Math 454: Analysis 3 – Theory of Lebesgue Measure

Definitions, Theorems, and Results from the Class

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1. Introduction

Definition 1 (Riemann Integral). Let $[a,b]$ be a bounded, closed interval and $f : [a,b] \to \mathbb{R}$ be a bounded function. Then, f is **Riemann Integrable** if

$$\underline{\int_a^b} f = \overline{\int_a^b} f \tag{1}$$

where

$$\underline{\int_a^b} f := \sup \left\{ \sum_{i=1}^{n} \inf_{]x_{i-1},x_i[} f \cdot (x_i - x_{i-1}) \;\middle|\; a = x_0 < \cdots < x_n = b \right\} \tag{2}$$

$$\overline{\int_a^b} f := \inf \left\{ \sum_{i=1}^{n} \sup_{]x_{i-1},x_i[} f \cdot (x_i - x_{i-1}) \;\middle|\; a = x_0 < \cdots < x_n = b \right\} \tag{3}$$

Theorem 1. Every continuous function $f : [a,b] \to \mathbb{R}$ is Riemann Integrable.

Definition 2 (Length). $\forall I \subseteq \mathbb{R}$, I an interval, we call the **length of I** to be the number:

$$\ell(I) := \begin{cases} b - a; & I = [a,b], [a,b[,]a,b], \text{ or }]a,b[\\ \infty & I \text{ is unbounded} \end{cases} \tag{4}$$

Definition 3 (Outer Measure). $\forall A \subseteq \mathbb{R}$, the **outer measure** of A, denoted by $m^*(A)$ is given by:

$$m^*(A) := \inf \left\{ \sum_{k=1}^{\infty} \ell(I_k) \;\middle|\; (I_k) \text{ open, bounded intervals s.t. } A \subseteq \bigcup_{k=1}^{\infty} I_k \right\} \tag{5}$$

Proposition 1. $A \subseteq \mathbb{R}$ is countable $\Rightarrow m^*(A) = 0$

Proposition 2 (Monotonicity of outer measure). If $A \subseteq \mathbb{B}$, then $m^*(A) \leq m^*(B)$.

Proposition 3. For every interval $I \subseteq \mathbb{R}$, $m^*(I) = \ell(I)$.

Proposition 4 (Translation invariance of outer measure). $\forall A \subseteq \mathbb{R}$, $y \in \mathbb{R}$, define $A + y := \{x + y \mid x \in A\}$. Then, $m^*(A) = m^*(A + y)$.

Proposition 5 (Countable Subadditivity of outer measure). $\forall (A_k)_{k \in \mathbb{N}}$ subsets of $\mathbb{R}$:

$$m^* \left(\bigcup_{k=1}^{\infty} A_k \right) \leq \sum_{k=1}^{\infty} m^*(A_k) \tag{6}$$

where the A_k's are not necessarily disjoint.

Definition 4 (Lebesgue Measure). A set $A \subseteq \mathbb{R}$ is **measurable** if $\forall B \subseteq \mathbb{R}$,

$$m^*(B) = m^*(B \cap A) + m^*(B \setminus A) \tag{7}$$

The only non-trival part of the definition to check is $m^*(B) \geq m^*(B \cap A) + m^*(B \setminus A)$, since the other inequality follows from the subadditivity of outer measure. We can also restrict B to the class of all finite-outer-measure sets, since the inequality is trivial for infinite-outer-measure sets.

Proposition 6. If $m^*(A) = 0$, then A is measurable.

Proposition 7. $\forall A \subseteq \mathbb{R}$, A measurable, $\Rightarrow \mathbb{R} \setminus A$ is measurable.

Theorem 2 (Excision Property). $\forall A_1, A_2 \subseteq \mathbb{R}$ mesurable, $A_2 \subseteq A_1$, and $m(A_2) < \infty$, then:

$$m(A_1 \setminus A_2) = m(A_1) - m(A_2) \tag{8}$$

Proposition 8. $\forall (A_k)_{k\in\mathbb{N}}$ measurable, we have:

(i) $\bigcup_{k=1}^{\infty} A_k$ is measurable and $\bigcap_{k=1}^{\infty} A_k$ is measurable.

(ii) **(Countable Additivity of Measure)**. If $A_i \cap A_j = \emptyset \ \forall i \neq j$, then:

$$m\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{i=1}^{\infty} m(A_k) \tag{9}$$

Proposition 9 (Continuity of Lebesgue Measure). Let $(A_k)_{k\in\mathbb{N}}$ be sequence of measurable sets. Then:

(i) If $A_k \subseteq A_{k+1} \forall k$ (increasing sequence of sets), then:

$$m\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{k\to\infty} m(A_k) \tag{10}$$

(ii) If $A_{k+1} \subseteq A_k \forall k$ (decreasing sequence of sets), then:

$$m\left(\bigcap_{k=1}^{\infty} A_k\right) = \lim_{k\to\infty} m(A_k) \tag{11}$$

Proposition 10 (Translation Invariance of Measurable Sets). $\forall A \subseteq \mathbb{R}$ measurable, and $\forall y \in \mathbb{R}$ fixed, $A + y$ is measurable.

