Computers and Operations Research 105 (2019) 226–236

Contents lists available at ScienceDirect

Computers and Operations Research

journal homepage: www.elsevier.com/locate/cor

Survey in Operations Research and Management Science

The Weapon-Target Assignment Problem

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ARTICLE INFO

Article history:
Received 4 April 2018
Revised 18 September 2018
Accepted 22 October 2018
Available online 30 October 2018

Keywords:
Weapon Target Assignment Problem
Static WTA
Dynamic WTA

ABSTRACT

Research addressing the Weapon Target Assignment (WTA) Problem, the problem of assigning weapons to targets while considering their effective probability of kill, began with Manne's seminal work in 1958. In the years following, improved modeling and solution techniques have been developed, along with improvements in computing power, which have enabled researchers to consider more complex variants of the problem, to include models with fewer assumptions and models in which time is a parameter. Herein, we review the various model formulations, exact algorithms, and heuristic algorithms for the static and dynamic WTA. We place the formulations into a comparable form and use this form to provide insight into the evolution of the defense-related WTA problem. The solution methods are comparatively analyzed and an analysis of the influence of past work is conducted. More recent developments are introduced and discussed.

1. Introduction

Projectile weapons have been a consistent threat of hostilities throughout history. Military advantage has always been aided by the capacity to inflict damage from a distance. In the 20th century, missile technology advanced to the point that an adversary had the potential to attack a protected asset from great distances. To neutralize this stand off threat, the concept of air defense evolved. However, as the ability to reduce a missile threat increased, so too did the quantity and quality of missiles available, and research into the effective allocation of air defense resources emerged.

Originally introduced into the field of operations research by Manne (1958), the Weapon Target Assignment (WTA) Problem, or Missile Allocation Problem (MAP) as it is sometimes known, seeks to assign available interceptors to incoming missiles so as to minimize the probability of a missile destroying a protected asset. While much of the literature on the WTA focuses on the defensive perspective, some have considered the offensive perspective (Sikanen, 2008), wherein the objective is to maximize the probability of destroying enemy protected assets.

There are two distinct categories of the WTA: the Static WTA (SWTA) and the Dynamic WTA (DWTA). Originally modeled by Manne (1958), the SWTA defines a scenario wherein a known number of incoming missiles (targets) are observed and a finite number of interceptors (weapons), with known probabilities of successfully destroying the targets (probabilities of kill), are available for a single exchange. The solution to the SWTA informs the defense on how many of each weapon type to shoot at each target. In the SWTA, no subsequent engagements are considered since time is not a dimension considered in the problem.

By contrast, the DWTA includes time as a dimension. Variants of the DWTA include the two stage DWTA and the shoot-look-shoot DWTA. The two stage DWTA replicates the SWTA in its first stage, but includes a second stage wherein a number of targets of various types are known only to a probability distribution. In this variant, the solution to the DWTA informs the defense on how to allocate the weapons in the first stage and how many to reserve for the second stage in order to minimize the probability of destruction. The shoot-look-shoot variant also replicates the SWTA, however it enables the defense to observe which targets may have survived the engagement (leakers) and allows for a subsequent engagement opportunity. The solution to this variant similarly informs the defense on how to allocate the weapons and how many weapons to reserve to reengage any leakers.

The WTA has been solved to optimality with exact algorithms. However, as Lloyd and Witsenhausen (1986) showed that the WTA is NP-Complete, the majority of solution techniques seek to find near optimal solutions in real-time, or "fast enough to provide an engagement solution before the oncoming targets reached their goals" (Leboucher et al., 2013). These real-time solution techniques are products of heuristic algorithms or are solved using exact algorithms applied to transformations of the formulation.

The rest of this paper proceeds as follows. In Section 2, we review the various formulations for both the SWTA and DWTA. We examine the basic formulations of each and explore the trans-

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https://doi.org/10.1016/j.cor.2018.10.015
0305-0548/Published by Elsevier Ltd.

formations which have been implemented. We also review novel formulations which have sought to model and solve the problem in unique settings. In Section 3, we review the exact algorithms that have been used to solve the SWTA and DWTA. Some of these algorithms provide optimal solutions to the original formulations whereas others refer to the transformed formulations identified in Section 2. In Section 4, we review the heuristic and metaheuristic solution techniques for the SWTA and DWTA. In Section 5, we discuss the state of the WTA and present a metric with which we focused this examination of the literature.

2. Formulations

There have been many different formulations of the WTA. Early literature sought to transform the nonlinear formulation from Manne (1958) due to the computational limitations with nonlinear programming. As computational power increased, transformations which were better suited to global optimization tools emerged. Burr et al. (1985) introduced the DWTA which captured the value of subsequent engagements. Similar to the SWTA, variations to the original DWTA occur throughout the literature.

Herein, we examine some of the formulations for both the SWTA and DWTA. For purposes of clarity in both formulation and in presentation, we map the formulations presented by their authors into the terms of the formulation developed by Manne (1958). Namely, variables that are shared between multiple formulations are defined as follows:

p_{ij} : the probability weapon$i$ destroys target j
q_{ij} : the probability weapon i fails to destroys target j
V_j : the destructive value of target j
x_{ij} : the number of weapons of type i assigned to target j
K : the number of protected assets
a_k : the value of asset k
n : the number of targets
m : the number of weapon types
w_i : the number of weapons of type i
c_{ij} : a cost parameter for assigning a weapon of type i to target j
$\mathcal{F}$: the set of feasible assignments
γ_{jk} : the probability target j destroys asset k
s_j : the maximum number of weapons that can be assigned to target j
t : the number of stages

2.1. SWTA formulations

The original formulation as defined by Manne (1958) considers a scenario where a defender has w_i of $i = 1, \ldots, m$ weapon types with which to defend against $j = 1, \ldots, n$ targets. Each weapon type i has a probability p_{ij} of killing target j and each target j has a destructive value V_j. With decision variables x_{ij} indicating the number of weapons of type i to assign to target j, the SWTA is formulated:

$$\begin{aligned} \min & \sum_{j=1}^{n} V_j \prod_{i=1}^{m} \left(1 - p_{ij}\right)^{x_{ij}} \\ \text{s.t.} & \sum_{j=1}^{n} x_{ij} \le w_i, \quad \text{for } i = 1, \ldots, m \\ & x_{ij} \in \mathbb{Z}_+, \quad \text{for } i = 1, \ldots, m, \; j = 1, \ldots, n\ . \end{aligned}$$

However, it is common to write the formulation in terms of the probability of survival $q_{ij} = 1 - p_{ij}$

$$\textbf{S1} \quad \begin{aligned} \min & \sum_{j=1}^{n} V_j \prod_{i=1}^{m} q_{ij}^{x_{ij}} \\ \text{s.t.} & \sum_{j=1}^{n} x_{ij} \le w_i, \quad \text{for } i = 1, \ldots, m \\ & x_{ij} \in \mathbb{Z}_+, \quad \text{for } i = 1, \ldots, m, \; j = 1, \ldots, n\ . \end{aligned}$$

The nonlinear objective function in **S1** seeks those assignments to minimize the expected value of survival. The assignments are integer and the total number of weapon i cannot exceed the number of weapons on hand, w_i. This formulation is used frequently, (e.g., (Ahuja et al., 2007),(Lemus and David, 1963),(Lee et al., 2002a),(denBroeder et al., 1959)) and is often the initial formulation used when implementing a transformation.

