CS 70 Discrete Mathematics and Probability Theory
Fall 2017 Satish Rao and Kannan Ramchandran

HW 3

Sundry

Before you start your homework, write down your team. Who else did you work with on this homework? List names and email addresses. (In case of homework party, you can also just describe the group.) How did you work on this homework? Working in groups of 3-5 will earn credit for your "Sundry" grade.

Please copy the following statement and sign next to it:

I certify that all solutions are entirely in my words and that I have not looked at another student's solutions. I have credited all external sources in this write up.

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1 Build-Up Error?

What is wrong with the following "proof"?

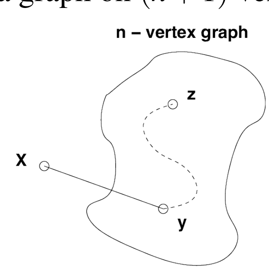
False Claim: If every vertex in an undirected graph has degree at least 1, then the graph is connected.

Proof: We use induction on the number of vertices $n \geq 1$.

Base case: There is only one graph with a single vertex and it has degree 0. Therefore, the base case is vacuously true, since the if-part is false.

Inductive hypothesis: Assume the claim is true for some $n \geq 1$.

Inductive step: We prove the claim is also true for $n+1$. Consider an undirected graph on n vertices in which every vertex has degree at least 1. By the inductive hypothesis, this graph is connected. Now add one more vertex x to obtain a graph on $(n+1)$ vertices, as shown below.



All that remains is to check that there is a path from x to every other vertex z. Since x has degree at least 1, there is an edge from x to some other vertex; call it y. Thus, we can obtain a path from x to z by adjoining the edge $\{x,y\}$ to the path from y to z. This proves the claim for $n+1$.

Solution:

The mistake is in the argument that *"every $(n+1)$-vertex graph with minimum degree 1 can be obtained from an n-vertex graph with minimum degree 1 by adding 1 more vertex"*. Instead of starting by considering an arbitrary $(n+1)$-vertex graph, this proof only considers an $(n+1)$-vertex graph that you can make by starting with an n-vertex graph with minimum degree 1. As a counterexample, consider a graph on four vertices $V = \{1,2,3,4\}$ with two edges $E = \{\{1,2\},\{3,4\}\}$. Every vertex in this graph has degree 1, but there is no way to build this 4-vertex graph from a 3-vertex graph with minimum degree 1.

More generally, this is an example of *build-up error* in proof by induction. Usually this arises from a faulty assumption that every size $n+1$ graph with some property can be "built up" from a size n graph with the same property. (This assumption is correct for some properties, but incorrect for others, such as the one in the argument above.)

One way to avoid an accidental build-up error is to use a *"shrink down, grow back"* process in the inductive step: start with a size $n+1$ graph, remove a vertex (or edge), apply the inductive hypothesis $P(n)$ to the smaller graph, and then add back the vertex (or edge) and argue that $P(n+1)$ holds.

Let's see what would have happened if we'd tried to prove the claim above by this method. In the inductive step, we must show that $P(n)$ implies $P(n+1)$ for all $n \geq 1$. Consider an $(n+1)$- vertex graph G in which every vertex has degree at least 1. Remove an arbitrary vertex v, leaving an n-vertex graph G' in which every vertex has degree... uh-oh! The reduced graph G' might contain a vertex of degree 0, making the inductive hypothesis $P(n)$ inapplicable! We are stuck —— and properly so, since the claim is false!

2 Proofs in Graphs

Please prove or disprove the following claims.

(a) In Old California, all roads were one way streets. Suppose Old California had n cities ($n \geq 2$) such that for every pair of cities X and Y, either X had a road to Y or Y had a road to X. Prove or disprove that there existed a city which was reachable from every other city by traveling through at most 2 roads.

[*Hint:* Induction]

(b) In lecture, we have shown that a connected undirected graph has an Eulerian tour if and only if every vertex has even degree.

