

Chapter 11: Topological Spaces (General Properties)

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Abstract

This document contains a summary of all the key definitions, results, and theorems from class. There are probably typos, and so I would be grateful if you brought those to my attention :-).

Syllabus: L^p space, duality, weak convergence, Young, Holder, and Minkowski inequalities, point-set topology, topological space, dense sets, completeness, compactness, connectedness, path-connectedness, separability, Tychonoff theorem, Stone-Weierstrass Theorem, Arzela-Ascoli, Baire category theorem, open mapping theorem, closed graph theorem, uniform boundedness principle, Hahn Banach theorem.

11.1. OPEN SETS, CLOSED SETS, BASES, AND SUB-BASES

Definition 1 (Open Sets). Let X be a non-empty set. A **topology** \mathcal{T} for X is a collection of subsets of X , called **open sets**, possessing the following properties:

- (i) The entire set X and the empty set \emptyset are open.
- (ii) The finite intersection of open sets are open.
- (iii) The union of any collection of open sets is open.

A non-empty set X , together with a topology on X , is called a **topological space**. For a point $x \in X$, an open set that contains x is called a **neighbourhood** of x .

Proposition 1. A subset $E \subseteq X$ is open \iff for each $x \in E$, there exists a neighbourhood of x that is contained in E .

Example 1 (Metric Topology). Let (X, ρ) be a metric space. Let $\mathcal{O} \subseteq X$ be open if for all $x \in \mathcal{O}$, \exists an open ball at x that is contained in \mathcal{O} . This collection of open sets forms a topology; we call this the **metric topology** induced by ρ .

Example 2 (Discrete Topology). This topology is “too much.” Let X be a non-empty subset. Let $\mathcal{T} := \mathcal{P}(X)$. Then, every set containing a point is a neighbourhood of that point. This is induced by the discrete metric.

Example 3 (Trivial Topology). Let X be non-empty. Define $\mathcal{T} := \{X, \emptyset\}$. The only neighbourhood of any point is the whole set X .

Definition 2 (Topological Subspaces). Let (X, \mathcal{T}) be a topological space and let E be a non-empty subset of X . The inherited topology \mathcal{S} for E is the set of all sets of the form $E \cap \mathcal{O}$, where $\mathcal{O} \in \mathcal{T}$. The topological space (E, \mathcal{S}) is called a **subspace** of (X, \mathcal{T}) .

Definition 3 (Base for the Topology). The building blocks of a topology is called a **base**. Let (X, \mathcal{T}) be a topological space. For a point $x \in X$, a collection of neighbourhoods of x , B_x , is called a **base for the topology at x** if \forall neighbourhoods \mathcal{U} of x , there exists a set B in the collection B_x for which $B \subseteq \mathcal{U}$.

A collection of open sets \mathcal{B} is called a **base for the topology \mathcal{T}** provided it contains a base for the topology at each point.

A base for a topology completely determines a topology, alongside \emptyset and X .

Proposition 2. For a non-empty set X , let \mathcal{B} be a collection of subsets of X . Then, \mathcal{B} is a base for a topology for $X \iff$:

(i) \mathcal{B} covers X . That is:

$$X = \bigcup_{B \in \mathcal{B}} B \quad (11.1)$$

(ii) If $B_1, B_2 \in \mathcal{B}$, and $x \in B_1 \cap B_2$, then there is a set $B_3 \in \mathcal{B}$ for which $x \in B_3 \subseteq B_1 \cap B_2$.

The unique topology that has \mathcal{B} as its base consists of \emptyset and unions of sub-collections of \mathcal{B} .

Definition 4 (Product Topology). Let (X, \mathcal{T}) and (Y, \mathcal{S}) be two topological spaces. In the cartesian product $X \times Y$, consider the collection of sets \mathcal{B} containing the products $\mathcal{O}_1 \times \mathcal{O}_2$, where \mathcal{O}_1 is open in X and \mathcal{O}_2 is open in Y . Then, \mathcal{B} is a base for a topology on $X \times Y$, which we call the **product topology**.

Definition 5 (Sub-base). Let (X, \mathcal{T}) be a topological space. The collection of \mathcal{S} of \mathcal{T} that covers X is called a **sub-base** for the topology \mathcal{T} provided intersections of finite collections of \mathcal{S} are a base for \mathcal{T} .

Definition 6 (Closure). Let $E \subseteq X$ be a subset of a topological space. A point $x \in E$ is called a **point of closure** of E if every neighbourhood of x contains a point in E . The collection of the points of closure of E is called the **closure** of E , denoted \overline{E} .

Proposition 3. Let X be a topological space, $E \subseteq X$. Then, \overline{E} is closed. Moreover, \overline{E} is the smallest closed subset of X containing E in the sense that if F is closed and $E \subseteq F$, then $\overline{E} \subseteq F$.

