CS 70 Discrete Mathematics and Probability Theory
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DIS 12A

1 Bayesian Darts

You play a game of darts with your friend. You are better than he is, and the distances of your darts to the center of the target are i.i.d. Uniform$[0,1]$ whereas his are i.i.d. Uniform$[0,2]$. To make the game fair, you agree that you will throw one dart and he will throw two darts. The dart closest to the center wins the game. What is the probability that you will win? *Note*: The distances *from the center of the board* are uniform.

Solution:

Let X be the distance of your closest dart to the center and Y that of the closest of your friend's darts. Then, for $x \in [0,1]$ and $y \in [0,2]$,

$$\mathbb{P}[X > x] = 1 - x \quad \text{and} \quad \mathbb{P}[Y > y] = \left(1 - \frac{y}{2}\right)^2.$$

Hence,

$$f_X(x) = \frac{\mathrm{d}}{\mathrm{d}x} F_X(x) = \frac{\mathrm{d}}{\mathrm{d}x}(1 - \mathbb{P}[X > x]) = -\frac{\mathrm{d}}{\mathrm{d}x}(1 - x) = 1, \qquad x \in [0,1].$$

Also,

$$\mathbb{P}[Y > X \mid X = x] = \left(1 - \frac{x}{2}\right)^2.$$

Thus,

$$\begin{aligned}\mathbb{P}[Y > X] &= \int_0^1 \mathbb{P}(Y > X \mid X = x) f_X(x)\,\mathrm{d}x = \int_0^1 \left(1 - \frac{x}{2}\right)^2 f_X(x)\,\mathrm{d}x = \mathbb{E}\left[\left(1 - \frac{X}{2}\right)^2\right] \\ &= \mathbb{E}\left[1 - X + \frac{X^2}{4}\right] = 1 - \frac{1}{2} + \frac{1}{12} = \frac{7}{12},\end{aligned}$$

since $\mathbb{E}[X] = 1/2$ and $\mathbb{E}[X^2] = \int_0^1 x^2\,\mathrm{d}x = 1/3$.

2 Exponential Median

What is the expected value of the median of three i.i.d. exponential variables with parameter λ?

Solution:

The minimum of exponential variables is another exponential variable with the sum of the originals' parameters as its parameter. Denote M as the median of the exponential variables, X_1, X_2, X_3. Suppose $\min(X_1, X_2, X_3) = m > 0$. Then, viewing the exponential variables as lightbulbs running simultaneously, the median is the next lightbulb to go out, which is the minimum of the next two exponential variables, which is an exponential with parameter 2λ. Use the memoryless property. Therefore $\mathbb{E}[M \mid \min(X_1, X_2, X_3) = m] = m + 1/(2\lambda)$. Therefore

$$\mathbb{E}[M] = \frac{1}{3\lambda} + \frac{1}{2\lambda} = \frac{5}{6\lambda}.$$

You can also try finding the density of the median. The density of the minimum is $3\lambda\,\mathrm{e}^{-3\lambda t}$ for $t > 0$. Let Y_1 and Y_2 denote the lifetimes of the light bulbs which survive after the minimum dies. Conditioned on $\min\{X_1, X_2, X_3\} = s > 0$, the memoryless property of exponentials implies that $\min\{Y_1, Y_2\}$ has the density $2\lambda\,\mathrm{e}^{-2\lambda(t-s)}$ for $t > s$.

$$\begin{aligned}\mathbb{P}(M \in [t, t+\mathrm{d}t]) &= \Big(\int_0^t 3\lambda\,\mathrm{e}^{-3\lambda s} \cdot 2\lambda\,\mathrm{e}^{-2\lambda(t-s)}\,\mathrm{d}s\Big)\mathrm{d}t = 6\lambda^2\,\mathrm{e}^{-2\lambda t}\,\mathrm{d}t \int_0^t \mathrm{e}^{-\lambda s}\,\mathrm{d}s \\ &= 6\lambda(\mathrm{e}^{-2\lambda t} - \mathrm{e}^{-3\lambda t})\,\mathrm{d}t\end{aligned}$$

The density is $6\lambda(\mathrm{e}^{-2\lambda t} - \mathrm{e}^{-3\lambda t})$ for $t > 0$, which should give the same expectation.

