TASKS FOR MATH 589

Instructions:

1. Complete **at least five tasks** in total, with **at least one task** chosen from each chapter.

2. For each of the tasks you have chosen, follow the specific instruction (if any) and try your best to give a complete, rigorous, self-contained and well-explained solution.
One standard you may follow in preparing your exposition: a last-year undergraduate or first-year graduate student in math/stat (without previous knowledge of Math 589 topics) should be able to follow all the steps in your solution.

3. Give explicit references to all the results (outside of the course materials of Math 587/589) that you have used without proof in your solutions, including the names of the results and the sources (book, chapter/section/page number, webpage, etc). You may use any reference that you consider proper in completing these tasks, but do not copy from the reference in any substantial way (e.g., copying word by word for a great length).

4. This file will be updated as the course progresses. Please check regularly for updates.

1. Convergence of Measures and Central Limit Theorem

Task 1. Prove the **Law of the Iterated Logarithm** for the sum of i.i.d. random variables.

Assume that $\{X_n : n \geq 1\}$ is a sequence of i.i.d. random variables on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with $\mathbb{E}[X_1] = 0$ and $\mathbb{E}\left[X_1^2\right] = 1$. For every $n \geq 1$, $S_n := \sum_{j=1}^n X_j$, $\Lambda_n := \sqrt{2n \ln\ln(n \vee 3)}$. Then,

$$\limsup_{n\to\infty} \frac{S_n}{\Lambda_n} = 1 = -\liminf_{n\to\infty} \frac{S_n}{\Lambda_n} \text{ almost surely.}$$

Task 2. Fill up all the detailed in the proof of the following theorem (*See the note of Lecture 3 on Jan 13*).

Theorem: Given μ, ν two probability measures on $\left(\mathbb{R}^d, \mathscr{B}\left(\mathbb{R}^d\right)\right)$. If $\hat{\mu}(\xi) = \hat{\nu}(\xi)$ for every $\xi \in \mathbb{R}^d$, then $\mu = \nu$.

Instruction: Follow the procedures outlined below and give a complete proof of this theorem. The first step was completed in the lecture (Lemma 1 in the lecture note), where we see that a probability measure μ is determined by $\langle \varphi, \mu \rangle$ for all $\varphi \in C_b\left(\mathbb{R}^d\right)$, so you do not have to repeat this step in your solution. Your work should focus on steps 2&3 (Lemma 2&3 in the lecture note).

(1) For Lemma 2, first prove the following claim, which is "the approximation theory by mollification":

Claim. Assume that, for $\varepsilon > 0$, $\rho_\varepsilon(x) := \left(\frac{1}{2\pi\varepsilon}\right)^{d/2} e^{-\frac{|x|^2}{2\varepsilon}}$ for every $x \in \mathbb{R}^d$ (i.e., ρ_ε is the density function of $\gamma_{0,\varepsilon I}$). Given $\varphi \in C_b\left(\mathbb{R}^d\right)$, define

$$x \in \mathbb{R}^d \mapsto \varphi_\varepsilon(x) := \varphi \star \rho_\varepsilon(x) = \int_{\mathbb{R}^d} \varphi(x-y)\,\rho_\varepsilon(y)\,dy.$$

φ_ε is called a *mollification* of φ. Prove that for every $\varepsilon > 0$, $\varphi_\varepsilon \in C^\infty\left(\mathbb{R}^d\right)$, $\|\varphi_\varepsilon\|_u \leq \|\varphi\|_u$ and $\lim_{\varepsilon \searrow 0} \varphi_\varepsilon(x) = \varphi(x)$ at every $x \in \mathbb{R}^d$. Moreover, the convergence of φ_ε to φ is uniform on every compact set.

Then, based on the claim, construct a sequence $\{\varphi_k : k \geq 1\} \subseteq C_c^\infty\left(\mathbb{R}^d\right)$ such that $\|\varphi_k\|_u \leq \|\varphi\|_u$ for every $k \geq 1$, and $\lim_{k\to\infty} \varphi_k(x) = \varphi(x)$ at every $x \in \mathbb{R}^d$.

(2) For Lemma 3, which is "a generalization of Plancherel's Theorem". You may use the *Lebesgue Differentiation Theorem* and *Plancherel's theorem for functions* without proof.

Task 3. Prove the second part of **Prokhorov's Theorem**.

Let $\{\mu_n : n \geq 1\}$ be a sequence of probability measures on $\left(\mathbb{R}^d, \mathscr{B}\left(\mathbb{R}^d\right)\right)$. If $\{\mu_n : n \geq 1\}$ is tight, then there exists a subsequence $\left\{\mu_{n_k} : k \geq 1\right\}$ and a probability measure μon $\left(\mathbb{R}^d, \mathscr{B}\left(\mathbb{R}^d\right)\right)$ such that $\mu_{n_k} \Rightarrow \mu$ as $k \to \infty$.

Instruction: You may use the *Riesz Representation Theorem* and the *Stone-Weierstrass Theorem* without proof.

2. Infinitely Divisible Laws, Lévy Processes and Brownian Motion

Task 4. Prove the "only if" statement of the **Lévy-Khinchine Formula**.

If $\mu \in \mathscr{I}\left(\mathbb{R}^d\right)$, then there exists $m \in \mathbb{R}^d$, $C = \left(C_{i,j}\right)_{d\times d}$ non-negative definite matrix and $M \in \mathfrak{M}_2\left(\mathbb{R}^d\right)$ such that for every $\xi \in \mathbb{R}^d$,

$$\hat{\mu}(\xi) = \exp\left[\mathrm{i}(\xi, m) - \frac{1}{2}(\xi, C\xi) + \int_{\mathbb{R}^d} \left(e^{\mathrm{i}(\xi,y)} - 1 - \mathbb{I}_{B(0,1)}(y)\,\mathrm{i}(\xi,y)\right) M(dy)\right].$$

Instruction: Observe that $\ell_{\hat{\mu}}(\xi) = \lim_{n\to\infty} n\left(\widehat{\mu_{\frac{1}{n}}}(\xi) - 1\right)$. Consider the operator A on $C_c^\infty\left(\mathbb{R}^d\right)$ where for every $\varphi \in C_c^\infty\left(\mathbb{R}^d\right)$,

$$A\varphi := \lim_{n\to\infty} n\left(\left\langle \varphi, \mu_{\frac{1}{n}} \right\rangle - \varphi(0)\right).$$

Study the structure and the property of such an operator A. You may use the *Riesz Representation Theorem* without proof.