HOMEWORK 8

Sundry

Before you start your homework, write down your team. Who else did you work with on this homework? List names and email addresses. (In case of homework party, you can also just describe the group.) How did you work on this homework? Working in groups of 3-5 will earn credit for your "Sundry" grade.

Please copy the following statement and sign next to it:

I certify that all solutions are entirely in my words and that I have not looked at another student's solutions. I have credited all external sources in this write up.

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1 Fermat's Wristband

Let p be a prime number and let k be a positive integer. We have beads of k different colors, where any two beads of the same color are indistinguishable.

(a) We place p beads onto a string. How many different ways are there construct such a sequence of p beads of k different colors?

(b) How many sequences of p beads on the string have at least two colors?

(c) Now we tie the two ends of the string together, forming a wristband. Two wristbands are equivalent if the sequence of colors on one can be obtained by rotating the beads on the other. (For instance, if we have $k = 3$ colors, red (R), green (G), and blue (B), then the length $p = 5$ necklaces RGGBG, GGBGR, GBGRG, BGRGG, and GRGGB are all equivalent, because these are all rotated versions of each other.)

How many non-equivalent wristbands are there now? Again, the p beads must not all have the same color. (Your answer should be a simple function of k and p.)

[*Hint*: Think about the fact that rotating all the beads on the wristband to another position produces an identical wristband.]

(d) Use your answer to part (c) to prove Fermat's little theorem.

Solution:

(a) k^p. For each of the p beads, there are k possibilities for its colors. Therefore, by the first counting principle, there are k^p different sequences.

(b) $k^p - k$. You can have k sequences of a beads with only one color.

(c) Since p is prime, rotating any sequence by less than p spots will produce a new sequence. As in, there is no number x smaller than p such that rotating the beads by x would cause the pattern to look the same. So, every pattern which has more than one color of beads can be rotated to form $p-1$ other patterns. So the total number of patterns equivalent with some bead sequence is p. Thus, the total number of non-equivalent patterns are $(k^p - k)/p$.

(d) $(k^p - k)/p$ must be an integer, because from the previous part, it is the number of ways to count something. Hence, $k^p - k$ has to be divisible by p, i.e., $k^p \equiv k \pmod p$, which is Fermat's Little Theorem.

2 Sampling

Suppose you have balls numbered $1, \ldots, n$, where n is a positive integer ≥ 2, inside a coffee mug. You pick a ball uniformly at random, look at the number on the ball, replace the ball back into the coffee mug, and pick another ball uniformly at random.

(a) What is the probability that the first ball is 1 and the second ball is 2?

(b) What is the probability that the second ball's number is strictly less than the first ball's number?

(c) What is the probability that the second ball's number is exactly one greater than the first ball's number?

(d) Now, assume that after you looked at the first ball, you did *not* replace the ball in the coffee mug (instead, you threw the ball away), and then you drew a second ball as before. Now, what are the answers to the previous parts?

Solution:

(a) Out of n^2 pairs of balls that you could have chosen, only one pair $(1,2)$ corresponds to the event we are interested in, so the probability is $1/n^2$.

(b) The probability that the two balls have the same number is $n/n^2 = 1/n$, so the probability that the balls have different numbers is $1 - 1/n = (n-1)/n$. By symmetry, it is equally likely for the first ball to have a greater number and for the second ball to have a greater number, so we take the probability $(n-1)/n$ and divide it by two to obtain $(n-1)/(2n)$.

(c) Again, there are n^2 pairs of balls that we could have drawn, but there are $n-1$ pairs of balls which correspond to the event we are interested in: $\{(1,2), (2,3), \ldots, (n-1, n)\}$. So, the probability is $(n-1)/n^2$.

(d) There are a total of $n(n-1)$ pairs of balls that we could have drawn, and only the pair $(1,2)$ corresponds to the event that we are interested in, so the probability is $1/(n(n-1))$.

The probability that the two balls are the same is now 0, but the symmetry described earlier still applies, so the probability that the second ball has a smaller number is $1/2$.