A simpler version of **S1** is given by denBroeder et al. (1959), who assumes that all weapons have the same probability of kill for target j, $p_{ij} = p_j \; \forall i = 1, \ldots, m$. His formulation differs from **S1** in the objective, which is

$$\textbf{S2} \quad \min \sum_{j=1}^{n} V_j q_j^{x_j}$$

This formulation simplifies **S1** and is easily optimized by a greedy assignment technique. However, its assumption of homogeneity greatly reduces the applicability of the formulation.

Kwon et al. (1999) utilize a similar model to **S1** but reformulate the problem into an integer program with a linear objective function and nonlinear constraints. They use a negative cost parameter, c_{ij}, for assigning weapon i to target j which they seek to minimize as follows:

$$\begin{aligned} \min & \sum_{(i,j)\in A} c_{ij} x_{ij} \\ \text{s.t.} & \sum_{\{j=1,\ldots,n \mid (i,j)\in A\}} x_{ij} \le w_i \quad \text{for } i = 1, \ldots, m \\ & 1 - \prod_{\{i=1,\ldots,m \mid (i,j)\in A\}} \left(1 - p_{ij}\right)^{x_{ij}} \ge d_j \quad \text{for } j = 1, \ldots, n \\ & x_{ij} \le u_{ij} \quad \forall (i,j) \in \mathcal{F}, \\ & x_{ij} \ge 0 \quad \forall (i,j) \in \mathcal{F} \end{aligned}$$

where d_j is the minimum desired probability of kill for target j, u_{ij} is an upper bound on the number of weapons i that can be assigned to target j, and $\mathcal{F}$ is the set of all feasible assignments. Kwon et al. (1999) then multiply a large number θ to a logarithmic transformation of the nonlinear constraint and round down to the largest integer contained in order to generate the following linear approximation, where $a_{ij} = \lfloor -\theta \ln(1 - p_{ij}) \rfloor > 0$ and $b_j = \lfloor -\theta \ln(1 - d_j) \rfloor > 0$

$$\textbf{S3} \quad \begin{aligned} \min & \sum_{(i,j)\in A} c_{ij} x_{ij} \\ \text{s.t.} & \sum_{\{j=1,\ldots,n \mid (i,j)\in A\}} x_{ij} \le w_i \quad \text{for } i = 1, \ldots, m \\ & \sum_{\{i=1,\ldots,m \mid (i,j)\in A\}} a_{ij} x_{ij} \ge b_j \quad \text{for } j = 1, \ldots, n \\ & x_{ij} \le u_{ij} \quad \forall (i,j) \in A, \\ & x_{ij} \ge 0 \quad \forall (i,j) \in A \end{aligned}$$

This formulation is linear and is computationally simpler than **S1** and a solution is more easily attained. Because the formulation is an approximation, however, its solution is not guaranteed to be optimal for **S1**.

A different transformation to **S1** is put forth by Ahuja et al. (2007) by applying a logarithmic transformation to the objective. Letting $d_{ij} = -\ln(q_{ij})$, their formulation becomes

$$\textbf{S4} \quad \min \sum_{j=1}^{n} V_j 2^{-y_j}$$
$$\text{s.t.} \sum_{j=1}^{n} x_{ij} \le w_i \quad \text{for } i = 1, \ldots, m$$
$$\sum_{i=1}^{m} d_{ij} x_{ij} = y_j \quad \text{for } j = 1, \ldots, n$$
$$x_{ij} \in \mathbb{Z}_+ \quad \text{for } i = 1, \ldots, m, \; j = 1, \ldots, n$$
$$y_j \ge 0 \quad \text{for } j = 1, \ldots, n$$

With this transformation, Ahuja et al. (2007) have an objective which is the sum of separable convex functions. They utilize this transformation to model the SWTA as a network flow problem, which is addressed later in Section 2.1. Further, as is shown by Kline et al. (2017b), utilizing **S4** within a commercial global optimization solver such as BARON is more reliable than when utilizing **S1**, which has roughly a 21% false optimality rate.

Others simplify the problem by limiting the number of weapons of each type to $w_i = 1$, making the problem a binary program. Li et al. (2009) propose the objective function

$$\textbf{S5} \quad \min \sum_{j=1}^{n} V_j \prod_{i=1}^{m} (1 - p_{ij} x_{ij}),$$

with an added constraint which limits x_{ij} to a binary decision variable. **S1** can be transformed to **S5** by setting the number of weapon types to the total number of weapons. That is, if $w_i = 3$ and $m = 5$, the problem could be transformed for **S5** by setting $w_i = 1$ and $m = 15$. This increases the number of decision variables of the problem, though the transformation to a binary program allows for more efficient solution techniques.

A more simplified formulation is put forth by Rosenberger et al. (2005), who model the SWTA as a knapsack problem. They define a positive cost parameter c_j, which is earned when assignment j is selected. Their model assumes that no two weapons can be assigned to the same target and is

$$\textbf{S6} \quad \max \sum_{j \in J} c_j x_j$$
$$\text{s.t.} \sum_{j \in S_i} x_j \le 1 \quad i = 1, \ldots, m$$
$$\sum_{j \in T_j} x_j \le 1 \quad j = 1, \ldots, n$$
$$x_j = \begin{cases} 1 & \text{assignment j is selected} \\ 0 & \text{otherwise} \end{cases}$$

In the first constraint, the set S_i is the subset of all feasible assignments of which weapon i is assigned. Similarly, the set T_j in the second constraint is the subset of all feasible assignments which assigns a weapon to target j. While simpler than **S1**, this formulation, like **S2**, carries more assumptions which limit its ability to model and solve complex missile defense problems.

Malcolm (2004) proposed a formulation with the same binary decision variables in which the objective is similar in structure to **S5**. He shows that, when weapon assignments are restricted to exactly one target and $m = n$, the objective can be written as

$$\textbf{S7} \quad \min \quad -\sum_{j=1}^{n} V_j \left(\sum_{i=1}^{m} x_{ij} p_{ij} \right)$$
$$\text{s.t.} \sum_{j=1}^{n} x_{ij} = 1 \quad \text{for } i = 1, \ldots, m$$
$$\sum_{i=1}^{m} x_{ij} = 1 \quad \text{for } j = 1, \ldots, n$$
$$x_{ij} = \begin{cases} 1 & \text{weapon } j \text{ is assigned to target } i \\ 0 & \text{otherwise} \end{cases}$$
$$\text{for } i = 1, \ldots, m, \; j = 1, \ldots, n$$

This allows for solution techniques which exploit the special structure of the formulation, but is only of use under certain rigid situations.

Two additional variants to the SWTA formulations are also in the literature. One, defined by Shang et al. (2007), considers the value, a_k, of a protected asset $k = 1, \ldots, K$ and the probability with which a target j will destroy this asset γ_{jk}. Given the probability that weapon i will destroy target j, p_{ij}, they formulate

$$\textbf{S8} \quad \min \sum_{k=1}^{K} a_k \prod_{j=1}^{n_k} \left[\gamma_{jk} \prod_{i=1}^{W} (1 - p_{ij} x_{ij}) \right]$$
$$\text{s.t.} \sum_{k=1}^{K} n_k = n$$
$$\sum_{j=1}^{n} x_{ij} = 1, \quad i = 1, \ldots, m$$
$$x_{ij} \in \{0, 1\}, \quad i = 1, \ldots, m, \; j = 1, \ldots, n$$

where each protected asset has incoming targets $1, \ldots, n_k$ and there are a total of n targets. This formulation considers the importance of different protected assets, which is relevant in a missile defense problem, but it adds complexity to the problem.

