

# MATH 454: ANALYSIS 3 – THEORY OF LEBESGUE MEASURE

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## 1. INTRODUCTION

**Definition 1** (Riemann Integral). Let  $[a, b]$  be a bounded, closed interval and  $f : [a, b] \rightarrow \mathbb{R}$  be a bounded function. Then,  $f$  is **Riemann Integrable** if

$$\int_a^b f = \overline{\int_a^b f} \quad (1)$$

where

$$\int_a^b f := \sup \left\{ \sum_{i=1}^n \inf_{x_{i-1}, x_i[} f \cdot (x_i - x_{i-1}) \mid a = x_0 < \dots < x_n = b \right\} \quad (2)$$

$$\overline{\int_a^b f} := \inf \left\{ \sum_{i=1}^n \sup_{x_{i-1}, x_i[} f \cdot (x_i - x_{i-1}) \mid a = x_0 < \dots < x_n = b \right\} \quad (3)$$

**Theorem 1.** Every continuous function  $f : [a, b] \rightarrow \mathbb{R}$  is Riemann Integrable.

**Definition 2** (Length).  $\forall I \subseteq \mathbb{R}$ ,  $I$  an interval, we call the **length of  $I$**  to be the number:

$$\ell(I) := \begin{cases} b - a; & I = [a, b], [a, b[, ]a, b], \text{ or } ]a, b[ \\ \infty & I \text{ is unbounded} \end{cases} \quad (4)$$

**Definition 3** (Outer Measure).  $\forall A \subseteq \mathbb{R}$ , the **outer measure** of  $A$ , denoted by  $m^*(A)$  is given by:

$$m^*(A) := \inf \left\{ \sum_{k=1}^{\infty} \ell(I_k) \mid (I_k) \text{ open, bounded intervals s.t. } A \subseteq \bigcup_{k=1}^{\infty} I_k \right\} \quad (5)$$

**Proposition 1.**  $A \subseteq \mathbb{R}$  is countable  $\Rightarrow m^*(A) = 0$

**Proposition 2** (Monotonicity of outer measure). If  $A \subseteq B$ , then  $m^*(A) \leq m^*(B)$ .

**Proposition 3.** For every interval  $I \subseteq \mathbb{R}$ ,  $m^*(I) = \ell(I)$ .

**Proposition 4** (Translation invariance of outer measure).  $\forall A \subseteq \mathbb{R}$ ,  $y \in \mathbb{R}$ , define  $A + y := \{x + y \mid x \in A\}$ . Then,  $m^*(A) = m^*(A + y)$ .

**Proposition 5** (Countable Subadditivity of outer measure).  $\forall (A_k)_{k \in \mathbb{N}}$  subsets of  $\mathbb{R}$ :

$$m^* \left( \bigcup_{k=1}^{\infty} A_k \right) \leq \sum_{k=1}^{\infty} m^*(A_k) \quad (6)$$

where the  $A_k$ 's are not necessarily disjoint.

**Definition 4** (Lebesgue Measure). A set  $A \subseteq \mathbb{R}$  is **measurable** if  $\forall B \subseteq \mathbb{R}$ ,

$$m^*(B) = m^*(B \cap A) + m^*(B \setminus A) \quad (7)$$

The only non-trivial part of the definition to check is  $m^*(B) \geq m^*(B \cap A) + m^*(B \setminus A)$ , since the other inequality follows from the subadditivity of outer measure. We can also restrict  $B$  to the class of all finite-outer-measure sets, since the inequality is trivial for infinite-outer-measure sets.

**Proposition 6.** If  $m^*(A) = 0$ , then  $A$  is measurable.

**Proposition 7.**  $\forall A \subseteq \mathbb{R}$ ,  $A$  measurable,  $\Rightarrow \mathbb{R} \setminus A$  is measurable.

**Theorem 2** (Excision Property).  $\forall A_1, A_2 \subseteq \mathbb{R}$  measurable,  $A_2 \subseteq A_1$ , and  $m(A_2) < \infty$ , then:

$$m(A_1 \setminus A_2) = m(A_1) - m(A_2) \quad (8)$$

**Proposition 8.**  $\forall (A_k)_{k \in \mathbb{N}}$  measurable, we have:

- (i)  $\bigcup_{k=1}^{\infty} A_k$  is measurable and  $\bigcap_{k=1}^{\infty} A_k$  is measurable.
- (ii) **(Countable Additivity of Measure)**. If  $A_i \cap A_j = \emptyset \forall i \neq j$ , then:

$$m\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{i=1}^{\infty} m(A_k) \quad (9)$$

**Proposition 9** (Continuity of Lebesgue Measure). Let  $(A_k)_{k \in \mathbb{N}}$  be sequence of measurable sets. Then:

- (i) If  $A_k \subseteq A_{k+1} \forall k$  (increasing sequence of sets), then:

$$m\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{k \rightarrow \infty} m(A_k) \quad (10)$$

- (ii) If  $A_{k+1} \subseteq A_k \forall k$  (decreasing sequence of sets), and  $m(A_1) < \infty$ , then:

$$m\left(\bigcap_{k=1}^{\infty} A_k\right) = \lim_{k \rightarrow \infty} m(A_k) \quad (11)$$

**Proposition 10** (Translation Invariance of Measurable Sets).  $\forall A \subseteq \mathbb{R}$  measurable, and  $\forall y \in \mathbb{R}$  fixed,  $A + y$  is measurable.