There are a total of $n(n-1)$ pairs of balls that we could have drawn, and we are interested in the $n-1$ pairs $(1,2),(2,3),\ldots,(n-1,n)$ as before. Thus, the probability that the second ball is one greater than the first ball is $1/n$.

3 Easter Eggs

You made the trek to Soda for a Spring Break-themed homework party, and every attendee gets to leave with a party favor. You're given a bag with 20 chocolate eggs and 40 (empty) plastic eggs. You pick 5 eggs without replacement.

(a) What is the probability that the first egg you drew was a chocolate egg?

(b) What is the probability that the second egg you drew was a chocolate egg?

(c) Given that the first egg you drew was an empty plastic one, what is the probability that the fifth egg you drew was also an empty plastic egg?

Solution:

(a) $\mathbb{P}(\text{chocolate egg}) = \frac{20}{60} = \frac{1}{3}$.

(b) Long calculation using Total Probability Rule: let C_i denote the event that the ith egg is chocolate, and P_i denote the event that the ith egg is plastic. We have

$$\begin{aligned}\mathbb{P}(C_2) &= \mathbb{P}(C_1 \cap C_2) + \mathbb{P}(P_1 \cap C_2)\\ &= \mathbb{P}(C_1)\mathbb{P}(C_2 \mid C_1) + \mathbb{P}(P_1)\mathbb{P}(C_2 \mid P_1)\\ &= \frac{1}{3}\cdot\frac{19}{59} + \frac{2}{3}\cdot\frac{20}{59}\\ &= \frac{1}{3}.\end{aligned} \tag{1}$$

Short calculation: By symmetry, this is the same probability as part (a), $1/3$. This is because we don't know what type of egg was picked on the first draw, so the distribution for the second egg is the same as that of the first.

(c) By symmetry, since we don't know any information about the 2nd, 3rd, or 4th eggs, $\mathbb{P}(\text{5th egg = plastic} \mid \text{1st egg = plastic}) = \mathbb{P}(\text{2nd egg = plastic} \mid \text{1st egg = plastic}) = 39/59$.

4 Parking Lots

Some of the CS 70 staff members founded a start-up company, and you just got hired. The company has twelve employees (including yourself), each of whom drive a car to work, and twelve parking spaces arranged in a row. You may assume that each day all orderings of the twelve cars are equally likely.

(a) On any given day, what is the probability that you park next to Professor Rao, who is working there for the summer?

(b) What is the probability that there are exactly three cars between yours and Professor Rao's?

(c) Suppose that, on some given day, you park in a space that is not at one of the ends of the row. As you leave your office, you know that exactly five of your colleagues have left work before you. Assuming that you remember nothing about where these colleagues had parked, what is the probability that you will find both spaces on either side of your car unoccupied?

Solution:

(a) There are 11 ways to choose two adjacent spaces, and 2 ways to choose which of those spaces is yours and which of those spaces is Professor Rao's. After the positions of your car and Professor Rao's car are determined, the remaining cars can park in any of 10! arrangements. Therefore, the probability is

$$\frac{11 \cdot 2 \cdot 10!}{12!} = \frac{2}{12} = \frac{1}{6}.$$

(b) Again we have to count the number of arrangements and divide by 12!. Either your car is parked to the left of Professor Rao's or to his right. In the first case your car can be in any of these spots (assuming they're numbered from left to right) $1, 2, \ldots, 8$, and Professor Rao's car will always be in the spot whose number is higher by 4. This gives your car and Professor Rao's car 8 ways to be parked with your car being on the left. By symmetry, there are also 8 ways in which your car is to the right of Professor Rao's car with three cars in between. So in total there are 16 ways for your car and Professor Rao's car to be parked with 3 spaces in between. After the spaces for these two cars are determined, the remaining ones can park in any of the 10! arrangements. So the total number of arrangements is $16 \times 10!$. The probability is therefore $(16 \times 10!)/12! = 16/(12 \cdot 11) = 4/33$.

