| = r\}}. \tag{1.38}$$

The norm of this space is given by,

$$\boxed{||f||_{C^{r,\gamma}} \coloneqq ||f||_{C^r} + \sup_{|\alpha|=r} [D^\alpha f]_{C^{0,r}}.} \tag{1.39}$$

Remark 1.40. If $f \in C^{0,\gamma}(\Omega)$, Ω bounded, then $f \in C^{0,\alpha}(\Omega)$ for all $0 < \alpha \leq \gamma$

Remark 1.41. **(Rademacher's Theorem)**. If $f \in C^{0,1}$, then f is differentiable a.e.

1.4. **Integration Theorems.**

Theorem 1.42 (Monotone Convergence Theorem)**.** *If $f_n \uparrow f$ pointwise for almost every x, then*

$$\boxed{\lim_{n\to\infty} \int_\Omega f_n(x)dx = \int_\Omega f(x)dx}. \tag{1.43}$$

Theorem 1.44 (Fatou's Lemma)**.** *Let $\{f_n\}$ be a sequence of measurable functions that are all positive. Then,*

$$\boxed{\int_\Omega \liminf_{n\to\infty} f_n(x) \leq \liminf_{n\to\infty} \int_\Omega f_n(x)dx} \tag{1.45}$$

Theorem 1.46 (Dominated Convergence Theorem)**.** *Assume that $\{f_n\}$ are measurable, $f_n \to f$ pointwise a.e. Then, if $|f_n(x)| \leq g(x)$ for all $n \in \mathbb{N}$ for almost every x, where $g \in L^1(\Omega)$, then*

$$\boxed{\lim_{n\to\infty} \int_\Omega f_n(x)dx = \int_\Omega f(x)dx} \tag{1.47}$$

This is the theorem that you use when you want to differentiate under integrals.

Theorem 1.48. *The space $C_c^0(\mathbb{R}^n)$ is dense in $L^1(\mathbb{R}^n)$.*

Theorem 1.49 (Fubini-Tonelli)**.** *For all* $f : X \times Y \to \mathbb{R}^n$,

$$\boxed{\int_X \int_Y |f(x,y)|dydx = \int_Y \int_X |f(x,y)|dxdy = \int_{X\times Y} |f(x,y)|d(x,y).} \tag{1.50}$$

If, moreover, $f \in L^1(X \times Y)$,

$$\int_X \int_Y f(x,y)dydx = \int_Y \int_X f(x,y)dxdy = \int_{X\times y} f(x,y)d(x,y) \tag{1.51}$$

1.5. **Elementary L^p Spaces.**

Definition 1.52 (L^p)**.** Fix $1 \leq p < \infty$, let $\Omega \subseteq \mathbb{R}^n$. Then, we define the L^p space to be

$$\boxed{L^p(\Omega) \coloneqq \{f : \Omega \to \mathbb{R} \mid f \text{ measurable } |f|^p \in L^1(\Omega)\}} \tag{1.53}$$

with the following norm,

$$||f||_{L^p} \coloneqq \left[\int_\Omega |f(x)|^p\right]^{1/p}. \tag{1.54}$$

Definition 1.55 (L^∞)**.** We define L^∞ to be:

$$L^\infty(\Omega) \coloneqq \{f : \Omega \to \mathbb{R} \mid f \text{ measurable }, \exists\, C \text{ s.t. } |f(x)| \leq C \text{ a.e. }\}, \tag{1.56}$$

with the following norm,

$$||f||_{L^\infty} = ||f||_\infty = \inf\{c \mid |f(x)| \leq c \text{ } a.e.\} \tag{1.57}$$

This definition implies that $f(x) \leq ||f||_\infty$ almost everywhere. Below are some fundamental tools that we'll be using

Theorem 1.58 (Hölder's Inequality)**.** *Let* $1 \leq p, p' \leq \infty$. *If* $f \in L^p(\Omega)$, $g \in L^{p'}(\Omega)$ *and* $1/p + 1/p' = 1$, *then* $fg \in L^1$ *and*

$$\boxed{\int |fg|dx \leq ||f||_p||g||_{p'}} \tag{1.59}$$

Theorem 1.60 (Minkowski's Inequality)**.** *For all* $p \in [1, \infty]$,

$$\boxed{||f + g||_p \leq ||f||_p + ||g||_p.} \tag{1.61}$$

As a consequence of Minkowski's Inequality, L^p is a vector space.

