Using Lie Groups to Solve Differential Equations

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June 6, 2023

1 Introduction and Motivation

In this report, we introduce a way to formally solve differential equations using symmetries. To motivate this, recall how we derived the fundamental solution of the heat equation in Math 580: the first step was to identify symmetries or invariants of the equation $\Delta u - u_t = 0$ [4]. In particular, we noticed that if a function $u(x,t)$ satisfied the heat equation, then so did $v(x,t) := u(\lambda x, \lambda^2 t)$ for $\lambda > 0$. We called this invariance *invariance under parabolic scaling*, which led to the following Ansatz for the solution: $\Phi(x,t) = w\left(\frac{|x|}{\sqrt{t}}\right)$. This assumption led to a simpler differential equation to solve, and solving that led to the heat kernel,

$$\Phi(x,t) = \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{x^2}{4t}\right). \tag{1}$$

This style of reasoning can be formalized, and applied to other differential equations; the purpose of this report is to investigate one of the ways we can do this. Generally speaking, symmetry methods are among the most powerful tools to study differential equations [1]. Sophus Lie, a Norwegian mathematician, initiated the study of Lie groups in the 1800s. He hoped to develop a theory analogous to Evariste Galois' theory of using group theory to study polynomial equations.

The main goal of this project is to briefly introduce the theoretical justification behind symmetry methods, and apply the theory to recover the fundamental solution of the heat equation (1). To that end, in Section 2 we develop the necessary theoretical tools, which will be Lie theory; in Section 3, we present the specific tools needed to apply Lie theory to differential equations; finally, in Section 4, we illustrate this technique by applying this machinery to solve the heat equation. To ensure this report is self-contained, we introduce the basics of smooth manifolds in the appendices. Throughout this report, we follow Carl Ohrnell's notes [2] closely; we also draw from Peter Olver book *Applications of Lie Groups to Differential Equations* [1] and from John Lee's *Smooth Manifolds* [3] (see the Bibliography 5 for full details).

2 Background and Context (Preliminary Lie Theory)

In this section, we introduce the requisite Lie theory, and establish a crucial correspondence between Lie groups (the objects we will use to capture a symmetry of a differential equation, but are complicated), and Lie algebras (which have a pleasant vector space structure). Lie groups are objects that combine the algebraic properties of a group with the topological and differentiable structure of a smooth manifold.

Definition 1 (Lie Group)**.** A **Lie Group** G is a smooth manifold equipped with a group structure such that the group multiplication, $\mu : G \times G \to G$, $(x,y) \mapsto xy$ and the inversion $\iota : G \to G, x \mapsto x^{-1}$ are smooth maps.

Some canonical examples of Lie groups include invertible matrices $GL(n,\mathbb{R})$, the real numbers $\mathbb{R}^n$, and the circle group $S^1 \subseteq \mathbb{C}$ (all under the usual group operations). We are interested in Lie groups as acting as transformations on manifolds; in particular, for applications we are often only interested in their local actions, since sometimes the group action is only defined for part of the group or part of the manifold.

Definition 2 (Local Group of a Transformation)**.** Let M be a smooth manifold. A **local group of transformations** acting on M is given by a (local) Lie group G, an open subset $\mathcal{U} \subseteq G \times M$ and a map ψ:

$$\begin{aligned}\psi : \{e\} \times M \subseteq \mathcal{U} \to \mathcal{U},\\ \psi(g, x) \mapsto g \cdot x,\end{aligned}$$

such that

1. **(Associativity)**: If $(h, x) \in \mathcal{U}$, $(g, h \cdot x) \in \mathcal{U}$, then $(gh, x) \in \mathcal{U}$ and $g \cdot (h, x) = (gh) \cdot x$.
2. **(Identity)**: for all $x \in M$, $e \cdot x = x$.
3. **(Inverse)** If $(g, x) \in \mathcal{U}$, then $(g^{-1}, g \cdot x) \in \mathcal{U}$ and $g^{-1} \cdot (g \cdot x) = x$.

We'll denote a group G acting on a manifold M by $G \curvearrowright M$.

