Winter 2020 Semester (Results, Definitions, and Theorems) Lecture: 09

Chapter 9: Metric Spaces (General Properties)

Class: Math 455 (Analysis 4) Date: 9 February 2020 Shereen Elaidi

Abstract

This document contains a summary of all the key definitions, results, and theorems from class. There are probably typos, and so I would be grateful if you brought those to my attention :-).

Syllabus: L^p space, duality, weak convergence, Young, Holder, and Minkowski inequalities, point-set topology, topological space, dense sets, completeness, compactness, connectedness, path-connectedness, separability, Tychnoff theorem, Stone-Weierstrass Theorem, Arzela-Ascoli, Baire category theorem, open mapping theorem, closed graph theorem, uniform boudnedness principle, Hahn Banch theorem.

This section was not covered in class, but since we have homework on this chapter I figured having this as a review from analysis 2 might be helpful. Also, there are a few terms/results that I don't think we covered in analysis 2.

9.1. Examples of Metric Spaces

Definition 1 (Metric Space)**.** Let X be a non-empty set. A function $\rho : X \times X \to \mathbb{R}$ is called a **metric** if $\forall\ x, y \in X$:

(i) $\rho(x,y) \geq 0$
(ii) $\rho(x,y) = 0 \iff x = y$
(iii) $\rho(x,y) = \rho(y,x)$
(iv) $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$ (**Triangle Inequality**).

A non-empty set together with a metric, denoted (X,ρ) is called a **metric space**.

Definition 2 (Discrete Metric)**.** For any non-empty set X, the **discrete metric** ρ is defined by setting $\rho(x,y) = 0$ if $x = y$ and $\rho(x,y) = 1$ if $x \neq y$.

Definition 3 (Metric Subspace)**.** For any metric space (X,ρ), let $Y \subseteq X$ be non-empty. Then, the restriction of ρ to $Y \times Y$ defines a metric on Y. We define this induced metric space as a **metric subspace**.

Example 9.1.1 (Examples of metric spaces)**.** The following are examples of metric spaces:

(i) Every non-empty subset of a Euclidean space.
(ii) $L^p(E)$, where $E \subseteq \mathbb{R}$ is a measurable set.
(iii) $C[a,b]$.

Definition 4 (Product Metric)**.** For metric spaces (X_1,ρ_1) and (X_2,ρ_2), we define the **product metric** τ on the cartesian product $X_1 \times X_2$ by setting, for (x_1,x_2) and (y_1,y_2) in $X_1 \times X_2$:

$$\tau((x_1,x_2),(y_1,y_2)) := \{[\rho_1(x_1,x_2)]^2 + [\rho_2(y_1,y_2)]^2\}^{1/2} \tag{9.1}$$

Definition 5. Two metrics ρ and σ on a set X are said to be **equivalent** if there are positive numbers c_1 and c_2 such that $\forall\, x_1, x_2 \in X$,

$$c_1\sigma(x_1, x_2) \leq \rho(x_1, x_2) \leq c_2\sigma(x_1, x_2)$$

Definition 6 (Isometry)**.** A mapping $f : (X, \rho) \to (Y, \sigma)$ between two metric spaces is called an **isometry** provided that f is surjective and $\forall x_1, x_2 \in X$:

$$\sigma(f(x_1), f(x_2)) = \rho(x_1, x_2) \tag{9.2}$$

We say that two metric spaces are **isometric** if there is an isometry from one to another.

9.2. Open Sets, Closed Sets, and Convergent Sequences

Definition 7 (Open Ball)**.** Let (X, ρ) be a metric space. For a point $x \in X$ and $r > 0$, the set:

$$B(x, r) := \left\{x' \in X \mid \rho(x', x) < r\right\} \tag{9.3}$$

is called the **open ball** centred at x of radius r. A subset $\mathcal{O} \subseteq X$ is said to be **open** if $\forall x \in \mathcal{O}$, there exists an open ball centred at x and contained in $\mathcal{O}$. For a point $x \in X$, an open set containing x is called a **neighbourhood** of x.

Proposition 1. Let X be a metric space. The whole set X and the empty set $\emptyset$ are open. The intersection of any two open sets is open. The union of any collection of open sets is open.

Proposition 2. Let X be a subspace of a metric space Y and $E \subseteq X$. Then, E is **open in** X $\iff$ $E = X \cap \mathcal{O}$, where $\mathcal{O}$ is open in Y.

Definition 8 (Closure)**.** For a subset $E \subseteq X$, a point $x \in X$ is called a **point of closure** of E provided that every neighbourhood of x contains a point in E. The collection of the points of closure of E is called the **closure** of E and is denoted by $\overline{E}$.

Proposition 3. For $E \subseteq X$, where X is a metric space, its closure $\overline{E}$ is closed. Moreover, $\overline{E}$ is the smallest closed subset of X containing E in the sense that if F is closed and if $E \subseteq F$, then $\overline{E} \subseteq F$.

