Math 455: Analysis IV Summary
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Key Results, Theorems, Definitions, etc.
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Abstract

This document contains a summary of all the key definitions, results, and theorems from class. There are probably typos, and so I would be grateful if you brought those to my attention :-).

Syllabus: L^p space, duality, weak convergence, Young, Holder, and Minkowski inequalities, point-set topology, topological space, dense sets, completeness, compactness, connectedness, path-connectedness, separability, Tychnoff theorem, Stone-Weierstrass Theorem, Arzela-Ascoli, Baire category theorem, open mapping theorem, closed graph theorem, uniform boudnedness principle, Hahn Banch theorem.

CONTENTS

1. L^p SPACES: COMPLETENESS AND APPROXIMATION

1.1. NORMED VECTOR SPACES

Definition 1 (ℓ^p space). Let $(a_1, a_n, ...)$ be a sequence. Then, the ℓ^p-space is:

$$\ell^p := \left\{ (a_1, a_2, ...) \mid \sum_{n=1}^{\infty} |a_n|^p < +\infty \right\} \tag{1}$$

Theorem 1 (Riesz-Fisher). $L^p(X)$ is complete.

Definition 2 (L^p space). Let E be a measurable set and let $1 \le p < \infty$. Then, $L^p(E)$ is the collection of measurable functions f for which $|f|^p$ is Lebesgue integrable over E.

Definition 3 (Equivalent Functions). Let $\mathcal{F}$ be the collection of all measurable extended real-valued functions on E that are finite a.e. on E. Define two functions f and g to be equivalent, and write $f \sim g$ if $g(x) = f(x)$ a.e. on E.

Definition 4 (Essentially Bounded). We call a function $f \in \mathcal{F}$ to be **essentially bounded** if there exists some $M \ge 0$, called the **essential upper bound** for f, for which

$$|f(x)| \le M$$

for almost every $x \in E$. $L^\infty(E)$ is the collection of equivalence classes $[f]$ for which f is essentially bounded.

Definition 5 (Norm). Let X be a linear space. A real-valued functional $|| \cdot ||$ on X is called a **norm** provided that for each f and g in X and each real number α,

(1) (The Triangle Inequality).

$$||f + g|| \le ||f|| + ||g||$$

(2) (Positive Homogeneity).

$$||\alpha f|| = |\alpha| ||f||$$

(3) (Non-Negativity).

$$||f|| \ge 0 \text{ and } ||f|| = 0 \text{ if and only if } f = 0$$

Definition 6 (Normed Linear Space). X is said to be a **normed linear space** if X is equipped with a norm.

Definition 7 (Essential Supremum). Let $f \in L^\infty(E)$. $||f||_\infty$ is called the **essential supremum** and is defined as:

$$||f||_\infty := \{M \mid M \text{ is an essential upper bound for } f\}$$

Theorem: $|| \cdot ||_\infty$ is a norm on $L^\infty(E)$.

1.2. THE INEQUALITIES OF YOUNG, HÖLDER, AND MINKOWSKI

Definition 8 (p-norm)**.** Let E be a measurable set, $1 < p < \infty$, and let $f \in L^p(E)$. Then, define the **p-norm** to be:

$$||f||_p := \left[\int_E |f|^p \right]^{\frac{1}{p}} \tag{2}$$

Definition 9 (Conjugate)**.** The **conjugate** of a number $p \in]1, \infty[$ is the number $q = p/(p-1)$, which is the unique number $q \in]1, \infty[$ for which

$$\frac{1}{p} + \frac{1}{q} = 1 \tag{3}$$

The conjugate of 1 is defined to be ∞ and the conjugate of ∞ is defined to be 1.

Definition 10 (Young's Inequality)**.** For $1 < p < \infty$, q the conjugate of p, and any two positive numbers a and b, we have:

$$ab \le \frac{a^p}{p} + \frac{b^q}{q} \tag{4}$$

Theorem 2 (Hölder's Inequality)**.** Let $E \subseteq \mathbb{R}$ be measurable, $1 \le p < \infty$, and q the conjugate of p. If f belongs to $L^p(E)$, and g belongs to $L^q(E)$, then their product $f \cdot g$ is integrable over E and:

$$\int_E |f \cdot g| \le ||f||_p \cdot ||g||_q. \tag{5}$$

Moreover, if $f \ne 0$, then the function defined as:

$$f^* := ||f||_p^{1-p} \cdot \operatorname{sgn}(f) \cdot |f|^{p-1} \tag{6}$$

belongs to $L^q(E)$,

$$\int_E f \cdot f^* = ||f||_p \text{ and } ||f^*||_q = 1$$

We call f^* defined as above to be called the **conjugate function** of f.

Theorem 3 (Minkowski's Inequality)**.** Let E be a measurable set and $1 \le p \le \infty$. If the functions f and g belong to $L^p(E)$, then so does their sum $f + g$. Moreover,

$$||f + g||_p \le ||f||_p + ||g||_p \tag{7}$$

Theorem 4 (Cauchy-Schwarz Inequality)**.** Let E be a measurable set and let f and g be measurable functions over E for which f^2 and g^2 are integrable over E. Then, $f \cdot g$ is integrable over E and

$$\int_E |f \cdot g| \le \sqrt{\int_E f^2} \cdot \sqrt{\int_E g^2} \tag{8}$$

Corollary 1. Let E be a measurable set and $1 < p < \infty$. Suppose $\mathcal{F}$ is a family of functions in $L^p(E)$ that is bounded in $L^p(E)$ in the sense that there is a constant M for which

$$||f||_p \le M \text{ for all } f \in \mathcal{F}$$

Then, the family $\mathcal{F}$ is uniformly integrable over E.

Corollary 2. Let E be a measurable set of finite measure and $1 \le p_1 < p_2 \le \infty$. Then, $L^{p_2}(E) \subseteq L^{p_1}(E)$. Furthermore,

$$||f||_{p_1} \le c||f||_{p_2}$$

for all f in $L^{p_2}(E)$, where $c = [m(E)]^{\frac{p_2 - p_1}{q_1 p_2}}$ if $p_2 < \infty$ and $c = [m(E)]^{\frac{1}{p_1}}$ if $p_2 = \infty$.

1.3. L^p IS COMPLETE: THE REISZ-FISCHER THEOREM

Definition 11 (Converge). A sequence $\{f_n\}$ in a linear space X normed by $||\cdot||$ is said to **converge to** f **in** X provided:

$$\lim_{n\to\infty} ||f - f_n|| = 0$$

Definition 12 (Cauchy). A sequence $\{f_n\}$ in a linear space X that is normed by $||\cdot||$ is said to be **Cauchy in** X provided for each $\varepsilon > 0$, there exists a $N \in \mathbb{N}$ such that

$$||f_n - f_m|| < \varepsilon \ \forall \ m, n \geq N \tag{9}$$

Definition 13 (Complete). A normed linear space X is called **complete** if every Cauchy sequence in X converges to a function in X. A complete normed linear space is called a **Banach space**.

Proposition 1. Let X be a normed linear space. Then, every convergent sequence in X is Cauchy. Moreover, a Cauchy sequence in X converges if it has a convergent subsequence.

