MATH 567: FUNCTIONAL ANALYSIS (FALL 2020 SEMESTER)

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This class is about linear functional analysis. This has a lot in common with linear algebra in infinite-dimensional spaces. We can think of this as infinite-dimensional linear algebra. There are two main applications of this: (a) geometry and topology in infinite dimensions and (b) solving PDEs. Recall that in regular linear algebra, we used those tools to solve linear systems. The infinite-dimensional equivalent to this is a PDE. In this class, we'll focus on the second application of functional analysis.

1. Basic Functional Analysis

This corresponds to Chapters 4 and 5 of the textbook.

1.1. Banach Spaces and General Topology.

Let X be a vector space. Recall that this means that it is closed under addition and scalar multiplication.

Definition 1.1 (Norm)**.** A **norm** on a vector space X, $||\cdot|| : X \to [0,\infty[$, satisfies the following three properties:

1. $||x|| = 0 \iff x = 0$.
2. **(Homogeneity)**: $||\lambda x|| = |\lambda|||x||$ for each $x \in X$, $\lambda \in \mathbb{R}$.
3. **(Triangle Inequality):** $||x+y|| \leq ||x|| + ||y||$.

Definition 1.2 (Completeness / Banach Space)**.** X is **complete** if every Cauchy sequence converges. $(X, ||\cdot||)$ is a **Banach space** if it is a complete normed vector space.

Definition 1.3 (Dense Subset)**.** $Y \subseteq X$ is **dense** if

1. $\overline{Y} = X$ (one thing we need to note: $\overline{Y}$ is the closure, but we need to ask ourselves "in which topology"?).

This is equivalent to:

$$\forall \varepsilon > 0,\ \forall x \in X,\ \exists y \in Y \ \text{ s.t. } ||x-y|| < \varepsilon.$$

And also equivalent to,

$$\forall x \in X,\ \exists \{y_n\} \subseteq Y \ \text{ s.t. } y_n \to y \in X.$$

Definition 1.4 (Strong Topology)**.** The **strong topology** is the topology induced by the norm, $||\cdot||$ (the open sets are characterized by the balls, $B_r := \{x \mid ||x|| < r\}$). In this topology, the definitions of density given above are equivalent.

Definition 1.5 (Separable)**.** X is **separable** if $\exists$ a countable dense subset.

We have the following equivalent definitions of compactness.

Definition 1.6 (Compactness 1)**.** $E \subseteq X$ is **compact** if every open cover of E admits a finite subcover.

Definition 1.7 (Compactness 2)**.** Every sequence has a convergent sub-sequence.

Definition 1.8 (Compactness 3)**.** For any sequence $\{x_n\} \subseteq E$, there exists $\{x_{n_k}\}$ and $x^* \in E$ such that $x_{n_k} \to x^* \in E$.

Definition 1.9 (Pre-Compact)**.** $E \subseteq X$ is **pre-compact** if $\overline{E}$ is compact.

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1.2. **Euclidean Space $\mathbb{R}^n$.** Let $x \in \mathbb{R}^n$. This is denoted by $(x_1, ..., x_n)$. Then, recall,

$$||x|| = ||x||_{\ell^2} = \left(\sum_{j=1}^{n} x_j^2\right)^{1/2}.$$

We also have these other typical norms on Euclidean space:

$$\begin{aligned} ||x||_{\ell^1} &= \sum_{j=1}^{n} |x_j| \\ ||x||_{\ell^p} &= \left(\sum_{j=1}^{n} |x_j|^p\right)^{1/p} \\ ||x||_{\ell^\infty} &= \max_{1\leq j\leq n} |x_j|. \end{aligned}$$

Definition 1.10 (Equivalent Norms)**.** We say that two norms, $|\cdot|$ and $||\cdot||$, are equivalent if and only if there exist two constants a and b such that

$$\boxed{a||x|| \leq |x| \leq b||x|| \qquad \forall x \in X.} \tag{1.11}$$

In words, this is saying that you can't be big on one norm but small in another. These norms are comparable; they are bounded by constants on either side.

