Chapter 8: The L^p Spaces: Duality and Weak Convergence

Class: Math 455 (Analysis 4) Date: 7 January 2020 Shereen Elaidi

Abstract

This document contains a summary of all the key definitions, results, and theorems from class. There are probably typos, and so I would be grateful if you brought those to my attention :-).

Syllabus: L^p space, duality, weak convergence, Young, Holder, and Minkowski inequalities, point-set topology, topological space, dense sets, completeness, compactness, connectedness, path-connectedness, separability, Tychnoff theorem, Stone-Weierstrass Theorem, Arzela-Ascoli, Baire category theorem, open mapping theorem, closed graph theorem, uniform boudnedness principle, Hahn Banch theorem.

8.1. RIESZ REPRESENTATION THEOREM FOR THE DUAL OF L^p, $1 \leq p < \infty$

Definition 1 (Linear Functional)**.** A **linear functional** on a linear space X is a real-valued function T on T such that for g and g in X and α and β real numbers,

$$T(\alpha \cdot g + \beta \cdot h) = \alpha \cdot T(g) + \beta \cdot T(h) \tag{8.1}$$

Definition 2 (Bounded)**.** For a normed linear space X, a linear functional T on X is said to be **bounded** provided there is an $M \geq 0$ for which

$$|T(f)| \leq M \cdot ||f|| \text{ for all } f \in X \tag{8.2}$$

The infimum of all such M is called the **norm** of T and is denoted by $||T||_*$.

Proposition 1 (Continuity Property of a Bounded Linear Functional)**.** Let T be a bounded linear functional on the normed space X. Then, if $\{f_n\} \to f$ in X, then $\{T(f_n)\} \to \{T(f)\}$.

Proposition 2. Let X be a normed vector space. Then, the collection of bounded linear functionals on X is a linear space which is normed by $|| \cdot ||_*$. This normed vector space is called the **dual space** of X, and is denoted by X^*.

Proposition 3. Let $E \subseteq \mathbb{R}$ be measurable, $1 \leq p < \infty$, q the conjugate of p, $g \in L^q(E)$. Define the functional T on $L^p(E)$ by:

$$T(f) := \int_E g \cdot f \ \forall f \in L^p(E) \tag{8.3}$$

Then, T is a bounded linear functional on $L^p(E)$ and $||T||_* = ||g||_q$.

Proposition 4. Let T, S be bounded linear functionals on the normed vector space X. If $T = S$ on a dense subset X_0 of X, then $T = S$.

Lemma 1. Let $E \subseteq \mathbb{R}$ be measurable, $1 \leq p < \infty$. Suppose that g is integrable over E and there exists a $M \geq 0$ for which

$$\left| \int_E g \cdot f \right| \leq M||f||_p \ \forall f \in L^p(E), \ f \text{ simple}$$

Then, $g \in L^q(E)$, where q is the conjugate of p. Moreover, $||g||_q \leq M$.

Theorem 2. Let $[a, b]$ be a closed, bounded interval, and $1 \leq p < \infty$. Suppose that T is a bounded linear functional on $L^p[a, b]$. Then, there is a functional $g \in L^q[a, b]$, where q is the conjugate of p, for which:

$$T(f) = \int_a^b g \cdot f \ \forall f \in L^p[a, b] \tag{8.4}$$

Theorem 3 (Riesz-Representation Theorem for the Dual of $L^p(E)$)**.** Let $E \subseteq \mathbb{R}$ be measurable, $1 \leq p < \infty$, and q the conjugate of p. For all $g \in L^q(E)$, define the bounded linear functional $\mathcal{R}_g$ on $L^p(E)$ by:

$$\mathcal{R}_g := \int_E g \cdot f \ \forall f \in L^p(E) \tag{8.5}$$

Then, for each bounded linear functional T on $L^p(E)$, there exists a unique $g \in L^q(E)$ for which

(i) $\mathcal{R}_g = T$ and
(ii) $||T||_* = ||g||_q$

8.2. Weak Sequential Convergence in L^p

Definition 3 (Converge Weakly)**.** Let X be a normed vector space. A sequence $\{f_n\}$ in X is said to **converge weakly** in X to f provided that

$$\lim_{n\to\infty} T(f_n) = T(f) \ \forall T \in X^* \tag{8.6}$$

we write

$$\{f_n\} \rightharpoonup f$$

to mean that f and each f_n belong to X and $\{f_n\}$ converges weakly in X to f.

Definition 4. Let $E \subseteq \mathbb{R}$ be measurable, $1 \leq p < \infty$, q the conjugate of p. Then, $\{f_n\} \rightharpoonup f$ in $L^p(E)$ $\iff$

$$\lim_{n\to\infty} \int_E g \cdot f_n = \int_E g \cdot f \ \forall g \in L^q(E) \tag{8.7}$$

Theorem 4. Let $E \subseteq \mathbb{R}$ be measurable, $1 \leq p < \infty$. Suppose that $\{f_n\} \rightharpoonup f$ in $L^p(E)$. Then:

$$\{f_n\} \text{ is bounded and } ||f||_p \leq \liminf ||f_n||_p$$

Corollary 1. Let $E \subseteq \mathbb{R}$ be measurable, $1 \leq p < \infty$, q the conjugate of p. Suppose $\{f_n\}$ converges weakly to f in $L^p(E)$ and $\{g_n\}$ converges strongly to $g \in L^q(E)$. Then:

$$\lim_{n\to\infty} \int_E g_n \cdot f_n = \int_E g \cdot f \tag{8.8}$$

Definition 5 (Linear Span)**.** Let X be a normed vector space, and let $S \subseteq X$. Then, the **linear span of** S is the vector space consisting of all linear functionals of the form:

$$f = \sum_{k=1}^{n} \alpha_k \cdot f_k \tag{8.9}$$

where each $\alpha_k \in \mathbb{R}$ and $f_k \in S$. It is the set of all *finite linear combinations of elements in S*. We care about this since L^p is an infinite dimensional space, so we want to find a way to approximate it with finitely many elements.

