CALCULUS: SINGLE VARIABLE, MULTIVARIABLE, DIFFERENTIAL EQUATIONS, AND VECTOR CALCULUS SUMMARY

SHEREEN ELAIDI

Abstract. The purpose of this document is to review Calculus. The content here should be equivalent to Math 140, Math 141, Math 222, and Math 248/358 at McGill.

Contents

1. Single Variable Calculus

1.1. Limits and Derivatives.

- **Precise Definition of a Limit**: Let f be a function defined on an open interval $]a, c[$ that contains the number a. Then, we say that the limit of $f(x)$ as x approaches a is L, and we write $\lim_{x\to a} f(x) = L$ if for every $\varepsilon > 0$, $\exists$ a $\delta > 0$ such that $0 < |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon$.

Date: 8 June 2020.

- Heuristically, this means that if any small interval $]L-\varepsilon, L+\varepsilon[$ is given around L, then we can find an interval $]a-\delta, a+\delta[$ around a such that f maps the points in $]a-\delta, a+\delta[$ (except possibly a) into the interval $]L-\varepsilon, L+\varepsilon[$.

- **Continuous**: A function f is said to be continuous at a number $a \in \mathbb{R}$ if $\lim_{x\to a} f(x) = f(a)$.
- **Intermediate Value Theorem**: Let f be continuous on the interval $[a,b]$ and let N be any number between $f(a)$ and $f(b)$ where $f(a) \neq f(b)$. Then, there exists a number $c \in]a,b[$ for which $f(c) = N$.
- **Tangent Line**: The tangent line to the curve $y = f(x)$ at the point $P = (a, f(a))$ is the line through P with the slope

$$m = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \iff m = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \tag{1}$$

- **Velocity / Instantaneous Velocity**: the instantaneous velocity $v(a)$ at the time $t = a$ is the limit of the average velocities:

$$v(a) := \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \tag{2}$$

- **Derivative**: The **derivative** of a function f at a number $a \in \mathbb{R}$, denoted by $f'(a)$, is

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = f'(a) \tag{3}$$

1.2. **Differentiation Rules.**

- Derivative of a constant function: $\frac{d}{dx}[c] = 0$.
- **Power Rule**: if $n \in \mathbb{R}$, $\frac{d}{dx}[x^n] = nx^{n-1}$. One can prove this using geometric series.
- **Constant Multiple Rule**: if $c \in \mathbb{R}$ and f differentiable, then $\frac{d}{dx}[cf(x)] = cf'(x)$.
- **Constant Sum Rule**: if f, g are differentiable, then $\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$.
- The rate of change of any exponential function is proportional to the function itself: for $f(x) := b^x$:

$$f'(x) = f'(0)b^x \tag{4}$$

- **Derivative of the Natural Exponential Function**:

$$\frac{d}{dx}[e^x] = e^x \tag{5}$$

- **Product Rule**: if f, g are differentiable, then:

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)] \tag{6}$$

- **Quotient Rule**: If f, g are differentiable, then:

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2} \tag{7}$$

- **Derivatives of Trigonometric Functions**:
  - $\frac{d}{dx}[\sin(x)] = \cos(x)$, $\frac{d}{dx}[\csc(x)] = -\csc(x)\cot(x)$
  - $\frac{d}{dx}[\cos(x)] = -\sin(x)$, $\frac{d}{dx}[\sec(x)] = \sec(x)\tan(x)$
  - $\frac{d}{dx}[\tan(x)] = \sec^2(x)$, $\frac{d}{dx}[\cot(x)] = -\csc^2(x)$.
- **Chain Rule**: If g is differentiable at x and if f is differentiable at $g(x)$, then the composite function $F := f \circ g$ defined by $F(x) := f(g(x))$ is differentiable at x and F' is given by the product:

$$F'(x) = f'(g(x)) \cdot g'(x) \tag{8}$$

or, in Leibnitz notation,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \tag{9}$$

- **Method of Implicit Differentiation**: Differentiating both sides of the equation with respect to x, and then solving the resulting equation for y'.
  - Application: finding the derivatives of inverse trigonometric functions:
    - $\frac{d}{dx}\left[\arcsin(x)\right] = \frac{1}{\sqrt{1-x^2}}$, $\frac{d}{dx}\left[\operatorname{arcsec}(x)\right] = \frac{-1}{x\sqrt{x^2-1}}$
    - $\frac{d}{dx}\left[\arccos(x)\right] = \frac{-1}{\sqrt{1-x^2}}$, $\frac{d}{dx}\left[\operatorname{arcsec}(x)\right] = \frac{1}{x\sqrt{x^2-1}}$
    - $\frac{d}{dx}\left[\arctan(x)\right] = \frac{1}{1+x^2}$, $\frac{d}{dx}\left[\operatorname{arccot}(x)\right] = \frac{-1}{x^2+1}$
  - Application: derivatives of logarithmic functions, $y = \log_b(x)$ and $y = \ln(x)$.
    - $\frac{d}{dx}\left[\log_b(x)\right] = \frac{1}{x\ln(b)}$
    - $\frac{d}{dx}\left[\ln(x)\right] = \frac{1}{x}$
  - **Method of Logarithmic Differentiation**: the calculation of complex functions involving products, quotients, or powers can be simplified by taking logarithms.
- **Hyperbolic Trigonometric Functions**: hyperbolic functions ~ hyperbola like trigonometric functions ~ circle. They are defined as:
  - $\sinh(x) := \frac{e^x-e^{-x}}{2}$, $\operatorname{csch}(x) := \frac{1}{\sinh(x)}$
  - $\cosh(x) := \frac{e^x+e^{-x}}{2}$, $\operatorname{sech}(x) := \frac{1}{\cosh(x)}$
  - $\tanh(x) := \frac{\sinh(x)}{\cosh(x)}$, $\coth(x) := \frac{\cosh(x)}{\sinh(x)}$
  - Applications: whenever an entity such as light, velocity, electricity, or radioactivity is gradually absorbed or extinguished.
  - Hyperbolic identities:
    - $\sinh(-x) = -\sinh(x)$, $\cosh(-x) = \cosh(x)$
    - $\cosh^2(x) - \sinh^2(x) = 1$, $1 - \tanh^2(x) = \operatorname{sech}^2(x)$
    - $\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y)$
    - $\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$
  - Derivatives of Hyperbolic Functions:
    - $\frac{d}{dx}\left[\sinh(x)\right] = \cosh(x)$, $\frac{d}{dx}\left[\operatorname{csch}(x)\right] = -\operatorname{csch}(x)\coth(x)$,
    - $\frac{d}{dx}\left[\cosh(x)\right] = \sinh(x)$, $\frac{d}{dx}\left[\operatorname{sech}(x)\right] = -\operatorname{sech}(x)\tanh(x)$,
    - $\frac{d}{dx}\left[\tanh(x)\right] = \operatorname{sech}^2(x)$, $\frac{d}{dx}\left[\coth(x)\right] = -\operatorname{csch}^2(x)$.
  - **Inverse Hyperbolic Functions**:
    - $\operatorname{arcsinh}(x) := \ln(x + \sqrt{x^2+1})$ for $x \in \mathbb{R}$.
    - $\operatorname{arccosh}(x) := \ln(x + \sqrt{x^2-1})$ for $x \geq 1$.
    - $\operatorname{arctanh}(x) := \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ for $x \in [-1,1]$
  - Derivatives of Inverse Hyperbolic Functions:
    - $\frac{d}{dx}\left[\operatorname{arcsinh}(x)\right] = \frac{1}{\sqrt{1+x^2}}$, $\frac{d}{dx}\left[\operatorname{arccsch}(x)\right] = \frac{-1}{|x|\sqrt{x^2+1}}$
    - $\frac{d}{dx}\left[\operatorname{arccosh}(x)\right] = \frac{1}{\sqrt{x^2-1}}$, $\frac{d}{dx}\left[\operatorname{arcsech}(x)\right] = \frac{-1}{x\sqrt{1-x^2}}$
    - $\frac{d}{dx}\left[\operatorname{arctanh}(x)\right] = \frac{1}{1-x^2}$, $\frac{d}{dx}\left[\operatorname{arccoth}(x)\right] = \frac{1}{1-x^2}$