Definition 3 (Orbit)**.** The **orbit** of G through a point x_0 is the set of all points that can be reached by applying sequences of group transformations from G starting at the point $x_0 \in M$.

Assuming some technical conditions[1] are met, the orbits of a group are sub-manifolds of M. The following example illustrates these two definitions.

Example 1 (Circle Group)**.** The circle group is a Lie group. We can study how it acts on $M = \mathbb{R}^2$. Elements of the circle group rotate points by an angle θ:

$$g_\theta \cdot \begin{bmatrix} x \\ y \end{bmatrix} := \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \tag{2}$$

The orbits of this group action are precisely the circles in $\mathbb{R}^2$. We can write this as the zero level set of the following function,

$$f(x, y) = x^2 + y^2 - c. \tag{3}$$

We'll now introduce what it means for a group G to be a symmetry group of an equation. We will modify this definition later to define what it means to be a symmetry group of a differential equation.

Definition 4 (Symmetry Group)**.** Let $G \curvearrowright M$, and $f : M \to \mathbb{R}$. We say that G is a **symmetry group** of $f(x) = 0$ if and only if the **solution set of** f, denoted by

$$S_f := \{x \in M \mid f(x) = 0\}, \tag{4}$$

is invariant under G. This is a sub-manifold of M.

Now we need to begin working on establishing the correspondence between Lie groups and simpler objects that we can study using linear algebra: Lie algebras. To do this, we'll need a few more definitions.

Definition 5 (Left Translation)**.** Let $(G, \cdot)$ be a Lie group. Let $x \in G$. The **left-translation** is defined as:

$$\begin{aligned}L_x : G \to G\\ y \mapsto x \cdot y.\end{aligned}$$

This is a diffeomorphism when viewed as a map on the underlying smooth manifold, with inverse being left-translation by $x^{-1} \in G$. This is important to note, since this will allow us to view the map that the left-translation induces on the tangent space of G.

The following special kinds of vector fields will be the building blocks of the Lie algebra.

[1] We require that the group action is *regular*. This is not important for our purposes, but this is explained in Chapter 5 of

Definition 6 (Left Invariant)**.** A vector field $\mathbf{X}$ on a Lie group G is said to be **left-invariant** if it is preserved under the differential of the left-translation map,

$$dL_x(\mathbf{X}) = \mathbf{X}, \tag{5}$$

for every $x \in G$. We'll denote by $L(G)$ the set of all left-invariant vector fields on a Lie group G.

Observe that $L(G)$ is closed under linear combinations, which gives it a vector space structure. This means that we can study $L(G)$ using linear algebra techniques, so $L(G)$ seems like a tractable object to study. Moreover, elements of $L(G)$ are uniquely determined by their value at the identity of G, denoted by $\mathbf{X}_e$, since we can obtain the value of the vector field at any other point $x \in G$ by left-translating $\mathbf{X}_e$ to x. These crucial observations give us an identification between all left-invariant vector fields of a Lie group and the tangent space at the identity; this identification is made precise in the theorem below.

Theorem 1 (Lie Algebra-Lie Group Correspondence)**.** *The space of all left-invariant vector fields of a Lie group $L(G)$ is isomorphic to the tangent space at the identity $T_e(G)$.*

Using this theorem, and the vector space structure on $L(G)$, we are prepared to define the Lie algebra associated with a Lie group.

Definition 7 (Lie Algebra)**.** Let G be a Lie group. The **Lie Algebra** of G, denoted by $\mathfrak{g}$, is the space of all left-invariant vector fields on G, with the Lie bracket as the multiplication operation. This turns $\mathfrak{g}$ into an algebra.

The Lie algebra is a *much* more manageable object to study than a Lie group, and thanks to the Lie Algebra-Lie Group correspondence, we can study the Lie algebra instead. There is one more correspondence, namely between the **one-parameter subgroups of G** and a Lie algebra $\mathfrak{g}$, but before doing this we need a few more definitions.

Definition 8 (Integral Curve)**.** Let $\mathbf{X}$ be a vector field on M. An **integral curve** of $\mathbf{X}$ is a differentiable curve $\gamma : J \to M$ whose velocity at each point is equal to the value of the vector field $\mathbf{X}$ at each point:

$$\gamma'(\varepsilon) = \mathbf{X}(\gamma(\varepsilon)). \tag{6}$$

The integral curve gives rise to the flow generated by a vector field.

