CS 70 Discrete Mathematics and Probability Theory
Fall 2017 Kannan Ramchandran and Satish Rao

Discussion 08A

1 Birthdays

Suppose you record the birthdays of a large group of people, one at a time until you have found a match, i.e., a birthday that has already been recorded. (Assume there are 365 days in a year.)

(a) What is the probability that after the first 3 people's birthdays are recorded, no match has occurred (i.e. each person has a unique birthday)?

(b) What is the probability that the first 3 people all share the same birthday?

(c) What is the probability that it takes more than 20 people for a match to occur?

(d) What is the probability that it takes exactly 20 people for a match to occur?

(e) Suppose instead that you record the birthdays of a large group of people, one at a time, until you have found a person whose birthday matches your own birthday. What is the probability that it takes exactly 20 people for this to occur?

Solution:

(a) $\frac{364}{365} \cdot \frac{363}{365}$.

(b) $\left(\frac{1}{365}\right)^2$.

(c)

$$\begin{aligned}\mathbb{P}[\text{it takes more than 20 people}] &= \mathbb{P}[\text{20 people don't have the same birthday}]\\ &= \frac{365!/(365-20)!}{365^{20}} = \boxed{\mathbf{\frac{365!}{345!365^{20}}}} \approx .589.\end{aligned}$$

Another explanation that does not use counting:

The first person can have any birthday. The second person must have a different birthday from the first person, which occurs with probability $364/365$. The third person must have a different birthday from the first two people, which occurs with probability $363/365$. Generalizing, the ith person must have a different birthday from the first $i-1$ people, which occurs with probability $(365-(i-1))/365$. Hence,

$$\begin{aligned}\mathbb{P}[\text{it takes more than 20 people}] &= \frac{365-19}{365} \times \frac{365-18}{365} \times \cdots \times \frac{363}{365} \times \frac{364}{365} \\ &\approx .589.\end{aligned}$$

(d) The probability that it takes exactly 20 people is the probability that the first 19 people have different birthdays **and** the 20th person shares a birthday with one of the first 19 people.

How many total ways can the birthdays be chosen for 20 people? 365^{20}.

How many ways can the birthdays be chosen so the first 19 have different birthdays and the 20^{th} person shares a birthday with the first 19? Well, the first person has 365 choices, the second has 364 choices left, and so on until the nineteenth person has $(365-19+1) = 347$ choices left. Then, the 20^{th} person has 19 choices for his birthday. So in total, there are $365 \cdot 364 \cdots 348 \cdot 347 \cdot 19 = (365!/346!) \cdot 19$ ways of getting what we want. So

$$\mathbb{P}[\text{it takes exactly 20 people}] = \frac{365 \cdot 364 \cdots 348 \cdot 347 \cdot 19}{365^{20}} = \boxed{\frac{\mathbf{365! \cdot 19}}{\mathbf{346! 365^{20}}}} \approx .032.$$

Another explanation that does not use counting:

As before, the ith person must have a different birthday from the first $i-1$ people, with probability $(365-(i-1))/365$, for $i = 1, \ldots, 19$. The 20th person must share a birthday with one of the first 19 people (who all have distinct birthdays), so the probabiltiy is $19/365$. Hence,

$$\begin{aligned}\mathbb{P}[\text{it takes exactly 20 people}] &= \frac{19}{365} \times \frac{365-18}{365} \times \cdots \times \frac{363}{365} \times \frac{364}{365} \\ &\approx .032.\end{aligned}$$

(e) The probability that it takes exactly 20 people is the probability that the first 19 people don't have your birthday and the 20th person has your birthday.