Proposition 11. (i) Every interval in $\mathbb{R}$ is Lebesuge measurable.

(ii) Every open set and every closed set is Lebesgue measurable.

Theorem 3 (Characterisation of Measurable Sets). Let $A \subseteq \mathbb{R}$. Then, TFAE:

(i) A is measurable.

(ii) **(Outer Approximation of Measurable Sets by Open Sets)**. $\forall \varepsilon > 0$, $\exists\ O_\varepsilon \subseteq \mathbb{R}$ open such that $A \subseteq O_\varepsilon$ and $m^*(O_\varepsilon \setminus A) < \varepsilon$.

(iii) **(Approximation by G_δ sets)**. $\exists (O_n)_{n\in\mathbb{N}}$ open such that $A \subseteq G$ and $m^*(G \setminus A) = 0$, where $G := \bigcap_{n\in\mathbb{N}} O_n$. The countable intersection of open sets is a **$\mathbf{G}_\delta$-set**.

(iv) **(Inner Approximation of Measurable Sets by Closed Sets)** $\forall \varepsilon > 0$, $\exists\ F_\varepsilon \subseteq \mathbb{R}$ closed such that $F_\varepsilon \subseteq A$ and $m^*(A \setminus F_\varepsilon) < \varepsilon$.

(v) **(Approximation by F_σ sets)**. $\exists (F_n)_{n\in\mathbb{N}}$ closed such that $F \subseteq A$ and $m^*(A \setminus F) = 0$, where $F := \bigcup_{n\in\mathbb{N}} F_n$. The countable intersection of closed sets is a **$\mathbf{F}_\sigma$-set**.

Theorem 4 (Vitali). $\forall A \subseteq \mathbb{R}$, if $m^*(A) < 0$, then $\exists B \subseteq A$ that is not measurable.

Definition 5 (Cantor Set)**.** The **Cantor Set** is recursively defined as:

$$C := \bigcap_{k=1}^{\infty} C_k \tag{12}$$

Where:

$$C_1 := \left[0, \frac{1}{3}\right] \bigcup \left[\frac{2}{3}, 1\right]$$

and for $k \geq 2$:

$$C_k := \bigcup_{j=1}^{2^k} I_{k,j} \ \forall j \in \{1, .., 2^{k-1}\}$$

Where $I_{k,2j-1}$ and $I_{k,2j}$ are the first and second thirds of the interval $I_{k-1,j}$.

Theorem 5. C is closed, uncountable, and $m^*(C) = 0$.

Definition 6 (σ-algebra)**.** A collection of sets $\mathcal{C}$ is called a **σ-algebra** if the following are true:

(i) $\mathbb{R} \in \mathcal{C}$.

(ii) $\forall C_1, C_2 \in \mathcal{C}$, $C_1 \setminus C_2 \in \mathcal{C}$ (stable under complementation).

(iii) $\forall (C_k)_{k \in \mathbb{N}} \in \mathcal{C}$, we have that:

$$\bigcup_{k=1}^{\infty} C_k \in \mathcal{C}$$

(stable under countable unions).

Proposition 12. Any intersection of $\sigma-$algebras is a $\sigma-$ algebra.

Definition 7 (Borel Sets)**.** A **Borel set** is a set that is in the intersection of all the sigma algebras containing the open sets. The **Borel sigma algebra** os the smallest sigma algebra containing all the open sets. (Alternatively, it is the sigma algebra generated by the open sets).

Proposition 13. There exists a subset of the Cantor Set which is not Borel. Thus, the set of measurable sets is indeed bigger than the smallest sigma algebra.

Definition 8 (Cantor Lebesgue Function)**.** The **Cantor-Lebesgue Function** is the function $\varphi : [0,1] \to [0,1]$ defined as:

$$\varphi(x) := \frac{i}{2k} \tag{13}$$

if $x \in J_{k,i}$, where $J_{k,i}$ is the i-th interval in $[0,1] \setminus C_k$, $\forall i \in \{1, ..., 2^k - 1\}$, and $\forall y \in [0,1] \setminus C$:

$$\varphi(y) := \begin{cases} \varphi(0) := 0 \\ \varphi(y) := \sup\{\varphi(x) \mid x \in [0, y[\setminus C\} \end{cases} \tag{14}$$

Proposition 14. φ is an increasing and continuous function.