Karasakal (2008) does not consider a target value but rather treats each target to be of identical destructive capacity. He defines the set of feasible solutions as $\mathcal{F}$, limits the number of weapons that can be assigned to target j as s_j, and defines a formulation

$$\textbf{S9} \quad \max \prod_{j=1,\ldots,n} \left[1 - \prod_{\{i=1,\ldots,m | (i,j) \in \mathcal{F}\}} (1 - p_{ij})^{x_{ij}} \right]$$
$$\text{s.t.} \sum_{\{j=1,\ldots,n | (i,j) \in \mathcal{F}\}} x_{ij} \le w_i \quad \text{for } i = 1, \ldots, m$$
$$\sum_{\{i=1,\ldots,m | (i,j) \in \mathcal{F}\}} x_{ij} \le s_j \quad \text{for } j = 1, \ldots, n$$
$$0 \le x_{ij} \le u_{ij}, \quad \forall i, j \in \mathcal{F} \text{ and } x_{ij} \text{ is integer}$$

This formulation treats all protected assets and targets as having equal value and simply seeks to maximize the expected destruction to incoming targets.

2.2. DWTA formulations

2.2.1. Shoot-look-shoot

There are two variants of the DWTA, each of which have unique formulations. The first variant is the shoot-look-shoot scenario, wherein weapons are assigned to targets in a first engagement and a subsequent engagement allows assigning remaining weapons to any surviving targets. This problem was discussed by Eckler and Burr (1972), who do not define a model but define the probability that n targets are destroyed over t stages, which is equivalent to the probability that at least n weapons do not fail over t stages, as

$$P(t) = \sum_{i=n}^{t} \binom{t}{i} (1 - p)^{t-i} p^i, \tag{1}$$

for a problem wherein all probabilities of kill are the same and all targets are of the same value. Eckler and Burr (1972) identify that the most desired strategy to the t stage problem under this assumption will be equivalent to finding the number of weapons to use in each stage which minimize the number of stages necessary to achieve some acceptable value of $P(t)$.

Soland (1987) provides a model for the Eckler and Burr (1972) scenario with the assumption that all weapons have the same probabilities of kill and all targets have the same value. Given a nondecreasing function $g(n_q)$ which defines the expected fraction of targets destroyed, where n_q is the number of unintercepted targets, the state of the system $S(n_q, d, t)$ defines the fraction of targets destroyed given n_q targets, d weapons, and t remaining engagements. The state space is bound by

$$S(0, d, t) = 0, \quad \text{for } d = 0, 1, \ldots, D, \ t = 1, \ldots, T,$$
$$S(n_q, d, 0) = g(n_q), \quad \text{for } n_q = 0, 1, \ldots, n, \ d = 0, 1, \ldots, D.$$

He defines the transition probability that j targets survive having used i weapons as $P(j|n_q, i, d, t)$, where

$$P(j|n_q, i, d, t) = \binom{n_q}{j} q_t^{jI} (1 - q_t^I)^{n_q - j}.$$

I defines the spread of weapons to targets, or $I = \frac{i}{n_q}$ in the case that $\frac{i}{n_q}$ is integer. If it is not integer, then $n_q + n_q \left\lfloor \frac{i}{n_q} \right\rfloor - i$ of the targets receive $\left\lfloor \frac{i}{n_q} \right\rfloor$ weapons and the remaining $i - n_q \left\lfloor \frac{i}{n_q} \right\rfloor$ targets receive $\left\lceil \frac{i}{n_q} \right\rceil$ weapons. His model seeks to minimize the number of weapons required to ensure that the expected number of surviving targets is less than some "nondecreasing maximum damage function f" (Soland, 1987)

D1 $\min D$

$$\text{s.t.} \ \ S(n_q, D, T) \le f(n_q), \quad n_q = 1, \ldots, n.$$

Hosein and Athans (1989) provide a different model than Soland (1987), but with the same underlying assumptions. They define $n_k(t)$ as the number of targets aimed at protected asset k at stage t, or the number of surviving targets after $t-1$ engagements. They compute the probability that the number of targets surviving into the second stage is $j^{(2)}$ given the assignment in the first stage is $x^{(1)}$, for all $i = 0, 1, \ldots, m$ and $j = 0, 1, \ldots, n(1)$ as $P\big(n(2) = j^{(2)} | x^{(1)}\big)$. Defining $J_s^*(n(2), m_2)$ as the optimal solution to the second stage with m_2 weapons available, they define the formulation

D2 $\max\limits_{x^{(1)} \in \mathbb{Z}_+^K} J_d = \mathop{\mathbb{E}}\limits_{n(2)} [J_s^*(n(2), m_2)]$

$$\text{s.t.} \quad |x^{(1)}| + m_2 = m.$$

The objective is the expected value of the optimal solution in the second stage, thus the optimal solution to the problem is to find the number of weapons to use in the first stage, m_1, and assign them to the appropriate targets, $x^{(1)}$, in such a way that the second stage can be solved to optimality given the number of surviving targets and the number of unused weapons, m_2.

A different approach, which considers the available windows in which targets can be engaged, is proposed by Leboucher et al. (2013). He computes random paths of randomly located targets using Bézier curves, which allow for the calculation of the time to impact for each target and the earliest point at which each weapon can engage the target. For each weapon-target pairing, he computes:

$$f_1(E_{i/j}) = EFF_{i/j}, \quad (i \in I), (j \in J)$$
$$f_2(E_{i/j}) = LFF_{i/j} - EFF_{i/j}, \quad (i \in I), (j \in J)$$
$$f_3(E_{i/j}) = d(P_{j_{out}}, P_{i_0})$$

where $EFF_{i/j}$ is the earliest feasible fire time for weapon i to target j, $LFF_{i/j}$ is the latest feasible fire time for weapon i to target j, and $d(P_{j_{out}}, P_{i_0})$ is the Euclidean distance that the weapon, i_0, must fly over the protected area to intercept target j.