**Proposition 11.** (i) Every interval in  $\mathbb{R}$  is Lebesgue measurable.

- (ii) Every open set and every closed set is Lebesgue measurable.

**Theorem 3** (Characterisation of Measurable Sets). Let  $A \subseteq \mathbb{R}$ . Then, TFAE:

- (i)  $A$  is measurable.
- (ii) **(Outer Approximation of Measurable Sets by Open Sets)**.  $\forall \varepsilon > 0$ ,  $\exists O_\varepsilon \subseteq \mathbb{R}$  open such that  $A \subseteq O_\varepsilon$  and  $m^*(O_\varepsilon \setminus A) < \varepsilon$ .
- (iii) **(Approximation by  $G_\delta$  sets)**.  $\exists (O_n)_{n \in \mathbb{N}}$  open such that  $A \subseteq G$  and  $m^*(G \setminus A) = 0$ , where  $G := \bigcap_{n \in \mathbb{N}} O_n$ . The countable intersection of open sets is a  **$G_\delta$ -set**.
- (iv) **(Inner Approximation of Measurable Sets by Closed Sets)**  $\forall \varepsilon > 0$ ,  $\exists F_\varepsilon \subseteq \mathbb{R}$  closed such that  $F_\varepsilon \subseteq A$  and  $m^*(A \setminus F_\varepsilon) < \varepsilon$ .
- (v) **(Approximation by  $F_\sigma$  sets)**.  $\exists (F_n)_{n \in \mathbb{N}}$  closed such that  $F \subseteq A$  and  $m^*(A \setminus F) = 0$ , where  $F := \bigcup_{n \in \mathbb{N}} F_n$ . The countable union of closed sets is a  **$F_\sigma$ -set**.

**Theorem 4** (Vitali).  $\forall A \subseteq \mathbb{R}$ , if  $m^*(A) > 0$ , then  $\exists B \subseteq A$  that is not measurable.

**Definition 5** (Cantor Set). The **Cantor Set** is recursively defined as:

$$C := \bigcap_{k=1}^{\infty} C_k \quad (12)$$

Where:

$$C_1 := \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$$

and for  $k \geq 2$ :

$$C_k := \bigcup_{j=1}^{2^k} I_{k,j} \quad \forall j \in \{1, \dots, 2^{k-1}\}$$

Where  $I_{k,2j-1}$  and  $I_{k,2j}$  are the first and second thirds of the interval  $I_{k-1,j}$ .

**Theorem 5.**  $C$  is closed, uncountable, and  $m^*(C) = 0$ .

**Definition 6** ( $\sigma$ -algebra). A collection of sets  $\mathcal{C}$  is called a  $\sigma$ -**algebra** if the following are true:

- (i)  $\mathbb{R} \in \mathcal{C}$ .
- (ii)  $\forall C_1, C_2 \in \mathcal{C}, C_1 \setminus C_2 \in \mathcal{C}$  (stable under complementation).
- (iii)  $\forall (C_k)_{k \in \mathbb{N}} \in \mathcal{C}$ , we have that:

$$\bigcup_{k=1}^{\infty} C_k \in \mathcal{C}$$

(stable under countable unions).

**Proposition 12.** Any intersection of  $\sigma$ -algebras is a  $\sigma$ -algebra.

**Definition 7** (Borel Sets). A **Borel set** is a set that is in the intersection of all the sigma algebras containing the open sets. The **Borel sigma algebra** is the smallest sigma algebra containing all the open sets. (Alternatively, it is the sigma algebra generated by the open sets).

**Proposition 13.** There exists a subset of the Cantor Set which is not Borel. Thus, the set of measurable sets is indeed bigger than the smallest sigma algebra.

**Definition 8** (Cantor Lebesgue Function). The **Cantor-Lebesgue Function** is the function  $\varphi : [0, 1] \rightarrow [0, 1]$  defined as:

$$\varphi(x) := \frac{i}{2^k} \tag{13}$$

if  $x \in J_{k,i}$ , where  $J_{k,i}$  is the  $i$ -th interval in  $[0, 1] \setminus C_k$ ,  $\forall i \in \{1, \dots, 2^k - 1\}$ , and  $\forall y \in [0, 1] \setminus C$ :

$$\varphi(y) := \begin{cases} \varphi(0) := 0 \\ \varphi(y) := \sup \{ \varphi(x) \mid x \in [0, y] \setminus C \} \end{cases} \tag{14}$$

**Proposition 14.**  $\varphi$  is an increasing and continuous function.

## 2. LEBESGUE MEASURABLE FUNCTIONS

**Proposition 15** (Lebesgue Measurable Function). Let  $A \subseteq \mathbb{R}$  be a measurable set and  $f : A \rightarrow \mathbb{R}$ . Then, TFAE:

- (i)  $\forall c \in \mathbb{R}, f^{-1}([c, +\infty])$  is measurable.
- (ii)  $\forall c \in \mathbb{R}, f^{-1}([c, +\infty])$  is measurable.
- (iii)  $\forall c \in \mathbb{R}, f^{-1}([-\infty, c])$  is measurable.
- (iv)  $\forall c \in \mathbb{R}, f^{-1}([-\infty, c])$  is measurable.