Definition 9 (Flow)**.** Let $\mathbf{X}$ be a vector field. Define the following map: $\theta : \mathbb{R} \to M$, $(\varepsilon, p) \mapsto \gamma_p(\varepsilon)$. θ is called the **flow** generated the vector field $\mathbf{X}$. We call $\mathbf{X}$ the **infinitesimal generator** of θ. We can also write the flow as,

$$\gamma_{\mathbf{X}}(\varepsilon) = \exp(\varepsilon \mathbf{X})x. \tag{7}$$

We also call the task of computing the flow generated by a vector field **exponentiating** the vector field.

The final proposition establishes the final correspondence, namely the one between the Lie algebra of G, $\mathfrak{g}$, and one-parameter subgroups of G.

Proposition 1. Let $\mathbf{X} \in L(G)$ be a left-invariant vector field on a Lie group G. The flow generated by X, which is given by:

$$g_\varepsilon = \exp(\varepsilon \mathbf{X})e = \exp(\varepsilon \mathbf{X}), \tag{8}$$

is defined for all $\varepsilon \in \mathbb{R}$. This forms a one-parameter subgroup of G with the following properties:

$$g_{\varepsilon+\delta} = g_\varepsilon \cdot g_\delta,\ g_0 = e,\ g_\varepsilon^{-1} = g_{-\varepsilon}. \tag{9}$$

On the other hand, any connected one-dimensional subgroup of G is generated by a left-invariant vector field.

The following example will illustrate how we can move between vector fields and flows.

Example 2. Consider the following vector field $\mathbf{X} = \frac{\partial}{\partial x} + 2\frac{\partial}{\partial y} - t\frac{\partial}{\partial t}$. We'll use the following notation to denote the components of the integral curve: $\varphi(\varepsilon) = (\varphi_x(\varepsilon), \varphi_y(\varepsilon), \varphi_t(\varepsilon))$. We need to find the integral curve it generates, which amounts to solving the following system of differential equations:

$$\varphi_x'(\varepsilon) = 1,\ \varphi_y'(\varepsilon) = 2,\ \varphi_t'(\varepsilon) = -t.$$

This gives us the following general solutions:

$$\varphi_x(\varepsilon) = \varepsilon + c_1,\ \varphi_y(\varepsilon) = 2\varepsilon + c_2,\ \varphi_t(\varepsilon) = c_3 e^{-\varepsilon}.$$

If the initial conditions are (x, y, t), the flow is given by $\varphi(\varepsilon) = (\varepsilon + x, 2\varepsilon + y, te^{\varepsilon})$.

To summarize, we have taken a Lie group G that acts on a manifold M. This is a very complicated object to study, so we've reduced it to a linear algebra problem by identifying it with the algebra of all left-invariant vector fields $L(G)$. Moreover, we've identified the left-invariant vector fields with one-parameter subgroups of G: flows. Practically, this means that we can find the transformation group by solving for the integral curves. Combining this with a modified symmetry condition for differential equations is ultimately how this method to solve differential equations works. This will be explained in the next section.

3 Application of Lie Groups to Differential Equations

We are now ready to apply Lie theory to finding symmetry groups of differential equations. Analogous to how we defined the solution set of an equation to be a certain submanifold, we need a suitable notion for what a solution set to a differential equation is. One subtlety is that since we are working with differential equations, we need to treat all the relevant partial derivatives as variables in their own right; this is made precise with the nth jet space.

Set-up: we have an nth order differential equation, given by

$$F(x, u^{(n)}) = 0, \tag{10}$$

where $x = (x^1, ..., x^p)$ is the vector of the p independent variables, and $u^{(n)}$ is all the partial derivatives of u up to and including order n.

Definition 10 (Jet Space)**.** Let $x \in \Omega \subseteq \mathbb{R}^p$, $u \in U \subseteq \mathbb{R}$. Let U_k be the space containing all of the partial derivatives of u up to order k. We define the **jet space of order** n as the following:

$$M^{(n)} := \Omega \times U \times U_1 \times ... \times U_n. \tag{11}$$

This space is also called the n**th prolongation** of the space of variables $M \cong \Omega \times U$, since we need to prolong the space of variables to include information about the partial derivatives of u.