Theorem 1.62 (Riesz-Fischer)**.** L^p *is a Banach space for all* $p \in [1, \infty]$.

Proof. **Case # 1:** $p = \infty$. Let $\{f_n\} \subseteq L^\infty$ be Cauchy. Hence, for all $k \in \mathbb{N}$, there exists an N_k such that for all $n, m \geq N_k$,

$$||f_n - f_m||_\infty < \frac{1}{k}. \tag{1.63}$$

Then, there exists a null set E_k such that $\forall n, m \geq N_k$.

$$|f_n(x) - f_m(x)| \leq \frac{1}{k}.$$

for all $x \in \Omega \setminus E_k$, $\{f_n\} \subseteq \mathbb{R}$ is a Cauchy sequence. Since $\mathbb{R}$ is complete, there exists an $f(x) \in \mathbb{R}$ such that

$$f_n(x) \to f(x) \qquad x \in \Omega \setminus E.$$

So, in particular, $\forall m \geq N_k$,

$$|f_n(x) - f(x)| \leq \frac{1}{k} \ \forall x \in \Omega \setminus E.$$

We can then take the supremum,

$$\sup_{x\in\Omega\setminus E} |f_m(x) - f(x)| \leq \frac{1}{k}.$$

Extend f to be whatever on E:

$$\begin{aligned}&\Rightarrow ||f-f_m||_\infty \leq \frac{1}{k}, \qquad n \geq N_k,\\ &\Rightarrow f_n \to f \text{ in } L^\infty.\end{aligned}$$

□

Also, $f = (f - f_n) + f_n$. We have that $(f-f_n) \in L^\infty$ and $f_n \in L^\infty$. Hence, $f \in L^\infty$ since L^∞ i a vector space. Hence, we have proven that L^p is a Banach space for $p = \infty$.

Case # 2: $1 \leq p < \infty$. Similarly, $\{f_n\} \subseteq L^p$ be Cauchy. Choose a subsequence such that,

$$||f_{n_{k+1}} - f_{n_k}||_{L^p} < \frac{1}{2^k} \qquad \forall k \geq 1.$$

Then,

$$\left|\left|\sum_{k=1}^N |f_{n_{k+1}} - f_{n_k}|\right|\right|_{L^p} \leq \sum_{k=1}^N \left(\frac{1}{2^k}\right) < 1.$$

Define,

$$v(x) \coloneqq \lim_{N\to\infty} \sum_{k=1}^N |f_{n_{k+1}}(x) - f_{n_k}(x)|.$$

(Possibly infinite, but always positive). By Fatou's Lemma,

$$\int_\Omega |v|^p dx \leq \liminf_{N\to\infty} \int_\Omega \left(\sum_{k=1}^N |f_{n_{k+1}}(x) - f_{n_k}(x)|\right)^p dx \leq 1.$$

Hence, $v \in L^p$ which implies that $|v(x)| < \infty$ a.e. and,

$$f_{n_{k+1}}(x) + \sum_{k=1}^\infty (f_{n_{k+1}}(x) - f_{n_k}(x)) \qquad (*)$$

converges almost everywhere for x. Observe that the partial sums of the above in $(*)$ are just $f_{n_{k+1}}(x)$ (telescoping series):

$$f(x) \coloneqq \lim_{k\to\infty} f_{n_k}(x),$$

which we already knew converges for a.e. x and extend this to be whatever on a null set. Claim: $f \in L^p$ and $||f_n - f||_{L^p} \to 0$. By Fatou's Lemma, for k sufficiently large,

$$\int_\Omega |f - f_{n_k}|^p dx \leq \liminf_{j\to\infty} \int_\Omega |f_{n_j} - f_{n_k}|^p \leq \frac{\varepsilon}{2} \qquad \text{(since Cauchy)}$$