Definition 14. Let X be a linear space normed by $||\cdot||$. A sequence $\{f_n\}$ in X is said to be **rapidly Cauchy** if there is a convergent series of positive numbers $\sum_{k=1}^{\infty} \varepsilon_k$ for which

$$||f_{k+1} - f_k|| \leq \varepsilon_k^2 \text{ for all } k$$

Proposition 2. Let X be a normed linear space. Then, every rapidly Cauchy sequence in X is Cauchy. Furthermore, every Cauchy sequence has a rapidly Cauchy subsequence.

Proposition 3. Let E be a measurable set and $1 \leq p \leq \infty$. Then, every rapidly Cauchy sequence in $L^p(E)$ converges with respect to the $L^p(E)$ norm and pointwise a.e. on E to a function in $L^p(E)$.

Theorem 5 (Riesz-Fischer Theorem). Let E be a measurable set and $1 \leq p \leq \infty$. Then $L^p(E)$ is a Banach space. Moreover, if $\{f_n\} \to f$ in $L^p(E)$, a subsequence of $\{f_n\}$ converges pointwise a.e. on E to f.

Theorem 6. Let E be a measurable set and $1 \leq p < \infty$. Suppose $\{f_n\}$ is a sequence in $L^p(E)$ that converges pointwise a.e. on E to the function f which belongs to $L^p(E)$. Then:

$$\{f_n\} \to f \text{ in } L^p(E) \iff \lim_{n\to\infty} \int_E |f_n|^p = \int_E |f|^p$$

Definition 15 (Tight). A family $\mathcal{F}$ of measurable functions on E is said to be **tight** over E provided that for each $\varepsilon > 0$, there exists a subset E_0 of E of finite measure for which

$$\int_{E\setminus E_0} |f| < \varepsilon \text{ for all } f \in \mathcal{F}$$

Theorem 7. Let E be a measurable set and let $1 \leq p < \infty$. Suppose $\{f_n\}$ is a sequence in $L^p(E)$ that converges pointwise a.e. on E to the function f which belongs to $L^p(E)$. Then, $\{f_n\} \to f$ in $L^p(E) \iff \{|f_n|^p\}$ is uniformly integrable and tight over E.

1.4.   APPROXIMATION AND SEPARABILITY

Definition 16 (Dense). Let X be a normed linear space with norm $||\cdot||$. Given two subsets $\mathcal{F}$ and $\mathcal{G}$ of X with $\mathcal{F} \subseteq \mathcal{G}$, we say that $\mathcal{F}$ is **dense** in $\mathcal{G}$ provided for each function g in $\mathcal{G}$ and $\varepsilon > 0$, there is a function $f \in \mathcal{F}$ for which $||f - g|| < \varepsilon$.

Proposition 4. Let E be a measurable set and let $1 \leq p \leq \infty$. Then, the subspace of simple functions in $L^p(E)$ is dense in $L^p(E)$.

Proposition 5. Let $[a, b]$ be a closed, bounded interval and $1 \leq p < \infty$. Then, the subspace of step functions on $[a, b]$ is dense in $L^p[a, b]$.

Definition 17 (Separable). A normed linear space X is said to be **separable** provided there is a countable subset that is dense in X.

Theorem 8. Let E be a measurable set and $1 \leq p < \infty$. Then, the normed linear space $L^p(E)$ is separable.

Theorem 9. Suppose E is measurable and let $1 \leq p < \infty$. Then, $C_c(E)$ (the set of all continuous functions with compact support on E) is dense in $L^p(E)$.

1.5.   RESULTS FROM THE HOMEWORK

(1) (When Hölder's inequality $\rightarrow$ equality): There is equality in Hölder's Inequality $\iff$ there exists constants α, β, both of which non-zero, for which:

$$\alpha|f|^p = \beta|g|^q$$

a.e. on E.

(2) (Extension of Hölder's Inequality for 3 functions): Let $E \subseteq \mathbb{R}$ be measurable, let $1 \leq p < \infty$, $1 \leq q < \infty$, $1 \leq r < \infty$ such that:

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$$

If $f \in L^p(E)$, $q \in L^q(E)$, and $h \in L^r(E)$, then $fgh \in L^($E)$ and:

$$\int_E |fgh| \leq ||f||_p ||g||_q ||h||_r$$

(3) For $1 \leq p \leq \infty$, q conjugate of p, $f \in L^p(E)$:

$$||f||_p = \max_{g \in L^q(E), ||g||_q \leq 1} \int_E fg$$

(4) (L^p dominated convergence theorem): Let $\{f_n\}$ be a sequence of measurable functions that converge pointwise a.e. on E to f. For $1 \leq p < \infty$, suppose $\exists$ a function $g \in L^p(E)$ such that $\forall \, n \in \mathbb{N}$, $|f_n| \leq g$ a.e. on E. Then, $\{f_n\} \rightarrow f$ in $L^p(E)$.

(5) Assume $1 \leq p < \infty$, if $E \subseteq \mathbb{R}$ has finite measure, $1 \leq p < \infty$, and $\{f_n\}$ is a sequence of measurable functions which converge pointwise a.e. on E to f, then $\{f_n\} \rightarrow f$ in $L^p(E)$ if $\exists$ a $\theta > 0$ such that $\{f_n\}$ belongs to and is <u>bounded</u> as a subset of $L^{p+\theta}(E)$.

(6) The space c of all convergent sequences of real numbers and the space c_0 of all sequences which converge to zero are Banach spaces with respect to the ℓ^∞ norm.

(7) Let $E \subseteq \mathbb{R}$ be measurable, $1 \leq p \leq \infty$, q the conjugate of p, and $\mathcal{S}$ a dense subset of $L^q(E)$. If $g \in L^p(E)$ and $\int_E g \cdot g = 0$ for all $g \in \mathcal{S}$, then $g = 0$.

(8) (Separability of ℓ^p): For $1 \leq p < \infty$, ℓ^p is separable. ℓ^∞ is <u>not</u> separable.

2. L^p Spaces: Duality and Weak Convergence

2.1. Riesz Representation Theorem for the Dual of L^p, $1 \leq p < \infty$

Definition 18 (Linear Functional). A **linear functional** on a linear space X is a real-valued function T on X such that for f and g in X and α and β real numbers,

$$T(\alpha \cdot g + \beta \cdot h) = \alpha \cdot T(g) + \beta \cdot T(h) \tag{10}$$

Definition 19 (Bounded). For a normed linear space X, a linear functional T on X is said to be **bounded** provided there is an $M \geq 0$ for which

$$|T(f)| \leq M \cdot ||f|| \text{ for all } f \in X \tag{11}$$

The infimum of all such M is called the **norm** of T and is denoted by $||T||_*$.

Proposition 6 (Continuity Property of a Bounded Linear Functional). Let T be a bounded linear functional on the normed space X. Then, if $\{f_n\} \to f$ in X, then $\{T(f_n)\} \to \{T(f)\}$.

Proposition 7. Let X be a normed vector space. Then, the collection of bounded linear functionals on X is a linear space which is normed by $|| \cdot ||_*$. This normed vector space is called the **dual space** of X, and is denoted by X^*.

Proposition 8. Let $E \subseteq \mathbb{R}$ be measurable, $1 \leq p < \infty$, q the conjugate of p, $g \in L^q(E)$. Define the functional T on $L^p(E)$ by:

$$T(f) := \int_E g \cdot f \; \forall f \in L^p(E) \tag{12}$$

Then, T is a bounded linear functional on $L^p(E)$ and $||T||_* = ||g||_q$.

Proposition 9. Let T, S be bounded linear functionals on the normed vector space X. If $T = S$ on a dense subset X_0 of X, then $T = S$.

