

# STAT 485/859: Design and Analysis of Experiments (TR 10:00-11:15 CL 247)

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# Outline

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- **INSTRUCTOR:**

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Office hours: TR 9:00-10:00, 12:30-14:30 or by appointment.

- **TEXT:** Douglas C. Montgomery: **Design and Analysis of Experiments**, 7th

Edition, John Wiley & Sons, Inc. 2009.

- **PREREQUISITE:**

STAT 351 and STAT 354 with grades of at least 60%.

- **COURSE CONTENT:**

Analysis of a single factor (analysis of variance; fixed effect models; random effects models); Randomized block designs and Latin squares; Factorial designs. Basically we will cover Chapters 1-7. Perhaps we may treat other interesting topic as time permits. I reserve the right to add or delete sections to the list.

- **EXAMS/ASSIGNMENTS/LAB:**

There will be regular assignments usually every one or two week(s) that are to be handed in and will be graded. There is one midterm test and a final examination to be scheduled as follows:

Midterm test: October 29, 2009 Thursday, 10:00-11:15;

Final exam: Dec 17, 2009, Thursday, 9:00-12:00

- **MARKING:**

The final mark for this course will be a weighted average of your test results, assignment results and lab. Assignments will count 20%(25%), midterm test will count 20%, lab will count 10%(0%), and the final exam will count 50% (55%).

**David Donoho,**

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**Stanford professor, NAS member, MaCarthy Genius Award winner, the world's #1 cited author in mathematical science in last decade(including math, applied math, and statistics)**

**"In no other field can you be (a) part mathematician, (b) part computer hacker, (c) part scientist, and (d) part ethical conscience of the whole world."**

**Bradley Efron**

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**Stanford professor, NAS member, MaCarthy Genius award winner, Wilks medalist, inventor of BOOTSTRAP.**

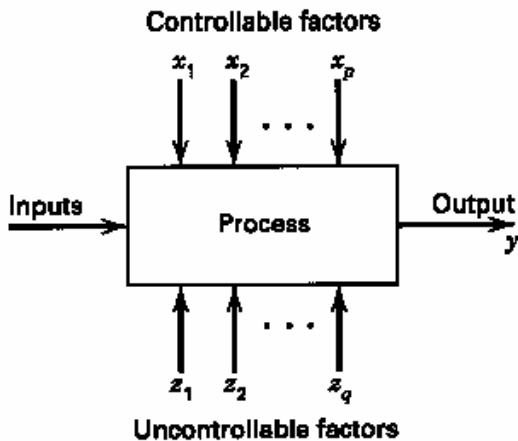
**"From the time I was a little boy until my senior year in college I wanted to be a mathematician. Then I learned that I really wanted to be a 19th century mathematician, the kind who does a little theory, a lot of computation, and some consulting with real scientists. The field of statistics has allowed me to do all three things, in whatever proportions I desired."**

# Chapter 1. Introduction

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## 1.1 Strategy of Experimentation

- Experiment: An experiment is a test or a series of tests
- Objectives of experiments:
  - ▶ Determine which  $x$  are most influential on the response variable  $y$ ;
  - ▶ Determine where to set the  $x$ 's so that  $y$  is almost always near the desired nominal value;
  - ▶ Determine where to set the  $x$ 's so that variability in  $y$  is small;
  - ▶ Determine where to set the influential  $x$ 's so that the effect of the uncontrollable variables  $z_1, z_2, \dots, z_q$  are minimized.
- The general approach to planning and conducting the experiment is called the strategy of experimentation.



**Figure 1-1** General model of a process or system.

## 1.2 Some typical application of experimental design

The application of experimental design techniques early in process development can result in

- ① Improved process yields
- ② Reduced variability and closer conformance to nominal or target requirements
- ③ Reduced development time
- ④ Reduced overall costs.

### 1.3 Basic principles:

Statistical design of experiments refers to the process of planning the experiment so that appropriate data will be collected, resulting in valid and objective conclusions.