Proposition 4. A subset of a topological space X is open \iff its complement is closed.

Proposition 5. Let X be a topological space. Then, (a) \emptyset and X are closed, (b) the union of a finite collection of closed sets is closed, (c) the intersection of any collection of closed sets in X is closed.

11.2. SEPARATION PROPERTIES

Motivation: Separation properties for a topology allow us to discriminate between which topologies discriminate between certain disjoint pairs of sets, which will then allow us to study a robust collection of cts real-valued functions on X .

Definition 7 (Neighbourhood). A **neighbourhood** of K for a subset $K \subseteq X$ is an open set that contains K .

Definition 8 (Separated by Neighbourhoods). We say that two disjoint sets A and B in X can be separated by disjoint neighbourhoods provided that there exists neighbourhoods of A and B , respectively, that are disjoint.

Definition 9 (Separation Properties of Topological Spaces). . In the order of most general to least general, they are:

- (i) **Tychonoff Separation Property:** For each two points $u, v \in X$, there exists a neighbourhood of u that does not contain v and a neighbourhood of v that does not contain u .
- (ii) **Hausdorff Separation Property:** Each two points in X can be separated by disjoint neighbourhoods.

- (iii) **Regular Separation Property**: Tychonoff + each closed set and a point not in the set can be separated by disjoint neighbourhoods.
- (iv) **Normal Separation Property**: Tychonoff + each two disjoint closed sets can be separated by disjoint neighbourhoods.

Proposition 6. A topological space is Tychonoff \iff every set containing a single point, $\{x\}$, is closed.

Proposition 7. Every metric space is normal.

Lemma 1. F is closed $\iff \text{dist}(x, F) > 0 \forall x \notin F$.

Proposition 8. Let X be a Tychonoff topological space. Then, X is normal \iff whenever \mathcal{U} is a neighbourhood of a closed subset F of X , there is another neighbourhood of F whose closure is contained in \mathcal{U} . that is, there is an open set \mathcal{O} for which:

$$F \subseteq \mathcal{O} \subseteq \overline{\mathcal{O}} \subseteq \mathcal{U} \quad (11.2)$$

11.3. COUNTABILITY AND SEPARABILITY

Definition 10 (Converge, Limit). A sequence $\{x_n\}$ in a topological space X is said to **converge** to the point $x \in X$ if for each neighbourhood \mathcal{U} of x , there exists an index $N \in \mathbb{N}$ such that if $n \geq N$, then x_n belongs to \mathcal{U} . This point is called a **limit** of the sequence.

Definition 11 (First and Second Countable). A topological space X is **first countable** if there is a countable base at each point. A space X is said to be **second countable** if there is a countable base for the topology.

Example 4. Every metric space is first countable.

Proposition 9. Let X be a first countable topological space. For a subset $E \subseteq X$, a point $x \in X$ is called a point of closure of E \iff it is a limit of a sequence in E . Thus, a subset E of X is closed \iff whenever a sequence in E converges to $x \in X$, we have that $x \in E$.

Definition 12 (Dense/Separable). A subset $E \subseteq X$ is **dense** in X if every open set in X contains a point of E . We call X **separable** if it has a countable dense subset.

Definition 13 (Metrisable). A topological space X is said to be **metrisable** if the topology is induced by the metric.

Theorem 2. Let X be a second countable topological space. Then, X is metrisable \iff it is normal.

11.4. CONTINUOUS MAPPINGS BETWEEN TOPOLOGICAL SPACES

Definition 14 (Continuous). For topological spaces (X, \mathcal{T}) , (Y, \mathcal{S}) , a mapping $f : X \rightarrow Y$ is said to be **continuous** at the point x_0 in X if, for every neighbourhood \mathcal{O} of $f(x_0)$, there is a neighbourhood \mathcal{U} of x_0 for which $f(\mathcal{U}) \subseteq \mathcal{O}$. We say that f is continuous provided it is continuous at each point in X .

Proposition 10. A mapping $f : X \rightarrow Y$ between topological spaces X and Y is continuous \iff for any open subset \mathcal{O} in Y , its inverse image under f , $f^{-1}(\mathcal{O})$, is an open subset of X .

Proposition 11. The composition of continuous mappings between topological spaces, when defined, is continuous.

Definition 15 (Stronger). Given two topologies \mathcal{T}_1 and \mathcal{T}_2 for a set X , if $\mathcal{T}_2 \subseteq \mathcal{T}_1$, then we say that \mathcal{T}_2 is **weaker** than \mathcal{T}_1 , and that \mathcal{T}_1 is **stronger** than \mathcal{T}_2 .