Lemma 10. Let $E \subseteq \mathbb{R}$ be measurable, $1 \leq p < \infty$. Suppose that g is integrable over E and there exists a $M \geq 0$ for which

$$\left| \int_E g \cdot f \right| \leq M ||f||_p \; \forall f \in L^p(E), \; f \text{ simple}$$

Then, $g \in L^q(E)$, where q is the conjugate of p. Moreover, $||g||_q \leq M$.

Theorem 11. Let $[a, b]$ be a closed, bounded interval, and $1 \leq p < \infty$. Suppose that T is a bounded linear functional on $L^p[a, b]$. Then, there is a functional $g \in L^q[a, b]$, where q is the conjugate of p, for which:

$$T(f) = \int_a^b g \cdot f \; \forall f \in L^p[a, b] \tag{13}$$

Theorem 12 (Riesz-Representation Theorem for the Dual of $L^p(E)$). Let $E \subseteq \mathbb{R}$ be measurable, $1 \leq p < \infty$, and q the conjugate of p. For all $g \in L^q(E)$, define the bounded linear functional $\mathcal{R}_g$ on $L^p(E)$ by:

$$\mathcal{R}_g := \int_E g \cdot f \; \forall f \in L^p(E) \tag{14}$$

Then, for each bounded linear functional T on $L^p(E)$, there exists a unique $g \in L^q(E)$ for which

(1) $\mathcal{R}_g = T$ and
(2) $||T||_* = ||g||_q$

2.2.   Weak Sequential Convergence in L^p

Definition 20 (Converge Weakly)**.** Let X be a normed vector space. A sequence $\{f_n\}$ in X is said to **converge weakly** in X to f provided that

$$\lim_{n\to\infty} T(f_n) = T(f) \ \forall T \in X^* \tag{15}$$

we write

$$\{f_n\} \rightharpoonup f$$

to mean that f and each f_n belong to X and $\{f_n\}$ converges weakly in X to f.

Definition 21. Let $E \subseteq \mathbb{R}$ be measurable, $1 \le p < \infty$, q the conjugate of p. Then, $\{f_n\} \rightharpoonup f$ in $L^p(E)$ $\iff$

$$\lim_{n\to\infty} \int_E g \cdot f_n = \int_E g \cdot f \ \forall g \in L^q(E) \tag{16}$$

Theorem 13. Let $E \subseteq \mathbb{R}$ be measurable, $1 \le p < \infty$. Suppose that $\{f_n\} \rightharpoonup f$ in $L^p(E)$. Then:

$$\{f_n\} \text{ is bounded and } ||f||_p \le \liminf ||f_n||_p$$

Corollary 3. Let $E \subseteq \mathbb{R}$ be measurable, $1 \le p < \infty$, q the conjugate of p. Suppose $\{f_n\}$ converges weakly to f in $L^p(E)$ and $\{g_n\}$ converges strongly to $g \in L^q(E)$. Then:

$$\lim_{n\to\infty} \int_E g_n \cdot f_n = \int_E g \cdot f \tag{17}$$

Definition 22 (Linear Span)**.** Let X be a normed vector space, and let $S \subseteq X$. Then, the **linear span of** S is the vector space consisting of all linear functionals of the form:

$$f = \sum_{k=1}^{n} \alpha_k \cdot f_k \tag{18}$$

where each $\alpha_k \in \mathbb{R}$ and $f_k \in S$. It is the set of all *finite linear combinations of elements in* S. We care about this since L^p is an infinite dimensional space, so we want to find a way to approximate it with finitely many elements.

Proposition 10 (Characterisation of Weak Convergence in $L^p(E)$)**.** Let $E \subseteq \mathbb{R}$ be measurable, $1 \le p < \infty$, q the conjugate of P. Assume that $\mathcal{F} \subseteq L^q(E)$ whose linear span is dense in $L^q(E)$. Let $\{f_n\}$ be a bounded sequence in $L^p(E)$, and let $f \in L^p(E)$. Then, $\{f_n\} \rightharpoonup f$ in $L^p(E)$ $\iff$

$$\lim_{n\to\infty} \int_E f_n \cdot g = \int_E f \cdot g \ \forall g \in \mathcal{F} \tag{19}$$

Theorem 14. Let $E \subseteq \mathbb{R}$ be measurable, $1 \le p < \infty$. Suppose that $\{f_n\}$ is a bounded sequence in $L^p(E)$ and f belongs to $L^p(E)$. Then, $\{f_n\} \rightharpoonup f$ in $L^p(E)$ $\iff$ $\forall$ measurable sets $A \subseteq E$:

$$\lim_{n\to\infty} \int_A f_n = \int_A f \tag{20}$$

if $p > 1$, then it is sufficient to consider sets A of finite measure.

Theorem 15. Let $[a, b]$ be a closed and bounded interval, $1 < p < \infty$. Suppose that $\{f_n\}$ is a bounded sequence in $L^p[a, b]$ and $f \in L^p[a, b]$. Then, $\{f_n\} \rightharpoonup f$ in $L^p(E)$ in $L^p[a, b]$ $\iff$

$$\lim_{n\to\infty} \left[\int_a^x f_n \right] = \int_a^x f \ \forall x \in [a, b] \tag{21}$$

Lemma 16 (Riemann-Lebesgue Lemma; used in Fourier Series :-)). Let $I = [-\pi, \pi]$, $1 \leq p < \infty$. $\forall n \in \mathbb{N}$, define $f_n(x) := \sin(nx)$ for $x \in I$. Then, $\{f_n\}$ converges weakly in $L^p(I)$ to $f \equiv 0$.

Theorem 17. Let $E \subseteq \mathbb{R}$ be measurable, $1 < p < \infty$. Suppose that $\{f_n\}$ is a bounded sequence in $L^p(E)$ that converges pointwise a.e. on E to f. Then, $\{f_n\} \rightharpoonup f$ in $L^p(E)$.

     This theorem was used in the proof but was not covered in Analysis 3:

> **Theorem 18** (Vitali Convergence Theorem). Let $E \subseteq \mathbb{R}$ be measurable and of finite measure. Suppose that the sequence of functions $\{f_n\}$ is uniformly integrable over E. Then, if $\{f_n\} \to f$ pointwise a.e. on E, then f is integrable over E and $\lim_{n \to \infty} \int_E f_n = f$.

Theorem 19 (Radon-Riesz Theorem). Let $E \subseteq \mathbb{R}$ be measurable, $1 < p < \infty$. Suppose that $\{f_n\} \rightharpoonup f$ in $L^p(E)$. Then:

$$\{f_n\} \to f \text{ in } L^p(E) \iff \lim_{n \to \infty} ||f_n||_p = ||f||_p \tag{22}$$

Corollary 4. (Not Covered in Class): Let $E \subseteq \mathbb{R}$ be measurable and $1 < p < \infty$. Suppose that $\{f_n\} \rightharpoonup f$ in $L^p(E)$. Then, a subsequence of $\{f_n\}$ converges strongly to $f \iff ||f||_p = \liminf ||f_n||_p$.

2.3. WEAK SEQUENTIAL COMPACTNESS ("COMPACTNESS FOUND!")

Theorem 20. Let $E \subseteq \mathbb{R}$ be measurable, $1 < p < \infty$. Then, every bounded sequence in $L^p(E)$ has a subsequence that converges weakly in $L^p(E)$ to a function in $L^p(E)$.

