Math 458: Differential Geometry

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1.1 Implicit and Inverse Function Theorems

2 Manifolds in $\mathbb{R}^3$

The aim of this part of the course is to build up to integration on manifolds and the invariant Stokes' theorem. The main purpose of this sections is to develop *coordinate-free* calculus, which clarifies the essence of what is happening (sometimes coordinates can be noisy).

2.1 Definitions

2.2 Smooth Maps from $M^m \to N^n$

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3 Curves

There are two subsets of differential geometry: classical differential geometry and global differential geometry. The objective of **classical differential geometry** is to study the local properties of curves and surfaces. The objective of **global differential geometry** is to study the influence of local properties on global behaviour.

3.1 Definitions

Definition 1 (Parameterised Differentiable Curve)**.** A **parameterised differentiable curve** is a differentiable map $\alpha : I \to \mathbb{R}^3$ of an open interval $I =]a, b[$ of the real line $\mathbb{R}$ into $\mathbb{R}^3$. The image of α is called the **trace** of α.

Some examples of parameterised curves include:

- The helix: $\alpha(t) = (a\cos(t), a\sin(t), bt)$ for $t \in \mathbb{R}$.
- The map $\alpha : \mathbb{R} \to \mathbb{R}^2$, $t \in \mathbb{R}$, is a parameterised differentiable curve.

Definition 2 (Norm on $\mathbb{R}^3$)**.** Let $u = (u_1, u_2, u_3) \in \mathbb{R}^3$. The **norm** of u is:

$$||u|| := \sqrt{u_1^2 + u_2^2 + u_3^3}$$

Definition 3 (Inner Product on $\mathbb{R}^3$)**.** Let $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ belong to $\mathbb{R}^3$ and let $\theta \in [0, \pi]$ be the angle formed between u, v. The **inner product** is defined by:

$$u \cdot v := ||u||||v|| \cos(\theta) \tag{1}$$

It satisfies the following properties:

1. If u, v are non-zero, then $u \cdot v = 0 \iff u \perp v$.
2. $u \cdot v = v \cdot u$.
3. $\lambda(u \cdot v) = \lambda u \cdot v = u \cdot \lambda v$.
4. $u(v + w) = u \cdot v + u \cdot w$.

If we have made a choice of basis, then we can formulate the dot product in terms of the components of the vectors as:

$$u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3 \tag{2}$$

3.1.1 Regular Curves and Arclength

In differential geometry, it is essential that our curves have a tangent line at every point. This motivates the following definition.

Definition 4 (Regular Curve)**.** A parameterised differentiable curve $\alpha : I \to \mathbb{R}^3$ is **regular** if $\alpha'(t) \neq 0$ $\forall t \in I$.

Definition 5 (Arc length)**.** Given $t_0 \in I$, the **arc length** of a regular parameterised curve $\alpha : I \to \mathbb{R}^3$ from t_0 to t is defined to be:

$$s(t) := \int_{t_0}^{t} |a'(t)| dt$$

where

$$|\alpha'(t)| := \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}$$

Since we only restrict our attention to regular surfaces, $a'(t) \neq 0$ for all t, and so the arlength function is a differentiable function of t and $ds/dt = |a'(t)|$ (by the Fundamental Theorem of Calculus). Arc length parameterisations make life simpler.

3.1.2 The Vector Product in $\mathbb{R}^3$

Definition 6 (Vector Product)**.** Let $u, v \in \mathbb{R}^3$. Then, the **vector product** of u, v is the unique vector $u \wedge v$ in $\mathbb{R}^3$ characterised by:

$$(u \wedge v) \cdot w = \det(u, v, w) \ \ \forall w \in \mathbb{R}^3$$

this is more commonly known as:

$$u \wedge v = \det \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}$$

where $\hat{\imath}, \hat{\jmath}, \hat{k}$ are the standard basis vectors in $\mathbb{R}^3$.

Properties of the Vector Product

1. (Anti-Commutativity): $u \wedge v = -v \wedge u$.
2. (Linear Dependence): $\forall\ \alpha,\ \beta \in \mathbb{R}$:

$$(\alpha u + \beta v) \wedge v = \alpha u \wedge v + \beta w \wedge v$$

3. $u \wedge v = 0 \iff u$ and v are linearly dependent.
4. $(u \wedge v) \cdot u = 0$, $(u \wedge v) \cdot v = 0$ (this implies that the vector product is normal to the plane generated by u and v).