Which implies,

$$\begin{aligned}&\Rightarrow f - f_{n_k} \in L^p(\Omega)\\ &\Rightarrow f(x) = \underbrace{(f(x) - f_{n_k}(x))}_{\in L^p} + \underbrace{f_{n_k}}_{\in L^p}(x)\\ &\Rightarrow f \in L^p.\end{aligned}$$

Break at the subsequence, which means that the limiting guy is in L^p. Also, for all $n \geq N$, $n_k \geq N$,

$$\begin{aligned}||f_n(x) - f(x)||_{L^p} &\leq ||f_n(x) - f_{n_k}(x)||_{L^p} + ||f_{n_k}(x) - f(x)||_{L^p}\\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}\\ &= \varepsilon.\end{aligned}$$

Hence, $f_n \to f$ in L^p.

Corollary 1.64. *Let $\{f_n\} \subseteq L^p$ and let $f \in L^p$ such that $||f_n - f||_{L^p} \to 0$, then there exists a subsequence such that,*

$$f_{n_k}(x) \to f(x) \qquad on \Omega.$$

Proof. Hidden in Reisz-Fischer. □

Theorem 1.65. *$C_c(\mathbb{R}^N)$ is dense in $L^p(\mathbb{R}^N)$ for all $p \in [1, \infty[$.*

Proof. We'll work with the truncation operator. It's a function $T_n : \mathbb{R} \to R$ defined by,

$$T_n r := \begin{cases} r & \text{if } |r| \leq n, \\ \frac{nr}{|r|} & \text{if } |r| \geq n. \end{cases}$$

Claim: for all $f \in L^p(\mathbb{R}^N)$ and for all $\varepsilon > 0$ there exist a $g \in L^\infty(\mathbb{R}^N)$ and a compact set $K \subseteq \mathbb{R}^N$ such that,

$$\text{supp}(g) \subseteq K \text{ and } ||f - g||_{L^p} < \varepsilon.$$

Let $f_n := T_n(f)\chi_{B(0,n)}$. Note that $f_n - f \to 0$ a.e. Then,

$$|f_n - f| \leq 2|f| \in L^p.$$

By the DCT, $||f_n - f||_{L^p} \to 0$. Thus, let $g(x) = f_n(x)$ for n large. So, $g \in L^p(\mathbb{R}^N)$ and is compactly supported. Hence, by inclusions in L^p and Hölder's inequality, we obtain:

$$g \in L^1(\mathbb{R}^N).$$

Thus, for all $\delta > 0$, by the density in L^p, there exists a $g \in C_c^0$ such that

$$||g - g_1||_{L^1} < \delta.$$

WLOG, we may assume that $||g_1||_\infty \leq ||g||_\infty$ (by replacing g_1 for $T_n g_1$ for n large). Since $p \in]1, \infty[$,

$$\begin{aligned} ||g - g_1||_{L^p} &= \left(\int |g - g_1|^p \right)^{1/p} \\ &= \left(\int |g - g_1||g - g_1|^{p-1} \right)^{1/p} \\ &= ||g - g_1||_\infty^{(p-1)/p} ||g - g_1||_{L^1}^{1/p} \\ &= \delta^{1/p} ||g - g_1||_\infty^{1-1/p} \\ &= 2||g||_{L^\infty}^{1-1/p} \delta^{1/p}. \end{aligned}$$

