Math 254: Analysis 1

Definitions, Theorems, and Results from the Class (Fall 2018)

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Abstract

The purpose of this document is to review analysis 1.

1. Introduction

Random things we proved to get a handle on how to prove things:

- $\cap_{x \in [0,1]} [0, x] = \{0\}$.
- $2^n < n!$
- Let X and Y be sets. Consider the following family of sets:
$$\{V_i \mid i \in I, V_i \subseteq Y\}$$
then, $f^{-1}(\cup_{i \in I} V_i) = \cup_{i \in I} f^{-1}(V_i)$.
- $5^n - 1$ is divisible by 4 $\forall n \geq 1$.
- **Bernoulli's Inequality**: $\forall n \in \mathbb{N}$, $x \in \mathbb{R}$, $x \geq -1$, one has:
$$(1 + x)^n \geq 1 + nx \tag{1}$$
- Every non-empty subset of the natural numbers has a smallest element.

Definition 1 (Cartesian Product). Let A and B be two sets. Then, their **Cartesian Product** is defined as:

$$A \times B := \{(a, b) \mid a \in A \wedge b \in B\} \tag{2}$$

Definition 2 (Function). Let D, E be sets. A **function** f from D to E is a subset of the cartesian product $D \times E$ such that $\forall x \in D$, $\exists_1 \; t \in E$ such that $(x, y) \in f$. In symbols, we define:

$$f(A) := \{f(x) \mid x \in A\} \tag{3}$$

Proposition 1 (Properties of Functions). Let $f : D \to E$ be a function and let $A, B \subseteq D$. Then, consider the following:

- $f(A \cup B) = f(A) \cup f(B)$ [well behaved with respect to unions]
- $f(A \cap B) \subseteq f(A) \cap f(B)$.

Definition 3 (Pre-Image). Let $f : D \to E$, $A \subseteq E$. Then, the **pre-image** is defined as:

$$f^{-1}(A) := \{x \in D \mid f(x) \in A\} \tag{4}$$

Proposition 2. Let $f : D \to E$, $A, B \subseteq E$. Then:

- $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$
- $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$

Definition 4 (Injective)**.** Let $f : D \to E$. f is said to be **injective** if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$.

Definition 5 (Surjective)**.** Let $f : D \to E$. f is said to be **surjective** if $\forall y \in E$, $\exists x \in D$ such that $f(x) = y$.

Definition 6 (Bijective)**.** $f : D \to E$ is called **bijective** if it is surjective and injective.

Definition 7. If $f : D \to E$ is bijective, then we can define the **inverse** function $f^{-1} : E \to D$ as follows:

$$f^{-1}(y) := x \tag{5}$$

where x is a uniquely determined point in D with $f(x) = y$.

1.1 Countability of Finite Sets

Definition 8 (Cardinality)**.** Let $S = \{a_1, ..., a_n\}$. Then, the **cardinality** of S, in symbols $|S|$, is the number of elements in a set S.

Theorem 1. Let A, B be finite sets. Then, $|A| \leq |B| \iff$ there exists a function $f : A \to B$ which is injective.

Theorem 2. Let A, B be finite sets. Then, $|A| \geq |B| \iff \exists$ a surjective map from $A \to B$.

Theorem 3. Let A, B be finite sets. Then, $|A| = |B| \iff \exists$ a bijective map $f : A \to B$.

Definition 9. Let A and B be sets, not necessarily finite. We then say that A and B have the **same cardinality**, in symbols,

$$|A| = |B| \tag{6}$$

if $\exists$ a bijective map $f : A \to B$.

Theorem 4 (Cantor's Theorem)**.** Let A and B be sets. If $|A| \leq |B|$ and if $|B| \leq |A|$, then $|A| = |B|$.

Definition 10 (Countability)**.** We say that a set A with $|A| = |\mathbb{N}|$ is **countably infinite**. A set which is either finite or countably infinite is called **countable**.

Theorem 5 (Arithmetic-Geometric Inequality)**.** $\forall n \geq 1$ and for all $x_1, ..., x_n > 0$, the following holds:

$$\frac{x_1 + ... + x_n}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n} \tag{7}$$

Lemma 6. Let $n \in \mathbb{N}$ and let $x_1, ..., x_n > 0$. If $x_1 \cdots x_n = 1$, then:

$$x_1 + ... + x_n \geq n \tag{8}$$

Theorem 7. Let $S \subseteq \mathbb{N}$. Then, there are only two possibilities:

(i) S is finite.
(ii) S is countably infinite.

Lemma 8. Let $a_1 < a_2 < \cdots$ be a strictly increasing sequence of natural numbers. Then, we can say something about the growth rate:

$$a_n \geq n \tag{9}$$

$\forall n \in \mathbb{N}$.