Theorem 21 (Helly's Theorem). Let X be a *SEPARABLE* normed vector space and $\{T_n\}$ a sequence in its dual space X^* that is bounded; that is, $\exists$ a $M > 0$ for which

$$|T_n(f)| \leq M \cdot ||f|| \ \forall f \in X, \ \forall n \in \mathbb{N}$$

Then, there is a subsequence $\{T_{n_k}\}$ of $\{T_n\}$ and $T \in X^*$ for which

$$\lim_{k \to \infty} T_{n_k}(f) = T(f) \ \forall f \in X \tag{23}$$

Definition 23 (Weakly Sequentially Compact (Compact in the "weak topology"). Let X be a normed vector space. Then, a subset $K \subseteq X$ is **weakly sequentially compact** in X provided that every sequence $\{f_n\}$ in K has a subsequence that converges weakly to $f \in K$.

Theorem 22 (The Unit Ball is Weakly Sequentially Compact). Let $E \subseteq \mathbb{R}$ be measurable, $1 < p < \infty$. Define:

$$B_1 := \{f \in L^p(E) \mid ||f||_p \leq 1\}. \tag{24}$$

B_1 is weakly sequentially compact in $L^p(E)$.

2.4. RESULTS FROM THE HOMEWORK

(1) (Reisz-Representation Theorem for the Dual of ℓ^p): Let $1 \leq p < \infty$, q the conjugate of p. Then for all $\{g_n\} \in \ell^q$, define the bounded linear functional $\mathcal{R}_g$ on ℓ^p by:

$$\mathcal{R}_g := T(\{f_n\}) = \sum_{n=1}^{\infty} g_n f_n \tag{25}$$

$\forall \{f_n\} \in \ell^p$. Then, for each bounded linear functional T on ℓ^p, there exists a unique $\{g_n\} \in \ell^q$ for which:

(1) $\mathcal{R}_g = T$

(2) $||T||_* = ||\{g_n\}||_q$

(2) Let c be the vector space of all real sequences that converge to a real number and let c_0 be the subspace of c comprising of all sequences that converge to zero. Norm each vector space with the ℓ^∞ norm. Then, $c^* = \ell^1$ and $c_0^* = \ell^1$.

(3) Assume that h is a continuous function defined on all of $\mathbb{R}$ that is periodic with period T and $\int_0^T h = 0$. Let $[a, b]$ be a closed + bounded interval. For each $n \in \mathbb{N}$, define $f_n(x) := h(nx)$. Define $f \equiv 0$ on $[a, b]$. Then, $\{f_n\}$ converges weakly to f in $L^p[a, b]$.

(4) Let $1 < p < \infty$, assume $f_0 \in L^p(\mathbb{R})$. For each $n \in \mathbb{N}$, define $f_n(x) := f_0(x - n)$. Define $f \equiv 0$ on $\mathbb{R}$. Then, $\{f_n\}$ converges weakly to f in $L^p(\mathbb{R})$. Not true for $p = 1$!

(5) For $1 \leq p < \infty$, for each $n \in \mathbb{N}$, let $e_n \in \ell^p$ be the standard basis sequence. If $p > 1$, then $\{e_n\}$ converges weakly to zero in ℓ^p, but no subsequence converges strongly to zero. $\{e_n\}$ does not converge at all in ℓ^1.

(6) (Uniform Boundedness Principle): Let $E \subseteq \mathbb{R}$ be measurable, $1 \leq p < \infty$, and q the conjugate of p. Suppose that $\{f_n\}$ is a sequence in $L^p(E)$ for which for each $g \in L^q(E)$, the sequence $\{\int_E g f_n\}$ is bounded. Show that $\{f_n\}$ is bounded in $L^p(E)$.

(7) $\{x^n\}$ in $C[0, 1]$ fails to have a strongly convergent subsequence. Suitably modify this to work in any $C[a, b]$ by:

$$f_n := \left(\frac{x - a}{b - a} \right)^n$$

(8) In ℓ^p, $1 < p < \infty$, every bounded sequence in ℓ^p has a weakly convergent subsequence.

(9) Let X be a normed vector space, and let $\{T_n\}$ be a sequence in X^* for which there exists an $M \geq 0$ such that $||T_n||_* \leq M$ for all $n \in \mathbb{N}$. Let $\mathcal{S} \subseteq X$ be a dense subset such that $\{T_n(g)\}$ is Cauchy for all $g \in \mathcal{S}$. Then:

(1) $\{T_n(g)\}$ is Cauchy for all $g \in X$.

(2) The limiting functional is linear and bounded.

(10) Helly's theorem fails when $X = L^\infty[0, 1]$. To see why, consider a sequence of linear functionals induced by the Rademacher functions.

3.   METRIC SPACES

This section was not covered in class, but since we have homework on this chapter I figured having this as a review from analysis 2 might be helpful. Also, there are a few terms/results that I don't think we covered in analysis 2.

3.1.   EXAMPLES OF METRIC SPACES

Definition 24 (Metric Space). Let X be a non-empty set. A function $\rho : X \times X \to \mathbb{R}$ is called a **metric** if $\forall \, x, y \in X$:

(1) $\rho(x, y) \geq 0$

(2) $\rho(x, y) = 0 \iff x = y$

(3) $\rho(x, y) = \rho(y, x)$

(4) $\rho(x,z) \le \rho(x,y) + \rho(y,z)$ (**Triangle Inequality**).

A non-empty set together with a metric, denoted (X,ρ) is called a **metric space**.

Definition 25 (Discrete Metric). For any non-empty set X, the **discrete metric** ρ is defined by setting $\rho(x,y) = 0$ if $x = y$ and $\rho(x,y) = 1$ if $x \ne y$.

Definition 26 (Metric Subspace). For any metric space (X,ρ), let $Y \subseteq X$ be non-empty. Then, the restriction of ρ to $Y \times Y$ defines a metric on Y. We define this induced metric space as a **metric subspace**.

Example 3.1 (Examples of metric spaces). The following are examples of metric spaces:

(1) Every non-empty subset of a Euclidean space.
(2) $L^p(E)$, where $E \subseteq \mathbb{R}$ is a measurable set.
(3) $C[a,b]$.

Definition 27 (Product Metric). For metric spaces (X_1, ρ_1) and (X_2, ρ_2), we define the **product metric** τ on the cartesian product $X_1 \times X_2$ by setting, for (x_1, x_2) and (y_1, y_2) in $X_1 \times X_2$:

$$\tau((x_1, x_2), (y_1, y_2)) := \{[\rho_1(x_1, x_2)]^2 + [\rho_2(y_1, y_2)]^2\}^{1/2} \tag{26}$$

Definition 28. Two metrics ρ and σ on a set X are said to be **equivalent** if there are positive numbers c_1 and c_2 such that $\forall\, x_1, x_2 \in X$,

$$c_1 \sigma(x_1, x_2) \le \rho(x_1, x_2) \le c_2 \sigma(x_1, x_2)$$

Definition 29 (Isometry). A mapping $f : (X,\rho) \to (Y,\sigma)$ between two metric spaces is called an **isometry** provided that f is surjective and $\forall x_1, x_2 \in X$:

$$\sigma(f(x_1), f(x_2)) = \rho(x_1, x_2) \tag{27}$$

We say that two metric spaces are **isometric** if there is an isometry from one to another.