Using these three parameters for each pairing, Leboucher et al. (2013) creates a cost matrix for all of the possible assignments

$$H = \begin{bmatrix} E_{1/1} & E_{2,1} & \cdots & E_{|I|/1} \\ E_{1/2} & E_{2,2} & \cdots & E_{|I|/2} \\ \vdots & \vdots & \ddots & \vdots \\ E_{1/|J|} & E_{2,|J|} & \cdots & E_{|I|/|J|} \end{bmatrix},$$

in which the cost of an assignment $H(E_{i/j})$ is

$$H(E_{i/j}) = \alpha_1 f_1(E_{i/j}) + \alpha_2 f_2(E_{i/j}) + \alpha_3 f_3(E_{i/j}),$$

where $\alpha_1 + \alpha_2 + \alpha_3 = 1$ and $(\alpha_1, \alpha_2, \alpha_3) \in [0, 1]^3$. He presents a formulation to determine the assignments

D3a $\min \sum\limits_{i=1}^{m} \sum\limits_{j=1}^{n} H(E_{i/j}) x_{ij}$

$$\text{s.t.} \quad \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} \le m$$

and defines, for each possible firing sequence (FS), three parameters. First, given the time at which target j is engaged, FT_j, the parameter measuring the firing time is $f_4(FS) = \sum_{j=1}^{T} FT_j$. Second, a parameter identifying any constraint violation is $f_5(FS) = \sum_{i=1}^{m} c_i$, where c_i is 1 if the assignment of weapon i violates a constraint and 0 otherwise. Lastly, the parameter representing idle time of the system, given the time at which weapon i is fired, FT_i, is $f_6(FS) \sum_{i=1}^{m-1} (FT_{i+1} - FT_i)$. Leboucher et al. (2013) present a formulation whose solution gives the optimal firing sequence of the assignment solution

D3b $\min F(FS)$

$$\text{s.t.} \quad F(FS) = \begin{cases} (f_4(FS) + 1) * f_6(FS) & f_5(FS) = 0 \\ \infty & f_5(FS) \neq 0 \end{cases}.$$

The formulation presented by Leboucher et al. (2013) enables observation of surviving targets following an engagement which can be reengaged in a subsequent iteration and can be used to solve a shoot-look-shoot problem.

2.2.2. 2-stage

The second variant of the DWTA is the 2 stage, or more generally the multi-stage, problem, which differs from the shoot-look-shoot in that it does not allow the reacquisition of leakers. That is, in the shoot-look-shoot problem, a given number of targets are repeatedly engaged until all have been destroyed or a limit to the number of iterations is met. In the 2 stage problem the given number of targets is only engaged once before a subsequent stage occurs. In the second stage, the number and type of incoming targets is known only to a probability distribution.

Chang et al. (1987) model the T stage WTA by considering the value of each stage as defined by the formulation **S1** and taking the sum over the T stages.

D4 $\min\limits_{x_{ij}(t)} \ \mathbb{E}\left[\sum\limits_{t=1}^{T} \sum\limits_{j \in \mathcal{A}_t} V_j(t) \prod\limits_{i=1}^{m} \left(1 - p_{ij}(t)\right)^{x_{ij}(t)} \right]$

$$\text{s.t.} \quad \mathcal{A}_{t+1} = (\mathcal{A}_t \cup \mathcal{L}_t) \cap \mathcal{K}_t'$$
$$M_i(t) = M_i(t-1) - \sum_{j=1}^{n} x_{ij}(t-1) \quad i = 1, \ldots, m$$

$$\sum_{j\in\mathcal{A}_t} x_{ij}(t) \le M_i \quad i=1,\dots,m$$
$$x_{ij}(t)\in\mathbb{Z}_+ \quad i=1,\dots,m,\ j=1,\dots,n$$

where $\mathcal{A}_t$ is the set of targets in stage t, $M_i(t)$ is the number of interceptors of type i at stage t, $\mathcal{L}_t$ is the set of new targets observed in stage t and $\mathcal{K}_t$ is the set of targets killed in stage t.

Burr et al. (1985) present a formulation for a multi-stage problem in which all weapons have the same probability of kill for all equally valued targets. That is, a known number of targets arrives in the first stage and a number arrive in each of the number of subsequent stages, both of which are known only to a probability distribution. Given the attack strategy $a(k)$, which identifies the number of targets aimed at asset k, and a defense strategy $d(k, j)$, which defines how many weapons to shoot at the target j threatening asset k, Burr et al. (1985) formulate the problem

$$\textbf{D5}\quad \min_{d(k,j)} \sum_{k=1}^{K}\sum_{j=1}^{a(k)} d(k,j)$$
$$\text{s.t.}\quad V(d,a) \le \frac{V}{n}\sum_{k=1}^{K} a(k),\ \text{for all } a$$

where V is the sum of all protected asset values, n is the total number of targets, and $V(d, a)$ is the expected damage to the protected asset given a targets and deployment strategy d

$$V(d,a) = \sum_{k=1}^{K} a_k\left(1-\prod_{j=1}^{a(k)}\left(1-q^{d(k,j)}\right)\right).$$

This formulation seeks to ensure that the expected damage to the protected assets is less than the total value of all protected assets.

Murphey (2000) formulates the multi-stage problem by defining $n(t)$ as the number of targets which arrive at time $t=1,\dots,T$ and $c(t)$ as a nondecreasing function which represents a cost of waiting. With the assumption that all weapons have the same probability of kill for target j, his formulation is

$$\textbf{D6}\quad \min \sum_{t=1}^{T} c(t)\sum_{j=1}^{n(t)} V_j q_j^{x_j(t)}$$
$$\text{s.t.}\quad \sum_{t=1}^{T}\sum_{j=1}^{n(t)} x_j(t) = m,$$
$$V_j \in \mathbb{V}\in\mathbb{R}_+^n \quad j=1,\dots,n(T)$$
$$x^{(t)}\in\mathbb{Z}_+^{n(t)} \quad t=1,\dots,T$$

where m is the total number of weapons and $\mathbb{V}$ is the set of all target values. This formulation is seeking to minimize the value of the assignments and the inclusion of a non-decreasing cost of waiting function will bias the solution to make earlier assignments unless these assignments are to targets of a very small value relative to those of later assignments.

Xin et al. (2011) allow for different probabilities of kill for each weapon to each target and further allow for different probabilities of kill between stages. Their formulation for stages $t=1,\dots,T$ is

$$\textbf{D7}\quad \min \sum_{k=1}^{K(t)} V_k \prod_{j=1}^{n(t)}\left(1-\gamma_{jk}\prod_{h=t}^{T}\prod_{i=1}^{m(t)}\left(1-p_{ij}(h)\right)^{x_{ij}(h)}\right)$$
$$\text{s.t.}\quad \sum_{j=1}^{n(t)} x_{ij}(t)\le n_i \quad \text{for } i=1,\dots,m,\ t=1,\dots,T$$
$$\sum_{i=1}^{m(t)} x_{ij}(t)\le s_j \quad \text{for } j=1,\dots,n,\ t=1,\dots,T$$
$$\sum_{t=1}^{T}\sum_{j=1}^{n(t)} x_{ij}(t)\le w_i \quad \text{for } i=1,\dots,m$$
$$x_{ij}(t)\le f_{ij}(t) \quad \text{for } i=1,\dots,m,\ j=1,\dots,n,\ t=1,\dots,T$$

where

$K(t)$: number of existing assets at time t
$n(t)$: number of existing targets at time t
$m(t)$: number of available weapons at time t
a_k : the value of asset k
γ_{jk} : the probability target j destroys asset k
$p_{ij}(t)$: the probability weapon i destroys target j at time t
n_i : maximum number of targets weapon i can shoot at each stage
s_j : maximum number of weapons that can be assigned to target j at each stage
w_i : total number of weapons of type i
$f_{ij}(t)$: 1 if weapon i can be assigned to target j, 0 otherwise.

This is one of the more complex and realistic models that can be found within the literature. It allows for expansion into a shoot-look-shoot problem and considers many parameters which are relevant to modeling missile defense. However, as the complexity is higher than other formulations, finding solutions is more computationally expensive than for simpler formulations.

