Chapter 11: Topological Spaces (General Properties)

Class: Math 455 (Analysis 4) Date: 9 February 2020 Shereen Elaidi

Abstract

This document contains a summary of all the key definitions, results, and theorems from class. There are probably typos, and so I would be grateful if you brought those to my attention :-).

Syllabus: L^p space, duality, weak convergence, Young, Holder, and Minkowski inequalities, point-set topology, topological space, dense sets, completeness, compactness, connectedness, path-connectedness, separability, Tychnoff theorem, Stone-Weierstrass Theorem, Arzela-Ascoli, Baire category theorem, open mapping theorem, closed graph theorem, uniform boudnedness principle, Hahn Banch theorem.

11.1. Open Sets, Closed Sets, Bases, and Sub-bases

Definition 1 (Open Sets)**.** Let X be a non-empty set. A **topology** $\mathcal{T}$ for X is a collection of subsets of X, called **open sets**, posessing the following properties:

(i) The entire set X and the empty set $\emptyset$ are open.
(ii) The finite intersection of open sets are open.
(iii) The union of any collection of open sets is open.

A non-empty set X, together with a topology on X, is called a **topological space**. For a point $x \in X$, an open set that contains x is called a **neighbourhood** of x.

Proposition 1. A subset $E \subseteq X$ is open $\iff$ for each $x \in E$, there exists a neighbourhood of x that is contained in E.

Example 1 (Metric Topology)**.** Let (X, ρ) be a metric space. Let $\mathcal{O} \subseteq X$ be open if for all $x \in \mathcal{O}$, $\exists$ an open ball at x that is contained in $\mathcal{O}$. This collection of open sets forms a topology; we call this the **metric topology** induced by ρ.

Example 2 (Discrete Topology)**.** This topology is "too much." Let X be a non-empty subset. Let $\mathcal{T} := \mathcal{P}(X)$. Then, every set containing a point is a neighbourhood of that point. This is induced by the discrete metric.

Example 3 (Trivial Topology)**.** Let X be non-empty. Define $\mathcal{T} := \{X, \emptyset\}$. The only neighbourhood of any point is the whole set X.

Definition 2 (Topological Subspaces)**.** Let $(X, \mathcal{T})$ be a topological space and let E be a non-empty subset of X. The inherited topology $\mathcal{S}$ for E is the set of all sets of the form $E \cap \mathcal{T}$, where $\mathcal{O} \in \mathcal{T}$. The topological space $(E, \mathcal{S})$ is called a **subspace** of $(X, \mathcal{T})$.

Definition 3 (Base for the Topology)**.** The building blocks of a topology is called a **base**. Let $(X, \mathcal{T})$ be a topological space. For a point $x \in X$, a collection of neighbourhoods of x, B_x, is called a **base for the topology at** X if $\forall$ neighbourhoods $\mathcal{U}$ of x, there exists a set B in the collection B_x for which $B \subseteq \mathcal{U}$.

A collection of open sets $\mathcal{B}$ is called a **base for the topology** $\mathcal{T}$ provided it contains a base for the topology at each point.

A base for a topology completely determines a topology, alongside $\emptyset$ and X.

Proposition 2. For a non-empty set X, let $\mathcal{B}$ be a collection of subsets of X. Then, $\mathcal{B}$ is a base for a topology for $X \iff$:

(i) $\mathcal{B}$ covers X. That is:

$$X = \bigcup_{B \in \mathcal{B}} B \tag{11.1}$$

(ii) If $B_1, B_2 \in \mathcal{B}$, and $x \in B_1 \cap B_2$, then there is a set $B_3 \in \mathcal{B}$ for which $x \in B_3 \subseteq B_1 \cap B_2$.

The unique topology that has $\mathcal{B}$ as its base consists of $\emptyset$ and unions of sub-collections of $\mathcal{B}$.

