STAT 485/859: Design and Analysis of Experiments (TR 13:00-14:15 CL 312)

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Outline

- **INSTRUCTOR:**
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  Office hours: MWF 14:30-16:30 or by appointment.
- **TEXT:** Douglas C. Montgomery: **Design and Analysis of Experiments,** 8th Edition, John Wiley & Sons, Inc. 2013.
- **PREREQUISITE:**
  STAT 354.
- **COURSE CONTENT:**
  Analysis of a single factor (analysis of variance; fixed effect models; random effects models); Randomized block designs and Latin squares; Factorial designs. Basically we will cover Chapters 1-7. Perhaps we may treat other interesting topic as time permits. I reserve the right to add or delete sections to the list.

- **EXAMS/ASSIGNMENTS/LAB:**

  There will be regular assignments usually every one or two week(s) that are to be handed in and will be partially graded. There are one midterm test and a final examination to be scheduled as follows:
  Midterm Test : Oct. 24, Thursday, 13:00-14:15;
  Final Exam: Dec 12, Thursday, 14:00-17:00 - Location TBA .

- **MARKING:**

  The final mark for this course will be a weighted average of your test results, assignment results and attendance. For undergraduates, assignments will count 20%, one midterm tests will count 20%, attendance will count 10% and the final exam will count 50%. For graduates, assignments will count 20%, two midterm tests will count 20%, and the final exam will count 60%. To pass this course students must pass the final exam.

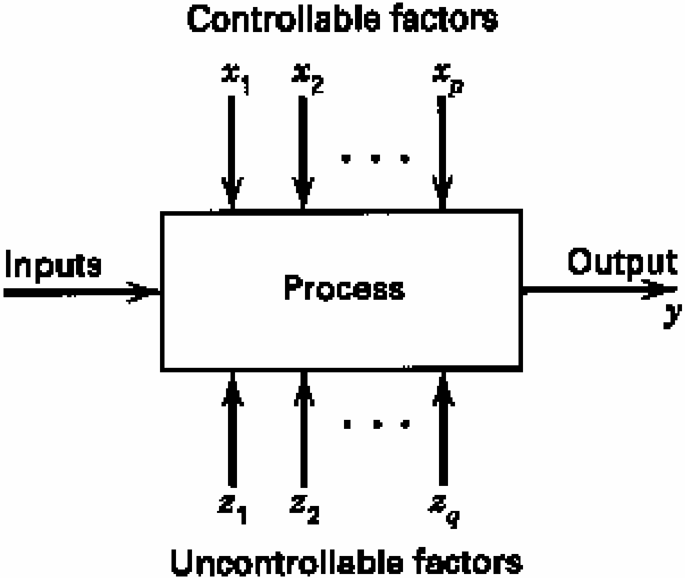
- **THE STATEMENT ON CHEATING:**

  Cheating will not be tolerated in this class. By "cheating" I mean submitting work that is not your own. This includes plagiarism (copying off of another student or from another source) or having another person write your assignments or tests. If I suspect a student of cheating, their assignment or test will be sent to the Associate Dean Academic for the Faculty of Science who will then contact the student and deal with the situation. Typical consequences for cheating can be found at: http://www.uregina.ca/arts/student-resources/avoiding-academic-misconduct/penalty- guidelines.html. Read the entry in the undergraduate calendar about student behaviour (Section 5.13), the policies listed in the calendar will be followed in this class.

Chapter 1. Introduction

1.1 Strategy of Experimentation

- Experiment: An experiment is a test or a series of tests
- Objectives of experiments:
  - Determine which x are most influential on the response variable y;
  - Determine where to set the x's so that y is almost always near the desired nominal value;
  - Determine where to set the x's so that variability in y is small;
  - Determine where to set the influential x's so that the effect of the uncontrollable variables $z_1, z_2, ..., z_q$ are minimized.
- The general approach to planning and conducting the experiment is called the strategy of experimentation.


Controllable factors
x_1
x_2
x_p
Inputs
Process
Output
y
z_1
z_2
z_q
Uncontrollable factors

1.2 Some typical application of experimental design

The application of experimental design techniques early in process development can result in

1. Improved process yields
2. Reduced variability and closer conformance to nominal or target requirements
3. Reduced development time
4. Reduced overall costs.