A model proposed by Khosla (2001) considers the required time for weapon system control and defines the following terms:

n : Number of current threats
m : Number of current weapon systems
T : Total number of time points in time interval
$TV(j)$: Threat value of threat j
$OW(i,j)$: Option weight of weapon system i for threat j
$LB(i,j)$: Begin launch time for weapon system i for threat j
$LE(i,j)$: End launch time for weapon system i for threat j
$GT(i,j)$: Guidance time interval for interceptor for weapon system i to engage threat j
$IR(i)$: Inventory resource of weapon system i (number of interceptors)
$GR(i)$: Guidance resource capacity of weapon i

where $GT(i, j)$ defines the amount of time the guidance system must be allocated to weapon i in targeting target j and $GR(i)$ is the number of guidance systems available. The option weight $OW(i, j)$ serves to add a benefit to preferred pairings; a bias for weapon i to be assigned to target j.

Khosla (2001) defines a mixed integer program with only a few of the considerations discussed thus far, proposing that expanded models including additional time constraints such as reload time. Using a decision variable $L(i,j,t)=1$ if t denotes the launch time for an interceptor from weapon system i to engage threat j and 0 otherwise, he models this problem, with a weight factor $\alpha\in[0,1]$, as

$$\textbf{D8}\quad \max \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{t=1}^{T}[\alpha TV(j)+(1-\alpha)OW(i,j)]L(i,j,t)$$
$$\text{s.t.}\quad \sum_{i=1}^{m}\sum_{t=1}^{T} L(i,j,t)\le 1,\quad \text{for } j=1,\dots,n$$

$$\sum_{j=1}^{n}\sum_{t=1}^{T} L(i,j,t) \leq IR(i), \quad \text{for } i = 1, \dots, m$$

$$If \;\; L(i,j,t) = 1, \;\; R(i,j,t') = 1, \;\; \forall\, t \leq t' \leq t + GT(i,j)$$

$$\sum_{j=1}^{n}\sum_{t=1}^{T} R(i,j,t) \leq GR(i), \quad \text{for } i = 1, \dots, m$$

In this model, Khosla (2001) ensures that only one weapon system is assigned to each target, the total number of interceptors does not exceed the inventory, and the guidance time required for the assignments does not exceed the number of guidance systems. He nests this model into a framework which updates the number of targets after each completed iteration and, as such, can be used in a 2 stage scenario where the number of targets in subsequent stages is stochastically determined.

This model simplifies some of the parameters and considerations posed in **D7**, but includes a discretized time step which provides the firing sequence inherent to, but previously not considered, missile defense. However, due to the exponential growth of the number of decision variables with the increase in resolution, Khosla (2001) identifies that even modest sized problems are very computationally expensive.

Ahner and Parson (2015) address the SWTA formulation, originally proposed by Murphey (2000), to compute the value of the first stage and includes in the objective the expected value of the second stage, with the maximum number of weapons, $b < M$, that can be used in any stage, as follows

$$\textbf{D9} \quad \min_{x} \left\{ \sum_{j=1}^{n_1} V_j^{(1)} \left(1 - p_j^{(1)}\right)^{x_j^{(1)}} + \mathop{\mathbb{E}}_{\omega \in \Omega}\left[Z_2\left(x^{(2)}, \omega^j\right)\right] \right\}$$

$$\text{s.t.} \quad x^{(1)} \leq b$$

$$x_j \in \mathbb{Z}_+, \;\; \text{for } j = 1, \dots, n$$

where $x_j^{(1)}$ defines the number of weapons fired at target j in the first stage and the total number of weapons fired in the first stage is $x^{(1)} = \sum_{j=1}^{n_1} x_j^{(1)}$. Further, the second stage, Z_2, is a function of the remaining weapons, $x^{(2)} = M - x^{(1)}$, and a random occurrence, $\omega \in \Omega$ of the number of targets and the type of each target.

$$Z_2\left(x^{(2)}, \omega^j\right) = \min_{x^{(2)}} \left\{ \sum_{j=1}^{n_2(\omega)} V_j^{(2)}(\omega)\left(1 - p_j^{(2)}\right)^{x_j^{(2)}} \right\}$$

$$\text{s.t.} \quad \sum_{j=1}^{n_1} x_j^{(1)} + \sum_{j=1}^{n_2(\omega^j)} x_j^{(2)} = M$$

$$x^{(2)} \leq b$$

$$x_j^{(2)} \in \mathbb{Z}_+, \;\; \text{for } j = 1, \dots, n$$

Unlike other formulations, this formulation accounts for the uncertainty of a second stage which must be considered when allocating available weapons to the first stage. However, it is a homogeneous model which assumes that all weapons have the same probability of kill for a target j.

Sikanen (2008) models the DWTA wherein all weapons have the same probability of kill for all targets

$$\textbf{D10} \quad \max_{x_{i1},\dots,x_{iT}} \quad \sum_{t=1}^{T}\sum_{j=1}^{n(t)} p_j(t)\lambda_j^t V_j x_{jt}$$

$$\text{s.t.} \quad \sum_{t=1}^{T} x_{jt} \leq 1 \quad \text{for } j = 1, \dots, n(t)$$

$$\sum_{t=1}^{T}\sum_{j=1}^{n(t)} x_{jt} \leq m$$

$$x_{jt} \in \{0, 1\} \quad \text{for } j = 1, \dots, n(t), \; t = 1, \dots, T$$

where there are T stages, $n(t)$ targets per stage, and m available weapons. Further, the value of target j is V_j and its time discount factor at stage t is λ_j^t. Given that x_{jt} is a binary decision variable, the expression is the sum over all stages of the sum over all assignments. The time discounting factor imposes a bias on earlier assignments. The product of the probability, target value, time discount factor, and decision variable result in either a value of 0 or the time discounted expected value of the assignment. Though a superior assignment may occur as the target is closer to the protected asset ($p_j(t)$ is a function of time), the value of the target decreases since a miss will reduce the ability to reengage.

3. Exact algorithms

There are few cases in the literature of exact algorithmic solutions to the WTA. The problem suffers due to its complexity as an NP-Complete problem (Lloyd and Witsenhausen, 1986) and, like routing problems, are simply hard to solve. For the SWTA, the number of possible permutations of assigning m weapons to n targets is n^m, assuming that all weapons must be assigned, which, as the number of weapons and targets increases, grows exponentially and searching all possible solutions quickly becomes computationally intractable. Because the DWTA includes either a shoot-look-shoot or multiple stage (or both) framework, it further increases the number of permutations. The literature implementing exact solution techniques generally fall into one of two categories: small problems and problems wherein assumptions reduce the complexity.

3.1. SWTA

denBroeder et al. (1959) showed the first optimal solution technique in their Maximum Marginal Return (MMR) algorithm. Assuming that the probability of kill for any weapon to target j is the same, they showed that an optimal solution can be found by assigning $x_{ij} = 1$ where $\{i, j\} \in \arg\max(V_j p_{ij})$ and then updating $V_j = V_j(1 - p_{ij})$ and $p(i, \cdot) = p(\cdot, j) = 0$ and repeating the process until all weapons have been assigned. Further, when the probabilities of kill are the same for all weapons to all targets, $p_{i_1 j_1} = p_{i_2 j_2}, \; \forall i_1, i_2 \in I, \; j_1, j_2 \in J$, the optimal solution is found by dividing the weapons evenly across all targets (Hosein et al., 1988).

