CS 70 Discrete Mathematics and Probability Theory
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DIS 11B

1 Uniform Distribution

You have two fidget spinners, each having a circumference of 10. You mark one point on each spinner as a needle and place each of them at the center of a circle with values in the range $[0,10)$ marked on the circumference. If you spin both (independently) and let X be the position of the first spinner's mark and Y be the position of the second spinner's mark, what is the probability that $X \geq 5$, given that $Y \geq X$?

Solution:

First we write down what we want and expand out the conditioning:

$$\mathbb{P}[X \geq 5 \mid Y \geq X] = \frac{\mathbb{P}[Y \geq X \cap X \geq 5]}{\mathbb{P}[Y \geq X]}.$$

$\mathbb{P}[Y \geq X] = 1/2$ by symmetry. To find $\mathbb{P}[Y \geq X \cap X \geq 5]$, it helps a lot to just look at the picture of the probability space and use the continuous uniform law $\mathbb{P}[A] = (\text{area of } A)/(\text{area of } \Omega)$. We are interested in the relative area of the region bounded by $x < y < 10$, $5 < x < 10$ to the entire square bounded by $0 < x < 10$, $0 < y < 10$.

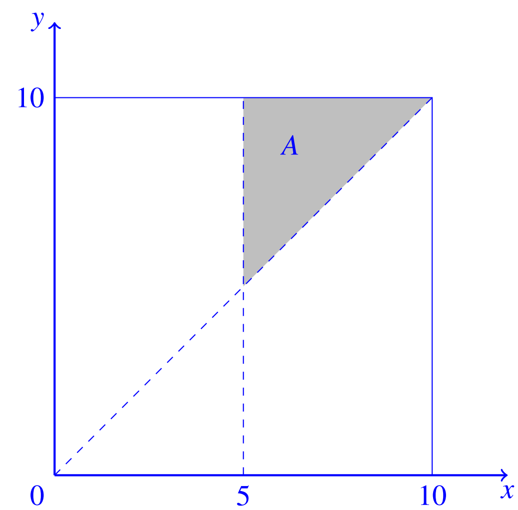

Figure 1: Joint probability density for the spinner.

$$\mathbb{P}[Y \geq X \cap X \geq 5] = \frac{5 \cdot 5/2}{10 \cdot 10} = \frac{1}{8}.$$

So $\mathbb{P}[X \geq 5 \mid Y \geq X] = (1/8)/(1/2) = 1/4$.

2 Lunch Meeting

Alice and Bob agree to try to meet for lunch between 12 PM and 1 PM at their favorite sushi restaurant. Being extremely busy, they are unable to specify their arrival times exactly, and can say only that each of them will arrive (independently) at a time that is uniformly distributed within the hour. In order to avoid wasting precious time, if the other person is not there when they arrive they agree to wait exactly fifteen minutes before leaving. What is the probability that they will actually meet for lunch?

Solution:

Let the random variable A be the time that Alice arrives and the random variable B be the time when Bob arrives. Consider Figure 2, plotting the space of all outcomes (a,b): The shaded region

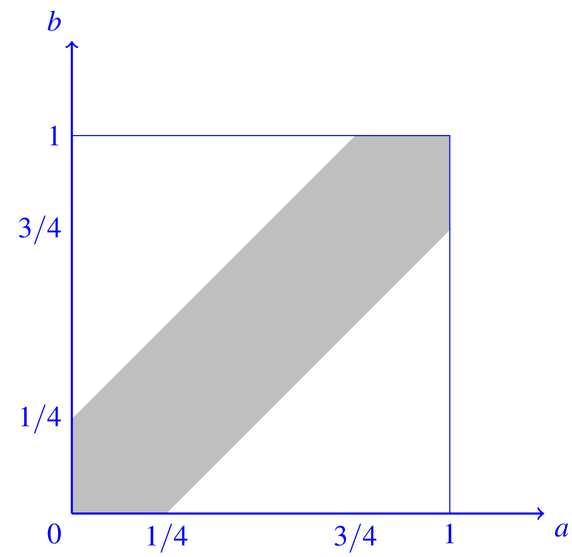

Figure 2: Visualization of joint probability density.

is the set of values (a,b) for which Alice and Bob will actually meet for lunch. Since all points in this square are equally likely, the probably they meet is the ratio of the shaded area to the area of the square. If the area of the square is 1, then the area of the shaded region is

$$1 - 2 \times \left[\frac{1}{2} \times \left(\frac{3}{4}\right)^2\right] = \frac{7}{16},$$

since the area of the white triangle on the upper-left is $(1/2) \cdot (3/4)^2$, and the white triangle on the lower-right has the same area. Therefore, the probability that Alice and Bob actually meet is $7/16$.

3 First Exponential to Die

Let X and Y be Exponential(λ_1) and Exponential(λ_2) respectively, independent. What is

$$\mathbb{P}\big(\min(X,Y) = X\big),$$

the probability that the first of the two to die is X?

Solution:

Recall that the CDF of an exponential is $\mathbb{P}[X \le x] = 1 - \exp(-\lambda x)$ for $x \ge 0$.

$$\mathbb{P}\big(\min(X,Y) = X\big) = \mathbb{P}(Y > X) = \int_0^\infty \mathbb{P}(Y > X \mid X = x) f_X(x)\,\mathrm{d}x = \int_0^\infty \mathrm{e}^{-\lambda_2 x} \cdot \lambda_1 \,\mathrm{e}^{-\lambda_1 x}\,\mathrm{d}x$$
$$= -\frac{\lambda_1}{\lambda_1 + \lambda_2}\,\mathrm{e}^{-(\lambda_1+\lambda_2)x}\Big|_{x=0}^{\infty} = \frac{\lambda_1}{\lambda_1 + \lambda_2}.$$