2. Lebesgue Measurable Functions

Proposition 15 (Lebesgue Measurable Function)**.** Let $A \subseteq \mathbb{R}$ be a measurable set and $f : A \to \mathbb{R}$. Then, TFAE:
(i) $\forall c \in \mathbb{R}$, $f^{-1}(]c, +\infty])$ is measurable.
(ii) $\forall c \in \mathbb{R}$, $f^{-1}([c, +\infty])$ is measurable.
(iii) $\forall c \in \mathbb{R}$, $f^{-1}([-\infty, c[)$ is measurable.
(iv) $\forall c \in \mathbb{R}$, $f^{-1}([-\infty, c])$ is measurable.
If any of the above conditions are met, then we say that f is **measurable**.

Proposition 16. Let $A \subseteq \mathbb{R}$ be measurable, and let $f : A \to \overline{\mathbb{R}}$. Then:

(i) f measurable $\Rightarrow \forall B \subseteq \mathbb{R}$, B a Borel Set, $f^{-1}(B)$ is measurable. (The inverse image of Borel sets are measurable sets).

(ii) If f is finite-valued, i.e., $f(A) \subsetneq \overline{\mathbb{R}}$, then we get a *characterisation of measurable functions*: f measurable $\iff \forall B \subseteq \mathbb{R}$, B Borel, $f^{-1}(B)$ is measurable.

Proposition 17. Let $A \subseteq \mathbb{R}$ be measurable and $f : A \to \mathbb{R}$ be continuous. Then, f is measurable.

Definition 9 (Almost Everywhere)**.** Let $x \in \mathbb{R}$ be measurable, and let $P(x)$ be a statement depending on $x \in A$. We say that $P(x)$ is **true almost everywhere in** A (abbreviated as a.e $x \in A$) if $m(\{x \in A \mid P(x) \text{ is false}\}) = 0$

Proposition 18. Let $f : A \to \overline{\mathbb{R}}$ be a measurable function. Let $g : A \to \overline{\mathbb{R}}$ be such that $f = g$ a.e. in A. Then, g is measurable.

Proposition 19. Let $(A_n)_{n\in\mathbb{N}}$ be disjoint, measurable sets and let $A = \bigcup_{n\in\mathbb{N}}$. Let $(f_n)_{n\in\mathbb{N}}$, $f_n : A_n \to \overline{\mathbb{R}}$ be measurable. Then, the function:

$$\begin{aligned} f :=& A \to \overline{\mathbb{R}} \\ & x \mapsto f_n(x) \end{aligned}$$

is measurable.

Definition 10 (Characteristic Function)**.** $\forall B \subseteq A$, B measurable, the **characteristic function of** B is the function $\chi_B : A \to \mathbb{R}$:

$$\chi_B := \begin{cases} x \mapsto 1; & x \in B \\ x \mapsto 0; & x \notin B \end{cases} \tag{15}$$

Definition 11 (Simple Functions)**.** $f : A \to \mathbb{R}$ is a **simple function** if $f(A)$ is a finite set. This means that f is a sum of characteristic functions; $\exists\ a_1 < a_2 < ... < a_N \in \mathbb{R}$ such that $f(A) = \{a_1, ..., a_N\}$. Letting $A_k := f^{-1}(\{a_k\})$, we have:

$$f = \sum_{k=1}^{N} a_k \chi_k$$

This representation is unique and is called the **canonical representation of** f.

Proposition 20 (Properties of Measurable Functions)**.** Let $A \subseteq \mathbb{R}$ be measurable. Then:
(i) $\forall B \subseteq A$ measurable, $f|_B$ is measurable.
(ii) $\forall B \subseteq \mathbb{R}$ Borel, if $f : B \to \mathbb{R}$ is continuous, $g : A \to B$ is measurable, then $f \circ g$ is measurable.

   (i) Note that we need f to be continuous, since we need the inverse image to preserve the Borel property.
(iii) $\forall f : A \to \overline{\mathbb{R}}$, $g : A \to \mathbb{R}$, $f + g$ is measurable.
   (i) Note that we need g not into $\overline{\mathbb{R}}$ since we need to avoid the $\infty - \infty$ case.
(iv) $\forall f, g : A \to \mathbb{R}$ measurable, $f \cdot g$ is measurable. (No $\overline{\mathbb{R}}$ to avoid the $\infty \cdot 0$ case).
(v) $\forall f_1, ..., f_n$, $f_n : A \to \mathbb{R}$ measurable,
   (i) $\max\{f_1, ...f_n\}$
   (ii) $\min\{f_1, ..., f_n\}$

   are measurable.

