CS 70 Discrete Mathematics and Probability Theory
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HW 10

Sundry

Before you start your homework, write down your team. Who else did you work with on this homework? List names and email addresses. (In case of homework party, you can also just describe the group.) How did you work on this homework? Working in groups of 3-5 will earn credit for your "Sundry" grade.

Please copy the following statement and sign next to it:

I certify that all solutions are entirely in my words and that I have not looked at another student's solutions. I have credited all external sources in this write up.

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1 How Many Queens?

You shuffle a standard 52-card deck, before drawing the first three cards from the top of the pile. Let X denote the number of queens you draw.

(a) What is $\mathbb{P}(X = 0)$?

(b) What is $\mathbb{P}(X = 1)$?

(c) What is $\mathbb{P}(X = 2)$?

(d) What is $\mathbb{P}(X = 3)$?

(e) Do the answers you computed in parts (a) through (d) add up to 1, as expected?

(f) Compute $\mathbb{E}(X)$ from the definition of expectation.

(g) Suppose we define indicators X_i, $1 \leq i \leq 3$, where X_i is the indicator variable that equals 1 if the ith card is a queen and 0 otherwise. Compute $\mathbb{E}(X)$.

(h) Are the X_i indicators independent? Does this affect your solution to part (g)?

Solution:

(a) We must draw 3 non-queen cards in a row, so the probability is

$$\mathbb{P}(X=0) = \frac{48}{52} \cdot \frac{47}{51} \cdot \frac{46}{50} = \frac{4324}{5525}.$$

Alternatively, every 3-card hand is equally likely, so we can use counting. There are $\binom{52}{3}$ total 3-card hands, and $\binom{48}{3}$ hands with only non-queen cards, which gives us the same result.

$$\mathbb{P}(X=0) = \frac{\binom{48}{3}}{\binom{52}{3}} = \frac{4324}{5525}$$

(b) We will continue to use counting. The number of hands with exactly one queen amounts to the number of ways to choose 1 queen out of 4, and 2 non-queens out of 48.

$$\mathbb{P}(X=1) = \frac{\binom{4}{1}\binom{48}{2}}{\binom{52}{3}} = \frac{1128}{5525}$$

(c) Choose 2 queens out of 4, and 1 non-queen out of 48.

$$\mathbb{P}(X=2) = \frac{\binom{4}{2}\binom{48}{1}}{\binom{52}{3}} = \frac{72}{5525}$$

(d) Choose 3 queens out of 4.

$$\mathbb{P}(X=3) = \frac{\binom{4}{3}}{\binom{52}{3}} = \frac{1}{5525}$$

(e) We check:

$$\mathbb{P}(X=0)+\mathbb{P}(X=1)+\mathbb{P}(X=2)+\mathbb{P}(X=3) = \frac{4324+1128+72+1}{5525} = 1$$

(f) From the definition, $\mathbb{E}(X) = \sum_{k=0}^{3} k\mathbb{P}(X=k)$, so

$$\mathbb{E}(X) = 0 \cdot \frac{4324}{5525} + 1 \cdot \frac{1128}{5525} + 2 \cdot \frac{72}{5525} + 3 \cdot \frac{1}{5525} = \frac{3}{13}.$$

(g) We know that $\mathbb{E}(X_i) = \mathbb{P}(\text{card } i \text{ is an queen}) = 1/13$, so

$$\mathbb{E}(X) = \mathbb{E}(X_1)+\mathbb{E}(X_2)+\mathbb{E}(X_3) = \frac{1}{13}+\frac{1}{13}+\frac{1}{13} = \frac{3}{13}.$$

Notice how much faster it was to compute the expectation using indicators!

(h) No, they are not independent. As an example:

$$\mathbb{P}(X_1=1)\mathbb{P}(X_2=1) = \frac{1}{13} \cdot \frac{1}{13} = \frac{1}{169}$$

However,

$$\mathbb{P}(X_1=1, X_2=1) = \mathbb{P}(\text{the first and second cards are both queens}) = \frac{4}{52} \cdot \frac{3}{51} = \frac{1}{221}.$$

Even though the indicators are not independent, this does not change our answer for part (g). Linearity of expectation *always* holds, which makes it an extremely powerful tool.