Choosing δ sufficiently small,

$$\leq \varepsilon.$$

By Minkowski, $g \in C_c^0(\mathbb{R}^N)$,

$$||f - g_1||_{L^p} \leq ||f - g||_{L^p} + ||g - g_1||_{L^p} \leq 2\varepsilon,$$

as desired. Hence, $C_c(\mathbb{R}^N)$ is dense in $L^p(\mathbb{R}^N)$. □

Theorem 1.66. *The vector space $L^p(\mathbb{R}^N)$ is separable.*

Proof. Define the following,

$$\mathcal{R} := \left\{ \prod_{k=1}^N]a_k, b_k[, \ a_k, b_k \in \mathbb{Q} \text{ rational rectangles.} \right\}.$$

And let,

$$\mathcal{E} := \{\text{finite linear combination of elements of } \chi_\mathcal{R}, R \in \mathcal{R}, \text{ with rational coefficients.}\}$$

(You can think of this as a vector space over the rationals $\mathbb{Q}$). **Claim:** $\mathcal{E}$ is dense in Ł^p. Given an $f \in L^p(\mathbb{R}^N)$, $\varepsilon > 0$ we know that there is a $f_1 \in C_c^0(\mathbb{R}^N)$ such that,

$$||f - f_1|| < \varepsilon.$$

Let $\text{supp}(f_1) \subseteq R \subseteq \mathcal{R}$. Now, for all $\delta > 0$, build an $f_2 \in \mathcal{E}$ such that $||f_1 - f_2||_\infty < \delta$. Indeed, re-write,

$$R := \bigcup_{i=1}^N R_i \text{ where } R_i \in \mathcal{R} \text{ and } \forall i, \text{osc}_{R_i} f_1 = \sup_{R_i} f_1 - \inf_{R_i} f_1 < \delta.$$

Hence,

$$f_2 = \sum_{i=1}^N q_i \chi_{R_i} \text{ with } q_i \in \mathbb{Q}, q \approx f_1|_{R_i}.$$

Which implies,

$$||f_1 - f_2||_{L^\infty} \leq \delta.$$

So,

$$\begin{aligned}||f_1 - f_2||_{L^p} &\leq ||f_1 - f_2||_\infty |R|^{1/p} \text{ (compact support)}\\ &\leq \delta |R|^{1/p}\\ &< \delta \text{ for } \delta \text{ chosen.}\end{aligned}$$

Which implies that,

$$\begin{aligned}||f - f_2||_{L^p} &\leq ||f - f_1||_{L^p} + ||f - f_2||_{L^p}\\ &\leq \varepsilon + \varepsilon = 2\varepsilon,\end{aligned}$$

as asserted. □

Remark 1.67. These results are more general. In particular, if Ω is separable, then $L^p(\Omega)$ is separable.

1.6. **Convolutions and Mollifers.**

Definition 1.68 (Convolution)**.** Let f and g be functions. Their **convolution** is defined as:

$$(f * g)(x) := \int_{\mathbb{R}^N} f(x-y)g(y)dy = \int_{\mathbb{R}^N} g(x-y)f(y)dy = (g * f)(x).$$

Theorem 1.69 (Young's Inequality)**.** *If $f \in L^1$, $g \in L^p$, then,*

$$\boxed{||f * g||_{L^p} \leq ||f||_{L^1} ||g||_{L^p}}$$

Hence, $f * g \in L^p$ and hence $f * g$ is defined almost everywhere.

Proposition 1.70. *Let f, g be functions. Then,*

$$supp(f * g) \subseteq \overline{sup(f) + sup(g)}$$

Remark 1.71. If f, g are both compactly supported, then $f * g$ is compactly supported. If only one is compactly supported, you cannot say anything.

Definition 1.72 (L^p_{loc})**.** Let f be a function. $f \in L^p_{\text{loc}}$ if $f\chi_K \in L^p$ for all K compact, $K \subseteq \Omega$.

Remark 1.73. By using some sort of a Hölder-estimate, we can show that $f \in L^p_{\text{loc}} \Rightarrow f \in L^1_{\text{loc}}$.