Definition 12 (Uniform and pointwise convergence)**.** Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of measurable functions, $f_n A :\to \overline{\mathbb{R}}$, and $f : A \to \overline{\mathbb{R}}$. We say that:

(i) $\{f_n\}_{n\in\mathbb{N}}$ converges **pointwise** to f in $B \subseteq A$ if

$$\forall x \in B, \lim_{n\to\infty} f_n(x) = f(x)$$

(ii) $\{f_n\}_{n\in\mathbb{N}}$ converges **uniformly** to f in $B \subseteq A$ if

$$\lim_{n\to\infty} \sup_B |f_n - f| = 0$$

Proposition 21. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of measurable functions, $f_n : A \to \overline{\mathbb{R}}$ converging pointwise almost everywhere in A to a function $f : A \to \mathbb{R}$. Then, $f : A \to \mathbb{R}$ is measurable.

Proposition 22 (Simple Approximation Lemma)**.** Let $f : A \to \mathbb{R}$ be measurable and bounded everywhere (i.e., $\exists$ a $M > 0$ such that $|f| < M$ in A). Then, $\forall \varepsilon > 0$, $\exists\ \psi_\varepsilon$, $\varphi_\varepsilon : A \to \mathbb{R}$ simple functions such that

$$\varphi_\epsilon \leq f \leq \psi_\varepsilon < \varphi_\varepsilon + \varepsilon$$

in A. In particular, the φ_ε and the ψ_ε converge uniformly to f in A.

Theorem 6 (Simple Approximation Theorem)**.** Let $f : A \to \overline{\mathbb{R}}$ on a measurable set A. Then, f is measurable $\iff$ there exist simple functions $(\varphi_n)_{n\in\mathbb{N}}$ such that:

(i) $(\varphi_n)_{n\in\mathbb{N}}$ converges pointwise to f.
(ii) $|\varphi_n| \leq |f|$ in A $\forall n \in \mathbb{N}$.

Moreover, if $f \geq 0$ in A, we can choose φ_n such that $\varphi_n \geq 0$ and $\varphi_{n+1} \geq \varphi_n$ $\forall n \in \mathbb{N}$.

Theorem 7 (Egoroff's Theorem)**.** Let $A \subseteq \mathbb{R}$ be a measurable set, and assume that $m(A) < \infty$. Let $(f_n)_{n\in\mathbb{N}}$, $f_n : A \to \mathbb{R}$ be a sequence of measurable functions converging pointwise to $f : A \to \mathbb{R}$ (not $\overline{\mathbb{R}}$!!). Then, $\forall \varepsilon > 0$, $\exists F_\varepsilon \subseteq A$ closed such that:

(i) $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly on F_ε.
(ii) $m(A \setminus F_\varepsilon) < \varepsilon$.

Theorem 8 (Lusin's Theorem)**.** Let $f : A \to \mathbb{R}$ be measurable (not into $\overline{\mathbb{R}}$!). Then, $\forall \varepsilon > 0$, $\exists\ F_\varepsilon \subseteq A$ closed such that:

(i) f is continuous on F_ε.
(ii) $m(A \setminus F_\varepsilon) < \varepsilon$.

3. The Lebesgue Integral

Definition 13 (Integral – Case of Simple Functions on a Set of Finite Measure)**.** Let $\psi : A \to \mathbb{R}$ be a simple function. Let $\psi = \sum_{k=1}^{N} a_k \chi_{A_k}$ be its canonical representation. We define the **integral** of ψ over A and denote $\int_A \psi$ and $\int_A \psi(x)dx$ to be the number:

$$\int_A \psi := \sum_{k=1}^{N} a_k m(A_k) \tag{16}$$

For every $B \subseteq A$ measurable, we denote $\int_B \psi = \int_B \psi|_B$. Here, the measure of A must be finite.

Definition 14 (Integral – Case of Measurable, Bounded Functions on a Set of Finite Measure)**.** Let $A \subseteq \mathbb{R}$ be a measurable set such that $m(A) < \infty$, and let $f : A \to \mathbb{R}$ be a bounded function. We say that f is **integrable over** A if:

$$\underline{\int_A} f = \overline{\int_A} f \tag{17}$$

where

$$\underline{\int_A} f := \sup \left\{ \int_A \varphi \mid \varphi \text{simple } , \varphi \leq f \text{ on } A \right\}$$
$$\overline{\int_A} f := \inf \left\{ \int_A \varphi \mid \varphi \text{simple } , f \leq \varphi \text{ on } A \right\}$$

We then denote $\int_A f = \int_A f(x)dx = \underline{\int_A} f = \overline{\int_A} f$ and we call this number the **integral** of f over A. For every $B \subseteq A$ measurable, we denote:

$$\int_B f = \int_B f|_B$$

Theorem 9. If $f : [a, b] \to \mathbb{R}$ is Riemann Integrable, then f is Lebesgue Integrable.

Theorem 10. Let $f : A \to \mathbb{R}$, $m(A) < \infty$, be a measurable and bounded function. Then, f is integrable.