Malcolm (2004) developed and solved **S7**, where he defined his constraint coefficient matrix A as totally unimodular. This ensures that every vertex of the convex polytope that defines the feasible solution space is an integer solution. As such, he uses the Simplex Method to quickly find the optimal solution.

Smaller problems were solved through an exhaustive search algorithm for **S1** by Johansson and Falkman (2009). In comparing the objective function value of every feasible solution, they show that a problem with 9 weapons and 8 targets took 13 min to run to completion and that adding one additional target took 43.7 min to run to completion, which they present to illustrate the combinatorial explosion in run time as a function of problem size.

Several cases of using a branch and bound algorithm are found in the literature. Rosenberger et al. (2005) solved **S6** for up to 8 weapons and 4 targets. Ahuja et al. (2007) implemented three lower bounding strategies to increase the efficiency of fathoming nodes: a generalized network flow solution, an MMR solution, and a minimum cost flow solution. Kline (2017) developed a branch and bound algorithm to solve **S1** and was able to find optimal solutions for up to 10 weapons and 10 targets. Beyond this, the size of the problem precluded convergence within 7 days of computation.

Karasakal (2008) utilizes linear integer programming techniques to find optimal solutions to two linear transformations of **S9**.

Bogdanowicz (2012) develops and utilizes an algorithm by which he searches through known effective weapon-target pairings to find an optimal solution. Utilizing the Joint Munition Effectiveness Manual (JMEM), he defines the desired minimal effect of any one pairing to reduce the number of sets through which he searches for an optimal set of pairings, given the number and type of weapons and targets.

3.2. DWTA

Burr et al. (1985) puts forth an optimal algorithm for **D4** given a scenario wherein one target per stage is observed and a defender will assign weapons for up to $k-1$ stages, after which he will surrender the protected asset. Given a maximum total expected damage of 1, the defender must limit the expected damage per stage to no greater than $r = \frac{1}{k}$, where each weapon has a probability of kill of $p = 1-q$. His algorithm is to set the minimum number of weapons to ensure damage does not exceed r for the first stage as

$$d(1) = \left\lceil \frac{\ln(r)}{\ln(q)} \right\rceil$$

and all subsequent stages as

$$d(k') = \left\lceil \frac{\ln\left(1 - \frac{1-rk'}{\prod_{i=1}^{k'-1}(1-q^{d(i)})}\right)}{\ln(q)} \right\rceil,$$

where $1 < k' \leq k-1$.

Soland (1987) gives an optimal solution to the shoot-look-shoot model **D1** in which the number of weapons i to assign to the total number of targets a in the first stage is simply $\left\lfloor \frac{i}{a} \right\rfloor$ and to preserve the remaining $i - \left\lfloor \frac{i}{a} \right\rfloor$ weapons for the surviving targets. If the problem allows for more than two stages, he iteratively performs this allocation, utilizing the largest integer contained in the fraction of available weapons to surviving targets in the immediate stage and preserving the remaining weapons for the subsequent stage.

Hosein (1989) proves that the optimal solution to **D2** is to spread the number of weapons used for each stage t, m_t, as evenly as possible, which is similar to Soland (1987). He therefore seeks to optimize over the decision variables m_t the minimum value of the final stage.

Ahner and Parson (2015) generate an optimal strategy for **D9** through the implementation of the Concave Adaptive Value Estimation (CAVE) algorithm with a modified MMR algorithm which they call the MMR Plus Algorithm. The CAVE algorithm estimates the value of second stage assignments by utilizing random realizations of the number of targets in the second stage and iteratively updating the subgradient of a concave value estimation, the CAVE function. Their MMR Plus algorithm assigns weapons to known targets in the first stage and, by comparing marginal returns of assignments to the CAVE function, indicates how many weapons to preserve for the second stage. Though the CAVE Algorithm is an approximation technique, Ahner and Parson (2015) prove the convergence to the optimal solution in the DWTA wherein all weapons have the same probabilities of kill to target j.

Sikanen (2008) uses dynamic programming to solve **D10**. He uses a backwards induction process to recursively define the policy which will optimize the problem.

4. Heuristic algorithms

Due to the computational complexity of the WTA, much of the literature focuses on heuristic algorithms which provide real time solutions rather than guaranteed optimal solutions. Many of these are of well known heuristic algorithms, such as the very large scale neighborhood (VLSN) search or the Genetic Algorithm (GA), but others are of new design, seeking to exploit the special structure of the WTA.

4.1. SWTA

The heuristic algorithms applied to the SWTA often fall into one of several groups. Herein, we will explore some of the varying approaches within these groups.

4.1.1. MMR

Kolitz (1988) implemented the MMR algorithm and, unlike denBroeder et al. (1959), did not assume that all weapons had the same probability of kill for any target j, but rather that each weapon's probability of kill for any target j was independent.

Julstrom (2009), Madni and Andrecut (2009), and Gelenbe et al. (2010) implement the MMR algorithm as a comparative benchmark in testing their heuristic approaches. Ahuja et al. (2007) utilize the MMR algorithm as one of three lower bounding schemes for their branch and bound algorithm.

4.1.2. Genetic algorithms

There have been several implementations of the GA in the SWTA, each with a minor adjustment yet the same in structure and execution. Metler et al. (1990) was the first to implement the GA for the SWTA. Lee et al. (2002b), Zhihua et al. (2009), Lee and Lee (2005), Bogdanowicz et al. (2013), Li et al. (2009), Fu et al. (2006), Lee et al. (2003), Lu et al. (2006), and Wu et al. (2008) are among the many subsequent researchers that utilized the GA for the SWTA.

4.1.3. VLSN

The very large scale neighborhood (VLSN) search metaheuristic is used by Ahuja et al. (2007) and Lee (2010) to improve upon informed feasible solutions. Their VLSN algorithms execute a heuristic search to efficiently find a quality solution and then they define local search neighborhoods within which to search for superior solutions.

4.1.4. Ant Colony Optimization

The Ant Colony Optimization (ACO) is another heuristic that is frequently implemented. It was first used by Lee et al. (2002a), and Yanxia et al. (2008), Lee and Lee (2003), Shang (2003), Shang et al. (2007), Shang (2008), Huang and LI (2005), and Su et al. (2008) among others have used the ACO to solve the SWTA.

4.1.5. Other heuristic algorithms

Other techniques that do not fall into more generalized groupings have been demonstrated to efficiently find quality solutions to the SWTA. Day (1966) solves an integer relaxed NLP and utilizes rounding schemes. Wacholder (1989) implemented neural networks to find robust solutions. Ahuja et al. (2007) used a network flow based construction heuristic to find near optimal solutions to some of the larger problems in the literature. Tokgöz and Bulkan (2013) compared the results of GA, Simulated Annealing (SA), Variable Neighborhood Search (VNS), and Tabu Search algorithms. Johansson and Falkman (2010) use Particle Swarm Optimization (PSO) and compare computational results to the GA, MMR, and exhaustive search algorithms. Similarly, Zeng et al. (2006) compares PSO with GA and a GA improved by greedy eugenics. Kwon et al. (1999) solves **S3** using a Lagrangian relaxation Branch and Bound Algorithm. Kline (2017) implemented the filtered beam search heuristic on **S1**, developed a heuristic based upon the optimal solution to the quiz problem and improved on these initial solutions using a metaheuristic which iteratively blocked assignment pairings which may have prevented superior solutions (Kline et al., 2017a) and also developed a heuristic with similarities to the Hungarian Algorithm (Kline et al., 2017b). See Hill and Pohl (2010) for a description of GA, SA, ACO, Tabu Search, and PSO.

