CS 70 Discrete Mathematics and Probability Theory
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Discussion 08B

1 Disease Diagnosis

You have a high fever and go to the doctor to identify the cause. 1% of the people have Ebola, 10% of the people have the flu, and 89% have neither. Assume that no person has both. Suppose that 100% of the Ebola people have a high fever, 30% of the flu people have a high fever, and 2% of the people who have neither, have a high fever. Is it more likely that you have Ebola, the flu, or neither?

Solution:

Let A be the event that the patient has Ebola, B be the event that the patient has flu, and C be the event that the patient has neither. The event of having a fever is D. We want to compare $\mathbb{P}(A \mid D)$, $\mathbb{P}(B \mid D)$, and $\mathbb{P}(C \mid D)$. We find each value using Bayes rule.

$$\begin{aligned}\mathbb{P}(A \mid D) &= \frac{\mathbb{P}(D \mid A)\mathbb{P}(A)}{\mathbb{P}(D \mid A)\mathbb{P}(A) + \mathbb{P}(D \mid B)\mathbb{P}(B) + \mathbb{P}(D \mid C)\mathbb{P}(C)} \\ &= \frac{1 \times 0.01}{1 \times 0.01 + 0.1 \times 0.3 + 0.89 \times 0.02} \\ &= 0.173\end{aligned} \tag{1}$$

$$\begin{aligned}\mathbb{P}(B \mid D) &= \frac{\mathbb{P}(D \mid B)\mathbb{P}(B)}{\mathbb{P}(D \mid A)\mathbb{P}(A) + \mathbb{P}(D \mid B)\mathbb{P}(B) + \mathbb{P}(D \mid C)\mathbb{P}(C)} \\ &= \frac{0.1 \times 0.3}{1 \times 0.01 + 0.1 \times 0.3 + 0.89 \times 0.02} \\ &= 0.519\end{aligned} \tag{2}$$

$$\begin{aligned}\mathbb{P}(C \mid D) &= \frac{\mathbb{P}(D \mid C)\mathbb{P}(C)}{\mathbb{P}(D \mid A)\mathbb{P}(A) + \mathbb{P}(D \mid B)\mathbb{P}(B) + \mathbb{P}(D \mid C)\mathbb{P}(C)} \\ &= \frac{0.89 \times 0.02}{1 \times 0.01 + 0.1 \times 0.3 + 0.89 \times 0.02} \\ &= 0.308\end{aligned} \tag{3}$$

So flu is the most likely.

2 Let's Talk Probability

(a) When is $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$ true? What is the general rule that always holds?

(b) When is $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$ true? What is the general rule that always holds?

(c) If A and B are disjoint, are they independent?

(d) On the space of a fair roll of a six-sided die, find three events, each of which is independent of the intersection of the other two, such that they are not mutually independent.

(e) If we roll 2 dice, what is the probability that the first roll is a 3? What is the probability that the first roll is a 3 if we know that the sum of the dice is 6?

Solution:

(a) In general, we know $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$. This is the Inclusion-Exclusion Principle. Therefore if A and B are disjoint, such that $\mathbb{P}(A \cap B) = 0$, then $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$ holds.

(b) $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$ holds if and only if A and B are independent (by definition).

(c) No, if two events are disjoint, we cannot conclude they are independent. Consider a roll of a fair six-sided die. Let A be the event that we roll a 1, and let B be the event that we roll a 2. Certainly A and B are disjoint, as $\mathbb{P}(A \cap B) = 0$. But these events are not independent: $\mathbb{P}(B \mid A) = 0$, but $\mathbb{P}(B) = 1/6$.

Since disjoint events have $\mathbb{P}(A \cap B) = 0$, we can see that the only time when A and B are independent is when either $\mathbb{P}(A) = 0$ or $\mathbb{P}(B) = 0$.

