Calculus: Single Variable, Multivariable, Differential Equations, and Vector Calculus Summary

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8 June 2020

Abstract

The purpose of this document is to review Calculus. The content here should be equivalent to Math 140, Math 141, Math 222, and Math 248/358 at McGill.

Contents

1 Single Variable Calculus

1.1 Limits and Derivatives

- **Precise Definition of a Limit**: Let f be a function defined on an open interval $]a, c[$ that contains the number a. Then, we say that the limit of $f(x)$ as x approaches a is L, and we write $\lim_{x\to a} f(x) = L$ if for every $\varepsilon > 0$, $\exists$ a $\delta > 0$ such that $0 < |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon$.
  - Heuristically, this means that if any small interval $]L-\varepsilon, L+\varepsilon[$ is given around L, then we can find an interval $]a-\delta, a+\delta[$ around a such that f maps the points in $]a-\delta, a+\delta[$ (except possibly a) into the interval $]L-\varepsilon, L+\varepsilon[$.
- **Continuous**: A function f is said to be continuous at a number $a \in \mathbb{R}$ if $\lim_{x\to a} f(x) = f(a)$.
- **Intermediate Value Theorem**: Let f be continuous on the interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$ where $f(a) \neq f(b)$. Then, there exists a number $c \in]a, b[$ for which $f(c) = N$.
- **Tangent Line**: The tangent line to the curve $y = f(x)$ at the point $P = (a, f(a))$ is the line through P with the slope

$$m = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} \iff m = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \tag{1}$$

- **Velocity / Instantaneous Velocity**: the instantaneous velocity $v(a)$ at the time $t = a$ is the limit of the average velocities:

$$v(a) := \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \tag{2}$$

- **Derivative**: The **derivative** of a function f at a number $a \in \mathbb{R}$, denoted by $f'(a)$, is

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = f'(a) \tag{3}$$

1.2 Differentiation Rules

- Derivative of a constant function: $\frac{d}{dx}[c] = 0$.
- **Power Rule**: if $n \in \mathbb{R}$, $\frac{d}{dx}[x^n] = nx^{n-1}$. One can prove this using geometric series.
- **Constant Multiple Rule**: if $c \in \mathbb{R}$ and f differentiable, then $\frac{d}{dx}[cf(x)] = cf'(x)$.
- **Constant Sum Rule**: if f, g are differentiable, then $\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$.
- The rate of change of any exponential function is proportional to the function itself: for $f(x) := b^x$:

$$f'(x) = f'(0)b^x \tag{4}$$

- **Derivative of the Natural Exponential Function**:

$$\frac{d}{dx}[e^x] = e^x \tag{5}$$

- **Product Rule**: if f, g are differentiable, then:

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)] \tag{6}$$

- **Quotient Rule**: If f, g are differentiable, then:

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2} \tag{7}$$

- **Derivatives of Trigonometric Functions**:
  - $\frac{d}{dx}[\sin(x)] = \cos(x)$, $\frac{d}{dx}[\csc(x)] = -\csc(x)\cot(x)$
  - $\frac{d}{dx}[\cos(x)] = -\sin(x)$, $\frac{d}{dx}[\sec(x)] = \sec(x)\tan(x)$
  - $\frac{d}{dx}[\tan(x)] = \sec^2(x)$, $\frac{d}{dx}[\cot(x)] = -\csc^2(x)$.
- **Chain Rule**: If g is differentiable at x and if f is differentiable at $g(x)$, then the composite function $F := f \circ g$ defined by $F(x) := f(g(x))$ is differentiable at x and F' is given by the product:

$$F'(x) = f'(g(x)) \cdot g'(x) \tag{8}$$

or, in Leibniz notation,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \tag{9}$$

- **Method of Implicit Differentiation**: Differentiating both sides of the equation with respect to x, and then solving the resulting equation for y'.
  - Application: finding the derivatives of inverse trigonometric functions:
    * $\frac{d}{dx}[\arcsin(x)] = \frac{1}{\sqrt{1-x^2}}$, $\frac{d}{dx}[\text{arcsec}(x)] = \frac{-1}{x\sqrt{x^2-1}}$
    * $\frac{d}{dx}[\arccos(x)] = \frac{-1}{\sqrt{1-x^2}}$, $\frac{d}{dx}[\text{arcsec}(x)] = \frac{1}{x\sqrt{x^2-1}}$
    * $\frac{d}{dx}[\arctan(x)] = \frac{1}{1+x^2}$, $\frac{d}{dx}[\text{arccot}(x)] = \frac{-1}{x^2+1}$
  - Application: derivatives of logarithmic functions, $y = \log_b(x)$ and $y = \ln(x)$.
    * $\frac{d}{dx}[\log_b(x)] = \frac{1}{x\ln(b)}$
    * $\frac{d}{dx}[\ln(x)] = \frac{1}{x}$
  - **Method of Logarithmic Differentiation**: the calculation of complex functions involving products, quotients, or powers can be simplified by taking logarithms.
- **Hyperbolic Trigonometric Functions**: hyperbolic functions $\sim$ hyperbola like trigonometric functions $\sim$ circle. They are defined as:
  - $\sinh(x) := \frac{e^x - e^{-x}}{2}$, $\text{csch}(x) := \frac{1}{\sinh(x)}$
  - $\cosh(x) := \frac{e^x + e^{-x}}{2}$, $\text{sech}(x) := \frac{1}{\cosh(x)}$
  - $\tanh(x) := \frac{\sinh(x)}{\cosh(x)}$, $\coth(x) := \frac{\cosh(x)}{\sinh(x)}$
  - Applications: whenever an entity such as light, velocity, electricity, or radioactivity is gradually absorbed or extinguished.
  - Hyperbolic identities:
    * $\sinh(-x) = -\sinh(x)$, $\cosh(-x) = \cosh(x)$
    * $\cosh^2(x) - \sinh^2(x) = 1$, $1 - \tanh^2(x) = \text{sech}^2(x)$
    * $\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y)$
    * $\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$
  - Derivatives of Hyperbolic Functions:
    * $\frac{d}{dx}[\sinh(x)] = \cosh(x)$, $\frac{d}{dx}[\text{csch}(x)] = -\text{csch}(x)\coth(x)$,
    * $\frac{d}{dx}[\cosh(x)] = \sinh(x)$, $\frac{d}{dx}[\text{sech}(x)] = -\text{sech}(x)\tanh(x)$,
    * $\frac{d}{dx}[\tanh(x)] = \text{sech}^2(x)$, $\frac{d}{dx}[\coth(x)] = -\text{csch}^2(x)$.
  - **Inverse Hyperbolic Functions**:
    * $\text{arcsinh}(x) := \ln(x + \sqrt{x^2+1})$ for $x \in \mathbb{R}$.
    * $\text{arccosh}(x) := \ln(x + \sqrt{x^2-1})$ for $x \geq 1$.
    * $\text{arctanh}(x) := \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ for $x \in [-1, 1]$
  - Derivatives of Inverse Hyperbolic Functions:

* $\frac{d}{dx}\left[\operatorname{arcsinh}(x)\right] = \frac{1}{\sqrt{1+x^2}}$, $\frac{d}{dx}\left[\operatorname{arccsch}(x)\right] = \frac{-1}{|x|\sqrt{x^2+1}}$
* $\frac{d}{dx}\left[\operatorname{arccosh}(x)\right] = \frac{1}{\sqrt{x^2-1}}$, $\frac{d}{dx}\left[\operatorname{arcsech}(x)\right] = \frac{-1}{x\sqrt{1-x^2}}$
* $\frac{d}{dx}\left[\operatorname{arctanh}(x)\right] = \frac{1}{1-x^2}$, $\frac{d}{dx}\left[\operatorname{arccoth}(x)\right] = \frac{1}{1-x^2}$

1.3 Applications of Differentiation

- **Extreme Value Theorem**: Let f be continuous on the closed and bounded interval $[a, b]$. Then, f attains an absolute maximum value $f(x)$ and an absolute minimum value $f(d)$ at some numbers $c, d \in [a, b]$.
- **Fermat's Theorem**: If f has a local maximum or minimum at c, and if $f'9x)$ exists, then $f'(x) = 0$.
- **Closed Interval Method**: To find the absolute maximum and minimum values of a continuous function f on a closed interval $[a, b]$,
  1. Find the values of f at the critical points of f in the open interval $]a, b[$.
  2. Compute $f(a)$ and $f(b)$.
  3. The max between (1) and (2) is the absolute max; the min between (1) and (2) is the absolute min.
- **Rolle's Theorem**: Let $f : [a, b] \to \mathbb{R}$ satisfy:
  1. f is continuous on $[a, b]$
  2. f is differentiable on $]a, b[$
  3. $f(a) = f(b)$.

