Math 458: Differential Geometry

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1.1 Implicit and Inverse Function Theorems

2 Manifolds in $\mathbb{R}^3$

The aim of this part of the course is to build up to integration on manifolds and the invariant Stokes' theorem. The main purpose of this sections is to develop *coordinate-free* calculus, which clarifies the essence of what is happening (sometimes coordinates can be noisy).

2.1 Definitions

2.2 Smooth Maps from $M^m \to N^n$

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3 Curves

There are two subsets of differential geometry: classical differential geometry and global differential geometry. The objective of **classical differential geometry** is to study the local properties of curves and surfaces. The objective of **global differential geometry** is to study the influence of local properties on global behaviour.

3.1 Definitions

Definition 1 (Parameterised Differentiable Curve)**.** A **parameterised differentiable curve** is a differentiable map $\alpha : I \to \mathbb{R}^3$ of an open interval $I =]a, b[$ of the real line $\mathbb{R}$ into $\mathbb{R}^3$. The image of α is called the **trace** of α.

Some examples of parameterised curves include:

- The helix: $\alpha(t) = (a\cos(t), a\sin(t), bt)$ for $t \in \mathbb{R}$.
- The map $\alpha : \mathbb{R} \to \mathbb{R}^2$, $t \in \mathbb{R}$, is a parameterised differentiable curve.

Definition 2 (Norm on $\mathbb{R}^3$)**.** Let $u = (u_1, u_2, u_3) \in \mathbb{R}^3$. The **norm** of u is:

$$||u|| := \sqrt{u_1^2 + u_2^2 + u_3^3}$$

Definition 3 (Inner Product on $\mathbb{R}^3$)**.** Let $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ belong to $\mathbb{R}^3$ and let $\theta \in [0, \pi]$ be the angle formed between u, v. The **inner product** is defined by:

$$u \cdot v := ||u||||v|| \cos(\theta) \tag{1}$$

It satisfies the following properties:

1. If u, v are non-zero, then $u \cdot v = 0 \iff u \perp v$.
2. $u \cdot v = v \cdot u$.
3. $\lambda(u \cdot v) = \lambda u \cdot v = u \cdot \lambda v$.
4. $u(v + w) = u \cdot v + u \cdot w$.

If we have made a choice of basis, then we can formulate the dot product in terms of the components of the vectors as:

$$u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3 \tag{2}$$

3.1.1 Regular Curves and Arclength

In differential geometry, it is essential that our curves have a tangent line at every point. This motivates the following definition.

Definition 4 (Regular Curve)**.** A parameterised differentiable curve $\alpha : I \to \mathbb{R}^3$ is **regular** if $\alpha'(t) \neq 0$ $\forall t \in I$.

Definition 5 (Arc length)**.** Given $t_0 \in I$, the **arc length** of a regular parameterised curve $\alpha : I \to \mathbb{R}^3$ from t_0 to t is defined to be:

$$s(t) := \int_{t_0}^{t} |a'(t)| dt$$

where

$$|\alpha'(t)| := \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}$$

Since we only restrict our attention to regular surfaces, $a'(t) \neq 0$ for all t, and so the arlength function is a differentiable function of t and $ds/dt = |a'(t)|$ (by the Fundamental Theorem of Calculus). Arc length parameterisations make life simpler.

3.1.2 The Vector Product in $\mathbb{R}^3$

Definition 6 (Vector Product)**.** Let $u, v \in \mathbb{R}^3$. Then, the **vector product** of u, v is the unique vector $u \wedge v$ in $\mathbb{R}^3$ characterised by:

$$(u \wedge v) \cdot w = \det(u, v, w) \quad \forall w \in \mathbb{R}^3$$

this is more commonly known as:

$$u \wedge v = \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}$$

where $\hat{i}, \hat{j}, \hat{k}$ are the standard basis vectors in $\mathbb{R}^3$.