2 Who Has More Sisters?

Out of all families in the world with $n > 0$ children, you sample one at random and observe X, the total number of sisters that the male children in this family have, and Y, the total number of sisters that the female children in this family have (for example, if $n = 3$ and there are two males and one female, then $X = 2$ and $Y = 0$).

Assuming that each child born into the world has an equal chance of being male or female, find expressions for $\mathbb{E}(X)$ and $\mathbb{E}(Y)$ in terms of n (these expressions should not involve summations). Based on your expressions, do males have more sisters or females have more sisters, on average?

[Hint: Define a random variable B to denote the number of boys, find an expression for X as a function of B, and apply linearity of expectation. Use a similar approach for girls.]

Solution:

Let B be the number of boys. Each boy has $n-B$ sisters, so boys have $B(n-B)$ sisters in total. Using linearity of expectation,

$$\mathbb{E}(X) = \mathbb{E}\big(B(n-B)\big) = \mathbb{E}(Bn - B^2) = n\,\mathbb{E}(B) - \mathbb{E}(B^2).$$

Note that B follows the distribution $\text{Binomial}(n,p)$ where $p = 1/2$. So $\mathbb{E}(B) = np = n/2$ and $\text{var}(B) = np(1-p) = n/4$. From the definition of variance, we also have

$$\begin{aligned}\mathbb{E}(B^2) - \mathbb{E}(B)^2 &= \text{var}(B)\\ \mathbb{E}(B^2) &= \text{var}(B) + \mathbb{E}(B)^2 = \frac{n}{4} + \left(\frac{n}{2}\right)^2 = \frac{n}{4} + \frac{n^2}{4}.\end{aligned}$$

Substituting into our expression for $\mathbb{E}(X)$, we obtain

$$\mathbb{E}(X) = n\,\mathbb{E}(B) - \mathbb{E}(B^2) = n \cdot \frac{n}{2} - \frac{n}{4} - \frac{n^2}{4} = \frac{n^2}{4} - \frac{n}{4} = \frac{n(n-1)}{4}.$$

Let G be the number of girls. Each girl has $G-1$ sisters, so girls have $G(G-1)$ sisters in total. Using linearity of expectation,

$$\mathbb{E}(Y) = \mathbb{E}\big(G(G-1)\big) = \mathbb{E}(G^2 - G) = \mathbb{E}(G^2) - \mathbb{E}(G).$$

Note that G follows exactly the same distribution as B, so $\mathbb{E}(G^2) = n^2/4 + n/4$ and $\mathbb{E}(G) = n/2$. Thus

$$\mathbb{E}(Y) = \mathbb{E}(G^2) - \mathbb{E}(G) = \frac{n^2}{4} + \frac{n}{4} - \frac{n}{2} = \frac{n^2}{4} - \frac{n}{4} = \frac{n(n-1)}{4}.$$

We expect boys and girls to have the same number of sisters!

Alternate Solution: We can solve this problem using indicators. For $i = 1, \ldots, n$, define X_i to be the number of sisters that child i has *if that child is male*, and 0 otherwise. Likewise, define Y_i to be the number of sisters that child i has *if that child is female*, and 0 otherwise. Then $X = \sum_{i=1}^{n} X_i$ and $Y = \sum_{i=1}^{n} Y_i$.

Now, every child i has $n-1$ siblings. Notice that among these siblings, we expect $1/2$ to be female, *regardless of the gender of child i!* Therefore child i has, on expectation, $(n-1)/2$ siblings. Child i also has a $1/2$ chance of being male or female, so

$$\mathbb{E}[X_i] = \mathbb{E}[Y_i] = \frac{n-1}{4}.$$

We can conclude that

$$\mathbb{E}[X] = \mathbb{E}\Big[\sum_{i=1}^{n} X_i\Big] = \sum_{i=1}^{n} \mathbb{E}[X_i] = n\,\mathbb{E}[X_i] = \frac{n(n-1)}{4}.$$

and

$$\mathbb{E}[Y] = \mathbb{E}\Big[\sum_{i=1}^{n} Y_i\Big] = \sum_{i=1}^{n} \mathbb{E}[Y_i] = n\,\mathbb{E}[Y_i] = \frac{n(n-1)}{4}.$$

which concludes the proof.

