

Math 455: Analysis IV Summary
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Key Results, Theorems, Definitions, etc.
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Abstract

This document contains a summary of all the key definitions, results, and theorems from class. There are probably typos, and so I would be grateful if you brought those to my attention :-).

Syllabus: L^p space, duality, weak convergence, Young, Holder, and Minkowski inequalities, point-set topology, topological space, dense sets, completeness, compactness, connectedness, path-connectedness, separability, Tychnoff theorem, Stone-Weierstrass Theorem, Arzela-Ascoli, Baire category theorem, open mapping theorem, closed graph theorem, uniform boundedness principle, Hahn Banach theorem.

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1. L^p SPACES: COMPLETENESS AND APPROXIMATION

1.1. NORMED VECTOR SPACES

Definition 1 (ℓ^p space). Let (a_1, a_n, \dots) be a sequence. Then, the ℓ^p -space is:

$$\ell^p := \left\{ (a_1, a_2, \dots) \mid \sum_{n=1}^{\infty} |a_n|^p < +\infty \right\} \quad (1)$$

Theorem 1 (Riesz-Fisher). $L^p(X)$ is complete.

Definition 2 (L^p space). Let E be a measurable set and let $1 \leq p < \infty$. Then, $L^p(E)$ is the collection of measurable functions f for which $|f|^p$ is Lebesgue integrable over E .

Definition 3 (Equivalent Functions). Let \mathcal{F} be the collection of all measurable extended real-valued functions on E that are finite a.e. on E . Define two functions f and g to be equivalent, and write $f \sim g$ if $g(x) = f(x)$ a.e. on E .

Definition 4 (Essentially Bounded). We call a function $f \in \mathcal{F}$ to be **essentially bounded** if there exists some $M \geq 0$, called the **essential upper bound** for f , for which

$$|f(x)| \leq M$$

for almost every $x \in E$. $L^\infty(E)$ is the collection of equivalence classes $[f]$ for which f is essentially bounded.

Definition 5 (Norm). Let X be a linear space. A real-valued functional $\|\cdot\|$ on X is called a **norm** provided that for each f and g in X and each real number α ,

(i) (The Triangle Inequality).

$$\|f + g\| \leq \|f\| + \|g\|$$

(ii) (Positive Homogeneity).

$$\|\alpha f\| = |\alpha| \|f\|$$

(iii) (Non-Negativity).

$$\|f\| \geq 0 \text{ and } \|f\| = 0 \text{ if and only if } f = 0$$

Definition 6 (Normed Linear Space). X is said to be a **normed linear space** if X is equipped with a norm.

Definition 7 (Essential Supremum). Let $f \in L^\infty(E)$. $\|f\|_\infty$ is called the **essential supremum** and is defined as:

$$\|f\|_\infty := \{M \mid M \text{ is an essential upper bound for } f\}$$

Theorem: $\|\cdot\|_\infty$ is a norm on $L^\infty(E)$.

1.2. THE INEQUALITIES OF YOUNG, HÖLDER, AND MINKOWSKI

Definition 8 (p-norm). Let E be a measurable set, $1 < p < \infty$, and let $f \in L^p(E)$. Then, define the **p-norm** to be:

$$\|f\|_p := \left[\int_E |f|^p \right]^{\frac{1}{p}} \quad (2)$$

Definition 9 (Conjugate). The **conjugate** of a number $p \in]1, \infty[$ is the number $q = p/(p-1)$, which is the unique number $q \in]1, \infty[$ for which

$$\frac{1}{p} + \frac{1}{q} = 1 \quad (3)$$

The conjugate of 1 is defined to be ∞ and the conjugate of ∞ is defined to be 1.

Definition 10 (Young's Inequality). For $1 < p < \infty$, q the conjugate of p , and any two positive numbers a and b , we have:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \quad (4)$$

Theorem 2 (Hölder's Inequality). Let $E \subseteq \mathbb{R}$ be measurable, $1 \leq p < \infty$, and q the conjugate of p . If f belongs to $L^p(E)$, and g belongs to $L^q(E)$, then their product $f \cdot g$ is integrable over E and:

$$\int_E |f \cdot g| \leq \|f\|_p \cdot \|g\|_q. \quad (5)$$

Moreover, if $f \neq 0$, then the function defined as:

$$f^* := \|f\|_p^{1-p} \cdot \text{sgn}(f) \cdot |f|^{p-1} \quad (6)$$

belongs to $L^q(E)$,

$$\int_E f \cdot f^* = \|f\|_p \text{ and } \|f^*\|_q = 1$$

We call f^* defined as above to be called the **conjugate function** of f .

Theorem 3 (Minkowski's Inequality). Let E be a measurable set and $1 \leq p \leq \infty$. If the functions f and g belong to $L^p(E)$, then so does their sum $f + g$. Moreover,

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p \quad (7)$$

Theorem 4 (Cauchy-Schwarz Inequality). Let E be a measurable set and let f and g be measurable functions over E for which f^2 and g^2 are integrable over E . Then, $f \cdot g$ is integrable over E and

$$\int_E |f \cdot g| \leq \sqrt{\int_E f^2} \cdot \sqrt{\int_E g^2} \quad (8)$$

Corollary 1. Let E be a measurable set and $1 < p < \infty$. Suppose \mathcal{F} is a family of functions in $L^p(E)$ that is bounded in $L^p(E)$ in the sense that there is a constant M for which

$$\|f\|_p \leq M \text{ for all } f \in \mathcal{F}$$

Then, the family \mathcal{F} is uniformly integrable over E .

Corollary 2. Let E be a measurable set of finite measure and $1 \leq p_1 < p_2 \leq \infty$. Then, $L^{p_2}(E) \subseteq L^{p_1}(E)$. Furthermore,

$$\|f\|_{p_1} \leq c \|f\|_{p_2}$$

for all f in $L^{p_2}(E)$, where $c = [m(E)]^{\frac{p_2-p_1}{p_1 p_2}}$ if $p_2 < \infty$ and $c = [m(E)]^{\frac{1}{p_1}}$ if $p_2 = \infty$.