There are two aspects to any experimental problem: the design of the experiment and the statistical analysis of the data

Basic principles of experimental design:

- 1 Replication:
  - \* Sample size (improving precision of effect estimation, estimation of error or background noise);
  - \* Replication versus repeat measurements?
- 2 Randomization;
  - \* Running the trial in an experiment in random order
  - \* Notion of balancing out effects of "lurking" variables
- 3 Blocking;
  - \* Dealing with nuisance factors



## 1.4 Guidelines for designing experiments

- 1 Recognition of and statement of the problem.
- 2 Choice of factors, levels and ranges.
- 3 Selection of the response variables.
- 4 Choice of experimental design.
- 5 Performing the experiment.
- 6 Statistical analysis of the data.
- 7 Conclusions and recommendations.

## 1.5 A brief history of statistical design

- The **agricultural** origins, 1918-1940s
  - \* R. A. Fisher and his co-workers
  - \* Profound impact on agricultural science
  - \* Factorial designs, ANOVA
- The **first industrial** era, 1951- late 1970s
  - \* Box and Wilson, response surfaces
  - \* Applications in the chemical & process industries
- The **second industrial** era, late 1970s-1990
  - \* Quality improvement initiatives in many companies
  - \* Taguchi and robust parameter design, process robustness
- The **modern** era, begin 1990.

## 1.6 Summary

- ❶ Use your nonstatistical knowledge of the problem
- ❷ Keep the design and analysis as simple as possible.
- ❸ Recognize the difference between practical and statistical significance
- ❹ Experiments are usually iterative.

# Chapter 2. Simple Comparative Experiments

## 2.1 Introduction

Table 2-1 Tension Bond Strength Data for the Portland Cement Formulation Experiment

	Modified Mortar	Unmodified Mortar
$j$	$y_{1j}$	$y_{2j}$
1	16.85	17.50
2	16.40	17.63
3	17.21	18.25
4	16.35	18.00
5	16.52	17.86
6	17.04	17.75
7	16.96	18.22
8	17.15	17.90
9	16.59	17.96
10	16.57	18.15

- Factor formulation : treatment or Level of factor
- Dot Diagram

## 2.2 Basic statistical concepts

- Run; experiment error or statistical error (noise);
- Random variable: discrete continuous;

Graphical description of variability:

- Dot diagram; find the location or central tendency; spread
- Box plot
- histogram;

Probability Distributions

- Distribution function; density function;
- Mean, variance, expected values

## 2.3 Sampling and sampling distributions

- Random samples, sample mean, and sample variance;  
random sample:  $y_1, y_2, \dots, y_n$  sample mean:

$$\bar{y} = \sum_{i=1}^n y_i / n$$

sample variance:

$$S^2 = \sum_{i=1}^n (y_i - \bar{y})^2 / (n - 1)$$

Standard deviation:  $S = \sqrt{S^2}$

- Properties of the sample mean and variance; estimator; estimate: particular value of estimator Good point estimator:
  - 1 Unbiased
  - 2 Minimum variance
- degree of freedom:  $n - 1$  is the number of d.f.

## 2.3 Sampling and sampling distributions (cont'd)

- Normal and other sampling distributions;

Normal distribution  $N(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(x - \mu)^2}{2\sigma^2}\right\}, \quad -\infty < x < \infty$$

$$N(0, 1) : Z = \frac{y - \mu}{\sigma}$$

- Central Limit Theorem:

### Theorem

*The central limit Theorem: If  $y_1, y_2, \dots, y_n$  is a sequence of  $n$  independent and identically distributed random variables with  $E(y_i) = \mu$  and  $V(y_i) = \sigma^2$  (both finite) and  $x = y_1 + y_2 + \dots + y_n$ , then the limiting form of the distribution of*

$$Z_n = \frac{x - n\mu}{\sqrt{n\sigma^2}}$$

*as  $n \rightarrow \infty$ , is the standard normal distribution.*

## 2.3 Sampling and sampling distributions (cont'd)

- Chi-square ( $\chi^2$ ) distribution

If  $z_1, z_2, \dots, z_k$  are normally and independently distributed random variables with mean 0 and variance 1, (NID(0,1)), then the random variable  $x = z_1^2 + z_2^2 + \dots + z_k^2$  follows the  $\chi^2$  distribution with  $k$  degrees of freedom. The density function of  $\chi^2$  is

$$f(x) = \frac{1}{2^{k/2} \Gamma(\frac{k}{2})} x^{(k/2)-1} e^{-x/2} \quad x > 0$$

with mean and variance  $\mu = k$  and  $\sigma^2 = 2k$ . Suppose that  $y_1, y_2, \dots, y_n$  is a random sample from an  $N(\mu, \sigma^2)$  distribution. Then

$$\frac{SS}{\sigma^2} = \frac{\sum_{i=1}^n (y_i - \bar{y})^2}{\sigma^2} \sim \chi_{n-1}^2$$



## 2.3 Sampling and sampling distributions (cont'd)

- $t$  (Student) Distribution: If  $z$  and  $\chi_k^2$  are independent standard normal and chi-square random variables respectively, the random variable

$$t_k = \frac{z}{\sqrt{\chi_k^2/k}}$$

follows the  $t$  distribution with  $k$  degrees of freedom, denoted  $t_k$ . The density function of  $t$  is

$$f(t) = \frac{\Gamma[(k+1)/2]}{\sqrt{k\pi}\Gamma(k/2)} \frac{1}{[(t^2/k) + 1]^{(k+1)/2}} \quad -\infty < t < +\infty$$

with mean and variance  $\mu = 0$  and  $\sigma^2 = k/(k-2)$  for  $k > 2$ , respectively. If  $y_1, y_2, \dots, y_n$  is a random sample from the  $N(\mu, \sigma^2)$  distribution, then the quantity

$$t = \frac{\bar{y} - \mu}{S/\sqrt{n}}$$

is distributed as  $t$  with  $n - 1$  degrees of freedom.

## 2.3 Sampling and sampling distributions (cont'd)

- $F$  distribution: If  $\chi_u^2$  and  $\chi_v^2$  are two independent chi-square random variables with  $u$  and  $v$  degrees of freedom, respectively, then the ratio

$$F_{u,v} = \frac{\chi_u^2/u}{\chi_v^2/v}$$

follows the  $F$  distribution with  $u$  – *numerator* degrees of freedom and  $v$  – *denominator* degrees of freedom. The probability distribution of  $F_{u,v}$  is

$$h(x) = \frac{\Gamma(\frac{u+v}{2}) (\frac{u}{v})^{u/2} x^{(u/2)-1}}{\Gamma(\frac{u}{2}) \Gamma(\frac{v}{2}) [(\frac{u}{v})x + 1]^{(u+v)/2}} \quad 0 < x < +\infty$$

If  $y_{11}, y_{12}, \dots, y_{1n_1}$  is a random sample of  $n_1$  observations from the first population  $N(\mu_1, \sigma^2)$ , and  $y_{21}, y_{22}, \dots, y_{2n_2}$  is a random sample of  $n_2$  observations from the second population  $N(\mu_2, \sigma^2)$ , then

$$\frac{S_1^2}{S_2^2} \sim F_{n_1-1, n_2-1}$$

where  $S_1^2$  and  $S_2^2$  are two sample variances.

## 2.4 Inference About The Differences In Means, Randomized Designs

### 2.4.1 Hypothesis testing

Suppose that the factor contains two treatments (or levels). Let  $y_{11}, y_{12}, \dots, y_{1n_1}$  represent the  $n_1$  observations from the first factor level and  $y_{21}, y_{22}, \dots, y_{2n_2}$  represent the  $n_2$  observations from the second factor level. Also, suppose that the samples are drawn at random from two independent normal populations.

