DIS 1A

1 Perfect Square

A *perfect square* is an integer n of the form $n = m^2$ for some integer m. Prove that every odd perfect square is of the form $8k+1$ for some integer k.

Solution:

We will proceed with a direct proof. Let $n = m^2$ for some integer m. Since n is odd, m is also odd, i.e., of the form $m = 2l+1$ for some integer l. Then, $m^2 = 4l^2+4l+1 = 4l(l+1)+1$. Since one of l and $l+1$ must be even, $l(l+1)$ is of the form $2k$ and $n = m^2 = 8k+1$.

2 Pigeonhole Principle

Prove the following statement: If you put $n+1$ balls into n bins, however you want, then at least one bin must contain at least two balls. This is known as the *pigeonhole principle*.

Solution:

We will use a proof by contradiction. Suppose this is not the case. Then all the bins would contain at most one ball. Then the maximum number of balls we could have would be n, but this is a contradiction since we have $n+1$ balls.

3 Numbers of Friends

Prove that if there are $n \geq 2$ people at a party, then at least 2 of them have the same number of friends at the party.

Solution:

We will prove this by contradiction. Suppose the contrary that everyone has a different number of friends at the party. Since the number of friends that each person can have ranges from 0 to $n-1$, we conclude that for every $i \in \{0,1,\ldots,n-1\}$, there is exactly one person who has exactly i friends at the party. In particular, there is one person who has $n-1$ friends (i.e., friends with everyone), and there is one person who has 0 friends (i.e., friends with no one), which is a contradiction.

4 Induction

Prove the following using induction:

(a) For all natural numbers $n > 2$, $2^n > 2n+1$.

(b) For all positive integers n, $1^3 + 3^3 + 5^3 + \cdots + (2n-1)^3 = n^2(2n^2-1)$.

(c) For all positive natural numbers n, $(5/4)8^n + 3^{3n-1}$ is divisible by 19.

Solution:

(a) The inequality is true for $n = 3$ because $8 > 7$. Let the inequality be true for $n = m$. Then,

$$2^{m+1} = 2 \cdot 2^m > 2 \cdot (2m+1) = 4m+2.$$

Since $2m > 1$, we have $4m+2 > 2m+3$, which completes the inductive step.

(b) For $n = 1$, the statement is $1 = 1$, which is true. Assume that it holds for $n = m$. Then,

$$\begin{aligned}\sum_{k=1}^{m+1}(2k-1)^3 &= \sum_{k=1}^{m}(2k-1)^3 + (2m+1)^3 = m^2(2m^2-1) + (2m+1)^3 \\ &= 2m^4 + 8m^3 + 11m^2 + 6m + 1 = (m+1)^2(2(m+1)^2-1).\end{aligned}$$

(c) For $n = 1$, the statement is "$10+9$ is divisible by 19", which is true. Assume that it holds for $n = m$. Then,

$$8 \cdot \frac{5}{4} \cdot 8^m + 27 \cdot 3^{3m-1} = 8\left(\frac{5}{4} \cdot 8^m + 3^{3m-1}\right) + 19 \cdot 3^{3m-1}.$$

The first term is divisible by the inductive hypothesis and the second term is clearly divisible by 19.