CS 70 Discrete Mathematics and Probability Theory
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HW 9

Sundry

Before you start your homework, write down your team. Who else did you work with on this homework? List names and email addresses. (In case of homework party, you can also just describe the group.) How did you work on this homework? Working in groups of 3-5 will earn credit for your "Sundry" grade.

Please copy the following statement and sign next to it:

I certify that all solutions are entirely in my words and that I have not looked at another student's solutions. I have credited all external sources in this write up.

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1 Identity Theft

A group of n friends go to the gym together, and while they are playing basketball, they leave their bags against the nearby wall. An evildoer comes, takes the student ID cards from the bags, randomly rearranges them, and places them back in the bags, one ID card per bag. What is the probability that no one receives his or her own ID card back? [*Hint*: Use the generalized inclusion-exclusion principle.]

Then, find an approximation for the probability as $n \to \infty$. You may, without proof, refer to the power series for e^x:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

Solution:

We are looking for the probability of the event that no one receives his or her own ID card back. It is easier to consider the complement of the above event, which is the event that at least one person receives his or her ID card back. Let A_i, $i = 1, \ldots, n$, be the event that the ith friend receives his or her own ID card back, so the event we are considering now is $A_1 \cup \cdots \cup A_n$. We will compute this probability using the generalized inclusion-exclusion formula.

- First, we add $\mathbb{P}(A_1)+\cdots+\mathbb{P}(A_n)$. Here, $\mathbb{P}(A_i)$ is the probability that the ith friend receives his or her own ID card back, which is $1/n$. So, we add $n \cdot (1/n) = 1$.
- Next, we subtract $\sum_{(i,j)} \mathbb{P}(A_i \cap A_j)$, where the sum runs over all $(i,j) \in \{1,\dots,n\}^2$ with $i<j$. Note that $\mathbb{P}(A_i \cap A_j)$ is the probability that both friend i and friend j receive their own ID cards back, which has probability $(n-2)!/n!$. (To see this, observe that once we have decided that friends i and j will receive their own ID cards back, there are $(n-2)!$ ways to permute the ID cards of the $n-2$ other friends, and there are $n!$ total permutations of the n ID cards.) So, we subtract $\sum_{(i,j)}(n-2)!/n!$, but the summation has $\binom{n}{2}$ terms, so we subtract a total of
$$\binom{n}{2}\frac{(n-2)!}{n!} = \frac{n!}{2!(n-2)!} \cdot \frac{(n-2)!}{n!} = \frac{1}{2!}.$$
- At the kth step of the inclusion-exclusion process, we add $(-1)^{k+1}\sum_{(i_1,\dots,i_k)} \mathbb{P}(A_{i_1} \cap \cdots \cap A_{i_k})$, where the k-tuples in the summation range over all $(i_1,\dots,i_k) \in \{1,\dots,n\}^k$ with $i_1 < \cdots < i_k$. To compute $\mathbb{P}(A_{i_1} \cap \cdots \cap A_{i_k})$, note that we have decided that k friends will receive their own ID cards back, the remaining $n-k$ ID cards can be permuted in $(n-k)!$ ways, and there are $n!$ total permutations, so the probability is $(n-k)!/n!$. The summation has a total of $\binom{n}{k}$ terms, so we add
$$(-1)^{k+1}\binom{n}{k}\frac{(n-k)!}{n!} = (-1)^{k+1}\frac{n!}{k!(n-k)!} \cdot \frac{(n-k)!}{n!} = (-1)^{k+1}\frac{1}{k!}.$$

Now, adding up all of these probabilities together, we have
$$\mathbb{P}(A_1 \cup \cdots \cup A_n) = \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \cdots + \frac{(-1)^{n+1}}{n!}.$$
Recall that $A_1 \cup \cdots \cup A_n$ is the *complement* of the event we were originally interested in. So,
$$\mathbb{P}(\text{no friends receive their own ID cards back}) = 1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + \frac{(-1)^n}{n!} = \sum_{k=0}^{n}\frac{(-1)^k}{k!}.$$
Recall the power series for e^x:
$$\mathrm{e}^x = \sum_{k=0}^{\infty}\frac{x^k}{k!}.$$
Therefore, we have the approximation (which gets better as $n \to \infty$):
$$\mathbb{P}(\text{no friends receive their own ID cards back}) \approx \frac{1}{\mathrm{e}} \approx 0.368.$$

2 Mario's Coins

Mario owns three identical-looking coins. One coin shows heads with probability $1/4$, another shows heads with probability $1/2$, and the last shows heads with probability $3/4$.