- A Model of the Data

$$y_{ij} = \mu_i + \epsilon_{ij}, \quad i = 1, 2; j = 1, 2, \dots, n_i$$

- Statistical Hypothesis

A statistical hypothesis is statement either about the parameters of a probability distribution or the parameters of a model.

$$H_0 : \mu_1 = \mu_2$$

null hypothesis

$$H_1 : \mu_1 \neq \mu_2$$

alternative hypothesis

## 2.4 Inference About The Differences In Means, Randomized Designs (cont'd)

- Test Statistic, Critical Region (or Rejection Region)

Type I error  $\alpha = P(\text{type I error}) = P(\text{reject } H_0 | H_0 \text{ is true})$

Type II error  $\beta = P(\text{type I error}) = P(\text{fail to reject } H_0 | H_0 \text{ is false})$

Power  $= 1 - \beta = P(\text{reject } H_0 | H_0 \text{ is false})$

$\alpha$  is called the significant level of the test.

- The Two-Sample t-Test:

$$t_0 = \frac{\bar{y}_1 - \bar{y}_2}{S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \quad S_p^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}$$

- The use of P-Values in Hypothesis Testing
- Computer Solution
- Checking Assumptions in the t-Test
- Normal Probability Plot

## 2.4 Inference About The Differences In Means, Randomized Designs (cont'd)

**Example** Now consider the Portland cement data, From these data we find that

Modified Mortar	$\bar{y}_1 = 16.76 \text{ kgf/cm}^2$	$S_1^2 = 0.10$	$S_1 = 0.316$	$n_1 = 10$
Unmodified Mortar	$\bar{y}_2 = 17.04 \text{ kgf/cm}^2$	$S_2^2 = 0.061$	$S_2 = 0.248$	$n_2 = 10$

we test the hypothesis:

$$H_0 : \mu_1 = \mu_2, \quad H_1 : \mu_1 \neq \mu_2$$

If we choose  $\alpha = 0.05$ ,  $t_{\alpha/2, 18} = 2.101$  where  $n_1 + n_2 - 2 = 18$  is the degrees of freedom. Next we compute the value of test statistic  $T_0$ . At first, we have that

$$S_p^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2} = \frac{9(0.100) + 9(0.061)}{10 + 10 - 2} = 0.081, \quad S_p = 0.284$$

and the test statistic is

$$t_0 = \frac{\bar{y}_1 - \bar{y}_2}{S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{16.76 - 17.04}{0.284 \sqrt{\frac{1}{10} + \frac{1}{10}}} = \frac{-0.28}{0.127} = -2.20$$

Since  $t_0 = -2.20 < -t_{0.025, 18} = -2.101$ , we reject  $H_0$  and conclude that the mean tension bond strengths of the two formulations of Portland cement mortar are different at the significance level  $\alpha = 0.05$ .

## 2.4 Inference About The Differences In Means, Randomized Designs (cont'd)

### 2.4.2 Choice of Sample Size

Suppose that we are testing the hypothesis

$$H_0 : \mu_1 = \mu_2 \quad \text{vs} \quad H_1 : \mu \neq \mu_2$$

and the mean are not equal so that  $\delta = \mu_1 - \mu_2$ .

The probability of type II error depends on the true difference in mean  $\delta$ . A graph of  $\beta$  versus  $\delta$  for a [articular sample size is called the **operating characteristic curve** or **O.C. curve** for the test.

$$d = \frac{|\mu_1 - \mu_2|}{2\sigma} = \frac{|\delta|}{2\sigma}$$

Furthermore, the sample size used to construct the curve is actually  $n^* = 2n - 1$ . Note that

- 1 The greater the difference in means  $\mu_1 - \mu_2$ , the smaller the probability of type II error for a given sample size and  $\alpha$ .
- 2 As the sample size gets large, the probability of type II error gets smaller for a given difference in means and  $\alpha$ .