  Then, there exists a number $c \in]a, b[$ such that $f'(c) = 0$.
- **Mean Value Theorem**: Let $f : [a, b] \to \mathbb{R}$ satisfy:
  1. f is continuous on $[a, b]$
  2. f is differentiable on $]a, b[$

  Then, there exists a number $c \in]a, b[$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \iff f(b) - f(a) = f'(c)[b - a] \tag{10}$$

- **Theorem** (Consequence of MVT): If $f'(x) = 0 \ \forall x \in]a, b[$, then f is constant on $]a, b[$.
  - **Corollary**: If $f'(x) = g'(x) \ \forall x \in]a, b[$, then $f - g$ is constant on $]a, b[$, i.e., $\exists c \in \mathbb{R}$ such that $f(x) = g(x) + c$.
- **L'Hopital's Rule**: Suppose f and g are differentiable and that $g(x) \neq 0$ on an open interval containing a. Suppose that

$$\lim_{x\to a} f(x) = 0 \text{ and } \lim_{x\to a} g(x) = 0 \tag{11}$$

  or

$$\lim_{x\to a} f(x) = \pm\infty \text{ and } \lim_{x\to a} f(x) = \pm\infty \tag{12}$$

  then

$$\lim_{x\to a} \left(\frac{f(x)}{g(x)}\right) = \lim_{x\to a} \left(\frac{f'(x)}{g'(x)}\right) \tag{13}$$

- **Antiderivative**: A function F is called an **anti-derivative** of f on an interval I if $F'(x) = f(x)$ $\forall x \in I$.

1.4 Integrals

- **Area**: The **area** A of a region S that lies under the graph of a continuous function f is the limit of the sum of the approximating rectangles
$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} [f(x_1)\Delta x + ... + f(x_n)\Delta x] \tag{14}$$
- **Definite Integral**: Let $f : [a, b] \to \mathbb{R}$. Divide $[a, b]$ into n subintervals of equal width $\Delta x = \frac{b-a}{n}$. Let $a = x_0 < x_1 < ... < x_n = b$ be the endpoints and let $x_1^*, ..., x_n^*$ be any sample points in these subintervals such that $x_i^* \in [x_{i-1}, x_i]$. Then, the definite integral of f from a to b is:
$$\int_a^b f(x)dx := \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x \tag{15}$$
provided that the limit exists and is the same for all possible choices of sample points. If it does exist, then we say that f is **integrable** on $[a, b]$.
- Formulae for the sums of positive integers:
  - $\sum_{i=1}^n i = \frac{n(n+1)}{2}$
  - $\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$
  - $\sum_{i=1}^n i^3 = \left[\frac{n(n+1)}{2}\right]^2$
- **Fundamental Theorem of Calculus** connects differential calculus and integral calculus. Deals with equations of the form
$$g(x) = \int_a^x f(t)dt \tag{16}$$
  - **Fundamental Theorem of Calculus Part 1**: let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. Then, the function g defined by
  $$g(x) := \int_a^x f(t)dt \tag{17}$$
  is continuous on $[a, b]$ and differentiable on $]a, b[$. Moreover, $g'(x) = f(x)$.
  - **Fundamental Theorem of Calculus Part 2**: If f is continuous on $[a, b]$, then
  $$\int_a^b f(x)dx = F(b) - F(a) \tag{18}$$
  where F is any anti-derivative of f.
  - Alternative expression for the FoC Part 1:
  $$\frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x) \tag{19}$$
- **Table of Integration Formulae**:
  - $\int x^n dx = \frac{x^{n+1}}{n+1}$ for $n \neq -1$.
  - $\int \frac{1}{x} dx = \ln|x|$
  - $\int e^x dx = e^x$
  - $\int b^x dx = \frac{b^x}{\ln b}$
  - $\int \sin(x)dx = -\cos(x)$
  - $\int \cos(x)dx = \sin(x)$
  - $\int \sec^2(x)dx = \tan(x)$
  - $\int \csc^2(x)dx = -\cot(x)$
  - $\int \sec(x)\tan(x)dx = \sec(x)$
  - $\int \csc(x)\cot(x)dx = -\csc(x)$
  - $\int \sec(x)dx = \ln|\sec(x) + \tan(x)|$
  - $\int \csc(x)dx = \ln|\csc(x) - \cot(x)|$
  - $\int \tan(x)dx = \ln|\sec(x)|$
  - $\int \cot(x)dx = \ln|\sin(x)|$
  - $\int \sinh(x)dx = \cosh(x)$
  - $\int \cosh(x)dx = \sinh(x)$
  - $\int \frac{1}{x^2+a^2} dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right)$
  - $\int \frac{1}{\sqrt{a^2-x^2}} dx = \arcsin\left(\frac{x}{a}\right)$ $a > 0$
  - $\int \frac{1}{x^2-a^2} dx = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|$
  - $\int \frac{1}{\sqrt{x^2\pm a^2}} dx = \ln|x + \sqrt{x^2 \pm a^2}|$

1.5 Applications of Integration

- The **average value** of f on the interval $[a, b]$ is:

$$f_{avg} := \frac{1}{b-a}\int_a^b f(x)dx \tag{20}$$

- **Mean Value Theorem for Integrals**: If f is continuous on $[a, b]$ then there exists a $c \in [a, b]$ such that

$$f(x) = f_{avg} = \frac{1}{b-a}\int_a^b f(x)dx \iff \int_a^b f(x)dx = f(c)(b-a) \tag{21}$$

1.6 Integration Techniques

- **Integration by Parts**:

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx \tag{22}$$

- **Trigonometric Integrals**:
  1. Strategy for evaluating $\int \sin^m(x)\cos^n(x)dx$:
     (a) If n is odd: save one cosine, use $\cos^2(x) = 1 - \sin^2(x)$ to express the remaining factors in terms of sine:

$$\int \sin^m(x)\cos^{2k+1}(x)dx = \int \sin^m(x)(1-\sin^2(x))^k\cos(x)dx \tag{23}$$

     and make the substitution $u = \sin(x)$.
     (b) If m is odd: save one sine, use $\sin^2(x) = 1 - \cos^2(x)$ to express the remaining factors in terms of cosine:

$$\int \sin^{2k+1}(x)\cos^n(x)dx = \int (1-\cos^2(x))^k\cos^n(x)\sin(x)dx \tag{24}$$

     and make the substitution $u = \cos(x)$.
     (c) If sine and cosine are even, then use the half-angle identities:

$$\sin^2(x) = \frac{1}{2}(1-\cos(2x)) \text{ and } \cos^2(x) = \frac{1}{2}(1+\cos(2x)) \tag{25}$$

     A helpful identity is $\sin(x)\cos(x) = \frac{1}{2}\sin(2x)$.
  2. Strategy for evaluating $\int \tan^m(x)\sec^n(x)dx$:
     (a) If n is even: save one secant squared, use the identity $\sec^2(x) = 1 + \tan^2(x)$ to express the remaining factors in terms of $\tan(x)$:

$$\int \tan^m(x)\sec^{2k}(x)dx = \int \tan^m(x)(1+\tan^2(x))^{k-1}\sec^2(x)dx \tag{26}$$

     and make the substitution $u = \tan(x)$.
     (b) If m is odd: save one $\sec(x)\tan(x)$, use $\tan^2(x) = \sec^2(x) - 1$ to express the remaining factors in terms of $\sec(x)$:

$$\int \tan^{2k+1}(x)\sec^n(x)dx = \int (\sec^2(x)-1)^k\sec^{n-1}(x)\sec(x)\tan(x)dx \tag{27}$$

     substitute $u = \sec(x)$.