1.3 Basic principles:

Statistical design of experiments refers to the process of planning the experiment so that appropriate data will be collected, resulting in valid and objective conclusions.

There are two aspects to any experimental problem: the design of the experiment and the statistical analysis of the data
Basic principles of experimental design:

1. Replication:

   * Sample size (improving precision of effect estimation, estimation of error or background noise);

   * Replication versus repeat measurements?

2. Randomization;

   * Running the trial in an experiment in random order

   * Notion of balancing out effects of "lurking" variables

3. Blocking;

   * Dealing with nuisance factors

1.4 Guidelines for designing experiments

1. Recognition of and statement of the problem.
2. Choice of factors, levels and ranges.
3. Selection of the response variables.
4. Choice of experimental design.
5. Performing the experiment.
6. Statistical analysis of the data.
7. Conclusions and recommendations.

1.5 A brief history of statistical design

- The **agricultural** origins, 1918-1940s
  * R. A. Fisher and his co-workers
  * Profound impact on agricultural science
  * Factorial designs, ANOVA
- The **first industrial** era, 1951- late 1970s
  * Box and Wilson, response surfaces
  * Applications in the chemical & process industries
- The **second industrial** era, late 1970s-1990
  * Quality improvement initiatives in many companies
  * Taguchi and robust parameter design, process robustness
- The **modern** era, begin 1990.

1.6 Summary

1. Use your nonstatistical knowledge of the problem
2. Keep the design and analysis as simple as possible.
3. Recognize the difference between practical and statistical significance
4. Experiments are usually iterative.

Chapter 2. Simple Comparative Experiments

2.1 Introduction

Table 2-1 Tension Bond Strength Data for the Portland Cement Formulation Experiment

	Modified Mortar	Unmodified Mortar
j	y_{1j}	y_{2j}
1	16.85	17.50
2	16.40	17.63
3	17.21	18.25
4	16.35	18.00
5	16.52	17.86
6	17.04	17.75
7	16.96	18.22
8	17.15	17.90
9	16.59	17.96
10	16.57	18.15

- Factor formulation : treatment or level of factor
- Dot Diagram

2.2 Basic statistical concepts

- Run; experiment error or statistical error (noise);
- Random variable: discrete continuous;

Graphical description of variability:

- Dot diagram; find the location or central tendency; spread
- Box plot
- histogram;

Probability Distributions

- Distribution function; density function;
- Mean, variance, expected values

2.3 Sampling and sampling distributions

- Random samples, sample mean, and sample variance;

  random sample: $y_1, y_2, ..., y_n$ sample mean:

$$\bar{y} = \sum_{i=1}^{n} y_i / n$$

  sample variance:

$$S^2 = \sum_{i=1}^{n} (y_i - \bar{y})^2 / (n-1)$$

  Standard deviation: $S = \sqrt{S^2}$

- Properties of the sample mean and variance; estimator; estimate: particular value of estimator

  Good point estimator:

  1. Unbiased
  2. Minimum variance

- degree of freedom: $n-1$ is the number of d.f.

- Normal and other sampling distributions;
  Normal distribution $N(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\{-\frac{(x-\mu)^2}{2\sigma^2}\}, \quad -\infty < x < \infty$$

  $N(0,1) : Z = \frac{y-\mu}{\sigma}$
- Central Limit Theorem:

Theorem

The central limit Theorem: If $y_1, y_2, ..., y_n$ is a sequence of n independent and identically distributed random variables with $E(y_i) = \mu$ and $V(y_i) = \sigma^2$ (both finite) and $x = y_1 + y_2 + ... + y_n$, then the limiting form of the distribution of

$$z_n = \frac{x - n\mu}{\sqrt{n\sigma^2}}$$

as $n \to \infty$, is the standard normal distribution.

