MATH 314: Advanced Calculus
Meeting 1

1. Suggested Exercises

2. Notes In the first meeting, I thought it would be a good idea to review vector fields, line integrals, the fundamental theorem of line integrals, and Green's theorem. Then, we can work on exercises.

Line integrals come from trying to compute the work a force does on a trajectory. We can model forces using a **vector fields**, so the first task is to understand what those are. Generally speaking, a vector field assigns to every point $(x, y) \in \mathbb{R}$ or $(x, y, z) \in \mathbb{R}^3$ a vector.

Definition 1 (Vector Field)**.** Let $D \subseteq \mathbb{R}^2$ be a set. A **vector field on** $\mathbb{R}^2$ is defined as a function $\mathbf{F}$ which assigns to each point $(x, y) \in D$ a two-dimensional vector $\mathbf{F}(x, y)$.

Since for each point, $\mathbf{F}(x, y)$ is a vector in $\mathbb{R}^2$, we can write it out component-wise since $\mathbf{i}$ and $\mathbf{j}$ are the standard basis vectors of $\mathbb{R}^2$:

$$\begin{aligned} \mathbf{F}(x, y) &= P(x, y)\mathbf{i} + Q(x, y)\mathbf{j} \\ &= \langle P(x, y), Q(x, y)\rangle. \end{aligned}$$

P and Q are scalar functions of x and y (scalar since they take on values in $\mathbb{R}$). Generally, when given a vector field, it's a good idea to sketch out some representative vectors by evaluating $\mathbf{F}(x, y)$ at some points $(x, y) \in \mathbb{R}^2$.

One special type of vector field is a **gradient field**. Recall: if we have some scalar function f of two variables, $f : \mathbb{R}^2 \to \mathbb{R}$, then it's gradient, which we denote by ∇f, is defined as:

$$\nabla f(x, y) := f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}. \tag{1}$$

Note that this is actually a vector field on $\mathbb{R}^2$: to every point in $\mathbb{R}^2$, it associates to it a vector, namely, the gradient of f at that point. This notion leads us to our first special type of a vector field.

Definition 2 (Conservative Vector Field)**.** Let $\mathbf{F}$ be a vector field. We say that $\mathbf{F}$ is a **conservative vector field** if there is some scalar function f such that $\mathbf{F} = \nabla f$. When such an f exists, we call it a **potential function** for $\mathbf{F}$.

Now that we have a notion of what a vector field is, we can define a line integral. Recall how we did regular, single-variable integrals: we integrated some function f over some interval $[a, b]$. Line integrals are similar to it, except instead of an interval, we integrate over some curve C. Suppose we have the following parametric equations for the curve C:

$$x = x(t),\ y = y(t),\ a \leq t \leq b \iff \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}.$$

Suppose $f : \mathbb{R}^2 \to \mathbb{R}$. The idea is similar in spirit to the Riemann sum which we used to define the Riemann integral: we chop up the curve into small arcs with length Δs, evaluate f at representative points, and take the limit as $\Delta s \to 0$.

Definition 3 (Line Integral of f along C)**.** Suppose f is defined on a smooth curve C with parametric equations given above. Then, the **line integral of** f **along** C is defined as:

$$\int_C f(x, y)ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s. \tag{2}$$

Remembering the formula for a length of a curve gives us the following formula which we can use to evaluate a line integral:

$$\int_C f(x,y)ds = \int_a^b f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}dt. \tag{3}$$

We can extend this to now doing line integrals over vector fields. To understand this, it's best to think about work. Recall that the work, W, is:

$$W = \text{force} \times \text{distance} = \mathbf{F} \cdot \Delta\mathbf{r}.$$

Now, what if the force actually depends on where in space we are (that is, the force is actually a vector field), and we want to know how much work is done by the force along some trajectory C? What we do is cut up the curve into very small segments, and do the above dot product on each segment. We will get an integral:

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \lim_{\Delta r_i \to 0} \sum_i \mathbf{F} \cdot \Delta\mathbf{r}_i.$$

We can re-write $\Delta\mathbf{r}_i = \frac{\Delta\mathbf{r}_i}{\Delta t}\Delta t$. $\frac{\Delta\mathbf{r}_i}{\Delta t}$ is nothing more than the velocity vector $\frac{d\mathbf{r}}{dt}$. So, we can explicitly compute the line integral as follows:

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_{t_1}^{t_2} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}dt. \tag{4}$$

There is another way we can compute line-integrals. Recall that the force $\mathbf{F}$ is a vector fields, so we can consider it's components, $\langle P(x,y), Q(x,y)\rangle$. We can think of $d\mathbf{r}$ as the vector with components $\langle dx, dy\rangle$. Then, we can write:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P(x,y)dx + Q(x,y)dy.$$

However, we still do not how to compute the above. We need to exploit the fact that along the curve, x and y are related to each other by some parameter. So the technique is to parameterize C with some parameter, say, t, substitute in that parameter. Then, we'd obtain regular one-dimensional integral in t which we can solve using standard techniques. Let's see how we can do this with an example.