Proposition 23 (Properties of the Integral)**.** Let $f, g : A \to \mathbb{R}$ be measurable and bounded. Then:

(i) $\forall \alpha, \beta \in \mathbb{R}$, $\alpha f + \beta g$ is measurable and bounded, and:

$$\int_A (\alpha f + \beta g) = \alpha \int_A f + \beta \int_A g$$

(ii) (Monotonicity): if $f \leq g$ on A, then:

$$\int_A f \leq \int_A g$$

(iii) $|\int_A f|$ is measurable and bounded, and $|\int_A f| \leq \int_A |f|$.

(iv) $\forall B \subseteq \mathbb{R}$ measurable, $f \cdot \chi_B$ is measurable, bounded, and

$$\int f\chi_B = \int_B f$$

(v) $\forall A_1, A_2$ measurable and disjoint,

$$\int_{A_1 \cup A_2} f = \int_{A_1} f + \int_{A_2} f$$

In particular, if $m(A_2) = 0$, then:

$$\int_{A_2} f = 0 \text{ and so } \int_{A_1 \cup A_2} f = \int_{A_1} f$$

Lemma 11 (Independence of Representation)**.** Let $n \in \mathbb{N}$ and let $a_1, ..., a_n \in \mathbb{R}$ and $A_1, ..., A_n \subseteq A$, where $m(A) < \infty$, be measurable and disjoint. Then:

$$\int \sum_{k=1}^{n} a_k \chi_{A_k} = \sum_{k=1}^{n} a_k m(A_k)$$

Theorem 12 (Bounded Convergence Theorem)**.** Let $A \subseteq \mathbb{R}$ be measurable, $m(A) < \infty$. Let $(f_n)_{n \in \mathbb{N}}$, $f_n : A \to \mathbb{R}$ be a sequence of measurable functions on A such that:

(i) (Uniformly bounded) $\exists$ an $M > 0$ such that $\forall n \in \mathbb{N}$, $|f_n| \leq M$ on A.
(ii) (Pointwise Convergence) $\exists$ $f : A \to \mathbb{R}$ such that $\forall x \in A$, $\lim_{n\to\infty} f_n(x) = f(x)$

Then, f is bounded and measurable, and we can interchange the limits as so:

$$\lim_{n\to\infty} \int_A f_n = f$$

Definition 15 (Integral in the case of a Non-Negative, Measurable Function on a Set of Possibly Infinite Measure)**.** Let $A \subseteq \mathbb{R}$ be measurable, possibly of infinite measure, and let $f : A \to [0, \infty]$ be measurable. We call the **integral** of f over A and denote $\int_A f = \int_A f(x)dx$ the number defined as

$$\int_A f := \sup \left\{ \int_B h \mid B \subseteq A,\ m(B) < \infty,\ h : B \to \mathbb{R} \text{ measurable, bd, } 0 \leq h \leq f \text{ on } B \right\} \tag{18}$$

For every $B \subseteq A$, we denote $\int_B f = \int_B f|_B$. If $\int_A f < \infty$, we say that f is **integrable** over A.

Proposition 24 (Properties of the Integral)**.** Let $f, g : A \to [0, \infty]$ be measurable. Then:

(i) $\forall \alpha, \beta \geq 0$, $\alpha f + \beta g$ is non-negative and measurable and :

$$\int (\alpha f + \beta g) = \alpha \int f + \beta \int g$$

(ii) $f \leq g$ on $A \Rightarrow \int_A f \leq \int_A g$.
(iii) If $|f| < \infty$, then $\forall B \subseteq A$ measurable, $\chi_B f$ is non-negative, measurable, and

$$\int_A \chi_B f = \int_B f$$

(iv) $\forall A_1, A_2 \subseteq A$ disjoint, measurable. Then:

$$\int_{A_1 \cup A_2} f = \int_{A_1} f + \int_{A_2} f$$

If, moreover, $m(A_2) = 0$, then

$$\int_{A_2} f = 0 \text{ and so } \int_{A_1 \cup A_2} f = \int_{A_1} f$$

Theorem 13 (Chebyshev's Inequality)**.** Let f be measurable, non-negative. Then, $\forall \lambda > 0$, then:

$$m(f^{-1}([\lambda, +\infty])) \leq \frac{1}{\lambda} \int_A f$$

Corollary 1. Let f be a non-negative, measurable function on A. Then, $f = 0$ a.e. in $A \iff \int_A f = 0$.

Corollary 2. Let f be non-negative, measurable on A. If f is integrable over A, then $f < \infty$ a.e. in A.

Lemma 14 (Fatou's Lemma)**.** Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of non-negative, measurable functions on $A \subseteq \mathbb{R}$. Then $\liminf_{n\to\infty} f_n$ is measurable and

$$\int_A \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int_A f_n \tag{19}$$

In particular, if $(\int_A f_n)_{n\in\mathbb{N}}$ is bounded by $M < \infty$, then $\liminf_{n\to\infty} f_n$ is integrable and $\int_A \liminf_{n\to\infty} f_n \leq M$.