1.3. Applications of Differentiation.

- **Extreme Value Theorem**: Let f be continuous on the closed and bounded interval $[a,b]$. Then, f attains an absolute maximum value $f(x)$ and an absolute minimum value $f(d)$ at some numbers $c, d \in [a,b]$.
- **Fermat's Theorem**: If f has a local maximum or minimum at c, and if $f'9x)$ exists, then $f'(x) = 0$.
- **Closed Interval Method**: To find the absolute maximum and minimum values of a continuous function f on a closed interval $[a,b]$,
  (1) Find the values of f at the critical points of f in the open interval $]a,b[$.
  (2) Compute $f(a)$ and $f(b)$.
  (3) The max between (1) and (2) is the absolute max; the min between (1) and (2) is the absolute min.
- **Rolle's Theorem**: Let $f : [a,b] \to \mathbb{R}$ satisfy:

(1) f is continuous on $[a, b]$
(2) f is differentiable on $]a, b[$
(3) $f(a) = f(b)$.
Then, there exists a number $c \in]a, b[$ such that $f'(c) = 0$.

- **Mean Value Theorem**: Let $f : [a, b] \to \mathbb{R}$ satisfy:
  (1) f is continuous on $[a, b]$
  (2) f is differentiable on $]a, b[$
  Then, there exists a number $c \in]a, b[$ such that
  $$f'(c) = \frac{f(b) - f(a)}{b - a} \iff f(b) - f(a) = f'(c)[b - a] \tag{10}$$
- **Theorem** (Consequence of MVT): If $f'(x) = 0\ \forall x \in]a, b[$, then f is constant on $]a, b[$.
  - **Corollary**: If $f'(x) = g'(x)\ \forall x \in]a, b[$, then $f - g$ is constant on $]a, b[$, i.e., $\exists c \in \mathbb{R}$ such that $f(x) = g(x) + c$.
- **L'Hopital's Rule**: Suppose f and g are differentiable and that $g(x) \neq 0$ on an open interval containing a. Suppose that
  $$\lim_{x\to a} f(x) = 0 \text{ and } \lim_{x\to a} g(x) = 0 \tag{11}$$
  or
  $$\lim_{x\to a} f(x) = \pm\infty \text{ and } \lim_{x\to a} f(x) = \pm\infty \tag{12}$$
  then
  $$\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \lim_{x\to a}\left(\frac{f'(x)}{g'(x)}\right) \tag{13}$$
- **Antiderivative**: A function F is called an **anti-derivative** of f on an interval I if $F'(x) = f(x)$ $\forall x \in I$.

1.4. Integrals.

- **Area**: The **area** A of a region S that lies under the graph of a continuous function f is the limit of the sum of the approximating rectangles
  $$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} [f(x_1)\Delta x + ... + f(x_n)\Delta x] \tag{14}$$
- **Definite Integral**: Let $f : [a, b] \to \mathbb{R}$. Divide $[a, b]$ into n subintervals of equal width $\Delta x = \frac{b-a}{n}$. Let $a = x_0 < x_1 < ... < x_n = b$ be the endpoints and let $x_1^*, ..., x_n^*$ be any sample points in these subintervals such that $x_i^* \in [x_{i-1}, x_i]$. Then, the definite integral of f from a to b is:
  $$\int_a^b f(x)dx := \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x \tag{15}$$
  provided that the limit exists and is the same for all possible choices of sample points. If it does exist, then we say that f is **integrable** on $[a, b]$.
- Formulae for the sums of positive integers:
  - $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
  - $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
  - $\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$
- **Fundamental Theorem of Calculus** connects differential calculus and integral calculus. Deals with equations of the form
  $$g(x) = \int_a^x f(t)dt \tag{16}$$

- **Fundamental Theorem of Calculus Part 1**: let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$. Then, the function g defined by

$$g(x) := \int_a^x f(t)dt \tag{17}$$

is continuous on $[a,b]$ and differentiable on $]a,b[$. Moreover, $g'(x) = f(x)$.
- **Fundamental Theorem of Calculus Part 2**: If f is continuous on $[a,b]$, then

$$\int_a^b f(x)dx = F(b) - F(a) \tag{18}$$

where F is any anti-derivative of f.
- Alternative expression for the FoC Part 1:

$$\frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x) \tag{19}$$

- **Table of Integration Formulae**:
  - $\int x^n dx = \frac{x^{n+1}}{n+1}$ for $n \neq -1$.
  - $\int e^x dx = e^x$
  - $\int \sin(x)dx = -\cos(x)$
  - $\int \sec^2(x)dx = \tan(x)$
  - $\int \sec(x)\tan(x)dx = \sec(x)$
  - $\int \sec(x)dx = \ln|\sec(x) + \tan(x)|$
  - $\int \tan(x)dx = \ln|\sec(x)|$
  - $\int \sinh(x)dx = \cosh(x)$
  - $\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right)$
  - $\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|$
  - $\int \frac{1}{x}dx = \ln|x|$
  - $\int b^x dx = \frac{b^x}{\ln b}$
  - $\int \cos(x)dx = \sin(x)$
  - $\int \csc^2(x)dx = -\cot(x)$
  - $\int \csc(x)\cot(x)dx = -\csc(x)$
  - $\int \csc(x)dx = \ln|\csc(x) - \cot(x)|$
  - $\int \cot(x)dx = \ln|\sin(x)|$
  - $\int \cosh(x)dx = \sinh(x)$
  - $\int \frac{1}{\sqrt{a^2-x^2}}dx = \arcsin\left(\frac{x}{a}\right)$ $a > 0$
  - $\int \frac{1}{\sqrt{x^2 \pm a^2}}dx = \ln|x + \sqrt{x^2 \pm a^2}|$

1.5. **Applications of Integration.**

- The **average value** of f on the interval $[a,b]$ is:

$$f_{avg} := \frac{1}{b-a}\int_a^b f(x)dx \tag{20}$$

- **Mean Value Theorem for Integrals**: If f is continuous on $[a,b]$ then there exists a $c \in [a,b]$ such that

$$f(x) = f_{avg} = \frac{1}{b-a}\int_a^b f(x)dx \iff \int_a^b f(x)dx = f(c)(b-a) \tag{21}$$

1.6. **Integration Techniques.**

- **Integration by Parts**:

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx \tag{22}$$

- **Trigonometric Integrals**:
  (1) Strategy for evaluating $\int \sin^m(x)\cos^n(x)dx$:
    (a) If n is odd: save one cosine, use $\cos^2(x) = 1 - \sin^2(x)$ to express the remaining factors in terms of sine:

$$\int \sin^m(x)\cos^{2k+1}(x)dx = \int \sin^m(x)(1-\sin^2(x))^k\cos(x)dx \tag{23}$$

    and make the substitution $u = \sin(x)$.
    (b) If m is odd: save one sine, use $\sin^2(x) = 1 - \cos^2(x)$ to express the remaining factors in terms of cosine:

$$\int \sin^{2k+1}(x)\cos^n(x)dx = \int (1-\cos^2(x))^k\cos^n(x)\sin(x)dx \tag{24}$$

    and make the substitution $u = \cos(x)$.

(c) If sine and cosine are even, then use the half-angle identities:

$$\sin^2(x) = \frac{1}{2}(1-\cos(2x)) \text{ and } \cos^2(x) = \frac{1}{2}(1+\cos(2x)) \tag{25}$$

A helpful identity is $\sin(x)\cos(x) = \frac{1}{2}\sin(2x)$.

