MATH 133 Midterm # 1 Review
Midterm Exam Date: October 7 - October 9

Some study advice:

1. Go through the tutorial questions, and try to re-solve them on your own. Verify with the solutions posted on myCourses, and if you have any doubts visit office hours!
2. Don't try to memorize solutions from the in-class problems, homeworks, or tutorials. Work to understand why.
3. Draw things out! Geometrically seeing things helps you build intuition, and try to practice drawing by hand as calculators are not permitted on exams.
4. To help build intuition, generate examples (and counter-examples) of subspaces, bases, linearly independent and dependent sets, etc. on your own.

0. Preliminaries

- Notation:
  - "$\in$": we use this notation to denote that an element is in a set. E.g.: suppose $A = \{1, 2, 3, 4\}$. Then, we write $1 \in A$.
  - "$\subseteq$": we use this to say that a set is inside another set. E.g.: Suppose $A = \{1, 2, 3, 4\}$. Then we write $\{1, 2\} \subseteq A$.
- Mathematical terminology:
  - A **set** is just a collection of items. We do not demand anything else (no need to span, be closed under anything, etc.).
  - Vectors will be denoted by $\vec{v}$:

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n \text{ and } v_1, v_2, ..., v_n \in \mathbb{R}.$$

1. Linear Systems of Equations

- Suppose I have the system of linear equations:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m &= b_n. \end{aligned}$$

  - The **matrix of coefficients** for the above system is given by:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \ddots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}.$$

  - The **augmented** matrix for the above system is given by:

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1m} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2m} & b_2 \\ \vdots & \ddots & \cdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} & b_n \end{array}\right].$$

- When performing **Gaussian elimination** to solve a system of linear equations, we can use the following three elementary row operations on the augmented matrix:
  1. Exchanging two rows: $R_i \leftrightarrow R_j$.
  2. Multiplying a row by a scalar $k \in \mathbb{R}$: $R_i \to kR_i$.
  3. Adding one row to another: $R_i \to R_i + R_j$.

  The goal is to bring the augmented matrix to **row echelon form (REF)** or **reduced row echelon form (RREF)**:
  1. A matrix is in **row echelon form** if:
     (a) All rows consisting of only zeros are at the bottom.
     (b) The **leading coefficient** of a non-zero row is strictly to the right of the leading coefficient to the row above it.
  2. A matrix is in **reduced row echelon form** if:
     (a) It's in *row echelon form*
     (b) Leading entry in each non-zero row is 1 (called a **leading one**).
     (c) Each column containing a leading one has zeroes in all other entries.

- A system of linear equations can be either **consistent** or **inconsistent**.
  - **Consistent**: there is *at least* one solution (one unique solution or infinitely many solutions). I *cannot* have 2 solutions, 3 solutions, 133 solutions, etc.; only one or infinitely many – why?
  - **Inconsistent**: there are no solutions.

- Variables can be either **free** or **leading** variables.
  - **Leading variables**: variables that correspond to leading ones when in RREF. The other variables are **free variables**.
  - The **rank** of a coefficient matrix is the number of leading ones when the matrix is in RREF.

- **Vector equations of lines**. Suppose I have a line given by the following "vector equation":

$$L(t) = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} + t \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}. \tag{1}$$

  This line is taking the span of the vector $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$, and "shifting" it to the vector $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$. In other words, this is a line that goes through the point (b_1, b_2) with slope $\frac{a_2}{a_1}$. Convince yourself that this is true; don't just memorize this!

2. Span and Linear (In)dependence

- **Linear Combination**: a linear combination of $\vec{v}_1, ..., \vec{v}_n$ is an expression of the form $c_1\vec{v}_1 + ... + c_n\vec{v}_n$ where $c_1, ..., c_n \in \mathbb{R}$.
  - A **linear relation** between $\vec{v}_1, ..., \vec{v}_n$ is an expression of the form:

$$c_1\vec{v}_1 + ... + c_n\vec{v}_n = 0.$$

- The **span** of vectors $\vec{v}_1, ..., \vec{v}_n$, denoted $\text{span}(\vec{v}_1, ..., \vec{v}_n)$, is the set of all possible linear combinations of these vectors. Informally, it's the set of all vectors I can "reach" with $\vec{v}_1, ..., \vec{v}_n$. **Common mistake:** Note that a set, in general, is *not* the same as the span of vectors in the set, i.e.:

$$\{v_1, v_2\} \neq \text{span}(v_1, v_2).$$

- A **homogeneous system** of linear equations is one where all the constant terms are zero. The augmented matrix looks like:

$$\left[\begin{array}{cccc|c} * & * & \cdots & * & 0 \\ * & * & \cdots & * & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & \cdots & * & 0 \end{array}\right].$$