(c) We know that 5 spots from the 11 that are not occupied by your car have been freed. All choices of these 5 spots are equally likely to happen. So we can count the arrangements of the 5 free spots to compute the probability. In total there are $\binom{11}{5}$ possible ways to choose the free spots. To count to number of ways that free both spaces next to your car, we pick those two spaces and then choose 3 more spots from the remaining 9. So there are $\binom{9}{3}$ such arrangements. Therefore the desired probability is $\binom{9}{3}/\binom{11}{5}$ which is equal to

$$\frac{9!/(3!6!)}{11!/(5!6!)} = \frac{5 \times 4}{11 \times 10} = \frac{2}{11}.$$

5 Calculate These... or Else

(a) A straight is defined as a 5 card hand such that the card values can be arranged in consecutive ascending order, i.e. $\{8,9,10,J,Q\}$ is a straight. Values do not loop around, so $\{Q,K,A,2,3\}$ is not a straight. However, an ace counts as both a low card and a high card, so both $\{A,2,3,4,5\}$ and $\{10,J,Q,K,A\}$ are considered straights. When drawing a 5 card hand, what is the probability of drawing a straight from a standard 52-card deck?

(b) What is the probability of drawing a straight or a flush? (A flush is five cards of the same suit.)

(c) When drawing a 5 card hand, what is the probability of drawing at least one card from each suit?

(d) Two distinct squares are chosen at random on 8×8 chessboard. What is the probability that they share a side?

(e) 8 rooks are placed randomly on an 8×8 chessboard. What is the probability none of them are attacking each other? (Two rooks attack each other if they are in the same row, or in the same column.)

Solution:

(a) The probability space is uniform over all possible 5-card hands, so we can use counting to solve this problem. There are $\binom{52}{5}$ possible hands, so that is our denominator. To count the number of possible straights, note that there are 4 choices of suit for each of the cards for a total of 4^5 suit choices. Also, observe that once we pick a starting card for the straight, the rest of the cards are determined (e.g. if we choose 3 as the first card, then our straight must be $\{3,4,5,6,7\}$). Therefore, we need to multiply by the number of possible starting cards.

$$\frac{10 \cdot 4^5}{\binom{52}{5}} \approx 0.00394.$$

(b) From part (a), we already know the probability of drawing a straight. We count the number of flushes in the following way: there are 4 choices for the suit, and $\binom{13}{5}$ choices for the cards within the suit, for a total of $4\binom{13}{5}$ flushes. Therefore, the probability of a flush is $4\binom{13}{5}/\binom{52}{5}$.

However, by the inclusion-exclusion principle, we must subtract the number of straight flushes. We now count the number of straight flushes: there are 4 choices for the suit and 10 choices for the starting card, for a total of 40 straight flushes.

The probability of drawing a straight or a flush is therefore:

$$\frac{10 \cdot 4^5 + 4 \cdot \binom{13}{5} - 40}{\binom{52}{5}} \approx 0.00591.$$

(c) $13^4 \cdot 48$ counts twice the total number of combinations of 1 card from each suit. So the final probability is $13^4 \cdot 24/\binom{52}{5} \approx 0.264$.

(d) In 64 squares, there are:

(1) 4 at-corner squares, each shares ONLY 2 sides with other squares.

(2) $6 \cdot 4 = 24$ side squares, each shares ONLY 3 sides with other squares.

(3) $6 \cdot 6 = 36$ inner squares, each shares 4 sides with other squares.

Notice that the three cases are mutually exclusive and we cannot choose the same square twice. So we just sum up the probabilities.

$$\frac{4}{64} \cdot \frac{2}{63} + \frac{24}{64} \cdot \frac{3}{63} + \frac{36}{64} \cdot \frac{4}{63} = \frac{1}{18} \approx 0.0556.$$

Alternatively, there are $\binom{64}{2}$ total pairs of squares, and for every pair of adjacent squares there is a unique edge associated with the pair (the edge that they share). Therefore, we can count the total number of edges in the chessboard, which is $8 \cdot 7 \cdot 2$ (if we only look at the horizontal edges, there are 8 edges per row, and 7 rows of edges, and then we multiply by 2 for the vertical edges too). Thus the probability is $8 \cdot 7 \cdot 2/\binom{64}{2} = 1/18$ as before.

