DIS 11A

1 Binomial Variance

Throw n balls into m bins uniformly at random. For a specific ball i, what is the variance of the number of roommates it has (i.e. the number of other balls that it shares its bin with)?

Solution:

When we concentrate on the bin that ball i is in, we care about how many of the other $n-1$ balls land in that same bin. Notice that when determining whether the balls land in this bin, each ball is independent of every other ball, which makes variance much easier to compute as we can take advantage of linearity of variance. This is a binomial distribution, with $n-1$ trials and probability $1/m$ of success for each trial, where success is defined as having the ball land in the same bin as ball i.

Therefore, the variance is

$$(n-1)\left(\frac{1}{m}\right)\left(1-\frac{1}{m}\right).$$

2 Continuous Intro

(a) Is

$$f(x) = \begin{cases} 2x, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

a valid density function? Why or why not? Is it a valid CDF? Why or why not?

(b) Calculate $\mathbb{E}[X]$ and $\operatorname{var}(X)$ for X with the density function

$$f(x) = \begin{cases} 1/\ell, & 0 \le x \le \ell, \\ 0, & \text{otherwise.} \end{cases}$$

(c) Suppose X and Y are independent and have densities

$$f_X(x) = \begin{cases} 2x, & 0 \le x \le 1, \\ 0, & \text{otherwise}, \end{cases}$$

$$f_Y(y) = \begin{cases} 1, & 0 \le y \le 1, \\ 0, & \text{otherwise}. \end{cases}$$

What is their joint distribution?

(d) Calculate $\mathbb{E}[XY]$ for the above X and Y.

Solution:

(a) Yes; it is non-negative and integrates to 1. No; a CDF should go to 1 as x goes to infinity and be non-decreasing.

(b) $\mathbb{E}[X] = \int_{x=0}^{\ell} x \cdot (1/\ell)\,\mathrm{d}x = \ell/2$. $\mathbb{E}[X^2] = \int_{x=0}^{\ell} x^2 \cdot (1/\ell)\,\mathrm{d}x = \ell^2/3$.
$\mathrm{var}(X) = \ell^2/3 - \ell^2/4 = \ell^2/12$.

This is known as the continuous uniform distribution over the interval $[0,\ell]$, sometimes denoted $\mathrm{Uniform}[0,\ell]$.

(c) Note that due to independence,

$$\begin{aligned} f_{X,Y}(x,y)\,\mathrm{d}x\,\mathrm{d}y &= \mathbb{P}(X \in [x, x+\mathrm{d}x], Y \in [y, y+\mathrm{d}y]) = \mathbb{P}(X \in [x, x+\mathrm{d}x])\mathbb{P}(Y \in [y, y+\mathrm{d}y]) \\ &\approx f_X(x) f_Y(y)\,\mathrm{d}x\,\mathrm{d}y \end{aligned}$$

so their joint distribution is $f(x,y) = 2x$ on the unit square $0 \le x \le 1$, $0 \le y \le 1$.

(d) $\mathbb{E}[XY] = \int_{x=0}^{1} \int_{y=0}^{1} xy \cdot 2x\,\mathrm{d}y\,\mathrm{d}x = \int_{x=0}^{1} x^2\,\mathrm{d}x = 1/3$.
Alternatively, since X and Y are independent, we can compute $\mathbb{E}[XY] = \mathbb{E}[X]\,\mathbb{E}[Y]$. Note that

$$\mathbb{E}[X] = \int_0^1 x \cdot 2x\,\mathrm{d}x = \frac{2}{3}x^3\Big|_0^1 = \frac{2}{3},$$

and $\mathbb{E}[Y] = 1/2$ since the density of Y is symmetric around $1/2$. Hence,

$$\mathbb{E}[XY] = \mathbb{E}[X]\,\mathbb{E}[Y] = \frac{1}{3}.$$

3 Continuous Computations

Let X be a continuous random variable whose pdf is cx^3 (for some constant c) in the range $0 \le x \le 1$, and is 0 outside this range.

(a) Find c.

(b) Find $\mathbb{P}[1/3 \le X \le 2/3 \mid X \le 1/2]$.

(c) Find $\mathbb{E}(X)$.

(d) Find $\text{var}(X)$.

Solution:

(a) Since our total probability must be equal to 1,

$$\int_0^1 cx^3 \, \mathrm{d}x = 1 = \frac{1}{4}cx^4 \Big|_{x=0}^1 = \frac{c}{4},$$

so $c = 4$.

(b)

$$\mathbb{P}\left[\frac{1}{3} \le X \le \frac{2}{3} \,\middle|\, X \le \frac{1}{2}\right] = \frac{\mathbb{P}[1/3 \le X \le 2/3 \cap X \le 1/2]}{\mathbb{P}[X \le 1/2]} = \frac{\mathbb{P}[1/3 \le X \le 1/2]}{\mathbb{P}[X \le 1/2]}$$
$$= \frac{\int_{1/3}^{1/2} 4x^3 \, \mathrm{d}x}{\int_0^{1/2} 4x^3 \, \mathrm{d}x} = \frac{[x^4]_{x=1/3}^{1/2}}{[x^4]_{x=0}^{1/2}} = \frac{(1/2)^4 - (1/3)^4}{(1/2)^4} = \frac{65}{81}.$$

(c)

$$\mathbb{E}(X) = \int_0^1 x \cdot 4x^3 \, \mathrm{d}x = \int_0^1 4x^4 \, \mathrm{d}x = \left[\frac{4}{5}x^5\right]_{x=0}^1 = \frac{4}{5}.$$

(d)

$$\text{var}(X) = \int_0^1 x^2 \cdot 4x^3 \, \mathrm{d}x - \mathbb{E}(X)^2 = \int_0^1 4x^5 \, \mathrm{d}x - \left(\frac{4}{5}\right)^2 = \left[\frac{2}{3}x^6\right]_{x=0}^1 - \frac{16}{25} = \frac{2}{75}.$$