If any of the above conditions are met, then we say that  $f$  is **measurable**.

**Proposition 16.** Let  $A \subseteq \mathbb{R}$  be measurable, and let  $f : A \rightarrow \overline{\mathbb{R}}$ . Then:

- (i)  $f$  measurable  $\Rightarrow \forall B \subseteq \mathbb{R}, B$  a Borel Set,  $f^{-1}(B)$  is measurable. (The inverse image of Borel sets are measurable sets).
- (ii) If  $f$  is finite-valued, i.e.,  $f(A) \subseteq \mathbb{R}$ , then we get a *characterisation of measurable functions*:  $f$  measurable  $\iff \forall B \subseteq \mathbb{R}, B$  Borel,  $f^{-1}(B)$  is measurable.

**Proposition 17.** Let  $A \subseteq \mathbb{R}$  be measurable and  $f : A \rightarrow \mathbb{R}$  be continuous. Then,  $f$  is measurable.

**Definition 9** (Almost Everywhere). Let  $x \in \mathbb{R}$  be measurable, and let  $P(x)$  be a statement depending on  $x \in A$ . We say that  $P(x)$  is **true almost everywhere in  $A$**  (abbreviated as a.e.  $x \in A$ ) if  $m(\{x \in A \mid P(x) \text{ is false}\}) = 0$

**Proposition 18.** Let  $f : A \rightarrow \overline{\mathbb{R}}$  be a measurable function. Let  $g : A \rightarrow \overline{\mathbb{R}}$  be such that  $f = g$  a.e. in  $A$ . Then,  $g$  is measurable.

**Proposition 19.** Let  $(A_n)_{n \in \mathbb{N}}$  be disjoint, measurable sets and let  $A = \bigcup_{n \in \mathbb{N}} A_n$ . Let  $(f_n)_{n \in \mathbb{N}}$ ,  $f_n : A_n \rightarrow \overline{\mathbb{R}}$  be measurable. Then, the function:

$$\begin{aligned} f &:= A \rightarrow \overline{\mathbb{R}} \\ x &\mapsto f_n(x) \end{aligned}$$

is measurable.

**Definition 10** (Characteristic Function).  $\forall B \subseteq A$ ,  $B$  measurable, the **characteristic function of  $B$**  is the function  $\chi_B : A \rightarrow \mathbb{R}$ :

$$\chi_B := \begin{cases} x \mapsto 1; & x \in B \\ x \mapsto 0; & x \notin B \end{cases} \quad (15)$$

**Definition 11** (Simple Functions).  $f : A \rightarrow \mathbb{R}$  is a **simple function** if  $f(A)$  is a finite set. This means that  $f$  is a sum of characteristic functions;  $\exists a_1 < a_2 < \dots < a_N \in \mathbb{R}$  such that  $f(A) = \{a_1, \dots, a_N\}$ . Letting  $A_k := f^{-1}(\{a_k\})$ , we have:

$$f = \sum_{k=1}^N a_k \chi_k$$

This representation is unique and is called the **canonical representation of  $f$** .

**Proposition 20** (Properties of Measurable Functions). Let  $A \subseteq \mathbb{R}$  be measurable. Then:

- (i)  $\forall B \subseteq A$  measurable,  $f|_B$  is measurable.
- (ii)  $\forall B \subseteq \mathbb{R}$  Borel, if  $f : B \rightarrow \mathbb{R}$  is continuous,  $g : A \rightarrow B$  is measurable, then  $f \circ g$  is measurable.
  - (i) Note that we need  $f$  to be continuous, since we need the inverse image to preserve the Borel property.
- (iii)  $\forall f : A \rightarrow \overline{\mathbb{R}}$ ,  $g : A \rightarrow \mathbb{R}$ ,  $f + g$  is measurable.
  - (i) Note that we need  $g$  not into  $\overline{\mathbb{R}}$  since we need to avoid the  $\infty - \infty$  case.
- (iv)  $\forall f, g : A \rightarrow \mathbb{R}$  measurable,  $f \cdot g$  is measurable. (No  $\overline{\mathbb{R}}$  to avoid the  $\infty \cdot 0$  case).
- (v)  $\forall f_1, \dots, f_n, f_n : A \rightarrow \mathbb{R}$  measurable,
  - (i)  $\max\{f_1, \dots, f_n\}$
  - (ii)  $\min\{f_1, \dots, f_n\}$
 are measurable.

**Definition 12** (Uniform and pointwise convergence). Let  $\{f_n\}_{n \in \mathbb{N}}$  be a sequence of measurable functions,  $f_n A \rightarrow \mathbb{R}$ , and  $f : A \rightarrow \overline{\mathbb{R}}$ . We say that:

- (i)  $\{f_n\}_{n \in \mathbb{N}}$  converges **pointwise** to  $f$  in  $B \subseteq A$  if

$$\forall x \in B, \lim_{n \rightarrow \infty} f_n(x) = f(x)$$

- (ii)  $\{f_n\}_{n \in \mathbb{N}}$  converges **uniformly** to  $f$  in  $B \subseteq A$  if

$$\lim_{n \rightarrow \infty} \sup_B |f_n - f| = 0$$

**Proposition 21.** Let  $\{f_n\}_{n \in \mathbb{N}}$  be a sequence of measurable functions,  $f_n : A \rightarrow \overline{\mathbb{R}}$  converging pointwise almost everywhere in  $A$  to a function  $f : A \rightarrow \mathbb{R}$ . Then,  $f : A \rightarrow \mathbb{R}$  is measurable.

