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| Winter 2020 Semester (Results, Definitions, and Theorems) | Lecture: 09 |

# Chapter 9: Metric Spaces (General Properties)

Class: Math 455 (Analysis 4) Date: 9 February 2020 Shereen Elaidi

**Abstract**

This document contains a summary of all the key definitions, results, and theorems from class. There are probably typos, and so I would be grateful if you brought those to my attention :-).

Syllabus: $L^p$ space, duality, weak convergence, Young, Holder, and Minkowski inequalities, point-set topology, topological space, dense sets, completeness, compactness, connectedness, path-connectedness, separability, Tychnoff theorem, Stone-Weierstrass Theorem, Arzela-Ascoli, Baire category theorem, open mapping theorem, closed graph theorem, uniform boudnedness principle, Hahn Banch theorem.

**This section was not covered in class, but since we have homework on this chapter I figured having this as a review from analysis 2 might be helpful. Also, there are a few terms/results that I don't think we covered in analysis 2.**

## 9.1. Examples of Metric Spaces

**Definition 1** (Metric Space)**.** Let $X$ be a non-empty set. A function $\rho : X \times X \to \mathbb{R}$ is called a **metric** if $\forall\ x, y \in X$:

(i) $\rho(x, y) \geq 0$
(ii) $\rho(x, y) = 0 \iff x = y$
(iii) $\rho(x, y) = \rho(y, x)$
(iv) $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$ (**Triangle Inequality**).

A non-empty set together with a metric, denoted $(X, \rho)$ is called a **metric space**.

**Definition 2** (Discrete Metric)**.** For any non-empty set $X$, the **discrete metric** $\rho$ is defined by setting $\rho(x, y) = 0$ if $x = y$ and $\rho(x, y) = 1$ if $x \neq y$.

**Definition 3** (Metric Subspace)**.** For any metric space $(X, \rho)$, let $Y \subseteq X$ be non-empty. Then, the restriction of $\rho$ to $Y \times Y$ defines a metric on $Y$. We define this induced metric space as a **metric subspace**.

**Example 9.1.1** (Examples of metric spaces)**.** The following are examples of metric spaces:

(i) Every non-empty subset of a Euclidean space.
(ii) $L^p(E)$, where $E \subseteq \mathbb{R}$ is a measurable set.
(iii) $C[a, b]$.

**Definition 4** (Product Metric)**.** For metric spaces $(X_1, \rho_1)$ and $(X_2, \rho_2)$, we define the **product metric** $\tau$ on the cartesian product $X_1 \times X_2$ by setting, for $(x_1, x_2)$ and $(y_1, y_2)$ in $X_1 \times X_2$:

$$\tau((x_1, x_2), (y_1, y_2)) := \{[\rho_1(x_1, x_2)]^2 + [\rho_2(y_1, y_2)]^2\}^{1/2} \tag{9.1}$$

**Definition 5.** Two metrics $\rho$ and $\sigma$ on a set $X$ are said to be **equivalent** if there are positive numbers $c_1$ and $c_2$ such that $\forall\ x_1, x_2 \in X$,

$$c_1\sigma(x_1, x_2) \leq \rho(x_1, x_2) \leq c_2\sigma(x_1, x_2)$$

**Definition 6** (Isometry)**.** A mapping $f : (X, \rho) \to (Y, \sigma)$ between two metric spaces is called an **isometry** provided that $f$ is surjective and $\forall x_1, x_2 \in X$:

$$\sigma(f(x_1), f(x_2)) = \rho(x_1, x_2) \tag{9.2}$$

We say that two metric spaces are **isometric** if there is an isometry from one to another.

## 9.2. Open Sets, Closed Sets, and Convergent Sequences

**Definition 7** (Open Ball)**.** Let $(X, \rho)$ be a metric space. For a point $x \in X$ and $r > 0$, the set:

$$B(x, r) := \left\{x' \in X \mid \rho(x', x) < r\right\} \tag{9.3}$$

is called the **open ball** centred at $x$ of radius $r$. A subset $\mathcal{O} \subseteq X$ is said to be **open** if $\forall x \in \mathcal{O}$, there exists an open ball centred at $x$ and contained in $\mathcal{O}$. For a point $x \in X$, an open set containing $x$ is called a **neighbourhood** of $x$.