Properties of the Vector Product

1. (Anti-Commutativity): $u \wedge v = -v \wedge u$.
2. (Linear Dependence): $\forall\ \alpha,\ \beta \in \mathbb{R}$:

$$(\alpha u + \beta v) \wedge v = \alpha u \wedge v + \beta w \wedge v$$

3. $u \wedge v = 0 \iff u$ and v are linearly dependent.
4. $(u \wedge v) \cdot u = 0$, $(u \wedge v) \cdot v = 0$ (this implies that the vector product is normal to the plane generated by u and v).

3.2 Frenet-Serret Frame

Definition 7 (Curvature)**.** Let $\alpha : I \to \mathbb{R}^3$ be a curve parameterised by arclength $s \in I$. The number $||\alpha''(s)|| = \kappa(s)$ is called the **curvature** of α at s.

It's straightforward to check that $\kappa(s) = 0 \iff \alpha(s) = us + v$ (i.e., the curve is actually a straight line). When $\kappa(s) \neq 0$, the **unit normal** $n(s)$ in the direction $\alpha''(s)$ is well-defined and is given by:

$$\alpha''(s) := \kappa(s) \cdot n(s)$$

The orthogonality of $n(s)$ to $\alpha'(s)$ can be verified by differentiating both sides of $\alpha'(s) \cdot \alpha'(s) = 1$ since $||\alpha'(s)|| = 1$.

Definition 8 (Osculating Plane at s)**.** The **osculating plane** at s is the plane determined by the unit tangent and normal vectors, $\alpha'(s)$, and $n(s)$.

Definition 9 (Binormal Vector at s, $b(s)$)**.** The **binormal vector** as s is defined as $t(s) \wedge n(s)$, where $t(s)$ is the unit tangent at s. The magnitude of this vector, $||b(s)||$, measures how rapidly the curve pulls away from the osculating plane at s in a neighbourhood of s.

Definition 10 (Torsion)**.** Let $\alpha : I \to \mathbb{R}^3$ be a curve parameterised by arclength s such that $\alpha''(s) \neq 0$, $s \in I$. The number $\tau(s)$ defined by $b'(s) := \tau(s)n(s)$ is called the **torsion** of α at s. We have the following useful characterisation:

$$\alpha \text{ is a plane curve} \iff \tau \equiv 0$$

Thus, torsion measures how much a curve *fails* to be a plane curve.

Collecting the orthogonal unit vectors $t(s), n(s), b(s)$ gives us the **Frenet Trihedron** at s. Using the above definitions gives us the **Frenet Formulae**, which is a set of differential equations:

$$t' = \kappa n \tag{3}$$

$$n' = -\kappa t - \tau b \tag{4}$$

$$b' = \tau n \tag{5}$$

- The tb plane is called the **rectifying plane**
- The nb plane is called the **normal plane**
- κ and τ completely describe a curve's behaviour.
- Bending $\sim$ curvature; twising $\sim$ torsion.

The Frenet-Serret frame can be concisely expressed as a skew-symmetric matrix:

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \cdot \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{6}$$

Theorem 1 (Fundamental Theorem of the Local Theory of Curves)**.** Given differentiable functions $\kappa(s) > 0$ and $\tau(s)$, $s \in I$, there exists a regular parameterised curve $\alpha : I \to \mathbb{R}^3$ such that s is the arclength, $\kappa(s)$ is the curvature, and $\tau(s)$ is the torsion of α. Moreover, any other curve $\widetilde{\alpha}$ satisfying the same conditions differ from α by a rigid motion.

Definition 11 (Rigid Motion)**.** A **rigid motion** means that $\exists$ an orthogonal map ρ of $\mathbb{R}^3$ with positive determinant and a vector c such that $\widetilde{\alpha} = \rho \circ \alpha + c$.

Without loss of generality, we can assume curves to be parameterised by arclength, since we can always re parameterise a parameterised curve by arclength:
Let $\alpha : I \to \mathbb{R}^3$ be a regular parameterised curve. Then, it is possible to obtain a curve $\beta : J \to \mathbb{R}^3$ that is parameterised by arc length with the same trace as α:

$$s = s(t) = \int_{t_0}^{t} |\alpha'(t)| dt$$

where $t, t_0 \in I$.

3.3 Global Properties of Curves

3.3.1 The Isoparametric Inequality

This is related to the following isoparametric question:

Q: Of all the simple closed curves in the plane with a given length, which bounds the largest area?