Proposition 1.74. *Let f, g be functions. If $f \in C^0_c(\mathbb{R}^N)$, $g \in L^1_{loc}(\mathbb{R}^N)$. Then, $(f * g)(x)$ is defined for every x and $(f * g) \in C(\mathbb{R}^N)$.*

Proof. We have that for every $x \in \mathbb{R}^N$,

$$\begin{aligned}\left|\int f(x-y)g(y)dy\right| &= \left|\int g(x-y)f(y)dy\right|\\ &\leq ||f||_\infty \int_K |g(x-y)|dy \text{ (by compactly supported)}\\ &\leq ||f||_\infty ||g||^1_L(\tilde{K}) \text{ (since } g \in L^1_{\text{loc}})\\ &< \infty\end{aligned}$$

Since $\tilde{K}$ is compact. Now suppose that $x_n \to x$ (which means that $|x_n - x| \leq B_1$ for all $n \geq N$). Then, since supp(f) is compact, there exists a compact set such that,

$$|f(x_n - y) - f(x-y)| \leq \varepsilon_n \chi_K(y)$$

(by the uniform continuity and by taking $\varepsilon_n \to 0$). So, now it's obvious,

$$|(f * g)(x_n) - (f * g)(x)| \leq \varepsilon_n \int_K |g(y)|dy \to 0,$$

where the last limit follows from the fact that $g \in L^1_{\text{loc}}$. □

Mollification is approximating a function by a smooth function. We define a **mollifier** by:

$$\rho(x) := \begin{cases} C\exp\left(-\frac{1}{1-|x|^2}\right) & \text{if } |x| \leq 1, \\ 0 & \text{if } |x| > 1 \end{cases}$$

This is continuous and smooth, with C chosen based on the dimensions such that,

$$\int_{\mathbb{R}^N} \rho(x)dx = 1.$$

Note that $\rho \in C_c^\infty(\mathbb{R}^N)$. We define:

$$\rho_h(x) := \frac{\rho(x/h)}{h^N} \qquad u_h(x) := (\rho_h * u)(x) = \frac{1}{h^N}\int_\Omega \rho\left(\frac{x-y}{h}\right)u(y)dy.$$

Proposition 1.75. *Let $u \in C_c^0(\Omega)$. Then, $u_h \in C_c^\infty(\Omega)$ and if $u < \operatorname{dist}(\operatorname{supp}, \partial\Omega)$, then $u_h \to u$ uniformly on Ω as $h \to 0$.*

Proof. Let $h < \operatorname{dist}(\operatorname{supp}, \partial\Omega)$. Observe that by the dominated convergence theorem,

$$\partial_i u_h(x) = \int_\Omega \partial_i\rho\left(\frac{x-y}{h}\right)u(y)dy,$$

where $\partial_i\rho\left(\frac{x-y}{h}\right)$ is smooth with compact support, and hence the integral is finite. This implies that $\rho \in C_c^\infty(\Omega)$ and hence $u_h \in C_c^\infty(\Omega)$. Observe,

$$\begin{aligned} &\frac{1}{h^N}\int_{\mathbb{R}^N} \rho\left(\frac{y}{h}\right)dy = 1 \\ \Rightarrow &\frac{1}{h^N}\int_{B(0,h)} \rho\left(\frac{y}{h}\right)dy = 1 \\ \Rightarrow &\frac{1}{h^N}\int_{B(0,h)} \rho\left(\frac{x-y}{h}\right)dy = 1 \ \forall x. \end{aligned}$$

Hence,

$$u_h(x) - u(x) = \frac{1}{h^N}\int_\Omega \rho\left(\frac{x-y}{h}\right)[u(y)-u(x)]dy \text{ (multiplying by 1 in a smart way)}$$

