Math 475: Partial Differential Equations

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1 Introduction

Q: Where do PDE's come from? PDEs are used in all types of math modelling; they translate phenomena coming from physics, biology, chemistry, etc. into mathematical terms.

Q: What goes into a PDE model?

Math model = General Physical Laws (balancing forces and conservation of quantities) + Constitutive Relations (Laws specific to the environment, e.g. Fick's Law of Diffusion)

A PDE $\leftrightarrow$ a physical description of a system.

Definition 1 (PDE)**.** A **PDE** is a mathematical relation involving partial derivatives, i.e., if $\mathbf{u} : \mathbb{R}^n \to \mathbb{R}$, $\mathbf{x} = (x_1, ..., x_n)$ a real-valued function of several variables, then any kth order PDE can be expressed as:

$$F(D^k\mathbf{u}, D^{k-1}\mathbf{u}, ..., D\mathbf{u}, \mathbf{u}, \mathbf{x}) = 0 \text{ in } \Omega \tag{1}$$

where $\Omega \subseteq \mathbb{R}^n$ is the domain or region where the PDE holds, and F is a function of the placeholders.

Definition 2 (C^k and $C^k(\Omega)$)**.** A function $\mathbf{u} : \mathbb{R}^n \to \mathbb{R}$ is C^k at a point $\mathbf{x} \in \mathbb{R}^n$ if every kth-order partial derivative is continuous as a function of $\mathbb{R}^n$ at x. If $u \in C^2(\mathbb{R}^n)$, then in advanced calculus it was proven that D^2u is a symmetric matrix.

Example 1 (Examples of PDEs)**.** 1. **Heat Equation**:

$$u_t - k\Delta u = u_t - k\sum_{i=1}^{n} u_{x_i x_i} = 0 \tag{2}$$

where $k > 0$ is a constant. Then the solution has a space- and time- component:

$$u(\mathbf{x}, t) = u : \mathbb{R}^{n+1} \to \mathbb{R} \tag{3}$$

2. **Laplace's Equation**:

$$-\Delta u = 0 \tag{4}$$

$u(x)$ is the steady state of a solution to the heat equation, since $u_t = 0 \Rightarrow$ "constant in time." The solution u is a function $u : \mathbb{R}^n \to \mathbb{R}$.

3. **Wave Equation**:

$$
\begin{aligned}
u_{tt} - \Delta u &= 0 \\
u : \mathbb{R}^{n+1} &\to \mathbb{R}
\end{aligned}
$$

$u(x,t)$ is the displacement of an object with wave-like behaviour at location $\mathbf{x}$ and time t. Ex: position of a guitar string.

4. **Transport Equation**:

$$u_t + cu_x = 0,\ c \in \mathbb{R}, u : \mathbb{R}^2 \to \mathbb{R} \text{ (space, time)}$$

$u(x,t)$ can be the density of a pollutant at location $\mathbf{x}$ and time t.

5. **Reaction-Diffusion**:

$$
\begin{aligned}
u_t - k\Delta u &= f(u) \\
u : \mathbb{R}^{n+1} &\to \mathbb{R} \\
f : \mathbb{R} &\to \mathbb{R}
\end{aligned}
$$

$u(\mathbf{x},t)$ can be the temperature at location $(\mathbf{x},t)$ subject to enhancement by $f(u)$ (for example, fire spreading).

6. **Burger's Equation**:

$$u_t - uu_x = vu_{xx} \tag{5}$$

$v > 0$ represents the viscosity. $u : \mathbb{R}^2 \to \mathbb{R}$, $u(x,t)$ is the concentration of a material in a fluid flow with convection.

1.1 Domains and Boundary Conditions

Definition 3 (C^1 domain)**.** $\Omega \subseteq \mathbb{R}^n$ is a **C^1 domain** if $\partial\Omega$ can locally be expressed as a graph of a C^1 function. This means...

1. $\partial\Omega$ has no corners $\Rightarrow$ smooth.
2. $\forall p \in \partial\Omega$, there exists a well-defined and unique tangent plane (whose slope is given by the derivative), which implies that there exists a well-defined inward and outward normal vector.
3. Inward and outward normal vectors move continuously along $\partial\Omega$.

1.1.1 What are the main boundary conditions?

1. **Dirichlet Boundary Conditions**: these prescribe what u is on $\partial\Omega$. We assume that $u \in C^k(\Omega) \cap C(\partial\Omega)$.
2. **Neumann Boundary Conditions**: these prescribe what the normal derivative of u on $\partial\Omega$ is. The meaning behind this is: *how does u change along the boundary*? It is specifying:

$$\frac{\partial u}{\partial n} = \nabla u \cdot n(x) = X \text{ on } \partial\Omega \tag{6}$$

3. **Robin boundary conditions**: a combination of the above:

$$\frac{\partial u}{\partial n} + \alpha u = X \text{ on } \partial\Omega \tag{7}$$

In general, in PDEs, we will not be able to identify an explicit solution. Thus, we care about the following four fundamental issues. The first three ensure that we have a *well-posed problem*, and the final one is important when we cannot obtain an explicit solution.

1. *Existence*: is there a solution?
2. *Uniqueness*: is there exactly one solution to the PDE?
3. *Stability*: does the solution depend continuously on the data?
4. *Qualitative Properties*: If I cannot find an explicit solution, but I know that it exists, what else can I tell you about the solution? Some questions we are interested in studying are:
   a) Does $u(\mathbf{x},t) \to 0$ as $t \to \infty$?
   b) What is $\max_\Omega |u(\mathbf{x})|$?

1.2 Classification of PDEs

To do: draw out the chart