Proposition 5 (Characterisation of Weak Convergence in $L^p(E)$)**.** Let $E \subseteq \mathbb{R}$ be measurable, $1 \leq p < \infty$, q the conjugate of P. Assume that $\mathcal{F} \subseteq L^q(E)$ whose linear span is dense in $L^q(E)$. Let $\{f_n\}$ be a bounded sequence in $L^p(E)$, and let $f \in L^p(E)$. Then, $\{f_n\} \rightharpoonup f$ in $L^p(E) \iff$

$$\lim_{n\to\infty} \int_E f_n \cdot g = \int_E f \cdot g \;\forall g \in \mathcal{F} \tag{8.10}$$

Theorem 5. Let $E \subseteq \mathbb{R}$ be measurable, $1 \leq p < \infty$. Suppose that $\{f_n\}$ is a bounded sequence in $L^p(E)$ and f belongs to $L^p(E)$. Then, $\{f_n\} \rightharpoonup f$ in $L^p(E) \iff \forall$ measurable sets $A \subseteq E$:

$$\lim_{n\to\infty} \int_A f_n = \int_A f \tag{8.11}$$

if $p > 1$, then it is sufficient to consider sets A of finite measure.

Theorem 6. Let $[a, b]$ be a closed and bounded interval, $1 < p < \infty$. Suppose that $\{f_n\}$ is a bounded sequence in $L^p[a, b]$ and $f \in L^p[a, b]$. Then, $\{f_n\} \rightharpoonup f$ in $L^p(E)$ in $L^p[a, b] \iff$

$$\lim_{n\to\infty} \left[\int_a^x f_n\right] = \int_a^x f \;\forall x \in [a, b] \tag{8.12}$$

Lemma 7 (Riemann-Lebesgue Lemma; used in Fourier Series :-))**.** Let $I = [-\pi, \pi]$, $1 \leq p < \infty$. $\forall n \in \mathbb{N}$, define $f_n(x) := \sin(nx)$ for $x \in I$. Then, $\{f_n\}$ converges weakly in $L^p(I)$ to $f \equiv 0$.

Theorem 8. Let $E \subseteq \mathbb{R}$ be measurable, $1 < p < \infty$. Suppose that $\{f_n\}$ is a bounded sequence in $L^p(E)$ that converges pointwise a.e. on E to f. Then, $\{f_n\} \rightharpoonup f$ in $L^p(E)$.

This theorem was used in the proof but was not covered in Analysis 3:

> **Theorem 9** (Vitali Convergence Theorem)**.** Let $E \subseteq \mathbb{R}$ be measurable and of finite measure. Suppose that the sequence of functions $\{f_n\}$ is uniformly integrable over E. Then, if $\{f_n\} \to f$ pointwise a.e. on E, then f is integrable over E and $\lim_{n\to\infty} \int_E f_n = f$.

Theorem 10 (Radon-Riesz Theorem)**.** Let $E \subseteq \mathbb{R}$ be measurable, $1 < p < \infty$. Suppose that $\{f_n\} \rightharpoonup f$ in $L^p(E)$. Then:

$$\{f_n\} \to f \text{ in } L^p(E) \iff \lim_{n\to\infty} ||f_n||_p = ||f||_p \tag{8.13}$$

Corollary 2. (Not Covered in Class): Let $E \subseteq \mathbb{R}$ be measurableand $1 < p < \infty$. Suppose that $\{f_n\} \rightharpoonup f$ in $L^p(E)$. Then, a subsequence of $\{f_n\}$ converges strongly to $f \iff ||f||_p = \liminf ||f_n||_p$.

8.3. Weak Sequential Compactness ("Compactness Found!")

Theorem 11. Let $E \subseteq \mathbb{R}$ be measurable, $1 < p < \infty$. Then, every bounded sequence in $L^p(E)$ has a subsequence that converges weakly in $L^p(E)$ to a function in $L^p(E)$.

Theorem 12 (Helly's Theorem)**.** Let X be a *SEPARABLE* normed vector space and $\{T_n\}$ a sequence in its dual space X^* that is bounded; that is, $\exists$ a $M > 0$ for which

$$|T_n(f)| \leq M \cdot ||f|| \;\forall f \in X, \;\forall n \in \mathbb{N}$$

Then, there is a subsequence $\{T_{n_k}\}$ of $\{T_n\}$ and $T \in X^*$ for which

$$\lim_{k\to\infty} T_{n_k}(f) = T(f) \;\forall f \in X \tag{8.14}$$

Definition 6 (Weakly Sequentially Compact (Compact in the "weak topology")**.** Let X be a normed vector space. Then, a subset $K \subseteq X$ is **weakly sequentially compact** in X provided that every sequence $\{f_n\}$ in K has a subsequence that converges weakly to $f \in K$.

Theorem 13 (The Unit Ball is Weakly Sequentially Compact)**.** Let $E \subseteq \mathbb{R}$ be measurable, $1 < p < \infty$. Define:

$$B_1 := \{f \in L^p(E) \mid ||f||_p \leq 1\}. \tag{8.15}$$

B_1 is weakly sequentially compact in $L^p(E)$.