Hence,

$$\begin{aligned} |u_h(x) - u(x)| &= \left|\frac{1}{h^N}\int_{|x-y|\leq h} \rho\left(\frac{x-y}{h}\right)|u(y)-u(x)|dy\right| \\ &\leq \sup_{|x-y|\leq h}|u(y)-u(x)|\frac{1}{h^N}\int_{|x-y|\leq h}\rho\left(\frac{x-y}{h}\right)dy \\ &= \sup_{|x-y|\leq h}|u(y)-u(x)|. \end{aligned}$$

Invoking the uniform continuity of $u \in C_c^0$, we can bound $\sup_{|x-y|\leq h}|u(y)-u(x)| \leq u(h)$. Hence,

$$\sup_{x\in\Omega}|u_h(x) - u(x)| \to 0 \text{ as } h \to 0.$$

□

Theorem 1.76. *Assume that $f \in L^p(\mathbb{R}^N)$ with $1 \leq p < \infty$. Then,*

$$(\rho_h * f) \to f \text{ as } h \to 0 \text{ in } L^p.$$

Proof. Fix an $\varepsilon > 0$. We know that there exists an $f_1 \in C_c^0(\mathbb{R}^N)$, $||f - f_1||_L^P < \varepsilon$. Also, since $f_1 \in C_c^0(\mathbb{R}^N)$, we know that

$$(\rho_h * f_1) \to f_1 \text{ uniformly.}$$

We also have that

$$\begin{aligned} \operatorname{supp}(\rho_h * f_1) &\subseteq \overline{B(0,h) + \operatorname{supp}(f_1)} \\ &\subseteq \underbrace{\overline{B(0,1) + \operatorname{supp}(f_1)}}_{\text{compact}} \end{aligned}$$

Hence,

$$||(\rho_h * f_1) - f_1||_{L^p} \to 0.$$

Thus,

$$(\rho_h * f) - f = (\rho_h * (f - f_1)) + [(\rho_h * f_1) - f_1] + f_1 - f.$$

By the triangle inequality and Young's inequality,

$$||(\rho_h * f) - f||_{L^p} \leq 2\underbrace{||f - f_1||_{L^p}}_{:=(1)} + \underbrace{||(\rho_h * f_1) - f_1||_{L^p}}_{:=(2)}$$

Where (1) is small by density and (2) is small because we just did it. Hence,

$$\limsup_{h\to 0} ||(\rho_h * f) - f||_{L^p} \leq \varepsilon$$
$$\Rightarrow \lim_{h\to 0} ||(\rho_h * f) - f||_{L^p} = 0$$

□

Corollary 1.77. *Let $\Omega \subseteq \mathbb{R}^N$ (possibly all of $\mathbb{R}^N$) with $1 \leq p < \infty$. Then, $(\rho_h * f) \to f$ in $L^P(\mathbb{R}^N)$ as $h \to 0$.*

Proof. Given an $f \in L^p(\Omega)$, we extend to $\overline{f} \in L^p(\mathbb{R}^n)$ by:

$$\overline{f}(x) := \begin{cases} f(x) & x \in \Omega, \\ 0 & x \in \mathbb{R}^n \setminus \Omega. \end{cases}$$

Let $\{K_N\} \subseteq \mathbb{R}^N$ be a compact set such that $\bigcup_{n=1}^{\infty} K_N = \Omega$. (Remark: if $\Omega \subseteq \mathbb{R}^N$ is bounded, then $\text{dist}(K_n \cap \Omega^c) > 2/n$). Now let $g_n := \overline{f}\chi_{K_n}$. This is compactly supported. Also, $f_n := \rho_{1/n} * g_n$ is compactly supported. Hence,

$$\text{supp}(f_n) \subseteq \overline{B(0, 1/n) + K_n} \subseteq \Omega,$$

and $f_n \in C_c^\infty(\Omega)$ for all $n \in \mathbb{N}$. Also,

$$\begin{aligned} ||f_n - f||_{L^p(\Omega)} &= ||f_n - \overline{f}||_{L^p(\mathbb{R}^N)} \\ &\leq ||(\rho_{1/n} * g_n) - (\rho_{1/n} * \overline{f})||_{L^p(\mathbb{R}^N)} + ||(\rho_{1/n} * \overline{f}) - \overline{f}||_{L^p(\mathbb{R}^N)} \text{ (Minkowski and triangle inequality)} \end{aligned}$$