(a) Mario randomly picks a coin and flips it. He then picks one of the other two coins and flips it. Let X_1 and X_2 be indicators of the 1st and 2nd flips showing heads. Are X_1 and X_2 independent? If so, prove it; if not, provide a counterexample.

(b) Mario randomly picks a single coin and flips it twice. Let Y_1 and Y_2 be indicators of the 1st and 2nd flips showing heads. Are Y_1 and Y_2 independent? If so, prove it; if not, provide a counterexample.

(c) Mario arranges his three coins in a row. He flips the coin on the left, which shows heads. He then flips the coin in the middle, which shows heads. Finally, he flips the coin on the right. What is the probability that it also shows heads?

Solution:

(a) X_1 and X_2 are not independent. Intuitively, the value of X_1 gives some information about the first coin that was chosen; this provides some information about the second coin that was chosen (since the first and second coins can't be the same coin), which directly influences the value of X_2.

There are many possible counterexamples. One example would be the following.

$$\mathbb{P}(X_1 = 1) = \left(\frac{1}{3}\right)\left(\frac{1}{4}\right) + \left(\frac{1}{3}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{3}\right)\left(\frac{3}{4}\right) = \frac{1}{2}$$

By symmetry, $\mathbb{P}(X_2 = 1) = \mathbb{P}(X_1 = 1)$, so

$$\mathbb{P}(X_1 = 1)\mathbb{P}(X_2 = 1) = \frac{1}{4}.$$

But if we consider the probability that both $X_1 = 1$ and $X_2 = 1$, we have

$$\begin{aligned}\mathbb{P}(X_1 = 1, X_2 = 1) &= \frac{1}{6}\Big[\left(\frac{1}{4}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{4}\right)\left(\frac{3}{4}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{4}\right) + \\ &\qquad \left(\frac{1}{2}\right)\left(\frac{3}{4}\right) + \left(\frac{3}{4}\right)\left(\frac{1}{4}\right) + \left(\frac{3}{4}\right)\left(\frac{1}{2}\right)\Big] \\ &= \frac{22}{96} = \frac{11}{48}\end{aligned}$$

which is not equal to $1/4$, violating the definition of independence.

(b) Y_1 and Y_2 are not independent. Intuitively, the value of Y_1 gives some information about the coin that was picked, which directly influences the value of Y_2.

There are many possible counterexamples. One example would be the following.

$$\mathbb{P}(Y_1 = 1) = \left(\frac{1}{3}\right)\left(\frac{1}{4}\right) + \left(\frac{1}{3}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{3}\right)\left(\frac{3}{4}\right) = \frac{1}{2}$$

By symmetry, $\mathbb{P}(Y_2 = 1) = \mathbb{P}(Y_1 = 1)$, so

$$\mathbb{P}(Y_1 = 1)\mathbb{P}(Y_2 = 1) = \frac{1}{4}$$

But if we consider the probability that both $Y_1 = 1$ and $Y_2 = 1$, we have

$$\mathbb{P}(Y_1 = 1, Y_2 = 1) = \left(\frac{1}{3}\right)\left(\frac{1}{4}\right)^2 + \left(\frac{1}{3}\right)\left(\frac{1}{2}\right)^2 + \left(\frac{1}{3}\right)\left(\frac{3}{4}\right)^2 = \frac{14}{48} = \frac{7}{24}$$

which is not equal to $1/4$, violating the definition of independence.

(c) Let A be the coin with bias $1/4$, B be the fair coin, and C be the coin with bias $3/4$. There are six orderings, each with probability $1/6$: ABC, ACB, BAC, BCA, CAB, and CBA. Thus

$$\begin{aligned}
&\mathbb{P}(\text{Third coin shows heads} \mid \text{First two coins show heads}) \\
&= \frac{\mathbb{P}(\text{All three coins show heads})}{\mathbb{P}(\text{First two coins show heads})} \\
&= \frac{(1/4)(1/2)(3/4)}{\sum_{\text{Orderings}} \mathbb{P}(\text{First two coins show heads} \mid \text{Ordering})\mathbb{P}(\text{Ordering})} \\
&= \frac{(1/4)(1/2)(3/4)}{(1/6)\sum_{\text{Orderings}} \mathbb{P}(\text{First two coins show heads} \mid \text{Ordering})} \\
&= \frac{(1/4)(1/2)(3/4)}{(1/6)((1/4)(1/2)+(1/4)(3/4)+(1/2)(1/4)+(1/2)(3/4)+(3/4)(1/4)+(3/4)(1/2))} \\
&= \frac{3/32}{11/48} = \frac{9}{22}.
\end{aligned}$$

3 Combinatorial Coins

Allen and Alvin are flipping coins for fun. Allen flips a fair coin k times and Alvin flips $n-k$ times. In total there are n coin flips.

