

Math 254: Analysis 1

Definitions, Theorems, and Results from the Class (Fall 2018)

Shereen Elaidi

Abstract

The purpose of this document is to review analysis 1.

1. INTRODUCTION

Random things we proved to get a handle on how to prove things:

- $\cap_{x \in [0,1]} [0, x] = \{0\}$.
- $2^n < n!$
- Let X and Y be sets. Consider the following family of sets:

$$\{V_i \mid i \in I, V_i \subseteq Y\}$$

then, $f^{-1}(\cup_{i \in I} V_i) = \cup_{i \in I} f^{-1}(V_i)$.

- $5^n - 1$ is divisible by 4 $\forall n \geq 1$.
- **Bernoulli's Inequality**: $\forall n \in \mathbb{N}, x \in \mathbb{R}, x \geq -1$, one has:

$$(1 + x)^n \geq 1 + nx \quad (1)$$

- Every non-empty subset of the natural numbers has a smallest element.

Definition 1 (Cartesian Product). Let A and B be two sets. Then, their **Cartesian Product** is defined as:

$$A \times B := \{(a, b) \mid a \in A \wedge b \in B\} \quad (2)$$

Definition 2 (Function). Let D, E be sets. A **function** f from D to E is a subset of the cartesian product $D \times E$ such that $\forall x \in D, \exists_1 t \in E$ such that $(x, t) \in f$. In symbols, we define:

$$f(A) := \{f(x) \mid x \in A\} \quad (3)$$

Proposition 1 (Properties of Functions). Let $f : D \rightarrow E$ be a function and let $A, B \subseteq D$. Then, consider the following:

- $f(A \cup B) = f(A) \cup f(B)$ [well behaved with respect to unions]
- $f(A \cap B) \subseteq f(A) \cap f(B)$.

Definition 3 (Pre-Image). Let $f : D \rightarrow E, A \subseteq E$. Then, the **pre-image** is defined as:

$$f^{-1}(A) := \{x \in D \mid f(x) \in A\} \quad (4)$$

Proposition 2. Let $f : D \rightarrow E, A, B \subseteq E$. Then:

- $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$
- $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$

Definition 4 (Injective). Let $f : D \rightarrow E$. f is said to be injective if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$.

Definition 5 (Surjective). Let $f : D \rightarrow E$. f is said to be surjective if $\forall y \in E, \exists x \in D$ such that $f(x) = y$.

Definition 6 (Bijective). $f : D \rightarrow E$ is called bijective if it is surjective and injective.

Definition 7. If $f : D \rightarrow E$ is bijective, then we can define the inverse function $f^{-1} : E \rightarrow D$ as follows:

$$f^{-1}(y) := x \quad (5)$$

where x is a uniquely determined point in D with $f(x) = y$.

1.1 Countability of Finite Sets

Definition 8 (Cardinality). Let $S = \{a_1, \dots, a_n\}$. Then, the cardinality of S , in symbols $|S|$, is the number of elements in a set S .

Theorem 1. Let A, B be finite sets. Then, $|A| \leq |B| \iff$ there exists a function $f : A \rightarrow B$ which is injective.

Theorem 2. Let A, B be finite sets. Then, $|A| \geq |B| \iff \exists$ a surjective map from $A \rightarrow B$.

Theorem 3. Let A, B be finite sets. Then, $|A| = |B| \iff \exists$ a bijective map $f : A \rightarrow B$.

Definition 9. Let A and B be sets, not necessarily finite. We then say that A and B have the same cardinality, in symbols,

$$|A| = |B| \quad (6)$$

if \exists a bijective map $f : A \rightarrow B$.

Theorem 4 (Cantor's Theorem). Let A and B be sets. If $|A| \leq |B|$ and if $|B| \leq |A|$, then $|A| = |B|$.

Definition 10 (Countability). We say that a set A with $|A| = |\mathbb{N}|$ is countably infinite. A set which is either finite or countably infinite is called countable.

Theorem 5 (Arithmetic-Geometric Inequality). $\forall n \geq 1$ and for all $x_1, \dots, x_n > 0$, the following holds:

$$\frac{x_1 + \dots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \dots x_n} \quad (7)$$

Lemma 6. Let $n \in \mathbb{N}$ and let $x_1, \dots, x_n > 0$. If $x_1 \dots x_n = 1$, then:

$$x_1 + \dots + x_n \geq n \quad (8)$$

Theorem 7. Let $S \subseteq \mathbb{N}$. Then, there are only two possibilities:

- (i) S is finite.
- (ii) S is countably infinite.