1.3. L^p IS COMPLETE: THE REISZ-FISCHER THEOREM

Definition 11 (Converge). A sequence $\{f_n\}$ in a linear space X normed by $\|\cdot\|$ is said to **converge to f in X** provided:

$$\lim_{n \rightarrow \infty} \|f - f_n\| = 0$$

Definition 12 (Cauchy). A sequence $\{f_n\}$ in a linear space X that is normed by $\|\cdot\|$ is said to be **Cauchy** in X provided for each $\varepsilon > 0$, there exists a $N \in \mathbb{N}$ such that

$$\|f_n - f_m\| < \varepsilon \quad \forall m, n \geq N \quad (9)$$

Definition 13 (Complete). A normed linear space X is called **complete** if every Cauchy sequence in X converges to a function in X . A complete normed linear space is called a **Banach space**.

Proposition 1. Let X be a normed linear space. Then, every convergent sequence in X is Cauchy. Moreover, a Cauchy sequence in X converges if it has a convergent subsequence.

Definition 14. Let X be a linear space normed by $\|\cdot\|$. A sequence $\{f_n\}$ in X is said to be **rapidly Cauchy** if there is a convergent series of positive numbers $\sum_{k=1}^{\infty} \varepsilon_k$ for which

$$\|f_{k+1} - f_k\| \leq \varepsilon_k^2 \text{ for all } k$$

Proposition 2. Let X be a normed linear space. Then, every rapidly Cauchy sequence in X is Cauchy. Furthermore, every Cauchy sequence has a rapidly Cauchy subsequence.

Proposition 3. Let E be a measurable set and $1 \leq p \leq \infty$. Then, every rapidly Cauchy sequence in $L^p(E)$ converges with respect to the $L^p(E)$ norm and pointwise a.e. on E to a function in $L^p(E)$.

Theorem 5 (Riesz-Fischer Theorem). Let E be a measurable set and $1 \leq p \leq \infty$. Then $L^p(E)$ is a Banach space. Moreover, if $\{f_n\} \rightarrow f$ in $L^p(E)$, a subsequence of $\{f_n\}$ converges pointwise a.e. on E to f .

Theorem 6. Let E be a measurable set and $1 \leq p < \infty$. Suppose $\{f_n\}$ is a sequence in $L^p(E)$ that converges pointwise a.e. on E to the function f which belongs to $L^p(E)$. Then:

$$\{f_n\} \rightarrow f \text{ in } L^p(E) \iff \lim_{n \rightarrow \infty} \int_E |f_n|^p = \int_E |f|^p$$

Definition 15 (Tight). A family \mathcal{F} of measurable functions on E is said to be **tight** over E provided that for each $\varepsilon > 0$, there exists a subset E_0 of E of finite measure for which

$$\int_{E \setminus E_0} |f| < \varepsilon \text{ for all } f \in \mathcal{F}$$

Theorem 7. Let E be a measurable set and let $1 \leq p < \infty$. Suppose $\{f_n\}$ is a sequence in $L^p(E)$ that converges pointwise a.e. on E to the function f which belongs to $L^p(E)$. Then, $\{f_n\} \rightarrow f$ in $L^p(E) \iff \{|f_n|^p\}$ is uniformly integrable and tight over E .

1.4. APPROXIMATION AND SEPARABILITY

Definition 16 (Dense). Let X be a normed linear space with norm $\|\cdot\|$. Given two subsets \mathcal{F} and \mathcal{G} of X with $\mathcal{F} \subseteq \mathcal{G}$, we say that \mathcal{F} is **dense** in \mathcal{G} provided for each function g in \mathcal{G} and $\varepsilon > 0$, there is a function $f \in \mathcal{F}$ for which $\|f - g\| < \varepsilon$.

Proposition 4. Let E be a measurable set and let $1 \leq p \leq \infty$. Then, the subspace of simple functions in $L^p(E)$ is dense in $L^p(E)$.

Proposition 5. Let $[a, b]$ be a closed, bounded interval and $1 \leq p < \infty$. Then, the subspace of step functions on $[a, b]$ is dense in $L^p[a, b]$.

Definition 17 (Separable). A normed linear space X is said to be **separable** provided there is a countable subset that is dense in X .

Theorem 8. Let E be a measurable set and $1 \leq p < \infty$. Then, the normed linear space $L^p(E)$ is separable.

Theorem 9. Suppose E is measurable and let $1 \leq p < \infty$. Then, $C_c(E)$ (the set of all continuous functions with compact support on E) is dense in $L^p(E)$.

2. L^p SPACES: DUALITY AND WEAK CONVERGENCE

2.1. RIESZ REPRESENTATION THEOREM FOR THE DUAL OF L^p , $1 \leq p < \infty$

Definition 18 (Linear Functional). A **linear functional** on a linear space X is a real-valued function T on X such that for g and h in X and α and β real numbers,

$$T(\alpha \cdot g + \beta \cdot h) = \alpha \cdot T(g) + \beta \cdot T(h) \quad (10)$$

Definition 19 (Bounded). For a normed linear space X , a linear functional T on X is said to be **bounded** provided there is an $M \geq 0$ for which

$$|T(f)| \leq M \cdot \|f\| \text{ for all } f \in X \quad (11)$$

The infimum of all such M is called the **norm** of T and is denoted by $\|T\|_*$.

Proposition 6 (Continuity Property of a Bounded Linear Functional). Let T be a bounded linear functional on the normed space X . Then, if $\{f_n\} \rightarrow f$ in X , then $\{T(f_n)\} \rightarrow \{T(f)\}$.

Proposition 7. Let X be a normed vector space. Then, the collection of bounded linear functionals on X is a linear space which is normed by $\|\cdot\|_*$. This normed vector space is called the **dual space** of X , and is denoted by X^* .

Proposition 8. Let $E \subseteq \mathbb{R}$ be measurable, $1 \leq p < \infty$, q the conjugate of p , $g \in L^q(E)$. Define the functional T on $L^p(E)$ by:

$$T(f) := \int_E g \cdot f \quad \forall f \in L^p(E) \quad (12)$$

Then, T is a bounded linear functional on $L^p(E)$ and $\|T\|_* = \|g\|_q$.