## 2.4 Inference About The Differences In Means, Randomized Designs (cont'd)

**Example** Consider the Portland cement mortar. Suppose that a difference in means is  $0.5 \text{ kgf/cm}^2$ . Then,

$$d = \frac{|\mu_1 - \mu_2|}{2\sigma} = \frac{0.5}{2\sigma} = \frac{0.25}{\sigma}.$$

Assume that  $\sigma = 0.25$ . Then we have  $d = 1$ . If we wish to reject the null hypothesis 95 percent of the time when  $\mu_1 - \mu_2 = 0.5$ , then  $\beta = 0.05$  and from the figure we find  $n^* = 2n - 1 = 16$  and the required sample size is

$$n = \frac{n^* + 1}{2} = \frac{16 + 1}{2} = 8.5 = 9$$

## 2.4 Inference About The Differences In Means, Randomized Designs (cont'd)

### 2.4.3 Confidence Intervals

Sometimes the interest of experimenters is the interval estimate on the difference in means  $\mu_1 - \mu_2$ .

To define a confidence interval, we need to find two statistics  $L$  and  $U$  such that

$$P(L \leq \theta \leq U) = 1 - \alpha$$

where  $\theta$  is an unknown parameter and  $1 - \alpha$  is the confidence coefficient. The interval

$$L \leq \theta \leq U$$

is called a  $100(1 - \alpha)$  **percent confidence interval** for the parameter  $\theta$ .  $L$  is lower limit;  $U$  is upper limit. For example, in order to obtain the confidence interval for  $\mu_1 - \mu_2$  in Data 1.2.1, we use the statistic:

$$\frac{\bar{y}_1 - \bar{y}_2 - (\mu_1 - \mu_2)}{S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \sim t_{n_1 + n_2 - 2}$$



## 2.4 Inference About The Differences In Means, Randomized Designs (cont'd)

The obtained  $100(1 - \alpha)$  percent confidence interval is

$$\begin{aligned} y_1 - y_2 - t_{\alpha/2, n_1+n_2-2} S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} &\leq \mu_1 - \mu_2 \\ &\leq y_1 - y_2 + t_{\alpha/2, n_1+n_2-2} S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} \end{aligned}$$

The 95% confidence interval for the difference in mean tension bond strength for the formulation of Portland cement mortar is

$$\begin{aligned} 16.76 - 17.04 - (2.101)0.284 \sqrt{\frac{1}{10} + \frac{1}{10}} &\leq \mu_1 - \mu_2 \\ &\leq 16.76 - 17.04 + (2.101)0.284 \sqrt{\frac{1}{10} + \frac{1}{10}} \\ -0.28 - 0.27 &\leq \mu_1 - \mu_2 \leq -0.28 + 0.27 \\ -0.55 &\leq \mu_1 - \mu_2 \leq -0.01 \end{aligned}$$

## 2.4 Inference About The Differences In Means, Randomized Designs (cont'd)

### 2.4.4 The Case Where $\sigma_1^2 \neq \sigma_2^2$

If we are testing

$$H_0 : \mu_1 = \mu_2 \quad \text{v.s.} H_1 \mu_1 \neq \mu_2$$

and can not reasonably assume that the variances  $\sigma_1^2$  and  $\sigma_2^2$  are equal, then the two-sample  $t$ -test must be modified slightly. The test statistic becomes

$$t_0 = \frac{\bar{y}_1 - \bar{y}_2}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}}$$

This statistic is well approximated by  $t$  if we use

$$\nu = \frac{\left( \frac{S_1^2}{n_1} + \frac{S_2^2}{n_2} \right)^2}{\frac{(S_1^2/n_1)^2}{n_1-1} + \frac{(S_2^2/n_2)^2}{n_2-1}}$$

as the degrees of freedom.