Theorem 15 (Monotone Convergence Theorem)**.** Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of non-negative, measurable functions on $A \subseteq \mathbb{R}$ such that $\forall n \in \mathbb{N}$, $f_n \leq f_{n+1}$ (so that the $\lim_{n\to\infty} f_n(x)$ exists in $[0, \infty]$ $\forall x \in A$ and $\lim_{n\to\infty} \int_A f_n$ exists in $[0, \infty]$), then

$$\int_A \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int_A f_n \tag{20}$$

Corollary 3. Let $(U_n)_{n\in\mathbb{N}}$ be a sequence of non-negative, measurable functions on $A \subseteq \mathbb{R}$. Then:

$$\int_A \sum_{n=1}^{\infty} U_n = \sum_{n=1}^{\infty} \int_A U_n \tag{21}$$

Definition 16 (Integral in the case of Possibly Sign-Changing Functions)**.** We say that a measurable function $f : A \to \overline{\mathbb{R}}$ is **integrable** over A if $f_+ := \max\{f, 0\}$ and $f_- := \max\{-f, 0\}$ are integrable. We then denote:

$$\int_A f := \int_A f_+ - \int_A f_- \tag{22}$$

$\forall B \subseteq A$ measurable, $\int_B f = \int_B f|_B$.

Proposition 25. f is Lebesgue integrable $\iff$ $|f|$ is Lebesgue integrable.

Proposition 26. Let f, g be integrable over $A \subseteq \mathbb{R}$. Then:

(i) $\forall \alpha, \beta \geq 0$, $\alpha f + \beta g$ is non-negative and measurable and :

$$\int (\alpha f + \beta g) = \alpha \int f + \beta \int g$$

(ii) $f \leq g$ on $A \Rightarrow \int_A f \leq \int_A g$.

(iii) $\forall B \subseteq A$ measurable, $\chi_B f$ is non-negative, measurable, and

$$\int_A \chi_B f = \int_B f$$

(iv) $\forall A_1, A_2 \subseteq A$ disjoint, measurable. Then:

$$\int_{A_1 \cup A_2} f = \int_{A_1} f + \int_{A_2} f$$

If, moreover, $m(A_2) = 0$, then

$$\int_{A_2} f = 0 \text{ and so } \int_{A_1 \cup A_2} f = \int_{A_1} f$$

Theorem 16 (Dominated Convergence Theorem)**.** Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions on $A \subseteq \mathbb{R}$ such that

(i) (Uniformly bounded) $\exists$ a g integrable over A so that $\forall n \in \mathbb{N}$ $|f_n| \leq g$.
(ii) (Pointwise convergence) $\exists f : A \to \overline{\mathbb{R}}$ such that $f_n \to f$ pointwise a.e. in A.

Then, the functions f_n and f are integrable and

$$\int_A f = \lim_{n\to\infty} \int_A f_n$$

Corollary 4 (Countable Additivity of Lebesgue Integration)**.** Let f be integrable over $A \subseteq \mathbb{R}$ and let $(A_n)_{n\in\mathbb{N}}$ be measurable, disjoint subsets of A. Then:

$$\int_{\cup_{n=1}^{\infty} A_n} f = \sum_{n=1}^{\infty} \int_{A_n} f \tag{23}$$

Corollary 5 (Continuity of Lebesgue Integration)**.** Let $(X, \mathcal{F}, \mu)$ be a measure space and let f be integrable over $A \subseteq X$. Then, if:

(i) If $(A_n)_{n\in\mathbb{N}}$ is an increasing sequence of measurable subsets of A (that is, $A_n \subseteq A_{n+1}$ $\forall n \in \mathbb{N}$), then:

$$\int_{\cup_{n\in\mathbb{N}} A_n} f d\mu = \lim_{n\to\infty} \int_{A_n} f d\mu$$

(ii) If $(A_n)_{n\in\mathbb{N}}$ is a decreasing sequence of measurable subsets of A (that is, $A_{n+1} \subseteq A_n$ $\forall n \in \mathbb{N}$), then:

$$\int_{\cap_{n\in\mathbb{N}} A_n} f d\mu = \lim_{n\to\infty} \int_{A_n} f d\mu$$

4. Integration and Differentiation

Definition 17 (Differentiable)**.** A function f is **differentiable** if $D_*(f) = D^*(f) < \infty$, where

$$D_*(f) := \liminf_{t\to 0} \frac{f(x+t) - f(x)}{t}$$
$$D^*(f) := \limsup_{t\to 0} \frac{f(x+t) - f(x)}{t}$$

Theorem 17 (Monster Theorem)**.** Every monotone function $f : [a, b] \to \mathbb{R}$ is differentiable a.e. in $[a, b]$. Furthermore, f' is integrable over $[a, b]$ and:

(i) If f is increasing, then $\int_a^b f' \leq f(b) - f(a)$.
(ii) If f is decreasing, then $\int_a^b f' \geq f(b) - f(a)$.