3.2 Frenet-Serret Frame

Definition 7 (Curvature)**.** Let $\alpha : I \to \mathbb{R}^3$ be a curve parameterised by arclength $s \in I$. The number $||\alpha''(s)|| = \kappa(s)$ is called the **curvature** of α at s.

It's straightforward to check that $\kappa(s) = 0 \iff \alpha(s) = us + v$ (i.e., the curve is actually a straight line). When $\kappa(s) \neq 0$, the **unit normal** $n(s)$ in the direction $\alpha''(s)$ is well-defined and is given by:

$$\alpha''(s) := \kappa(s) \cdot n(s)$$

The orthogonality of $n(s)$ to $\alpha'(s)$ can be verified by differentiating both sides of $\alpha'(s) \cdot \alpha'(s) = 1$ since $||\alpha'(s)|| = 1$.

Definition 8 (Osculating Plane at s)**.** The **osculating plane** at s is the plane determined by the unit tangent and normal vectors, $\alpha'(s)$, and $n(s)$.

Definition 9 (Binormal Vector at s, $b(s)$)**.** The **binormal vector** as s is defined as $t(s) \wedge n(s)$, where $t(s)$ is the unit tangent at s. The magnitude of this vector, $||b(s)||$, measures how rapidly the curve pulls away from the osculating plane at s in a neighbourhood of s.

Definition 10 (Torsion)**.** Let $\alpha : I \to \mathbb{R}^3$ be a curve parameterised by arclength s such that $\alpha''(s) \neq 0$, $s \in I$. The number $\tau(s)$ defined by $b'(s) := \tau(s)n(s)$ is called the **torsion** of α at s. We have the following useful characterisation:

$$\alpha \text{ is a plane curve} \iff \tau \equiv 0$$

Thus, torsion measures how much a curve *fails* to be a plane curve.

Collecting the orthogonal unit vectors $t(s), n(s), b(s)$ gives us the **Frenet Trihedron** at s. Using the above definitions gives us the **Frenet Formulae**, which is a set of differential equations:

$$t' = \kappa n \tag{3}$$
$$n' = -\kappa t - \tau b \tag{4}$$
$$b' = \tau n \tag{5}$$

- The tb plane is called the **rectifying plane**
- The nb plane is called the **normal plane**
- κ and τ completely describe a curve's behaviour.
- Bending $\sim$ curvature; twising $\sim$ torsion.

The Frenet-Serret frame can be concisely expressed as a skew-symmetric matrix:

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \cdot \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{6}$$

Theorem 1 (Fundamental Theorem of the Local Theory of Curves)**.** Given differentiable functions $\kappa(s) > 0$ and $\tau(s)$, $s \in I$, there exists a regular parameterised curve $\alpha : I \to \mathbb{R}^3$ such that s is the arclength, $\kappa(s)$ is the curvature, and $\tau(s)$ is the torsion of α. Moreover, any other curve $\widetilde{\alpha}$ satisfying the same conditions differ from α by a rigid motion.

Definition 11 (Rigid Motion)**.** A **rigid motion** means that $\exists$ an orthogonal map ρ of $\mathbb{R}^3$ with positive determinant and a vector c such that $\widetilde{\alpha} = \rho \circ \alpha + c$.

Without loss of generality, we can assume curves to be parameterised by arclength, since we can always re parameterise a parameterised curve by arclength:
Let $\alpha : I \to \mathbb{R}^3$ be a regular parameterised curve. Then, it is possible to obtain a curve $\beta : J \to \mathbb{R}^3$ that is parameterised by arc length with the same trace as α:

$$s = s(t) = \int_{t_0}^{t} |\alpha'(t)| dt$$

where $t, t_0 \in I$.

3.3 Global Properties of Curves

3.3.1 The Isoparametric Inequality

This is related to the following isoparametric question:

Q: Of all the simple closed curves in the plane with a given length, which bounds the largest area?

We will use the following formula for the area A bounded by a positively oriented simple closed curve $\alpha(t) = (x(t), y(t))$:

$$A = -\int_a^b y(t)x'(t)dt = \int_a^b x(t)y'(t)dt = \frac{1}{2}(xy' - yx')dt$$

Theorem 2 (The Isoparametric Inequality)**.** Let C be a simple closed plane curve with length ℓ and let A be the area of the region bounded by C. Then:

$$\ell^2 - 4\pi A \geq 0 \tag{7}$$

where equality holds $\iff$ C is a circle.