We will use the following formula for the area A bounded by a positively oriented simple closed curve $\alpha(t) = (x(t), y(t))$:

$$A = -\int_a^b y(t)x'(t)dt = \int_a^b x(t)y'(t)dt = \frac{1}{2}(xy' - yx')dt$$

Theorem 2 (The Isoparametric Inequality)**.** Let C be a simple closed plane curve with length ℓ and let A be the area of the region bounded by C. Then:

$$\ell^2 - 4\pi A \geq 0 \tag{7}$$

where equality holds $\iff$ C is a circle.

3.3.2 Cauchy Crofton Formula

Theorem 3 (Cauchy Crofton Formula)**.** Let C be a regular plane curve with length ℓ. The measure of the set of straight lines, counted with multiplicities (**multiplicity** is the number of intersection points of a line with C), which meet C is equal to 2ℓ.

Definition 12 (Rigid Motion in $\mathbb{R}^2$)**.** A **rigid motion** in $\mathbb{R}^2$ is a map $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by $(\overline{x}, \overline{y}) \to (x, y)$, where:

$$x = a + \overline{x}\cos(\varphi) - \overline{y}\sin(\varphi)$$
$$y = b + \overline{x}\sin(\varphi) + \overline{y}\cos(\varphi)$$

Proposition 1. Let $f(x, y)$ be a continuous function defined in $\mathbb{R}^2$. For any set $S \subseteq \mathbb{R}^2$, define the **area A of S** by:

$$A(S) := \iint_S f(x, y)dxdy \tag{8}$$

Assume that A is invariant under rigid motions; that is, if S is a set and $\overline{S} = F^{-1}(S)$, where F is a rigid motion, then if:

$$A(\overline{S}) = \iint_{\overline{S}} f(\overline{x}, \overline{y})d\overline{x}d\overline{y} = \iint_S f(x, y)dxdy = A(S)$$

Then, $f(x, y)$ is a constant.

4 Surfaces

4.1 Definitions

Motivation: we want to define a regular surface to be something that is nice enough for us to extend the usual notions of calculus to.

Definition 13 (Regular Surface)**.** A subset $S \subseteq \mathbb{R}^3$ is called a **regular surface** if, $\forall\ p \in S$, there exists a neighbourhood $V \subseteq \mathbb{R}^3$ and a map $\mathbb{X} : U \to V \cap S$ of an open set $V \subseteq \mathbb{R}^2$ onto $V \cap S \subseteq \mathbb{R}^3$ for which the following conditions hold:

1. $\mathbb{X}$ is differentiable; that is, if we write

$$\mathbb{X}(u, v) = (x(u, v), y(u, v), z(u, v))$$

   for $(u, v) \in U$, then the functions $x(u, v)$, $y(u, v)$ and $z(u, v)$ have continuous partial derivatives of all orders in U.
2. $\mathbb{X}$ is a **homeomorphism**: there exists an inverse $\mathbb{X}^{-1} : V \cap S \to U$, which is continuous.
3. (Regularity Condition): $\forall q \in U$, the differential $d\mathrm{x}_q : \mathbb{R}^2 \to \mathbb{R}^3$ is bijective.

Then, the mapping $\mathbb{X}$ is called a **parameterisation** or a **system of local coordinates** in a neighbourhood of p. The neighbourhood $V \cap S$ of p is called a **coordinate neighbourhood**.

4.2 Regular Surfaces

Example 1 (The Unit Sphere is a Regular Surface)**.** The Unit Sphere is a regular surface. It's parametrised by:

$$S^2 := \{(x, y, z) \in \mathbb{R}^2 \mid x^2 + y^2 + z^2 = 1\}$$

In the textbook, they check all three conditions from the above definition. Since this can be quite exhausting, we want some propositions that simplify the task of determining if a surface is regular or not. This is the aim of this section.

Proposition 2. If $f : U \subseteq \mathbb{R}^2 \to \mathbb{R}$, U open, is a differentiable, then the graph of f, that is, the subset of $\mathbb{R}^3$ given by $(x, y, f(x, y))$ for $(x, y) \in U$, is a regular surface.