**Proposition 22** (Simple Approximation Lemma). Let  $f : A \rightarrow \mathbb{R}$  be measurable and bounded everywhere (i.e.,  $\exists$  a  $M > 0$  such that  $|f| < M$  in  $A$ ). Then,  $\forall \varepsilon > 0$ ,  $\exists \psi_\varepsilon, \varphi_\varepsilon : A \rightarrow \mathbb{R}$  simple functions such that

$$\varphi_\varepsilon \leq f \leq \psi_\varepsilon < \varphi_\varepsilon + \varepsilon$$

in  $A$ . In particular, the  $\varphi_\varepsilon$  and the  $\psi_\varepsilon$  converge uniformly to  $f$  in  $A$ .

**Theorem 6** (Simple Approximation Theorem). Let  $f : A \rightarrow \overline{\mathbb{R}}$  on a measurable set  $A$ . Then,  $f$  is measurable  $\iff$  there exist simple functions  $(\varphi_n)_{n \in \mathbb{N}}$  such that:

- (i)  $(\varphi_n)_{n \in \mathbb{N}}$  converges pointwise to  $f$ .
- (ii)  $|\varphi_n| \leq |f|$  in  $A \forall n \in \mathbb{N}$ .

Moreover, if  $f \geq 0$  in  $A$ , we can choose  $\varphi_n$  such that  $\varphi_n \geq 0$  and  $\varphi_{n+1} \geq \varphi_n \forall n \in \mathbb{N}$ .

**Theorem 7** (Egoroff's Theorem). Let  $A \subseteq \mathbb{R}$  be a measurable set, and assume that  $m(A) < \infty$ . Let  $(f_n)_{n \in \mathbb{N}}$ ,  $f_n : A \rightarrow \mathbb{R}$  be a sequence of measurable functions converging pointwise to  $f : A \rightarrow \mathbb{R}$  (not  $\overline{\mathbb{R}}$ !!). Then,  $\forall \varepsilon > 0$ ,  $\exists F_\varepsilon \subseteq A$  closed such that:

- (i)  $\{f_n\}_{n \in \mathbb{N}}$  converges uniformly on  $F_\varepsilon$ .
- (ii)  $m(A \setminus F_\varepsilon) < \varepsilon$ .

**Theorem 8** (Lusin's Theorem). Let  $f : A \rightarrow \mathbb{R}$  be measurable (not into  $\overline{\mathbb{R}}$ !). Then,  $\forall \varepsilon > 0$ ,  $\exists F_\varepsilon \subseteq A$  closed such that:

- (i)  $f$  is continuous on  $F_\varepsilon$ .
- (ii)  $m(A \setminus F_\varepsilon) < \varepsilon$ .

### 3. THE LEBESGUE INTEGRAL

**Definition 13** (Integral – Case of Simple Functions on a Set of Finite Measure). Let  $\psi : A \rightarrow \mathbb{R}$  be a simple function. Let  $\psi = \sum_{k=1}^N a_k \chi_{A_k}$  be its canonical representation. We define the **integral** of  $\psi$  over  $A$  and denote  $\int_A \psi$  and  $\int_A \psi(x)dx$  to be the number:

$$\int_A \psi := \sum_{k=1}^N a_k m(A_k) \quad (16)$$

For every  $B \subseteq A$  measurable, we denote  $\int_B \psi = \int_B \psi|_B$ . Here, the measure of  $A$  must be finite.

**Definition 14** (Integral – Case of Measurable, Bounded Functions on a Set of Finite Measure). Let  $A \subseteq \mathbb{R}$  be a measurable set such that  $m(A) < \infty$ , and let  $f : A \rightarrow \mathbb{R}$  be a bounded function. We say that  $f$  is **integrable over  $A$**  if:

$$\underline{\int_A} f = \overline{\int_A} f \quad (17)$$

where

$$\underline{\int_A} f := \sup \left\{ \int_A \varphi \mid \varphi \text{ simple, } \varphi \leq f \text{ on } A \right\}$$

$$\overline{\int_A} f := \inf \left\{ \int_A \varphi \mid \varphi \text{ simple, } f \leq \varphi \text{ on } A \right\}$$

We then denote  $\int_A f = \int_A f(x)dx = \underline{\int_A} f = \overline{\int_A} f$  and we call this number the **integral** of  $f$  over  $A$ . For every  $B \subseteq A$  measurable, we denote:

$$\int_B f = \int_B f|_B$$

**Theorem 9.** If  $f : [a, b] \rightarrow \mathbb{R}$  is Riemann Integrable, then  $f$  is Lebesgue Integrable.

**Theorem 10.** Let  $f : A \rightarrow \mathbb{R}$ ,  $m(A) < \infty$ , be a measurable and bounded function. Then,  $f$  is integrable.