Proposition 9. Let T, S be bounded linear functionals on the normed vector space X . If $T = S$ on a dense subset X_0 of X , then $T = S$.

Lemma 10. Let $E \subseteq \mathbb{R}$ be measurable, $1 \leq p < \infty$. Suppose that g is integrable over E and there exists a $M \geq 0$ for which

$$\left| \int_E g \cdot f \right| \leq M \|f\|_p \quad \forall f \in L^p(E), \text{ } f \text{ simple}$$

Then, $g \in L^q(E)$, where q is the conjugate of p . Moreover, $\|g\|_q \leq M$.

Theorem 11. Let $[a, b]$ be a closed, bounded interval, and $1 \leq p < \infty$. Suppose that T is a bounded linear functional on $L^p[a, b]$. Then, there is a functional $g \in L^q[a, b]$, where q is the conjugate of p , for which:

$$T(f) = \int_a^b g \cdot f \quad \forall f \in L^p[a, b] \quad (13)$$

Theorem 12 (Riesz-Representation Theorem for the Dual of $L^p(E)$). Let $E \subseteq \mathbb{R}$ be measurable, $1 \leq p < \infty$, and q the conjugate of p . For all $g \in L^q(E)$, define the bounded linear functional \mathcal{R}_g on $L^p(E)$ by:

$$\mathcal{R}_g := \int_E g \cdot f \quad \forall f \in L^p(E) \quad (14)$$

Then, for each bounded linear functional T on $L^p(E)$, there exists a unique $g \in L^q(E)$ for which

- (i) $\mathcal{R}_g = T$ and
- (ii) $\|T\|_* = \|g\|_q$

2.2. WEAK SEQUENTIAL CONVERGENCE IN L^p

Definition 20 (Converge Weakly). Let X be a normed vector space. A sequence $\{f_n\}$ in X is said to **converge weakly** in X to f provided that

$$\lim_{n \rightarrow \infty} T(f_n) = T(f) \quad \forall T \in X^* \quad (15)$$

we write

$$\{f_n\} \rightharpoonup f$$

to mean that f and each f_n belong to X and $\{f_n\}$ converges weakly in X to f .

Definition 21. Let $E \subseteq \mathbb{R}$ be measurable, $1 \leq p < \infty$, q the conjugate of p . Then, $\{f_n\} \rightharpoonup f$ in $L^p(E)$ \iff

$$\lim_{n \rightarrow \infty} \int_E g \cdot f_n = \int_E g \cdot f \quad \forall g \in L^q(E) \quad (16)$$

Theorem 13. Let $E \subseteq \mathbb{R}$ be measurable, $1 \leq p < \infty$. Suppose that $\{f_n\} \rightharpoonup f$ in $L^p(E)$. Then:

$$\{f_n\} \text{ is bounded and } \|f\|_p \leq \liminf \|f_n\|_p$$

Corollary 3. Let $E \subseteq \mathbb{R}$ be measurable, $1 \leq p < \infty$, q the conjugate of p . Suppose $\{f_n\}$ converges weakly to f in $L^p(E)$ and $\{g_n\}$ converges strongly to $g \in L^q(E)$. Then:

$$\lim_{n \rightarrow \infty} \int_E g_n \cdot f_n = \int_E g \cdot f \quad (17)$$

Definition 22 (Linear Span). Let X be a normed vector space, and let $S \subseteq X$. Then, the **linear span** of S is the vector space consisting of all linear functionals of the form:

$$f = \sum_{k=1}^n \alpha_k \cdot f_k \quad (18)$$

where each $\alpha_k \in \mathbb{R}$ and $f_k \in S$. It is the set of all *finite linear combinations of elements in S* . We care about this since L^p is an infinite dimensional space, so we want to find a way to approximate it with finitely many elements.

Proposition 10 (Characterisation of Weak Convergence in $L^p(E)$). Let $E \subseteq \mathbb{R}$ be measurable, $1 \leq p < \infty$, q the conjugate of p . Assume that $\mathcal{F} \subseteq L^q(E)$ whose linear span is dense in $L^q(E)$. Let $\{f_n\}$ be a bounded sequence in $L^p(E)$, and let $f \in L^p(E)$. Then, $\{f_n\} \rightharpoonup f$ in $L^p(E) \iff$

$$\lim_{n \rightarrow \infty} \int_E f_n \cdot g = \int_E f \cdot g \quad \forall g \in \mathcal{F} \quad (19)$$

Theorem 14. Let $E \subseteq \mathbb{R}$ be measurable, $1 \leq p < \infty$. Suppose that $\{f_n\}$ is a bounded sequence in $L^p(E)$ and f belongs to $L^p(E)$. Then, $\{f_n\} \rightharpoonup f$ in $L^p(E) \iff \forall$ measurable sets $A \subseteq E$:

$$\lim_{n \rightarrow \infty} \int_A f_n = \int_A f \quad (20)$$

if $p > 1$, then it is sufficient to consider sets A of finite measure.

Theorem 15. Let $[a, b]$ be a closed and bounded interval, $1 < p < \infty$. Suppose that $\{f_n\}$ is a bounded sequence in $L^p[a, b]$ and $f \in L^p[a, b]$. Then, $\{f_n\} \rightharpoonup f$ in $L^p(E)$ in $L^p[a, b] \iff$

$$\lim_{n \rightarrow \infty} \left[\int_a^x f_n \right] = \int_a^x f \quad \forall x \in [a, b] \quad (21)$$

Lemma 16 (Riemann-Lebesgue Lemma; used in Fourier Series :-)). Let $I = [-\pi, \pi]$, $1 \leq p < \infty$. $\forall n \in \mathbb{N}$, define $f_n(x) := \sin(nx)$ for $x \in I$. Then, $\{f_n\}$ converges weakly in $L^p(I)$ to $f \equiv 0$.

Theorem 17. Let $E \subseteq \mathbb{R}$ be measurable, $1 < p < \infty$. Suppose that $\{f_n\}$ is a bounded sequence in $L^p(E)$ that converges pointwise a.e. on E to f . Then, $\{f_n\} \rightharpoonup f$ in $L^p(E)$.