3.3.2 Cauchy Crofton Formula

Theorem 3 (Cauchy Crofton Formula)**.** Let C be a regular plane curve with length ℓ. The measure of the set of straight lines, counted with multiplicities (**multiplicity** is the number of intersection points of a line with C), which meet C is equal to 2ℓ.

Definition 12 (Rigid Motion in $\mathbb{R}^2$)**.** A **rigid motion** in $\mathbb{R}^2$ is a map $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by $(\overline{x}, \overline{y}) \to (x, y)$, where:

$$\begin{aligned} x &= a + \overline{x}\cos(\varphi) - \overline{y}\sin(\varphi) \\ y &= b + \overline{x}\sin(\varphi) + \overline{y}\cos(\varphi) \end{aligned}$$

Proposition 1. Let $f(x, y)$ be a continuous function defined in $\mathbb{R}^2$. For any set $S \subseteq \mathbb{R}^2$, define the **area A of S** by:

$$A(S) := \iint_S f(x, y)dxdy \tag{8}$$

Assume that A is invariant under rigid motions; that is, if S is a set and $\overline{S} = F^{-1}(S)$, where F is a rigid motion, then if:

$$A(\overline{S}) = \iint_{\overline{S}} f(\overline{x}, \overline{y})d\overline{x}d\overline{y} = \iint_S f(x, y)dxdy = A(S)$$

Then, $f(x, y)$ is a constant.

4 Surfaces

4.1 Definitions

Motivation: we want to define a regular surface to be something that is nice enough for us to extend the usual notions of calculus to.

Definition 13 (Regular Surface)**.** A subset $S \subseteq \mathbb{R}^3$ is called a **regular surface** if, $\forall\, p \in S$, there exists a neighbourhood $V \subseteq \mathbb{R}^3$ and a map $\mathbb{X} : U \to V \cap S$ of an open set $V \subseteq \mathbb{R}^2$ onto $V \cap S \subseteq \mathbb{R}^3$ for which the following conditions hold:

1. $\mathbb{X}$ is differentiable; that is, if we write

   $$\mathbb{X}(u, v) = (x(u, v), y(u, v), z(u, v))$$

   for $(u, v) \in U$, then the functions $x(u, v)$, $y(u, v)$ and $z(u, v)$ have continuous partial derivatives of all orders in U.
2. $\mathbb{X}$ is a **homeomorphism**: there exists an inverse $\mathbb{X}^{-1} : V \cap S \to U$, which is continuous.
3. (Regularity Condition): $\forall q \in U$, the differential $d\mathrm{x}_q : \mathbb{R}^2 \to \mathbb{R}^3$ is bijective.

Then, the mapping $\mathbb{X}$ is called a **parameterisation** or a **system of local coordinates** in a neighbourhood of p. The neighbourhood $V \cap S$ of p is called a **coordinate neighbourhood**.

4.2 Regular Surfaces

Example 1 (The Unit Sphere is a Regular Surface)**.** The Unit Sphere is a regular surface. It's parametrised by:

$$S^2 := \{(x, y, z) \in \mathbb{R}^2 \mid x^2 + y^2 + z^2 = 1\}$$

In the textbook, they check all three conditions from the above definition. Since this can be quite exhausting, we want some propositions that simplify the task of determining if a surface is regular or not. This is the aim of this section.

Proposition 2. If $f : U \subseteq \mathbb{R}^2 \to \mathbb{R}$, U open, is a differentiable, then the graph of f, that is, the subset of $\mathbb{R}^3$ given by $(x, y, f(x, y))$ for $(x, y) \in U$, is a regular surface.

Before introducing the second proposition, we will first need to define critical points, critical values, and regular values for differentiable maps.

Definition 14 (Critical Point)**.** Given a differentiable map $F : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ defined in an open set $U \subseteq \mathbb{R}^n$, we say that $p \in U$ is a **critical point** of F id the differential $\mathrm{d}F_p : \mathbb{R}^n \to \mathbb{R}^m$ is not a surjective mapping. The image $F(p) \in \mathbb{R}^m$ of a critical point is called a **critical value** of F. A point $\mathbb{R}^m$ which is not a critical value is called a **regular value**.