3. Important product identities to remember:
   (a) $\sin(A)\cos(B) = \frac{1}{2}[\sin(A-B)+\sin(A+B)]$
   (b) $\sin(A)\sin(B) = \frac{1}{2}[\cos(A-B)+\cos(A+B)]$
   (c) $\cos(A)\cos(B) = \frac{1}{2}[\cos(A-B)+\cos(A+B)]$

- **Trigonometric Substitution**
  - $\sqrt{a^2-x^2} \to x = a\sin(\theta)$, $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to 1-\sin^2(\theta) = \cos^2(\theta)$.
  - $\sqrt{a^2+x^2} \to x = a\tan(\theta)$, $\theta \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\to 1+\tan^2(\theta) = \sec^2(\theta)$.
  - $\sqrt{x^2-a^2} \to x = a\sec(\theta)$, $\theta \in \left[0, \frac{\pi}{2}\right[\cup \left[\pi, \frac{3\pi}{2}\right[\to \sec^2(\theta)-1 = \tan^2(\theta)$.

- **Partial Fractions**:
  1. Case I: Denominator $Q(x)$ is a product of distinct linear factors:
  $$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + ... + \frac{A_k}{a_kx+b_k} \tag{28}$$
  2. Case II: Denominator $Q(x)$ is a product of linear factors, some of which are repeated r times:
  $$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + ... + \frac{A_k}{(a_1x+b_1)^r} \tag{29}$$
  3. Case III: $Q(x)$ contains irreducible quadratic factors, none of which is repeated. Then, expression will have a term of the form
  $$\frac{Ax+B}{ax^2+bx+c} \tag{30}$$
  which can be integrated by completing the square and using the formula:
  $$\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C \tag{31}$$
  4. Case IV: $Q(x)$ contains a repeated irreducible factor. Then, the expression will have a term of the form:
  $$\frac{A_1x+B_1}{ax^2+bx+c} + ... + \frac{A_rx+B_r}{(ax^2+bx+c)^r} \tag{32}$$

- General Strategy for Integrating:
  1. Simplify the integrand if possible using algebraic manipulation and trigonometric identities.
  2. Look for obvious substitutions.
  3. Classify integrand according to its form
     (a) Trigonometric functions
     (b) Rational functions ($\to$ partial fractions)
     (c) Integration by parts
     (d) Radicals
        i. $\sqrt{\pm x^2 \pm a^2} \to$ trigonmetric substitution
        ii. $(ax+b)^{1/n} \to$ rationalising substitution $u = (ax+b)^{1/n}$

- **Improper Integral**: if in the definite integral, $\int_a^n f(x)dx$, either $[a,b]$ is an unbounded interval or $f(x)$ has an infinite discontinuity in $[a,b]$

1.7 Further Applications of Integration

- **Arc-length formula**: If f' is continuous on $[a, b]$, then the length of the curve $y = f(x)$, $a \leq x \leq b$ is:

$$L = \int_a^b \sqrt{1 + [f'(x)]^2} \tag{33}$$

- **Arc-Length Function**: If a smooth curve C has the equation $y = f(x)$, $a \leq x \leq b$, let $s(x)$ be the distance along C from the initial point $P_0(a, f(a))$ to the point $Q(x, f(x))$. Then, s is a function given by:

$$s(x) = \int_a^x \sqrt{1 + [f'(t)]^2} dt \tag{34}$$

1.8 Parameter Equations and Polar Coordinates

Motivation: some curves are best handeled when both x and y are given as a function of a third variable t: $x = f(t)$, $y = g(t)$.

- Suppose f, g are differentiable functions and suppose we want to find the tangent line at a point on the parametric curve $x = f(t)$, $y = g(t)$, where y is also a differentiable function of x. If $\frac{dx}{dt} \neq 0$, then the slope of the parametric curve is given by:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \tag{35}$$

  - We can consider $\frac{d^2y}{d^2x}$:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} \tag{36}$$

  - **Areas**: if a curve is traced out by the parametric equations $x = f(t)$ and $y = g(t)$ for $t \in [\alpha, \beta]$, then using the substitution rule for integrals one has the following formula:

$$\mathcal{A} = \int_a^b y dx = \int_\alpha^\beta g(t) f'(t) dt \tag{37}$$

  - **Arc Length** if a curve C is described by parametric equations $x = f(t)$, $y = g(t)$, $\alpha \leq t \leq \beta$, where f' and g' are continuous on $[a, b]$ and C is traversed exactly once as t travels from α to β, then the length of C is:

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \tag{38}$$

  - **Surface Area**: similar to the conditions in the previous theorem, the surface area of a curve obtained by rotating it about the x-axis is given by:

$$S = \int_\alpha^\beta 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \tag{39}$$

- Equations to convert between cartesian and polar coordinates:

$$x = r\cos(\theta) \;\; y = r\sin(\theta) \tag{40}$$

$$r^2 = x^2 + y^2 \;\; \tan(\theta) = \frac{y}{x} \tag{41}$$

- If $r = f(\theta)$ is a polar curve, then we can find the tangent line to a polar curve by regarding θ as a parameter:

$$x = f(\theta)\cos(\theta)$$
$$y = f(\theta)\sin(\theta)$$

and the tangent is given by:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin(\theta) + r\cos(\theta)}{\frac{dr}{d\theta}\cos(\theta) - r\sin(\theta)} \tag{42}$$

- Areas and lengths in polar coordinates
  - The area of a sector of a circle: $A = \frac{1}{2}r^2\theta$. The area of a polar region $\mathcal{R}$:

$$A(\mathcal{R}) = \int_a^b \frac{1}{2}[f(\theta)]^2 d\theta \tag{43}$$

  - The arc-length of a polar curve with the equation $r = f(\theta)$, $a \leq \theta \leq b$ is:

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta \tag{44}$$

1.9 Infinite Sequences and Series

- A **Sequence** is a list of numbers written in a definite order

$$a_1, a_2, a_3, ... \tag{45}$$

- A sequence has a **limit** L and we write

$$\lim_{n\to\infty} a_n = L$$

if we can make the terms a_n as close to L as we'd like by taking n sufficiently large. if $\lim_{n\to\infty} a_n$ exists, then we say that $\{a_n\}$ is **convergent**. Else, it is **divergent**.
  - A sequence $\{a_n\}$ has a limit L if $\forall\ \varepsilon > 0$, $\exists$ an $N \in \mathbb{N}$ such that $\forall n \geq N$, one has that $|a_n - L| < \varepsilon$.

- **Squeeze Theorem**: if $a_n \leq b_n \leq c_n$ $\forall n \geq n_0$, and if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.
  - If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.
  - If $\lim_{n\to\infty} a_n = L$ and if f is a continuous function at L, then

$$\lim_{n\to\infty} f(a_n) = f(L)$$

- The sequence $\{r^n\}$ is convergent if $r \in]-1, 1]$ and divergent for all other values of r. If $r \in]-1, 1]$:

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } r \in]-1, 1[\\ 1 & \text{if } r = 1 \end{cases}$$

- **Monotonic Sequence Theorem**: every bounded, monotonic sequence converges.