(e) $8!/\binom{64}{8} \approx 9.11 \cdot 10^{-6}$. This counts safe arrangements (8 choices in first row, 7 in second row, etc.) over total arrangements, and since this is a uniform probability space, this gives the probability no rooks are threatening one another.

6 Balls and Bins, All Day Every Day

You throw n balls into n bins uniformly at random, where n is a positive even integer.

(a) What is the probability that exactly k balls land in the first bin, where k is an integer $0 \leq k \leq n$?

(b) What is the probability p that at least half of the balls land in the first bin? (You may leave your answer as a summation.)

(c) Using the union bound, give a simple upper bound, in terms of p, on the probability that some bin contains at least half of the balls.

(d) What is the probability, in terms of p, that at least half of the balls land in the first bin, or at least half of the balls land in the second bin?

(e) After you throw the balls into the bins, you walk over to the bin which contains the first ball you threw, and you randomly pick a ball from this bin. What is the probability that you pick up the first ball you threw? (Again, leave your answer as a summation.)

Solution:

(a) The probability that a particular ball lands in the first bin is $1/n$. We need exactly k balls to land in the first bin, which occurs with probability $(1/n)^k$, and we need exactly $n-k$ balls to land in a different bin, which occurs with probability $(1-1/n)^{n-k}$, and there are $\binom{n}{k}$ ways to choose which of the n balls land in first bin. Thus, the probability is $\binom{n}{k}(1/n)^k(1-1/n)^{n-k}$.

(b) This is the summation over $k = n/2, \ldots, n$ of the probabilities computed in the first part, i.e., $\sum_{k=n/2}^{n} \binom{n}{k} (1/n)^k (1-1/n)^{n-k}$.

(c) The event that some bin has at least half of the bins is the union of the events A_k, $k = 1, \ldots, n$, where A_k is the event that bin k has at least half of the balls. By the union bound, $\mathbb{P}(\bigcup_{i=1}^{n} A_k) \leq \sum_{i=1}^{n} \mathbb{P}(A_k) = np$.

(d) The probability that the first bin has at least half of the balls is p; similarly, the probability that the second bin has at least half of the balls is also p. There is overlap between these two events, however: the probability that the first bin has at least half of the balls *and* the second bin has at least half of the balls is $\binom{n}{n/2} n^{-n}$: there are n^n total possible configurations for the n balls to land in the bins, but if we require exactly $n/2$ of the balls to land in the first bin and the remaining balls to land in the second bin, there are $\binom{n}{n/2}$ ways to choose which balls land in the first bin. By the principle of inclusion-exclusion, our desired probability is $p + p - \binom{n}{n/2} n^{-n} = 2p - \binom{n}{n/2} n^{-n}$.

(e) Condition on the number of balls in the bin. First we calculate the probability $\mathbb{P}(A_k)$, where A_k is the event that the bin contains k balls and $k \in \{1, \ldots, n\}$ (note that $k \neq 0$ since we know at least one ball has landed in this bin). A_k is the event that, in addition to the first ball you threw, an additional $k-1$ of the other $n-1$ balls landed in this bin, which by the reasoning in Part (a) has probability $\mathbb{P}(A_k) = \binom{n-1}{k-1} (1/n)^{k-1} (1-1/n)^{n-k}$. If we let B be the event that we pick up the first ball we threw, then $\mathbb{P}(B \mid A_k) = 1/k$ since we are equally likely to pick any of the k balls in the bin. Thus the overall probability we are looking for is, by an application of the law of total probability,

$$\mathbb{P}(B) = \sum_{k=1}^{n} \mathbb{P}(A_k \cap B) = \sum_{k=1}^{n} \mathbb{P}(A_k) \mathbb{P}(B \mid A_k) = \sum_{k=1}^{n} \frac{1}{k} \binom{n-1}{k-1} \left(\frac{1}{n}\right)^{k-1} \left(1 - \frac{1}{n}\right)^{n-k}.$$