3.2.   OPEN SETS, CLOSED SETS, AND CONVERGENT SEQUENCES

Definition 30 (Open Ball). Let (X,ρ) be a metric space. For a point $x \in X$ and $r > 0$, the set:

$$B(x,r) := \{x' \in X \mid \rho(x',x) < r\} \tag{28}$$

is called the **open ball** centred at x of radius r. A subset $\mathcal{O} \subseteq X$ is said to be **open** if $\forall x \in \mathcal{O}$, there exists an open ball centred at x and contained in $\mathcal{O}$. For a point $x \in X$, an open set containing x is called a **neighbourhood** of x.

Proposition 11. Let X be a metric space. The whole set X and the empty set $\emptyset$ are open. The intersection of any two open sets is open. The union of any collection of open sets is open.

Proposition 12. Let X be a subspace of a metric space Y and $E \subseteq X$. Then, E is **open in X** $\iff$ $E = X \cap \mathcal{O}$, where $\mathcal{O}$ is open in Y.

Definition 31 (Closure). For a subset $E \subseteq X$, a point $x \in X$ is called a **point of closure** of E provided that every neighbourhood of x contains a point in E. The collection of the points of closure of E is called the **closure** of E and is denoted by $\overline{E}$.

Proposition 13. For $E \subseteq X$, where X is a metric space, its closure $\overline{E}$ is closed. Moreover, $\overline{E}$ is the smallest closed subset of X containing E in the sense that if F is closed and if $E \subseteq F$, then $\overline{E} \subseteq F$.

Definition 32 (Converge). A sequence $\{x_n\}$ in a metric space (X, ρ) is said to **converge** to the point $x \in x$ provided that:

$$\lim_{n \to \infty} \rho(x_n, x) = 0$$

that is, $\forall\, \varepsilon > 0$, $\exists$ an index N such that $\forall n \geq N$, $\rho(x_n, x) < \varepsilon$.

Proposition 14. Let ρ and σ be equivalent metrics on a non-empty set X. Then, a subset X is open in a metric space $(X, \rho) \iff$ it is open in (X, σ).

3.3.   Continuous Mappings Between Metric Spaces

Definition 33 (Continuous). A mapping f from a metric space X to a metric space Y is continuous at the point $x \in X$ if $\forall\, \{x_n\} \in X$, if $\{x_n\} \to x$, then $\{f(x_n)\} \to f(x)$. f is said to be **continuous** if it is continuous at every point in X.

Proposition 15 (ε-δ criteria for continuity). A mapping from a metric space (X, ρ) to a metric (Y, σ) is continuous at the point $x \in X \iff \forall\, \varepsilon > 0$, $\exists\, \delta > 0$ such that if $\rho(x, x') < \delta$, then $\sigma(f(x), f(x')) < \varepsilon$. That is:

$$f(B(x, \delta)) \subseteq B(f(x), \varepsilon) \tag{29}$$

Proposition 16. A mapping f from a metric space X to a metric space Y is continuous $\iff \forall$ open subsets $\mathcal{O} \subseteq Y$, the inverse image under f of $\mathcal{O}$, $f^{-1}(\mathcal{O})$, is an open subset of X.

Proposition 17. The composition of continuous mappings between metric spaces, when defined, is continuous.

Definition 34 (Uniformly Continuous). A mapping from a metric space (X, ρ) to a metric space (Y, σ) is said to be **uniformly continuous** if $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ such that $\forall u, v \in X$, if $\rho(u, v) < \delta$, $\sigma(f(u), f(v)) < \varepsilon$.

Definition 35 (Lipschitz). A mapping $f : (X, \rho) \to (Y, \sigma)$ is said to be **Lipschitz** if $\exists$ a $c \geq 0$ such that $\forall\, u, v \in X$:

$$\sigma(f(u), f(v)) \leq c\rho(u, v)$$

3.4.   Complete Metric Spaces

Definition 36 (Cauchy). A sequence $\{x_n\}$ in a metric space (X, ρ) is said to be a **Cauchy sequence** if $\forall \varepsilon > 0$, there exists a $N \in \mathbb{N}$ such that if $m, n \geq N$, then $\rho(x_n, x_m) < \varepsilon$.

Definition 37 (Complete). A metric space X is said to be **complete** if every Cauchy sequence in X converges to a point in X.

Proposition 18. Let $[a, b]$ be a closed and bounded interval of real numbers. Then, $C[a, b]$ with the metric induced by the max norm is complete.

Proposition 19 (Characterisation of Complete Subspaces of Metric Spaces). Let $E \subseteq X$, where X is a complete metric space. Then, the metric subspace E is complete $\iff E$ is a closed subset of X.

Theorem 23. The following are complete metric spaces:

(1) Every non-empty closed subset of $\mathbb{R}^n$.
(2) $E \subseteq \mathbb{R}$ measurable, $1 \leq p \leq \infty$, each non-empty closed subset of $L^p(E)$.
(3) Each non-empty closed subset of $C[a, b]$.

Definition 38 (Diameter)**.** Let E be a non-empty subset of a metric space (X, ρ). We define the **diameter** of E, denoted by $\operatorname{diam}(E)$, by:

$$\operatorname{diam}(E) := \sup \{\rho(x, y) \mid x, y \in E\} \tag{30}$$

We say that E is **bounded** if it has finite diameter.

Definition 39 (Contracting Sequence)**.** A decreasing sequence $\{E_n\}$ of non-empty subsets of X is called a **contracting sequence** if:

$$\lim_{n \to \infty} \operatorname{diam}(E_n) = 0 \tag{31}$$

Theorem 24 (Cantor Intersection Theorem)**.** Let X be a metric space. Then, X is complete $\iff$ whenever $\{F_n\}$ is a contracting sequence of non-empty closed subsets of X, there is a point $x \in X$ for which:

$$\bigcap_{n=1}^{\infty} F_n = \{x\} \tag{32}$$

Theorem 25. Let (X, ρ) be a metric space. Then, there is a complete metric space $(\widetilde{X}, \tilde{\rho})$ for which X is a dense subset of $\widetilde{X}$ and

$$\rho(u, v) = \tilde{\rho}(u, v) \; \forall \; u, v \in X \tag{33}$$

we call such a space the **completion** of (X, ρ).

3.5.   COMPACT METRIC SPACES

Definition 40 (Compact Metric Space)**.** A metric space X is called **compact** if every open cover of X has a finite sub-cover. A subset $K \subseteq X$ is compact if K, considered as a metric subspace of X, is compact.

    <u>**Formulation of compactness in terms of closed sets:**</u> Let $\mathcal{T}$ be a collection of open subsets of a metric space X. Define $\mathcal{F}$ to be the collection of the complements of elements in $\mathcal{T}$. Since the elements of $\mathcal{T}$ are open, the elements of $\mathcal{F}$ are closed. Thus, $\mathcal{T}$ is a cover $\iff$ the elements of $\mathcal{F}$ have *empty intersection*. By deMorgan's law, we can formulate compactness in terms of closed sets as:

    A metric space X is compact $\iff$ every collection of closed sets with empty intersection has a finite sub-collection whose intersection is non-empty.

This property is called the **finite intersection property**.

Definition 41 (Finite Intersection Property)**.** A collection of sets $\mathcal{F}$ is said to have the **finite intersection property** if any finite sub-collection of $\mathcal{F}$ has a non-empty intersection.