Theorem 1.12. *All norms on $\mathbb{R}^n$ are equivalent (all norms in finite dimensions are equivalent).*

Proof. Let $||\cdot||$ be the Euclidean norm, and let $|\cdot|$ be another norm. Let $\{e_i\}_{i=1}^n$ be the standard basis of $\mathbb{R}^n$; recall that this is $e_i = (0, ..., 1, ..., 0)$ where the 1 is in the ith slot. Since this is a basis, for $x \in X$:

$$x = \sum_{i=1}^{n} x_i e_i.$$

By the reverse triangle inequality,

$$\begin{aligned} ||x| - |y|| &\leq |x - y| \\ &= \left|\sum_{i=1}^{n} (x_i - y_i) e_i\right| \\ &\leq \sum_{i=1}^{n} |x_i - y_i||e_i| \\ &\leq \underbrace{\left(\sum_{i=1}^{n} |e_i|^2\right)^{1/2}}_{:=C} \left(\sum_{i=1}^{n} |x_i - y_i|^2\right)^{1/2} \qquad \text{(Cauchy-Schwarz)} \\ &\leq C||x - y|| \qquad (*), \end{aligned}$$

where C is some number. Norms are continuous; $x \mapsto ||x||$ is continuous $S = \{x \mid ||x|| = 1\}$ (the unit ball). By $(*)$, $x \mapsto |x|$ is continuous on S. S is closed and bounded on $\mathbb{R}^n$ which means that S is compact. By the extreme value theorem, this means that there exist two constants $a, b \in \mathbb{R}$ such that

$$a \leq |x| \leq b \qquad \forall x \in S. \tag{1.13}$$

Observe that $|x| = 0 \iff x = 0$, which implies that $a > 0$. For any $y \in \mathbb{R}^n$, let $x := \frac{y}{||y||} \in S$. Then,

$$a \leq \left|\frac{y}{||y||}\right| \leq b \iff a \leq \frac{1}{||y||}|y| \leq b \iff a||y|| \leq |y| \leq b||y|| \qquad \forall y \in \mathbb{R}^n \setminus \{0\}.$$

The case of $y = 0$ is straightforward. This proves that any norm in a finite-dimensional vector space are equivalent. Note that this proof rests on the fact that we have a basis. ☐

Remark 1.14. $\mathbb{R}^n$ is separable in any norm. The typical countable dense subset of $\mathbb{R}^n$ is $\mathbb{Q}^n$. We will see in infinite-dimensions that all norms are not equivalent.

1.3. **The Spaces of C^r, $C^{r,\gamma}$ of Continuous Functions.**

Definition 1.15 (C^0)**.** Let $\Omega \subseteq \mathbb{R}^n$ be open. Then,

$$C^0(\Omega) \coloneqq \{f \mid \Omega \to \mathbb{R} \text{ s.t. } f \text{ is continuous on } \Omega\}$$
$$C^0(\overline{\Omega}) \coloneqq \{f \mid \overline{\Omega} \to \mathbb{R} \text{ s.t. } f \text{ is continuous on } \overline{\Omega}\}.$$

This implies that $f \in C^0(\overline{\Omega})$ is bounded and uniformly continuous.

Definition 1.16 ($||\cdot||_\infty$)**.** The standard norm on $C^0(\Omega)$ is

$$||u||_\infty \coloneqq \sup_{x\in\Omega} |u(x)| \leftrightarrow \text{ uniform convergence.} \tag{1.17}$$

Proposition 1.18. *(1) $(C^0(\Omega), ||\cdot||_\infty)$ is a Banach space.*
(2) If $\Omega \subseteq \mathbb{R}^n$ is bounded, then $C^0(\overline{\Omega})$ is separable.

We will only give a sketch of the proof.

Proof. (1) The uniform limit of continuous functions is continuous.
(2) Follows from the Weierstrass approximation theorem: polynomials are dense in $C^0(\overline{\Omega})$; then, consider the polynomials with rational coefficients.