Example 1. Consider the vector field $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$. Suppose we want to integrate the following parametric curve C along this vector field:

$$x = t,\ y = t^2,\ 0 \le t \le 1.$$

Then, we write:

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C P(x,y)dx + Q(x,y)dy \\ &= \int_C -ydx + xdy.\end{aligned}$$

Now we substitute: $x = t \iff dx = dt$ and $y = t^2 \iff dy = 2tdt$. We integrate from $t = 0$ to $t = 1$.

We obtain:

$$\begin{aligned}
&= \int_0^1 (-t^2)dt + t(2t)dt \\
&= \int_0^1 t^2 dt \\
&= \left[\frac{t}{3}\right]_{t=0}^{t=1} \\
&= \frac{1}{3}.
\end{aligned}$$

Helpful Tip: We can parameterize a line-segment which begins at $\mathbf{r}_0$ and ends at $\mathbf{r}_1$ with the following:

$$\mathbf{r}(t) = (1-t)\mathbf{r}_0 + t\mathbf{r}_1. \tag{5}$$

Another way we can compute the line integral over a vector field is the following:

$$W = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt. \tag{6}$$

Next, we move onto the **Fundamental Theorem for Line Integrals**. Recall in Calculus II, we had the following relation for the integral of a derivative:

$$\int_a^b F'(x)dx = F(a) - F(b). \tag{7}$$

In other words, all that we do is evaluate F at the boundary points of the interval $[a, b]$. Heuristically, we can think of the gradient of a function as a sort of derivative. It turns out that we have an analogous statement for line integrals over gradient fields. We call this the **Fundamental Theorem for Line Integrals**

Theorem 1 (Fundamental Theorem for Line Integrals)**.** *Suppose C is a smooth curve given by the vector function $\mathbf{r}(t)$ for $a \leq t \leq b$. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable whose gradient is continuous on C. Then,*

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)). \tag{8}$$

Recall that conservative vector fields can be written as the gradient of some scalar function. This means that for any conservative vector field, if we want to compute the line integral of it over some curve, we actually only need its value at the endpoints of C. This has a lot of interesting implications. Namely, we obtain a notion of *path-independence*: recall in a previous example, we stated that it actually does matter which trajectory you take to get the same terminal point. But this theorem tells us if the vector field is conservative, it actually does not matter.

Also, what if the terminal point is the same as the starting point? In this case, we say that the curve C is **closed**: $\mathbf{r}(a) = \mathbf{r}(b)$. The Fundamental Theorem for Line Integrals tells us then that the line integral over a closed curve is zero. These two observations lead to the following theorem.

Theorem 2. *$\int_C \mathbf{F} \cdot d\mathbf{r}$ is path-independent in D $\iff$ $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path C in D.*

If we have special assumptions on D, it turns out that path-independence of line integrals over vector fields is actually a characterization of being conservative.

Theorem 3. *Suppose* $\mathbf{F}$ *is a vector field that's continuous on an open, connected region* D. *Suppose that* $\int_C \mathbf{F}\cdot d\mathbf{r}$ *is path-independent in* D. *Then,* $\mathbf{F}$) *is a conservative vector field on* D.

By now, you are probably wondering if there's a simple way to determine if a vector field is conservative or not. By the equality of second partial derivatives for C^1 functions, we get the following theorem:

Theorem 4. *Suppose* D *is an open, simply-connected region. Suppose* $\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ *with* P *and* Q *having continuous first-order partial derivatives on the domain* D. *Then, if*

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \tag{9}$$

throughout D, *then* $\mathbf{F}$ *is conservative.*

1 Exercises

Problem 1. Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F}(x,y,z) = xy\mathbf{i}+yz\mathbf{j}+zx\mathbf{k}$ and C is the twisted cube parameterized by:

$$x = t,\ y = t^2,\ z = t^3,\ 0 \leq t \leq 1.$$

Solution: In vector form, we have:

$$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}.$$

Differentiating, we obtain:

$$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}.$$

Finally,

$$\mathbf{F}(\mathbf{r}(t)) = t^3\mathbf{i} + t^5\mathbf{j} + t^4\mathbf{k}$$
$$\Rightarrow \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t) = t^3 + 2t^6 + 3t^6 = t^3 + 5t^6.$$

Finally, we integrate:

$$\int_C \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)dt = \int_0^1 t^3 + 5t^6 dt = \left[\frac{t^4}{4} + 5\frac{t^7}{7}\right]_{t=0}^{t=1} = \frac{27}{28}.$$

Problem 2. Compute the following line integral:

$$\int_C xy^4 ds,$$

where C is the right half of the circle $x^2 + y^2 = 16$.