- Chi-square (χ^2) distribution

  If $z_1, z_2, \cdots, z_k$ are normally and independently distributed random variables with mean 0 and variance 1, (NID(0,1)), then the random variable $x = z_1^2 + z_2^2 + \cdots + z_k^2$ follows the χ^2 distribution with k degrees of freedom. The density function of χ^2 is

$$f(x) = \frac{1}{2^{k/2}\Gamma(\frac{k}{2})} x^{(k/2)-1} e^{-x/2} \qquad x > 0$$

  with mean and variance $\mu = k$ and $\sigma = 2k$. Suppose that $y_1, y_2, \cdots, y_n$ is a random sample from an $N(\mu, \sigma^2)$ distribution. Then

$$\frac{SS}{\sigma^2} = \frac{\sum_{i=1}^{n}(y_i - \bar{y})^2}{\sigma^2} \sim \chi^2_{n-1}$$

- t (Student) Distribution: If z and χ_k^2 are independent standard normal and chi-square random variables respectively, the random variable

$$t_k = \frac{z}{\sqrt{\chi_k^2/k}}$$

follows the t *distribution with* k *degrees of freedom*, denoted t_k. The density function of t is

$$f(t) = \frac{\Gamma[(k+1)/2]}{\sqrt{k\pi}\Gamma(k/2)} \frac{1}{[(t^2/k)+1]^{(k+1)/2}} \qquad -\infty < t < +\infty$$

with mean and variance $\mu = 0$ and $\sigma^2 = k/(k-2)$ for $k > 2$, respectively. If $y_1, y_2, \cdots, y_n$ is a random sample from the $N(\mu, \sigma^2)$ distribution, then the quantity

$$t = \frac{\bar{y} - \mu}{S/\sqrt{n}}$$

is distributed as t with $n-1$ degrees of freedom.

- F distribution: If χ_u^2 and χ_v^2 are two independent chi-square random variables with u and v degrees of freedom, respectively, then the ratio

$$F_{u,v} = \frac{\chi_u^2/u}{\chi_v^2/v}$$

follows the F distribution with $u - numerator$ degrees of freedom and $v - denominator$ degrees of freedom. The probability distribution of $F_{u,v}$ is

$$h(x) = \frac{\Gamma(\frac{u+v}{2})(\frac{u}{v})^{u/2}x^{(u/2)-1}}{\Gamma(\frac{u}{2})\Gamma(\frac{v}{2})[(\frac{u}{v})x+1]^{(u+v)/2}} \qquad 0 < x < +\infty$$

If $y_{11}, y_{12}, \cdots, y_{1n_1}$ is a random sample of n_1 observations from the first population $N(\mu_1, \sigma^2)$, and $y_{21}, y_{22}, \cdots, y_{2n_2}$ is a random sample of n_2 observations from the second population $N(\mu_2, \sigma^2)$, then

$$\frac{S_1^2}{S_2^2} \sim F_{n_1-1,n_2-1}$$

where S_1^2 and S_2^2 are two sample variances.

2.4.1 Hypothesis testing

Suppose that the factor contains two treatments (or levels). Let $y_{11}, y_{12}, ..., y_{1n_1}$ represent the n_1 observations from the first factor level and $y_{21}, y_{22}, ..., y_{2n_1}$ represent the n_2 observations from the second factor level. Also, suppose that the samples are drawn at random from two independent normal populations.

- A Model of the data

$$y_{ij} = \mu_i + \epsilon_{ij}, \qquad i = 1, 2; j = 1, 2, ..., n_i$$

- Statistical Hypothesis

  A statistical hypothesis is statement either about the parameters of s probability distribution or the parameters of a models.

$$\begin{aligned} H_0 &: \mu_1 = \mu_2 && \text{null hypothesis} \\ H_1 &: \mu_1 \neq \mu_2 && \text{alternative hypothesis} \end{aligned}$$

- Test Statistic, Critical Region (or Rejection Region)

Type I error	$\alpha = P(\text{type I error}) = P(\text{reject } H_0 \mid H_0 \text{ is true})$
Type II error	$\beta = P(\text{type I error}) = P(\text{fail to reject } H_0 \mid H_0 \text{ is false})$
	$\text{Power} = 1 - \beta = P(\text{reject } H_0 \mid H_0 \text{ is false})$

  α is called the significant level of the test.