Proposition 12. Let X be a non-empty set and let \mathcal{S} be a collection of subsets of X that covers X . The collection of subsets of X consisting of intersections of finite collections of \mathcal{S} is a base for a topology \mathcal{T} of X . It is the weakest topology containing \mathcal{S} in the sense that if \mathcal{T}' is any other topology for X containing \mathcal{S} , then $\mathcal{T} \subseteq \mathcal{T}'$.

Definition 16 (Weak Topology). Let X be a non-empty set and $\mathcal{F} := \{f_\alpha \mid X \rightarrow X_\alpha\}_{\alpha \in \Lambda}$ a collection of mappings, where each X_α is a topological space. The weakest topology for X that contains the collection of sets

$$\{f_\alpha^{-1}(\mathcal{O}_\alpha) \mid f_\alpha \in \mathcal{F}, \mathcal{O}_\alpha \text{ open in } X_\alpha\} \quad (11.3)$$

is called the **weak topology for X induced by \mathcal{F}** .

Proposition 13. Let X be a non-empty set, $\mathcal{F} := \{f_\lambda \mid X \rightarrow X_\lambda\}_{\lambda \in \Lambda}$ a collection of mappings where each X_λ is a topological space. The weak topology for X induced by \mathcal{F} is the topology on X that has the fewest number of sets covering the topologies on X for which each mapping $f_\lambda : X \rightarrow X_\lambda$ is continuous.

Definition 17 (Homeomorphism). A mapping from a topological space $X \rightarrow Y$ is said to be a **homeomorphism** if it is bijective and has a continuous inverse $f^{-1} : Y \rightarrow X$. Two topological spaces are said to be **homeomorphic** if there exists a homeomorphism between them. The notion of homeomorphism induces a notion of an equivalence relation between spaces.

11.5. COMPACT TOPOLOGICAL SPACES

Definition 18 (Cover). A collection of sets $\{E_\lambda\}_{\lambda \in \Lambda}$ is said to be a **cover** of a set E if $E \subseteq \bigcup_{\lambda \in \Lambda} E_\lambda$.

Definition 19 (Compact). A topological space X is said to be **compact** if every open cover of X has a finite sub-cover. A subset $K \subseteq X$ is compact if K , considered as a topological space with the subspace topology inherited from X , is compact.

Proposition 14. A topological space X is compact \iff every collection of closed subsets of X that possesses the finite intersection property has non-empty intersection.

Proposition 15. A closed subset K of a compact topological space is compact.

Proposition 16. A compact subspace K of a Hausdorff topological space is a closed subset of X .

Definition 20 (Sequentially Compact). A topological space X is said to be **sequentially compact** if every sequence in X has a subsequence that converges to a point in X .

Proposition 17. Let X be a second countable topological space. Then, X is compact \iff it is sequentially compact.

Theorem 3. A compact Hausdorff space is normal.

Proposition 18. A continuous one-to-one mapping f of a compact space X onto a Hausdorff space Y is a homeomorphism.

Proposition 19. The continuous image of a compact topological space is compact.

Corollary 1. A continuous real-valued function on a compact topological space takes on a minimum and maximum functional value.

Definition 21 (Countably Compact). A topological space is **countably compact** if every countable open cover has a finite subcover.

11.6. CONNECTED TOPOLOGICAL SPACE

Definition 22 (Separated). Two non-empty subsets of a topological space **separate** X if they are disjoint and their union is X .

Definition 23 (Connected). A topological space which cannot be separated by open sets is said to be **connected**. A subset $E \subseteq X$ is **connected** if there do NOT exist open subsets $\mathcal{O}_1, \mathcal{O}_2$ of X for which:

$$\begin{aligned}\mathcal{O}_1 \cap E &\neq \emptyset \\ \mathcal{O}_2 \cap E &\neq \emptyset \\ E &\subseteq \mathcal{O}_1 \cup \mathcal{O}_2, \\ E \cap \mathcal{O}_1 \cap \mathcal{O}_2 &= \emptyset\end{aligned}$$

Proposition 20. Let f be a continuous mapping of a connected space X to a topological space Y . Then, its image $f(X)$ is connected.

Proposition 21. For A set $C \in \mathbb{R}$, the following are equivalent.

- (i) C is an interval.
- (ii) C is convex.
- (iii) C is connected.

Definition 24 (Intermediate Value Property). A topological space X has the **intermediate value property** if the image of any continuous real-valued function on X is an interval.

Proposition 22. A topological space has the intermediate value property \iff it is connected.

Definition 25 (Arcwise connected). A topological space X is **arcwise connected** if, for each pair $u, v \in X$, there exists a continuous map $f : [0, 1] \rightarrow X$ for which $f(0) = u$ and $f(1) = v$. Note:

- (i) Connected \iff arcwise connected in \mathbb{R}^n .
- (ii) Arcwise connected \Rightarrow connected (in general)
- (iii) There exist connected but non-arcwise connected spaces (in general).