(5)

(3)

2) There is no norm that makes $C^\infty(\bar\Omega)$ a Banach Space. However, there are various subspaces of $C^\infty(\bar\Omega)$ that are Banach spaces. For example, for a fixed sequence $C = \{c_n\}_{n=1}^\infty$, define the norm,

$$\|f\|_c := \sum_{n=1}^{\infty} c_n \|f\|_{C^n(\bar\Omega)}$$

Show that the subspace

$$E := \{f \mid \|f\|_c < \infty\}$$

is a Banach Space.

We know that if a subsequence of a Cauchy Sequence in some normed vector space vector $f$, then the mother sequence will converge to $f$. So, let $\{f_n\} \subseteq E$ be a Cauchy sequence in $E$. Then, $\forall n \in \mathbb{N}$, $\|f_n\|_c < \infty$. Since $C^\infty(\Omega) = \bigcap_{r=1}^{\infty} C^r(\Omega)$, $\{f_n\}$ is Cauchy in all of the $C^n(\bar\Omega)$ for $n \in \mathbb{N}$. We'll construct a subsequence out of $\{f_n\}$ as follows:

$\underline{n=1}$: $\{f_n\}$ is Cauchy in $C^1(\bar\Omega)$. Since $C^1(\bar\Omega)$ is complete, $\{f_n\} \to f \in C^1(\bar\Omega)$. Hence, $\exists\, n_1 \in \mathbb{N}$ s.t. $\forall k \geq n_1$,

$$\|f_{n_k} - f\|_{C^1(\bar\Omega)} < \frac{\varepsilon}{c_{n_1} 2^1}$$

Let the first element of our subsequence be $f_{n_1}$.

$\underline{n=2}$: $\{f_n\}$ is Cauchy in $C^2(\bar\Omega)$. Since $C^2(\bar\Omega)$ is complete, $\{f_n\} \to f \in C^2(\bar\Omega)$; it's the same $f$ as in $C^1(\bar\Omega)$. Hence, $\exists\, n_2 \in \mathbb{N}$ s.t. $\forall k \geq n_2$,

$$\|f_{n_k} - f\|_{C^2(\bar\Omega)} < \frac{\varepsilon}{c_{n_2} 2^2}$$

Let the second element of that subsequence be $f_{n_2}$.

Inductively continue this construction. Consider $C^n(\bar\Omega)$ for some $n \in \mathbb{N}$. By the completeness of $C^n(\bar\Omega)$, $\{f_n\} \to f \in C^n(\bar\Omega)$. Hence, $\exists\, n_n \in \mathbb{N}$ s.t. $\forall k \geq n_n$,

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$$\|f_{n_k} - f\| < \frac{\varepsilon}{C_{n_k} 2^n}.$$

Let the nth element of the subsequence be $f_{n_n}$. We have inductively chosen a subsequence of the $\{f_n\}$ s.t

$$\|f_{n_k} - f\|_{C^n(\Omega)} \leq \frac{\varepsilon}{C_{n_k} 2^n}.$$

Hence,

$$\|f_{n_k} - f\|_C = \|f_{n_k} + f_{m_k} - f_{m_k} - f\|_C$$

$$\leq \underbrace{\|f_{n_k} - f_{m_k}\|_C}_{\substack{\text{can be made small}\\ \text{since a subsequence}\\ \text{of a Cauchy Seq.}\\ \text{remains Cauchy!}}} + \underbrace{\|f_{m_k} - f\|_C}_{\text{can be made small by } (*)}$$

$$\leq \varepsilon + \sum_{k=1}^{\infty} C_{n_k} \frac{\varepsilon}{C_{n_k} 2^n}$$

$$= \varepsilon + \varepsilon \sum_{k=1}^{\infty} \left(\tfrac{1}{2}\right)^n \leftarrow \text{convergent geometric series}$$

$$= \varepsilon + \varepsilon\left(\tfrac{1}{1/2}\right) = 3\varepsilon \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

We've shown that $f_{n_k} \to f$, but we need to show that $f \in E$. However, this follows from the following fact from class:

$$\boxed{C^\infty(\Omega) := \bigcap_{n=1}^{\infty} C^n(\Omega)}$$

However, since the limiting function $f$ is the same $\forall n \in \mathbb{N}$ (otherwise it would violate the uniqueness of limits in a normed vector space),

$$f \in C^n(\Omega) \quad \forall n \in \mathbb{N} \quad \Leftrightarrow \quad f \in \bigcap_{n=1}^{\infty} C^n(\Omega) =: C^\infty(\Omega)$$

which proves that $f \in C^\infty(\Omega)$.