Similar to the last problem, there are 364 choices for the first person's birthday to be different than yours, 364 for the second person, and so on until the nineteenth person has 364 choices. Then, the 20^{th} person has exactly 1 choice to have your birthday. So the total number of ways to get what we want is $364^{19} \cdot 1$. There are 365^{20} possibilities total. So

$$\mathbb{P}[\text{it takes exactly 20 people}] = \boxed{\frac{\mathbf{364^{19}}}{\mathbf{365^{20}}}} \approx .0026.$$

Another explanation that does not use counting:

Each of the 19 people who do not share your birthday do so with probability $364/365$, and the last person must share your birthday with probability $1/365$. Hence,

$$\begin{aligned}\mathbb{P}[\text{it takes exactly 20 people}] &= \frac{364}{365} \times \frac{364}{365} \times \cdots \times \frac{364}{365} \times \frac{1}{365} \\ &= \frac{364^{19} \times 1}{365^{20}} \\ &\approx 0.0026.\end{aligned}$$

2 Rain and Wind

The local weather channel just released a statistic for the months of November and December. It said that the probability that it would rain on a windy day is 0.3 and the probability that it would rain on a non-windy day is 0.8. The probability of a day being windy is 0.2. As a student in EECS 70, you are curious to play around with these numbers. Find the probability that:

(a) A given day is both windy and rainy.

(b) A given day is rainy.

(c) For a given pair of days, exactly one of the two days is rainy.

(d) A given day that is non-rainy is also non-windy.

Solution:

(a) Let R be the event that it rains on a given day and W be the event that a given day is windy. We are given $\mathbb{P}(R \mid W) = 0.3$, $\mathbb{P}(R \mid W^c) = 0.8$ and $\mathbb{P}(W) = 0.2$. Then probability that a given day is both rainy and windy is $\mathbb{P}(R \cap W) = \mathbb{P}(R \mid W)\mathbb{P}(W) = 0.3 \times 0.2 = 0.06$.

(b) Probability that it rains on a given day is $\mathbb{P}(R) = \mathbb{P}(R \mid W)\mathbb{P}(W) + \mathbb{P}(R \mid W^c)\mathbb{P}(W^c) = 0.3 \times 0.2 + 0.8 \times 0.8 = 0.7$.

(c) Let R_1 and R_2 be the events that it rained on day 1 and day 2 respectively. Since the days are independent, $\mathbb{P}(R_1) = \mathbb{P}(R_2) = \mathbb{P}(R)$. The required probability is $\mathbb{P}(R_1)\mathbb{P}(R_2^c) + \mathbb{P}(R_1^c)\mathbb{P}(R_2) = 2 \cdot 0.7 \cdot 0.3 = 0.42$.

(d) Probability that a non-rainy day is non-windy is

$$\mathbb{P}(W^c \mid R^c) = \frac{\mathbb{P}(W^c \cap R^c)}{\mathbb{P}(R^c)} = \frac{\mathbb{P}(R^c \mid W^c)(W^c)}{\mathbb{P}(R^c)} = \frac{0.2 \times 0.8}{0.3} = \frac{8}{15}.$$

3 Lie Detector

A lie detector is known to be $4/5$ reliable when the person is guilty and $9/10$ reliable when the person is innocent. If a suspect is chosen from a group of suspects of which only $1/100$ have ever committed a crime, and the test indicates that the person is guilty, what is the probability that he is innocent?

Solution:

Let A denote the event that the test indicates that the person is guilty, and B the event that the person is innocent. Note that

$$\mathbb{P}[B] = \frac{99}{100}, \quad \mathbb{P}[\overline{B}] = \frac{1}{100}, \quad \mathbb{P}[A \mid B] = \frac{1}{10}, \quad \mathbb{P}[A \mid \overline{B}] = \frac{4}{5}.$$

Using the Bayesian Inference Rule, we can compute the desired probability as follows:

$$\mathbb{P}[B \mid A] = \frac{\mathbb{P}[B]\mathbb{P}[A \mid B]}{\mathbb{P}[B]\mathbb{P}[A \mid B] + \mathbb{P}[\overline{B}]\mathbb{P}[A \mid \overline{B}]} = \frac{(99/100)(1/10)}{(99/100)(1/10) + (1/100)(4/5)} = \frac{99}{107}$$