- The Two-Sample t-Test:

$$t_0 = \frac{\bar{y}_1 - \bar{y}_2}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \qquad S_p^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}$$

- The use of P-Values in Hypothesis Testing
- Computer Solution
- Checking Assumptions in the t-Test
- Normal Probability Plot

Example

Now consider the Portland cement data, From these data we find that

Modified Mortar	$\bar{y}_1 = 16.76\text{kgf/cm}^2$	$S_1^2 = 0.10$	$S_1 = 0.316$	$n_1 = 10$
Unmodified Mortar	$\bar{y}_2 = 17.04\text{kgf/cm}^2$	$S_2^2 = 0.061$	$S_2 = 0.248$	$n_2 = 10$

we test the hypothesis:

$$H_0 : \mu_1 = \mu_2, \qquad H_1 : \mu_1 \neq \mu_2$$

If we choose $\alpha = 0.05$, $t_{\alpha/2,18} = 2.101$ where $n_1 + n_2 = 2 = 18$ is the degrees of freedom. Next we compute the value of test statistic T_0. At first, we have that

$$S_p^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2} = \frac{9(0.100) + 9(0.061)}{10 + 10 - 2} = 0.081, \quad S_p = 0.284$$

and the test statistic is

$$t_0 = \frac{\bar{y}_1 - \bar{y}_2}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{16.76 - 17.04}{0.284\sqrt{\frac{1}{10} + \frac{1}{10}}} = \frac{-0.28}{0.127} = -2.20$$

Since $t_0 = -2.20 < -t_{0.025,18} = -2.101$, we reject H_0 and conclude that the mean tension bond strengths of the two formulations of Portland cement mortar are different at the significance level $\alpha = 0.05$.

2.4.2 Choice of Sample Size

Suppose that we are testing the hypothesis

$$H_0 : \mu_1 = \mu_2 \qquad vs \qquad H_1 : \mu \neq \mu_2$$

and the mean are not equal so that $\delta = \mu_1 - \mu_2$.
The probability of type II error depends on the true difference in mean δ. A graph of β versus δ for a [articular sample size is called the **operating characteristic curve** or **O.C. curve** for the test.

$$d = \frac{|\mu_1 - \mu_2|}{2\sigma} = \frac{|\delta|}{2\sigma}$$

Furthermore, the sample size used to construct the curve is actually $n* = 2n - 1$. Note that

1. The greater the difference in means $\mu_1 - \mu_2$, the smaller the probability of type II error for a given sample size and α.
2. As the sample size gets large, the probability of type II error gets smaller for a given difference in means and α.

Example

Consider the Portland cement mortar. Suppose that a difference in means is 0.5 kgf/cm^2. Then,

$$d = \frac{|\mu_1 - \mu_2|}{2\sigma} = \frac{0.5}{2\sigma} = \frac{0.25}{\sigma}.$$

Assume that $\sigma = 0.25$ Then we have $d = 1$. If we wish to reject the null hypothesis 95 percent of the time when $\mu_1 - \mu_2 = 0.5$, then $\beta = 0.05$ and from the figure we find $n^* = 2n - 1 = 16$ and the required sample size is

$$n = \frac{n * +1}{2} = \frac{16 + 1}{2} = 8.5 = 9$$

2.4.3 Confidence Intervals

Sometimes the interest of experimenters is the interval estimate on the difference in means $\mu_1 - \mu_2$.
To define a confidence interval, we need to find two statistics L and U such that

$$P(L \leq \theta \leq U) = 1 - \alpha$$

where θ is an unknown parameter and $1 - \alpha$ is the confidence coefficient.
The interval

$$L \leq \theta \leq U$$

is called a $100(1 - \alpha)$ **percent confidence interval** for the parameter θ. L is lower limit; U is upper limit. For example, in order to obtain the confidence interval for $\mu_1 - \mu_2$ in Data 1.2.1, we use the statistic:

$$\frac{\bar{y}_1 - \bar{y}_2 - (\mu_1 - \mu_2)}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \sim t_{n_1+n_2-2}$$

The obtained $100(1-\alpha)$ percent confidence interval is

$$y_1 - y_2 - t_{\alpha/2,n_1+n_2-2} S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} \le \mu_1 - \mu_2$$
$$\le y_1 - y_2 + t_{\alpha/2,n_1+n_2-2} S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