**Proposition 23** (Properties of the Integral). Let  $f, g : A \rightarrow \mathbb{R}$  be measurable and bounded. Then:

- (i)  $\forall \alpha, \beta \in \mathbb{R}$ ,  $\alpha f + \beta g$  is measurable and bounded, and:

$$\int_A (\alpha f + \beta g) = \alpha \int_A f + \beta \int_A g$$

- (ii) (Monotonicity): if  $f \leq g$  on  $A$ , then:

$$\int_A f \leq \int_A g$$

- (iii)  $|\int_A f|$  is measurable and bounded, and  $|\int_A f| \leq \int_A |f|$ .

(iv)  $\forall B \subseteq \mathbb{R}$  measurable,  $f \cdot \chi_B$  is measurable, bounded, and

$$\int f \chi_B = \int_B f$$

(v)  $\forall A_1, A_2$  measurable and disjoint,

$$\int_{A_1 \cup A_2} f = \int_{A_1} f + \int_{A_2} f$$

In particular, if  $m(A_2) = 0$ , then:

$$\int_{A_2} f = 0 \text{ and so } \int_{A_1 \cup A_2} f = \int_{A_1} f$$

**Lemma 11** (Independence of Representation). Let  $n \in \mathbb{N}$  and let  $a_1, \dots, a_n \in \mathbb{R}$  and  $A_1, \dots, A_n \subseteq A$ , where  $m(A) < \infty$ , be measurable and disjoint. Then:

$$\int \sum_{k=1}^n a_k \chi_{A_k} = \sum_{k=1}^n a_k m(A_k)$$

**Theorem 12** (Bounded Convergence Theorem). Let  $A \subseteq \mathbb{R}$  be measurable,  $m(A) < \infty$ . Let  $(f_n)_{n \in \mathbb{N}}$ ,  $f_n : A \rightarrow \mathbb{R}$  be a sequence of measurable functions on  $A$  such that:

- (i) (Uniformly bounded)  $\exists$  an  $M > 0$  such that  $\forall n \in \mathbb{N}$ ,  $|f_n| \leq M$  on  $A$ .
- (ii) (Pointwise Convergence)  $\exists f : A \rightarrow \mathbb{R}$  such that  $\forall x \in A$ ,  $\lim_{n \rightarrow \infty} f_n(x) = f(x)$

Then,  $f$  is bounded and measurable, and we can interchange the limits as so:

$$\lim_{n \rightarrow \infty} \int_A f_n = \int_A f$$

**Definition 15** (Integral in the case of a Non-Negative, Measurable Function on a Set of Possibly Infinite Measure). Let  $A \subseteq \mathbb{R}$  be measurable, possibly of infinite measure, and let  $f : A \rightarrow [0, \infty]$  be measurable. We call the **integral** of  $f$  over  $A$  and denote  $\int_A f = \int_A f(x) dx$  the number defined as

$$\int_A f := \sup \left\{ \int_B h \mid B \subseteq A, m(B) < \infty, h : B \rightarrow \mathbb{R} \text{ measurable, bd, } 0 \leq h \leq f \text{ on } B \right\} \quad (18)$$

For every  $B \subseteq A$ , we denote  $\int_B f = \int_B f|_B$ . If  $\int_A f < \infty$ , we say that  $f$  is **integrable** over  $A$ .

**Proposition 24** (Properties of the Integral). Let  $f, g : A \rightarrow [0, \infty]$  be measurable. Then:

- (i)  $\forall \alpha, \beta \geq 0$ ,  $\alpha f + \beta g$  is non-negative and measurable and :

$$\int (\alpha f + \beta g) = \alpha \int f + \beta \int g$$

- (ii)  $f \leq g$  on  $A \Rightarrow \int_A f \leq \int_A g$ .

- (iii) If  $|f| < \infty$ , then  $\forall B \subseteq A$  measurable,  $\chi_B f$  is non-negative, measurable, and

$$\int_A \chi_B f = \int_B f$$

- (iv)  $\forall A_1, A_2 \subseteq A$  disjoint, measurable. Then:

$$\int_{A_1 \cup A_2} f = \int_{A_1} f + \int_{A_2} f$$

If, moreover,  $m(A_2) = 0$ , then

$$\int_{A_2} f = 0 \text{ and so } \int_{A_1 \cup A_2} f = \int_{A_1} f$$

**Theorem 13** (Chebyshev's Inequality). Let  $f$  be measurable, non-negative. Then,  $\forall \lambda > 0$ , then:

$$m(f^{-1}([\lambda, +\infty])) \leq \frac{1}{\lambda} \int_A f$$

**Corollary 1.** Let  $f$  be a non-negative, measurable function on  $A$ . Then,  $f = 0$  a.e. in  $A \iff \int_A f = 0$ .

**Corollary 2.** Let  $f$  be non-negative, measurable on  $A$ . If  $f$  is integrable over  $A$ , then  $f < \infty$  a.e. in  $A$ .

**Lemma 14** (Fatou's Lemma). Let  $(f_n)_{n \in \mathbb{N}}$  be a sequence of non-negative, measurable functions on  $A \subseteq \mathbb{R}$ . Then  $\liminf_{n \rightarrow \infty} f_n$  is measurable and

$$\int_A \liminf_{n \rightarrow \infty} f_n \leq \liminf_{n \rightarrow \infty} \int_A f_n \quad (19)$$

In particular, if  $(\int_A f_n)_{n \in \mathbb{N}}$  is bounded by  $M < \infty$ , then  $\liminf_{n \rightarrow \infty} f_n$  is integrable and  $\int_A \liminf_{n \rightarrow \infty} f_n \leq M$ .