- **Series**: Given a series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + ...$$

let s_n denote the nth partial sum:

$$s_n := \sum_{i=1}^{n} a_i = a_1 + ... + a_n$$

if the *sequence* $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = s \in \mathbb{R}$, then the series $\sum_{n=1}^{\infty} a_n$ is convergent.
- **Geometric Series**: is an important example of an infinite series.

$$a + ar + ar^2 + ... + ar^{n-1} + ... = \sum_{n=1}^{\infty} ar^{n-1} \ (a \neq 0) \tag{46}$$

  - If $r = 1$, then $s_n = na \to \infty$.
  - If $r \neq 1$, then $s_n = \frac{a(1-r^n)}{1-r}$. If $r \in]-1, 1[$, then $r^n \to 0$ as $n \to \infty$, and so

$$\lim_{n\to\infty} s_n = \frac{a}{1-r} \tag{47}$$

    otherwise the geometric series diverges.
- **Harmonic Series** is defined as $\sum_{n=1}^{\infty} \frac{1}{n}$. It's divergent.
- If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.
  - **Test for Divergence**: if $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
- **Integral Test**: Suppose f is a continuous, positive, decreasing function on $[1, \infty[$ and let $a_n = f(n)$. Then, the series $\sum_{n=1}^{\infty} a_n$ is convergent $\iff$ the improper integral $\int_1^{\infty} f(x)dx$ is convergent.
  - **p-series**: the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges $\iff$ $p > 1$.
- **Remainder Estimate for the Integral Test**: Suppose that $f(k) = a_k$, where f is a continuous, decreasing, positive function for $x \geq n$ and suppose that $\sum a_n$ is convergent. If $R_n := S - S_n$, then

$$\int_{n+1}^{\infty} f(x)dx \leq R_n \leq \int_{n}^{\infty} f(x)dx \tag{48}$$

- **Comparison Test**: Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.
  1. If $\sum b_n$ is convergent and $a_n \leq b_n \ \forall n$, then $\sum a_n$ converges.
  2. If $\sum b_n$ is divergent and $b_n \leq a_n \ \forall n$, then $\sum a_n$ diverges.
- **Limit Comparison Test**: Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where $c \in]0, \infty[$, then both series have the same behaviour; i,e, either both series converge or both diverge.
- **Alternating Series Test**: If the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ for $b_n > 0$ satisfies:
  1. $b_{n+1} \leq b_n \ \forall n \in \mathbb{N}$
  2. $\lim_{n\to\infty} b_n = 0$

  then, the series converges.

- **Absolutely Convergent**: A series $\sum a_n$ is called **absolutely convergent** if the series of absolute values $\sum |a_n|$ is convergent.
- **Conditionally Convergent**: A series $\sum a_n$ is called **conditionally convergent** if its convergent but not absolutely convergent.
- If $\sum a_n$ is absolutely convergent then it is convergent.
- **Ratio Test** Let $\{a_n\}$ be a sequence.
  1. If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then $\sum_{n=1}^{\infty} a_n$ absolutely converges (and thus converges).
  2. If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| - L = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.
  3. If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L = 1$ then the Ratio test is inconclusive.
- **Root Test** Let $\{a_n\}$ be a sequence.
  1. If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series absolutely converges (and therefore converges).
  2. If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or if $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$ then the series diverges.
  3. If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L = 1$, then the root test will be inconclusive.

1.10 Strategy for Testing Series

1. If the series is of the form $\sum \frac{1}{n^p}$, then apply the p-series rule.
2. If the series is of the form $\sum ar^{n-1}$ or $\sum ar^n$, then apply the geometric series rule.
3. If the series is similar to a p-series or a geometric series, then use a comparison test.
4. If $\lim_{n\to\infty} a_n \neq 0$, use the divergence test to conclude that the series diverges.
5. If the series is of the form $\sum(-1)^{n-1}b_n$ or $\sum(-1)^n b_n$, then use the alternating series test.
6. If the series has factorials in it, consider applying a ratio test.
7. If the series is of the form $(b_n)^n$, then consider the root test.
8. If $a_n = f(n)$ where $\int_1^{\infty} f(x)dx$ is easily evaluated, then consider the integral test.

1.11 Power Series

- **Power Series**: a power series is of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + ... \tag{49}$$

  A series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 \tag{50}$$

  is called a **power series in** $(x-a)$
- **Theorem**: for a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, there are only three possibilities:
  1. The series converges only when $x = a$.
  2. The series converges for all x.
  3. $\exists$ an $R > 0$ such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.
- **Theorem (Term-by-term Differentiation and Integration)**: If the power series $\sum c_n(x-a)^n$ has a radius of convergence $R > 0$, then the function defined by:

$$f(x) := c_0 + c_1(x-a) + c_2(x-a)^2 + ... = \sum_{n=0}^{\infty} c_n(x-a)^n \tag{51}$$

  is differentiable (and thus continuous) on the interval $]a-R, a+R[$ and:

1. $f'(x) = c_0 + 2c_s(x-a) + 3c_3(x-a)^2 + ... = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$.
2. $\int f(x)dx = c + c_0(x-a) + c_a\frac{(c-a)^2}{2} + c_2\frac{(x-a)^3}{3} + ...$

$$\int f(x)dx = c + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} \tag{52}$$

The radius of convergence of both (1) and (2) remain R.

- **Theorem (Taylor Series Representation)**: If f has a power series representation (expansion) at a, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n \tag{53}$$

for $|x-a| < R$, then the coefficients are given by the formula:

$$c_n = \frac{f^{(n)}(a)}{n!} \tag{54}$$

Then, the **Taylor Series** for f about a is given by:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \tag{55}$$

For the special case of $a = 0$, then the above becomes:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + ... \tag{56}$$

- **Theorem (Remainder)**: Define the remainder of the Taylor series by $R_n := f(x) - T_n(x)$. If $f(x) = T_n(x) + R_n(x)$, where T_n is the nth degree Taylor polynomial of f at a and

$$\lim_{n\to\infty} R_n(x) = 0 \tag{57}$$

for $|x-a| < R$, then f is equal to the sum of its Taylor series on the interval $|x-a| < R$.
- The following theorem is often used when trying to show that $\lim_{n\to\infty} R_n = 0$ for a specific function f:
  - **Taylor's Inequality**: If $|f^{(n+1)}(x)| \leq M \ \forall \ |x-a| \leq d$, then the remainder $R_n(x)$ of the Taylor series satisfies the following inequality:

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \tag{58}$$

for $|x-a| \leq d$.

- **Important MacLaurin Series and their Radii of Convergence**:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + ... \ (R = 1) \tag{59}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ... \ (R = \infty) \tag{60}$$

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}... \ (R = \infty) \tag{61}$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ... \ (R = \infty) \tag{62}$$

$$\arctan(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)} = 1 - \frac{x^3}{3} + \frac{x^5}{5!} - \frac{x^7}{7!} + ... \ (R = 1) \tag{63}$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ... \ (R = 1) \tag{64}$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!} x^3 + \frac{k(k-1)(k-2)}{3!} x^3 + ... \ (R = 1) \tag{65}$$

2 Multi Variable Calculus

2.1 Vectors and Geometry of Space

- If θ is the angle between vectors $\mathbf{a}$ and $\mathbf{b}$, then $\mathbf{a} \cdot \mathbf{b} = ||\mathbf{a}||||\mathbf{b}|| \cos(\theta)$
  - Two vectors are orthogonal $\iff$ $\mathbf{a} \cdot \mathbf{b} = 0$.
  - **Scalar Projection of b onto a**:

$$\text{comp}_{\mathbf{a}}(\mathbf{b}) : \frac{\mathbf{a} \cdot \mathbf{b}}{||\mathbf{a}||} \tag{66}$$

  - **Vector projection of b onto a**:

$$\text{proj}_{\mathbf{a}}(\mathbf{b}) := \left(\frac{\mathbf{a} \cdot \mathbf{b}}{||\mathbf{a}||} \right) \frac{\mathbf{a}}{||\mathbf{a}||} \tag{67}$$