Definition 18 (Bounded Variation)**.** We say that a function $f : [a, b] \to \mathbb{R}$ is of **bounded variation** if $\mathrm{TV}(f) < \infty$, where:

$$\mathrm{TV}(f) := \sup \left\{ \sum_{k=0}^{N-1} |f(x_{k+1}) - f(x_k)| \mid a = x_0 < x_1 < \ldots < x_N = b \right\} \tag{24}$$

$\mathrm{TV}(f)$ is called the **total variation** of f.

Proposition 27. Let $f : [a, b] \to \mathbb{R}$ and let $c \in]a, b[$. Then:

$$\mathrm{TV}(f) = \mathrm{TV}(f|_{[a,c]}) + \mathrm{TV}(f|_{[c,b]})$$

Theorem 18 (Characterisation of Functions of Bounded Variation)**.** A function $f : [a, b] \to \mathbb{R}$ is of bounded variation $\iff$ it can be written as the difference of two increasing functions. In particular, every function $f : [a, b] \to \mathbb{R}$ is differentiable a.e. in $[a, b]$ and f' is integrable over $[a, b]$.

Definition 19 (Absolutely Continuous)**.** We say that a function $f : [a, b] \to \mathbb{R}$ is **absolutely continuous** if $\forall \varepsilon > 0$, $\exists\, \delta_\varepsilon > 0$ such that $\forall$ finite collections of open, bounded intervals that are disjoint $]a_1, b_1[, \ldots,]a_N, b_N[$, if

$$\sum_{k=1}^{N} |b_k - a_k| < \delta \Rightarrow \sum_{k=1}^{N} |f(b_k) - f(a_k)| < \varepsilon$$

Theorem 19. Every absolutely continuous function $f : [a, b] \to \mathbb{R}$ can be written as the difference of two increasing and absolutely continuous functions. In particular, it is of bounded variation.

Theorem 20. Let $f : [a, b] \to \mathbb{R}$. Then:

(i) If f is absolutely continuous on $[a, b]$ $\forall x \in [a, b]$, then

$$\int_{[a,x]} f' = f(x) - f(a)$$

(ii) Conversely, if $\exists$ a g integrable over $[a, b]$ such that $\forall x \in [a, b]$, $\int_{[a,x]} g = f(x) - f(a)$, then f is absolutely continuous and $f' = g$ a.e. in $[a, b]$.

Lemma 21. Let h be integrable over $[a, b]$. Then, $h = 0$ a.e. in $[a, b]$ $\iff$ $\forall\ x < y \in]a, b[$

$$\int_{]x,y[} h = 0$$

Corollary 6. Let $f : [a, b] \to \mathbb{R}$ be monotone. Then, f is absolutely continuous on $[a, b]$ $\iff$

$$\int_{]a,b[} f' = f(b) - f(a)$$

Corollary 7. Every function $f : [a, b] \Rightarrow \mathbb{R}$ of bounded variation can be written as $f = f_{\text{abs}} + f_{\text{sing}}$, where f_{abs} is absolutely continuous and $f'_{\text{sing}} = 0$ a.e. in $]a, b[$.

5. Lebesgue Measure and Integration in $\mathbb{R}^d$, $d \geq 2$

Definition 20 (Outer Measure)**.** Let $A \subseteq \mathbb{R}^d$. We define the **outer measure** of A as:

$$m^*(A) := \inf \left\{ \sum_{k=1}^{\infty} \mathrm{Vol}(R_k) \mid R_k =]a_{k_1}, b_{k_1}[\times \cdots \times]a_{k_d}, b_{k_d}[\text{ open, bd rectangles covering } A \right\} \tag{25}$$

where

$$\mathrm{Vol}(R_k) := \prod_{i=1}^{d} (b_{k_i} - a_{k_i})$$

Proposition 28. Every open set $\mathcal{O} \subseteq \mathbb{R}^d$ can be written as a countable union of almost disjoint closed cubes.