The justification for the next proposition comes from the inverse function theorem.

Proposition 3. If $f : U \subseteq \mathbb{R}^3 \to \mathbb{R}$ is a differentiable function and $a \in f(U)$ is a regular value of f, then $f^{-1}(a)$ is a regular surface in $\mathbb{R}^3$.

Example 2 (Ellipsoid)**.** The ellipsoid is given by:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Since it is the set $f^{-1}(0)$ where

$$f(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$$

and f is a differentiable function and 0 is a regular value of f.

Definition 15 (Connected)**.** A surface $S \subseteq \mathbb{R}^3$ is **connected** if any two of its points can be joined by a continuous curve in S.

The next proposition is a very useful property that follows from the intermediate value theorem:

Definition 16. If $f : S \subseteq \mathbb{R}^3 \to \mathbb{R}$ is a non-zero continuous function defined on a connected surface S, then f does not change sign on S.

4.3 Differentiable Functions on Surfaces

4.4 Tangent Plane

The third condition of a regular surface guarantees that for any fixed point $p \in S$, the set of tangent vectors to the parameterised curves of S passing through p constitutes a plane.

Proposition 4. Let $\mathbb{X} : U \subseteq \mathbb{R}^2 \to S$ be a parameterisation of a regular surface S and let $q \in U$. The vector subspace of dimension 2:

$$\mathrm{d}x_q(\mathbb{R}^2) \subseteq \mathbb{R}^3 \tag{9}$$

coincides with the set of tangent vectors to S at $\mathbb{X}(q)$.

This plane does not depend on the parameterisation $\mathbb{X}$ and it is called the **tangent plane** to S at p and is denoted by $T_p(S)$. A choice of parameterisation $\mathbb{X}$ induces a basis on $T_p(S)$:

$$\{(\partial\mathbb{X}/\partial u)(q), (\partial\mathbb{X}/\partial v)(q)\}$$

The next proposition states that a map between two regular surfaces induces a map between the tangent planes, which we can think of as the differential of the map.

Proposition 5. Let S_1, S_2 be regular surfaces and let $\varphi : V \subseteq S_1 \to S_2$ be a differentiable mapping of an open set V of S_1 into S_2. Then, tangent vectors $w \in T_p(S_1)$ are the velocity vectors $\alpha'(0)$ of a differentiable parameterised curve $\alpha :]-\varepsilon, \varepsilon[\to V$ with $\alpha(0) = p$. If we define $\beta := \varphi \circ \alpha$, then $\beta'(0)$ is a vector of $T_{\varphi(p)}(S_2)$. Given a w, the vector $\beta'(0)$ does not depend on the choice of α and the map $\mathrm{d}\varphi_p : T_p(S_1) \to T_{\varphi(p)}(S_2)$ defined by $\mathrm{d}\varphi_p(w) = \beta'(0)$ is linear.

Before moving onto the next proposition, we first need to define what a local diffeomorphism is. The aim is to build up to a generalisation of the standard inverse function theorem from calculus.

Definition 17 (Local Diffeomorphism)**.** A mapping $\varphi : U \subseteq S_1 \to S_2$ is called a **local diffeomorphism** at $p \in U$ if there is a neighbourhood $V \subseteq U$ of p such that $\varphi|_U$ is a diffeomorphism onto an open set $\varphi(V) \subseteq S_2$.

Proposition 6. If S_1 and S_2 are regular surfaces and $\varphi : U \subseteq S_1 \to S_2$ is a differentiable mapping of an open set $U \subseteq S_1$ such that the differential $\mathrm{d}\varphi_p$ of φ at $p \in U$ is an isomorphism, then φ is a local diffeomorphism at p.

For any point on a regular surface, we can find two unit normal vectors. By fixing a parameterisation $\mathbb{X} : U \subseteq \mathbb{R}^2 \to S$ for $p \in S$, we can make a definite choice of a unit normal at each point $q \in \mathbb{X}(U)$ by the following rule:

$$N(q) := \frac{\mathbb{X}_u \wedge x_v}{||x_u \wedge x_v||}(q) \tag{10}$$

This gives us a differentiable map $N : \mathbb{X}(U) \to \mathbb{R}^3$.