(2) Strategy for evaluating $\int \tan^m(x)\sec^n(x)dx$:

(a) If n is even: save one secant squared, use the identity $\sec^2(x) = 1+\tan^2(x)$ to express the remaining factors in terms of $\tan(x)$:

$$\int \tan^m(x)\sec^{2k}(x)dx = \int \tan^m(x)(1+\tan^2(x))^{k-1}\sec^2(x)dx \tag{26}$$

and make the substitution $u = \tan(x)$.

(b) If m is odd: save one $\sec(x)\tan(x)$, use $\tan^2(x) = \sec^2(x) - 1$ to express the remaining factors in terms of $\sec(x)$:

$$\int \tan^{2k+1}(x)\sec^n(x)dx = \int (\sec^2(x)-1)^k \sec^{n-1}(x)\sec(x)\tan(x)dx \tag{27}$$

substitute $u = \sec(x)$.

(3) Important product identities to remember:

(a) $\sin(A)\cos(B) = \frac{1}{2}[\sin(A-B)+\sin(A+B)]$

(b) $\sin(A)\sin(B) = \frac{1}{2}[\cos(A-B)+\cos(A+B)]$

(c) $\cos(A)\cos(B) = \frac{1}{2}[\cos(A-B)+\cos(A+B)]$

- **Trigonometric Substitution**
  - $\sqrt{a^2-x^2} \to x = a\sin(\theta)$, $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to 1-\sin^2(\theta) = \cos^2(\theta)$.
  - $\sqrt{a^2+x^2} \to x = a\tan(\theta)$, $\theta \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\to 1+\tan^2(\theta) = \sec^2(\theta)$.
  - $\sqrt{x^2-a^2} \to x = a\sec(\theta)$, $\theta \in \left[0, \frac{\pi}{2}\right[\cup \left[\pi, \frac{3\pi}{2}\right[\to \sec^2(\theta) - 1 = \tan^2(\theta)$.
- **Partial Fractions**:

(1) Case I: Denominator $Q(x)$ is a product of distinct linear factors:

$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \ldots + \frac{A_k}{a_kx+b_k} \tag{28}$$

(2) Case II: Denominator $Q(x)$ is a product of linear factors, some of which are repeated r times:

$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \ldots + \frac{A_k}{(a_1x+b_1)^r} \tag{29}$$

(3) Case III: $Q(x)$ contains irreducible quadratic factors, none of which is repeated. Then, expression will have a term of the form

$$\frac{Ax+B}{ax^2+bx+c} \tag{30}$$

which can be integrated by completing the square and using the formula:

$$\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C \tag{31}$$

(4) Case IV: $Q(x)$ contains a repeated irreducible factor. Then, the expression will have a term of the form:

$$\frac{A_1x+B_1}{ax^2+bx+c} + \ldots. + \frac{A_rx+B_r}{(ax^2+bx+c)^r} \tag{32}$$

- General Strategy for Integrating:

(1) Simplify the integrand if possible using algebraic manipulation and trigonometric identities.

(2) Look for obvious substitutions.

(3) Classify integrand according to its form

(a) Trigonometric functions

(b) Rational functions ($\to$ partial fractions)

(c) Integration by parts
(d) Radicals
 (i) $\sqrt{\pm x^2 \pm a^2} \to$ trigonmetric substitution
 (ii) $(ax+b)^{1/n} \to$ rationalising substitution $u = (ax+b)^{1/n}$

- **Improper Integral**: if in the definite integral, $\int_a^n f(x)dx$, either $[a,b]$ is an unbounded interval or $f(x)$ has an infinite discontinuity in $[a,b]$

1.7. **Further Applications of Integration.**

- **Arc-length formula**: If f' is continuous on $[a,b]$, then the length of the curve $y = f(x)$, $a \le x \le b$ is:

$$L = \int_a^b \sqrt{1 + [f'(x)]^2} \tag{33}$$

- **Arc-Length Function**: If a smooth curve C has the equation $y = f(x)$, $a \le x \le b$, let $s(x)$ be the distance along C from the initial point $P_0(a, f(a))$ to the point $Q(x, f(x))$. Then, s is a function given by:

$$s(x) = \int_a^x \sqrt{1 + [f'(t)]^2} dt \tag{34}$$

1.8. **Parameter Equations and Polar Coordinates. Motivation:** some curves are best handeled when both x and y are given as a function of a third variable t: $x = f(t)$, $y = g(t)$.

- Suppose f, g are differentiable functions and suppose we want to find the tangent line at a point on the parametric curve $x = f(t)$, $y = g(t)$, where y is also a differentiable function of x. If $\frac{dx}{dt} \neq 0$, then the slope of the parametric curve is given by:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \tag{35}$$

 - We can consider $\frac{d^2y}{d^2x}$:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} \tag{36}$$

 - **Areas**: if a curve is traced out by the parametric equations $x = f(t)$ and $y = g(t)$ for $t \in [\alpha, \beta]$, then using the substitution rule for integrals one has the following formula:

$$\mathcal{A} = \int_a^b y dx = \int_\alpha^\beta g(t) f'(t) dt \tag{37}$$

 - **Arc Length** if a curve C is described by parametric equations $x = f(t)$, $y = g(t)$, $\alpha \le t \le \beta$, where f' and g' are continuous on $[a,b]$ and C is traversed exactly once as t travels from α to β, then the length of C is:

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \tag{38}$$

 - **Surface Area**: similar to the conditions in the previous theorem, the surface area of a curve obtained by rotating it about the x-axis is given by:

$$S = \int_\alpha^\beta 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \tag{39}$$

- Equations to convert between cartesian and polar coordinates:

$$x = r\cos(\theta) \;\; y = r\sin(\theta) \tag{40}$$

$$r^2 = x^2 + y^2 \;\; \tan(\theta) = \frac{y}{x} \tag{41}$$

- If $r = f(\theta)$ is a polar curve, then we can find the tangent line to a polar curve by regarding θ as a parameter:

$$x = f(\theta)\cos(\theta)$$
$$y = f(\theta)\sin(\theta)$$

and the tangent is given by:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin(\theta) + r\cos(\theta)}{\frac{dr}{d\theta}\cos(\theta) - r\sin(\theta)} \tag{42}$$

- Areas and lengths in polar coordinates
  - The area of a sector of a circle: $A = \frac{1}{2}r^2\theta$. The area of a polar region $\mathcal{R}$:

$$A(\mathcal{R}) = \int_a^b \frac{1}{2}[f(\theta)]^2 d\theta \tag{43}$$

  - The arc-length of a polar curve with the equation $r = f(\theta)$, $a \le \theta \le b$ is:

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta \tag{44}$$

1.9. **Infinite Sequences and Series.**

- A **Sequence** is a list of numbers written in a definite order

$$a_1, a_2, a_3, ... \tag{45}$$

- A sequence has a **limit** L and we write

$$\lim_{n\to\infty} a_n = L$$

if we can make the terms a_n as close to L as we'd like by taking n sufficiently large. if $\lim_{n\to\infty} a_n$ exists, then we say that $\{a_n\}$ is **convergent**. Else, it is **divergent**.
  - A sequence $\{a_n\}$ has a limit L if $\forall\ \varepsilon > 0$, $\exists$ an $N \in \mathbb{N}$ such that $\forall n \ge N$, one has that $|a_n - L| < \varepsilon$.
- **Squeeze Theorem**: if $a_n \le b_n \le c_n$ $\forall n \ge n_0$, and if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.
  - If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.
  - If $\lim_{n\to\infty} a_n = L$ and if f is a continuous function at L, then

$$\lim_{n\to\infty} f(a_n) = f(L)$$

- The sequence $\{r^n\}$ is convergent if $r \in]-1,1]$ and divergent for all other values of r. If $r \in]-1,1]$:

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } r \in]-1,1[\\ 1 & \text{if } r = 1 \end{cases}$$

- **Monotonic Sequence Theorem**: every bounded, monotonic sequence converges.
- **Series**: Given a series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + ...$$

let s_n denote the nth partial sum:

$$s_n := \sum_{i=1}^{n} a_i = a_1 + ... + a_n$$

if the *sequence* $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = s \in \mathbb{R}$, then the series $\sum_{n=1}^{\infty} a_n$ is convergent.
- **Geometric Series**: is an important example of an infinite series.

$$a + ar + ar^2 + ... + ar^{n-1} + ... = \sum_{n=1}^{\infty} ar^{n-1}\ (a \ne 0) \tag{46}$$

- If $r = 1$, then $s_n = na \to \infty$.
- If $r \neq 1$, then $s_n = \frac{a(1-r^n)}{1-r}$. If $r \in]-1,1[$, then $r^n \to 0$ as $n \to \infty$, and so

$$\lim_{n\to\infty} s_n = \frac{a}{1-r} \tag{47}$$

otherwise the geometric series diverges.

- **Harmonic Series** is defined as $\sum_{n=1}^{\infty} \frac{1}{n}$. It's divergent.
- If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.
  - **Test for Divergence**: if $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
- **Integral Test**: Suppose f is a continuous, positive, decreasing function on $[1,\infty[$ and let $a_n = f(n)$. Then, the series $\sum_{n=1}^{\infty} a_n$ is convergent $\iff$ the improper integral $\int_1^{\infty} f(x)dx$ is convergent.
  - **p-series**: the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges $\iff$ $p > 1$.
- **Remainder Estimate for the Integral Test**: Suppose that $f(k) = a_k$, where f is a continuous, decreasing, positive function for $x \geq n$ and suppose that $\sum a_n$ is convergent. If $R_n := S - S_n$, then

$$\int_{n+1}^{\infty} f(x)dx \leq R_n \leq \int_n^{\infty} f(x)dx \tag{48}$$

- **Comparison Test**: Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.
  (1) If $\sum b_n$ is convergent and $a_n \leq b_n \ \forall n$, then $\sum a_n$ converges.
  (2) If $\sum b_n$ is divergent and $b_n \leq a_n \ \forall n$, then $\sum a_n$ diverges.
- **Limit Comparison Test**: Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where $c \in]0,\infty[$, then both series have the same behaviour; i,e, either both series converge or both diverge.

- **Alternating Series Test**: If the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ for $b_n > 0$ satisfies:
  (1) $b_{n+1} \leq b_n \ \forall n \in \mathbb{N}$
  (2) $\lim_{n\to\infty} b_n = 0$
  then, the series converges.
- **Absolutely Convergent**: A series $\sum a_n$ is called **absolutely convergent** if the series of absolute values $\sum |a_n|$ is convergent.
- **Conditionally Convergent**: A series $\sum a_n$ is called **conditionally convergent** if its convergent but not absolutely convergent.
- If $\sum a_n$ is absolutely convergent then it is convergent.
- **Ratio Test** Let $\{a_n\}$ be a sequence.
  (1) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then $\sum_{n=1}^{\infty} a_n$ absolutely converges (and thus converges).
  (2) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| - L = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.
  (3) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L = 1$ then the Ratio test is inconclusive.
- **Root Test** Let $\{a_n\}$ be a sequence.
  (1) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series absolutely converges (and therefore converges).
  (2) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or if $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$ then the series diverges.
  (3) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L = 1$, then the root test will be inconclusive.

1.10. Strategy for Testing Series.

(1) If the series is of the form $\sum \frac{1}{n^p}$, then apply the p-series rule.
(2) If the series is of the form $\sum ar^{n-1}$ or $\sum ar^n$, then apply the geometric series rule.
(3) If the series is similar to a p-series or a geometric series, then use a comparison test.
(4) If $\lim_{n\to\infty} a_n \neq 0$, use the divergence test to conclude that the series diverges.
(5) If the series is of the form $\sum(-1)^{n-1} b_n$ or $\sum(-1)^n b_n$, then use the alternating series test.
(6) If the series has factorials in it, consider applying a ratio test.

(7) If the series is of the form $(b_n)^n$, then consider the root test.
(8) If $a_n = f(n)$ where $\int_1^\infty f(x)dx$ is easily evaluated, then consider the integral test.

1.11. Power Series.

- **Power Series**: a power series is of the form
$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \ldots \tag{49}$$
A series of the form
$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 \tag{50}$$
is called a **power series in** $(x-a)$
- **Theorem**: for a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, there are only three possibilities:
  (1) The series converges only when $x = a$.
  (2) The series converges for all x.
  (3) $\exists$ an $R > 0$ such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.
- **Theorem (Term-by-term Differentiation and Integration)**: If the power series $\sum c_n(x-a)^n$ has a radius of convergence $R > 0$, then the function defined by:
$$f(x) := c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots = \sum_{n=0}^{\infty} c_n(x-a)^n \tag{51}$$
is differentiable (and thus continuous) on the interval $]a-R, a+R[$ and:
  (1) $f'(x) = c_0 + 2c_s(x-a) + 3c_3(x-a)^2 + \ldots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$.
  (2) $\int f(x)dx = c + c_0(x-a) + c_a\frac{(c-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \ldots.$
$$\int f(x)dx = c + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} \tag{52}$$
The radius of convergence of both (1) and (2) remain R.
- **Theorem (Taylor Series Representation)**: If f has a power series representation (expansion) at a, that is, if
$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n \tag{53}$$
for $|x-a| < R$, then the coefficients are given by the formula:
$$c_n = \frac{f^{(n)}(a)}{n!} \tag{54}$$
Then, the **Taylor Series** for f about a is given by:
$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \tag{55}$$
For the special case of $a = 0$, then the above becomes:
$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \ldots \tag{56}$$
- **Theorem (Remainder)**: Define the remainder of the Taylor series by $R_n := f(x) - T_n(x)$. If $f(x) = T_n(x) + R_n(x)$, where T_n is the nth degree Taylor polynomial of f at a and
$$\lim_{n\to\infty} R_n(x) = 0 \tag{57}$$
for $|x-a| < R$, then f is equal to the sum of its Taylor series on the interval $|x-a| < R$.
- The following theorem is often used when trying to show that $\lim_{n\to\infty} R_n = 0$ for a specific function f:

- **Taylor's Inequality**: If $|f^{(n+1)}(x)| \leq M \ \forall \ |x-a| \leq d$, then the remainder $R_n(x)$ of the Taylor series satisfies the following inequality:

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \tag{58}$$

for $|x-a| \leq d$.