Lemma 8. Let $a_1 < a_2 < \dots$ be a strictly increasing sequence of natural numbers. Then, we can say something about the growth rate:

$$a_n \geq n \quad (9)$$

$\forall n \in \mathbb{N}$.

Theorem 9. Let $f : \mathbb{N} \rightarrow S$ be surjective. Then, S is countable.

Theorem 10 (Cantor). The set \mathbb{Q} of all rational numbers is countably infinite.

Theorem 11. \mathbb{R} is uncountable (i.e, \mathbb{R} is infinite and there does not exist a bijection from \mathbb{N} to \mathbb{R}).

Definition 11 (Absolute Value). Let $x \in \mathbb{R}$. Then, the **absolute value** of x is defined as:

$$|x| := \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} \quad (10)$$

Note that $|x|$ is used to measure distances.

Proposition 3 (Properties of Absolute Value). (i) $\forall x \in \mathbb{R}, |x| \geq 0$ and $|x| = 0 \iff x = 0$.
(ii) $\forall x, y \in \mathbb{R}, |xy| = |x||y|$. Especially, $|-x| = |x|$, in this case you would simply set $y = -1$.
(iii) $\forall x \in \mathbb{R}, -|x| \leq x \leq |x|$.
(iv) Let $a > 0, x \in \mathbb{R}$. Then, $|x| \leq a \iff -a \leq x \leq a$.

Theorem 12 (Triangle Inequality). Let $x, y \in \mathbb{R}$. Then:

- (i) $|x + y| \leq |x| + |y|$
- (ii) $|x - y| \geq ||x| - |y||$
- (iii) Especially,
 - (i) $|x - y| \geq |x| - |y|$
 - (ii) $|x - y| \geq |y| - |x|$

Corollary 1. We also have,

- (i) $|x - y| \leq |x| + |y|$
- (ii) $|x + y| \geq |x| - |y|$ and $|x + y| \geq |y| - |x|$.

Corollary 2 (Generalisation of the Triangle Inequality).

$$|x_1 + x_2 + \dots + x_n| \leq |x_1| + |x_2| + \dots + |x_n| \quad (11)$$

Definition 12. ε -neighbourhood Let $x \in \mathbb{R}$ and let $\varepsilon > 0$ be fixed. Then, the **ε -neighbourhood** of x , $V_\varepsilon(x)$, to be:

$$\begin{aligned} V_\varepsilon(x) &:=]x - \varepsilon, x + \varepsilon[\\ &= \{y \in \mathbb{R} \mid |y - x| < \varepsilon\} \end{aligned}$$

Theorem 13. Let $x, y \in \mathbb{R}$, where $x \neq y$. Then, “ x and y can be separated by neighbourhoods”, i.e., $\exists a \varepsilon > 0$ such that $V_\varepsilon(x) \cap V_\varepsilon(y) \neq \emptyset$.

1.2 Supremum and Infimum

Definition 13 (Bounded From Above). Let $S \subseteq \mathbb{R}, S \neq \emptyset$. We say that S is **bounded from above** if \exists a $u \in \mathbb{R}$ such that $\forall s \in S, s \leq u$.

Definition 14 (Bounded from Below). Let $S \subseteq \mathbb{R}, S \neq \emptyset$. We say that S is **bounded from below** if \exists a $u \in \mathbb{R}$ such that $\forall s \in S, u \leq s$.

Definition 15 (Supremum/Least Upper Bound). Let $S \subseteq \mathbb{R}, S \neq \emptyset$. $u \in \mathbb{R}$ is called a **supremum** or **least upper bound**, denoted by $\sup S$, if:

- (i) u is an upper bound for S .

- (ii) If v is any other upper bound for S , then $u \leq v$.

If $u = \sup S \in S$, then we say that u is the **maximum element** of S .

Definition 16 (Infimum/Greatest Lower Bound). Let $S \subseteq \mathbb{R}$, $S \neq \emptyset$. $u \in \mathbb{R}$ is called a **infimum** or **greatest lower bound**, denoted by $\inf S$, if:

- (i) u is a lower bound.
(ii) If v is an arbitrary lower bound of S , then $v \leq u$.

If $u = \inf S \in S$, then we say that u is the **minimum element of S** .