3 Student Life

In an attempt to avoid having to do laundry often, Marcus comes up with a system. Every night, he designates one of his shirts as his dirtiest shirt. In the morning, he randomly picks one of his shirts to wear. If he picked the dirtiest one, he puts it in a dirty pile at the end of the day (a shirt in the dirty pile is not used again until it is cleaned). When Marcus puts his last shirt into the dirty pile, he finally does his laundry, and again designates one of his shirts as his dirtiest shirt (laundry isn't perfect) before going to bed. This process then repeats.

(a) If Marcus has n shirts, what is the expected number of days that transpire between laundry events? Your answer should be a function of n involving no summations.

(b) Say he gets even lazier, and instead of organizing his shirts in his dresser every night, he throws his shirts randomly onto one of n different locations in his room (one shirt per location), designates one of his shirts as his dirtiest shirt, and one location as the dirtiest location. In the morning, if he happens to pick the dirtiest shirt, *and* the dirtiest shirt was in the dirtiest location, then he puts the shirt into the dirty pile at the end of the day and does not use that location anymore (it is too dirty now). What is the expected number of days that transpire between laundry events now? Again, your answer should be a function of n involving no summations.

Solution:

(a) The number of days that it takes for him to throw a shirt into the dirty pile can be represented as a geometric RV. For the first shirt, this is the geometric RV with $p = 1/n$. We can see this by noticing that every day the probability of getting the dirtiest shirt remains $1/n$.

We'll call X_i the number of days that goes until he throws the ith shirt into the dirty pile. Since on the ith shirt, there are $n-i+1$ shirts left, we can get the parameter of X_i is $1/(n-i+1)$.

The number of days until he does his laundry is a sum of these variables. Therefore, we can get the following result:

$$\mathbb{E}[X] = \mathbb{E}\Big[\sum_{i=1}^{n} X_i\Big] = \sum_{i=1}^{n} \mathbb{E}[X_i] = \sum_{i=1}^{n} (n-i+1) = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

(b) For this part we can use a similar approach but the probability for X_i becomes $1/(n-i+1)^2$. This is because the dirtiest shirt falls into the dirtiest spot with probability $1/(n-i+1)$ and we pick it after that with probability $1/(n-i+1)$, so the probability of picking the dirtiest shirt from the dirtiest spot for the ith shirt is $1/(n-i+1)^2$. Using the same approach, we get the following sum:

$$\mathbb{E}[X] = \mathbb{E}\Big[\sum_{i=1}^{n} X_i\Big] = \sum_{i=1}^{n} \mathbb{E}[X_i] = \sum_{i=1}^{n} (n-i+1)^2 = \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

4 Soccer Practice

Messi and Ronaldo are practicing their penalty kicks. In each *round* of their training exercise, each player takes a penalty kick and either scores or misses. Assume (somewhat unrealistically) that each penalty is independent of every other penalty kick, with Messi missing each penalty with probability p_1 and Ronaldo missing each penalty with probability p_2. Show that the number of rounds until the first miss of *either* player is geometrically distributed with parameter $p_1 + p_2 - p_1 p_2$.

Solution:

Let X_1 denote the number of rounds until Messi's first miss, and X_2 denote the number of rounds until Ronaldo's first miss. We have that $X_1 \sim \text{Geometric}(p_1)$ and $X_2 \sim \text{Geometric}(p_2)$, where X_1, X_2 are independent r.v.'s.