- **Important MacLaurin Series and their Radii of Convergence**:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + ... \ (R = 1) \tag{59}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ... \ (R = \infty) \tag{60}$$

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} ... \ (R = \infty) \tag{61}$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ... \ (R = \infty) \tag{62}$$

$$\arctan(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)} = 1 - \frac{x^3}{3} + \frac{x^5}{5!} - \frac{x^7}{7!} + ... \ (R = 1) \tag{63}$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ... \ (R = 1) \tag{64}$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^3 + \frac{k(k-1)(k-2)}{3!}x^3 + ... \ (R = 1) \tag{65}$$

2. Multi Variable Calculus

2.1. Vectors and Geometry of Space.

- If θ is the angle between vectors $\mathbf{a}$ and $\mathbf{b}$, then $\mathbf{a} \cdot \mathbf{b} = ||\mathbf{a}|| ||\mathbf{b}|| \cos(\theta)$
  - Two vectors are orthogonal $\iff$ $\mathbf{a} \cdot \mathbf{b} = 0$.
  - **Scalar Projection of b onto a**:

$$\text{comp}_{\mathbf{a}}(\mathbf{b}) : \frac{\mathbf{a} \cdot \mathbf{b}}{||\mathbf{a}||} \tag{66}$$

  - **Vector projection of b onto a**:

$$\text{proj}_{\mathbf{a}}(\mathbf{b}) := \left(\frac{\mathbf{a} \cdot \mathbf{b}}{||\mathbf{a}||} \right) \frac{\mathbf{a}}{||\mathbf{a}||} \tag{67}$$

- **Cross Product**: if $\mathbf{a} := \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} := \langle b_1, b_2, b_3 \rangle$, then their **cross product** is:

$$\mathbf{a} \times \mathbf{b} := \det \begin{bmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} \tag{68}$$

  - If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, then

$$||\mathbf{a} \times \mathbf{b}|| := ||\mathbf{a}|| ||\mathbf{b}|| \sin(\theta) \tag{69}$$

  - Two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel $\iff$ $\mathbf{a} \times \mathbf{b} = 0$.
  - The volume of the parallelepiped spanned by the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ is $|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$

- A parametric equation for a line going through the point (x_0, y_0, z_0) parallel to the direction vector $\langle a, b, c\rangle$ are:

$$x = x_0 + at$$
$$y = y_0 + bt$$
$$z = z_0 + ct$$

- The scalar equation of the plane through the point $P_0(x_0, y_0, z_0)$ with the normal vector $\mathbf{n} := \langle a, b, c\rangle$ is:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \tag{70}$$

- The distance D from a point $P_1(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$ is:

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + z^2}} \tag{71}$$

- **Vector-Valued Function**: a function whose domain is the set of real numbers and whose range is a set of vectors.
  - The **limit** of a vector-valued function $\mathbf{r}$ is defined by taking the limits of the component functions as follows:

$$\lim_{t\to a} \mathbf{r}(t) := \langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t)\rangle \tag{72}$$

  - **Space-curve**: the set C of all points (x, y, z) in space where $x = f(t)$, $y = g(t)$, and $z = h(t)$, where $t \in I$, is called a **space-curve**.
- The **derivative** $\mathbf{r}'$ of a vector-valued function $\mathbf{r}$ is defined as:

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h\to 0} \frac{\mathbf{r}(t + h) - \mathbf{r}(t)}{h} \tag{73}$$

  - The **unit tangent** vector $\mathbf{T}(t)$ is defined as:

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{||\mathbf{r}'(t)||} \tag{74}$$

  - The **definite integral** of a vector-valued function is exactly what one would expect:

$$\int_a^b \mathbf{r}(t)dt = \left(\int_a^b f(t)dt\right)\hat{\mathbf{i}} + \left(\int_a^b g(t)dt\right)\hat{\mathbf{j}} + \left(\int_a^b h(t)dt\right)\hat{\mathbf{k}} \tag{75}$$

- The length L of a space-curve between the points a and b is parameterisation-independent and is given by:

$$L = \int_a^b ||\mathbf{r}'(t)||dt \tag{76}$$

  - The **arc-length function** of a curve, s, is defined as:

$$s(t) := \int_a^t ||\mathbf{r}'(u)||du \tag{77}$$

  We can use the above equation to parameterise a curve with respect to arc-length by differentiating both sides of the equation above with respect to t and applying the fundamental theorem of calculus:

$$\frac{ds}{dt} = ||\mathbf{r}'(t)|| \tag{78}$$

  Advantages of an arc-length parametrisation include: it arises naturally from the shape of the curve and it's coordinate-system independent.
- **Smooth**: a parameterisation $\mathbf{r}(t)$ is called **smooth** on an interval I if $\mathbf{r}'$ is continuous and $\mathbf{r}'(t) \neq 0$ on I.

- **Curvature**: the **curvature** of a curve is given by:

$$\kappa := \left\| \frac{d\mathbf{T}}{ds} \right\| \tag{79}$$

where $\mathbf{T}$ is the unit tangent vector. We have three other formulae for curvature:

(1)

$$\kappa(t) = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} \tag{80}$$

(2)

$$\kappa(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} \tag{81}$$

(3) For plane curves, write $\mathbf{r}(x) = x\hat{\mathbf{i}} + f(x)\hat{\mathbf{j}}$

$$\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x)^2]^{3/2}} \tag{82}$$

- When $\kappa(t) \neq 0$, one can define the **principle unit normal** $\mathbf{N}(t)$:

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} \tag{83}$$

  - **Normal plane**: the plane determined by the normal and binormal vectors. Consists of all lines orthogonal to the unit tangent vector $\mathbf{T}$.
  - **Osculating circle**: the plane determined by the vectors $\mathbf{T}$ and $\mathbf{N}$.
    * Closest plane to containing the part of the curve near P.
    * **Osculating circle**: the circle that lies on the osculating plane of C at P, has the same tangent as C at P, and lies on the concave side of C (towards where $\mathbf{N}$ is pointing). This best describes the behaviour of C near P.

- The **velocity vector** $\mathbf{v}(t)$ at time t:

$$\mathbf{v}(t) := \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} = \mathbf{r}'(t) \tag{84}$$

  - **Speed**: the magnitude of the velocity vector $\|\mathbf{v}(t)\|$.

$$\|\mathbf{v}(t)\| = \|\mathbf{r}'(t)\| = \frac{ds}{dt} \tag{85}$$

  - **Acceleration**: the derivative of the velocity

$$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t) \tag{86}$$

- Often times it can be useful to resolve the acceleration of a particle into its tangential and normal components:

$$\mathbf{a} = \underbrace{v'}_{:=\mathbf{a}_T} \mathbf{T} + \underbrace{\kappa v^2}_{:=\mathbf{a}_N} \mathbf{N} \tag{87}$$

where $v := \|\mathbf{r}'(t)\|$. One can re-write Equation (87) so that it only depends on $\mathbf{r}$, $\mathbf{r}'$, and $\mathbf{r}''$:

$$\mathbf{a}_T = \frac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{\|\mathbf{r}'(t)\|} \tag{88}$$

$$\mathbf{a}_N = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|} \tag{89}$$

2.2. Partial Derivatives.

- **Graph**: let f be a function in two variables with domain Ω. Then, the **graph** of f is the set of all points $(x, y, z) \in \mathbb{R}^3$ such that $z = f(x, y)$ for $(z, y) \in \Omega$.
- **Level Curves**: The **level curves** of a function f in two variables are the curves with equations $f(x, y) = k$, where $k \in \mathbb{R}$ is a constant.
- **Limit**: Let f be a function of two variables whose domain Ω includes points arbitrarily close to (a, b). Then, we say that the **limit of $f(x, y)$ as $(x, y) \to (a, b)$** is L and we write:

$$\lim_{(x,y)\to(a,b)} f(x, y) = L \tag{90}$$

  if $\forall\ \varepsilon > 0$, $\exists \delta > 0$ such that if $(x, y) \in \Omega$, $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$, then $|f(x, y) - L| < \varepsilon$.
- **Partial Derivative**: the partial derivative of f with respect to x at (a, b) is:

$$f_x(a, b) := \lim_{h\to 0} \frac{f(a+h) - f(a, b)}{h} \tag{91}$$

- **Claircut's Theorem**: suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xx} and f_{yy} are both continuous on D, then,

$$f_{xy}(a, b) = f_{yx}(a, b) \tag{92}$$

- **Tangent Plane**: Suppose f has continuous partial derivatives. An equation for the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is given by:

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \tag{93}$$

  - **Linearisation**: an equation for the tangent plane to the graph f at the point $(a, b, f(a, b))$ is given by:

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \tag{94}$$

    the graph of this tangent plane is called the **linearisation** of f at (a, b):

$$L(x, y) := f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \tag{95}$$

- **Differentiable**: If $z = f(x, y)$, then f is **differentiable** at (a, b) if Δz can be expressed in the form:

$$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y \tag{96}$$

  where $\varepsilon_1, \varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.
  - If the partial derivatives f_x and f_y exist near (a, b), and are continuous at (a, b), then f is differentiable at (a, b).
  - **Total Differential**: for a differentiable function of two variables $z = f(x, y)$, then the **total differential** is defined as:

$$\begin{aligned} dz &:= f_x(x, y)dx + f_y(x, y)dy \\ &= \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \end{aligned}$$

    and so in the language of differentials, we write:

$$f(x, y) \approx f(a, b) + dz$$

- **Chain Rule (1. Case)**: Suppose that $z = f(x, y)$ is a differentiable function of x and y, and suppose that both x and y are differentiable functions of t (i.e, $x = x(t)$, $y = y(t)$) so that $z = (f(x(t), y(t))$. Then, z is a differentiable function of t and:

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \tag{97}$$

- **Chain Rule (General Version)**: Suppose that u is a differentiable function of n variables $(x_1, ..., x_n)$ and each x_j is a differentiable function of m variables $t_1, ..., t_m$. Then, u is a function of $t_1, ..., t_m$ and one has:

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_i}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_i}\frac{\partial x_2}{\partial t_i} + ... + \frac{\partial u}{\partial x_i}\frac{\partial x_n}{\partial t_i} \tag{98}$$

$\forall i = 1, 2, ..., m$.

- **Implicit Function Theorem**: Let $y = f(x)$. If F is defined on a disc containing (a, b), where $F(a,b) = 0$ $F_y(a,b) \neq 0$, and F_y and F_x are continuous on the disc, then the equation $F(x,y) = 0$ defines y as a function of x near the point (a,b) whose derivative is given by:

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y} \tag{99}$$

Now, for $z = f(x,y)$, if z is implicitly given as a function by an equation of the form $F(x,y,z) = 0$, then the derivative of the implicitly defined function is:

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \tag{100}$$

$$\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} \tag{101}$$

- **Directional Derivative**: the **directional derivative** of f at (x_0, y_0) in the direction of the unit vector $\mathbf{u} = \langle a, b\rangle$ is:

$$D_u f(x_0, y_0) := \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h} \tag{102}$$

  - If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b\rangle$ and

$$D_f(x,y) = f_x(x,y)a + f_y(x,y)b \tag{103}$$

- **Gradient**: Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a function of two variables. Then, the **gradient** of f is the vector function ∇f defined by:

$$\begin{aligned}\nabla f(x,y) &:= \langle f_x(x,y), f_y(x,y)\rangle \\ &= \frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}}\end{aligned}$$

Using this notation, we can re-write the directional derivative as a dot product:

$$D_u f(x,y) = \nabla f(x,y) \cdot \mathbf{u}$$

Thus, we can interpret the directional derivative as the scalar projection of the gradient function onto $\mathbf{u}$.

- Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function of n variables. The maximum value of the directional derivative $D_u f(x)$ is $\|\nabla f(x)\|$; this occurs when $\mathbf{u}$ is parallel to $\nabla f(x)$.
- The **tangent planet to the level surface** $F(x,y,z) = k$ at the point $P(x_0, y_0, z_0)$ is the plane that passes through P with the normal vector $\nabla F(x_0, y_0, z_0)$. This is written as:

$$F_x(x_0,y_0,z_0)(x - x_0) + F_y(x_0,y_0,z_0)(y - y_0) + F_z(x_0,y_0,z_0)(z - z_0) = 0 \tag{104}$$

2.3. **Multiple Integrals.**

- **Double Integral**: the **double integral** of f over the rectangle $\mathcal{R}$ is:

$$\iint_{\mathcal{R}} f(x,y)dA := \lim_{m,n \to \infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x^*_{i,j}, y^*_{i,j})\Delta A \tag{105}$$

if the limit exists.

- If $f \geq 0$, then the volume V of the solid above the rectangle $\mathcal{R}$ and below the surface $z = f(x,y)$ is:

$$V = \iint_{\mathcal{R}} f(x,y)dA \tag{106}$$

- **Fubini's Theorem**: If f is continuous on the rectangle $\mathcal{R} := \{(x,y) \mid a \leq x \leq b, c \leq y \leq d\}$ then:

$$\iint_{\mathcal{R}} f(x,y)dA = \int_a^b \int_c^d f(x,y)dydx = \int_c^d \int_a^b f(x,y)dxdy \tag{107}$$

- **Average Value**: the **average value** of a function f of two variables defined on the rectangle $\mathcal{R}$ is defined to be:

$$f_{\text{avg}} := \frac{1}{A(\mathcal{R})} \iint_{\mathcal{R}} f(x,y)dA \tag{108}$$

- **Double Integral of f over D**: let D be a bounded region in $\mathbb{R}^n$. Define the following function F; let $\mathcal{R}$ be a rectangle such that $D \subseteq \mathcal{R}$ and

$$F(x,y) := \begin{cases} f(x,y) & (x,y) \in D \\ 0 & (x,y) \notin D \end{cases} \tag{109}$$

then, the **double integral of f over D** is given by:

$$\iint_D f(x,y)dA := \iint_{\mathcal{R}} F(x,y)dA \tag{110}$$

- We have various "types" of domains/regions:
  - **Type I**: A plane region D is Type I if it lies between the graphs of two continuous functions of x:

$$D := \{(x,y) \in \mathbb{R} \mid a \leq x \leq b, g_1(x) \leq y g_2(x)\} \tag{111}$$

  in this case, the double integral is given by:

$$\iint_D f(x,y)dA := \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx \tag{112}$$

  - **Type II**: a plane region of Type II is:

$$D := \{(x,y) \in \mathbb{R}^2 \mid c \leq y \leq d,\ h_1(y) \leq x \leq h_2(y)\} \tag{113}$$

  where $h_1, h_2 \in C(\mathbb{R})$. In this case, the double integral is given by:

$$\iint_D f(x,y)dA := \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dxdy$$

- If $m \leq f(x,y) \leq M \ \forall \ (x,y) \in D$, then:

$$mA(D) \leq \iint_D f(x,y)dA \leq MA(D) \tag{114}$$

- **Polar Rectangle**: $\mathcal{R} := \{(r,\theta) \mid a \leq r \leq b, \alpha \leq \theta \leq \beta\}$
  - **Change to Polar Coordinates in a Double Integral**: Let $f : \mathbb{R} \to \mathbb{R}$ be continuous, $\mathcal{R}$ a polar rectangle, f continuous. If $\mathcal{R}$ is given by $0 \leq a \leq r \leq b$, $\alpha \leq \theta\beta$, $0 \leq \beta - \alpha \leq 2\pi$. Then:

$$\iint_{\mathcal{R}} f(x,y)dA := \int_\alpha^\beta \int_a^b f(r\cos(\theta), r\sin(\theta))rdrd\theta \tag{115}$$

  if f is continuous on a polar region of the form $D := \{(r,\theta) \mid \alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta)\}$, then:

$$\iint_D f(x,y)dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos(\theta), r\sin(\theta))rdrd\theta \tag{116}$$

- **Surface Area**: the area of the surface $z = f(x,y)$ for $(x,y) \in D$ where $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are continuous is given by:

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right) + \left(\frac{\partial z}{\partial y}\right)} \tag{117}$$

2.4. **Triple Integrals.**

- **Triple Integrals over a Box**: The **triple integral** of f over the box B is:

$$\iiint_B f(x,y,z)dV := \lim_{\ell,m,n\to\infty} \sum_{i=1}^{\ell}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\Delta V \tag{118}$$

- **Cylindrical Coordinate System**: a point $P \in \mathbb{R}^3$ is represented by the ordered triple (r,θ,z); the equations to convert are given by:

$$x = r\cos(\theta) \qquad y = r\sin(\theta) \;\; z = z$$
$$r^2 = x^2 + y^2 + z^2 \qquad \tan(\theta) = y/z \;\; z = z$$

Often useful in problems involving symmetry about an axis.