By the linearity of convolution,

$$\leq ||\rho_{1/n} * (g_n - \overline{f})||_{L^p(\mathbb{R}^N)} + ||(\rho_{1/n} * \overline{f}) - \overline{f}||_{L^p(\mathbb{R}^N)}.$$

Apply Young's Inequality to the first term,

$$\leq ||g_n - \overline{f}||_{L^p(\mathbb{R}^N)} + ||(\rho_{1/n} * \overline{f}) - \overline{f}||_{L^p(\mathbb{R}^N)}.$$

Note that $g_n := \overline{f}\chi_{k_n}$, and hence by the dominated convergence theorem,

$$||g_n - \overline{f}||_{L^p(\mathbb{R}^N)} \to 0,$$

and by the last theorem,

$$||(\rho_{1/n} * \overline{f}) - \overline{f}||_{L^p(\mathbb{R}^N)} \to 0.$$

Combining everything together, we get

$$||f_n - f||_{L^p(\Omega)} \to 0.$$

Which proves that smooth functions with compact support are dense in L^p. □

1.7. **Hilbert Spaces.**

Definition 1.78 (Inner Product)**.** An **inner product** over a vector space X is a map $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{R}$ such that:

(1) $\langle \lambda x + \mu y, z \rangle = \lambda \langle x, z \rangle + \mu \langle y, z \rangle$ for all $\lambda, \mu \in \mathbb{R}$.
(2) $\langle y, x \rangle = \langle x, y \rangle$ for all $x, y \in X$.
(3) $\langle x, x \rangle \geq 0$ for all $x \in X$ and $\langle x, x \rangle = 0 \iff x = 0$.

An inner product generates a norm:

$$||x|| := \sqrt{\langle x, x, \rangle}.$$

And we have Cauchy-Schwarz:

$$\boxed{|\langle x, y\rangle| \leq ||x||||y||}$$

Proof of Cauchy-Schwarz:

Proof. Let $z := x - \frac{\langle x,y\rangle}{\langle y,y\rangle} y$. We have that,

$$\begin{aligned} 0 \leq ||z||^2 &= \left\langle x - \frac{\langle x, y\rangle}{\langle x, x\rangle} y, x - \frac{\langle x, y\rangle}{\langle y, y\rangle} \right\rangle \\ &= \langle x, x\rangle - \frac{\langle x, y\rangle^2}{\langle y, y\rangle} - \frac{\langle x, y\rangle^2}{\langle y, y\rangle} + \frac{\langle x, y\rangle^2 \langle y, y\rangle}{\langle y, y\rangle^2} \\ &= ||x||^2 - \frac{\langle x, y\rangle^2}{||y||^2} \end{aligned}$$

And hence,

$$\langle x, y\rangle^2 \leq ||x||^2 ||y||^2 \iff \boxed{.||\langle x, y\rangle|| \leq ||x||||y||}$$

□

Definition 1.79 (Hilbert Space)**.** A **Hilbert Space** $\mathcal{H}$ is a complete inner product space (with respect to the norm induced by the inner product). This satisfies the parallelogram law,

$$\boxed{||u + v||^2 + ||u - v||^2 = 2||u||^2 + 2||v||^2}$$

Examples of Hilbert Spaces you've encountered:

(1) ℓ^2 with the inner product $\langle x, y\rangle = \sum_{i=1}^{\infty} x_i y_i$.
(2) L^2 with inner product $\langle f, g\rangle := \int_\Omega f(x)g(x)dx$.