(a) Use a combinatorial proof to show that

$$\sum_{i=0}^{k} \binom{k}{k-i}\binom{n-k}{i} = \binom{n}{k}.$$

You may assume that $n-k \geq k$.

(b) Prove that the probability that Allen and Alvin flip the same number of heads is equal to the probability that there are a total of k heads.

Solution:

(a) On the right-hand side of the equation, we are calculating the total number of ways to choose k items from a group of n items. On the left-hand side, we first partition the whole group into two subgroups with k items and $n-k$ items. In order to choose k items in total from those two groups, we pick $k-i$ items from the first group and i items from the second group. Therefore, the total number of ways to do so is simply a summation with all possible values of i.

(b) Let's first figure out the probability that there are a total of k heads. Suppose X is a random variable that denotes the total number of heads in the n flips. It can be seen clearly that X follows a binomial distribution with parameter $p = 1/2$ and n. Then the probability that there are a total of k heads can be calculated using binomial distribution:

$$\mathbb{P}(X = k) = \binom{n}{k}\left(\frac{1}{2}\right)^n.$$

Now, let's do a mathematical derivation to prove that it's the same as the probability that Allen and Alvin flip the same number of heads. Let A be the random variable representing the number of heads Allen gets, and let B be the random variable representing the number of heads Alvin gets.

$$\begin{aligned}
\mathbb{P}(\text{Allen and Alvin flip the same number of heads}) &= \sum_i \mathbb{P}(A = i, B = i) \\
&= \sum_i \binom{k}{i}\left(\frac{1}{2}\right)^k \binom{n-k}{i}\left(\frac{1}{2}\right)^{n-k} \\
&= \sum_i \binom{k}{i}\binom{n-k}{i}\left(\frac{1}{2}\right)^n \\
&= \sum_i \binom{k}{k-i}\binom{n-k}{i}\left(\frac{1}{2}\right)^n \\
&= \binom{n}{k}\left(\frac{1}{2}\right)^n.
\end{aligned}$$

For the last equality, we used the identity from part (a), $\sum_i \binom{k}{k-i}\binom{n-k}{i} = \binom{n}{k}$. In order to use the identity from part (a), we need the assumption that $n-k \geq k$. However, since the probability of obtaining exactly k heads is the same as the probability of obtaining exactly $n-k$ heads for a fair coin, and Alvin and Allen are interchangeable in the problem, we can interchange $n-k$ and k if necessary to ensure that $n-k \geq k$, so the above derivation works in the case $n-k < k$ case as well.

4 Sinho's Dice

Sinho rolls three fair-sided dice. What is the PMF for the maximum of the three values rolled? [*Hint*: First find the CDF.]

Solution:

Let X denote the maximum of the three values rolled. We are interested in $\mathbb{P}(X = x)$, where $x = 1,2,3,4,5,6$. First, define X_1, X_2, X_3 to be the values rolled by the first, second, and third dice. These random variables are i.i.d. and uniformly distributed between 1 and 6 inclusive.

We first find the CDF of X. For $x = 1,2,3,4,5,6$:

$$\mathbb{P}(X \le x) = \mathbb{P}(X_1 \le x)\mathbb{P}(X_2 \le x)\mathbb{P}(X_3 \le x) = \left(\frac{x}{6}\right)\left(\frac{x}{6}\right)\left(\frac{x}{6}\right) = \left(\frac{x}{6}\right)^3$$

We can then derive the PDF by observing $\mathbb{P}(X = x) = \mathbb{P}(X \le x) - \mathbb{P}(X \le x-1)$:

$$\mathbb{P}(X = x) = \left(\frac{x}{6}\right)^3 - \left(\frac{x-1}{6}\right)^3 = \frac{3x^2 - 3x + 1}{216} = \begin{cases} \frac{1}{216}, & x = 1 \\ \frac{7}{216}, & x = 2 \\ \frac{19}{216}, & x = 3 \\ \frac{37}{216}, & x = 4 \\ \frac{61}{216}, & x = 5 \\ \frac{91}{216}, & x = 6 \end{cases}$$

One can confirm that $\sum_{x=1}^{6} \mathbb{P}(X = x) = 1$.