This theorem was used in the proof but was not covered in Analysis 3:

Theorem 18 (Vitali Convergence Theorem). Let $E \subseteq \mathbb{R}$ be measurable and of finite measure. Suppose that the sequence of functions $\{f_n\}$ is uniformly integrable over E . Then, if $\{f_n\} \rightarrow f$ pointwise a.e. on E , then f is integrable over E and $\lim_{n \rightarrow \infty} \int_E f_n = \int_E f$.

Theorem 19 (Radon-Riesz Theorem). Let $E \subseteq \mathbb{R}$ be measurable, $1 < p < \infty$. Suppose that $\{f_n\} \rightharpoonup f$ in $L^p(E)$. Then:

$$\{f_n\} \rightarrow f \text{ in } L^p(E) \iff \lim_{n \rightarrow \infty} \|f_n\|_p = \|f\|_p \quad (22)$$

Corollary 4. (Not Covered in Class): Let $E \subseteq \mathbb{R}$ be measurable and $1 < p < \infty$. Suppose that $\{f_n\} \rightharpoonup f$ in $L^p(E)$. Then, a subsequence of $\{f_n\}$ converges strongly to $f \iff \|f\|_p = \liminf \|f_n\|_p$.

2.3. WEAK SEQUENTIAL COMPACTNESS ("COMPACTNESS FOUND!")

Theorem 20. Let $E \subseteq \mathbb{R}$ be measurable, $1 < p < \infty$. Then, every bounded sequence in $L^p(E)$ has a subsequence that converges weakly in $L^p(E)$ to a function in $L^p(E)$.

Theorem 21 (Helly's Theorem). Let X be a *SEPARABLE* normed vector space and $\{T_n\}$ a sequence in its dual space X^* that is bounded; that is, \exists a $M > 0$ for which

$$|T_n(f)| \leq M \cdot \|f\| \quad \forall f \in X, \quad \forall n \in \mathbb{N}$$

Then, there is a subsequence $\{T_{n_k}\}$ of $\{T_n\}$ and $T \in X^*$ for which

$$\lim_{k \rightarrow \infty} T_{n_k}(f) = T(f) \quad \forall f \in X \quad (23)$$

Definition 23 (Weakly Sequentially Compact (Compact in the “weak topology”). Let X be a normed vector space. Then, a subset $K \subseteq X$ is **weakly sequentially compact** in X provided that every sequence $\{f_n\}$ in K has a subsequence that converges weakly to $f \in K$.

Theorem 22 (The Unit Ball is Weakly Sequentially Compact). Let $E \subseteq \mathbb{R}$ be measurable, $1 < p < \infty$. Define:

$$B_1 := \{f \in L^p(E) \mid \|f\|_p \leq 1\}. \quad (24)$$

B_1 is weakly sequentially compact in $L^p(E)$.

3. METRIC SPACES

This section was not covered in class, but since we have homework on this chapter I figured having this as a review from analysis 2 might be helpful. Also, there are a few terms/results that I don't think we covered in analysis 2.

3.1. EXAMPLES OF METRIC SPACES

Definition 24 (Metric Space). Let X be a non-empty set. A function $\rho : X \times X \rightarrow \mathbb{R}$ is called a **metric** if $\forall x, y \in X$:

- (i) $\rho(x, y) \geq 0$
- (ii) $\rho(x, y) = 0 \iff x = y$
- (iii) $\rho(x, y) = \rho(y, x)$
- (iv) $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$ (**Triangle Inequality**).

A non-empty set together with a metric, denoted (X, ρ) is called a **metric space**.

Definition 25 (Discrete Metric). For any non-empty set X , the **discrete metric** ρ is defined by setting $\rho(x, y) = 0$ if $x = y$ and $\rho(x, y) = 1$ if $x \neq y$.

Definition 26 (Metric Subspace). For any metric space (X, ρ) , let $Y \subseteq X$ be non-empty. Then, the restriction of ρ to $Y \times Y$ defines a metric on Y . We define this induced metric space as a **metric subspace**.

Example 3.1 (Examples of metric spaces). The following are examples of metric spaces:

- (i) Every non-empty subset of a Euclidean space.
- (ii) $L^p(E)$, where $E \subseteq \mathbb{R}$ is a measurable set.
- (iii) $C[a, b]$.

Definition 27 (Product Metric). For metric spaces (X_1, ρ_1) and (X_2, ρ_2) , we define the **product metric** τ on the cartesian product $X_1 \times X_2$ by setting, for (x_1, x_2) and (y_1, y_2) in $X_1 \times X_2$:

$$\tau((x_1, x_2), (y_1, y_2)) := \{[\rho_1(x_1, x_2)]^2 + [\rho_2(y_1, y_2)]^2\}^{1/2} \quad (25)$$

Definition 28. Two metrics ρ and σ on a set X are said to be **equivalent** if there are positive numbers c_1 and c_2 such that $\forall x_1, x_2 \in X$,

$$c_1\sigma(x_1, x_2) \leq \rho(x_1, x_2) \leq c_2\sigma(x_1, x_2)$$

Definition 29 (Isometry). A mapping $f : (X, \rho) \rightarrow (Y, \sigma)$ between two metric spaces is called an **isometry** provided that f is surjective and $\forall x_1, x_2 \in X$:

$$\sigma(f(x_1), f(x_2)) = \rho(x_1, x_2) \quad (26)$$

We say that two metric spaces are **isometric** if there is an isometry from one to another.

3.2. OPEN SETS, CLOSED SETS, AND CONVERGENT SEQUENCES

Definition 30 (Open Ball). Let (X, ρ) be a metric space. For a point $x \in X$ and $r > 0$, the set:

$$B(x, r) := \{x' \in X \mid \rho(x', x) < r\} \quad (27)$$

is called the **open ball** centred at x of radius r . A subset $\mathcal{O} \subseteq X$ is said to be **open** if $\forall x \in \mathcal{O}$, there exists an open ball centred at x and contained in \mathcal{O} . For a point $x \in X$, an open set containing x is called a **neighbourhood** of x .