Solution: We parameterize this as follows:

$$x = 4\cos(\theta),\ y = 4\sin(\theta), \frac{3\pi}{2} \leq \theta \leq \frac{\pi}{2}.$$

Carrying out the necessary computations, we obtain:

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{16\sin^2(\theta) + 16\cos^2(\theta)} = 4.$$

The integral becomes:

$$\begin{aligned}\int_C xy^4 ds &= \int_{\frac{3\pi}{2}}^{\frac{\pi}{2}} (4\cos(\theta))(4\sin(\theta))^4 4d\theta \\ &= 4^6 \int_{\frac{3\pi}{2}}^{\frac{\pi}{2}} \cos(\theta)(\sin(\theta))^4 dt.\end{aligned}$$

With the substitution $u = \sin(\theta)$, we obtain:

$$\begin{aligned}&= 4^6 \int_{-1}^{1} u^4 du \\ &= 4^6 \left[\frac{u^5}{5}\right]_{-1}^{1} \\ &= \boxed{4^6 \frac{2}{5}}\end{aligned}$$

This one is from your exam.

Problem 3. Consider the vector field $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ defined by:

$$\mathbf{F}(x, y) := (y^5 + 2x, 5xy^4 - 2). \tag{10}$$

Let C be the semi-circle $x^2 + y^2 = 1$, $x \geq 0$, oriented clock-wise. Evaluate the line-integral of $\mathbf{F}$ along the curve C, that is, the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$.

Solution. First, let's talk strategy: looking at this problem right now, we do not want to do it the brute-force way. We could, and it could be handy to check your answer on an exam if you have the time, but let's see if we can use any theorems. In particular, since part (1) told us to show that $\mathbf{F}$ is conservative, then that's a major hint to me that the next part should not be a brute-force computation. In particular, I want to use the fact that $\mathbf{F}$ is conservative to write it as a gradient of some function. This will totally simplify the computation, since then I can apply the Fundamental Theorem for Line Integrals to reduce the computation of a line integral to just evaluating that function at the end-points.

To that end, first I claim that

$$\mathbf{F} = \nabla f,$$

where $f(x, y) = xy^5 - 2y + x^2$. You can check this by computing ∇f and verifying that it does indeed equal $\mathbf{F}$. The fundamental theorem of line integrals says:

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)). \tag{11}$$

So, we still need to parameterize C. Draw a picture. The semi-circle $x^2 + y^2 = 1$, $x \geq 1$, oriented clock-wise, can be parameterized as follows:

$$x = \cos(\theta),\ y = \sin(\theta),\ \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}.$$

We have:

$$\begin{aligned}\mathbf{r}(b) = \mathbf{r}\left(\frac{3\pi}{2}\right) &= \cos\left(\frac{3\pi}{2}\right)\mathbf{i} + \sin\left(\frac{3\pi}{2}\right)\mathbf{j} = 0\mathbf{i} - 1\mathbf{j} \\ \mathbf{r}(a) = \mathbf{r}\left(\frac{\pi}{2}\right) &= \cos\left(\frac{\pi}{2}\right)\mathbf{i} + \sin\left(\frac{\pi}{2}\right)\mathbf{j} = 0\mathbf{i} + 1\mathbf{j}.\end{aligned}$$

$$\begin{aligned}f\left(\mathbf{r}\left(\frac{3\pi}{2}\right)\right) &= -2(-1) = 2. \\ f\left(\mathbf{r}\left(\frac{\pi}{2}\right)\right) &= -2.\end{aligned}$$

Applying the fundamental theorem of line integrals yields,

$$\boxed{\int_C \mathbf{F} \cdot d\mathbf{r} = f\left(\mathbf{r}\left(\frac{\pi}{2}\right)\right) - f\left(\mathbf{r}\left(\frac{3\pi}{2}\right)\right) = -4}$$

Problem 4. (Similar in spirit to the previous problem)

1. Say we have the following 3 pieces of information about some parametric curve:

   (a) $\mathbf{r}''(t) = \langle 6, 0, 0 \rangle$ for all t.