4.2. DWTA

Less attention has been given to the DWTA as compared to the SWTA. Thus, there are fewer heuristic algorithms shared among researchers. Often, hybrid heuristic algorithms are used to inform one another in execution.

Metler et al. (1990) propose three greedy heuristics, the first of which is simply the MMR algorithm. In the second heuristic, the expected value of each pairing is computed and the selection of a random number determines the assignment based upon a probability mass function for which assignments with higher expected values have higher probabilities. The third heuristic proposed by Metler et al. (1990) is called the ALIAS Algorithm. This algorithm first updates the value of a target in stage t by dividing the value of the group by the total number of targets of type j in stage t. It then assigns weapons to targets based upon an MMR procedure, updating the probabilities of kill and repeating until an assignment violates one of the constraints or the maximum number of iterations has occurred.

Chang et al. (1987) developed a heuristic algorithm for **D4** which utilizes a heuristic subroutine to solve the first stage. An iterative process then decrements the number of weapons to use in the first stage based upon its marginal contribution until the contribution is greater than some ϵ, at which point the number of weapons for the first stage is fixed and the second stage is considered. This process iterates until either all weapons have been assigned or all stages have been considered. As a subroutine to solve the first stage, Chang et al. (1987) use three different heuristics: the MMR, an iterative linear network programming algorithm, and a nonlinear network flow algorithm.

Murphey (2000) develops a decomposition algorithm to solve **D9**. In this heuristic, he solves the first stage by some heuristic algorithm, saving the first stage solution and expected second stage solution. After this he solves the second stage primal and dual formulation across all possible second stage target outcomes. He uses these solutions to define the expected objective function value of the second stage. If this value exceeds the expected second stage objective function value previously determined, he adds a cut to the problem and repeats the process.

Xin et al. (2010) solve **D7** using Virtual Permutation (VP), TS, GA, and ACO. In a subsequent work, they developed a rule-based heuristic to solve **D7** in which they consider the saturation of the constraints in order to inform the greedy selection process by which they assign weapons to targets in a stage t (Xin et al., 2011).

Leboucher et al. (2013) use a Hungarian Algorithm to solve the assignment pairings for **D3a** and uses a GA-PSO hybrid algorithm to solve **D3b** in order to determine the firing order of the assignments. Khosla (2001) uses a GA-SA hybrid algorithm to solve **D8**. Chen et al. (2009) implements a GA to solve **D7**. Bertsekas et al. (2000) uses Neuro-dynamic programming to obtain near optimal policies which he compares to optimal policies obtained through dynamic programming.

5. Discussion

5.1. Evolution of WTA

Research on the WTA has evolved since the work of Manne (1958) with developments in both the formulation of the problem and the solution techniques implemented. In the earliest works, reference to the limited capacity to solve large nonlinear problems (Day, 1966) resulted in attention on simplified formulations of the SWTA (denBroeder et al., 1959) and solution techniques capable given the computational capacity of the day (i.e., (Lemus and David, 1963), (Day, 1966)). Eckler and Burr (1972) proposed and discussed the possibility of solving dynamic variants of the SWTA but were unable to generate algorithms to solve such problems.

As computational power increased, so too did the ability to solve problems of increased complexity. Burr et al. (1985) modeled and solved one of the earliest DWTA problems, as did Chang et al. (1987), Soland (1987), and Hosein et al. (1988). Meanwhile, models of the SWTA with fewer assumptions were solved with novel approaches (i.e., Kwon et al., 1999; Metler et al., 1990; Wacholder, 1989).

This pattern continued into the 2000s, with model developments either capturing additional parameters which more closely resemble reality (i.e., Shang et al., 2007 and Karasakal, 2008) or models which enabled faster optimal or near optimal solutions (i.e., Malcolm, 2004, Ahuja et al., 2007, and Ahner and Parson, 2015). Once developed, these models were solved using newer approaches (i.e., Bertsekas et al., 2000; Kline et al., 2017a; Wu et al., 2008) or combinations of existing approaches which could be implemented efficiently (i.e., Ahuja et al., 2007; Lee et al., 2002a; Su et al., 2008, and Xin et al., 2010).

As computational power continues to grow, the WTA will likely continue to be the subject of research which improves upon existing solution techniques. Dynamic models which consider the time dependence of weapon utilization and target flight paths have been proposed (Khosla, 2001; Leboucher et al., 2013) but have received less attention than existing models. Improvements to the solution techniques in these models are yet to emerge, and as remarked by Khosla (2001), "in spite of the two-step approach [outlined in Khosla (2001)], each of the optimization problems still have a huge search space even for a modest number of threats, weapon systems, and time points." Methods of improving on the two-step approach are yet to emerge. Similarly, Leboucher et al. (2013) remarks on the exponential growth of the problem and proposes a two-step solution technique, adding that an additional problem is "to be able to quantify the quality of one proposed solution."

The future of the WTA will need to address the aforementioned difficulties of the scheduling-focused DWTA with techniques capable of exploiting the special structure of the problem. Additionally, there exist many parameters of the problem which are removed due to the increased computational complexity they would bring that could be introduced using novel modeling techniques.

5.2. Recent developments

While the focus of the research discussed heretofore focuses primarily on the static and dynamic allocation of interceptors to offensive missiles, recent research has provided different frameworks through which this problem is addressed. We briefly discuss these recent developments here.

5.2.1. Sensor Weapon Target Assignment Problem

Missile defense depends on the accuracy and reliability of sensors to identify the type and position of each incoming missile so as to appropriately defend a protected asset. Much of the literature disregards the allocation of sensors and assumes the defender's omniscience. However, different approaches concerning the consideration of a finite number of sensors are found within the literature.

Bogdanowicz et al. (2007) develop a model that seeks to maximize the sum of the benefits of assigning each sensor to each target and each weapon to each target. Zi-fen et al. (2011) combine the auction algorithm based technique developed by Bogdanowicz et al. (2007) to reduce the limitations that an imperfect network topology would introduce.

Others have considered the effect that sensors have on the probability of detecting incoming missiles. Jian and Chen (2015) models the damage probability of an interceptor as the probability that a sensor will identify the missile and the destructive capacity of the weapon with which the sensor is paired. Xin et al. (2018) extends this by modeling the probability of successful engagement as the product of the interceptor's probability of kill and the sensor's probability of detection.

5.2.2. Multi-objective programs

Each of the formulations presented in Section 2 seeks to maximize the probability of destruction of all of the incoming missiles in some capacity. While the DWTA formulations include parameters and constraints that promote the preservation of some of the interceptors for subsequent salvos or subsequent shots to a leaker, solving each of these formulations results in the consumption of all available resources. Though it is important to defend a protected asset, there may be situations in which it is beneficial to conserve interceptors. As such, there has been research into the simultaneous maximization of damage and minimization of shots.