Consider a connected graph G with n vertices which has exactly $2m$ vertices of odd degree, where $m > 0$. Prove or disprove that there are m walks that *together* cover all the edges of G

(i.e., each edge of G occurs in exactly one of the m walks, and each of the walks should not contain any particular edge more than once).

Solution:

(a) We prove this by induction on the number of cities n.

Base case For $n = 2$, there's always a road from one city to the other.

Inductive Hypothesis When there are k cities, there exists a city c that is reachable from every other city by traveling through at most 2 roads.

Inductive Step Consider the case where there are $k+1$ cities. Remove one of the cities d and all of the roads to and from d. Now there are k cities, and by our inductive hypothesis, there exists some city c which is reachable from every other city by traveling through at most 2 roads. Let A be the set of cities with a road to c, and B be the set of cities two roads away from c. The inductive hypothesis states that the set S of the k cities consists of $S = \{c\} \cup A \cup B$.

Now add back d and all roads to and from d. Between d and every city in S, there must be a road from one to the other. If there is at least one road from d to $\{c\} \cup A$, c would still be reachable from d with at most 2 road traversals. Otherwise, if all roads from $\{c\} \cup A$ point to d, d will be reachable from every city in B with at most 2 road traversals, because every city in B can take one road to go to a city in A, then take one more road to go to d. In either case there exists a city in the new set of $k+1$ cities that is reachable from every other city by traveling at most 2 roads.

(b) We split the $2m$ odd-degree vertices into m pairs, and join each pair with an edge, adding m more edges in total. (Here, we allow for the possibility of multi-edges, that is, pairs of vertices with more than one edge between them.) Notice that now all vertices in this graph are of even degree. Now by Euler's theorem the resulting graph has an Eulerian tour. Removing the m added edges breaks the tour into m walks covering all the edges in the original graph, with each edge belonging to exactly one walk.

3 Connectivity

Consider the following claims regarding connectivity:

(a) Prove: If G is a graph with n vertices such that for any two non-adjacent vertices u and v, it holds that $\deg u + \deg v \geq n-1$, then G is connected.

[*Hint:* Show something more specific: for any two non-adjacent vertices u and v, there must be a vertex w such that u and v are both adjacent to w.]

(b) Give an example to show that if the condition $\deg u + \deg v \geq n-1$ is replaced with $\deg u + \deg v \geq n-2$, then G is not necessarily connected.

(c) Prove: For a graph G with n vertices, if the degree of each vertex is at least $n/2$, then G is connected.

(d) Prove: If there are exactly two vertices with odd degrees in a graph, then they must be connected to each other (meaning, there is a path connecting these two vertices).

[*Hint:* Proof by contradiction.]

Solution:

(a) Consider non-adjacent u and v. Then, there must be a vertex w such that u and v are both adjacent to w. To see why, suppose this is not the case. Then, the set of neighbors of u and v has $n-1$ elements, but there are only $n-2$ other vertices. (This is the Pigeonhole Principle.) We have proven that for any non-adjacent u and v, there is a path $u \to w \to v$, and thus G is connected.

(b) Consider a graph with $n = 2m$ vertices which consists of two disconnected copies of K_m. For non-adjacent u, v, it holds that $\deg u + \deg v = (m-1) + (m-1) = 2m - 2 = n - 2$, but the graph is not connected.

(c) Suppose that G is not connected. There must be at least two connected components, say G_1 and G_2. One of them will have at most $n/2$ vertices, and the maximum degree in this subgraph will be at most $n/2 - 1$. A contradiction.

Notice that part (a) directly implies the claim in this part. If each vertex's degree is at least $n/2$, then for any two non-adjacent vertices u, v,

$$\deg u + \deg v \geq \frac{n}{2} + \frac{n}{2} = n > n - 1.$$

Then by part (a), the graph is connected.

(d) Suppose that they are not connected to each other. Then they must belong to two different connected components, say G_1 and G_2. Each of them will only have one vertex with odd degree. This leads to a contradiction since the sum of all degrees should be an even number.