- **Triple Integration in Polar Coordinates**: Suppose E is a Type I region whose projection D onto the xy-plane is described in polar coordinates:

$$E = \{(x,y,z) \mid (x,y) \in D \mid u_1(x,y) \le z \le u_2(x,y)\ \}$$
$$D = \{(r,\theta) \mid \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\}$$

Then,

$$\iint_E f(x,y,z)dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos(\theta), r\sin(\theta)}^{u_2(r\cos(\theta), r\sin(\theta)} f(r\cos(\theta), r\sin(\theta))rdzdrd\theta \tag{119}$$

- **Spherical Coordinates**: the spherical coordinates (ρ,θ,φ) of a point $p \in \mathbb{R}^3$ are given by:

$$x = \rho\sin(\varphi)\cos(\theta),\ y = \rho\sin(\theta)\sin(\theta),\ z = \rho\cos(\varphi)$$
$$\rho^2 = x^2 + y^2 + z^2$$

  – The formula for a triple integral in spherical coordinates is given by:

$$\iiint_E f(x,y,z)dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho\sin(\varphi)\cos(\theta), \rho\sin(\theta)\sin(\theta), \rho\cos(\varphi))\rho^2\sin(\varphi)d\rho d\theta d\varphi \tag{120}$$

- **Jacobian**: the **Jacobian** of a transformation T given by $x = g(u,v)$ and $y = h(u,v)$ is given by:

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} \tag{121}$$

  – Suppose T is a C^1-transformation whose Jacobian is non-zero and that T maps S in the uv-plane to region $\mathcal{R}$ in the xy-plane. Suppose that f is continuous on $\mathcal{R}$ and that both $\mathcal{R}$ and S are Type I or Type II regions. Moreover, assume that T is bijective, except for potentially on ∂S. Then:

$$\iint_{\mathcal{R}} f(x,y)dA = \iint_S f(x(u,v), y(u,v))|\det(\mathbf{Jac})|dudv \tag{122}$$

3. Vector Calculus

3.1. **Vector Fields.**

- **Vector Field on $\mathbb{R}^n$**: Let $D \subseteq \mathbb{R}^n$. A **vector field** on $\mathbb{R}^n$ is a function $\mathbf{F}$ that assigns to every point $(x_1,...,x_n) \in D$ an n-dimensional vector $\mathbf{F}(x_1,...,x_n)$. One can write this in terms of component functions, e.g. in $\mathbb{R}^2$:

$$\mathbf{F}(x,y) = P(x,y)\hat{\mathbf{i}} + Q(x,y)\hat{\mathbf{j}} + Q(x,y)\hat{\mathbf{k}} = \langle P(x,y), Q(x,y), Q(x,y)\rangle \tag{123}$$

  - **Gradient Vector Field**: if $f : \mathbb{R}^2 \to \mathbb{R}$, recall that ∇f is:

$$\nabla f(x, y) = f_x(x, y)\hat{\mathbf{i}} + f_y(x, y)\hat{\mathbf{j}} \tag{124}$$

    which means that ∇f is a vector field on $\mathbb{R}^2$ (we call this vector field a **gradient vector field**).
- **Conservative Vector Field**: A vector field $\mathbf{F}$ is a **conservative vector field** if there exists a scalar function f such that $\nabla f = \mathbf{F}$.

3.2. **Line Integrals.**

- **Line Integral**: parameterise a smooth curve C by

$$x = x(t), \;\; y = y(t) \;\; t \in [a, b]$$

  or, equivalently,

$$r(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}}$$

  then, the **line integral** of f along C is:

$$\int_C f(x, y) ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i \tag{125}$$

  the line integral is given by:

$$\int_C f(x, y) ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \tag{126}$$

  - A more compact notation for line integrals can be given by:

$$\int_a^b f(\mathbf{r}(t)) \|\mathbf{r}'(t)\| dt \tag{127}$$

- **Line Integrals over Vector Fields**: Let $\mathbf{F}$ be a continuous vector field defined on a smooth curve C be given by a vector function $\mathbf{r}(t)$, $t \in [a, b]$. Then, the **line integral of F along C** is:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b F(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt = \int_C \mathbf{F} \cdot \mathbf{T} ds \tag{128}$$

  Suppose a vector field $\mathbf{F}$ on $\mathbb{R}^3$ is given in compact-form by the equation $\mathbf{F} = P\hat{\mathbf{i}} + Q\hat{\mathbf{j}} + R\hat{\mathbf{k}}$. Then,

$$\int_C \mathbf{F} \cdot dr = \int_C P dx + Q dy + K dz \tag{129}$$

- **Fundamental Theorem for Line Integrals**: Let C be a smooth curve given by the vector function $\mathbf{r}(t)$ for $t \in [a, b]$. Let f be a differentiable function f of two or three variables whose gradient vector ∇f is continuous on C. Then:

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)) \tag{130}$$

- Path Independence
  - $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in $D \iff \int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path C in D.
  - Let D be an open, connected domain. Suppose that F is a vector field on D. If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path on D, then $\mathbf{F}$ is a conservative vector field on D, that is, $\exists$ an f such that $\nabla f = \mathbf{F}$.
  - Let $F(x, y) = P(x, y)\hat{\mathbf{i}} + Q(x, y)\hat{\mathbf{j}}$ be a conservative vector field, where P and Q have continuous first-order partial derivatives on a domain D. Then, throughout D, we have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \tag{131}$$

  - **Simple Curve**: a curve that doesn't intersect itself anywhere between its endpoints.
  - **Simply-Connected Region**: a connected region D such that every simple closed curve in D encloses points that are only in D.

- Let $\mathbf{F} = P\hat{\mathbf{i}} + Q\hat{\mathbf{j}}$ be a vector field of an open, simply-connected region D. Suppose that P and Q have continuous first-order partial derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \tag{132}$$

throughout D. Then, $\mathbf{F}$ is conservative.

3.3. **Green's Theorem.** Counterpart of the Fundamental Theorem of Calculus for double integrals.

- **Green's Theorem**: Let C be a positively-oriented, piece-wise smooth simple closed curve in the plane and let D be a region bounded by C. If P and Q have continuous partial derivatives on an open region containing D, then:

$$\oint_C Pdx + Qdy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \tag{133}$$

- Can be used to calculate areas:

$$A(D) = \oint_C xdy = -\oint_C ydx = \frac{1}{2} \oint_C xdy - ydx \tag{134}$$

3.4. **Curl and Divergence.** Each of the following operations resemble differentiation, but one produces a vector field and the other produces a scalar field.

