Chapter 11: Topological Spaces (General Properties)

Class: Math 455 (Analysis 4) Date: 9 February 2020 Shereen Elaidi

Abstract

This document contains a summary of all the key definitions, results, and theorems from class. There are probably typos, and so I would be grateful if you brought those to my attention :-).

Syllabus: L^p space, duality, weak convergence, Young, Holder, and Minkowski inequalities, point-set topology, topological space, dense sets, completeness, compactness, connectedness, path-connectedness, separability, Tychnoff theorem, Stone-Weierstrass Theorem, Arzela-Ascoli, Baire category theorem, open mapping theorem, closed graph theorem, uniform boudnedness principle, Hahn Banch theorem.

11.1. Open Sets, Closed Sets, Bases, and Sub-bases

Definition 1 (Open Sets). Let X be a non-empty set. A **topology** $\mathcal{T}$ for X is a collection of subsets of X, called **open sets**, posessing the following properties:

(i) The entire set X and the empty set $\emptyset$ are open.
(ii) The finite intersection of open sets are open.
(iii) The union of any collection of open sets is open.

A non-empty set X, together with a topology on X, is called a **topological space**. For a point $x \in X$, an open set that contains x is called a **neighbourhood** of x.

Proposition 1. A subset $E \subseteq X$ is open $\iff$ for each $x \in E$, there exists a neighbourhood of x that is contained in E.

Example 1 (Metric Topology). Let (X, ρ) be a metric space. Let $\mathcal{O} \subseteq X$ be open if for all $x \in \mathcal{O}$, $\exists$ an open ball at x that is contained in $\mathcal{O}$. This collection of open sets forms a topology; we call this the **metric topology** induced by ρ.

Example 2 (Discrete Topology). This topology is "too much." Let X be a non-empty subset. Let $\mathcal{T} := \mathcal{P}(X)$. Then, every set containing a point is a neighbourhood of that point. This is induced by the discrete metric.

Example 3 (Trivial Topology). Let X be non-empty. Define $\mathcal{T} := \{X, \emptyset\}$. The only neighbourhood of any point is the whole set X.

Definition 2 (Topological Subspaces). Let $(X, \mathcal{T})$ be a topological space and let E be a non-empty subset of X. The inherited topology $\mathcal{S}$ for E is the set of all sets of the form $E \cap \mathcal{T}$, where $\mathcal{O} \in \mathcal{T}$. The topological space $(E, \mathcal{S})$ is called a **subspace** of $(X, \mathcal{T})$.

Definition 3 (Base for the Topology). The building blocks of a topology is called a **base**. Let $(X, \mathcal{T})$ be a topological space. For a point $x \in X$, a collection of neighbourhoods of x, B_x, is called a **base for the topology at** X if $\forall$ neighbourhoods $\mathcal{U}$ of x, there exists a set B in the collection B_x for which $B \subseteq \mathcal{U}$.

A collection of open sets $\mathcal{B}$ is called a **base for the topology** $\mathcal{T}$ provided it contains a base for the topology at each point.

A base for a topology completely determines a topology, alongside $\emptyset$ and X.

Proposition 2. For a non-empty set X, let $\mathcal{B}$ be a collection of subsets of X. Then, $\mathcal{B}$ is a base for a topology for $X \iff$:

(i) $\mathcal{B}$ covers X. That is:

$$X = \bigcup_{B \in \mathcal{B}} B \tag{11.1}$$

(ii) If $B_1, B_2 \in \mathcal{B}$, and $x \in B_1 \cap B_2$, then there is a set $B_3 \in \mathcal{B}$ for which $x \in B_3 \subseteq B_1 \cap B_2$.

The unique topology that has $\mathcal{B}$ as its base consists of $\emptyset$ and unions of sub-collections of $\mathcal{B}$.

Definition 4 (Product Topology)**.** Let $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ be two topological spaces. In the cartesian product $X \times Y$, consider the collection of sets $\mathcal{B}$ containing the products $\mathcal{O}_1 \times \mathcal{O}_2$, where $\mathcal{O}_1$ is open in X and $\mathcal{O}_2$ is open in Y. Then, $\mathcal{B}$ is a base for a topology on $X \times Y$, which we call the **product topology.**

Definition 5 (Sub-base)**.** Let $(X, \mathcal{T})$ be a topological space. The collection of $\mathcal{S}$ of $\mathcal{T}$ that covers X is called a **sub-base** for the topology $\mathcal{T}$ provided intersections of finite collections of $\mathcal{S}$ are a base for $\mathcal{T}$.

Definition 6 (Closure)**.** Let $E \subseteq X$ be a subset of a topological space. A point $x \in E$ is called a **point of closure** of E if every neighbourhood of x contains a point in E. The collection of the points of closure of E is called the **closure** of E, denoted $\overline{E}$.

Proposition 3. Let X be a topological space, $E \subseteq X$. Then, $\overline{E}$ is closed. Moreover, $\overline{E}$ is the smallest closed subset of X containing E in the sense that if F is closed and $E \subseteq F$, then $\overline{E} \subseteq F$.

Proposition 4. A subset of a topological space X is open $\iff$ its complement is closed.

Proposition 5. Let X be a topological space. Then, (a) $\emptyset$ and X are closed, (b) the union of a finite collection of closed sets is closed, (c) the intersection of any collection of closed sets in X is closed.

11.2. Separation Properties

Motivation: Separation properties for a topology allow us to discriminate between which topologies discriminate between certain disjoint pairs of sets, which will then allow us to study a robust collection of cts real-valued functions on X.

Definition 7 (Neighbourhood)**.** A **neighbourhood** of K for a subset $K \subseteq X$ is an open set that contains K.

Definition 8 (Separated by Neighbourhoods)**.** We say that two disjoint sets A and B in X can be separated by disjoint neighbourhoods provided that there exists neighbourhoods of A and B, respectively, that are disjoint.

Definition 9 (Separation Properties of Topological Spaces)**.** . In the order of most general to least general, they are:

(i) **Tychonoff Separation Property**: For each two points $u, v \in X$, there exists a neighbourhood of u that does not contain v and a neighbourhood of v that does not contain u.
(ii) **Hausdorff Separation Property**: Each two points in X can be separated by disjoint neighbourhoods.

(iii) **Regular Separation Property**: Tychonoff + each closed set and a point not in the set can be separated by disjoint neighbourhoods.

(iv) **Normal Separation Property**: Tychonoff + each two disjoint closed sets can be separated by disjoint neighbourhoods.

Proposition 6. A topological space is Tychonoff $\iff$ every set containing a single point, $\{x\}$, is closed.

Proposition 7. Every metric space is normal.

Lemma 1. F is closed $\iff$ $\text{dist}(x, F) > 0 \; \forall \; x \notin F$.

Proposition 8. Let X be a Tychonoff topological space. Then, X is normal $\iff$ whenever $\mathcal{U}$ is a neighbourhood of a closed subset of F of X, there is another neighbourhood of F whose closure is contained in $\mathcal{U}$. that is, there is an open set $\mathcal{O}$ for which:

$$F \subseteq \mathcal{O} \subseteq \overline{\mathcal{O}} \subseteq \mathcal{U} \tag{11.2}$$