**Proposition 1.** Let $X$ be a metric space. The whole set $X$ and the empty set $\emptyset$ are open. The intersection of any two open sets is open. The union of any collection of open sets is open.

**Proposition 2.** Let $X$ be a subspace of a metric space $Y$ and $E \subseteq X$. Then, E is **open in** $X \iff E = X \cap \mathcal{O}$, where $\mathcal{O}$ is open in $Y$.

**Definition 8** (Closure)**.** For a subset $E \subseteq X$, a point $x \in X$ is called a **point of closure** of $E$ provided that every neighbourhood of $x$ contains a point in $E$. The collection of the points of closure of $E$ is called the **closure** of $E$ and is denoted by $\overline{E}$.

**Proposition 3.** For $E \subseteq X$, where $X$ is a metric space, its closure $\overline{E}$ is closed. Moreover, $\overline{E}$ is the smallest closed subset of $X$ containing $E$ in the sense that if $F$ is closed and if $E \subseteq F$, then $\overline{E} \subseteq F$.

**Definition 9** (Converge)**.** A sequence $\{x_n\}$ in a metric space $(X, \rho)$ is said to **converge** to the point $x \in x$ provided that:

$$\lim_{n\to\infty} \rho(x_n, x) = 0$$

that is, $\forall\ \varepsilon > 0$, $\exists$ an index $N$ such that $\forall n \geq N$, $\rho(x_n, x) < \varepsilon$.

**Proposition 4.** Let $\rho$ and $\sigma$ be equivalent metrics on a non-empty set $X$. Then, a subset $X$ is open in a metric space $(X, \rho) \iff$ it is open in $(X, \sigma)$.

## 9.3. Continuous Mappings Between Metric Spaces

**Definition 10** (Continuous)**.** A mapping $f$ from a metric space $X$ to a metric space $Y$ is continuous at the point $x \in X$ if $\forall\ \{x_n\} \in X$, if $\{x_n\} \to x$, then $\{f(x_n)\} \to f(x)$. $f$ is said to be **continuous** if it is continuous at every point in $X$.

**Proposition 5** ($\varepsilon$-$\delta$ criteria for continuity)**.** A mapping from a metric space $(X, \rho)$ to a metric $(Y, \sigma)$ is continuous at the point $x \in X \iff \forall\, \varepsilon > 0, \exists\, \delta > 0$ such that if $\rho(x, x') < \delta$, then $\sigma(f(x), f(x')) < \varepsilon$. That is:

$$f(B(x, \delta)) \subseteq B(f(x), \varepsilon) \tag{9.4}$$

**Proposition 6.** A mapping $f$ from a metric space $X$ to a metric space $Y$ is continuous $\iff \forall$ open subsets $\mathcal{O} \subseteq Y$, the inverse image under $f$ of $\mathcal{O}$, $f^{-1}(\mathcal{O})$, is an open subset of $X$.

**Proposition 7.** The composition of continuous mappings between metric spaces, when defined, is continuous.

**Definition 11** (Uniformly Continuous)**.** A mapping from a metric space $(X, \rho)$ to a metric space $(Y, \sigma)$ is said to be **uniformly continuous** if $\forall\, \varepsilon > 0, \exists\, \delta > 0$ such that $\forall u, v \in X$, if $\rho(u, v) < \delta$, $\sigma(f(u), f(v)) < \varepsilon$.

**Definition 12** (Lipschitz)**.** A mapping $f : (X, \rho) \to (Y, \sigma)$ is said to be **Lipschitz** if $\exists$ a $c \geq 0$ such that $\forall\, u, v \in X$:

$$\sigma(f(u), f(v)) \leq c\rho(u, v)$$

## 9.4. Complete Metric Spaces

**Definition 13** (Cauchy)**.** A sequence $\{x_n\}$ in a metric space $(X, \rho)$ is said to be a **Cauchy sequence** if $\forall \varepsilon > 0$, there exists a $N \in \mathbb{N}$ such that if $m, n \geq N$, then $\rho(x_n, x_m) < \varepsilon$.