The 95% confidence interval for the difference in mean tension bond strength for the formulation of Portland cement mortar is

$$16.76 - 17.04 - (2.101)0.284\sqrt{\frac{1}{10} + \frac{1}{10}} \le \mu_1 - \mu_2$$
$$\le 16.76 - 17.04 - (2.101)0.284\sqrt{\frac{1}{10} + \frac{1}{10}}$$
$$-0.28 - 0.27 \le \mu_1 - \mu_2 \le -0.28 + 0.27$$
$$-0.55 \le \mu_1 - \mu_2 \le -0.01$$

2.4.4 The Case Where $\sigma_1^2 \neq \sigma_2^2$

If we are testing

$$H_0 : \mu_1 = \mu_2 \qquad v.s. H_1 \mu_1 \neq \mu_2$$

and can not reasonably assume that the variances σ_1^2 and σ_2^2 are equal, then the two-sample t-test must be modified slightly. The test statistic becomes

$$t_0 = \frac{\bar{y}_1 - \bar{y}_2}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}}$$

This statistic is well approximated by t if we use

$$\nu = \frac{\left(\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}\right)^2}{\frac{(S_1^2/n_1)^2}{n_1 - 1} + \frac{(S_2^2/n_2)^2}{n_2 - 1}}$$

as the degrees of freedom.

2.4.5 The Case Where σ_1^2 and σ_2^2 Are Known

If the variances of both populations are known, then the hypotheses

$$H_0 : \mu_1 = \mu_2 \qquad v.s. H_1 \mu_1 \neq \mu_2$$

may be tested by using the statistic

$$Z_0 = \frac{\bar{y}_1 - \bar{y}_2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \sim N(0,1)$$

We reject H_0 if $|Z_0| \geq Z_{\alpha/2}$. The $100(1-\alpha)$ percent confidence interval on $\mu_1 - \mu_2$ where the variances are known is

$$\bar{y}_1 - \bar{y}_2 - Z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \leq \mu_1 - \mu_2 \leq \bar{y}_1 - \bar{y}_2 + Z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

2.4.6 Comparing a Single Mean to a Specified Value This this case the hypotheses are

$$H_0 : \mu = \mu_0 \qquad v.s. \quad H_1 : \mu \neq \mu_0$$

The test statistic is

$$Z_0 = \frac{\bar{y} - \mu_0}{\sigma/\sqrt{n}}$$

if the variance σ^2 is known. Otherwise, the test statistic is

$$t_0 = \frac{\bar{y} - \mu_0}{S/\sqrt{n}}.$$

The corresponding $100(1-\alpha)$ percent confidence intervals are

$$\bar{y} - Z_{\alpha/2}\sigma/\sqrt{n} \leq \mu \leq \bar{y} + Z_{\alpha/2}\sigma/\sqrt{n}$$

and

$$\bar{y} - t_{\alpha/2,n-1}S/\sqrt{n} \leq \mu \leq \bar{y} + t_{\alpha/2,n-1}S/\sqrt{n}$$

2.4.7 Summary

Table 2-3 Tests on Means with Variance Known

Hypothesis	Test statistic	Criteria for Rejection	P-value
$H_0 : \mu = \mu_0$ $H_1 : \mu \neq \mu_0$		$\lvert Z_0 \rvert > Z_{\alpha/2}$	$P = 2[1 - \Phi(Z_0)]$
$H_0 : \mu = \mu_0$ $H_1 : \mu < \mu_0$	$Z_0 = \frac{\bar{y}-\mu_0}{\sigma/\sqrt{n}}$	$Z_0 < -Z_\alpha$	$P = \Phi(Z_0)$
$H_0 : \mu = \mu_0$ $H_1 : \mu > \mu_0$		$Z_0 > Z_\alpha$	$P + 1 - \Phi(Z_0)$
$H_0 : \mu_1 = \mu_2$ $H_1 : \mu_1 \neq \mu_2$		$\lvert Z_0 \rvert > Z_{\alpha/2}$	$P = 2[1 - \Phi(Z_0)]$
$H_0 : \mu_1 = \mu_2$ $H_1 : \mu_1 < \mu_2$	$Z_0 = \frac{\bar{y}_1-\bar{y}_2}{\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}}$	$Z_0 < -Z_\alpha$	$P = \Phi(Z_0)$
$H_0 : \mu_1 = \mu_2$ $H_1 : \mu_1 > \mu_2$		$Z_0 > Z_\alpha$	$P = 1 - \Phi(Z_0)$