Definition 9 (Converge)**.** A sequence $\{x_n\}$ in a metric space (X, ρ) is said to **converge** to the point $x \in x$ provided that:

$$\lim_{n\to\infty} \rho(x_n, x) = 0$$

that is, $\forall\, \varepsilon > 0$, $\exists$ an index N such that $\forall n \geq N$, $\rho(x_n, x) < \varepsilon$.

Proposition 4. Let ρ and σ be equivalent metrics on a non-empty set X. Then, a subset X is open in a metric space (X, ρ) $\iff$ it is open in (X, σ).

9.3. Continuous Mappings Between Metric Spaces

Definition 10 (Continuous)**.** A mapping f from a metric space X to a metric space Y is continuous at the point $x \in X$ if $\forall\, \{x_n\} \in X$, if $\{x_n\} \to x$, then $\{f(x_n)\} \to f(x)$. f is said to be **continuous** if it is continuous at every point in X.

Proposition 5 (ε-δ criteria for continuity)**.** A mapping from a metric space (X, ρ) to a metric (Y, σ) is continuous at the point $x \in X \iff \forall\, \varepsilon > 0$, $\exists\, \delta > 0$ such that if $\rho(x, x') < \delta$, then $\sigma(f(x), f(x')) < \varepsilon$. That is:

$$f(B(x, \delta)) \subseteq B(f(x), \varepsilon) \tag{9.4}$$

Proposition 6. A mapping f from a metric space X to a metric space Y is continuous $\iff \forall$ open subsets $\mathcal{O} \subseteq Y$, the inverse image under f of $\mathcal{O}$, $f^{-1}(\mathcal{O})$, is an open subset of X.

Proposition 7. The composition of continuous mappings between metric spaces, when defined, is continuous.

Definition 11 (Uniformly Continuous)**.** A mapping from a metric space (X, ρ) to a metric space (Y, σ) is said to be **uniformly continuous** if $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ such that $\forall u, v \in X$, if $\rho(u, v) < \delta$, $\sigma(f(u), f(v)) < \varepsilon$.

Definition 12 (Lipschitz)**.** A mapping $f : (X, \rho) \to (Y, \sigma)$ is said to be **Lipschitz** if $\exists$ a $c \geq 0$ such that $\forall\, u, v \in X$:

$$\sigma(f(u), f(v)) \leq c\rho(u, v)$$

9.4. Complete Metric Spaces

Definition 13 (Cauchy)**.** A sequence $\{x_n\}$ in a metric space (X, ρ) is said to be a **Cauchy sequence** if $\forall \varepsilon > 0$, there exists a $N \in \mathbb{N}$ such that if $m, n \geq N$, then $\rho(x_n, x) < \varepsilon$.

Definition 14 (Complete)**.** A metric space X is said to be **complete** if every Cauchy sequence in X converges to a point in X.

Proposition 8. Let $[a, b]$ be a closed and bounded interval of real numbers. Then, $C[a, b]$ with the metric induced by the max norm is complete.

Proposition 9 (Characterisation of Complete Subspaces of Metric Spaces)**.** Let $E \subseteq X$, where X is a complete metric space. Then, the metric subspace E is complete $\iff$ E is a closed subset of X.

Theorem 1. The following are complete metric spaces:

(i) Every non-empty closed subset of $\mathbb{R}^n$.
(ii) $E \subseteq \mathbb{R}$ measurable, $1 \leq p \leq \infty$, each non-empty closed subset of $L^p(E)$.
(iii) Each non-empty closed subset of $C[a, b]$.

Definition 15 (Diameter)**.** Let E be a non-empty subset of a metric space (X, ρ). We define the **diameter** of E, denoted by $\text{diam}(E)$, by:

$$\text{diam}(E) := \sup\{\rho(x, y) \mid x, y \in E\} \tag{9.5}$$

We say that E is **bounded** if it has finite diameter.

Definition 16 (Contracting Sequence)**.** A decreasing sequence $\{E_n\}$ of non-empty subsets of X is called a **contracting sequence** if:

$$\lim_{n\to\infty} \text{diam}(E_n) = 0 \tag{9.6}$$

Theorem 2 (Cantor Intersection Theorem)**.** Let X be a metric space. Then, X is complete $\iff$ whenever $\{F_n\}$ is a contracting sequence of non-empty closed subsets of X, there is a point $x \in X$ for which:

$$\bigcap_{n=1}^{\infty} F_n = \{x\} \tag{9.7}$$

Theorem 3. Let (X, ρ) be a metric space. Then, there is a complete metric space $(\widetilde{X}, \tilde{\rho})$ for which X is a dense subset of $\widetilde{X}$ and

$$\rho(u, v) = \tilde{\rho}(u, v)\ \forall\ u, v \in X \tag{9.8}$$

we call such a space the **completion** of (X, ρ).

9.5. Compact Metric Spaces

Definition 17 (Compact Metric Space)**.** A metric space X is called **compact** if every open cover of X has a finite sub-cover. A subset $K \subseteq X$ is compact if K, considered as a metric subspace of X, is compact.

