(13)

7) Let $f \in L^P(\mathbb{R}^N)$ with $1 \leq p < \infty$. For every $r > 0$, set

$$f_r(x) := \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)\,dy$$

a) Prove that $f_r \in L^P(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ and that $f_r(x) \to 0$ as $|x| \to \infty$ (with r being fixed)

b) Prove that $f_r \to f$ in $L^P(\mathbb{R}^N)$ as $r \to 0$.

Hint: Write $f_r = (\varphi_r * f)$ for some appropriate φ.

a) • Claim: $f_r(x) \to 0$ as $|x| \to 0$ (with r fixed).
$C_c(\mathbb{R}^N)$ is dense in $L^P(\mathbb{R}^N)$ and $f_r \in L^P(\mathbb{R}^N)$ (first half of the question) $\Rightarrow \exists$ a sequence of $\{f_n\} \subseteq C_c(\mathbb{R}^N)$ s.t. $\|f_n - f\|_p \to 0$. However, convergence in $L^P \Rightarrow$ convergence in L^1, and so $|f_n - f| \to 0$. Hence,

$$\begin{aligned} |f_r(x)| &= |f_r(x) + f_n(x) - f_n(x)| \\ &= |(f_r(x) - f_n(x)) + f_n(x)| \\ &\leq |f_r(x) - f_n(x)| + |f_n(x)| \end{aligned}$$

Since $f_n \to f_r$, $\exists N_1 \in \mathbb{N}$ s.t. $\forall n \geq N_1$, $|f_r(x) - f_n(x)| < \varepsilon/2$.
$< \varepsilon/2 + |f_n(x)|$.

Since $\lim_{|x| \to 0} |f_n(x)| = 0$, choose x sufficiently large s.t. $|f_n(x)| < \varepsilon/2$,

$$\Rightarrow \lim_{|x| \to \infty} |f_r(x)| \leq \lim_{|x| \to \infty} [\varepsilon/2 + |f_n(x)|]$$
$$\leq \varepsilon/2 + \varepsilon/2 = \varepsilon$$

$$\Rightarrow \boxed{\lim_{|x| \to \infty} |f_r(x)| = 0}$$

• Claim: $f \in L^P(\mathbb{R}^N) \cap C(\mathbb{R}^N)$
• $f_r \in L^P(\mathbb{R}^N)$: follows from Young's Inequality; write f_r as the following convolution.

(14)

$$f_r = \frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)\,dy = \frac{1}{|B(x,r)|}\int_{\mathbb{R}^N} \chi_{B(x,r)}(y) f(y)\,dy$$

shift the ball so that it's centred at zero:

$$= \int_{\mathbb{R}^N} \frac{\chi_{B(0,r)}}{|B(x,r)|}(x-y)\, f(y)\,dy$$

$$= \underbrace{\frac{1}{|B(x,r)|}}_{\text{constant}} (\chi_{B(0,r)} * f)(x)$$

Young's Inequality Says: $f \in L^1,\ g \in L^p \Rightarrow \boxed{\|f*g\|_p \leq \|f\|_1 \|g\|_p}$

- $\chi_{B(0,r)}$ clearly belongs to L^1: $\int_{\mathbb{R}} \chi_{B(0,r)} = m(B(0,r)) < \infty$.
- $f \in L^p$ by assumption

$\Rightarrow (\chi_{B(0,r)} * f) \leq \|\chi_{B(0,r)}\|_1 \|f\|_p < \infty$

$\Rightarrow (\chi_{B(0,r)} * f) \in L^p$

$\Rightarrow \frac{1}{|B(x,r)|}(\chi_{B(0,r)} * f) \in L^p \quad \Rightarrow f_r \in L^p$

- $f_r \in C(\mathbb{R}^N)$: Let $\{x_n\}$ be a convergent sequence in $\mathbb{R}^N$. Applying the DCT, we obtain:

$$\lim_{n\to\infty} f_r(x_n) = \lim_{n\to\infty} \frac{1}{|B(x,r)|}\int_{\mathbb{R}^N} \chi_{B(x_n,r)} f(y)\,dy$$

$$= \frac{1}{|B(x,r)|}\int_{\mathbb{R}^N} \lim_{n\to\infty} \chi_{B(x_n,r)} f(y)\,dy$$

$\forall n \in \mathbb{N}$, $\chi_{B(x_n,r)} f(y) \leq |f(y)| \in L^p(\mathbb{R}^N)$

$$= f_r(x)$$

By the sequential definition of continuity, this proves that f_r is cts

~~b) By the density of $C_c(\mathbb{R}N)$ in $L^p(\mathbb{R}N)$, we can approach $\chi_{B(0,1)}$ by a sequence of mollifiers, call them $\varphi_r(x)$.~~
~~But, first we have that $|B(0,r)| \sim r^n$, and so $\frac{1}{|B(0,r)|} \sim \frac{1}{r^N}$~~
~~After doing a change of variables,~~

~~f_r~~

b) $f \in L^p(\mathbb{R}N) \Rightarrow$ we can approximate it by a sequence $\{g_n\} \subseteq C_c^0(\mathbb{R}N)$. We'll prove the result for $g \in C_c^0(\mathbb{R}N)$ and then generalize it to f by taking the limit.
WLOG, ~~assume that~~ well prove it in the case of $x=0$.

Claim, for g cts,

$$g_r := \frac{1}{|B(0,r)|}\int_{B(0,r)} g(y)\,dy \to g(0) \quad \text{pointwise.}$$

Since pointwise convergence $\Rightarrow$ L^p convergence, this will prove the claim. Since g is continuous at 0, $\forall \varepsilon>0, \exists \delta>0$ s.t. if $x \in B(0,\delta)$, then $|g(x)-g(0)|<\varepsilon$.
Then,

$$\left|\frac{1}{|B(x,r)|}\int_{B(0,r)} g\,dx - g(0)\right| = \left|\frac{1}{|B(x,r)|}\int_{B(0,r)} [g(x)-g(0)]\,dx\right|$$

$$\leq \frac{1}{|B(x,r)|}\int_{B(0,r)} |g(x)-g(0)|\,dx$$

In the limit as $r \to 0$, Since $\delta>0$, at one point $r<\delta$. Hence, for r sufficiently small,

$$\leq \frac{1}{|B(x,r)|}\int_{B(0,r)} \varepsilon\,dx$$

$$= \varepsilon \frac{\cancel{|B(x,r)|}}{\cancel{|B(x,r)|}}$$

$$= \varepsilon$$

$\Rightarrow g_r(x) \to g(x)$ pw $\Rightarrow g_r(x) \to g(x)$ uniformly.

Now for the general case. $C_c^0(\mathbb{R}^N)$ is dense in $L^p(\mathbb{R}^N)$, so approximate f by a sequence of functions g_n. Then,

$$\|f_r - f\|_p = \|f_r + g_r - g_r + g_n - g_n - f\|_p$$
$$= \|(f_r - g_r) + (g_n - g_r) + (g_n - f)\|_p$$

$$\leq \|f_r - g_r\|_p + \|g_n - g_r\| + \|g_n - f\|_p$$

We can make this small, by the Dominated convergence theorem

Just proven

can be made small for sufficiently large n

$$\leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3$$
$$= \varepsilon$$

$$\Rightarrow \boxed{f_r \to f \text{ in } L^p}$$