**Theorem 15** (Monotone Convergence Theorem). Let  $(f_n)_{n \in \mathbb{N}}$  be a sequence of non-negative, measurable functions on  $A \subseteq \mathbb{R}$  such that  $\forall n \in \mathbb{N}, f_n \leq f_{n+1}$  (so that the  $\lim_{n \rightarrow \infty} f_n(x)$  exists in  $[0, \infty] \forall x \in A$  and  $\lim_{n \rightarrow \infty} \int_A f_n$  exists in  $[0, \infty]$ ), then

$$\int_A \lim_{n \rightarrow \infty} f_n = \lim_{n \rightarrow \infty} \int_A f_n \quad (20)$$

**Corollary 3.** Let  $(U_n)_{n \in \mathbb{N}}$  be a sequence of non-negative, measurable functions on  $A \subseteq \mathbb{R}$ . Then:

$$\int_A \sum_{n=1}^{\infty} U_n = \sum_{n=1}^{\infty} \int_A U_n \quad (21)$$

**Definition 16** (Integral in the case of Possibly Sign-Changing Functions). We say that a measurable function  $f : A \rightarrow \overline{\mathbb{R}}$  is **integrable** over  $A$  if  $f_+ := \max\{f, 0\}$  and  $f_- := \max\{-f, 0\}$  are integrable. We then denote:

$$\int_A f := \int_A f_+ - \int_A f_- \quad (22)$$

$\forall B \subseteq A$  measurable,  $\int_B f = \int_B f|_B$ .

**Proposition 25.**  $f$  is Lebesgue integrable  $\iff |f|$  is Lebesgue integrable.

**Proposition 26.** Let  $f, g$  be integrable over  $A \subseteq \mathbb{R}$ . Then:

(i)  $\forall \alpha, \beta \geq 0, \alpha f + \beta g$  is non-negative and measurable and :

$$\int (\alpha f + \beta g) = \alpha \int f + \beta \int g$$

(ii)  $f \leq g$  on  $A \Rightarrow \int_A f \leq \int_A g$ .

(iii)  $\forall B \subseteq A$  measurable,  $\chi_B f$  is non-negative, measurable, and

$$\int_A \chi_B f = \int_B f$$

(iv)  $\forall A_1, A_2 \subseteq A$  disjoint, measurable. Then:

$$\int_{A_1 \cup A_2} f = \int_{A_1} f + \int_{A_2} f$$

If, moreover,  $m(A_2) = 0$ , then

$$\int_{A_2} f = 0 \text{ and so } \int_{A_1 \cup A_2} f = \int_{A_1} f$$

**Theorem 16** (Dominated Convergence Theorem). Let  $(f_n)_{n \in \mathbb{N}}$  be a sequence of measurable functions on  $A \subseteq \mathbb{R}$  such that

(i) (Uniformly bounded)  $\exists$  a  $g$  integrable over  $A$  so that  $\forall n \in \mathbb{N} |f_n| \leq g$ .

(ii) (Pointwise convergence)  $\exists f : A \rightarrow \overline{\mathbb{R}}$  such that  $f_n \rightarrow f$  pointwise a.e. in  $A$ .

Then, the functions  $f_n$  and  $f$  are integrable and

$$\int_A f = \lim_{n \rightarrow \infty} \int_A f_n$$

**Corollary 4** (Countable Additivity of Lebesgue Integration). Let  $f$  be integrable over  $A \subseteq \mathbb{R}$  and let  $(A_n)_{n \in \mathbb{N}}$  be measurable, disjoint subsets of  $A$ . Then:

$$\int_{\bigcup_{n=1}^{\infty} A_n} f = \sum_{n=1}^{\infty} \int_{A_n} f \quad (23)$$

**Corollary 5** (Continuity of Lebesgue Integration). Let  $(X, \mathcal{F}, \mu)$  be a measure space and let  $f$  be integrable over  $A \subseteq X$ . Then, if:

- (i) If  $(A_n)_{n \in \mathbb{N}}$  is an increasing sequence of measurable subsets of  $A$  (that is,  $A_n \subseteq A_{n+1} \forall n \in \mathbb{N}$ ), then:

$$\int_{\bigcup_{n \in \mathbb{N}} A_n} f d\mu = \lim_{n \rightarrow \infty} \int_{A_n} f d\mu$$

- (ii) If  $(A_n)_{n \in \mathbb{N}}$  is a decreasing sequence of measurable subsets of  $A$  (that is,  $A_{n+1} \subseteq A_n \forall n \in \mathbb{N}$ ), then:

$$\int_{\bigcap_{n \in \mathbb{N}} A_n} f d\mu = \lim_{n \rightarrow \infty} \int_{A_n} f d\mu$$

#### 4. INTEGRATION AND DIFFERENTIATION

**Definition 17** (Differentiable). A function  $f$  is **differentiable** if  $D_*(f) = D^*(f) < \infty$ , where

$$D_*(f) := \liminf_{t \rightarrow 0} \frac{f(x+t) - f(x)}{t}$$

$$D^*(f) := \limsup_{t \rightarrow 0} \frac{f(x+t) - f(x)}{t}$$

**Theorem 17** (Monster Theorem). Every monotone function  $f : [a, b] \rightarrow \mathbb{R}$  is differentiable a.e. in  $[a, b]$ . Furthermore,  $f'$  is integrable over  $[a, b]$  and:

- (i) If  $f$  is increasing, then  $\int_a^b f' \leq f(b) - f(a)$ .  
(ii) If  $f$  is decreasing, then  $\int_a^b f' \geq f(b) - f(a)$ .