**Definition 14** (Complete)**.** A metric space $X$ is said to be **complete** if every Cauchy sequence in $X$ converges to a point in $X$.

**Proposition 8.** Let $[a, b]$ be a closed and bounded interval of real numbers. Then, $C[a, b]$ with the metric induced by the max norm is complete.

**Proposition 9** (Characterisation of Complete Subspaces of Metric Spaces)**.** Let $E \subseteq X$, where $X$ is a complete metric space. Then, the metric subspace $E$ is complete $\iff$ $E$ is a closed subset of $X$.

**Theorem 1.** The following are complete metric spaces:

(i) Every non-empty closed subset of $\mathbb{R}^n$.
(ii) $E \subseteq \mathbb{R}$ measurable, $1 \leq p \leq \infty$, each non-empty closed subset of $L^p(E)$.
(iii) Each non-empty closed subset of $C[a, b]$.

**Definition 15** (Diameter)**.** Let $E$ be a non-empty subset of a metric space $(X, \rho)$. We define the **diameter** of $E$, denoted by $\operatorname{diam}(E)$, by:

$$\operatorname{diam}(E) := \sup \{\rho(x, y) \mid x, y \in E\} \tag{9.5}$$

We say that $E$ is **bounded** if it has finite diameter.

**Definition 16** (Contracting Sequence)**.** A decreasing sequence $\{E_n\}$ of non-empty subsets of $X$ is called a **contracting sequence** if:

$$\lim_{n \to \infty} \operatorname{diam}(E_n) = 0 \tag{9.6}$$

**Theorem 2** (Cantor Intersection Theorem)**.** Let $X$ be a metric space. Then, $X$ is complete $\iff$ whenever $\{F_n\}$ is a contracting sequence of non-empty closed subsets of $X$, there is a point $x \in X$ for which:

$$\bigcap_{n=1}^{\infty} F_n = \{x\} \tag{9.7}$$

**Theorem 3.** Let $(X, \rho)$ be a metric space. Then, there is a complete metric space $(\widetilde{X}, \tilde{\rho})$ for which $X$ is a dense subset of $\widetilde{X}$ and

$$\rho(u, v) = \tilde{\rho}(u, v) \ \forall \ u, v \in X \tag{9.8}$$

we call such a space the **completion** of $(X, \rho)$.

## 9.5. Compact Metric Spaces

**Definition 17** (Compact Metric Space)**.** A metric space $X$ is called **compact** if every open cover of $X$ has a finite sub-cover. A subset $K \subseteq X$ is compact if $K$, considered as a metric subspace of $X$, is compact.

**Formulation of compactness in terms of closed sets:** Let $\mathcal{T}$ be a collection of open subsets of a metric space $X$. Define $\mathcal{F}$ to be the collection of the complements of elements in $\mathcal{T}$. Since the elements of $\mathcal{T}$ are open, the elements of $\mathcal{F}$ are closed. Thus, $\mathcal{T}$ is a cover $\iff$ the elements of $\mathcal{F}$ have *empty intersection*. By deMorgan's law, we can formulate compactness in terms of closed sets as:

> A metric space $X$ is compact $\iff$ every collection of closed sets with empty intersection has a finite sub-collection whose intersection is non-empty.

This property is called the **finite intersection property**.

**Definition 18** (Finite Intersection Property)**.** A collection of sets $\mathcal{F}$ is said to have the **finite intersection property** if any finite sub-collection of $\mathcal{F}$ has a non-empty intersection.

**Proposition 10** (Compactness in terms of closed sets)**.** A metric space $X$ is compact $\iff$ every collection $\mathcal{F}$ of closed subsets of $X$ with the finite intersection property has a non-empty intersection.

**Definition 19** (Totally Bounded)**.** A metric space $X$ is **totally bounded** if $\forall \ \varepsilon > 0$, the space $X$ can be covered by a finite number of open balls of radius $\varepsilon$. A subset $E \subseteq X$ is said to be **totally bounded** if $E$, a s a subspace of the metric space $X$, is totally bounded.