4.5 First Fundamental Form: Area

Motivation: the natural inner product on $\mathbb{R}^3$ induces on each regular surface $S \subseteq \mathbb{R}^3$'s tangent plane $T_p(S)$ an inner product, $\langle \cdot, \cdot \rangle_p$. The aim of the First Fundamental Form is to express how a surface inherits the natural inner product of $\mathbb{R}^3$. This allows us to make metric measurements of the surface, such as lengths of curves, angles of tangent vectors, and areas of regions without referring to the ambient space in which they reside.

Definition 18 (First Fundamental Form)**.** Let w_1, $w_2 \in T_p(S) \subseteq \mathbb{R}^3$. Then, the quadratic form given by $I_p : T_p(S) \to \mathbb{R}$:

$$I_p(w) := \langle w, w \rangle_p = ||w||^2 > 0 \tag{11}$$

is called the **First Fundamental Form** of the regular surface $S \subseteq \mathbb{R}^3$ at $p \in S$.

4.5.1 Deriving the First Fundamental Form Given a Basis and a Parameterisation

Let $\mathbb{X}(u, v)$ be a parametrisation. We will now express the first fundamental form in the basis $\{\mathbb{X}_u, \mathbb{X}_v\}$ associated to a parameterisation $\mathbb{X}(u, v)$ at p. Recall that a tangent vector $w \in T_p(S)$ is equivalent to a tangent vector to a parameterised curve $\alpha(t) = \mathbb{X}(u(t), v(t))$ for $t \in]-\varepsilon, +\varepsilon[$ for which $p = \alpha(0) = \mathbb{X}(u_0, v_0)$.

From the definition of the first fundamental form, we have:

$$\begin{aligned} I_p(\alpha'(0)) &= \langle \alpha'(0), \alpha'(0) \rangle_p \\ &= \langle \mathbb{X}_u u' + \mathbb{X}_v v', \mathbb{X}_u u' + \mathbb{X}_v v' \rangle_p \\ &= \langle \mathbb{X}_u, \mathbb{X}_u \rangle_p (u')^2 + 2\langle \mathbb{X}_u, \mathbb{X}_v \rangle u'v' + \langle \mathbb{X}_v, \mathbb{X}_v \rangle_p (v')^2 \end{aligned}$$

If we define

$$E(u_0, v_0) := \langle \mathbb{X}_u, \mathbb{X}_u \rangle_p$$
$$F(u_0, v_0) := \langle \mathbb{X}_u, \mathbb{X}_v \rangle_p$$
$$G(u_0, v_0) := \langle \mathbb{X}_v, \mathbb{X}_v \rangle_p$$

then the first fundamental form can be expressed as:

$$I_p = E(u')^2 + 2Fu'v' + G(v')^2$$

4.5.2 Examples of First Fundamental Forms

1. Recall that the **plane** going through $p_0 = (x_0, y_0, z_0)$ containing the orthonormal vectors $w_1 = (a_1, a_2, a_3)$ and $w_2 = (b_1, b_2, b_3)$ is given by:

$$\mathbb{X}(u, v) = p_0 + uw_1 + vw_2$$

for $(u, v) \in \mathbb{R}^2$. Then, $E = 1$, $F = 0$, and $G = 1$.

2. The **cylinder** over the circle $x^2 + y^2 = 1$ parameterised by $\mathbb{X}(u, v) = (\cos(u), \sin(v), v)$ where $u \in]0, 2\pi[$ and $v \in \mathbb{R}$. Then: $E = \sin^2(u) + \cos^2(u) = 1$, $F = 0$, and $G = 1$.
3. The **Helicoid** is given by: $\mathbb{X}(u, v) := (v\cos(u), v\sin(u)au)$. $u \in]0, 2\pi[$, $v \in \mathbb{R}$. The first fundamental form is given by: $E = v^2 + a^2$, $F(u, v) = 0$, and $G(u, v) = 1$.

