3) Prove the generalized Hölder's Inequality: if $p_1, \dots, p_n$ are such that
$$\sum_{j=1}^{n} \frac{1}{p_j} = 1$$
and $f_j \in L^{p_j}(\Omega)$, then
$$\int_\Omega |f_1(x) \cdots f_n(x)|\, dx \le \|f_1\|_{L^1} \cdots \|f_n\|_{L^n}$$

We'll first make a lemma:

<u>Lemma</u> (Generalized Young's Inequality): Let $a_1, \dots, a_n \in \mathbb{R}$. Then:
$$\boxed{a_1 \cdots a_n \le \frac{a_1^{p_1}}{p_1} + \dots + \frac{a_n^{p_n}}{p_n}}$$

<u>Proof</u>: ~~Since $x \mapsto \ln(x)$ is concave,~~

$$\cancel{\ln\left(\frac{a_1^{p_1}}{p_1} + \dots + \frac{a_n^{p_n}}{p_n}\right) \ge \frac{1}{p_1} \ln(a_1^{p_1}) + \dots +}$$

$x \mapsto e^x$ is convex $\Rightarrow$
$$e^{\sum_{i=1}^{n} \frac{1}{p_i} a^{p_i}} \le \sum_{i=1}^{n} \frac{1}{p_i} e^{\ln(a_i) p^i}$$

Hence,
$$a_1 \cdots a_n = e^{\ln(a_1 \cdots a_n)} = e^{\sum_{i=1}^{n} \frac{1}{p_i} \ln(a_i)^{p_i}}$$
$$\le \sum_{i=1}^{n} \frac{1}{p_i} e^{\ln(a_i)^{p_i}} = \sum_{i=1}^{n} \frac{a_i^{p_i}}{p_i}$$

which is what we wanted to show.

WLOG, assume that $\|f_i\| = 1$ $\forall i = 1, \dots, n$. Since we can normalize any $f \in L^{p_i}(\Omega) \setminus \{0\}$ (Hölder's is trivial when $f=0$), we can make such an assumption. This means that we have to show that
$$\int_\Omega |f_1 \cdots f_n|\, dx \le 1$$

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By the lemma

$$|f_1 \cdots f_n| \leq \frac{|f_1|^p}{p_1} + \ldots + \frac{|f_n|^p}{p_n} \quad \text{a.e. on } \Omega.$$

By the monotonicity and linearity of integration, we have.

$$\int_\Omega |f_1 \cdots f_n| \, dx \leq \int_\Omega \left[\frac{|f_1|^{p_1}}{p_1} + \ldots + \frac{|f_n|^{p_n}}{p_n} \right] dx$$

$$= \frac{1}{p_1} \int_\Omega |f_1|^{p_1} dx + \ldots + \frac{1}{p_n} \int_\Omega |f_n|^{p_n} dx$$

$$= \sum_{i=1}^{n} \frac{1}{p_i} \qquad (\text{Since } \|f_i\| = 1 \ \forall i = 1, \ldots, n)$$

$$= 1 \qquad (\text{by assumption})$$