Math 458: Differential Geometry

Shereen Elaidi

Winter 2020 Term

Contents

1 Introduction

1.1 Implicit and Inverse Function Theorems

2 Manifolds in $\mathbb{R}^3$

The aim of this part of the course is to build up to integration on manifolds and the invariant Stokes' theorem. The main purpose of this sections is to develop *coordinate-free* calculus, which clarifies the essence of what is happening (sometimes coordinates can be noisy).

2.1 Definitions

2.2 Smooth Maps from $M^m \to N^n$

2.3 Change of Coordinates

2.4 Multi-Linear Algebra

2.5 Differential Forms in M^n

2.6 Change of Variables for Integrals in $\mathbb{R}^n$

2.7 Integrating a n-Form on M^n ($\int_M \omega$)

3 Curves

3.1 Definitions

3.2 Frenet-Serret Frame

3.3 Global Properties of Curves

3.3.1 The Isoperimetric Inequality

3.3.2 Cauchy Crofton Formula

4 Surfaces

4.1 Definitions

Motivation: we want to define a regular surface to be something that is nice enough for us to extend the usual notions of calculus to.

Definition 1 (Regular Surface)**.** A subset $S \subseteq \mathbb{R}^3$ is called a **regular surface** if, $\forall\, p \in S$, there exists a neighbourhood $V \subseteq \mathbb{R}^3$ and a map $\mathbb{X} : U \to V \cap S$ of an open set $V \subseteq \mathbb{R}^2$ onto $V \cap S \subseteq \mathbb{R}^3$ for which the following conditions hold:

1. $\mathbb{X}$ is differentiable; that is, if we write

$$\mathbb{X}(u, v) = (x(u, v), y(u, v), z(u, v))$$

for $(u, v) \in U$, then the functions $x(u, v)$, $y(u, v)$ and $z(u, v)$ have continuous partial derivatives of all orders in U.

2. $\mathbb{X}$ is a **homeomorphism**: there exists an inverse $\mathbb{X}^{-1} : V \cap S \to U$, which is continuous.
3. (Regularity Condition): $\forall q \in U$, the differential $d\mathrm{x}_q : \mathbb{R}^2 \to \mathbb{R}^3$ is bijective.

Then, the mapping $\mathbb{X}$ is called a **parameterisation** or a **system of local coordinates** in a neighbourhood of p. The neighbourhood $V \cap S$ of p is called a **coordinate neighbourhood**.

4.2 Differentiable Functions on Surfaces

4.3 Tangent Plane

4.4 First Fundamental Form: Area

5 The Gauss Map

5.1 Ruled Surfaces and Minimal Surfaces

6 The Intrinsic Geometry of Surfaces

6.1 Isometries and Conformal Maps