We also use the following definition of the minimum:

$$\min(x,y) = \begin{cases} x, & \text{if } x \leq y; \\ y, & \text{if } x > y. \end{cases}$$

Now, for all $k \in \{1,2,\dots\}$, $\min(X_1,X_2) = k$ is equivalent to $(X_1 = k) \cap (X_2 \geq k)$ or $(X_2 = k) \cap (X_1 > k)$. Hence,

$$\begin{aligned} \mathbb{P}[X = k] = \mathbb{P}[\min(X_1,X_2) = k] &= \mathbb{P}[(X_1 = k) \cap (X_2 \geq k)] + \mathbb{P}[(X_2 = k) \cap (X_1 > k)] \\ &= \mathbb{P}[X_1 = k] \cdot \mathbb{P}[X_2 \geq k] + \mathbb{P}[X_2 = k] \cdot \mathbb{P}[X_1 > k] \end{aligned}$$

(since X_1 and X_2 are independent)

$$= [(1-p_1)^{k-1} p_1](1-p_2)^{k-1} + [(1-p_2)^{k-1} p_2](1-p_1)^k$$

(since X_1 and X_2 are geometric)

$$= \big((1-p_1)(1-p_2)\big)^{k-1}\big(p_1+p_2(1-p_1)\big)$$
$$= (1-p_1-p_2+p_1p_2)^{k-1}(p_1+p_2-p_1p_2).$$

But this final expression is precisely the probability that a geometric RV with parameter $p_1+p_2-p_1p_2$ takes the value k. Hence $X \sim \text{Geometric}(p_1+p_2-p_1p_2)$, and $\mathbb{E}[X] = (p_1+p_2-p_1p_2)^{-1}$.

An alternative, slightly cleaner approach is to work with the *tail probabilities* of the geometric distribution, rather than with the usual point probabilities as above. In other words, we can work with $\mathbb{P}[X \geq k]$ rather than with $\mathbb{P}[X = k]$; clearly the values $\mathbb{P}[X \geq k]$ specify the values $\mathbb{P}[X = k]$ since $\mathbb{P}[X = k] = \mathbb{P}[X \geq k] - \mathbb{P}[X \geq (k+1)]$, so it suffices to calculate them instead. We then get the following argument:

$$\begin{aligned}
\mathbb{P}[X \geq k] &= \mathbb{P}[\min(X_1,X_2) \geq k] = \mathbb{P}[(X_1 \geq k) \cap (X_2 \geq k)] \\
&= \mathbb{P}[X_1 \geq k] \cdot \mathbb{P}[X_2 \geq k] && \text{since } X_1, X_2 \text{ are independent} \\
&= (1-p_1)^{k-1}(1-p_2)^{k-1} && \text{since } X_1, X_2 \text{ are geometric} \\
&= \big((1-p_1)(1-p_2)\big)^{k-1} = (1-p_1-p_2+p_1p_2)^{k-1}.
\end{aligned}$$

This is the tail probability of a geometric distribution with parameter $p_1+p_2-p_1p_2$, so we are done.

5 Boutique Store

(a) Consider a boutique store in a busy shopping mall. Every hour, a large number of people visit the mall, and each independently enters the boutique store with some small probability. The store owner decides to model X, the number of customers that enter her store during a particular hour, as a Poisson random variable with mean λ. Suppose that whenever a customer enters the boutique store, they leave the shop without buying anything with probability p. Assume that customers act independently, i.e. you can assume that they each simply flip a biased coin to decide whether to buy anything at all. Let us denote the number of customers that buy something as Y and the number of them that do not buy anything as Z (so $X = Y + Z$). What is the probability that $Y = k$ for a given k? How about $\mathbb{P}[Z = k]$? Prove that Y and Z are Poisson random variables themselves.

Hint: You can use the identity

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

(b) Prove that Y and Z are independent.

Solution:

(a) We consider all possible ways that the event $Y = k$ might happen: namely, $k + j$ people enter the store ($X = k + j$) and then exactly k of them choose to buy something. That is,