Now, consider the following one-parameter Lie group g_ε. It acts on M as:

$$(x, u) \mapsto g_\varepsilon \cdot (x, u) = (\tilde{x}, \tilde{u}) = (\tilde{x}(x, u, \varepsilon), \tilde{u}(x, u, \varepsilon)). \tag{12}$$

We want to eventually find all the g_ε, as these will be the symmetries of the differential equation. Thanks the correspondence between one-parameter subgroups of a Lie group, this flow has an infinitesimal generator, which we may write as

$$\mathbf{X} = \sum_{i=1}^{p} \xi(x, u)\frac{\partial}{\partial x^i} + \eta(x, u)\frac{\partial}{\partial u}. \tag{13}$$

Since $\mathbf{X}$ is the infinitesimal generator of the flow, this means that the components of $\mathbf{X}$ satisfy

$$\xi^i(x,u) = \frac{\partial \tilde{x}^i}{\partial \varepsilon},\ \eta(x,u) = \frac{\partial \tilde{u}^i}{\partial \varepsilon}. \tag{14}$$

How does G act on the jet space $M^{(n)}$? To determine this, we must also prolong the transformations themselves into a map between the jet spaces $g^{(n)} : M^{(n)} \to M^{(n)}$. The first step to do this is to define the notion of the *total derivative*, which treats x and $u^{(n)}$ as variables in their own right.

Definition 11. For a differential function $F(x, u^{(n)}) : M^{(n)} \to \mathbb{R}$, we define the **total derivative** as:

$$D_i F := \frac{\partial F}{\partial x^i} + u_i \frac{\partial F}{\partial u} + \sum_{j=1}^{p} u_{ij} \frac{\partial F}{\partial u_j} + ... \tag{15}$$

The main technical machinery in this section is the following proposition, which gives us a recursive formula to find the coefficients of the generating vector field of the one-parameter group of the transformation G.

Proposition 2 (Prolongation Formulae)**.** Let G be a one-parameter group of transformations acting on $M \simeq \mathbb{R}$ generated by the vector field

$$\mathbf{X} = \sum_{i=1}^{p} \xi^i(x,u) \frac{\partial}{\partial x^i} + \eta(x,u) \frac{\partial}{\partial u}. \tag{16}$$

The first prolongation of the generator of $G^{(1)} \curvearrowright M^{(1)}$ is given by

$$\mathbf{X}^{(1)} = \mathbf{X} + \sum_{i=1}^{p} \eta^i(x, u^{(i)}) \frac{\partial}{\partial u_i}. \tag{17}$$

The second prolongation of the generator $G^{(2)} \curvearrowright M^{(2)}$ is given by:

$$\mathbf{X}^{(2)} = \mathbf{X}^{(1)} + \sum_{i,j=1}^{p} \eta^{ij}(x, u^{(2)}) \frac{\partial}{\partial u_{ij}}, \tag{18}$$

up until the nth prolongation. Then, we can recursively compute the η functions using the following formulae:

$$\eta^i = D_i(\eta) - \sum_{i=1}^{p} u_k D_i(\xi^k), \tag{19}$$

$$\eta^{ij} = D_j(\eta^i) - \sum_{k=1}^{p} u_{ik} D_j(\xi^k). \tag{20}$$

Finally, we can formulate the symmetry condition for differential equations.

Theorem 2 (Symmetry Condition)**.** *Let G be a connected Lie group. We say that it is a symmetry group of a nth order differential equation $F(x, u^{(n)}) = 0$ if and only if*

$$\mathbf{X}^{(n)}(F(x, u^{(n)})) = 0, \tag{21}$$

whenever $F(x, u^{(n)}) = 0$ for every infinitesimal generator $\mathbf{X} \in \mathfrak{g}$.

4 The Heat Equation

To bring all this theory down to Earth, we will concretely illustrate this technique by applying these methods to solve the heat equation in one dimension.