4 Leaves in a Tree

A *leaf* in a tree is a vertex with degree 1.

(a) Consider a tree with $n \geq 3$ vertices. What is the largest possible number of leaves the tree could have? Prove that this maximum m is possible to achieve, and further that there cannot exist a tree with more than m leaves.

(b) Prove that every tree on $n \geq 2$ vertices must have at least two leaves.

Solution:

(a) We claim the maximum number of leaves is $n-1$. This is achieved when there is one vertex that is connected to all other vertices (this is called the *star graph*).

We now show that a tree on $n \geq 3$ vertices cannot have n leaves. Suppose the contrary that there is a tree on $n \geq 3$ vertices such that all its n vertices are leaves. Pick an arbitrary vertex x, and let y be its unique neighbor. Since x and y both have degree 1, the vertices x, y form a connected component separate from the rest of the tree, contradicting the fact that a tree is connected.

(b) We give a direct proof. Consider the longest path $\{v_0, v_1\}, \{v_1, v_2\}, \dots, \{v_{k-1}, v_k\}$ between two vertices $x = v_0$ and $y = v_k$ in the tree (here the length of a path is how many edges it uses, and if there are multiple longest paths then we just pick one of them). We claim that x and y must be leaves. Suppose the contrary that x is not a leaf, so it has degree at least two. This means x is adjacent to another vertex z different from v_1. Observe that z cannot appear in the path from x to y that we are considering, for otherwise there would be a cycle in the tree. Therefore, we can add the edge $\{z, x\}$ to our path to obtain a longer path in the tree, contradicting our earlier choice of the longest path. Thus, we conclude that x is a leaf. By the same argument, we conclude y is also a leaf.

The case when a tree has only two leaves is called the *path graph*, which is the graph on $V = \{1, 2, \dots, n\}$ with edges $E = \{\{1,2\}, \{2,3\}, \dots, \{n-1, n\}\}$.

5 Coloring Trees

(a) What is the minimum number of colors needed to color a tree? Prove your claim.

(b) Prove that all trees are bipartite.

[*Hint:* How does your answer to part (a) relate to this?]

Solution:

(a) The minimum number of colors needed to color a tree is 2. Prove this using induction.

When the tree is just a single vertex, we can indeed color it with a single color.

Assume that all trees with k vertices can be 2-colored. Now consider a tree T with $k+1$ vertices. We know that every tree must have at least 2 leaves, so remove one leaf u and the edge connected to u. The resulting graph $T - u$ is a tree with k vertices and can be 2-colored by our inductive hypothesis.

Now when we add u back to the graph, u has a single neighbor which is colored with one color. Simply color u with the other color. Thus, any tree with $k+1$ vertices can be 2-colored.

We have shown that any tree of any size can be colored with only 2 colors.

(b) Since every tree can be colored with 2 colors, every tree is bipartite. If the vertices can be colored such that no two vertices connected by an edge share the same color, this means that every edge connects a vertex of color c_0 with a vertex of color c_1. Thus, the tree can be split into two groups: all vertices colored c_0 and all vertices colored c_1. By the rules of coloring, every edge must connect a vertex in the first group with a vertex in the second group. Every tree can therefore be considered a bipartite graph.

6 Edge-Disjoint Paths in a Hypercube

Prove that between any two distinct vertices x, y in the n-dimensional hypercube graph, there are at least n edge-disjoint paths from x to y (i.e., no two paths share an edge, though they may share vertices).