$$
\begin{aligned}
\mathbb{P}[Y=k] &= \sum_{j=0}^{\infty} \mathbb{P}[X=k+j] \cdot \mathbb{P}[Y=k \mid X=k+j] \\
&= \sum_{j=0}^{\infty} \left(\frac{\lambda^{k+j}}{(k+j)!} \mathrm{e}^{-\lambda} \right) \cdot \left(\binom{k+j}{k} p^j (1-p)^k \right) \\
&= \sum_{j=0}^{\infty} \frac{\lambda^{k+j}}{(k+j)!} \mathrm{e}^{-\lambda} \cdot \frac{(k+j)!}{k!j!} p^j (1-p)^k = \frac{(\lambda(1-p))^k \mathrm{e}^{-\lambda}}{k!} \cdot \sum_{j=0}^{\infty} \frac{(\lambda p)^j}{j!} \\
&= \frac{(\lambda(1-p))^k \mathrm{e}^{-\lambda}}{k!} \cdot \mathrm{e}^{\lambda p} = \frac{(\lambda(1-p))^k \mathrm{e}^{-\lambda(1-p)}}{k!}.
\end{aligned}
$$

Hence, Y follows the Poisson distribution with parameter $\lambda(1-p)$. The case for Z is completely analogous:

$$\mathbb{P}[Z=k] = \frac{(\lambda p)^k \mathrm{e}^{-\lambda p}}{k!}$$

and Z follows the Poisson distribution with parameter λp.

(b) If Y and Z are independent, then $\mathbb{P}(Y=y, Z=z) = \mathbb{P}(Y=y) \cdot \mathbb{P}(Z=z)$:

$$
\begin{aligned}
\mathbb{P}(Y=y,Z=z) = \sum_{x=0}^{\infty} \mathbb{P}(X=x,Y=y,Z=z) &= \sum_{x=0}^{\infty} \mathbb{P}(Y=y,Z=z \mid X=x)\mathbb{P}(X=x) \\
&= \mathbb{P}(Y=y,Z=z \mid X=y+z)\mathbb{P}(X=y+z) = \frac{\cancel{(y+z)!}}{y!z!} p^z (1-p)^y \frac{\mathrm{e}^{-\lambda} \lambda^{y+z}}{\cancel{(y+z)!}} \\
&= \frac{\mathrm{e}^{-\lambda(1-p)} (\lambda(1-p))^y}{y!} \cdot \frac{\mathrm{e}^{-\lambda p} (\lambda p)^z}{z!} = \mathbb{P}(Y=y) \cdot \mathbb{P}(Z=z).
\end{aligned}
$$

6 Sum of Poisson Variables

Assume that you were given two independent Poisson random variables X_1, X_2. Assume that the first has mean λ_1 and the second has mean λ_2. Prove that $X_1 + X_2$ is a Poisson random variable with mean $\lambda_1 + \lambda_2$.

Hint: Recall the binomial theorem.

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Solution:

To show that $X_1 + X_2$ is a Poisson random variable with mean $\lambda_1 + \lambda_2$, we have show that

$$\mathbb{P}[(X_1+X_2)=i] = \frac{(\lambda_1+\lambda_2)^i}{i!} \mathrm{e}^{-(\lambda_1+\lambda_2)}.$$

We proceed as follows:

$$\begin{aligned}\mathbb{P}[(X_1+X_2)=i] &= \sum_{k=0}^{i}\mathbb{P}[X_1=k,X_2=(i-k)] = \sum_{k=0}^{i}\frac{\lambda_1^k}{k!}\,\mathrm{e}^{-\lambda_1}\cdot\frac{\lambda_2^{i-k}}{(i-k)!}\,\mathrm{e}^{-\lambda_2} \\ &= \mathrm{e}^{-\lambda_1}\,\mathrm{e}^{-\lambda_2}\sum_{k=0}^{i}\frac{1}{k!(i-k)!}\lambda_1^k\lambda_2^{i-k} = \frac{\mathrm{e}^{-\lambda_1}\,\mathrm{e}^{-\lambda_2}}{i!}\sum_{k=0}^{i}\frac{i!}{k!(i-k)!}\lambda_1^k\lambda_2^{i-k} \\ &= \frac{\mathrm{e}^{-(\lambda_1+\lambda_2)}}{i!}\sum_{k=0}^{i}\binom{i}{k}\lambda_1^k\lambda_2^{i-k} = \frac{\mathrm{e}^{-(\lambda_1+\lambda_2)}}{i!}(\lambda_1+\lambda_2)^i\end{aligned}$$

In the last line, we use the binomial expansion.