Proposition 20 (Compactness in terms of closed sets)**.** A metric space X is compact $\iff$ every collection $\mathcal{F}$ of closed subsets of X with the finite intersection property has a non-empty intersection.

Definition 42 (Totally Bounded)**.** A metric space X is **totally bounded** if $\forall \, \varepsilon > 0$, the space X can be covered by a finite number of open balls of radius ε. A subset $E \subseteq X$ is said to be **totally bounded** if E, a s a subspace of the metric space X, is totally bounded.

Definition 43 (ε-net)**.** Let E be a subset of a metric space X. A ε-**net** for R is a finite collection of open balls $\{B(x_k, \varepsilon)\}_{k=1}^{n}$ with centres $x_k \in X$ whose union covers E.

Proposition 21. A metric space E is totally bounded $\iff \forall \, \varepsilon > 0$, there is a finite ε-net for E.

Proposition 22. A subset of Euclidean space $\mathbb{R}^n$ is bounded $\iff$ it is totally bounded.

Definition 44 (Sequentially Compact)**.** A metric space X is **sequentially compact** if every sequence in X has a subsequence that converges to a point in X.

Theorem 26 (Characterisation of Compactness for a metric space)**.** . Let X be a metric space. Then, TFAE:

   (1) X is complete and totally bounded.
   (2) X is compact.
   (3) X is sequentially compact.

    The following three propositions of this chapter are just breaking down these equivalences, so I will not write them.

Theorem 27. Let $K \subseteq \mathbb{R}^n$. Then, TFAE:

   (1) K is closed and bounded.
   (2) K is compact.
   (3) K is sequentially compact.

Observe: The equivalence (1) $\iff$ (2) is the Heine-Borel theorem. The equivalence (2) $\iff$ (3) is the Bolzano-Weierstrass theorem.

Proposition 23. Let f be a continuous mapping from a compact metric space X to a compact metric space Y. Then, its image $f(X)$ is compact.

Theorem 28 (Extreme Value Theorem)**.** Let X be a metric space. Then, X is compact $\iff$ every continuous real-valued function on X attains a minimum and maximum value.

Definition 45 (Lebesgue Number)**.** Let X be a metric space, and let $\{\mathcal{O}_\lambda\}_{\lambda\in\Lambda}$ be an open cover of X. Thus, each $x \in X$ is contained in a member of the cover, $\mathcal{O}_\lambda$. Since $\mathcal{O}_\lambda$ is open, $\exists\, \varepsilon > 0$ such that:

$$B(x, \varepsilon) \subseteq \mathcal{O}_\lambda$$

In general, ε on X, but for compact metric spaces we can get *uniform control*. This ε that uniformly works is called the **Lebesgue number** for the cover $\{\mathcal{O}_\lambda\}_{\lambda\in\Lambda}$.

Lemma 29. Let $\{\mathcal{O}_\lambda\}_{\lambda\in\Lambda}$ be an open cover of a compact metric space X. Then, there is a number $\varepsilon > 0$ such that for each $x \in X$, the open ball $B(x, \varepsilon)$ is contained in some member of the cover.

Proposition 24. A continuous mapping from a compact space (X, ρ) to a metric space (Y, σ) is uniformly continuous.

3.6.   Separable Metric Spaces

Definition 46 (Dense & Separable)**.** A subset D of a metric space X is **dense** in X if every non-empty subset of X contains a point of D. A metric space is **separable** if there is a countable subset of X that is dense in X.

    The **Weierstrass Approximation Theorem** states that polynomials are dense in $C[a, b]$. So, $C[a, b]$ is separable, with the countable dense set being the set of polynomials with rational coefficients.

Proposition 25. A compact metric space is separable.

Proposition 26. A metric space X is separable $\iff$ there is a countable collection of $\{\mathcal{O}_n\}$ of open subsets of X such that any open subset of X is the union of a sub-collection of $\{\mathcal{O}_n\}$.

Proposition 27. Every subspace of a separable metric space is separable.

Theorem 30. Each of the following are separable metric spaces:

(1) Every non-empty subset of Euclidean space $\mathbb{R}^n$.
(2) $1 \leq p < \infty$, $L^p(E)$ and all non-empty subsets of $L^p(E)$.
(3) Each non-empty subset of $C[a,b]$.

3.7. Results from the Homework

(1) $\{(X_n, \rho_n)\}_{n=1}^{\infty}$ a countable collection of metric spaces. Then, the following is a metric on the Cartesian product:

$$\rho_*(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{\rho_n(x_n, y_n)}{1 + \rho_n(x_n + y_n)}$$

(2) A continuous mapping between metric spaces remains continuous if an equivalent metric is imposed on the domain and an equivalent metric is imposed on the domain.
(3) The distance function (from a point to a set) is continuous.
(4) $\{x \in X \mid \operatorname{dist}(x, E) = 0\} = \overline{E}$.
(5) (Sequential Definition of Uniform Continuity): For a mapping f of a metric space (X, ρ) to the metric space (Y, σ), f is uniformly continuous $\iff$ for all sequences $\{u_n\}$ and $\{v_n\}$ in X:

$$\text{if } \lim_{n \to \infty} \rho(u_n, v_n) = 0 \text{ then } \lim_{n \to \infty} \sigma(f(u_n), f(v_n)) = 0$$

(6) If X and Y are metric spaces, with Y complete, and f a uniformly continuous mapping from $E \subseteq X \to Y$, then f has a uniquely uniformly continuous extension mapping $\overline{f}$ of $\overline{E}$ to Y.
(7) Let $E \subseteq X$, X a compact metric space. Then, the metric subspace E is compact $\iff$ E is a closed subset of X.
(8) $E \subseteq X$, X complete. Then, E is totally bounded $\iff$ $\overline{E}$ is totally bounded.
(9) The closed unit ball in ℓ^2 is not compact.

4. Topological Spaces

4.1. Open Sets, Closed Sets, Bases, and Sub-bases

Definition 47 (Open Sets). Let X be a non-empty set. A **topology** $\mathcal{T}$ for X is a collection of subsets of X, called **open sets**, posessing the following properties:

(1) The entire set X and the empty set $\emptyset$ are open.
(2) The finite intersection of open sets are open.
(3) The union of any collection of open sets is open.

A non-empty set X, together with a topology on X, is called a **topological space**. For a point $x \in X$, an open set that contains x is called a **neighbourhood** of x.

Proposition 28. A subset $E \subseteq X$ is open $\iff$ for each $x \in E$, there exists a neighbourhood of x that is contained in E.

Example 1 (Metric Topology). Let (X, ρ) be a metric space. Let $\mathcal{O} \subseteq X$ be open if for all $x \in \mathcal{O}$, $\exists$ an open ball at x that is contained in $\mathcal{O}$. This collection of open sets forms a topology; we call this the **metric topology** induced by ρ.

Example 2 (Discrete Topology). This topology is "too much." Let X be a non-empty subset. Let $\mathcal{T} := \mathcal{P}(X)$. Then, every set containing a point is a neighbourhood of that point. This is induced by the discrete metric.

Example 3 (Trivial Topology). Let X be non-empty. Define $\mathcal{T} := \{X, \emptyset\}$. The only neighbourhood of any point is the whole set X.