Definition 4 (Product Topology)**.** Let $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ be two topological spaces. In the cartesian product $X \times Y$, consider the collection of sets $\mathcal{B}$ containing the products $\mathcal{O}_1 \times \mathcal{O}_2$, where $\mathcal{O}_1$ is open in X and $\mathcal{O}_2$ is open in Y. Then, $\mathcal{B}$ is a base for a topology on $X \times Y$, which we call the **product topology.**

Definition 5 (Sub-base)**.** Let $(X, \mathcal{T})$ be a topological space. The collection of $\mathcal{S}$ of $\mathcal{T}$ that covers X is called a **sub-base** for the topology $\mathcal{T}$ provided intersections of finite collections of $\mathcal{S}$ are a base for $\mathcal{T}$.

Definition 6 (Closure)**.** Let $E \subseteq X$ be a subset of a topological space. A point $x \in E$ is called a **point of closure** of E if every neighbourhood of x contains a point in E. The collection of the points of closure of E is called the **closure** of E, denoted $\overline{E}$.

Proposition 3. Let X be a topological space, $E \subseteq X$. Then, $\overline{E}$ is closed. Moreover, $\overline{E}$ is the smallest closed subset of X containing E in the sense that if F is closed and $E \subseteq F$, then $\overline{E} \subseteq F$.

Proposition 4. A subset of a topological space X is open $\iff$ its complement is closed.

Proposition 5. Let X be a topological space. Then, (a) $\emptyset$ and X are closed, (b) the union of a finite collection of closed sets is closed, (c) the intersection of any collection of closed sets in X is closed.

11.2. Separation Properties

Motivation: Separation properties for a topology allow us to discriminate between which topologies discriminate between certain disjoint pairs of sets, which will then allow us to study a robust collection of cts real-valued functions on X.

Definition 7 (Neighbourhood)**.** A **neighbourhood** of K for a subset $K \subseteq X$ is an open set that contains K.

Definition 8 (Separated by Neighbourhoods)**.** We say that two disjoint sets A and B in X can be separated by disjoint neighbourhoods provided that there exists neighbourhoods of A and B, respectively, that are disjoint.

Definition 9 (Separation Properties of Topological Spaces)**.** . In the order of most general to least general, they are:

(i) **Tychonoff Separation Property**: For each two points $u, v \in X$, there exists a neighbourhood of u that does not contain v and a neighbourhood of v that does not contain u.
(ii) **Hausdorff Separation Property**: Each two points in X can be separated by disjoint neighbourhoods.

(iii) **Regular Separation Property**: Tychonoff + each closed set and a point not in the set can be separated by disjoint neighbourhoods.
(iv) **Normal Separation Property**: Tychonoff + each two disjoint closed sets can be separated by disjoint neighbourhoods.

Proposition 6. A topological space is Tychonoff $\iff$ every set containing a single point, $\{x\}$, is closed.

Proposition 7. Every metric space is normal.

Lemma 1. F is closed $\iff$ $\text{dist}(x, F) > 0 \; \forall \; x \notin F$.

Proposition 8. Let X be a Tychonoff topological space. Then, X is normal $\iff$ whenever $\mathcal{U}$ is a neighbourhood of a closed subset of F of X, there is another neighbourhood of F whose closure is contained in $\mathcal{U}$. that is, there is an open set $\mathcal{O}$ for which:

$$F \subseteq \mathcal{O} \subseteq \overline{\mathcal{O}} \subseteq \mathcal{U} \tag{11.2}$$

11.3. Countability and Separability

Definition 10 (Converge, Limit)**.** A sequence $\{x_n\}$ in a topological space X is said to **converge** to the point $x \in X$ if for each neighbourhood $\mathcal{U}$ of x, there exists an index $N \in \mathbb{N}$ such that if $n \geq N$, then x_n belongs to $\mathcal{U}$. This point is called a **limit** of the sequence.

Definition 11 (First and Second Countable)**.** A topological space X is **first countable** if there is a countable base at each point. A space X is said to be **second countable** if there is a countable base for the topology.

Example 4. Every metric space is first countable.

Proposition 9. Let X be a first countable topological space. For a subset $E \subseteq X$, a point $x \in X$ is called a point of closure of E $\iff$ it is a limit of a sequence in E. Thus, a subset E of X is closed $\iff$ whenever a sequence in E converges to $x \in X$, we have that $x \in E$.