- **Cross Product**: if $\mathbf{a} := \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} := \langle b_1, b_2, b_3 \rangle$, then their **cross product** is:

$$\mathbf{a} \times \mathbf{b} := \det \begin{bmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} \tag{68}$$

  - If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, then

$$||\mathbf{a} \times \mathbf{b}|| := ||\mathbf{a}||||\mathbf{b}|| \sin(\theta) \tag{69}$$

  - Two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel $\iff$ $\mathbf{a} \times \mathbf{b} = 0$.
  - The volume of the parallelepiped spanned by the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ is $|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$

- A parametric equation for a line going through the point (x_0, y_0, z_0) parallel to the direction vector $\langle a, b, c \rangle$ are:

$$x = x_0 + at$$
$$y = y_0 + bt$$
$$z = z_0 + ct$$

- The scalar equation of the plane through the point $P_0(x_0, y_0, z_0)$ with the normal vector $\mathbf{n} := \langle a, b, c\rangle$ is:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \tag{70}$$

- The distance D from a point $P_1(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$ is:

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + z^2}} \tag{71}$$

- **Vector-Valued Function**: a function whose domain is the set of real numbers and whose range is a set of vectors.
  - The **limit** of a vector-valued function $\mathbf{r}$ is defined by taking the limits of the component functions as follows:

$$\lim_{t\to a} \mathbf{r}(t) := \langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t)\rangle \tag{72}$$

  - **Space-curve**: the set C of all points (x, y, z) in space where $x = f(t)$, $y = g(t)$, and $z = h(t)$, where $t \in I$, is called a **space-curve**.
- The **derivative** $\mathbf{r}'$ of a vector-valued function $\mathbf{r}$ is defined as:

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h\to 0} \frac{\mathbf{r}(t + h) - \mathbf{r}(t)}{h} \tag{73}$$

  - The **unit tangent** vector $\mathbf{T}(t)$ is defined as:

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{||\mathbf{r}'(t)||} \tag{74}$$

  - The **definite integral** of a vector-valued function is exactly what one would expect:

$$\int_a^b \mathbf{r}(t)dt = \left(\int_a^b f(t)dt\right)\hat{\mathbf{i}} + \left(\int_a^b g(t)dt\right)\hat{\mathbf{j}} + \left(\int_a^b h(t)dt\right)\hat{\mathbf{k}} \tag{75}$$

- The length L of a space-curve between the points a and b is parameterisation-independent and is given by:

$$L = \int_a^b ||\mathbf{r}'(t)||dt \tag{76}$$

  - The **arc-length function** of a curve, s, is defined as:

$$s(t) := \int_a^t ||\mathbf{r}'(u)||du \tag{77}$$

  We can use the above equation to parameterise a curve with respect to arc-length by differentiating both sides of the equation above with respect to t and applying the fundamental theorem of calculus:

$$\frac{ds}{dt} = ||\mathbf{r}'(t)|| \tag{78}$$

  Advantages of an arc-length parametrisation include: it arises naturally from the shape of the curve and it's coordinate-system independent.

- **Smooth**: a parameterisation $\mathbf{r}(t)$ is called **smooth** on an interval I if $\mathbf{r}'$ is continuous and $\mathbf{r}'(t) \neq 0$ on I.
- **Curvature**: the **curvature** of a curve is given by:

$$\kappa := \left|\left| \frac{d\mathbf{T}}{ds} \right|\right| \tag{79}$$

where $\mathbf{T}$ is the unit tangent vector. We have three other formulae for curvature:

1. $$\kappa(t) = \frac{||\mathbf{T}'(t)||}{||\mathbf{r}'(t)||} \tag{80}$$

2. $$\kappa(t) = \frac{||\mathbf{r}'(t) \times \mathbf{r}''(t)||}{||\mathbf{r}'(t)||^3} \tag{81}$$

3. For plane curves, write $\mathbf{r}(x) = x\hat{\mathbf{i}} + f(x)\hat{\mathbf{j}}$

$$\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x)^2]^{3/2}} \tag{82}$$

- When $\kappa(t) \neq 0$, one can define the **principle unit normal** $\mathbf{N}(t)$:

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{||\mathbf{T}'(t)||} \tag{83}$$

  - **Normal plane**: the plane determined by the normal and binormal vectors. Consists of all lines orthogonal to the unit tangent vector $\mathbf{T}$.
  - **Osculating circle**: the plane determined by the vectors $\mathbf{T}$ and $\mathbf{N}$.
    - Closest plane to containing the part of the curve near P.
    - **Osculating circle**: the circle that lies on the osculating plane of C at P, has the same tangent as C at P, and lies on the concave side of C (towards where $\mathbf{N}$ is pointing). This best describes the behaviour of C near P.
- The **velocity vector** $\mathbf{v}(t)$ at time t:

$$\mathbf{v}(t) := \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} = \mathbf{r}'(t) \tag{84}$$

  - **Speed**: the magnitude of the velocity vector $||\mathbf{v}(t)||$.

$$||\mathbf{v}(t)|| = ||\mathbf{r}'(t)|| = \frac{ds}{dt} \tag{85}$$

  - **Acceleration**: the derivative of the velocity

$$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t) \tag{86}$$

- Often times it can be useful to resolve the acceleration of a particle into its tangential and normal components:

$$\mathbf{a} = \underbrace{v'}_{:=\mathbf{a}_T} \mathbf{T} + \underbrace{\kappa v^2}_{:=\mathbf{a}_N} \mathbf{N} \tag{87}$$

where $v := ||\mathbf{r}'(t)||$. One can re-write Equation (87) so that it only depends on $\mathbf{r}$, $\mathbf{r}'$, and $\mathbf{r}''$:

$$\mathbf{a}_T = \frac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{||\mathbf{r}'(t)||} \tag{88}$$

$$\mathbf{a}_N = \frac{||\mathbf{r}'(t) \times \mathbf{r}''(t)||}{||\mathbf{r}'(t)||} \tag{89}$$

2.2 Partial Derivatives

- **Graph**: let f be a function in two variables with domain Ω. Then, the **graph** of f is the set of all points $(x, y, z) \in \mathbb{R}^3$ such that $z = f(x, y)$ for $(z, y) \in \Omega$.
- **Level Curves**: The **level curves** of a function f in two variables are the curves with equations $f(x, y) = k$, where $k \in \mathbb{R}$ is a constant.
- **Limit**: Let f be a function of two variables whose domain Ω includes points arbitrarily close to (a, b). Then, we say that the **limit of $f(x, y)$ as $(x, y) \to (a, b)$** is L and we write:

$$\lim_{(x,y)\to(a,b)} f(x, y) = L \tag{90}$$

  if $\forall\ \varepsilon > 0$, $\exists \delta > 0$ such that if $(x, y) \in \Omega$, $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$, then $|f(x, y) - L| < \varepsilon$.
- **Partial Derivative**: the partial derivative of f with respect to x at (a, b) is:

$$f_x(a, b) := \lim_{h\to 0} \frac{f(a + h) - f(a, b)}{h} \tag{91}$$

- **Claircut's Theorem**: suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xx} and f_{yy} are both continuous on D, then,

$$f_{xy}(a, b) = f_{yx}(a, b) \tag{92}$$

- **Tangent Plane**: Suppose f has continuous partial derivatives. An equation for the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is given by:

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \tag{93}$$

  - **Linearisation**: an equation for the tangent plane to the graph f at the point $(a, b, f(a, b))$ is given by:

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \tag{94}$$

    the graph of this tangent plane is called the **linearisation** of f at (a, b):

$$L(x, y) := f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \tag{95}$$

- **Differentiable**: If $z = f(x, y)$, then f is **differentiable** at (a, b) if Δz can be expressed in the form:

$$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y \tag{96}$$

  where $\varepsilon_1, \varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.
  - If the partial derivatives f_x and f_y exist near (a, b), and are continuous at (a, b), then f is differentiable at (a, b).
  - **Total Differential**: for a differentiable function of two variables $z = f(x, y)$, then the **total differential** is defined as:

$$\begin{aligned} dz &:= f_x(x, y)dx + f_y(x, y)dy \\ &= \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \end{aligned}$$

    and so in the language of differentials, we write:

$$f(x, y) \approx f(a, b) + dz$$

- **Chain Rule (1. Case)**: Suppose that $z = f(x, y)$ is a differentiable function of x and y, and suppose that both x and y are differentiable functions of t (i.e, $x = x(t)$, $y = y(t)$) so that $z = (f(x(t), y(t))$. Then, z is a differentiable function of t and:
$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \tag{97}$$
- **Chain Rule (General Version)**: Suppose that u is a differentiable function of n variables $(x_1, ..., x_n)$ and each x_j is a differentiable function of m variables $t_1, ..., t_m$. Then, u is a function of $t_1, ..., t_m$ and one has:
$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_i}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_i}\frac{\partial x_2}{\partial t_i} + ... + \frac{\partial u}{\partial x_i}\frac{\partial x_n}{\partial t_i} \tag{98}$$
$\forall i = 1, 2, ..., m$.
- **Implicit Function Theorem**: Let $y = f(x)$. If F is defined on a disc containing (a, b), where $F(a, b) = 0$ $F_y(a, b) \neq 0$, and F_y and F_x are continuous on the disc, then the equation $F(x, y) = 0$ defines y as a function of x near the point (a, b) whose derivative is given by:
$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y} \tag{99}$$
Now, for $z = f(x, y)$, if z is implicitly given as a function by an equation of the form $F(x, y, z) = 0$, then the derivative of the implicitly defined function is:
$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \tag{100}$$
$$\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} \tag{101}$$
- **Directional Derivative**: the **directional derivative** of f at (x_0, y_0) in the direction of the unit vector $\mathbf{u} = \langle a, b\rangle$ is:
$$D_u f(x_0, y_0) := \lim_{h\to 0}\frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h} \tag{102}$$
  - If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b\rangle$ and
$$D_f(x, y) = f_x(x, y)a + f_y(x, y)b \tag{103}$$
- **Gradient**: Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a function of two variables. Then, the **gradient** of f is the vector function ∇f defined by:
$$\begin{aligned}\nabla f(x, y) &:= \langle f_x(x, y), f_y(x, y)\rangle \\ &= \frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}}\end{aligned}$$
Using this notation, we can re-write the directional derivative as a dot product:
$$D_u f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$$
Thus, we can interpret the directional derivative as the scalar projection of the gradient function onto $\mathbf{u}$.
- Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function of n variables. The maximum value of the directional derivative $D_u f(x)$ is $||\nabla f(x)||$; this occurs when $\mathbf{u}$ is parallel to $\nabla f(x)$.
- The **tangent planet to the level surface** $F(x, y, z) = k$ at the point $P(x_0, y_0, z_0)$ is the plane that passes through P with the normal vector $\nabla F(x_0, y_0, z_0)$. This is written as:
$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0 \tag{104}$$

2.3 Multiple Integrals

- **Double Integral**: the **double integral** of f over the rectangle $\mathcal{R}$ is:

$$\iint_{\mathcal{R}} f(x,y)dA := \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{i,j}^*, y_{i,j}^*)\Delta A \tag{105}$$

  if the limit exists.

  - If $f \geq 0$, then the volume V of the solid above the rectangle $\mathcal{R}$ and below the surface $z = f(x,y)$ is:

$$V = \iint_{\mathcal{R}} f(x,y)dA \tag{106}$$

- **Fubini's Theorem**: If f is continuous on the rectangle $\mathcal{R} := \{(x,y) \mid a \leq x \leq b, c \leq y \leq d\}$ then:

$$\iint_{\mathcal{R}} f(x,y)dA = \int_a^b \int_c^d f(x,y)dydx = \int_c^d \int_a^b f(x,y)dxdy \tag{107}$$

- **Average Value**: the **average value** of a function f of two variables defined on the rectangle $\mathcal{R}$ is defined to be:

$$f_{\text{avg}} := \frac{1}{A(\mathcal{R})} \iint_{\mathcal{R}} f(x,y)dA \tag{108}$$

- **Double Integral of f over D**: let D be a bounded region in $\mathbb{R}^n$. Define the following function F; let $\mathcal{R}$ be a rectangle such that $D \subseteq \mathcal{R}$ and

$$F(x,y) := \begin{cases} f(x,y) & (x,y) \in D \\ 0 & (x,y) \notin D \end{cases} \tag{109}$$

  then, the **double integral of f over D** is given by:

$$\iint_D f(x,y)dA := \iint_{\mathcal{R}} F(x,y)dA \tag{110}$$

- We have various "types" of domains/regions:

  - **Type I**: A plane region D is Type I if it lies between the graphs of two continuous functions of x:

$$D := \{(x,y) \in \mathbb{R} \mid a \leq x \leq b, g_1(x) \leq y g_2(x)\} \tag{111}$$

    in this case, the double integral is given by:

$$\iint_D f(x,y)dA := \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx \tag{112}$$

  - **Type II**: a plane region of Type II is:

$$D := \{(x,y) \in \mathbb{R}^2 \mid c \leq y \leq d,\ h_1(y) \leq x \leq h_2(y)\} \tag{113}$$

    where $h_1, h_2 \in C(\mathbb{R})$. In this case, the double integral is given by:

$$\iint_D f(x,y)dA := \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dxdy$$

- If $m \leq f(x,y) \leq M \ \forall \ (x,y) \in D$, then:

$$mA(D) \leq \iint_D f(x,y)dA \leq MA(D) \tag{114}$$

- **Polar Rectangle**: $\mathcal{R} := \{(r,\theta) \mid a \leq r \leq b, \alpha \leq \theta \leq \beta\}$
  - **Change to Polar Coordinates in a Double Integral**: Let $f : \mathbb{R} \to \mathbb{R}$ be continuous, $\mathcal{R}$ a polar rectangle, f continuous. If $\mathcal{R}$ is given by $0 \leq a \leq r \leq b$, $\alpha \leq \theta\beta$, $0 \leq \beta - \alpha \leq 2\pi$. Then:

$$\iint_{\mathcal{R}} f(x,y)dA := \int_\alpha^\beta \int_a^b f(r\cos(\theta), r\sin(\theta))rdrd\theta \tag{115}$$

  if f is continuous on a polar region of the form $D := \{(r,\theta) \mid \alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta)\}$, then:

$$\iint_D f(x,y)dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos(\theta), r\sin(\theta))rdrd\theta \tag{116}$$

- **Surface Area**: the area of the surface $z = f(x,y)$ for $(x,y) \in D$ where $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are continuous is given by:

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right) + \left(\frac{\partial z}{\partial y}\right)} \tag{117}$$

2.4 Triple Integrals

- **Triple Integrals over a Box**: The **triple integral** of f over the box B is:

$$\iiint_B f(x,y,z)dV := \lim_{\ell,m,n\to\infty} \sum_{i=1}^{\ell} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\Delta V \tag{118}$$

- **Cylindrical Coordinate System**: a point $P \in \mathbb{R}^3$ is represented by the ordered triple (r,θ,z); the equations to convert are given by:

$$x = r\cos(\theta) \qquad y = r\sin(\theta) \ z = z$$
$$r^2 = x^2 + y^2 + z^2 \qquad \tan(\theta) = y/z \ z = z$$

  Often useful in problems involving symmetry about an axis.