Theorem 3 (Main PDE Argument)**.** *The fundamental solution of the heat equation,*

$$u_t = u_{xx}, \tag{22}$$

can be recovered using symmetry techniques.

Proof. The overall procedure, which can be applied to any differential equation, is as follows: (1) using the structure of the PDE, write down the generating vector field, and the formula for the nth prolongation of the vector field (where n is the highest order derivative present in the PDE), (2) Use the PDE to obtain a symmetry condition, which will be some relationship between the coefficients of the nth prolongation, (3) Use the recursive relationships in Proposition 2 to obtain a set of determining PDEs for the coefficients of the generating vector field, (4) Find the general solutions for this system of differential equations, then use the constants to obtain m linearly independent vector fields that span the Lie algebra, (5) Exponentiate the vector fields to obtain the symmetry groups, then finally (6) Use a known trivial solution, e.g. $u = c$, and the symmetry groups to build up more complicated solutions from the known solutions.

Consider (22). Since this PDE has two independent variables (x and t), the symmetry groups will be generated by the following vector field:

$$\mathbf{X} = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u}. \tag{23}$$

The objective is to figure out what the functions ξ, τ, and η are. Since the highest-order derivative present in (22) is u_{xx}, the solution manifold will lie in the second jet space $M^{(2)}$, since we must account for the group's actions on all the second-order partial derivatives. So the vector field will be of the following form:

$$\mathbf{X}^{(2)} = \mathbf{X} + \eta^x\frac{\partial}{\partial u_x} + \eta^t\frac{\partial}{\partial u_t} + \eta^{xx}\frac{\partial}{\partial u_{xx}} + \eta^{xt}\frac{\partial}{\partial u_{xt}} + \eta^{tt}\frac{\partial}{\partial u_{tt}}. \tag{24}$$

Using that the PDE says $F = u_t - u_{xx} = 0$, we obtain the symmetry condition by applying the second prolongation to F and demanding that it is equal to zero:

$$\mathbf{X}^{(2)}(F) = \mathbf{X}^{(2)}(u_t - u_{xx}) = \eta^t - \eta^{xx} = 0, \tag{25}$$

since the partial derivatives are with respect to the partial derivatives of u viewed as variables themselves. Next, we use the recursive relations in Proposition 2 to determine the coefficients of the second prolongation. We have,

$$\eta^t = D_t(\eta) - u_x D_t(\xi) - u_t D_t(\tau), \tag{26}$$

where D is the total derivative operator. This yields,

$$D_t(\eta) = \eta_t + u_t\eta_u. \tag{27}$$

The computations for $D_t(\xi)$ and $D_t(\tau)$ are identical. Substituting these into (26), after simplifications, yields:

$$\eta^t = \eta_t + \xi_t u_x + (\eta_u - \tau_t)u_t - \xi_u u_x u_t - \tau_u u_t^2. \tag{28}$$

Lee's Smooth Manifolds.

Carrying out identical computations for η^{xx} yields:

$$
\begin{aligned}
\eta^{xx} &= \eta_{xx} + (2\eta_{xx} - \xi_{xx})u_x - \tau_{xx}u_t + (\eta_{uu} - 2\xi_{xu})u_x^2 - 2\tau_{xu}u_xu_t - \xi_{uu}u_x^3 \\
&\quad - \tau_{uu}u_x^2u_t + (\eta_u - 2\xi_x)u_xx + 2\tau_xu_{xt} - 3\xi_uu_xu_{xx} - \tau_uu_tu_{xx} - 2\tau_uu_xu_{xt}.
\end{aligned}
$$

Plugging these values into the symmetry condition $\eta^t = \eta^{xx}$ and doing lots of tedious simplifications yields the following determining equations, read off by equating the coefficients of each monomial in the jet space.[2]