3 Expecting You to Integrate by Parts!

We derived an alternative form for the expected value of a non-negative integer-valued random variable Y, $\mathbb{E}[Y] = \sum_{i=1}^{\infty} \mathbb{P}(Y \geq i)$. In this problem, we will derive the continuous analog of this expression. Throughout this problem, assume X is a *continuous* non-negative random variable with PDF f_X and $\mathbb{E}[X] < \infty$.

(a) Write an expression for $\mathbb{P}(X \geq x)$ in terms of $f_X(x)$ for $x > 0$. This is called the *complementary cumulative distribution function* of X, of the CDF of X. For this problem, we denote this as $\bar{F}_X(x)$. What is $\bar{F}_X(0)$? How about $\bar{F}_X(x)$ as $x \to \infty$?

(b) Use integration by parts on $\mathbb{E}[X] = \int_0^\infty x f_X(x)\,\mathrm{d}x$ to derive the expression in question. [*Hint:* What is the antiderivative of f_X?]

Solution:

(a) The complementary cumulative distribution function is the *complement* of the CDF, so we know that $\mathbb{P}(X \geq x) = 1 - \mathbb{P}(X \leq x)$. Therefore, for $x > 0$,

$$\bar{F}_X(x) = 1 - \int_0^x f_X(t)\,\mathrm{d}t.$$

If we plug in $x = 0$, the integral is equal to 0, so $\bar{F}_X(0) = 1$. If we let $x \to \infty$, then the integral becomes 1, since we will have integrated the entire probability density function (PDF). Therefore $\bar{F}_X(x) \to 0$ as $x \to \infty$.

(b) We define the CDF, F_X, for $x > 0$, as

$$F_X(x) = \mathbb{P}(X \le x) = \int_0^x f_X(t)\,\mathrm{d}t.$$

Taking $u(x) = -x$ and $v'(x) = -f_X(x)$,

$$\begin{aligned}\mathbb{E}[X] &= \int_0^\infty -x \cdot \big(-f_X(x)\big)\,\mathrm{d}x = \big[-x\big(1 - F_X(x)\big)\big]_0^\infty - \int_0^\infty (-1) \cdot \big(1 - F_X(x)\big)\,\mathrm{d}x \\ &= [-x\bar{F}_X(x)]_0^\infty + \int_0^\infty \big(1 - F_X(x)\big)\,\mathrm{d}x = 0 + \int_0^\infty \bar{F}_X(x)\,\mathrm{d}x = \int_0^\infty \mathbb{P}(X \ge x)\,\mathrm{d}x.\end{aligned}$$

The first term of the third equation ends up being $\infty \cdot 0$ when we try to evaluate at $x = \infty$. To show that this is 0, we can break up the expectation into the part below x and the part above x, where $x > 0$:

$$\begin{aligned}\mathbb{E}[X] = \int_0^\infty t f(t)\,\mathrm{d}t &\ge \int_0^x t f(t)\,\mathrm{d}t + x\int_x^\infty f(t)\,\mathrm{d}t && \text{break up expectation} \\ &\ge \int_0^x t f(t)\,\mathrm{d}t + x\bar{F}_X(x) && \text{formula for } \bar{F}_X(x)\end{aligned}$$

Rearranging terms, we get:

$$x\bar{F}_X(x) \le \mathbb{E}[X] - \int_0^x t f(t)\,\mathrm{d}t$$

As $x \to \infty$, the second term on the RHS becomes exactly $\mathbb{E}[X]$, so their difference goes to 0, as desired.