Solution:

We use induction on $n \geq 1$. The base case $n = 1$ holds because in this case the graph only has two vertices $V = \{0,1\}$, and there is 1 path connecting them. Assume the claim holds for the $(n-1)$-dimensional hypercube. Let $x = x_1x_2\dots x_n$ and $y = y_1y_2\dots y_n$ be distinct vertices in the n-dimensional hypercube; we want to show there are at least n edge-disjoint paths from x to y. To do that, we consider two cases:

1. Suppose $x_i = y_i$ for some index $i \in \{1,\dots,n\}$. Without loss of generality (and for ease of explanation), we may assume $i = 1$, because the hypercube is symmetric with respect to the indices. Moreover, by interchanging the bits 0 and 1 if necessary, we may also assume $x_1 = y_1 = 0$. This means x and y both lie in the 0-subcube, where recall the 0-subcube (respectively, the 1-subcube) is the $(n-1)$-dimensional hypercube with vertices labeled $0z$ (respectively, $1z$) for $z \in \{0,1\}^{n-1}$.

   Applying the inductive hypothesis, we know there are at least $n-1$ edge-disjoint paths from x to y, and moreover, these paths all lie within the 0-subcube. Clearly these $n-1$ paths will still be edge-disjoint in the original n-dimensional hypercube. We have an additional path from x to y that goes through the 1-subcube as follows: go from x to x', then from x' to y' following any path in the 1-subcube, and finally go from y' back to y. Here $x' = 1x_2\dots x_n$ and $y' = 1y_2\dots y_n$ are the corresponding points of x and y in the 1-subcube. Since this last path does not use any edges in the 0-subcube, this path is edge-disjoint to the $n-1$ paths that we have found. Therefore, we conclude that there are at least n edge-disjoint paths from x to y.

2. Suppose $x_i \neq y_i$ for all $i \in \{1,\dots,n\}$. This means x and y are two opposite vertices in the hypercube, and without loss of generality, we may assume $x = 00\dots0$ and $y = 11\dots1$. We explicitly exhibit n paths $P_1,\dots,P_n$ from x to y, and we claim they are edge-disjoint.

   For $i \in \{1,\dots,n\}$, the i-th path P_i is defined as follows: start from the vertex x (which is all zeros), flip the i-th bit to a 1, then keep flipping the bits one by one moving rightward from position $i+1$ to n, then from position 1 moving rightward to $i-1$. For example, the path P_1 is given by

$$000\dots0 \;\rightarrow\; 100\dots0 \;\rightarrow\; 110\dots0 \;\rightarrow\; 111\dots0 \;\rightarrow\; \cdots \;\rightarrow\; 111\dots1$$

while the path P_2 is given by

$$000\ldots0 \rightarrow 010\ldots0 \rightarrow 011\ldots0 \rightarrow \cdots \rightarrow 011\ldots1 \rightarrow 111\ldots1$$

Note that the paths $P_1,\ldots,P_n$ don't share vertices other than $x = 00\ldots0$ and $y = 11\ldots1$, so in particular they must be edge-disjoint.

Alternative for Case 2:
We can also reduce case 2, in which x and y have no bits in common, to case 1.

Suppose that $x_i \neq y_i$ for all $i = 1,\ldots,n$. Let $\tilde{x}$ be x with the first bit flipped, so now $\tilde{x}$ and y both lie on a subcube together. From the inductive hypothesis, there are $n-1$ edge-disjoint paths from $\tilde{x}$ to y along the shared subcube (these paths only flip bits 2 through n because the paths lie entirely on a subcube, and these paths are of length at least $n-1$ since $\tilde{x}$ and y differ by that many bits). We would like to make these into edge-disjoint paths from y to x.

Starting at y, take the first of the $n-1$ paths, but before starting, flip the first bit. Then follow the first path to get a path from y to x. Then to use the second path starting at y, we travel one step along the second path, then flip the first bit, and then continue along the second path (carry out the sequence of bit flips in the second path, giving us another path from y to x). Continuing this way, we take the $(n-1)$st path, travel $n-2$ steps along the path, flip the first bit, then continue following the path.

It can be seen that this gives $n-1$ edge-disjoint paths from x to y. Where do we get the last path? Well we take the first path again, and now we follow the path for $n-1$ steps, and then flip the first bit – this gives us yet another edge-disjoint path for a total of n edge-disjoint paths! Cool!