## 2.4 Inference About The Differences In Means, Randomized Designs (cont'd)

### 2.4.5 The Case Where $\sigma_1^2$ and $\sigma_2^2$ Are Known

If the variances of both populations are known, then the hypotheses

$$H_0 : \mu_1 = \mu_2 \quad \text{v.s.} \quad H_1 : \mu_1 \neq \mu_2$$

may be tested by using the statistic

$$Z_0 = \frac{\bar{y}_1 - \bar{y}_2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \sim N(0, 1)$$

We reject  $H_0$  if  $|Z_0| \geq Z_{\alpha/2}$ . The  $100(1 - \alpha)$  percent confidence interval on  $\mu_1 - \mu_2$  where the variances are known is

$$\bar{y}_1 - \bar{y}_2 - Z_{\alpha/2} \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \leq \mu_1 - \mu_2 \leq \bar{y}_1 - \bar{y}_2 + Z_{\alpha/2} \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

## 2.4 Inference About The Differences In Means, Randomized Designs (cont'd)

**2.4.6 Comparing a Single Mean to a Specified Value** This this case the hypotheses are

$$H_0 : \mu = \mu_0 \quad v.s. H_1 \mu \neq \mu_0$$

The test statistic is

$$Z_0 = \frac{\bar{y} - \mu_0}{\sigma/\sqrt{n}}$$

if the variance  $\sigma^2$  is known. Otherwise, the test statistic is

$$t_0 = \frac{\bar{y} - \mu_0}{S/\sqrt{n}}$$

.The corresponding  $100(1 - \alpha)$  percent confidence intervals are

$$\bar{y} - Z_{\alpha/2}\sigma/\sqrt{n} \leq \mu \leq \bar{y} + Z_{\alpha/2}\sigma/\sqrt{n}$$

and

$$\bar{y} - t_{\alpha/2, n-1}S/\sqrt{n} \leq \mu \leq \bar{y} + t_{\alpha/2, n-1}S/\sqrt{n}$$

## 2.4 Inference About The Differences In Means, Randomized Designs (cont'd)

## 2.4.7 Summary

Table 2-3 Tests on Means with Variance Known

Hypothesis	Test statistic	Criteria for Rejection	P-value
$H_0 : \mu = \mu_0$ $H_1 : \mu \neq \mu_0$		$ Z_0  > Z_{\alpha/2}$	$P = 2[1 - \Phi(Z_0)]$
$H_0 : \mu = \mu_0$ $H_1 : \mu < \mu_0$	$Z_0 = \frac{\bar{y} - \mu_0}{\sigma/\sqrt{n}}$	$Z_0 < -Z_{\alpha}$	$P = \Phi(Z_0)$
$H_0 : \mu = \mu_0$ $H_1 : \mu > \mu_0$		$Z_0 > Z_{\alpha}$	$P = 1 - \Phi(Z_0)$
$H_0 : \mu_1 = \mu_2$ $H_1 : \mu_1 \neq \mu_2$		$ Z_0  > Z_{\alpha/2}$	$P = 2[1 - \Phi(Z_0)]$
$H_0 : \mu_1 = \mu_2$ $H_1 : \mu_1 < \mu_2$	$Z_0 = \frac{\bar{y}_1 - \bar{y}_2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$	$Z_0 < -Z_{\alpha}$	$P = \Phi(Z_0)$
$H_0 : \mu_1 = \mu_2$ $H_1 : \mu_1 > \mu_2$		$Z_0 > Z_{\alpha}$	$P = 1 - \Phi(Z_0)$