We can express arclength in terms of the terms of the functions of the first fundamental form. Let s be an arclength-parameterised curve $\alpha : I \to s$. Then, the arc-length is:

$$s(t) = \int_0^t |\alpha'(t)|dt = \int_0^t \sqrt{I(\alpha'(t))}dt$$

Substituting in the derivation gives us:

$$s(t) = \int_0^t \sqrt{E(u')^2 + 2Fu'v' + G(v')^2}dt$$

We can also represent angles of intersections of parameterised curves using the coefficients of the first fundamental form. Let $\alpha : I \to S$ and $\beta : I \to S$ be two parameterised curves. The angle θ at which they intersect at $t = t_0$ is given by:

$$\cos(\theta) = \frac{\langle a'(t_0), \beta'(t_0)\rangle}{||\alpha'(t_0)||||\beta'(t_0)||} \tag{12}$$

In terms of the coefficients of the first fundamental form, we have:

$$\cos(\theta) = \frac{\langle x_u, x_v\rangle}{||x_u||||x_v||} = \frac{F}{\sqrt{EG}}$$

A special type of parameterisation is called an **orthogonal parameterisation**, which is a parameterisation where the coordinate curves of a parameterisation are orthogonal. By the above, this happens if and only if $F(u, v) = 0$ for all $u, v \in S$. Moreover, from the arc length formula, an **element of arclength** is given by:

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2$$

One final classic example of computing first fundamental forms is that of a sphere. If we parameterise a sphere as:

$$\mathbb{X}(\theta, \varphi) = (\sin\theta\cos\varphi, \sin\theta, \sin\varphi, -\sin\theta)$$

Then, the coefficients of the first fundamental form become:

$$E(\theta, \varphi) = 1$$
$$F(\theta, \varphi) = 0$$
$$G(\theta, \varphi) = \sin^2(\theta)$$

Then, for a vector $w \in T_p(S)$ at the point p with the coordinates based on the basis associated to the parametrisation $\mathbb{X}(\theta, \varphi)$, we write:

$$w = a\mathbb{X}_\theta + b\mathbb{X}_\varphi$$

and so

$$||w||^2 = I(w) = Ea^2 + 2Fab + Gb^2 = a^2 + b^2 \sin^2 \theta$$

We can use the first fundamental form to compute areas.

Definition 19 (Area). Let $R \subseteq S$ be a bounded region of a regular surface contained in the coordinate neighbourhood of the parameterisation $\mathbb{X} : U \subseteq \mathbb{R}^2 \to S$. Then, the positive number:

$$A(R) := \iint_Q ||\mathbb{X}_u \wedge \mathbb{X}_v|| dudv$$

where $Q = \mathbb{X}^{-1}(R)$ is called the **area** of R. This is equivalent to, in terms of the first fundamental form:

$$= \iint_Q \sqrt{EG - F^2} dudv$$

5 The Gauss Map

Motivation: try to measure how rapidly a surface S pulls away from the tangent plane $T_p(S)$ in a neighbourhood of a point $p \in S \leftrightarrow$ measuring the rate of change at p of a unit normal vector field N on a neighbourhood of p. This gives rise to a linear map on $T_p(S)$ that is self-adjoint. This map happens to give us a lot of information about local properties of the surface S at p.

5.1 The Definition of the Gauss Map and its Fundamental Properties

- N is said to be a **differentiable field of unit normal vectors on** an open set $V \subseteq S$ if $N : V \to \mathbb{R}^3$ is a differentiable map which associates to each $q \in V$ a unit normal vector at q.
- A regular surface V is called **orientable** if it admits a differentiable field of unit normal vectors defined on the whole surface.
  - The Möbius strip is an example of a non-orientable surface.
  - The choice of such a field N is called an **orientation** of S.
  - Every surface is locally orientable.
  - Orientation is a global property in the sense that it involves the *whole* surface.

The Gauss map is the map which assigns unit normals to points on surfaces. We derived this map in homework 1.

Definition 20 (Gauss Map). Let $S \subseteq \mathbb{R}^3$ be a surface with orientation N. The map $N : S \to \mathbb{R}^3$ takes its values in the unit sphere:

$$S^2 := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\} \tag{13}$$

This map $N : S \to S^2$ as defined is called the **Gauss Map** of S.

The differential induced by the Gauss Map, $\mathrm{d}N_p : T_p(S) \to T_{N(p)}(S)$, is a linear map. Restricting the map to a parameterised curve $\alpha(t)$ in S provides for us a measure of how N pulls away from $N(p)$ in a neighbourhood of p. For curves, this information is encoded in the curvature, a scalar. For surfaces, the "notion" of curvature is encoded as a linear map.

Here are several examples of what dN would be for some surfaces.