Table 2-4 Tests on Means of Normal Distributions, with Variance Unknown

Hypothesis	Test statistic	Criteria for Rejection
$H_0: \mu = \mu_0$ $H_1: \mu \neq \mu_0$		$\lvert t_0 \rvert > t_{\alpha/2, n-1}$
$H_0: \mu = \mu_0$ $H_1: \mu < \mu_0$	$t_0 = \frac{\bar{y} - \mu_0}{S/\sqrt{n}}$	$t_0 < -t_{\alpha, n-1}$
$H_0: \mu = \mu_0$ $H_1: \mu > \mu_0$		$t_0 > t_{\alpha, n-1}$
$H_0: \mu_1 = \mu_2$ $H_1: \mu_1 \neq \mu_2$	if $\sigma_1^2 = \sigma_2^2$ $t_0 = \frac{\bar{y}_1 - \bar{y}_2}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$ $\nu = n_1 + n_2 - 2$	$\lvert t_0 \rvert > t_{\alpha/2, \nu}$
$H_0: \mu_1 = \mu_2$ $H_1: \mu_1 < \mu_2$	if $\sigma_1^2 \neq \sigma_2^2$ $t_0 = \frac{\bar{y}_1 - \bar{y}_2}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}}$	$t_0 < -t_{\alpha, \nu}$
$H_0: \mu_1 = \mu_2$ $H_1: \mu_1 > \mu_2$	$\nu = \frac{(\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2})^2}{\frac{(s_1^2/n_1)^2}{n_1 - 1} + \frac{(s_2^2/n_2)^2}{n_2 - 1}}$	$t_0 > t_{\alpha, \nu}$

2.5.1 The paired Comparison Problem In some simple comparative experiments, we can greatly improve the precision by making comparisons within matched pairs for experimental material.

Table 2-5 Data for the Hardness Testing Experiment

spaceman j	Tip 1 y_{1j}	Tip 2 y_{2j}	Difference d_j
1	7	6	1
2	3	3	0
3	3	5	-2
4	4	3	1
5	8	8	0
6	3	2	1
7	2	4	-2
8	9	9	0
9	5	4	0
10	4	5	-1

Consider an alternative experimental design. Assume that each specimen is large enough so that two hardness determinations may be made on it.
We may write a statistical model as

$$y_{ij} = \mu_i + \beta_j + \epsilon, \quad i = 1, 2; j = 1, 2, ...10$$

Note that if we compute the jth paired difference

$$d_j = y_{1j} - y_{2j}, \qquad j = 1, 2, ..., 10$$

the expected value of this difference is

$$\mu_d = E(d_i) = E(y_{1j}) - E(y_{2j}) = \mu_1 + \beta_j - (\mu_2 + \beta_j) = \mu_1 - \mu_2$$

Testing $H_0 : \mu_1 = \mu_2$ is equivalent to testing

$$H_0 : \mu_d = 0, \qquad H_1 : \mu_d \neq 0$$

The test statistic is

$$t_0 = \frac{\bar{d}}{S_d/\sqrt{n}}$$

For the data in Table 2.5, we have

$$\bar{d} = \frac{1}{n}\sum_{i=1}^{n} d_i = \frac{1}{10}(-1) = -0.10$$

$$S_d^2 = \frac{\sum_{i=1}^{n}(d_i - \bar{d})^2}{n-1} = \frac{\sum_{i=1}^{n} d)i^2 - \frac{1}{n}(\sum_{i=1}^{n} d_i)^2}{n-1} = \frac{13 - \frac{1}{10}(-1)^2}{10-1} = \frac{43}{30}$$

Suppose that $\alpha = 0.05$. Then the critical value is $t_{0.025,9} = 2.262$. Now the value of test statistic is

$$t_0 = \frac{\bar{d}}{S_d/\sqrt{n}} = \frac{-0.10}{\sqrt{43/(30*10)}} = -0.264135272$$

Since $|t_0| = 0.264 < t_{0.025,9} = 2.262$, we fail to reject the hypothesis $H_0 : \mu_d = 0$. That is, there is no evidence to indicate that the two tips produce different readings.