Proposition 11. Let X be a metric space. The whole set X and the empty set \emptyset are open. The intersection of any two open sets is open. The union of any collection of open sets is open.

Proposition 12. Let X be a subspace of a metric space Y and $E \subseteq X$. Then, E is **open in X** $\iff E = X \cap \mathcal{O}$, where \mathcal{O} is open in Y .

Definition 31 (Closure). For a subset $E \subseteq X$, a point $x \in X$ is called a **point of closure** of E provided that every neighbourhood of x contains a point in E . The collection of the points of closure of E is called the **closure** of E and is denoted by \overline{E} .

Proposition 13. For $E \subseteq X$, where X is a metric space, its closure \overline{E} is closed. Moreover, \overline{E} is the smallest closed subset of X containing E in the sense that if F is closed and if $E \subseteq F$, then $\overline{E} \subseteq F$.

Definition 32 (Converge). A sequence $\{x_n\}$ in a metric space (X, ρ) is said to **converge** to the point $x \in X$ provided that:

$$\lim_{n \rightarrow \infty} \rho(x_n, x) = 0$$

that is, $\forall \varepsilon > 0$, \exists an index N such that $\forall n \geq N$, $\rho(x_n, x) < \varepsilon$.

Proposition 14. Let ρ and σ be equivalent metrics on a non-empty set X . Then, a subset X is open in a metric space (X, ρ) \iff it is open in (X, σ) .

3.3. CONTINUOUS MAPPINGS BETWEEN METRIC SPACES

Definition 33 (Continuous). A mapping f from a metric space X to a metric space Y is continuous at the point $x \in X$ if $\forall \{x_n\} \in X$, if $\{x_n\} \rightarrow x$, then $\{f(x_n)\} \rightarrow f(x)$. f is said to be **continuous** if it is continuous at every point in X .

Proposition 15 (ε - δ criteria for continuity). A mapping from a metric space (X, ρ) to a metric (Y, σ) is continuous at the point $x \in X$ $\iff \forall \varepsilon > 0$, $\exists \delta > 0$ such that if $\rho(x, x') < \delta$, then $\sigma(f(x), f(x')) < \varepsilon$. That is:

$$f(B(x, \delta)) \subseteq B(f(x), \varepsilon) \quad (28)$$

Proposition 16. A mapping f from a metric space X to a metric space Y is continuous $\iff \forall$ open subsets $\mathcal{O} \subseteq Y$, the inverse image under f of \mathcal{O} , $f^{-1}(\mathcal{O})$, is an open subset of X .

Proposition 17. The composition of continuous mappings between metric spaces, when defined, is continuous.

Definition 34 (Uniformly Continuous). A mapping from a metric space (X, ρ) to a metric space (Y, σ) is said to be **uniformly continuous** if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\forall u, v \in X$, if $\rho(u, v) < \delta$, $\sigma(f(u), f(v)) < \varepsilon$.

Definition 35 (Lipschitz). A mapping $f : (X, \rho) \rightarrow (Y, \sigma)$ is said to be **Lipschitz** if \exists a $c \geq 0$ such that $\forall u, v \in X$:

$$\sigma(f(u), f(v)) \leq c\rho(u, v)$$

3.4. COMPLETE METRIC SPACES

Definition 36 (Cauchy). A sequence $\{x_n\}$ in a metric space (X, ρ) is said to be a **Cauchy sequence** if $\forall \varepsilon > 0$, there exists a $N \in \mathbb{N}$ such that if $m, n \geq N$, then $\rho(x_n, x_m) < \varepsilon$.

Definition 37 (Complete). A metric space X is said to be **complete** if every Cauchy sequence in X converges to a point in X .

Proposition 18. Let $[a, b]$ be a closed and bounded interval of real numbers. Then, $C[a, b]$ with the metric induced by the max norm is complete.

Proposition 19 (Characterisation of Complete Subspaces of Metric Spaces). Let $E \subseteq X$, where X is a complete metric space. Then, the metric subspace E is complete $\iff E$ is a closed subset of X .

Theorem 23. The following are complete metric spaces:

- (i) Every non-empty closed subset of \mathbb{R}^n .
- (ii) $E \subseteq \mathbb{R}$ measurable, $1 \leq p \leq \infty$, each non-empty closed subset of $L^p(E)$.
- (iii) Each non-empty closed subset of $C[a, b]$.

Definition 38 (Diameter). Let E be a non-empty subset of a metric space (X, ρ) . We define the **diameter** of E , denoted by $\text{diam}(E)$, by:

$$\text{diam}(E) := \sup \{\rho(x, y) \mid x, y \in E\} \quad (29)$$

We say that E is **bounded** if it has finite diameter.

Definition 39 (Contracting Sequence). A decreasing sequence $\{E_n\}$ of non-empty subsets of X is called a **contracting sequence** if:

$$\lim_{n \rightarrow \infty} \text{diam}(E_n) = 0 \quad (30)$$

Theorem 24 (Cantor Intersection Theorem). Let X be a metric space. Then, X is complete \iff whenever $\{F_n\}$ is a contracting sequence of non-empty closed subsets of X , there is a point $x \in X$ for which:

$$\bigcap_{n=1}^{\infty} F_n = \{x\} \quad (31)$$

Theorem 25. Let (X, ρ) be a metric space. Then, there is a complete metric space $(\tilde{X}, \tilde{\rho})$ for which X is a dense subset of \tilde{X} and

$$\rho(u, v) = \tilde{\rho}(u, v) \quad \forall u, v \in X \quad (32)$$

we call such a space the **completion** of (X, ρ) .

3.5. COMPACT METRIC SPACES

Definition 40 (Compact Metric Space). A metric space X is called **compact** if every open cover of X has a finite sub-cover. A subset $K \subseteq X$ is compact if K , considered as a metric subspace of X , is compact.

Formulation of compactness in terms of closed sets: Let \mathcal{T} be a collection of open subsets of a metric space X . Define \mathcal{F} to be the collection of the complements of elements in \mathcal{T} . Since the elements of \mathcal{T} are open, the elements of \mathcal{F} are closed. Thus, \mathcal{T} is a cover \iff the elements of \mathcal{F} have *empty intersection*. By deMorgan's law, we can formulate compactness in terms of closed sets as:

A metric space X is compact \iff every collection of closed sets with empty intersection has a finite sub-collection whose intersection is non-empty.