Monomial	**Determining Equation**	**Equation Number**
u_xu_{xt}	$-2\tau_u = 0$	(E1)
u_{xt}	$-2\tau_x = 0$	(E2)
u_{xx}^2	$-\tau_u = -\tau_u$	(E3)
$u_x^2u_{xx}$	$-\tau_{uu} = 0$	(E4)
u_xu_{xx}	$-\xi_u = -2\tau_{xu} - 3\xi_u$	(E5)
u_{xx}	$\eta_u - \tau_t = -\tau_{xx} + \eta_u - 2\xi_x$	(E6)
u_x^3	$-\xi_{uu} = 0$	(E7)
u_x^2	$2\xi_{xu} - \eta_{uu} = 0$	(E8)
u_x	$-\xi_t = 2\eta_{xu} - \xi_{xx}$	(E9)
1	$\eta_t = \eta_{xx}$	(E10)

These differential equations can be solved exactly, see Carl Ohrnell's notes for details [2]. We obtain the following set of general solutions:

$$
\begin{aligned}
\tau(t) &= c_1 + c_2t + c_3t^2 \\
\xi(x,t) &= \frac{1}{2}(c_2 + 2c_3t)x + c_4 + c_5t \\
\eta(x,t,u) &= \left(-\frac{1}{6}c_6 - \frac{1}{2}c_5x - \frac{1}{2}c_3t - \frac{1}{4}c_3x^2\right)u + \alpha(x,t).
\end{aligned}
$$

By plugging each coefficient into (23), and setting each $c_i = 0$ for $i \in \{1, ..., \hat{j}, ..., 6\}$ we get the vector fields $\mathbf{X}_j$:

$$
\begin{aligned}
\mathbf{X}_1 &= \frac{\partial}{\partial t} \\
\mathbf{X}_2 &= x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} \\
\mathbf{X}_3 &= 4xt\frac{\partial}{\partial x} + 4t^2\frac{\partial}{\partial t} - (x^2 + 2t)u\frac{\partial}{\partial u} \\
\mathbf{X}_4 &= \frac{\partial}{\partial x} \\
\mathbf{X}_5 &= 2t\frac{\partial}{\partial x} - xu\frac{\partial}{\partial u} \\
\mathbf{X}_6 &= u\frac{\partial}{\partial u}.
\end{aligned}
$$

These six vector fields span the Lie algebra, and we also get the following infinite-dimensional sub-algebra,

$$\mathbf{X}_\alpha = \alpha(x,t)\frac{\partial}{\partial u}, \tag{29}$$

where α is any solution of the heat equation. This shows up since during the computations we got that $\alpha_t(x,t) = \alpha_{xx}(x,t)$. Finally, we take the exponential of each of these seven vector fields to obtain the

[2]This table matches up with the one in Olver's book rather than Carl's notes, since I believe there is a typo in Carl's notes.

symmetry groups of the heat equation. I will show the computation for $\mathbf{X}_1 \to G_1$; the rest are done in a similar fashion. We need to compute the curve whose components satisfy,

$$\varphi_x'(\varepsilon) = 0,\ \varphi_t'(\varepsilon) = 1,\ \varphi_u'(\varepsilon) = 0. \tag{30}$$

This is a system of ODEs with initial condition (x, t, u). This yields the following particular solution:

$$\varphi(\varepsilon) = (x, t + \varepsilon, u). \tag{31}$$

This means the symmetry group G_1 is $(x, t + \varepsilon, u)$. This symmetry captures the time-invariance of the heat equation; explicitly, if $f(x, t)$ solves the heat equation, then so does $v(x, t) = f(x, t - \varepsilon)$. We have the six other symmetry groups below, as well as how they act on functions, i.e., if $u = f(x, t)$ is a solution, then so are the $u^{(i)}$:

Group label	**Group**	$u^{(i)}$ **is also a solution**
G_1	$(x, t + \varepsilon, u)$	$u^{(1)} = f(x, t - \varepsilon)$
G_2	$(e^{\varepsilon}x, e^{2\varepsilon}t, u)$	$u^{(2)} = f(e^{-\varepsilon}x, e^{-2\varepsilon}t)$
G_3	$\left(\frac{x}{1-4\varepsilon t}, \frac{t}{1-4\varepsilon t}, u\sqrt{1-4\varepsilon t}\exp\left(-\frac{\varepsilon x^2}{1-4\varepsilon t}\right)\right)$	$u^{(3)} = \frac{1}{\sqrt{1+4\varepsilon t}}\exp\left(\frac{-\varepsilon x^2}{1+4\varepsilon t}\right) f\left(\frac{x}{1+4\varepsilon t}, \frac{t}{1+4\varepsilon t}\right)$
G_4	$(x + \varepsilon, t, u)$	$u^{(4)} = f(x - \varepsilon, t).$
G_5	$(x + 2\varepsilon t, t, u\exp(-\varepsilon x - \varepsilon t^2))$	$u^{(5)} = \exp\left(-\varepsilon x + \varepsilon^2 t\right) f(x - 2\varepsilon t, t)$
G_6	$(x, t, e^{\varepsilon}u)$	$u^{(6)} = e^{\varepsilon} f(x, t)$
G_α	$(x, t, u + \varepsilon\alpha(x, t))$	$u^{(\alpha)} = f(x, t) + \varepsilon\alpha(x, t)$

We see that the symmetry G_2 gives us the parabolic invariance that was discussed in the Introduction, and G_α tells us about the principle of superposition. How can we use all of this to actually solve the heat equation? The idea is to take an easily known solution, and use the symmetry groups G_i to build more complicated solutions. We do this as follows: suppose $u = f(x)$ solves the PDE. Then, if G is a symmetry group of the PDE such that $g_\varepsilon \in G$, then $u = g_\varepsilon \cdot f$ is also a solution. Let's try to recover the fundamental solution (1). Concretely, let us take $u = c$ for any $c \in \mathbb{R} \setminus \{0\}$. This clearly satisfies the heat equation. For some ε first apply a symmetry from G_3:

$$\begin{aligned} u_1 &= g_\varepsilon \cdot c \\ &= \frac{c}{\sqrt{1 + 4\varepsilon t}}\exp\left(\frac{-\varepsilon x^2}{1 + 4\varepsilon t}\right). \end{aligned}$$

By symmetry, u_1 solves the heat equation. Now fix $c = \sqrt{\frac{\varepsilon}{\pi}}$. This gives us the following intermediate solution:

$$u_2 = \sqrt{\frac{\varepsilon}{\pi(1 + 4\varepsilon t)}}\exp\left(\frac{-\varepsilon x^2}{1 + 4\varepsilon t}\right). \tag{32}$$

Finally, to recover the fundamental solution, we'll use the time-invariance (G_1) to translate time by $-\frac{1}{4\varepsilon}$. This immediately gives us the heat kernel,

$$u_3 = \frac{1}{\sqrt{4\pi t}}\exp\left(-\frac{x^2}{4t}\right). \tag{33}$$

□

5 Conclusion and Future Work

One can obtain even more solutions by considering linear combinations of vector fields, which will generate even more symmetry groups. Moreover, this method can be applied to other PDEs studied in Math 580, including Burger's Equation, Laplace's Equation, and the wave equation. Furthermore, symmetry methods are closely related to conservation laws, see Chapter 4 of [1] for a discussion about this. Additionally, there are applications of these methods to numerical analysis (see [5], [6]). These are all very interesting extensions of this project that I intend to work on, as well as learning more Lie theory to understand the geometric proofs behind the auxiliary results.

Appendix A Requisite Differential Geometry

The central setting for the first part of this report are smooth manifolds. The definitions are nearly quoted from Olver's book [1], Lee's book [3], and Carl's notes [2].

Definition 12 (Smooth Manifold)**.** An m-dimensional **manifold** is a set M, together with a countable collection of subsets $U_\alpha \subseteq M$ called **coordinate charts**, and bijective functions $\varphi_\alpha : U_\alpha \to V_\alpha$ onto connected open subsets $V_\alpha \subseteq \mathbb{R}^n$, called **local coordinate maps**, which satisfy the following properties:

1. The open sets cover M.
2. M is a Hausdorff topological space.
3. On the overlap of any pair of coordinate charts $U_\alpha \cap U_\beta$, the transition functions
$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta),$$
are smooth C^∞ functions.

Since we aim to solve differential equations, we must do calculus; we can't do this on the manifold itself, we need to do it on a different space.