1. The **plane** has zero "curvature." Parameterise this plane by $ax + by + cz + d = 0$. Then, the unit normal vector is given by:

$$N = \frac{(a, b, c)}{\sqrt{a^2 + b^2 + c^2}}$$

   and is thus a constant. This means that $\mathrm{d}N = 0$.
2. The **unit sphere** is parameterised by:

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$$

   Fix an orientation on S^2 by choosing $N = (-x, -y, -z)$. Then, $\mathrm{d}N_p(v) = -v$ for $p \in S^2$, $v \in T_p(S^2)$.
3. The **cylinder over the unit circle** is parameterised by:

$$C = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$$

   Fix an orientation by choosing $N = (-x, -y, 0)$. For a $v \in T_p(C)$, there are two cases:

   a) If v is tangent to the cylinder and parallel to the z-axis, then $\mathrm{d}N(v) = 0 = 0v$.
   b) If v is tangent to the cylinder and parallel to the xy-plane, then $dN(w) = -w$.

   v and w are eigenvectors of dN with eigenvalues 0 and -1, respectively.
4. **Hyperbolic Paraboloid**: analyse the point $p = (0, 0, 0)$ of the hyperbolic paraboloid. Parameterise it by:

$$\mathbb{X}(u, v) = (u, v, v^2 - u^2)$$

   The normal vector is given by:

$$N = \left(\frac{u}{\sqrt{u^2 + v^2 + 1/4}}, \frac{-v}{\sqrt{u^2 + v^2 + 1/4}}, \frac{1}{2\sqrt{u^2 + v^2 + 1/4}}\right)$$

   and so at p, $\mathrm{d}N_p(u'(0), v'(0), 0) = (2u'(0), -2v'(0), 0)$ meaning that $(1, 0, 0)$ and $(0, 1, 0)$ are eigenvectors of dN_p with eigenvalues 2 and -2 respectively.

Before introducing the second fundamental form, we need to first define an self-adjoint map.

Definition 21 (Self-Adjoint)**.** We say that a linear map $A : V \to V$ is **self-adjoint** if $\langle Av, w\rangle = \langle v, Aw\rangle \; \forall \; v, w \in V$.

The following proposition is useful since it allows us to associate $\mathrm{d}N_p$ to a quadratic form Q in $T_p(S)$, which will be important for the second fundamental form. The quadratic form will be given by:

$$W(v) = \langle dN_p(v), v\rangle$$

for $v \in T_p(S)$.

Proposition 7. The differential of the Gauss Map, $\mathrm{d}N_p : T_p(S) \to T_p(S)$, is a self-adjoint linear map.

Definition 22 (The Second Fundamental Form)**.** The quadratic form II_p defined in $T_p(S)$ given by $II_p(v) = -\langle dN_p(V), v\rangle$ is called the **second fundamental form** of S at p.

Definition 23 (Normal Curvature)**.** Let C be a regular surface in S passing through $p \in S$, κ the curvature of C at p, and $\cos\theta = \langle n, N\rangle$ where n is the normal vector to C and N is the normal vector to S at p. Then, the number $k_n := k\cos\theta$ is called the **normal curvature** of $C \subseteq S$ at p.

Thus, k_n represents the length of the projection of the vector kn over the normal to the surface at the point $p \in C$.

Proposition 8 (Meusnier)**.** All of the curves lying on a surface S with the same tangent line at a given point $p \in S$ have the same normal curvatures.

- Gives meaning to the notion of "normal curvature along a given direction at p".
- **Normal section of S at p**: given a unit vector $v \in T_p(S)$, the intersection of S with the plane containing v and $N(p)$ is called the **normal section of S at p** along v.
- The curvature of a curve is equal to the absolute value of the normal curvature along v at p, where v is the tangent vector of the curve at p.
- So, Prop. 8 is saying that the absolute value of the normal curvature at p of a curve $\alpha(s)$ is equal to the curvature of the normal section of S at p along $\alpha'(0)$.

Examples of second fundamental forms for surfaces:

1. **Plane**: all normal sections are straight lines. So, all normal curvatures are zero. Thus, the second fundamental form is identically equal to zero at all points $\leftrightarrow \mathrm{d}N \equiv 0$.
2. **Sphere S^2**: Choose an orientation N. The normal sections through a point $p \in S^2$ are circles with radius 1. Thus, all normal curvatures are equal to 1, and so the second fundamental form is $II_p(v) = 1\ \forall p \in S^2$, $v \in T_p(S)$, $|v| = 1$.
3. **Cylinder**: normal sections vary from a circle perpendicular to the cylinder's axis to straight lines parallel to the axis, which means that normal curvature varies from 1 to 0.