**Definition 18** (Bounded Variation). We say that a function  $f : [a, b] \rightarrow \mathbb{R}$  is of **bounded variation** if  $\text{TV}(f) < \infty$ , where:

$$\text{TV}(f) := \sup \left\{ \sum_{k=0}^{N-1} |f(x_{k+1}) - f(x_k)| \mid a = x_0 < x_1 < \dots < x_N = b \right\} \quad (24)$$

$\text{TV}(f)$  is called the **total variation** of  $f$ .

**Proposition 27.** Let  $f : [a, b] \rightarrow \mathbb{R}$  and let  $c \in ]a, b[$ . Then:

$$\text{TV}(f) = \text{TV}(f|_{[a,c]}) + \text{TV}(f|_{[c,b]})$$

**Theorem 18** (Characterisation of Functions of Bounded Variation). A function  $f : [a, b] \rightarrow \mathbb{R}$  is of bounded variation  $\iff$  it can be written as the difference of two increasing functions. In particular, every function  $f : [a, b] \rightarrow \mathbb{R}$  is differentiable a.e. in  $[a, b]$  and  $f'$  is integrable over  $[a, b]$ .

**Definition 19** (Absolutely Continuous). We say that a function  $f : [a, b] \rightarrow \mathbb{R}$  is **absolutely continuous** if  $\forall \varepsilon > 0, \exists \delta_\varepsilon > 0$  such that  $\forall$  finite collections of open, bounded intervals that are disjoint  $]a_1, b_1[, \dots, ]a_N, b_N[$ , if

$$\sum_{k=1}^N |b_k - a_k| < \delta \implies \sum_{k=1}^N |f(b_k) - f(a_k)| < \varepsilon$$

**Theorem 19.** Every absolutely continuous function  $f : [a, b] \rightarrow \mathbb{R}$  can be written as the difference of two increasing and absolutely continuous functions. In particular, it is of bounded variation.

**Theorem 20.** Let  $f : [a, b] \rightarrow \mathbb{R}$ . Then:



(i) If  $f$  is absolutely continuous on  $[a, b] \forall x \in [a, b]$ , then

$$\int_{[a,x]} f' = f(x) - f(a)$$

(ii) Conversely, if  $\exists$  a  $g$  integrable over  $[a, b]$  such that  $\forall x \in [a, b]$ ,  $\int_{[a,x]} g = f(x) - f(a)$ , then  $f$  is absolutely continuous and  $f' = g$  a.e. in  $[a, b]$ .

**Lemma 21.** Let  $h$  be integrable over  $[a, b]$ . Then,  $h = 0$  a.e. in  $[a, b] \iff \forall x < y \in ]a, b[$

$$\int_{]x,y[} h = 0$$

**Corollary 6.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be monotone. Then,  $f$  is absolutely continuous on  $[a, b] \iff$

$$\int_{]a,b[} f' = f(b) - f(a)$$

**Corollary 7.** Every function  $f : [a, b] \rightarrow \mathbb{R}$  of bounded variation can be written as  $f = f_{\text{abs}} + f_{\text{sing}}$ , where  $f_{\text{abs}}$  is absolutely continuous and  $f'_{\text{sing}} = 0$  a.e. in  $]a, b[$ .

## 5. LEBESGUE MEASURE AND INTEGRATION IN $\mathbb{R}^d$ , $d \geq 2$

**Definition 20** (Outer Measure). Let  $A \subseteq \mathbb{R}^d$ . We define the **outer measure** of  $A$  as:

$$m^*(A) := \inf \left\{ \sum_{k=1}^{\infty} \text{Vol}(R_k) \mid R_k = ]a_{k_1}, b_{k_1}[ \times \cdots \times ]a_{k_d}, b_{k_d}[ \text{ open, bd rectangles covering } A \right\} \quad (25)$$

where

$$\text{Vol}(R_k) := \prod_{i=1}^d (b_{k_i} - a_{k_i})$$

**Proposition 28.** Every open set  $\mathcal{O} \subseteq \mathbb{R}^d$  can be written as a countable union of almost disjoint closed cubes.