Definition 48 (Topological Subspaces). Let $(X, \mathcal{T})$ be a topological space and let E be a non-empty subset of X. The inherited topology $\mathcal{S}$ for E is the set of all sets of the form $E \cap \mathcal{T}$, where $\mathcal{O} \in \mathcal{T}$. The topological space $(E, \mathcal{S})$ is called a **subspace** of $(X, \mathcal{T})$.

Definition 49 (Base for the Topology). The building blocks of a topology is called a **base**. Let $(X, \mathcal{T})$ be a topological space. For a point $x \in X$, a collection of neighbourhoods of x, B_x, is called a **base for the topology at** X if $\forall$ neighbourhoods $\mathcal{U}$ of x, there exists a set B in the collection B_x for which $B \subseteq \mathcal{U}$.

    A collection of open sets $\mathcal{B}$ is called a **base for the topology** $\mathcal{T}$ provided it contains a base for the topology at each point.

A base for a topology completely determines a topology, alongside $\emptyset$ and X.

Proposition 29. For a non-empty set X, let $\mathcal{B}$ be a collection of subsets of X. Then, $\mathcal{B}$ is a base for a topology for $X \iff$:

(1) $\mathcal{B}$ covers X. That is:

$$X = \bigcup_{B \in \mathcal{B}} B \tag{34}$$

(2) If $B_1, B_2 \in \mathcal{B}$, and $x \in B_1 \cap B_2$, then there is a set $B_3 \in \mathcal{B}$ for which $x \in B_3 \subseteq B_1 \cap B_2$.

The unique topology that has $\mathcal{B}$ as its base consists of $\emptyset$ and unions of sub-collections of $\mathcal{B}$.

Definition 50 (Product Topology). Let $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ be two topological spaces. In the cartesian product $X \times Y$, consider the collection of sets $\mathcal{B}$ containing the products $\mathcal{O}_1 \times \mathcal{O}_2$, where $\mathcal{O}_1$ is open in X and $\mathcal{O}_2$ is open in Y. Then, $\mathcal{B}$ is a base for a topology on $X \times Y$, which we call the **product topology.**

Definition 51 (Sub-base). Let $(X, \mathcal{T})$ be a topological space. The collection of $\mathcal{S}$ of $\mathcal{T}$ that covers X is called a **sub-base** for the topology $\mathcal{T}$ provided intersections of finite collections of $\mathcal{S}$ are a base for $\mathcal{T}$.

Definition 52 (Closure). Let $E \subseteq X$ be a subset of a topological space. A point $x \in E$ is called a **point of closure** of E if every neighbourhood of x contains a point in E. The collection of the points of closure of E is called the **closure** of E, denoted $\overline{E}$.

Proposition 30. Let X be a topological space, $E \subseteq X$. Then, $\overline{E}$ is closed. Moreover, $\overline{E}$ is the smallest closed subset of X containing E in the sense that if F is closed and $E \subseteq F$, then $\overline{E} \subseteq F$.

Proposition 31. A subset of a topological space X is open $\iff$ its complement is closed.

Proposition 32. Let X be a topological space. Then, (a) $\emptyset$ and X are closed, (b) the union of a finite collection of closed sets is closed, (c) the intersection of any collection of closed sets in X is closed.

4.2.   SEPARATION PROPERTIES

Motivation: Separation properties for a topology allow us to discriminate between which topologies discriminate between certain disjoint pairs of sets, which will then allow us to study a robust collection of cts real-valued functions on X.

Definition 53 (Neighbourhood)**.** A **neighbourhood** of K for a subset $K \subseteq X$ is an open set that contains K.

Definition 54 (Separated by Neighbourhoods)**.** We say that two disjoint sets A and B in X can be separated by disjoint neighbourhoods provided that there exists neighbourhoods of A and B, respectively, that are disjoint.

Definition 55 (Separation Properties of Topological Spaces)**.** . In the order of most general to least general, they are:

(1) **Tychonoff Separation Property**: For each two points $u, v \in X$, there exists a neighbourhood of u that does not contain v and a neighbourhood of v that does not contain u.
(2) **Hausdorff Separation Property**: Each two points in X can be separated by disjoint neighbourhoods.
(3) **Regular Separation Property**: Tychonoff + each closed set and a point not in the set can be separated by disjoint neighbourhoods.
(4) **Normal Separation Property**: Tychonoff + each two disjoint closed sets can be separated by disjoint neighbourhoods.

Proposition 33. A topological space is Tychonoff $\iff$ every set containing a single point, $\{x\}$, is closed.

Proposition 34. Every metric space is normal.

Lemma 31. F is closed $\iff$ $\text{dist}(x, F) > 0 \ \forall \ x \notin F$.

Proposition 35. Let X be a Tychonoff topological space. Then, X is normal $\iff$ whenever $\mathcal{U}$ is a neighbourhood of a closed subset of F of X, there is another neighbourhood of F whose closure is contained in $\mathcal{U}$. that is, there is an open set $\mathcal{O}$ for which:

$$F \subseteq \mathcal{O} \subseteq \overline{\mathcal{O}} \subseteq \mathcal{U} \tag{35}$$

4.3.   COUNTABILITY AND SEPARABILITY

Definition 56 (Converge, Limit)**.** A sequence $\{x_n\}$ in a topological space X is said to **converge** to the point $x \in X$ if for each neighbourhood $\mathcal{U}$ of x, there exists an index $N \in \mathbb{N}$ such that if $n \geq N$, then x_n belongs to $\mathcal{U}$. This point is called a **limit** of the sequence.

Definition 57 (First and Second Countable)**.** A topological space X is **first countable** if there is a countable base at each point. A space X is said to be **second countable** if there is a countable base for the topology.

Example 4. Every metric space is first countable.

Proposition 36. Let X be a first countable topological space. For a subset $E \subseteq X$, a point $x \in X$ is called a point of closure of E $\iff$ it is a limit of a sequence in E. Thus, a subset E of X is closed $\iff$ whenever a sequence in E converges to $x \in X$, we have that $x \in E$.

Definition 58 (Dense/Separable)**.** A subset $E \subseteq X$ is **dense** in X if every open set in X contains a point of E. We call X **separable** if it has a countable dense subset.

Definition 59 (Metrisable)**.** A topological space X is said to be **metrisable** if the topology is induced by the metric.

Theorem 32. Let X be a second countable topological space. Then, X is metrisable $\iff$ it is normal.

4.4. CONTINUOUS MAPPINGS BETWEEN TOPOLOGICAL SPACES

Definition 60 (Continuous). For topological spaces $(X, \mathcal{T})$, $(Y, \mathcal{S})$, a mapping $f : X \to Y$ is said to be **continuous** at the point x_0 in X if, for every neighbourhood $\mathcal{O}$ if $f(x_0)$, there is a neighbourhood $\mathcal{U}$ of x_0 for which $f(\mathcal{U}) \subseteq \mathcal{O}$. We say that f is continuous provided it is continuous at each point in X.

Proposition 37. A mapping $f : X \to Y$ between topological spaces X and Y is continuous $\iff$ for any open subset $\mathcal{O}$ in Y, its inverse image under f, $f^{-1}(\mathcal{O})$, is an open subset of X.

Proposition 38. The composition of continuous mappings between topological spaces, when defined, is continuous.

Definition 61 (Stronger). Given two topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ for a set X, if $\mathcal{T}_2 \subseteq \mathcal{T}_1$, then we say that $\mathcal{T}_2$ is **weaker** than $\mathcal{T}_1$, and that $\mathcal{T}_1$ is **stronger** than $\mathcal{T}_2$.