Definition 24 (Maximum Normal Curvature and Minimum Normal Curvature)**.** The **maximum normal curvature** k_1 and the **minimum normal curvature** k_2 are called the principle curvatures at p; the corresponding directions, that is, the directions given by the eigenvectors $\{\hat{e_1}, \hat{e_2}\}$, are called the **principal directions** at p.

Definition 25 (Lines of Curvature)**.** If a regular connected curve C in S is such that $\forall p \in C$, the tangent line of C is a principal direction at p, then C is said to be a **line of curvature** of S.

The following proposition gives us a necessary and sufficient condition for a connected regular curve to be a line of curvature.

Proposition 9. A necessary and sufficient condition for a connected regular curve C on S to be a line of curvature is that:

$$N'(t) = \lambda(t)\alpha'(t)$$

for any parameterisation $\alpha(t)$ of C, where $N(t) = N \circ \alpha(t)$ and $\lambda(t)$ is a differentiable function of t. In this case, $-\lambda(t)$ is called the **principle curvature along** $\alpha'(t)$.

This proposition can be used to easily compute the normal curvatures along a given direction in $T_p(S)$.

Definition 26 (Gaussian Curvature, Mean Curvature)**.** Let $p \in S$ and let $\mathrm{d}N_p T_p(S) \to T_p(S)$ be the differential of the Gauss map. The determinant $\det(dN_p)$ is the **Gaussian Curvature** κ of S at p. The value $1/2\mathrm{trace}(dN_p)$ is called the **mean curvature H of S at p**. In terms of principal curvatures, these quantities are:

$$\kappa = k_1 \cdot k_2$$
$$H = \frac{1}{2}(k_1 + k_2)$$

since k_1 and k_2 are the eigenvalues.

Definition 27 (Elliptic, Hyperbolic, Parabolic, Planar)**.** A point $p \in S$ is called:

- **Elliptic** if $\det(dN_p) > 0$
- **Hyperbolic** if $\det(dN_p) < 0$
- **Parabolic** if $\det(dN_p) = 0$ and $dN_p \neq 0$
- **Planar** if $dN_p \equiv 0$.

Examples of using this classification:

- Elliptic points: all points on a sphere, the point $(0,0,0)$ of the paraboloid $z = x^2 + ky^2$, $k > 0$.
- Hyperbolic points: the point $(0,0,0)$ of a hyperbolic paraboloid $z = y^2 - x^2$.
- Parabolic points: the points of a cylinder.

Definition 28 (Umbilical Points)**.** If at $p \in S$, $k_1 = k_2$, then p is called an **umbilical point** of S. The planar points $k_1 = k_2 = 0$ are called umbilical points. The points of a sphere are also umbilical points.

Proposition 10. If all the points of a connected surface S are umbilical points, then S is either (a) contained in a sphere or (b) contained in a plane.

Definition 29 (Asymptotic Direction or Curve)**.** Let $p \in S$.

1. An **asymptotic direction** of S at p is a direction of $T_p(S)$ for which the normal curvature is zero.
2. An **asymptotic curve** of S is a regular connected curve $C \subseteq S$ such that $\forall\ p \in S$, the tangent line of C at p is an asymptotic direction.

1. At an elliptic point, there are no asymptotic directions.
2. The Dupin indicatrix provides a useful geometric interpretation of the asymptotic directions.

Definition 30 (Dupin Indicatrix)**.** Let $p \in S$. Then, the **Dupin Indicatrix** at p is the set of vectors w of $T_p(S)$ such that $II_p(w) = \pm 1$.

Definition 31 (Conjugate Point)**.** Let $p \in S$ be a point. Two non-zero vectors $w_1, w_2 \in T_p(S)$ are **conjugate** if $\langle dN_p(w_1), w_2\rangle = \langle w_2, dN_p(w_2)\rangle = 0$. Two directions r_1, r_2 at p are **conjugate** if a pair of non-zero vectors w_1, w_2, are parallel to r_1, r_2, respectively, are conjugate.

5.2 Ruled Surfaces and Minimal Surfaces

6 The Intrinsic Geometry of Surfaces

6.1 Isometries and Conformal Maps