Definition 12 (Dense/Separable)**.** A subset $E \subseteq X$ is **dense** in X if every open set in X contains a point of E. We call X **separable** if it has a countable dense subset.

Definition 13 (Metrisable)**.** A topological space X is said to be **metrisable** if the topology is induced by the metric.

Theorem 2. Let X be a second countable topological space. Then, X is metrisable $\iff$ it is normal.

11.4. Continuous Mappings between Topological Spaces

Definition 14 (Continuous)**.** For topological spaces $(X, \mathcal{T})$, $(Y, \mathcal{S})$, a mapping $f : X \to Y$ is said to be **continuous** at the point x_0 in X if, for every neighbourhood $\mathcal{O}$ if $f(x_0)$, there is a neighbourhood $\mathcal{U}$ of x_0 for which $f(\mathcal{U}) \subseteq \mathcal{O}$. We say that f is continuous provided it is continuous at each point in X.

Proposition 10. A mapping $f : X \to Y$ between topological spaces X and Y is continuous $\iff$ for any open subset $\mathcal{O}$ in Y, its inverse image under f, $f^{-1}(\mathcal{O})$, is an open subset of X.

Proposition 11. The composition of continuous mappings between topological spaces, when defined, is continuous.

Definition 15 (Stronger)**.** Given two topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ for a set X, if $\mathcal{T}_2 \subseteq \mathcal{T}_1$, then we say that $\mathcal{T}_2$ is **weaker** than $\mathcal{T}_1$, and that $\mathcal{T}_1$ is **stronger** than $\mathcal{T}_2$.

Proposition 12. Let X be a non-empty set and let $\mathcal{S}$ be a collection of subsets of X that covers X. The collection of subsets of X consisting of intersections of finite collections of $\mathcal{S}$ is a base for a topology $\mathcal{T}$ of X. It is the weakest topology containing $\mathcal{S}$ in the sense that if $\mathcal{T}'$ is any other topology for X containing $\mathcal{S}$, then $\mathcal{T} \subseteq \mathcal{T}'$.

Definition 16 (Weak Topology)**.** Let X be a non-empty set and $\mathcal{F} := \{f_\alpha \mid X \to X_\alpha\}_{\alpha\in\Lambda}$ a collection of mappings, where each X_α is a topological space. The weakest topology for X that contains the collection of sets

$$\{f_\alpha^{-1}(\mathcal{O}_\alpha) \mid f_\alpha \in \mathcal{F},\ \mathcal{O}_\alpha \text{ open in } X_\alpha\} \tag{11.3}$$

is called the **weak topology for X induced by $\mathcal{F}$.**

Proposition 13. Let X be a non-empty set, $\mathcal{F} := \{f_\lambda \mid X \to X_\lambda\}_{\lambda\in\Lambda}$ a collection of mappings where each X_λ is a topological space. The weak topology for X induced by $\mathcal{F}$ is the topology on X that has the fewest number of sets covering the topologies on X for which each mapping $f_\lambda : X \to X_\lambda$ is continuous.

Definition 17 (Homeomorphism)**.** A mapping from a topological space $X \to Y$ is said to be a **homeomorphism** if it is bijective and has a continuous inverse $f^{-1} : Y \to X$. Two topological spaces are said to be **homeomorphic** if there exists a homeomorphism between them. The notion of homeomorphism induces a notion of an equivalence relation between spaces.

11.5. Compact Topological Spaces

Definition 18 (Cover)**.** A collection of sets $\{E_\lambda\}_{\lambda\in\Lambda}$ is said to be a **cover** of a set E if $E \subseteq \bigcup_{\lambda\in\Lambda} E_\lambda$.

Definition 19 (Compact)**.** A topological space X is said to be **compact** if every open cover of X has a finite sub-cover. A subset $K \subseteq X$ is compact if K, considered as a topological space with the subspace topology inherited from X, is compact.

Proposition 14. A topological space X is compact $\iff$ every collection of closed subsets of X that posesses the finite intersection property has non-empty intersection.

Proposition 15. A closed subset K of a compact topological space is compact.

Proposition 16. A compact subspace K of a Hausdorff topological space is a closed subset of X.