

DIS 0B

1 Writing in Propositional Logic

For each of the following sentences, translate the sentence into propositional logic using the notation introduced in class, and write its negation.

- (a) The square of a nonzero integer is positive.
- (b) There are no integer solutions to the equation $x^2 - y^2 = 10$.
- (c) There is one and only one real solution to the equation $x^3 + x + 1 = 0$.
- (d) For any two distinct real numbers, we can find a rational number in between them.

Solution:

- (a) We can rephrase the sentence as “if n is a nonzero integer, then $n^2 > 0$ ”, which can be written as

$$\forall n \in \mathbb{Z}, (n \neq 0) \implies (n^2 > 0)$$

or equivalently as

$$\forall n \in \mathbb{Z}, (n = 0) \vee (n^2 > 0).$$

The latter is easier to negate, and its negation is given by

$$\exists n \in \mathbb{Z}, (n \neq 0) \wedge (n^2 \leq 0).$$

- (b) The sentence is

$$\forall x, y \in \mathbb{Z}, x^2 - y^2 \neq 10.$$

The negation is

$$\exists x, y \in \mathbb{Z}, x^2 - y^2 = 10.$$

- (c) Let $p(x) = x^3 + x + 1$. The sentence can be read “there is a solution x to the equation $p(x) = 0$, and any other solution y is equal to x ”. Or,

$$\exists x \in \mathbb{R}, ((p(x) = 0) \wedge (\forall y \in \mathbb{R}, (p(y) = 0) \implies (x = y))).$$

Its negation is given by

$$\forall x \in \mathbb{R}, ((p(x) \neq 0) \vee (\exists y \in \mathbb{R}, (p(y) = 0) \wedge (x \neq y))).$$

- (d) The sentence can be read “if x and y are distinct real numbers, then there is a rational number z between x and y .” Or,

$$\forall x, y \in \mathbb{R}, (x \neq y) \implies (\exists z \in \mathbb{Q}, (x < z < y) \vee (y < z < x)).$$

Equivalently,

$$\forall x, y \in \mathbb{R}, (x = y) \vee (\exists z \in \mathbb{Q}, (x < z < y) \vee (y < z < x)).$$

Note that $x < z < y$ is mathematical shorthand for $(x < z) \wedge (z < y)$, so the above statement is equivalent to

$$\forall x, y \in \mathbb{R}, (x = y) \vee (\exists z \in \mathbb{Q}, ((x < z) \wedge (z < y)) \vee ((y < z) \wedge (z < x))).$$

Then the negation is

$$\exists x, y \in \mathbb{R}, (x \neq y) \wedge (\forall z \in \mathbb{Q}, ((z \leq x) \vee (z \geq y)) \wedge ((y \geq z) \vee (x \leq z))).$$

2 Implication

Which of the following implications are always true, regardless of P ? Give a counterexample for each false assertion.

- (a) $\forall x, \forall y, P(x, y) \implies \forall y, \forall x, P(x, y)$.
- (b) $\exists x, \exists y, P(x, y) \implies \exists y, \exists x, P(x, y)$.
- (c) $\forall x, \exists y, P(x, y) \implies \exists y, \forall x, P(x, y)$.
- (d) $\exists x, \forall y, P(x, y) \implies \forall y, \exists x, P(x, y)$.

Solution:

- (a) True. For all can be switched if they are adjacent; since $\forall x, \forall y$ and $\forall y, \forall x$ means for all x and y in our universe.
- (b) True. There exists can be switched if they are adjacent; $\exists x, \exists y$ and $\exists y, \exists x$ means there exists x and y in our universe.
- (c) False. Let $P(x, y)$ be $x < y$, and the universe for x and y be the integers. Or let $P(x, y)$ be $x = y$ and the universe be any set with at least two elements. In both cases, the antecedent is true and the consequence is false, those the entire implication statement is false.
- (d) True. The first statement says that there is an x , say x' where for every y , $P(x, y)$ is true. Thus, one can choose $x = x'$ for the second statement and that statement will be true again for every y . Note that the two statements are not equivalent as the converse of this is statement 3, which is false.

3 Necessary and Sufficient Conditions

- (a) Given implication $A \implies B$, A is a _____ condition for B .
- (b) Given implication $\neg A \implies \neg B$, A is a _____ condition for B .
- (c) Given implication $\neg B \implies \neg A$, A is a _____ condition for B .
- (d) Given implication $B \implies A$, A is a _____ condition for B .

Solution:

- (a) Sufficient
- (b) Necessary
- (c) Sufficient
- (d) Necessary