- **Curl**: Let $\mathbf{F} := P\hat{\mathbf{i}} + Q\hat{\mathbf{j}} + R\hat{\mathbf{k}}$ be a vector field on $\mathbb{R}^3$. Assume that the partial derivatives P, Q, and R all exist. Then, the **curl** of $\mathbf{F}$ is the vector field on $\mathbb{R}^3$ defined by:

$$\begin{aligned} \operatorname{curl}(F) &:= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \hat{\mathbf{i}} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \hat{\mathbf{j}} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \hat{\mathbf{k}} \\ &= \nabla \times \mathbf{F} \\ &= \det \begin{bmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{bmatrix} \end{aligned}$$

- If f is a function of three variables and has continuous, second-order partial derivatives, then:

$$\operatorname{curl}(\nabla f) = 0 \tag{135}$$

  - Conservative vector fields have $\mathbf{F} = \nabla f$, and so $\operatorname{curl}(\mathbf{F}) = 0$ for conservative vector fields.
  - Let $\mathbf{F}$ be a vector field defined on all of $\mathbb{R}^3$ whose component functions have continuous partial derivatives and $\operatorname{curl}(\mathbf{F}) = 0$. Then, $\mathbf{F}$ is a conservative vector field.
  - We say that a vector field $\mathbf{F}$ is irrotational if $\operatorname{curl}(\mathbf{F}) = 0$.
- **Divergence**: Let $\mathbf{F} := P\hat{\mathbf{i}} + Q\hat{\mathbf{j}} + R\hat{\mathbf{k}}$ be a vector field on $\mathbb{R}^3$ and assume that $\frac{\partial P}{\partial x}$, $\frac{\partial Q}{\partial y}$, and $\frac{\partial R}{\partial z}$ all exist. Then, the **divergence** of $\mathbf{F}$ is defined by:

$$\operatorname{div}(\mathbf{F}) := \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \tag{136}$$

$$= \nabla \cdot \mathbf{F} \tag{137}$$

- If $\mathbf{F} := P\hat{\mathbf{i}} + Q\hat{\mathbf{j}} + R\hat{\mathbf{k}}$ be a vector field on $\mathbb{R}^3$ and if P, Q, and R have continuous second-order partial derivatives, then

$$\operatorname{div}(\operatorname{curl}(\mathbf{F})) = 0 \tag{138}$$

  - A vector field $\mathbf{F}$ is called **incompressible** if $\operatorname{div}(\mathbf{F}) = 0$.
- We can use what we've built up here to formulate Green's Theorem in terms of vector forms.
  - The first vector form of Green's theorem is:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\operatorname{curl}(\mathbf{F})) \cdot \hat{\mathbf{k}} dA \tag{139}$$

– The second vector form of Green's Theorem is:

$$\oint_C \mathbf{F}\cdot\hat{\mathbf{n}}ds = \iint_D \operatorname{div}(\mathbf{F})dA \tag{140}$$

3.5. **Parametric Surfaces and their Areas.** A vector function $\mathbf{r}(u,v) := x(u,v)\hat{\mathbf{i}} + y(u,v)\hat{\mathbf{j}} + z(u,v)\hat{\mathbf{k}}$ can trace out a surface. We call this surface a **parametric surface**; consider the following tangent vectors:

$$\mathbf{r}_v = \frac{\partial x}{\partial v}(u_0,v_0)\hat{\mathbf{i}} + \frac{\partial y}{\partial v}(u_0,v_0)\hat{\mathbf{j}} + \frac{\partial z}{\partial v}(u_0,v_0)\hat{\mathbf{k}} \tag{141}$$

$$\mathbf{r}_u = \frac{\partial x}{\partial u}(u_0,v_0)\hat{\mathbf{i}} + \frac{\partial y}{\partial u}(u_0,v_0)\hat{\mathbf{j}} + \frac{\partial z}{\partial u}(u_0,v_0)\hat{\mathbf{k}} \tag{142}$$

If $\mathbf{r}_u \times \mathbf{r}_v \neq \mathbf{0}$, then the surface is **smooth** and the tangent plane is the plane containing the vectors $\mathbf{r}_u$ and $\mathbf{r}_v$.

Definition 1 (Surface Area)**.** Let S be a smooth parametric surface given by the equation:

$$\mathbf{r}(u,v) = x(u,v)\hat{\mathbf{i}} + y(u,v)\hat{\mathbf{j}} + z(u,v)\hat{\mathbf{k}} \tag{143}$$

for $(u,v)\in D$. Then, the **surface area** of S is:

$$A(S) := \iint_D \|\mathbf{r}_u \times \mathbf{r}_v\| dA \tag{144}$$

If, moreover, the surface S has the equation $z = f(x,y)$, $(x,y)\in D$ and $f\in C^1$, then:

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA \tag{145}$$

3.6. **Surface Integrals.**

Definition 2 (Surface Integral over S)**.** Suppose that a surface S has the following vector equation:

$$\mathbf{r}(u,v) = x(u,v)\hat{\mathbf{i}} + y(u,v)\hat{\mathbf{j}} + z(u,v)\hat{\mathbf{k}} \tag{146}$$

Then, the **surface integral of f over S** is:

$$\iint_S f(x,y,z)dS := \lim_{n,m\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(P_{ij}^*)\Delta S_{ij} \tag{147}$$

where

$$\Delta S_{ij} \approx \|\mathbf{r}_u \times \mathbf{r}_v\|\Delta u \Delta v$$

If the components are continuous and $\mathbf{r}_u$ and $\mathbf{r}_v$ are non-zero and non-parallel on $\operatorname{int}(D)$, then:

$$\iint_S f(x,y,z)dS = \iint_D f(\mathbf{r}(u,v))\|\mathbf{r}_u \times \mathbf{r}_v\| dA \tag{148}$$

If, moreover, S has an equation $z = g(x,y)$, then:

$$\iint_S f(x,y,z)dS = \iint_D f(x,y,g(x,y))\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA$$

3.7. **Oriented Surface.**

Definition 3 (Oriented Surface)**.** Let S be a surface with a with a tangent plane at every point $(x,y,z)\in S$. If it's possible to choose a unit normal vector $\mathbf{n}$ at every point $(x,y,z)\in S$ such that $\mathbf{n}$ varies continuously over S, then we say that S is an **orientable surface**.

- For a surface $z = g(x,y)$, the natural orientation will be given by the following unit normal vector:

$$\mathbf{n} = \frac{-\frac{\partial g}{\partial x}\hat{\mathbf{i}} - \frac{\partial g}{\partial y}\hat{\mathbf{j}} + \hat{\mathbf{k}}}{\sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}} \tag{149}$$

- For a smooth orientable surface S given in parametric form by a vector $\mathbf{r}(x,y)$, the unit normal points in the same direction as the position vector:

$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{||\mathbf{r}_u \times \mathbf{r}_v||} \tag{150}$$

3.8. **Surface Integrals over Vector Fields.**

Definition 4 (Surface Integral)**.** Let $\mathbf{F}$ be a continuous vector field defined on an oriented surface S with unit normal vector $\mathbf{n}$. Then, the **surface integral of F over** S is:

$$\iint_S \mathbf{F}\cdot \mathbf{dS} = \iint_S \mathbf{F}\cdot \mathbf{n} dS \tag{151}$$

(1) If S is given by a vector function $\mathbf{r}(u,v)$, then:

$$\iint_S \mathbf{F}\cdot \mathbf{dS} = \iint_D \mathbf{F}\cdot [\mathbf{r}_u \times \mathbf{r}_v] dA \tag{152}$$

(2) If S is given by a graph $z = g(x,y)$, then let $\mathbf{F} := (P,Q,R)$. Then, the surface integral is:

$$\iint_S \mathbf{F}\cdot \mathbf{dS} = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA \tag{153}$$

3.8.1. *Physical Interpretations: Electric Flux and Heat Flow.*

(1) Electricity: Let $\mathbf{E}$ be an electric field. Then, the meaning of the surface integral $\iint_S \mathbf{E}\cdot\mathbf{dS}$ is that it represents the **electric flux of E through** S.Then, **Gauss' Law** states that the net charge enclosed by a closed surface S is given by:

$$Q = \epsilon_0 \iint_S \mathbf{E}\cdot \mathbf{dS} \tag{154}$$

(2) Heat Flow: the **heat flow** is the vector field given by $\mathbf{F} := -k\nabla\mathbf{u}$, where k is a constant that encodes the conductivity of the material. Then, the rate of heat flow across the surface S in the body is given by:

$$\iint_S \mathbf{F}\cdot \mathbf{dS} = -k\iint_S \nabla\mathbf{u}\cdot \mathbf{dS} \tag{155}$$