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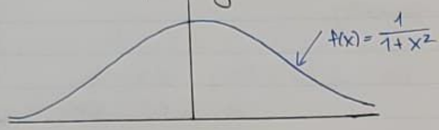
MATH 567: FUNCTIONAL ANALYSIS
ASSIGNMENT #1

1) Show that $(C_c^0(\mathbb{R}^N), \|\cdot\|_\infty)$ is not a Banach Space by constructing a Cauchy sequence which does not converge. What is the completion of $C_c^0(\mathbb{R}^N)$ under $\|\cdot\|_\infty$?

consider the following function:



It's clear that $f(x) \notin C_c^0(\mathbb{R}^N)$ $\forall x \in \mathbb{R}$, $f(x) > 0$. Hence, we can approach $f(x)$ with a sequence of compactly supported functions.

$$f_n := \left(\frac{1+\left(\frac{x}{n}\right)^2}{1+x^2}\right)\left(\chi_{[n,n]}\right)$$

It's clear that each $f_n \in C_c^0(\mathbb{R}^N)$ due to the $\chi_{[n,n]}$ term. It's also clear that $f_n \in C^0(\mathbb{R}^N)$ since $x = \pm n$, $\frac{1+\left(\frac{x}{n}\right)^2}{1+x^2} = 0$, which is the only part of f_n where continuity could be violated.

<u>Claim:</u> $\lim_{n\to\infty} f_n = f$

<u>Proof:</u>

$$\lim_{n\to\infty} f_n = \lim_{n\to\infty}\left(\frac{1+\left(\frac{x}{n}\right)^2}{1+x^2}\right)\lim_{n\to\infty}\left(\chi_{[n,n]}\right)$$

$$= \left(\frac{1-\lim_{n\to\infty}\left(\frac{x}{n}\right)^2}{1+x^2}\right)\chi_{\mathbb{R}} \quad \text{(Since the sequence } \{[-n,n]\}_{n\in\mathbb{N}}$$

$$= \frac{1-0}{1+x^2} \quad \text{(since } x \mapsto x^2 \text{ is cts)}$$

$$= \frac{1}{1+x^2}$$

$\Rightarrow \{f_n\} \subseteq C_c^0(\mathbb{R}^N)$ s.t. $\lim_{n\to\infty} f_n \notin C_c^0(\mathbb{R}^N)$

$\Rightarrow C_c^0(\mathbb{R}^N)$ is not complete.

<u>Claim:</u> the completion of $C_c^0(\mathbb{R}^N)$ under the $\|\cdot\|_\infty$ norm is continuous functions which eventually tend to 0, denote it by.

$$C^{00}(\mathbb{R}^N) := \{f \in C^0(\mathbb{R}^N) \mid \lim_{\|x\|\to 0} |f(x)| = 0\}$$

"$\subseteq$": $\{\text{completion of } C_c^0(\mathbb{R}^N)\} \subseteq C^{00}(\mathbb{R}^N)$

Let $\{f_n\}$ be a sequence in $C_c^0(\mathbb{R}^N)$. Let f be its limit. For a contradiction, assume that $f \notin C^{00}(\mathbb{R}^N)$. Then, $\exists\, \varepsilon_0 > 0$ s.t. $\forall x \in \mathbb{R}$, $|f(x)| > \varepsilon_0$. Since $f_n \to f$, let $N \in \mathbb{N}$, such that $\|f_n - f\|_\infty < \varepsilon_0/2$. Then, by the reverse triangle inequality,

$$\sup_{x\in\mathbb{R}} \|f_n - f\| < \varepsilon_0/2 \quad \Leftrightarrow \quad f_n(x) > \varepsilon_0/2 \quad \forall x \in \mathbb{R}$$
$$\Rightarrow f_n \notin C_c(\mathbb{R}^N) \quad ↯$$

"$\supseteq$": $C^{00}(\mathbb{R}^N) \subseteq \{\text{completion of } C_c^0(\mathbb{R}^N)\}$.

Let $f \in C^{00}(\mathbb{R}^N)$ be arbitrary. Then, we can approximate f by a series of compactly supported func as follows:

$$f_n(x) := \begin{cases} f(x) & x \in [-n, n] \\ \text{line from } (n, f(n)) \text{ to } (n+1, 0) & x \in]n, n+1] \\ \text{line from } (-n, f(n)) \text{ to } (-n-1, 0) & x \in]-n-1, -n] \\ 0 & x \in]-\infty, n-1[\cup]-n, \infty[\end{cases}$$

It's clear that each $f_n \in C_c^0(\mathbb{R}^N)$ (by construction, f_n is supported on $[-n-1, n+1]$). Moreover, the f_n clearly converge to f since $f \in C^{00}(\mathbb{R}^N)$ by construction and since the uniform limit of uniformly continuous functions is at least continuous

$$\boxed{\Rightarrow \text{Completion of } C_c^0(\mathbb{R}^N) = C^{00}(\mathbb{R}^N)}$$

completeness of ... Consider $C^n(\bar{\pi})$ ... some
$n_n \in \mathbb{N}$ s.t. $\forall k \geq n_n$, $C^n(\bar{\pi})$, $\{f_n\} \to f \in C^n(\bar{\pi})$.