- Vectors $\vec{v}_1, ..., \vec{v}_n \in \mathbb{R}^n$ are **linearly independent** if and only if the only linear relation between them is the trivial one, i.e.:

$$c_1\vec{v}_1 + c_2\vec{v}_2 + ... + c_n\vec{v}_n = 0 \iff c_1 = c_2 = ... = c_n = 0.$$

  Otherwise, the vectors are **linearly dependent**.
- A **subspace** of $\mathbb{R}^n$ is a *non-empty* subset $V \subseteq \mathbb{R}^n$ with the following properties:
  1. "Closed under addition": For *all* $\vec{v}, \vec{w} \in V$, $\vec{v} + \vec{w} \in V$, AND
  2. "Closed under scalar multiplication": For *all* $\vec{v} \in V$ and *all* $k \in \mathbb{R}$, $k \cdot \vec{v} \in V$.
- Facts about subspaces:
  - Note that (2) implies that $\vec{0} \in V$ if V is a subspace. If you are unsure, check it!
  - The span of vectors $\vec{v}_1, ..., \vec{v}_k \in \mathbb{R}^n$ forms a subspace of $\mathbb{R}^n$. If you are unsure, check it!
  - Say I want to know if a set of vectors $\vec{v}_1, ..., \vec{v}_n$ spans a subspace V. Then I take an arbitrary vector $\vec{v}_0 \in V$, and show that the system of equations $c_1\vec{v}_1 + ... + c_n\vec{v}_n = \vec{v}_0$ is *consistent*. If you are unsure why this works, check it!
- The vectors $\vec{v}_1, ..., \vec{v}_k \in V$ are said to be a **basis** of V if they:
  1. Span V AND
  2. are linearly independent.

  In other words, a basis for V is the *most efficient* way to span all of V.
  - The **dimension** of the set V is the number of vectors in a basis of V.
  - I need *at least* N vectors to span an N-dimensional space V. **Common mistake:** A common mistake in tutorials I've seen is students understanding why the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

    cannot span all of $\mathbb{R}^3$, but incorrectly thinking that if I had some non-zero entry in the 3rd row, i.e,:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ c_1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ c_2 \end{bmatrix}, \; c_1, c_2 \neq 0.$$

    then I can span all of $\mathbb{R}^3$ since the z-component is non-zero. This is totally false, since I am not given complete freedom with my c_1, c_2 since they must change with the other two components of the vector, which will still constrain me to a plane.
  - The **standard basis vectors of** $\mathbb{R}^n$ are written as $\vec{e}_i$ where $i = 1, 2, ..., n$. They are the vectors of zeroes with a 1 in the ith row:

$$\mathbb{R}^2: \; \vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\mathbb{R}^3: \vec{e}_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \; \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \; \vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

3. Linear Transformations

- An $m \times n$ matrix is a matrix with m rows and n columns.
- There are two ways to view matrix multiplication: the **column view** and the **row view**.
  1. **Column View**: the product of a matrix and a vector is a linear combination of the columns of the matrix. Let $\vec{c}_i$ denote the ith column of the matrix A.

$$A \cdot \vec{x} = \begin{bmatrix} \vec{c}_1 & \vec{c}_2 & \dots & \vec{c}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \vec{c}_1 \cdot x_1 + \vec{c}_2 \cdot x_2 + ... + \vec{c}_n \cdot x_n. \tag{2}$$

  2. **Row view**: the product of a matrix and a vector is the dot product of the rows with the vector.
     - **Dot product**: takes as input two vectors $\vec{v}_1$ and $\vec{v}_2$ and gives us as an output a scalar $\vec{v}_1 \cdot \vec{v}_2 \in \mathbb{R}$:

$$\vec{v}_1 \cdot \vec{v}_2 = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} := u_1 v_1 + u_2 v_2 + ... + u_n v_n.$$

  So the row view is given by:

$$A\vec{x} = \begin{bmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vdots \\ \vec{r}_n \end{bmatrix} \vec{x} = \begin{bmatrix} \vec{r}_1 \cdot \vec{x} \\ \vec{r}_2 \cdot \vec{x} \\ \vdots \\ \vec{r}_n \cdot \vec{x} \end{bmatrix}$$

- **Linear Transformation**: $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if:
  1. "T preserves vector addition": for every $\vec{x}, \vec{y} \in \mathbb{R}^n$, $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$.
  2. "T preserves scalar multiplication": for every $\vec{x} \in \mathbb{R}^n$, $c \in \mathbb{R}$, $T(c\vec{x}) = cT(\vec{x})$.

  $T : \mathbb{R}^{n} \to \mathbb{R}^{m}$ is a linear transformation is equivalent to T being induced by an $m \times n$ matrix A.