- **Triple Integration in Polar Coordinates**: Suppose E is a Type I region whose projection D onto the xy-plane is described in polar coordinates:

$$E = \{(x,y,z) \mid (x,y) \in D \mid u_1(x,y) \leq z \leq u_2(x,y) \ \}$$
$$D = \{(r,\theta) \mid \alpha \leq \theta \leq \beta, \ h_1(\theta) \leq r \leq h_2(\theta)\}$$

  Then,

$$\iint_E f(x,y,z)dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos(\theta),r\sin(\theta)}^{u_2(r\cos(\theta),r\sin(\theta)} f(r\cos(\theta), r\sin(\theta))rdzdrd\theta \tag{119}$$

- **Spherical Coordinates**: the spherical coordinates (ρ, θ, φ) of a point $p \in \mathbb{R}^3$ are given by:

$$x = \rho \sin(\varphi)\cos(\theta),\ y = \rho\sin(\theta)\sin(\theta),\ z = \rho\cos(\varphi)$$
$$\rho^2 = x^2 + y^2 + z^2$$

  - The formula for a triple integral in spherical coordinates is given by:

$$\iiint_E f(x,y,z)dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho\sin(\varphi)\cos(\theta), \rho\sin(\theta)\sin(\theta), \rho\cos(\varphi))\rho^2\sin(\varphi)d\rho d\theta d\varphi \tag{120}$$

- **Jacobian**: the **Jacobian** of a transformation T given by $x = g(u,v)$ and $y = h(u,v)$ is given by:

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} \tag{121}$$

  - Suppose T is a C^1-transformation whose Jacobian is non-zero and that T maps S in the uv-plane to region $\mathcal{R}$ in the xy-plane. Suppose that f is continuous on $\mathcal{R}$ and that both $\mathcal{R}$ and S are Type I or Type II regions. Moreover, assume that T is bijective, except for potentially on ∂S. Then:

$$\iint_\mathcal{R} f(x,y)dA = \iint_S f(x(u,v), y(u,v))|\det(\mathbf{Jac})|dudv \tag{122}$$

3 Vector Calculus

3.1 Vector Fields

- **Vector Field on $\mathbb{R}^n$**: Let $D \subseteq \mathbb{R}^n$. A **vector field** on $\mathbb{R}^n$ is a function $\mathbf{F}$ that assigns to every point $(x_1, ..., x_n) \in D$ an n-dimensional vector $\mathbf{F}(x_1, ..., x_n)$. One can write this in terms of component functions, e.g. in $\mathbb{R}^2$:

$$\mathbf{F}(x,y) = P(x,y)\hat{\mathbf{i}} + Q(x,y)\hat{\mathbf{j}} + Q(x,y)\hat{\mathbf{k}} = \langle P(x,y), Q(x,y), Q(x,y)\rangle \tag{123}$$

  - **Gradient Vector Field**: if $f : \mathbb{R}^2 \to \mathbb{R}$, recall that ∇f is:

$$\nabla f(x,y) = f_x(x,y)\hat{\mathbf{i}} + f_y(x,y)\hat{\mathbf{j}} \tag{124}$$

    which means that ∇f is a vector field on $\mathbb{R}^2$ (we call this vector field a **gradient vector field**).

- **Conservative Vector Field**: A vector field $\mathbf{F}$ is a **conservative vector field** if there exists a scalar function f such that $\nabla f = \mathbf{F}$.

3.2 Line Integrals

- **Line Integral**: parameterise a smooth curve C by

$$x = x(t),\ y = y(t)\ t \in [a,b]$$

  or, equivalently,

$$r(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}}$$

then, the **line integral** of f along C is:

$$\int_C f(x,y)ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i \tag{125}$$

the line integral is given by:

$$\int_C f(x,y)ds = \int_a^b f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}dt \tag{126}$$

– A more compact notation for line integrals can be given by:

$$\int_a^b f(\mathbf{r}(t))||\mathbf{r}'(t)||dt \tag{127}$$

- **Line Integrals over Vector Fields**: Let $\mathbf{F}$ be a continuous vector field defined on a smooth curve C be given by a vector function $\mathbf{r}(t)$, $t \in [a,b]$. Then, the **line integral of F along** C is:

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b F(\mathbf{r}(t))\cdot \mathbf{r}'(t)dt = \int_C \mathbf{F}\cdot \mathbf{T}ds \tag{128}$$

Suppose a vector field $\mathbf{F}$ on $\mathbb{R}^3$ is given in compact-form by the equation $\mathbf{F} = P\hat{\mathbf{i}} + Q\hat{\mathbf{j}} + R\hat{\mathbf{k}}$. Then,

$$\int_C \mathbf{F}\cdot dr = \int_C Pdx + Qdy + Kdz \tag{129}$$

- **Fundamental Theorem for Line Integrals**: Let C be a smooth curve given by the vector function $\mathbf{r}(t)$ for $t \in [a,b]$. Let f be a differentiable function f of two or three variables whose gradient vector ∇f is continuous on C. Then:

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)) \tag{130}$$

- Path Independence
  - $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of path in $D \iff \int_C \mathbf{F}\cdot d\mathbf{r} = 0$ for every closed path C in D.
  - Let D be an open, connected domain. Suppose that F is a vector field on D. If $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of path on D, then $\mathbf{F}$ is a conservative vector field on D, that is, $\exists$ an f such that $\nabla f = \mathbf{F}$.
  - Let $F(x,y) = P(x,y)\hat{\mathbf{i}} + Q(x,y)\hat{\mathbf{j}}$ be a conservative vector field, where P and Q have continuous first-order partial derivatives on a domain D. Then, throughout D, we have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \tag{131}$$

  - **Simple Curve**: a curve that doesn't intersect itself anywhere between its endpoints.
  - **Simply-Connected Region**: a connected region D such that every simple closed curve in D encloses points that are only in D.
- Let $\mathbf{F} = P\hat{\mathbf{i}} + Q\hat{\mathbf{j}}$ be a vector field of an open, simply-connected region D. Suppose that P and Q have continuous first-order partial derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \tag{132}$$

throughout D. Then, $\mathbf{F}$ is conservative.

3.3 Green's Theorem

Counterpart of the Fundamental Theorem of Calculus for double integrals.

- **Green's Theorem**: Let C be a positively-oriented, piece-wise smooth simple closed curve in the plane and let D be a region bounded by C. If P and Q have continuous partial derivatives on an open region containing D, then:

$$\oint_C Pdx + Qdy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \tag{133}$$

- Can be used to calculate areas:

$$A(D) = \oint_C xdy = -\oint_C ydx = \frac{1}{2}\oint_C xdy - ydx \tag{134}$$

3.4 Curl and Divergence

Each of the following operations resemble differentiation, but one produces a vector field and the other produces a scalar field.

- **Curl**: Let $\mathbf{F} := P\hat{\mathbf{i}} + Q\hat{\mathbf{j}} + R\hat{\mathbf{k}}$ be a vector field on $\mathbb{R}^3$. Assume that the partial derivatives P, Q, and R all exist. Then, the **curl** of $\mathbf{F}$ is the vector field on $\mathbb{R}^3$ defined by:

$$\begin{aligned}
\operatorname{curl}(F) &:= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \hat{\mathbf{i}} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \hat{\mathbf{j}} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \hat{\mathbf{k}} \\
&= \nabla \times \mathbf{F} \\
&= \det \begin{bmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{bmatrix}
\end{aligned}$$

- If f is a function of three variables and has continuous, second-order partial derivatives, then:

$$\operatorname{curl}(\nabla f) = 0 \tag{135}$$

  - Conservative vector fields have $\mathbf{F} = \nabla f$, and so $\operatorname{curl}(\mathbf{F}) = 0$ for conservative vector fields.
  - Let $\mathbf{F}$ be a vector field defined on all of $\mathbb{R}^3$ whose component functions have continuous partial derivatives and $\operatorname{curl}(\mathbf{F}) = 0$. Then, $\mathbf{F}$ is a conservative vector field.
  - We say that a vector field $\mathbf{F}$ is irrotational if $\operatorname{curl}(\mathbf{F}) = 0$.