   (b) $\mathbf{r}(0) = \langle 0, 3, 4 \rangle$.

   (c) $\mathbf{r}'(0) = \langle 0, 0, 1 \rangle$.

   Find an expression for $\mathbf{r}(t)$ for all t. Use this to find $\mathbf{r}(1)$.

2. Using this same curve that you found in (a), use an integral theorem to compute the line integral of the following vector field:

$$\mathbf{F}(x, y, z) = \pi \cos(\pi x)\mathbf{i} + (3y^2 + z)\mathbf{j} + (4z^3 + y)\mathbf{k}, \tag{12}$$

   along the path $\mathbf{r}(t)$ from $t = 0$ to $t = 1$.

Solution:
(1): By the condition for the second derivative, we know that for the first component, it must be a second-order polynomial, and the other two components need to be first-order or less. Since we know the second derivative is constantly equal to 6, this fixes the following form for $\mathbf{r}(t)$:

$$\mathbf{r}(t) = \langle 3t^2 + a, b_1 t + b_2, c_1 t + c_2 \rangle. \tag{13}$$

Since the first component of $\mathbf{r}'(0)$ is zero, this fixes $a = 0$. The first derivative of the second term is b_1. By the condition on the first derivative, this fixes $b_1 = 0$. By the condition on the value of $\mathbf{r}$ at 0, this fixes $b_2 = 3$. Now we move on to the final condition. Taking the derivative of the final component and comparing to the first derivative condition fixes $c_1 = 1$. Comparing to value of $\mathbf{r}$ at zero condition fixes $c_2 = 4$. Combining all this information together yields:

$$\boxed{\mathbf{r}(t) = \langle 3t^2, 3, t + 4 \rangle.} \tag{14}$$

You can, and should, verify that this is correct by manually checking that all three conditions are true for this $\mathbf{r}(t)$. Especially, now, $\mathbf{r}(1) = \langle 3, 3, 5 \rangle$.

(b): the problem asked us to compute the value of $\mathbf{r}$ at an end-point, so this hints to us that we should use the Fundamental Theorem of line integrals, since we are given $\mathbf{r}(0)$ anyways. We need to have that $\mathbf{F}$ is conservative: you can check that using the condition, or if we go ahead and find a potential function f for $\mathbf{F}$, that will also do it.

Similar to the exam question, we are looking for a function f which satisfies the following:

$$\frac{\partial f}{\partial x} = \pi \cos(\pi x)$$
$$\frac{\partial f}{\partial y} = 3y^2 + z$$
$$\frac{\partial f}{\partial z} = 4z^3 + y.$$

One can (should) check that the following choice for f satisfies this:

$$f(x, y, z) = \sin(\pi x) + yz + y^3 + z^4.$$

The Fundamental Theorem of Line Integrals says:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(1)) - f(\mathbf{r}(0)).$$

We evaluate the end-points:

$$\begin{aligned} f(\mathbf{r}(1)) &= \sin(3\pi) + 15 + 3^3 + 5^4 \\ &= 15 + 3^3 + 5^4. \\ f(\mathbf{r}(0)) &= 12 + 3^3 + 4^4. \end{aligned}$$

Finally,

$$f(\mathbf{r}(1)) - f(\mathbf{r}(0)) = 3 + 5^4 - 4^4.$$

So,

$$\boxed{\int_C \mathbf{F} \cdot d\mathbf{r} = 3 + 5^4 - 4^4}$$

I got this one off an old McGill exam. **Solution:** Let's talk strategy again. The curve in (1) looks horrendous, the ellipse doesn't look too bad. It turns out we won't need to compute anything for it, however. Looking at $\mathbf{F}$, it looks like it could be conservative. Let's see if we can find a potential. Indeed, one can (should) verify that $\nabla f = \mathbf{F}$, where

$$f(x, y) = e^{2x} \cos(\pi y).$$

This means solving (1) literally boils down to evaluating $f(x, y)$ at the end-points, which we are given in the problem (thanks to the Fundamental Theorem of Line Integrals). Indeed,

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C \nabla f \cdot d\mathbf{r} \\ &= f(1, 1) - f(0, 0) \\ &= \boxed{-e^2 - 1} \end{aligned}$$

For (2), it's even easier: we know that the line integral over a closed curve is zero, thanks to the fundamental theorem of line integrals. But going around the ellipse once is a closed curve, so this is zero.