This property is called the **finite intersection property**.

Definition 41 (Finite Intersection Property). A collection of sets \mathcal{F} is said to have the **finite intersection property** if any finite sub-collection of \mathcal{F} has a non-empty intersection.

Proposition 20 (Compactness in terms of closed sets). A metric space X is compact \iff every collection \mathcal{F} of closed subsets of X with the finite intersection property has a non-empty intersection.

Definition 42 (Totally Bounded). A metric space X is **totally bounded** if $\forall \varepsilon > 0$, the space X can be covered by a finite number of open balls of radius ε . A subset $E \subseteq X$ is said to be **totally bounded** if E , as a subspace of the metric space X , is totally bounded.

Definition 43 (ε -net). Let E be a subset of a metric space X . A ε -**net** for E is a finite collection of open balls $\{B(x_k, \varepsilon)\}_{k=1}^n$ with centres $x_k \in E$ whose union covers E .

Proposition 21. A metric space E is totally bounded $\iff \forall \varepsilon > 0$, there is a finite ε -net for E .

Proposition 22. A subset of Euclidean space \mathbb{R}^n is bounded \iff it is totally bounded.

Definition 44 (Sequentially Compact). A metric space X is **sequentially compact** if every sequence in X has a subsequence that converges to a point in X .

Theorem 26 (Characterisation of Compactness for a metric space). . Let X be a metric space. Then, TFAE:

- (i) X is complete and totally bounded.
- (ii) X is compact.
- (iii) X is sequentially compact.

The following three propositions of this chapter are just breaking down these equivalences, so I will not write them.

Theorem 27. Let $K \subseteq \mathbb{R}^n$. Then, TFAE:

- (i) K is closed and bounded.
- (ii) K is compact.
- (iii) K is sequentially compact.

Observe: The equivalence (1) \iff (2) is the Heine-Borel theorem. The equivalence (2) \iff (3) is the Bolzano-Weierstrass theorem.

Proposition 23. Let f be a continuous mapping from a compact metric space X to a compact metric space Y . Then, its image $f(X)$ is compact.

Theorem 28 (Extreme Value Theorem). Let X be a metric space. Then, X is compact \iff every continuous real-valued function on X attains a minimum and maximum value.

Definition 45 (Lebesgue Number). Let X be a metric space, and let $\{\mathcal{O}_\lambda\}_{\lambda \in \Lambda}$ be an open cover of X . Thus, each $x \in X$ is contained in a member of the cover, \mathcal{O}_λ . Since \mathcal{O}_λ is open, $\exists \varepsilon > 0$ such that:

$$B(x, \varepsilon) \subseteq \mathcal{O}_\lambda$$

In general, ε on X , but for compact metric spaces we can get *uniform control*. This ε that uniformly works is called the **Lebesgue number** for the cover $\{\mathcal{O}_\lambda\}_{\lambda \in \Lambda}$.

Lemma 29. Let $\{\mathcal{O}_\lambda\}_{\lambda \in \Lambda}$ be an open cover of a compact metric space X . Then, there is a number $\varepsilon > 0$ such that for each $x \in X$, the open ball $B(x, \varepsilon)$ is contained in some member of the cover.

Proposition 24. A continuous mapping from a compact space (X, ρ) to a metric space (Y, σ) is uniformly continuous.

3.6. SEPARABLE METRIC SPACES

Definition 46 (Dense & Separable). A subset D of a metric space X is **dense** in X if every non-empty subset of X contains a point of D . A metric space is **separable** if there is a countable subset of X that is dense in X .

The **Weierstrass Approximation Theorem** states that polynomials are dense in $C[a, b]$. So, $C[a, b]$ is separable, with the countable dense set being the set of polynomials with rational coefficients.

Proposition 25. A compact metric space is separable.

Proposition 26. A metric space X is separable \iff there is a countable collection of $\{\mathcal{O}_n\}$ of open subsets of X such that any open subset of X is the union of a sub-collection of $\{\mathcal{O}_n\}$.

Proposition 27. Every subspace of a separable metric space is separable.

Theorem 30. Each of the following are separable metric spaces:

- (i) Every non-empty subset of Euclidean space \mathbb{R}^n .
- (ii) $1 \leq p < \infty$, $L^p(E)$ and all non-empty subsets of $L^p(E)$.
- (iii) Each non-empty subset of $C[a, b]$.

4. TOPOLOGICAL SPACES

4.1. OPEN SETS, CLOSED SETS, BASES, AND SUB-BASES

Definition 47 (Open Sets). Let X be a non-empty set. A **topology** \mathcal{T} for X is a collection of subsets of X , called **open sets**, possessing the following properties:

- (i) The entire set X and the empty set \emptyset are open.
- (ii) The finite intersection of open sets are open.
- (iii) The union of any collection of open sets is open.

A non-empty set X , together with a topology on X , is called a **topological space**. For a point $x \in X$, an open set that contains x is called a **neighbourhood** of x .

Proposition 28. A subset $E \subseteq X$ is open \iff for each $x \in E$, there exists a neighbourhood of x that is contained in E .

Example 1 (Metric Topology). Let (X, ρ) be a metric space. Let $\mathcal{O} \subseteq X$ be open if for all $x \in \mathcal{O}$, \exists an open ball at x that is contained in \mathcal{O} . This collection of open sets forms a topology; we call this the **metric topology** induced by ρ .

Example 2 (Discrete Topology). This topology is “too much.” Let X be a non-empty subset. Let $\mathcal{T} := \mathcal{P}(X)$. Then, every set containing a point is a neighbourhood of that point. This is induced by the discrete metric.

Example 3 (Trivial Topology). Let X be non-empty. Define $\mathcal{T} := \{X, \emptyset\}$. The only neighbourhood of any point is the whole set X .