Formulation of compactness in terms of closed sets: Let $\mathcal{T}$ be a collection of open subsets of a metric space X. Define $\mathcal{F}$ to be the collection of the complements of elements in $\mathcal{T}$. Since the elements of $\mathcal{T}$ are open, the elements of $\mathcal{F}$ are closed. Thus, $\mathcal{T}$ is a cover $\iff$ the elements of $\mathcal{F}$ have *empty intersection*. By deMorgan's law, we can formulate compactness in terms of closed sets as:

> A metric space X is compact $\iff$ every collection of closed sets with empty intersection has a finite sub-collection whose intersection is non-empty.

This property is called the **finite intersection property**.

Definition 18 (Finite Intersection Property)**.** A collection of sets $\mathcal{F}$ is said to have the **finite intersection property** if any finite sub-collection of $\mathcal{F}$ has a non-empty intersection.

Proposition 10 (Compactness in terms of closed sets)**.** A metric space X is compact $\iff$ every collection $\mathcal{F}$ of closed subsets of X with the finite intersection property has a non-empty intersection.

Definition 19 (Totally Bounded)**.** A metric space X is **totally bounded** if $\forall\ \varepsilon > 0$, the space X can be covered by a finite number of open balls of radius ε. A subset $E \subseteq X$ is said to be **totally bounded** if E, a s a subspace of the metric space X, is totally bounded.

Definition 20 (ε-net)**.** Let E be a subset of a metric space X. A ε**-net** for R is a finite collection of open balls $\{B(x_k, \varepsilon)\}_{k=1}^{n}$ with centres $x_k \in X$ whose union covers E.

Proposition 11. A metric space E is totally bounded $\iff$ $\forall\ \varepsilon > 0$, there is a finite ε-net for E.

Proposition 12. A subset of Euclidean space $\mathbb{R}^n$ is bounded $\iff$ it is totally bounded.

Definition 21 (Sequentially Compact)**.** A metric space X is **sequentially compact** if every sequence X has a subsequence that converges to a point in X.

Theorem 4 (Characterisation of Compactness for a metric space)**.** . Let X be a metric space. Then, TFAE:

(i) X is complete and totally bounded.

(ii) X is compact.
(iii) X is sequentially compact.

The following three propositions of this chapter are just breaking down these equivalences, so I will not write them.

Theorem 5. Let $K \subseteq \mathbb{R}^n$. Then, TFAE:

(i) K is closed and bounded.
(ii) K is compact.
(iii) K is sequentially compact.

Observe: The equivalence (1) $\iff$ (2) is the Heine-Borel theorem. The equivalence (2) $\iff$ (3) is the Bolzano-Weierstrass theorem.

Proposition 13. Let f be a continuous mapping from a compact metric space X to a compact metric space Y. Then, its image $f(X)$ is compact.

Theorem 6 (Extreme Value Theorem)**.** Let X be a metric space. Then, X is compact $\iff$ every continuous real-valued function on X attains a minimum and maximum value.

Definition 22 (Lebesgue Number)**.** Let X be a metric space, and let $\{\mathcal{O}_\lambda\}_{\lambda\in\Lambda}$ be an open cover of X. Thus, each $x \in X$ is contained in a member of the cover, $\mathcal{O}_\lambda$. Since $\mathcal{O}_\lambda$ is open, $\exists\ \varepsilon > 0$ such that:

$$B(x, \varepsilon) \subseteq \mathcal{O}_\lambda$$

In general, ε on X, but for compact metric spaces we can get *uniform control*. This ε that uniformly works is called the **Lebesgue number** for the cover $\{\mathcal{O}_\lambda\}_{\lambda\in\Lambda}$.

Lemma 7. Let $\{\mathcal{O}_\lambda\}_{\lambda in \Lambda}$ be an open cover of a compact metric space X. Then, there is a number $\varepsilon > 0$ such that for each $x \in X$, the open ball $B(x, \varepsilon)$ is contained in some member of the cover.

Proposition 14. A continuous mapping from a compact space (X, ρ) to a metric space (Y, σ) is uniformly continuous.

9.6. Separable Metric Spaces

Definition 23 (Dense & Separable)**.** A subset D of a metric space X is **dense** in X if every non-empty subset of X contains a point of D. A metric space is **separable** if there is a countable subset of X that is dense in X.

The **Weierstrass Approximation Theorem** states that polynomials are dense in $C[a, b]$. So, $C[a, b]$ is separable, with the countable dense set being the set of polynomials with rational coefficients.

Proposition 15. A compact metric space is separable.

Proposition 16. A metric space X is separable $\iff$ there is a countable collection of $\{\mathcal{O}_n\}$ of open subsets of X such that any open subset of X is the union of a sub-collection of $\{\mathcal{O}_n\}$.

Proposition 17. Every subspace of a separable metric space is separable.

Theorem 8. Each of the following are separable metric spaces:

(i) Every non-empty subset of Euclidean space $\mathbb{R}^n$.
(ii) $1 \le p < \infty$, $L^p(E)$ and all non-empty subsets of $L^p(E)$.
(iii) Each non-empty subset of $C[a, b]$.