## 2.4 Inference About The Differences In Means, Randomized Designs (cont'd)

Table 2-4 Tests on Means of Normal Distributions, with Variance Unknown

Hypothesis	Test statistic	Criteria for Rejection
$H_0 : \mu = \mu_0$ $H_1 : \mu \neq \mu_0$		$ t_0  > t_{\alpha/2, n-1}$
$H_0 : \mu = \mu_0$ $H_1 : \mu < \mu_0$	$t_0 = \frac{\bar{y} - \mu_0}{S/\sqrt{n}}$	$t_0 < -t_{\alpha, n-1}$
$H_0 : \mu = \mu_0$ $H_1 : \mu > \mu_0$		$t_0 > t_{\alpha, n-1}$
$H_0 : \mu_1 = \mu_2$ $H_1 : \mu_1 \neq \mu_2$	if $\sigma_1^2 = \sigma_2^2$ $t_0 = \frac{\bar{y}_1 - \bar{y}_2}{S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$	$ t_0  > t_{\alpha/2, \nu}$
$H_0 : \mu_1 = \mu_2$ $H_1 : \mu_1 < \mu_2$	$\nu = n_1 + n_2 - 2$ if $\sigma_1^2 \neq \sigma_2^2$ $t_0 = \frac{\bar{y}_1 - \bar{y}_2}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}}$	$t_0 < -t_{\alpha, \nu}$
$H_0 : \mu_1 = \mu_2$ $H_1 : \mu_1 > \mu_2$	$\nu = \frac{(\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2})^2}{\frac{(S_1^2/n_1)^2}{n_1-1} + \frac{(S_2^2/n_2)^2}{n_2-1}}$	$t_0 > t_{\alpha, \nu}$

## 2.5 Inference About The Differences In Means, Paired Comparison Designs

**2.5.1 The paired Comparison Problem** In some simple comparative experiments, we can greatly improve the precision by making comparisons within matched pairs for experimental material.

Table 2-5 Data for the Hardness Testing Experiment

spaceman	Tip 1	Tip 2	Difference
$j$	$y_{1j}$	$y_{2j}$	$d_j$
1	7	6	1
2	3	3	0
3	3	5	-2
4	4	3	1
5	8	8	0
6	3	2	1
7	2	4	-2
8	9	9	0
9	5	4	0
10	4	5	-1

## 2.5 Inference About The Differences In Means, Paired Comparison Designs

Consider an alternative experimental design. Assume that each specimen is large enough so that two hardness determinations may be made on it.

We may write a statistical model as

$$y_{ij} = \mu_i + \beta_j + \epsilon, \quad i = 1, 2; j = 1, 2, \dots, 10$$

Note that if we compute the  $j$ th paired difference

$$d_j = y_{1j} - y_{2j}, \quad j = 1, 2, \dots, 10$$

the expected value of this difference is

$$\mu_d = E(d_j) = E(y_{1j}) - E(y_{2j}) = \mu_1 + \beta_j - (\mu_2 + \beta_j) = \mu_1 - \mu_2$$

Testing  $H_0 : \mu_1 = \mu_2$  is equivalent to testing

$$H_0 : \mu_d = 0, \quad H_1 : \mu_d \neq 0$$

The test statistic is

$$t_0 = \frac{\bar{d}}{S_d / \sqrt{n}}$$



2.5 Inference About The Differences In Means, Paired Comparison Designs For the data in Table 2.5, we have

$$\bar{d} = \frac{1}{n} \sum_{i=1}^n d_i = \frac{1}{10}(-1) = -0.10$$

$$S_d^2 = \frac{\sum_{i=1}^n (d_i - \bar{d})^2}{n-1} = \frac{\sum_{i=1}^n d_i^2 - \frac{1}{n}(\sum_{i=1}^n d_i)^2}{n-1} = \frac{13 - \frac{1}{10}(-1)^2}{10-1} = \frac{43}{30}$$

Suppose that  $\alpha = 0.05$ . Then the critical value is  $t_{0.025,9} = 2.262$ . Now the value of test statistic is

$$t_0 = \frac{\bar{d}}{S_d/\sqrt{n}} = \frac{-0.10}{\sqrt{43/(30 * 10)}} = -0.264135272$$

Since  $|t_0| = 0.264 < t_{0.025,9} = 2.262$ , we fail to reject the hypothesis  $H_0 : \mu_d = 0$ . That is, there is no evidence to indicate that the two tips produce different readings.