□

1.3.1. *Higher-Order Derivatives.* Recall some notation from advanced calculus:

$$\boxed{Du = \nabla u = \text{ gradient of } u = \begin{bmatrix} \partial_1 u \\ \vdots \\ \partial_n u \end{bmatrix}}. \tag{1.19}$$

We consider the **multi-index** $\alpha = (\alpha_1, ..., \alpha_n)$, where $|\alpha| \coloneqq \alpha_1 + ... + \alpha_n$, and $\forall k \in \mathbb{R}^n$, define $k^\alpha \coloneqq k_1^{\alpha_1} \cdots h_n^{\alpha_n}$. Then, in this notation,

$$\begin{aligned} D^\alpha u &= \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} u \\ &= \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \qquad \text{(partial derivative).} \end{aligned}$$

Definition 1.20 ($C^r(\Omega)$)**.**

$$C^r(\Omega) \coloneqq \{f \mid D^\alpha f \in C^0(\Omega) \ \forall \ |\alpha| < r\} \tag{1.21}$$

In words, this means that all partial derivatives less than or equal to r are continuous. Then, we can define the following space:

$$C^\infty(\Omega) \coloneqq \bigcap_{r=1}^{\infty} C^r(\Omega). \tag{1.22}$$

Definition 1.23 (Support of f)**.** The **support** of f is defined as the smallest closed set such that $f \equiv 0$ on $\mathbb{R}^n \setminus \operatorname{supp}(f)$.

$$\operatorname{supp}(f) \coloneqq \overline{\{x \mid f(x) \neq 0\}}. \tag{1.24}$$

Definition 1.25 (Compactly Contained)**.** A set $K \subset\subset \Omega$ means that $K \subseteq \Omega$ is compact. We say that K is **compactly contained** in Ω if $K \subset\subset \Omega$, Ω is bounded, and that there exists an $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq \Omega$ for all $x \in K$. This is equivalent to for all $x \in K$,

$$\exists \ \varepsilon > 0 \text{ s.t. } d(x, \partial\Omega) \coloneqq \inf_{y\in\partial\Omega} |x - y| > \varepsilon. \tag{1.26}$$

Definition 1.27 ($C_c^r(\Omega)$)**.**

$$C_c^r(\Omega) \coloneqq \{f \mid f \in C^r(\Omega), \operatorname{supp}(f) \subset\subset \Omega\}. \tag{1.28}$$

Definition 1.29 (Norm on $C^r(\overline{\Omega})$)**.** Let Ω be bounded. Then,

$$\boxed{||f||_{C^r} \coloneqq \sum_{|\alpha|\leq r} \sup_{x\in\Omega} |D^\alpha f(x)|.} \tag{1.30}$$

Proposition 1.31. *Let $\Omega \subseteq \mathbb{R}^n$ be bounded. Then, $C^r(\Omega)$ is a separable Banach space (in fact, all you need for separable is that it is bounded) for all $r < \infty$.*

Remarks 1.32. $C_c^r(\Omega)$ is not complete. $C^\infty(\Omega)$ is not complete. However, subspaces of C^∞ is still complete with some norm.

We also introduce,

Definition 1.33 (Hölder Continuous $C^{0,\gamma}(\Omega)$)**.** $f : \Omega \to \mathbb{R}$ is **Hölder Continuous** with exponent $\gamma \in [0, 1[$ if there exists a C such that

$$\boxed{|f(x) - f(y)| \leq C||x - y||^\gamma.} \tag{1.34}$$

If $\gamma = 1 \Rightarrow f$ is **Lipschitz Continuous**.

Also,

$$\boxed{[f]_{C^{0,\gamma}(\Omega)} := \sup_{x,y \in \Omega} \frac{|f(x) - f(y)|}{||x - y||^\gamma}}$$

is called the **Hölder seminorm**. This is nor a norm, but we can make it a norm:

Definition 1.35 ($|| \cdot ||_{C^{0,\gamma}}$)**.**

$$\boxed{||f||_{C^{0,\gamma}(\Omega)} := ||f||_\infty + [f]_{C^{0,\gamma}(\Omega)}.} \tag{1.36}$$

On the homework, you'll show that

$$(C^{0,\gamma}(\Omega), ||f||_{C^{0,\gamma}(\Omega)})$$

is complete.