For the next family of theorems, we are in the following set-up. Let $d_1, d_2 \in \mathbb{N}$ be such that $d_1 + d_2 = d$. For every $E \subseteq \mathbb{R}^d$ and $(x, y) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} = \mathbb{R}^d$. We denote

$$E_{x_0} := \left\{ y \in \mathbb{R}^{d_2} \mid (x_0, y) \in E \right\}$$
$$E_{y_0} := \left\{ x \in \mathbb{R}^{d_1} \mid (x, y_0) \in E \right\}$$

and $\forall\ f : E \to \mathbb{R}$

$$f_{x_0} := \begin{cases} E_{x_0} \to \overline{\mathbb{R}} \\ y \mapsto f(x_0, y) \end{cases}$$
$$f_{y_0} := \begin{cases} E_{y_0} \to \overline{\mathbb{R}} \\ x \mapsto f(x, y_0) \end{cases}$$

Theorem 22 (Fubini's Theorem in $\mathbb{R}^d$)**.** Let $f : \mathbb{R}^d \to \mathbb{R}$ be integrable over $\mathbb{R}$. Then:

(i) (Existence of the Integral I) For almost every $y \in \mathbb{R}^{d2}$, f_y is integrable over $\mathbb{R}^{d1}$.
(ii) (Existence of the Integral II) $y \mapsto \int_{\mathbb{R}^{d1}} f_y$ is integrable over $\mathbb{R}^{d2}$.
(iii) (Fubini's Theorem)

$$\int_{\mathbb{R}^{d2}} \left(\int_{\mathbb{R}^{d1}} f(x, y) dx \right) dy = \int_{\mathbb{R}^d} f \tag{26}$$

Theorem 23 (Tonelli's Theorem; Fubini's Theorem for non-negative measurable functions)**.** Let $f : \mathbb{R}^d \to [0, \infty]$ be measurable. Then:

(i) (Existence of the Integral I) For almost every $y \in \mathbb{R}^{d2}$, f_y is non-negative and measurable over $\mathbb{R}^{d1}$.
(ii) (Existence of the Integral II) $y \mapsto \int_{\mathbb{R}^{d1}} f_y$ is non-negative, measurable over $\mathbb{R}^{d2}$.
(iii) (Fubini's Theorem)

$$\int_{\mathbb{R}^{d2}} \left(\int_{\mathbb{R}^{d1}} f_y dx \right) dy = \int_{\mathbb{R}^d} f \tag{27}$$

Corollary 8. Let $E \subseteq \mathbb{R}^d$ be measurable. Then:

(i) For a.e. $y \in \mathbb{R}^{d2}$, E_y is measurable.

(ii) $y \mapsto m(E_y)$ is measurable.
(iii) $m(E) = \int_{\mathbb{R}^{d2}} m(E_y)dy$.

Corollary 9 (General Version of Tonelli's Theorem)**.** [1] Let $E \subseteq \mathbb{R}^d$ be measurable, and let $f : E \to [0, \infty]$ be measurable. Then:

(i) For almost every $y \in \mathbb{R}^{d2}$, f is non-negative and measurable on E_y.
(ii) $y \mapsto \int_{E_y} f_y$ is non-negative, measurable, on $\mathbb{R}^{d2}$.
(iii)

$$\int_{\mathbb{R}^{d2}} \int_{E_y} f_y = \int_E f$$

Corollary 10 (General Version of Fubini's Theorem)**.** Let $E \subseteq \mathbb{R}^d$ be measurable and let $f : E \to \overline{\mathbb{R}}$ be measurable. Then:

(i) For almost every $y \in \mathbb{R}^{d2}$, f_y is integrable on E_y.
(ii) $y \mapsto \int_{E_y} f_y$ is measurable on $\mathbb{R}^{d2}$.
(iii)

$$\int_{\mathbb{R}^{d2}} \int_{E_y} f_y = \int_E f$$

Theorem 24. Let E_1 and E_2 be measurable sets in $\mathbb{R}^{d1}$ and $\mathbb{R}^{d2}$ respectively. Then $E_1 \times E_2$ is measurable and

$$m(E_1 \times E_2) = \begin{cases} m(E_1) \times m(E_2) & \text{if } m(E_1) \neq 0 \ \wedge m(E_2) \neq 0 \\ 0 \text{ else} \end{cases}$$

Corollary 11. Let E_1, E_2 be two measurable sets, $E_1 \subseteq \mathbb{R}^{d1}$ and $E_2 \subseteq \mathbb{R}^{d2}$. Let $f : E_1 \to \overline{\mathbb{R}}$ be measurable. Then:

$$\widetilde{f} := \begin{cases} E_1 \times E_2 \to \overline{\mathbb{R}} \\ \widetilde{f}(x, y) = f(x) \end{cases}$$

is measurable as a function of $E_1 \times E_2$.

Theorem 25 (Formula for the Integral of a non-negative measurable function in terms of a region in $\mathbb{R}^d$)**.** Assume that $d_1 = d - 1$ and $d_2 = 1$. Let $E_1 \subseteq \mathbb{R}^{d-1}$ be measurable and consider $f : E_1 \to [0, \infty]$.

(i) f is measurable $\iff$ the set A:

$$A := \{(x, y) \in E_1 \times \mathbb{R} \mid 0 < y < f(x)\}$$

is measurable.
(ii) Moreover, if f is measurable, then

$$m(A) = \int_{E_1} f \tag{28}$$

[1] The difference between points i and ii here vs. Tonelli's theorem is that we cannot fix x here.