For the next family of theorems, we are in the following set-up. Let  $d_1, d_2 \in \mathbb{N}$  be such that  $d_1 + d_2 = d$ . For every  $E \subseteq \mathbb{R}^d$  and  $(x, y) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} = \mathbb{R}^d$ . We denote

$$E_{x_0} := \left\{ y \in \mathbb{R}^{d_2} \mid (x_0, y) \in E \right\}$$

$$E_{y_0} := \left\{ x \in \mathbb{R}^{d_1} \mid (x, y_0) \in E \right\}$$

and  $\forall f : E \rightarrow \mathbb{R}$

$$f_{x_0} := \begin{cases} E_{x_0} \rightarrow \overline{\mathbb{R}} \\ y \mapsto f(x_0, y) \end{cases}$$

$$f_{y_0} := \begin{cases} E_{y_0} \rightarrow \overline{\mathbb{R}} \\ x \mapsto f(x, y_0) \end{cases}$$

**Theorem 22** (Fubini's Theorem in  $\mathbb{R}^d$ ). Let  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  be integrable over  $\mathbb{R}$ . Then:

- (i) (Existence of the Integral I) For almost every  $y \in \mathbb{R}^{d_2}$ ,  $f_y$  is integrable over  $\mathbb{R}^{d_1}$ .
- (ii) (Existence of the Integral II)  $y \mapsto \int_{\mathbb{R}^{d_1}} f_y$  is integrable over  $\mathbb{R}^{d_2}$ .
- (iii) (Fubini's Theorem)

$$\int_{\mathbb{R}^{d_2}} \left( \int_{\mathbb{R}^{d_1}} f(x, y) dx \right) dy = \int_{\mathbb{R}^d} f \quad (26)$$

**Theorem 23** (Tonelli's Theorem; Fubini's Theorem for non-negative measurable functions). Let  $f : \mathbb{R}^d \rightarrow [0, \infty]$  be measurable. Then:

- (i) (Existence of the Integral I) For almost every  $y \in \mathbb{R}^{d_2}$ ,  $f_y$  is non-negative and measurable over  $\mathbb{R}^{d_1}$ .
- (ii) (Existence of the Integral II)  $y \mapsto \int_{\mathbb{R}^{d_1}} f_y$  is non-negative, measurable over  $\mathbb{R}^{d_2}$ .
- (iii) (Fubini's Theorem)

$$\int_{\mathbb{R}^{d_2}} \left( \int_{\mathbb{R}^{d_1}} f_y dx \right) dy = \int_{\mathbb{R}^d} f \quad (27)$$

**Corollary 8.** Let  $E \subseteq \mathbb{R}^d$  be measurable. Then:

- (i) For a.e.  $y \in \mathbb{R}^{d_2}$ ,  $E_y$  is measurable.
- (ii)  $y \mapsto m(E_y)$  is measurable.
- (iii)  $m(E) = \int_{\mathbb{R}^{d_2}} m(E_y) dy$ .

**Corollary 9** (General Version of Tonelli's Theorem).<sup>1</sup> Let  $E \subseteq \mathbb{R}^d$  be measurable, and let  $f : E \rightarrow [0, \infty]$  be measurable. Then:

- (i) For almost every  $y \in \mathbb{R}^{d_2}$ ,  $f$  is non-negative and measurable on  $E_y$ .
- (ii)  $y \mapsto \int_{E_y} f_y$  is non-negative, measurable, on  $\mathbb{R}^{d_2}$ .
- (iii)

$$\int_{\mathbb{R}^{d_2}} \int_{E_y} f_y = \int_E f$$

**Corollary 10** (General Version of Fubini's Theorem). Let  $E \subseteq \mathbb{R}^d$  be measurable and let  $f : E \rightarrow \overline{\mathbb{R}}$  be measurable. Then:

- (i) For almost every  $y \in \mathbb{R}^{d_2}$ ,  $f_y$  is integrable on  $E_y$ .
- (ii)  $y \mapsto \int_{E_y} f_y$  is measurable on  $\mathbb{R}^{d_2}$ .
- (iii)

$$\int_{\mathbb{R}^{d_2}} \int_{E_y} f_y = \int_E f$$

**Theorem 24.** Let  $E_1$  and  $E_2$  be measurable sets in  $\mathbb{R}^{d_1}$  and  $\mathbb{R}^{d_2}$  respectively. Then  $E_1 \times E_2$  is measurable and

$$m(E_1 \times E_2) = \begin{cases} m(E_1) \times m(E_2) & \text{if } m(E_1) \neq 0 \wedge m(E_2) \neq 0 \\ 0 & \text{else} \end{cases}$$

**Corollary 11.** Let  $E_1, E_2$  be two measurable sets,  $E_1 \subseteq \mathbb{R}^{d_1}$  and  $E_2 \subseteq \mathbb{R}^{d_2}$ . Let  $f : E_1 \rightarrow \overline{\mathbb{R}}$  be measurable. Then:

$$\tilde{f} := \begin{cases} E_1 \times E_2 \rightarrow \overline{\mathbb{R}} \\ \tilde{f}(x, y) = f(x) \end{cases}$$

is measurable as a function of  $E_1 \times E_2$ .

**Theorem 25** (Formula for the Integral of a non-negative measurable function in terms of a region in  $\mathbb{R}^d$ ). Assume that  $d_1 = d - 1$  and  $d_2 = 1$ . Let  $E_1 \subseteq \mathbb{R}^{d-1}$  be measurable and consider  $f : E_1 \rightarrow [0, \infty]$ .

- (i)  $f$  is measurable  $\iff$  the set  $A$ :

$$A := \{(x, y) \in E_1 \times \mathbb{R} \mid 0 < y < f(x)\}$$

is measurable.

- (ii) Moreover, if  $f$  is measurable, then

$$m(A) = \int_{E_1} f \quad (28)$$

<sup>1</sup>The difference between points i and ii here vs. Tonelli's theorem is that we cannot fix  $x$  here.