

Math 475: Partial Differential Equations

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1 Introduction

Q: Where do PDE's come from? PDEs are used in all types of math modelling; they translate phenomena coming from physics, biology, chemistry, etc. into mathematical terms.

Q: What goes into a PDE model?

Math model = General Physical Laws (balancing forces and conservation of quantities) +
Constitutive Relations (Laws specific to the environment, e.g. Fick's Law of Diffusion)

A PDE \leftrightarrow a physical description of a system.

Definition 1 (PDE). A **PDE** is a mathematical relation involving partial derivatives, i.e., if $\mathbf{u} : \mathbb{R}^n \rightarrow \mathbb{R}$, $\mathbf{x} = (x_1, \dots, x_n)$ a real-valued function of several variables, then any k th order PDE can be expressed as:

$$F(D^k \mathbf{u}, D^{k-1} \mathbf{u}, \dots, D\mathbf{u}, \mathbf{u}, \mathbf{x}) = 0 \text{ in } \Omega \quad (1)$$

where $\Omega \subseteq \mathbb{R}^n$ is the domain or region where the PDE holds, and F is a function of the placeholders.

Definition 2 (C^k and $C^k(\Omega)$). A function $\mathbf{u} : \mathbb{R}^n \rightarrow \mathbb{R}$ is C^k at a point $\mathbf{x} \in \mathbb{R}^n$ if every k th-order partial derivative is continuous as a function of \mathbb{R}^n at x . If $u \in C^2(\mathbb{R}^n)$, then in advanced calculus it was proven that $D^2 u$ is a symmetric matrix.

Example 1 (Examples of PDEs). 1. Heat Equation:

$$u_t - k\Delta u = u_t - k \sum_{i=1}^n u_{x_i x_i} = 0 \quad (2)$$

where $k > 0$ is a constant. Then the solution has a space- and time- component:

$$u(\mathbf{x}, t) = u : \mathbb{R}^{n+1} \rightarrow \mathbb{R} \quad (3)$$

2. Laplace's Equation:

$$-\Delta u = 0 \quad (4)$$

$u(x)$ is the steady state of a solution to the heat equation, since $u_t = 0 \Rightarrow$ "constant in time."
The solution u is a function $u : \mathbb{R}^n \rightarrow \mathbb{R}$.

3. Wave Equation:

$$u_{tt} - \Delta u = 0$$
$$u : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$$

$u(x, t)$ is the displacement of an object with wave-like behaviour at location \mathbf{x} and time t . Ex: position of a guitar string.

4. Transport Equation:

$$u_t + cu_x = 0, \quad c \in \mathbb{R}, u : \mathbb{R}^2 \rightarrow \mathbb{R} \text{ (space, time)}$$

$u(x, t)$ can be the density of a pollutant at location \mathbf{x} and time t .

5. Reaction-Diffusion:

$$u_t - k\Delta u = f(u)$$
$$u : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$$
$$f : \mathbb{R} \rightarrow \mathbb{R}$$

$u(\mathbf{x}, t)$ can be the temperature at location (\mathbf{x}, t) subject to enhancement by $f(u)$ (for example, fire spreading).

6. Burger's Equation:

$$u_t - uu_x = \nu u_{xx} \tag{5}$$

$\nu > 0$ represents the viscosity. $u : \mathbb{R}^2 \rightarrow \mathbb{R}$, $u(x, t)$ is the concentration of a material in a fluid flow with convection.

1.1 Domains and Boundary Conditions

Definition 3 (C^1 domain). $\Omega \subseteq \mathbb{R}^n$ is a C^1 domain if $\partial\Omega$ can locally be expressed as a graph of a C^1 function. This means...

1. $\partial\Omega$ has no corners \Rightarrow smooth.
2. $\forall p \in \partial\Omega$, there exists a well-defined and unique tangent plane (whose slope is given by the derivative), which implies that there exists a well-defined inward and outward normal vector.
3. Inward and outward normal vectors move continuously along $\partial\Omega$.

1.1.1 What are the main boundary conditions?

1. Dirichlet Boundary Conditions: these prescribe what u is on $\partial\Omega$. We assume that $u \in C^k(\Omega) \cap C(\partial\Omega)$.
2. Neumann Boundary Conditions: these prescribe what the normal derivative of u on $\partial\Omega$ is. The meaning behind this is: *how does u change along the boundary?* It is specifying:

$$\frac{\partial u}{\partial n} = \nabla u \cdot \mathbf{n}(x) = X \text{ on } \partial\Omega \tag{6}$$

3. Robin boundary conditions: a combination of the above:

$$\frac{\partial u}{\partial n} + \alpha u = X \text{ on } \partial\Omega \tag{7}$$

In general, in PDEs, we will not be able to identify an explicit solution. Thus, we care about the following four fundamental issues. The first three ensure that we have a *well-posed problem*, and the final one is important when we cannot obtain an explicit solution.

1. *Existence*: is there a solution?
2. *Uniqueness*: is there exactly one solution to the PDE?
3. *Stability*: does the solution depend continuously on the data?
4. *Qualitative Properties*: If I cannot find an explicit solution, but I know that it exists, what else can I tell you about the solution? Some questions we are interested in studying are:
 - a) Does $u(\mathbf{x}, t) \rightarrow 0$ as $t \rightarrow \infty$?
 - b) What is $\max_{\Omega} |u(\mathbf{x})|$?

1.2 Classification of PDEs

To do: draw out the chart