**Definition 20** ($\varepsilon$-net)**.** Let $E$ be a subset of a metric space $X$. A $\varepsilon$**-net** for $R$ is a finite collection of open balls $\{B(x_k, \varepsilon)\}_{k=1}^{n}$ with centres $x_k \in X$ whose union covers $E$.

**Proposition 11.** A metric space $E$ is totally bounded $\iff$ $\forall \ \varepsilon > 0$, there is a finite $\varepsilon$-net for $E$.

**Proposition 12.** A subset of Euclidean space $\mathbb{R}^n$ is bounded $\iff$ it is totally bounded.

**Definition 21** (Sequentially Compact)**.** A metric space $X$ is **sequentially compact** if every sequence in $X$ has a subsequence that converges to a point in $X$.

**Theorem 4** (Characterisation of Compactness for a metric space)**.** . Let $X$ be a metric space. Then, TFAE:

(i) $X$ is complete and totally bounded.

(ii) $X$ is compact.
(iii) $X$ is sequentially compact.

The following three propositions of this chapter are just breaking down these equivalences, so I will not write them.

**Theorem 5.** Let $K \subseteq \mathbb{R}^n$. Then, TFAE:

(i) $K$ is closed and bounded.
(ii) $K$ is compact.
(iii) $K$ is sequentially compact.

**Observe**: The equivalence (1) $\iff$ (2) is the Heine-Borel theorem. The equivalence (2) $\iff$ (3) is the Bolzano-Weierstrass theorem.

**Proposition 13.** Let $f$ be a continuous mapping from a compact metric space $X$ to a compact metric space $Y$. Then, its image $f(X)$ is compact.

**Theorem 6** (Extreme Value Theorem)**.** Let $X$ be a metric space. Then, $X$ is compact $\iff$ every continuous real-valued function on $X$ attains a minimum and maximum value.

**Definition 22** (Lebesgue Number)**.** Let $X$ be a metric space, and let $\{\mathcal{O}_\lambda\}_{\lambda\in\Lambda}$ be an open cover of $X$. Thus, each $x \in X$ is contained in a member of the cover, $\mathcal{O}_\lambda$. Since $\mathcal{O}_\lambda$ is open, $\exists\ \varepsilon > 0$ such that:

$$B(x, \varepsilon) \subseteq \mathcal{O}_\lambda$$

In general, $\varepsilon$ on $X$, but for compact metric spaces we can get *uniform control*. This $\varepsilon$ that uniformly works is called the **Lebesgue number** for the cover $\{\mathcal{O}_\lambda\}_{\lambda\in\Lambda}$.

**Lemma 7.** Let $\{\mathcal{O}_\lambda\}_{\lambda in \Lambda}$ be an open cover of a compact metric space $X$. Then, there is a number $\varepsilon > 0$ such that for each $x \in X$, the open ball $B(x, \varepsilon)$ is contained in some member of the cover.

**Proposition 14.** A continuous mapping from a compact space $(X, \rho)$ to a metric space $(Y, \sigma)$ is uniformly continuous.

## 9.6. Separable Metric Spaces

**Definition 23** (Dense & Separable)**.** A subset $D$ of a metric space $X$ is **dense** in $X$ if every non-empty subset of $X$ contains a point of $D$. A metric space is **separable** if there is a countable subset of $X$ that is dense in $X$.

The **Weierstrass Approximation Theorem** states that polynomials are dense in $C[a, b]$. So, $C[a, b]$ is separable, with the countable dense set being the set of polynomials with rational coefficients.

**Proposition 15.** A compact metric space is separable.

**Proposition 16.** A metric space $X$ is separable $\iff$ there is a countable collection of $\{\mathcal{O}_n\}$ of open subsets of $X$ such that any open subset of $X$ is the union of a sub-collection of $\{\mathcal{O}_n\}$.

**Proposition 17.** Every subspace of a separable metric space is separable.

**Theorem 8.** Each of the following are separable metric spaces:

(i) Every non-empty subset of Euclidean space $\mathbb{R}^n$.
(ii) $1 \leq p < \infty$, $L^p(E)$ and all non-empty subsets of $L^p(E)$.
(iii) Each non-empty subset of $C[a, b]$.