Theorem 9. Let $f : \mathbb{N} \to S$ be surjective. Then, S is countable.

Theorem 10 (Cantor)**.** The set $\mathbb{Q}$ of all rational numbers is countably infinite.

Theorem 11. $\mathbb{R}$ is uncountable (i.e, $\mathbb{R}$ is infinite and there does not exist a bijection from $\mathbb{N}$ to $\mathbb{R}$.

Definition 11 (Absolute Value)**.** Let $x \in \mathbb{R}$. Then, the **absolute value** of x is defined as:

$$|x| := \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} \tag{10}$$

Note that $|x|$ is used to measure distances.

Proposition 3 (Properties of Absolute Value)**.** (i) $\forall x \in \mathbb{R}$, $|x| \geq 0$ and $|x| = 0 \iff x = 0$.
(ii) $\forall x, y \in \mathbb{R}$, $|xy| = |x||y|$. Especially, $|-x| = |x|$, in this case you would simply set $y = -1$.
(iii) $\forall x \in \mathbb{R}$, $-|x| \leq x \leq |x|$.
(iv) Let $a > 0$, $x \in \mathbb{R}$. Then, $|x| \leq a \iff -a \leq x \leq a$.

Theorem 12 (Triangle Inequality)**.** Let $x, y \in \mathbb{R}$. Then:

(i) $|x + y| \leq |x| + |y|$
(ii) $|x - y| \geq ||x| - |y||$
(iii) Especially,
  (i) $|x - y| \geq |x| - |y|$
  (ii) $|x - y| \geq |y| - |x|$

Corollary 1. We also have,

(i) $|x - y| \leq |x| + |y|$
(ii) $|x + y| \geq |x| - |y|$ and $|x + y| \geq |y| - |x|$.

Corollary 2 (Generalisation of the Triangle Inequality)**.**

$$|x_1 + x_2 + ... + x_n| \leq |x_1| + |x_2| + ... + |x_n| \tag{11}$$

Definition 12. ε-neighbourhood Let $x \in \mathbb{R}$ and let $\varepsilon > 0$ be fixed. Then, the **ε-neighbourhood** of x, $V_\varepsilon(x)$, to be:

$$\begin{aligned} V_\varepsilon(x) &:=]x - \varepsilon, x + \varepsilon[\\ &= \{y \in \mathbb{R} \mid |y - x| < \varepsilon\} \end{aligned}$$

Theorem 13. Let $x, y \in \mathbb{R}$, where $x \neq y$. Then, "x and y can be separated by neighbourhoods", i.e., $\exists$ a $\varepsilon > 0$ such that $V_\varepsilon(x) \cap V_\varepsilon(y) \neq \emptyset$.

1.2 Supremum and Infimum

Definition 13 (Bounded From Above)**.** Let $S \subseteq \mathbb{R}$, $S \neq \emptyset$. We say that S is **bounded from above** if $\exists$ a $u \in \mathbb{R}$ such that $\forall s \in S$ $s \leq u$.

Definition 14 (Bounded from Below)**.** Let $S \subseteq \mathbb{R}$, $S \neq \emptyset$. We say that S is **bounded from below** if $\exists$ a $u \in \mathbb{R}$ such that $\forall s \in S$, $u \leq s$.

Definition 15 (Supremum/Least Upper Bound)**.** Let $S \subseteq \mathbb{R}$, $S \neq \emptyset$. $u \in \mathbb{R}$ is called a **supremum** or **least upper bound**, denoted by $\sup S$, if:

(i) u is an upper bound for S.

(ii) If v is any other upper bound for S, then $u \leq v$.

If $u = \sup S \in S$, then we say that u is the **maximum element** of S.

Definition 16 (Infimum/Greatest Lower Bound)**.** Let $S \subseteq \mathbb{R}$, $S \neq \emptyset$. $u \in \mathbb{R}$ is called a **infimum** or **greatest lower bound**, denoted by $\inf S$, if:

(i) u is a lower bound.
(ii) If v is an arbitrary lower bound of S, then $v \leq u$.

If $u = \inf S \in S$, then we say that u is the **minimum element of S**.

[Begin Tutorial]

Proposition 4. If $X_1, ..., X_{n+1}$ are countable sets, then so is $X_1 \times \cdots \times X_{n+1}$.

Definition 17 (Power Set)**.** Let X be a set, possibly empty. Then, the **power set of X**, denoted $\mathcal{P}(X)$, is defined as the set of all subsets of X:

$$\mathcal{P}(X) := \{A \mid A \subseteq X\} \tag{12}$$

Theorem 14 (Cantor's Theorem)**.** Let X be a set. Then, there does not exist a surjection $X \to \mathcal{P}(X)$, which means that $|X| < |\mathcal{P}(X)|$

Corollary 3 (Russel's Paradox)**.** The set of all sets does not exist.