Definition 48 (Topological Subspaces). Let (X, \mathcal{T}) be a topological space and let E be a non-empty subset of X . The inherited topology \mathcal{S} for E is the set of all sets of the form $E \cap \mathcal{O}$, where $\mathcal{O} \in \mathcal{T}$. The topological space (E, \mathcal{S}) is called a **subspace** of (X, \mathcal{T}) .

Definition 49 (Base for the Topology). The building blocks of a topology is called a **base**. Let (X, \mathcal{T}) be a topological space. For a point $x \in X$, a collection of neighbourhoods of x , B_x , is called a **base for the topology at x** if \forall neighbourhoods \mathcal{U} of x , there exists a set B in the collection B_x for which $B \subseteq \mathcal{U}$.

A collection of open sets \mathcal{B} is called a **base for the topology \mathcal{T}** provided it contains a base for the topology at each point.

A base for a topology completely determines a topology, alongside \emptyset and X .

Proposition 29. For a non-empty set X , let \mathcal{B} be a collection of subsets of X . Then, \mathcal{B} is a base for a topology for $X \iff$:

(i) \mathcal{B} covers X . That is:

$$X = \bigcup_{B \in \mathcal{B}} B \quad (33)$$

(ii) If $B_1, B_2 \in \mathcal{B}$, and $x \in B_1 \cap B_2$, then there is a set $B_3 \in \mathcal{B}$ for which $x \in B_3 \subseteq B_1 \cap B_2$.

The unique topology that has \mathcal{B} as its base consists of \emptyset and unions of sub-collections of \mathcal{B} .

Definition 50 (Product Topology). Let (X, \mathcal{T}) and (Y, \mathcal{S}) be two topological spaces. In the cartesian product $X \times Y$, consider the collection of sets \mathcal{B} containing the products $\mathcal{O}_1 \times \mathcal{O}_2$, where \mathcal{O}_1 is open in X and \mathcal{O}_2 is open in Y . Then, \mathcal{B} is a base for a topology on $X \times Y$, which we call the **product topology**.

Definition 51 (Sub-base). Let (X, \mathcal{T}) be a topological space. The collection of \mathcal{S} of \mathcal{T} that covers X is called a **sub-base** for the topology \mathcal{T} provided intersections of finite collections of \mathcal{S} are a base for \mathcal{T} .

Definition 52 (Closure). Let $E \subseteq X$ be a subset of a topological space. A point $x \in E$ is called a **point of closure** of E if every neighbourhood of x contains a point in E . The collection of the points of closure of E is called the **closure** of E , denoted \overline{E} .

Proposition 30. Let X be a topological space, $E \subseteq X$. Then, \overline{E} is closed. Moreover, \overline{E} is the smallest closed subset of X containing E in the sense that if F is closed and $E \subseteq F$, then $\overline{E} \subseteq F$.

Proposition 31. A subset of a topological space X is open \iff its complement is closed.

Proposition 32. Let X be a topological space. Then, (a) \emptyset and X are closed, (b) the union of a finite collection of closed sets is closed, (c) the intersection of any collection of closed sets in X is closed.

4.2. SEPARATION PROPERTIES

Motivation: Separation properties for a topology allow us to discriminate between which topologies discriminate between certain disjoint pairs of sets, which will then allow us to study a robust collection of continuous real-valued functions on X .

Definition 53 (Neighbourhood). A **neighbourhood** of K for a subset $K \subseteq X$ is an open set that contains K .

Definition 54 (Separated by Neighbourhoods). We say that two disjoint sets A and B in X can be separated by disjoint neighbourhoods provided that there exists neighbourhoods of A and B , respectively, that are disjoint.

Definition 55 (Separation Properties of Topological Spaces). . In the order of most general to least general, they are:

- (i) **Tychonoff Separation Property:** For each two points $u, v \in X$, there exists a neighbourhood of u that does not contain v and a neighbourhood of v that does not contain u .
- (ii) **Hausdorff Separation Property:** Each two points in X can be separated by disjoint neighbourhoods.
- (iii) **Regular Separation Property:** Tychonoff + each closed set and a point not in the set can be separated by disjoint neighbourhoods.
- (iv) **Normal Separation Property:** Tychonoff + each two disjoint closed sets can be separated by disjoint neighbourhoods.

Proposition 33. A topological space is Tychonoff \iff every set containing a single point, $\{x\}$, is closed.

Proposition 34. Every metric space is normal.

Lemma 31. F is closed $\iff \text{dist}(x, F) > 0 \ \forall \ x \notin F$.

Proposition 35. Let X be a Tychonoff topological space. Then, X is normal \iff whenever \mathcal{U} is a neighbourhood of a closed subset of F of X , there is another neighbourhood of F whose closure is contained in \mathcal{U} . that is, there is an open set \mathcal{O} for which:

$$F \subseteq \mathcal{O} \subseteq \overline{\mathcal{O}} \subseteq \mathcal{U} \quad (34)$$

4.3. COUNTABILITY AND SEPARABILITY

Definition 56 (Converge, Limit). A sequence $\{x_n\}$ in a topological space X is said to **converge** to the point $x \in X$ if for each neighbourhood \mathcal{U} of x , there exists an index $N \in \mathbb{N}$ such that if $n \geq N$, then x_n belongs to \mathcal{U} . This point is called a **limit** of the sequence.

Definition 57 (First and Second Countable). A topological space X is **first countable** if there is a countable base at each point. A space X is said to be **second countable** if there is a countable base for the topology.

Example 4. Every metric space is first countable.

Proposition 36. Let X be a first countable topological space. For a subset $E \subseteq X$, a point $x \in X$ is called a point of closure of E \iff it is a limit of a sequence in E . Thus, a subset E of X is closed \iff whenever a sequence in E converges to $x \in X$, we have that $x \in E$.

Definition 58 (Dense/Separable). A subset $E \subseteq X$ is **dense** in X if every open set in X contains a point of E . We call X **separable** if it has a countable dense subset.

Definition 59 (Metrisable). A topological space X is said to be **metrisable** if the topology is induced by the metric.

Theorem 32. Let X be a second countable topological space. Then, X is metrisable \iff it is normal.