Before introducing the second proposition, we will first need to define critical points, critical values, and regular values for differentiable maps.

Definition 14 (Critical Point)**.** Given a differentiable map $F : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ defined in an open set $U \subseteq \mathbb{R}^n$, we say that $p \in U$ is a **critical point** of F id the differential $\mathrm{d}F_p : \mathbb{R}^n \to \mathbb{R}^m$ is not a surjective mapping. The image $F(p) \in \mathbb{R}^m$ of a critical point is called a **critical value** of F. A point $\mathbb{R}^m$ which is not a critical value is called a **regular value**.

The justification for the next proposition comes from the inverse function theorem.

Proposition 3. If $f : U \subseteq \mathbb{R}^3 \to \mathbb{R}$ is a differentiable function and $a \in f(U)$ is a regular value of f, then $f^{-1}(a)$ is a regular surface in $\mathbb{R}^3$.

Example 2 (Ellipsoid)**.** The ellipsoid is given by:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Since it is the set $f^{-1}(0)$ where

$$f(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$$

and f is a differentiable function and 0 is a regular value of f.

Definition 15 (Connected)**.** A surface $S \subseteq \mathbb{R}^3$ is **connected** if any two of its points can be joined by a continuous curve in S.

The next proposition is a very useful property that follows from the intermediate value theorem:

Definition 16. If $f : S \subseteq \mathbb{R}^3 \to \mathbb{R}$ is a non-zero continuous function defined on a connected surface S, then f does not change sign on S.

4.3 Differentiable Functions on Surfaces

4.4 Tangent Plane

The third condition of a regular surface guarantees that for any fixed point $p \in S$, the set of tangent vectors to the parameterised curves of S passing through p constitutes a plane.

Proposition 4. Let $\mathbb{X} : U \subseteq \mathbb{R}^2 \to S$ be a parameterisation of a regular surface S and let $q \in U$. The vector subspace of dimension 2:

$$\mathrm{d}x_q(\mathbb{R}^2) \subseteq \mathbb{R}^3 \tag{9}$$

coincides with the set of tangent vectors to S at $\mathbb{X}(q)$.

This plane does not depend on the parameterisation $\mathbb{X}$ and it is called the **tangent plane** to S at p and is denoted by $T_p(S)$. A choice of parameterisation $\mathbb{X}$ induces a basis on $T_p(S)$:

$$\{(\partial\mathbb{X}/\partial u)(q), (\partial\mathbb{X}/\partial v)(q)\}$$

The next proposition states that a map between two regular surfaces induces a map between the tangent planes, which we can think of as the differential of the map.

Proposition 5. Let S_1, S_2 be regular surfaces and let $\varphi : V \subseteq S_1 \to S_2$ be a differentiable mapping of an open set V of S_1 into S_2. Then, tangent vectors $w \in T_p(S_1)$ are the velocity vectors $\alpha'(0)$ of a differentiable parameterised curve $\alpha :]-\varepsilon, \varepsilon[\to V$ with $\alpha(0) = p$. If we define $\beta := \varphi \circ \alpha$, then $\beta'(0)$ is a vector of $T_{\varphi(p)}(S_2)$. Given a w, the vector $\beta'(0)$ does not depend on the choice of α and the map $\mathrm{d}\varphi_p : T_p(S_1) \to T_{\varphi(p)}(S_2)$ defined by $\mathrm{d}\varphi_p(w) = \beta'(0)$ is linear.

Before moving onto the next proposition, we first need to define what a local diffeomorphism is. The aim is to build up to a generalisation of the standard inverse function theorem from calculus.

Definition 17 (Local Diffeomorphism)**.** A mapping $\varphi : U \subseteq S_1 \to S_2$ is called a **local diffeomorphism** at $p \in U$ if there is a neighbourhood $V \subseteq U$ of p such that $\varphi|_U$ is a diffeomorphism onto an open set $\varphi(V) \subseteq S_2$.

4.5 First Fundamental Form: Area

5 The Gauss Map

5.1 Ruled Surfaces and Minimal Surfaces

6 The Intrinsic Geometry of Surfaces

6.1 Isometries and Conformal Maps