2.5.2 Advantage of the Paired Comparison Design

The design actually used for this experiment is called the **paired comparison design**. It is a special case of **the randomized block design**. Note that we lose the $n-1$ degrees of freedom ($t_{0.025,18} = 2.101, t_{0.025,9} = 2.262$) but gain the smaller standard deviation S_d. ($S_p = 2.32, S_d = 1.20$).
Generally, S_p is always larger than S_d. In fact assuming that both population variance ares equal, then S_d is unbiased estimator of σ^2 but

$$E(S_p^2)^2 = \sigma^2 + \sum_{j=1}^{n} \beta_j^2$$

. That is, the block effects β_j inflate the variance estimate. This is why blocking serves as a noise reduction design techniques. We may also express the results of this experiment in terms of a confidence interval on $\mu_1 - \mu_2$. By using the paired data, a 95 percent confidence interval for $\mu_1 - \mu_2$ is

$$\bar{d} \pm t_{0.025,9} S_d \sqrt{n}, \quad -0.10 \pm (2.262)(1.20)\sqrt{10} \quad -0.10 \pm 0.86$$

Conversely using the pooled or independent analysis, 1 95 percent confidence interval for $\mu_1 - \mu_2$ is

$$\bar{Y}_1 - \bar{y}_2 \pm t_{0.025,18} S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}, \quad 4.80 - 4.90 \pm (2.101)(2.32)\sqrt{\frac{1}{10} + \frac{1}{10}}, \quad -0.10 \pm 2.18$$

2.6 Inference About the Variances of Normal Distributions

Suppose that we wish to test the hypothesis that the variance of normal population equals a constant, σ_0^2, that is, the hypotheses are

$$H0 : \sigma^2 = \sigma_0^2, \qquad H_1 : \sigma^2 \neq \sigma_0^2$$

The test statistic is

$$\chi_0^2 = \frac{SS}{\sigma_0^2} = \frac{(n-1)S^2}{\sigma_0^2}$$

and the $100(1-\alpha)$ percent confidence interval for σ^2 is

$$\frac{(n-1)S^2}{\chi^2_{\alpha/2,n-1}} \leq \sigma^2 \leq \frac{(n-1)S^2}{\chi^2_{1-\alpha/2,n-1}}$$

Table 2-7 Tests on Variance of Normal Distributions

Hypothesis	Test statistic	Criteria for Rejection
$H_0 : \sigma^2 = \sigma_0^2$ $H_1 : \sigma^2 \neq \sigma_0^2$		$\chi_0^2 > \chi^2_{\alpha/2,n-1}$ or $\chi_0^2 < \chi^2_{1-\alpha/2,n-1}$
$H_0 : \sigma^2 = \sigma_0^2$ $H_1 : \sigma^2 < \sigma_0^2$	$\chi_0^2 = \frac{(n-1)S^2}{\sigma_0^2}$	$\chi_0^2 < \chi^2_{1-\alpha,n-1}$
$H_0 : \sigma^2 = \sigma_0^2$ $H_1 : \sigma^2 > \sigma_0^2$		$\chi_0^2 > \chi^2_{\alpha,n-1}$
$H_0 : \sigma_1^2 = \sigma_2^2$ $H_1 : \sigma_1^2 \neq \sigma_2^2$	$F_0 = \frac{S_1^2}{S_2^2}$	$F_0 > F_{\alpha/2,n_1-1,n_2-1}$ or $F_0 < F_{1-\alpha/2,n_1-1,n_2-1}$
$H_0 : \sigma_1^2 = \sigma_2^2$ $H_1 : \sigma_1^2 < \sigma_2^2$	$F_0 = \frac{S_2^2}{S_1^2}$	$F_0 > F_{\alpha,n_2-1,n_1-1}$
$H_0 : \sigma_1^2 = \sigma_2^2$ $H_1 : \sigma_1^2 > \sigma_2^2$	$F_0 = \frac{S_1^2}{S_2^2}$	$F_0 > F_{\alpha,n_1-1,n_2-1}$