4.4. CONTINUOUS MAPPINGS BETWEEN TOPOLOGICAL SPACES

Definition 60 (Continuous). For topological spaces (X, \mathcal{T}) , (Y, \mathcal{S}) , a mapping $f : X \rightarrow Y$ is said to be **continuous** at the point x_0 in X if, for every neighbourhood \mathcal{O} of $f(x_0)$, there is a neighbourhood \mathcal{U} of x_0 for which $f(\mathcal{U}) \subseteq \mathcal{O}$. We say that f is continuous provided it is continuous at each point in X .

Proposition 37. A mapping $f : X \rightarrow Y$ between topological spaces X and Y is continuous \iff for any open subset \mathcal{O} in Y , its inverse image under f , $f^{-1}(\mathcal{O})$, is an open subset of X .

Proposition 38. The composition of continuous mappings between topological spaces, when defined, is continuous.

Definition 61 (Stronger). Given two topologies \mathcal{T}_1 and \mathcal{T}_2 for a set X , if $\mathcal{T}_2 \subseteq \mathcal{T}_1$, then we say that \mathcal{T}_2 is **weaker** than \mathcal{T}_1 , and that \mathcal{T}_1 is **stronger** than \mathcal{T}_2 .

Proposition 39. Let X be a non-empty set and let \mathcal{S} be a collection of subsets of X that covers X . The collection of subsets of X consisting of intersections of finite collections of \mathcal{S} is a base for a topology \mathcal{T} of X . It is the weakest topology containing \mathcal{S} in the sense that if \mathcal{T}' is any other topology for X containing \mathcal{S} , then $\mathcal{T} \subseteq \mathcal{T}'$.

Definition 62 (Weak Topology). Let X be a non-empty set and $\mathcal{F} := \{f_\alpha \mid X \rightarrow X_\alpha\}_{\alpha \in \Lambda}$ a collection of mappings, where each X_α is a topological space. The weakest topology for X that contains the collection of sets

$$\{f_\alpha^{-1}(\mathcal{O}_\alpha) \mid f_\alpha \in \mathcal{F}, \mathcal{O}_\alpha \text{ open in } X_\alpha\} \quad (35)$$

is called the **weak topology for X induced by \mathcal{F}** .

Proposition 40. Let X be a non-empty set, $\mathcal{F} := \{f_\lambda \mid X \rightarrow X_\lambda\}_{\lambda \in \Lambda}$ a collection of mappings where each X_λ is a topological space. The weak topology for X induced by \mathcal{F} is the topology on X that has the fewest number of sets covering the topologies on X for which each mapping $f_\lambda : X \rightarrow X_\lambda$ is continuous.

Definition 63 (Homeomorphism). A mapping from a topological space $X \rightarrow Y$ is said to be a **homeomorphism** if it is bijective and has a continuous inverse $f^{-1} : Y \rightarrow X$. Two topological spaces are said to be **homeomorphic** if there exists a homeomorphism between them. The notion of homeomorphism induces a notion of an equivalence relation between spaces.

4.5. COMPACT TOPOLOGICAL SPACES

Definition 64 (Cover). A collection of sets $\{E_\lambda\}_{\lambda \in \Lambda}$ is said to be a **cover** of a set E if $E \subseteq \bigcup_{\lambda \in \Lambda} E_\lambda$.

Definition 65 (Compact). A topological space X is said to be **compact** if every open cover of X has a finite sub-cover. A subset $K \subseteq X$ is compact if K , considered as a topological space with the subspace topology inherited from X , is compact.

Proposition 41. A topological space X is compact \iff every collection of closed subsets of X that possesses the finite intersection property has non-empty intersection.

Proposition 42. A closed subset K of a compact topological space is compact.

Proposition 43. A compact subspace K of a Hausdorff topological space is a closed subset of X .

Definition 66 (Sequentially Compact). A topological space X is said to be **sequentially compact** if every sequence in X has a subsequence that converges to a point in X .

Proposition 44. Let X be a second countable topological space. Then, X is compact \iff it is sequentially compact.

Theorem 33. A compact Hausdorff space is normal.

Proposition 45. A continuous one-to-one mapping f of a compact space X onto a Hausdorff space Y is a homeomorphism.

Proposition 46. The continuous image of a compact topological space is compact.

Corollary 5. A continuous real-valued function on a compact topological space takes on a minimum and maximum functional value.

Definition 67 (Countably Compact). A topological space is **countably compact** if every countable open cover has a finite subcover.

4.6. CONNECTED TOPOLOGICAL SPACE

Definition 68 (Separated). Two non-empty subsets of a topological space **separate** X if they are disjoint and their union is X .

Definition 69 (Connected). A topological space which cannot be separated by open sets is said to be **connected**. A subset $E \subseteq X$ is **connected** if there do NOT exist open subsets $\mathcal{O}_1, \mathcal{O}_2$ of X for which:

$$\begin{aligned}\mathcal{O}_1 \cap E &\neq \emptyset \\ \mathcal{O}_2 \cap E &\neq \emptyset \\ E &\subseteq \mathcal{O}_1 \cup \mathcal{O}_2, \\ E \cap \mathcal{O}_1 \cap \mathcal{O}_2 &= \emptyset\end{aligned}$$

Proposition 47. Let f be a continuous mapping of a connected space X to a topological space Y . Then, its image $f(X)$ is connected.

Proposition 48. For A set $C \in \mathbb{R}$, the following are equivalent.

- (i) C is an interval.
- (ii) C is convex.
- (iii) C is connected.

Definition 70 (Intermediate Value Property). A topological space X has the **intermediate value property** if the image of any continuous real-valued function on X is an interval.

Proposition 49. A topological space has the intermediate value property \iff it is connected.

Definition 71 (Arcwise connected). A topological space X is **arcwise connected** if, for each pair $u, v \in X$, there exists a continuous map $f : [0, 1] \rightarrow X$ for which $f(0) = u$ and $f(1) = v$. Note:

- (i) Connected \iff arcwise connected in \mathbb{R}^n .
- (ii) Arcwise connected \Rightarrow connected (in general)
- (iii) There exist connected but non-arcwise connected spaces (in general).