Definition 1.37 ($C^{r,\gamma}$)**.**

$$\boxed{C^{r,\gamma}(\Omega) := \{f \mid f \in C^r(\Omega) \text{ and } |D^\alpha f(x) - D^\alpha f(y)| \leq C||x - y||^\gamma \ \forall \ |\alpha| = r\}}. \tag{1.38}$$

The norm of this space is given by,

$$\boxed{||f||_{C^{r,\gamma}} := ||f||_{C^r} + \sup_{|\alpha|=r} [D^\alpha f]_{C^{0,r}}.} \tag{1.39}$$

Remark 1.40. If $f \in C^{0,\gamma}(\Omega)$, Ω bounded, then $f \in C^{0,\alpha}(\Omega)$ for all $0 < \alpha \leq \gamma$

Remark 1.41. **(Rademacher's Theorem)**. If $f \in C^{0,1}$, then f is differentiable a.e.

1.4. **Integration Theorems.**

Theorem 1.42 (Monotone Convergence Theorem)**.** *If $f_n \uparrow f$ pointwise for almost every x, then*

$$\boxed{\lim_{n\to\infty} \int_\Omega f_n(x)dx = \int_\Omega f(x)dx}. \tag{1.43}$$

Theorem 1.44 (Fatou's Lemma)**.** *Let $\{f_n\}$ be a sequence of measurable functions that are all positive. Then,*

$$\boxed{\int_\Omega \liminf_{n\to\infty} f_n(x) \leq \liminf_{n\to\infty} \int_\Omega f_n(x)dx} \tag{1.45}$$

Theorem 1.46 (Dominated Convergence Theorem)**.** *Assume that $\{f_n\}$ are measurable, $f_n \to f$ pointwise a.e. Then, if $|f_n(x)| \leq g(x)$ for all $n \in \mathbb{N}$ for almost every x, where $g \in L^1(\Omega)$, then*

$$\boxed{\lim_{n\to\infty} \int_\Omega f_n(x)dx = \int_\Omega f(x)dx} \tag{1.47}$$

This is the theorem that you use when you want to differentiate under integrals.

Theorem 1.48. *The space $C_c^0(\mathbb{R}^n)$ is dense in $L^1(\mathbb{R}^n)$.*

Theorem 1.49 (Fubini-Tonelli). *For all $f : X \times Y \to \mathbb{R}^n$,*

$$\boxed{\int_X \int_Y |f(x,y)|dydx = \int_Y \int_X |f(x,y)|dxdy = \int_{X\times Y} |f(x,y)|d(x,y).} \tag{1.50}$$

If, moreover, $f \in L^1(X \times Y)$,

$$\int_X \int_Y f(x,y)dydx = \int_Y \int_X f(x,y)dxdy = \int_{X\times y} f(x,y)d(x,y) \tag{1.51}$$

1.5. **Elementary L^p Spaces.**

Definition 1.52 (L^p)**.** Fix $1 \le p < \infty$, let $\Omega \subseteq \mathbb{R}^n$. Then, we define the L^p space to be

$$\boxed{L^p(\Omega) := \{f : \Omega \to \mathbb{R} \mid f \text{ measurable } |f|^p \in L^1(\Omega)\}} \tag{1.53}$$

with the following norm,

$$||f||_{L^p} := \left[\int_\Omega |f(x)|^p\right]^{1/p}. \tag{1.54}$$

Definition 1.55 (L^∞)**.** We define L^∞ to be:

$$L^\infty(\Omega) := \{f : \Omega \to \mathbb{R} \mid f \text{ measurable }, \exists\, C \text{ s.t. } |f(x)| \le C \text{ a.e. }\}, \tag{1.56}$$

with the following norm,

$$||f||_{L^\infty} = ||f||_\infty = \inf\{c \mid |f(x)| \le c \; a.e.\} \tag{1.57}$$

This definition implies that $f(x) \le ||f||_\infty$ almost everywhere. Below are some fundamental tools that we'll be using

Theorem 1.58 (Hölder's Inequality)**.** *Let $1 \le p, p' \le \infty$. If $f \in L^p(\Omega)$, $g \in L^{p'}(\Omega)$ and $1/p + 1/p' = 1$, then $fg \in L^1$ and*

$$\boxed{\int |fg|dx \le ||f||_p ||g||_{p'}} \tag{1.59}$$

Theorem 1.60 (Minkowski's Inequality)**.** *For all $p \in [1, \infty]$,*

$$\boxed{||f + g||_p \le ||f||_p + ||g||_p.} \tag{1.61}$$

As a consequence of [Minkowski's Inequality, L^p is a vector space.