Definition 13 (Tangent Vector)**.** Let $C \subset M$ be a smooth curve, parameterized by $\gamma : I \to M$, where $I \subseteq \mathbb{R}$. In local coordinates $x = (x^1, ..., x^n)$, each of the components of C are smooth functions: $\gamma(t) = (\gamma^1(t), ..., \gamma^m(t))$. The **tangent vector** at a point x on C is the component-wise derivative:

$$\dot{\gamma}(t) = \dot{\gamma}^1(t)\frac{\partial}{\partial x^1} + ... + \dot{\gamma}^m(t)\frac{\partial}{\partial x^m}.$$

Definition 14 (Tangent Space)**.** The **tangent space** to M at a point x, denoted T_xM, is the set of all possible tangent vectors at x.

If we select one tangent vector from T_xM for each x in a smooth way, we get a vector field.

Definition 15 (Vector Field)**.** A **smooth vector field X** is an assignment of a tangent vector $X_x \in T_xM$ to each point x in M that varies in a smooth way. In local coordinates, this looks like:

$$\mathbf{X} = \xi^1(x)\frac{\partial}{\partial x^1} + .. + \xi^m(x)\frac{\partial}{\partial x^m}, \tag{34}$$

where each $\xi_i(x)$ is a smooth function.

We can view the tangent vectors $\frac{\partial}{\partial x_i}$ as the basis vectors for the tangent space.

Definition 16 (Integral Curve)**.** An **integral curve** of a vector field **X** is a smooth parameterized curve $x = \gamma(t)$ whose tangent vector is equal to the value of the vector field **X** at that point:

$$\dot{\gamma(t)} = \mathbf{X}(\gamma(t)). \tag{35}$$

In local coordinates, computing an integral curve boils down to solving the following system of ODE's, for $i = 1, ..., m$:

$$\frac{dx^i}{dt} = \xi^i(x). \tag{36}$$

Every point x on the manifold M passes through a unique maximal integral curve, thanks to the theory of existence and uniqueness for systems of ODEs.

Next we want to consider maps between manifolds. First, we'll define what a diffeomorphism is.

Definition 17 (Diffeomorphism)**.** Let M, N be two manifolds, and $f : M \to N$ be a map. We say that f is a **diffeomorphism** if f is bijective, and if both f and f^{-1} are C^∞.

Maps between manifolds induce a map between the tangent spaces of the manifold, which tells us how tangent vectors get transported under maps. This map is called the differential.

Definition 18 (Differential)**.** Let $F : M \to M$ be a smooth map, $x = \gamma(t)$ a curve in M, and $F(x) = F(\gamma(t)) \in N$ is the image of the curve. We define the **differential** of F, denoted by dF, as the following linear map between the tangent spaces, $dF : T_x M \to T_{F(x)} N$, which we obtain by:

$$dF\left(\frac{d}{dt}\gamma(t)\right) := \frac{d}{dt}F(\gamma(t)). \tag{37}$$

To clarify things, we write (37) in local coordinates as:

$$dF\left(X_x\right) = dF\left(\sum_{i=1}^{m} \xi^i(x)\frac{\partial}{\partial x^i}\right) = \sum_{i=1}^{n} X(F^i(x))\frac{\partial}{\partial y^i}, \tag{38}$$

which boils down to the Jacobian matrix of F at the point x.

Finally, we define a suitable way to multiply two vector fields together, which will be the multiplication on the Lie Algebra that we will define in Section 2.

Definition 19. Let X and Y be two vector fields on a manifold M. We define their **Lie bracket** or **commutator** as the unique vector field that satisfies the following:

$$[X, Y](f) := X(Y(f)) - Y(X(f)). \tag{39}$$

Next, we'll define some definitions from Lie theory that are used in the main text. They are taken straight from Lee's *Smooth Manifolds.*

Definition 20 (Group Homomorphism)**.** Let G and H be Lie groups. A **Lie group homomorphism** is a smooth map $F : G \to H$ that is also a group homomorphism.

Definition 21 (One-parameter Subgroup)**.** A **one-parameter subgroup of** G is defined to be a Lie group homomorphism $\gamma : \mathbb{R} \to G$ where we treat $\mathbb{R}$ as a Lie group under standard vector addition.

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