Li et al. (2015) model the DWTA with an objective that simultaneously maximizes the expected damage and minimizes the ammunition consumption. They compare the performance of two solution techniques for this bi-objective program and later develop and compare a third technique (Li et al., 2017a). Li et al. (2017b) solve a similar formulation with a modified ant colony algorithm. Li et al. (2018) include a third objective which seeks to maximize the value of each weapon type and use a genetic algorithm to solve the multi-objective model.

5.2.3. Game theory approaches

While all of the research discussed thus far addresses the response to an adversary with no consideration of the adversary's reaction, there has been research on this game theory aspect to missile defense. In contrast to the discussion regarding the research utilizing sensors and the research of multi-objective programs, the research of game theory approaches does not conform to similar models.

Shan and Zhuang (2013) develop a model that considers the impact of defensive resource allocation in the face of strategically focused and non-strategically focused adversaries. Golany et al. (2015) develop a model that seeks to place defensive resources in order to defend multiple assets and extend this model with a superior solution technique in Golany et al. (2017). Similarly, Boardman et al. (2017) models such a scenario and considers interceptor probabilities of kill.

Shalumov and Shima (2017) models a scenario wherein the protected assets are maneuvering aircrafts. Their model considers the flight paths of the aircrafts, the trajectories of the missiles, and the probabilities of kill of the interceptors in order to best guide the aircrafts and their defensive actions. Within the simulation they run, Shalumov and Shima (2017) test different assignment algorithms within a small scale two agent game.

5.3. Alternate applications

The WTA literature informs research beyond missile defense. Often, WTA works are cited for their modeling or solution techniques, as they are applicable in many assignment problems with quantifiable rewards or costs and limited resources. Gülpınar et al. (2018) framed their model and solution technique for a dynamic resource allocation problem on much of the same literature that is outlined in Sections 2–4 of this survey. Çetin and Esen (2006) model and solve a media allocation problem with an objective function which, if V_j is the audience type value, p_{ij} is the probability that audience j views advertisement i, and decision variable x_{ij} is the number of advertisements of type i to assign to audience j, is the formulation **S1**. Onay et al. (2016) model neuromarketing with **S1** as an objective function where V_j is the value of the brain stimulus, p_{ij} is the probability that stimulus i affects the brain region j, and decision variable x_{ij} is the number of stimulants of type i to assign to brain region j. Another application using objective functions similar to **S1** is cancer treatment. The targeting of cancer cells with medication is modeled and solved by Çetin (2007) and Esen et al. (2008) using WTA research. Both Alighanbari (2004) and Bertuccelli and How (2011) model and solve unmanned aerial vehicle (UAV) assignment planning problems with static and dynamic WTA models. Lastly, Gelenbe et al. (2010) use WTA research to model and solve a problem of dispatching ambulances to emergencies with an objective function that is similar to formulations **S1** and **S3**.

5.4. Analysis of literature influence

Matlin (1970) put forth the first survey of the WTA literature, characterizing the problem with five components and defining elements of the problem which structured its subsequent research. Due to the high volume of literature at present, a strategy to focus the considered literature for this survey was necessary, else an exhaustive list of the literature would obfuscate the state of the WTA and how it came to be. We considered the relevance of any work in the literature to be a function of its usefulness to subsequent research and used a rate of citation metric as a tool to limit our discussion heretofore.

Table 1 shows that much of the work used in this survey with higher citation rate, given the current year of 2018, focuses on heuristic solutions to the SWTA. The entries within this table rate approximately 2 citations per year or more and demonstrate, by their consistent impact on research, the importance and substantial contribution they have made to the WTA literature.

At the same time, we find that this metric, while helpful in reducing the volume of literature to consider, can lead us to consider some works as less relevant due to the lower citation rate which is sensitive to original publication date. Despite this reduced rate, the works in Table 2 have a large number of citations and are foundational in much of the literature we consider highly relevant. As such, we include these works.

6. Conclusion

The WTA has a rich breadth of literature which serves to improve upon the theory and techniques necessary to efficiently solve these complex problems. Early works sought to find methods to transform the problem into a simpler form, assume many of the complexities away, or do both in order to generate a formulation which was manageable with the computational capacity of the day. The theories and techniques proposed by the earliest researchers, such as Manne (1958) and denBroeder et al. (1959), inform much of the current research and built a foundation upon which subsequent researchers were able to extend the theory and solution techniques of the WTA.

In this survey, we have provided nine static models and ten dynamic models for the WTA which have had an impact on the literature and have provided insights into the problem from a modeling perspective. Additionally, we have reviewed some of the exact algorithms, heuristic algorithms, and metaheuristic algorithms for

Table 1
WTA literature by citation density .

Author	Year	Citations	Citation rate	SWTA or DWTA	Exact or Heuristic
Lee, Z	2003	257	17.13	SWTA	Heuristic
Ahuja	2007	187	17	SWTA	Both
Lee, Z	2002	249	15.56	SWTA	Heuristic
Lee, Z	2005	117	9	SWTA	Heuristic
Lloyd	1986	230	7.19	SWTA	Exact
Xin	2011	50	7.14	DWTA	Heuristic
Xin	2010	55	6.88	DWTA	Heuristic
Karasakal	2008	57	5.7	SWTA	Exact
Ahner	2015	15	5	DWTA	Exact
Gelenbe	2010	38	4.75	SWTA	Heuristic
Chen	2009	42	4.67	DWTA	Heuristic
Bertsekas	2000	83	4.61	DWTA	Both
Rosenberger	2005	59	4.54	SWTA	Exact
Lee, M	2010	36	4.5	SWTA	Heuristic
Yanxia	2008	42	4.2	SWTA	Heuristic
Zeng	2006	50	4.17	SWTA	Heuristic
Wacholder	1989	119	4.10	SWTA	Heuristic
Lee, Z	2002	62	3.88	SWTA	Heuristic
Bogdanowicz	2013	16	3.2	SWTA	Heuristic
Eckler	1972	144	3.13	Both	Exact
Khosla	2001	52	3.06	DWTA	Heuristic
Murphey	2000	54	3	DWTA	Heuristic
Lee, Z	2003	43	2.87	SWTA	Heuristic
Johansson	2011	18	2.57	SWTA	Heuristic
Matlin	1970	115	2.40	SWTA	Exact
Madni	2009	21	2.33	SWTA	Heuristic
Hosein	1988	63	2.1	DWTA	Exact
As of September 2018					

Table 2
Included foundational WTA literature by citation count.

Author	Year	Citations	Citation rate	SWTA or DWTA	Exact or Heuristic
Manne	1958	118	1.97	SWTA	Exact
Hosein	1989	46	1.59	DWTA	Exact
denBroeder	1959	83	1.41	SWTA	Exact
Soland	1987	31	1	DWTA	Exact
Day	1966	53	1.02	SWTA	Heuristic
As of September 2018					

the static and dynamic WTA. Some of these algorithms are widely used in the literature, such as the branch and bound algorithm or the genetic algorithm. Others, such as the algorithm developed by Bogdanowicz (2012) or the rule based heuristic developed by Xin et al. (2010) were created to solve the WTA, efficiently exploiting the special structure of the problem.

The only consistent aspect of the WTA since its introduction into the field is its enduring relevance. As defensive strategies improve to enhance the capacity to mitigate the risk that ballistic missiles present, the technology of these ballistic missiles also improves. Additionally, while only minimally addressed here, many non-defensive applications will continue to benefit from the lively research surrounding the Weapon Target Assignment Problem.

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