Proposition 39. Let X be a non-empty set and let $\mathcal{S}$ be a collection of subsets of X that covers X. The collection of subsets of X consisting of intersections of finite collections of $\mathcal{S}$ is a base for a topology $\mathcal{T}$ of X. It is the weakest topology containing $\mathcal{S}$ in the sense that if $\mathcal{T}'$ is any other topology for X containing $\mathcal{S}$, then $\mathcal{T} \subseteq \mathcal{T}'$.

Definition 62 (Weak Topology). Let X be a non-empty set and $\mathcal{F} := \{f_\alpha \mid X \to X_\alpha\}_{\alpha \in \Lambda}$ a collection of mappings, where each X_α is a topological space. The weakest topology for X that contains the collection of sets

$$\{f_\alpha^{-1}(\mathcal{O}_\alpha) \mid f_\alpha \in \mathcal{F}, \ \mathcal{O}_\alpha \text{ open in } X_\alpha\} \tag{36}$$

is called the **weak topology for X induced by $\mathcal{F}$.**

Proposition 40. Let X be a non-empty set, $\mathcal{F} := \{f_\lambda \mid X \to X_\lambda\}_{\lambda \in \Lambda}$ a collection of mappings where each X_λ is a topological space. The weak topology for X induced by $\mathcal{F}$ is the topology on X that has the fewest number of sets covering the topologies on X for which each mapping $f_\lambda : X \to X_\lambda$ is continuous.

Definition 63 (Homeomorphism). A mapping from a topological space $X \to Y$ is said to be a **homeomorphism** if it is bijective and has a continuous inverse $f^{-1} : Y \to X$. Two topological spaces are said to be **homeomorphic** if there exists a homeomorphism between them. The notion of homeomorphism induces a notion of an equivalence relation between spaces.

4.5. COMPACT TOPOLOGICAL SPACES

Definition 64 (Cover). A collection of sets $\{E_\lambda\}_{\lambda \in \Lambda}$ is said to be a **cover** of a set E if $E \subseteq \bigcup_{\lambda \in \Lambda} E_\lambda$.

Definition 65 (Compact). A topological space X is said to be **compact** if every open cover of X has a finite sub-cover. A subset $K \subseteq X$ is compact if K, considered as a topological space with the subspace topology inherited from X, is compact.

Proposition 41. A topological space X is compact $\iff$ every collection of closed subsets of X that posesses the finite intersection property has non-empty intersection.

Proposition 42. A closed subset K of a compact topological space is compact.

Proposition 43. A compact subspace K of a Hausdorff topological space is a closed subset of X.

Definition 66 (Sequentially Compact). A topological space X is said to be **sequentially compact** if every sequence in X has a subsequence that converges to a point in X.

Proposition 44. Let X be a second countable topological space. Then, X is compact $\iff$ it is sequentially compact.

Theorem 33. A compact Hausdorff space is normal.

Proposition 45. A continuous one-to-one mapping f of a compact space X onto a Hausdorff space Y is a homeomorphism.

Proposition 46. The continuous image of a compact topological space is compact.

Corollary 5. A continuous real-valued function on a compact topological space takes on a minimum and maximum functional value.

Definition 67 (Countably Compact). A topological space is **countably compact** if every countable open cover has a finite subcover.

4.6. CONNECTED TOPOLOGICAL SPACE

Definition 68 (Separated). Two non-empty subsets of a topological space **separate** X if they are disjoint and their union is X.

Definition 69 (Connected). A topological space which cannot be separated by open sets is said to be **connected**. A subset $E \subseteq X$ is **connected** if there do NOT exist open subsets $\mathcal{O}_1$, $\mathcal{O}_2$ of X for which:

$$\mathcal{O}_1 \cap E \neq \emptyset$$
$$\mathcal{O}_2 \cap E \neq \emptyset$$
$$E \subseteq \mathcal{O}_1 \cup \mathcal{O}_2,$$
$$E \cap \mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$$

Proposition 47. Let f be a continuous mapping of a connected space X to a topological space Y. Then, its image $f(X)$ is connected.

Proposition 48. For A set $C \in \mathbb{R}$, the following are equivalent.

(1) C is an interval.
(2) C is convex.
(3) C is connected.

Definition 70 (Intermediate Value Property). A topological space X has the **intermediate value property** if the image of any continuous real-valued function on X is an interval.

Proposition 49. A topological space has the intermediate value property $\iff$ it is connected.

Definition 71 (Arcwise connected). A topological space X is **arcwise connected** if, for each pair $u, v \in X$, there exists a continuous map $f : [0,1] \to X$ for which $f(0) = u$ and $f(1) = v$. Note:

(1) Connected $\iff$ arcwise connected in $\mathbb{R}^n$.
(2) Arcwise connected $\Rightarrow$ connected (in general)
(3) There exist connected but non-arcwise connected spaces (in general).

4.7. RESULTS FROM HOMEWORK

(1) Let X be a topological space. Then, X is Hausdorff $\iff$ the diagonal $D := \{(x_1, x_2) \in X \times X \mid x_1 = x_2\}$ is closed as a subset of $X \times X$.
(2) The Moore plane is separable. The subspace $\mathbb{R} \times \{0\}$ is not separable. Thus, the Moore plane is not metrisable and not second countable.
(3) Let X and Y be topological spaces. Then, you can construct a continuous map from a Hausdorff space to a non-Hausdorff space, and you can do the same for a normal space to a non-normal space.

(4) If ρ_1 and ρ_2 are metrics on a set X that induce topologies $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively, then if they generate the same topology $\mathcal{T}_1 = \mathcal{T}_2$, then they are <u>NOT</u> necessarily equivalent. A counter example would be:

$$\rho_1 := |x - y|$$
$$\rho_2 := \frac{|x - y|}{1 + |x - y|}$$

The two types of models that we will implement are the multilayer perceptron model and the convolutional neural net. A multilayer perceptron, or MLP, has an input layer (where the data is fed into the network), one or more hidden layers, and an output layer. Each node, except for the input layer, has a non-linear activation function (such as ReLu or arctan). To train an MLP an algorithm called back-propagation is learned. Roughly speaking, back-propagation is a supervised learning algorithm where which computes the gradient of the loss function with respect to each of the weights of the neural net, and updates weights after each iteration to minimise the loss. Back-propagation computes the loss with respect to each weight using the chain rule from the final layer to the first layer. MLPs tend to be "fully-connected", meaning that each node is connected to every other node in the next layer.

Unlike a MLP, convolutional neural networks (CNN) are generally not fully-connected. They are inspired by the connectivity patterns in the human brain's visual cortex. Instead of general matrix multiplication, which MLPs use, convolutional neural networks use an operation called convolution. Mathematically, a **convolution** is an integral transform given by:

$$(f * g)(t) := \int_{\mathbb{R}} f(\tau)g(t - \tau)d\tau \tag{37}$$

Convolutional neural networks are frequently used in image recognition, natural language processing, and recommender systems.

In terms of the current "state of the art" methods for analysing the CIFAR-10 dataset, according to the paper "Big Transfer (BiT): General Visual Representation Learning", the team used the BiT algorithm to train a neural network and obtained a 99.4 % accuracy on the CIFAR-10 dataset. In short, this method uses the "BiT-HyperRule", which "select[s] the most important hyper-parameters for tuning as a simple function of the task's intrinsic image resolution and number of data points."