- **Divergence**: Let $\mathbf{F} := P\hat{\mathbf{i}} + Q\hat{\mathbf{j}} + R\hat{\mathbf{k}}$ be a vector field on $\mathbb{R}^3$ and assume that $\frac{\partial P}{\partial x}$, $\frac{\partial Q}{\partial y}$, and $\frac{\partial R}{\partial z}$ all exist. Then, the **divergence** of $\mathbf{F}$ is defined by:

$$\operatorname{div}(\mathbf{F}) := \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \tag{136}$$

$$= \nabla \cdot \mathbf{F} \tag{137}$$

- If $\mathbf{F} := P\hat{\mathbf{i}} + Q\hat{\mathbf{j}} + R\hat{\mathbf{k}}$ be a vector field on $\mathbb{R}^3$ and if P, Q, and R have continuous second-order partial derivatives, then

$$\operatorname{div}(\operatorname{curl}(\mathbf{F})) = 0 \tag{138}$$

  - A vector field $\mathbf{F}$ is called **incompressible** if $\operatorname{div}(\mathbf{F}) = 0$.

- We can use what we've built up here to formulate Green's Theorem in terms of vector forms.

- The first vector form of Green's theorem is:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\text{curl}(\mathbf{F})) \cdot \hat{\mathbf{k}} dA \tag{139}$$

- The second vector form of Green's Theorem is:

$$\oint_C \mathbf{F} \cdot \hat{\mathbf{n}} ds = \iint_D \text{div}(\mathbf{F}) dA \tag{140}$$

3.5 Parametric Surfaces and their Areas

4 Ordinary Differential Equations

4.1 Basic Concepts

- A **differential equation** is an equation that involves an unknown function and its derivatives. We say that a function is an **ordinary differential equation** if the unknown function depends only on one independent variable.
- The **order** of a differential equation is the order of the highest derivative appearing in the equation.
- A **solution** of an ODE in an unknown function y and an independent variable x on an interval I is a function $y(x)$ satisfying the ODE for all $x \in I$.

4.2 Classification of First Order Differential Equations

First of all, the two forms that we can see ODEs come in:

- **Standard Form**: $y' = f(x, y)$.
- **Differential Form**: $M(x, y)dx + N(x, y)dy = 0$.

And now, some types of first order differential equations that we will encounter:

- **Linear Equation**: If $f(x, y)$ can be written as:

$$f(x, y) = -p(x)y + q(x) \tag{141}$$

  then the ODE is **linear** and it can always be expressed as:

$$y' + p(x)y = q(x) \tag{142}$$

- **Bernoulli Equation**: A **Bernoulli differential equation** is an equation of the form:

$$y' + p(x)y = q(x)y^n \tag{143}$$

- **Homogeneous Equation**: A differential equation in standard form is said to be **homogeneous** if:

$$f(tx, ty) = f(x, y) \tag{144}$$

- **Separable Equation**: A differential equation given in differential form

$$M(x, y)dx + N(x, y)dy = 0$$

  is said to be **separable** if $M(x, y) = A(x)$ and $B(x, y) = B(y)$.
- **Exact Equations**: A differential equation in the differential form is called **exact** if:

$$\frac{\partial M(x, y)}{\partial y} = \frac{\partial N(x, y)}{\partial x} \tag{145}$$

- Separable ODE $\Rightarrow$ exact ODE.

4.3 Separable First Order Differential Equations

Suppose our ODE is of the form:

$$A(x)dx + B(y)dy = 0 \tag{146}$$

i.e., we have a separable first-order differential equation. Then, the solution is obtained by integrating:

$$\int A(x)dx + \int B(y)dy = c \tag{147}$$

where $c \in \mathbb{R}$ is our constant of integration. This is the **general solution**; if we have an initial value problem:

$$\begin{aligned} &A(x)dx + B(y)dy = 0 \\ &y(x_0) = y_0 \end{aligned}$$

then we can obtain the particular solution by first obtaining the general solution, and then applying the initial condition to solve for $c \in \mathbb{R}$.

4.3.1 Reduction of a Homogeneous Equation

Suppose we have a homogeneous equation (i.e, $f(xt, yt) = f(x, t)\ \forall t$. Then, there are two transformations which we can apply to obtain a separable equation:

1. Make the following substitution: $y = xv$. This has the corresponding derivative:
$$\frac{dy}{dx} = v + x\frac{dv}{dx} \tag{148}$$
Then, the ODE is separable in v and x.
2. Re-write the differential equation as:
$$\frac{dx}{dy} = \frac{1}{f(x, y)} \tag{149}$$
and make the following substitution for x $x := yu$, which gives the corresponding derivative:
$$\frac{dx}{dy} = u + y\frac{du}{dy}$$
and substitute back into the ODE.

4.4 Exact First-Order Differential Equations

We say than an ODE of the form

$$M(x, y)dx + N(x, y)dy = 0$$

is **exact** if there exists a function $g(x, y)$ such that

$$dg(x, y) = M(x, y)dx + N(x, y)dy \tag{150}$$

We can check to see if an ODE is exact by using the **test for exactness**: if $M(x, y)$ and $N(x, y)$ are continuous functions with continuous first partial derivatives on a rectangle in the xy-plane, then the ODE $M(x, y)dx + N(x, y)dy = 0$ is exact $\iff$:

$$\frac{\partial M(x, y)}{\partial y} = \frac{\partial N(x, y)}{\partial x} \tag{151}$$

Then, the solution to the ODE is given by first solving the equations:

$$\frac{\partial g(x,y)}{x} = M(x,y)$$
$$\frac{\partial g(x,y)}{y} = N(x,y)$$

for $g(x,y)$; the solution is then implicitly given by:

$$g(x,y) = c,\ c \in \mathbb{R} \tag{152}$$

4.4.1 Integrating Factors

We can transform an ODE into an exact ODE by using an **integrating factor** $I(x,y)$; we say that $I(x,y)$ is an **integrating factor** if the ODE

$$I(x,y)[M(x,y)dx + N(x,y)dy] = 0$$

is exact. Below, we have some common integrating factors:

1. If $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \equiv g(x)$, then:

$$I(x,y) = e^{\int g(x)dx}$$

2. If $\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \equiv h(y)$, then:

$$I(x,y) = e^{-\int h(y)dy}$$

Common Integrating Factors:

Group of Terms	Integrating Factor $I(x,y)$	Exact Differential $dg(x,y)$
$ydx - xdy$	$-\frac{1}{x^2}$	$\frac{xdy-ydx}{x^2} = d\left(\frac{x}{y}\right)$
$ydx - xdy$	$\frac{1}{y^2}$	$\frac{ydx=xdy}{y^2} = d\left(\frac{x}{y}\right)$
$ydx - xdy$	$-\frac{1}{xy}$	$\frac{xdy-ydx}{xy} = d\left(\ln\frac{y}{x}\right)$
$ydx - xdy$	$-\frac{1}{x^2+y^2}$	$\frac{xdy-ydx}{x^2+y^2} = d\left(\arctan\frac{y}{x}\right)$
$ydx + xdy$	$\frac{1}{xy}$	$\frac{ydx+xdy}{xy} = d(\ln(x,y))$
$ydx + xdy$	$\frac{1}{(xy)^n}$	$\frac{ydx+xdy}{(xy)^n} = d\left[\frac{-1}{(n-1)(xy)^{n-1}}\right]$
$ydy + xdx$	$\frac{1}{x^2+y^2}$	$\frac{ydy+xdx}{(x^2+y^2)^n} = d\left[\frac{1}{2}\ln(x^2+y^2)\right]$
$ydy + xdx$	$\frac{1}{(x^2+y^2)^n},\ n > 1$	$\frac{ydy+xdx}{(x^2+y^2)^n} = d\left[\frac{-1}{2(n-1)(x^2+y^2)^{n-1}}\right]$
$aydx + bxdy$	$a^{a-1}y^{b-1}$	$x^{a-1}y^{b-1}(aydx + bxdy) = d(x^a y^b)$

If $M = yf(xy)$ and $N = xg(xy)$, then

$$I(x,y) = \frac{1}{xM - yN} \tag{153}$$