Proposition 5. A binary sequence is a list of points

$$a_1, a_2, ..., a_n, ...$$

such that each $a_i \in \{0, 1\}$. Let $\mathcal{B}$ be the set of all binary sequences. Then, $\mathcal{B}$ is uncountable.

[End Tutorial]

Theorem 15. Let S be a non-empty and bounded set from above, with supremum $\sup S$. Define:

$$a + S := \{a + s \mid s \in S\}$$

Then, $a + S$ has a supremum which is given by:

$$\sup (a + S) = a + \sup S \tag{13}$$

Theorem 16. Let $S \neq \emptyset$, $S \subseteq \mathbb{R}$, S bounded from above with supremum $\sup S$. Let $k > 0$ and define:

$$k \cdot s := \{ks \mid s \in S\}$$

Then,

- If $k > 0$, $k \cdot S$ is bounded from above and

$$\sup k \cdot S = k \cdot \sup S \tag{14}$$

- if $k < 0$, then $k \cdot S$ is bounded from below and

$$\inf k \cdot S = k \cdot \sup S \tag{15}$$

AXIOM: we assume $\mathbb{R}$ is complete. This means that every non-empty subset $S \subseteq \mathbb{R}$ which is bounded from above has a supremum in $\mathbb{R}$.

Theorem 17 (Archimedean Property of $\mathbb{R}$)**.** Let $x \in \mathbb{R}$, $x > 0$. Then, $\exists n \in \mathbb{N}$ such that $n \geq x$.

Theorem 18. Let $x < y$, $x, y \in \mathbb{R}$. Then, $\exists r \in \mathbb{Q}$ such that $x < r < y$. I.e., this means that the rational numbers are **dense** in $\mathbb{R}$.

Theorem 19. The irrational numbers are dense in $\mathbb{R}$.

Definition 18. Let $I_1, , I_2, I_3, ...$ be intervals with the following property:

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$$

Then, we call the $I_1, I_2, I_3, ...$ a **nested sequence** of intervals.

Theorem 20 (Nested Interval Property)**.** Let $I_1 \supseteq I_2 \supseteq I_3 \cdots$ be a nested sequence of non-empty, closed and bounded (we call this compact) intervals, then:

$$\bigcap_{n \in \mathbb{N}} I_n \neq \emptyset \tag{16}$$

THE NESTED INTERVAL PROPERTY IS IN FACT EQUIVALENT TO COMPLETNEESS.

Corollary 4. $\mathbb{R}$ is uncountable.

[Begin Tutorial]

COMPLETENESS PROPERTY OF $\mathbb{R}$: Let X be a non-empty subset of $\mathbb{R}$ that is bounded from above. Then, X has a least upper bound, denoted by $\sup X$.

Proposition 6. Let $X \subseteq \mathbb{R}$.

(i) if X has a supremum, then X is non-empty and bounded from above.
(ii) if X has an infimum, then X is non-empty and bounded from below.

Proposition 7. Let X be a non-empty set and let s be an upper bound for X in $\mathbb{R}$. Then, the following statements are equivalent:

(i) $s = \sup S$
(ii) $\forall\ \varepsilon > 0$, $\exists\ x_\varepsilon \in X$ such that:

$$s - \varepsilon < x_\varepsilon \leq s \tag{17}$$

Proposition 8. Let X be a non-empty set and let v be a lower bound for X in $\mathbb{R}$. Then, the following statements are equivalent:

(i) $v = \inf S$
(ii) $\forall\ \varepsilon > 0$, $\exists\ x_\varepsilon \in X$ such that:

$$v \leq x_\varepsilon < v + \varepsilon \tag{18}$$

A useful application of the Archimedean property: $\forall\ \varepsilon > 0$, one has that $\exists$ an $m \in \mathbb{N}$ such that $0 < \frac{1}{m} < \varepsilon$.

Theorem 21 (Characterisation of Intervals)**.** Let $S \subseteq \mathbb{R}$ contain at least two points and assume that S satisfies the property:

$$x, y \in S \text{ and } x < y \ \Rightarrow [x, y] \subseteq S \tag{19}$$

then S is an interval.

Proposition 9 (Algebraic Properties of Sup and Inf)**.** Let A, B be non-empty subsets of $\mathbb{R}$ that are bounded from above. Suppose that both $x, y \in [0, \infty[$. Then:

(i) $\sup(A \cdot B) = \sup(A)\sup(B)$, where $A \cdot B := \{ab \mid a \in A, b \in B\}$.

[End Tutorial]