(d) Let A be the event that we roll a 1. Let B be the event that we roll a 2. Let C be the event that we roll a 3. Then $\mathbb{P}(A) = \mathbb{P}(B) = \mathbb{P}(C) = 1/6$, and $\mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C) = 1/216$. We know $\mathbb{P}(A \cap B \cap C) = 0 \neq \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C)$, so the events are not mutually independent.
However, each of the pairwise intersections is the empty set, such that $\mathbb{P}(A \cap B) = \mathbb{P}(B \cap C) = \mathbb{P}(A \cap C) = 0$, and every event is independent of the empty set. For example, $\mathbb{P}(A \cap (B \cap C)) = 0 = \mathbb{P}(A) \cdot \mathbb{P}(B \cap C)$, and likewise for the other pairs. Thus each event is independent of the intersection of the other two, but the three events are not mutually independent.

(e) With no prior information, the probability that the first roll is a 3 is $1/6$. Now let A be the event that the sum of the dice is 6, and B be the event that the first roll is a 3. The probability we wish to compute is:

$$\mathbb{P}(B \mid A) = \frac{\mathbb{P}(B \cap A)}{\mathbb{P}(A)} = \frac{1/36}{5/36} = \frac{1}{5}$$

Having additional information about the dice changes the probability that the first roll is a 3.

3 Simple Dice Roll

A die is rolled four times. What is the probability that we obtain exactly one 6?

Solution:

$$\binom{4}{1}\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^3 = 0.386.$$

4 Maybe Lossy Maybe Not

Let us say that Alice would like to send a message to Bob, over some channel. Alice has a message of length 4 and sends 5 packets.

(a) Packets are dropped with probability p. What is probability that Bob can successfully reconstruct Alice's message?

(b) Again, packets can be dropped with probability p. The channel may additionally corrupt 1 packet. Alice realizes this and sends 3 additional packets. What is the probability that Bob receives enough packets to successfully reconstruct Alice's message?

(c) Again, packets can be dropped with probability p. This time, packets may be corrupted with probability q. Consider the original scenario where Alice sends 5 packets for a message of length 4. What is probability that Bob can successfully reconstruct Alice's message?

Solution:

(a) Alice's message requires a polynomial of degree 3, which can be uniquely identified by 4 points. Thus, at least 4 points need to make it across the channel. The probability that Bob can recover the message is thus the probability that none of the packets are lost. Since the packets are lost with probability with probability p, we have the probability of losing 1 packet is

$$\binom{5}{1}(1-p)^4 p.$$

The probability of losing 0 packets is $(1-p)^5$. Thus, the probability of losing 0 or 1 packets is

$$\binom{5}{1}(1-p)^4 p + (1-p)^5.$$

This is the probability that Bob receives 4 packets, meaning he can successfully reconstruct the 3-degree polynomial.

(b) Bob needs $n+2k = 6$ packets to guarantee successful reconstruction of Alice's message. There are a total of 8 packets sent, so this guarantee occurs only if 0 packets, 1 packet or 2 packets are lost. The probability of 0 packets lost is

$$(1-p)^8.$$

The probability of one packet lost is

$$\binom{8}{1}p(1-p)^7.$$

The probability of two packets lost is

$$\binom{8}{2}p^2(1-p)^6.$$

Thus, the probability of success is

$$(1-p)^8+\binom{8}{1}p(1-p)^7+\binom{8}{2}p^2(1-p)^6.$$

(c) Again, Bob can reconstruct the message if none of the packets are corrupted. We use the same idea as in Part (a). The probability that none of the packets are corrupted is $(1-q)^5$. We know that *on top of* being uncorrupted, we can only at lose at most 1 packet. Thus, we can either lose one packet, which has probability

$$\binom{5}{1}p(1-p)^4.$$

Or, we can lose no packets, which has probability $(1-p)^5$. Yet another possibility is if exactly one packet is corrupted, but that packet is also dropped; in this case, we can recover the message, so long as no other packets are corrupted or dropped. This occurs with probability

$$\binom{5}{1}pq(1-p)^4(1-q)^4.$$

As a result, we have the following.

$$(1-q)^5\left(5p(1-p)^4+(1-p)^5\right)+5pq(1-p)^4(1-q)^4.$$