Equivalent Statements of the Axiom of Choice

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The following is an exersize from Paul Halmos's Naive Set Theory:

Show that the following are equivalent to the axiom of choice:

$$\text{Every partially ordered set has a maximal chain} \tag{1}$$

$$\text{Every chain in a partially ordered set is included in some maximal chain} \tag{2}$$

$$\text{A partially ordered set where each chain has a supremum has a maximal element} \tag{3}$$

Proof. Assuming that Zorn's Lemma is equivalent to the Axiom of Choice, it suffices to show that:

$$(1) \Rightarrow (2) \Rightarrow [\text{Zorn's Lemma}] \Rightarrow (3) \Rightarrow (1)$$

$(3) \Rightarrow (1)$

Let $(X, \leq)$ be an arbitrary partial ordering and let χ be all those chain in X:

$$\chi := \{\gamma \in \mathcal{P}(X) : \gamma \text{ is a chain}\}$$

We then define the ordering $(\chi, \subset)$ by inclusion in χ. Let A be a chain in χ meaning that it is a chain ordered by inclusion of chains in X. Consider $\bigcup A$. I claim that this is a least upper bound of A. To show that this is a least upper bound first note that for all $a \in A \;\; a \subset \bigcup A$. Furthermore $\bigcup A$ is a chain since if we were to fix $x, y \in \bigcup A$ then there would be A_x and $A_y \in A$ for which $x \in A_x$ and $y \in A_y$. Without loss of generality we can assume that $A_x \subset A_y$ it follows that $x, y \in A_y$ and since in particular A_y is a chain we have that $x \leq y$ or $y \leq x$ and these are the desired inequalities. It has now been shown that $\bigcup A$ is a chain that includes every element of A, therefore it is an upper bound and it remains to be shown that it is a least upper bound.

To show this, assume towards a contradiction that $\bigcup A$ is not a least upper bound. That implies that there exists some $B \in \chi$ such that B is a proper subset of $\bigcup A$ and B is an upper bound of A. Clearly we have that $B \subset \bigcup A$. In addition to this if we fix an arbitray $\gamma \in \bigcup A$ there must exist some $A_\gamma \in A$ for which $\gamma \in A_\gamma$. By the definition of B we have that $A_\gamma \subset B$ so then $\gamma \in B$. This yields that $\bigcup A \subset B$ which then allows us to conclude that $\bigcup A = B$ which is a contradiction with B being a proper subset of $\bigcup A$.

Since an arbitrary chain A in χ has a least upper bound (namely $\bigcup A$) we can apply (3) to conclude that there must be some maximal element in χ meaning that X has a maximal chain. This is exactly the statement of (1) and we are done.

$(1) \Rightarrow (2)$

Again letting $(X, \leq)$ be a partial ordering, let $(\chi, \subset)$ be the collection of chains in X ordered by inclusion. Fixing an arbitrary $A \in \chi$ define C_A to be all those chains in χ which include A:

$$C_A := \{\gamma \in \chi : A \subset \gamma\}$$

C_A is non empty since it contains A itself. Furthermore, C_A inherits the ordering from χ. Therefore we can apply (1) to conclude tha there exists some maximal chain Λ in C_A.

Note that Λ is a subest of C_A and is a chain with respect to the inclusion ordering on C_A. As previously shown $\bigcup \Lambda$ is itself a chain included in χ because it is a union of chains. I assert that $\bigcup \Lambda$ is a maximal chain.

Assume to get a contradiction that $\bigcup \Lambda$ is not a maximal chain. By this assumption there exists some B such that $\bigcup \Lambda$ is a proper subset of B. A is a subset of $\bigcup \Lambda$ since it is the union of chains which include A, therefore $A \subset B$. It follows that $B \in C_A$. Furthermore, since $\bigcup \Lambda$ is a proper subset of B we have that $B \notin \Lambda$. This is true because by definition of the proper subset, there must be some $b \in B$ such that $b \notin \bigcup \Lambda$. If B were to be in Λ then b would necessarily be an element of $\bigcup \Lambda$. Using this we can form $\Lambda \cup \{B\}$ this is a chain since every element of Λ is a subset of $\bigcup \Lambda$ and by transitivity is a subset of B. This construction contradicts the maximality of Λ since Λ is a proper subset of $\Lambda \cup \{B\}$.

We have constructed a maximal chain which includes A which is what is required by (2).

(2) $\Rightarrow$ [Zorn's Lemma]
Let $(X, \leq)$ be a partial ordering where every chain in X has a maximal element. We need to show that there exists a maximal element in X. Therefore, fix some arbitrary chain A in X. In this case, (2) applies and we see that A is included in some maximal chain Ξ. Since Ξ is a chain in X our assumption applies and it has a maximal element m. This element is a maximal element of m.

If m was not a maximal element of X then there must exist some m' such that $m \leq m'$ and $m \neq m'$. Since for every $\gamma \in \Xi$ we have that $\gamma \leq m$ and $m \leq m'$ we can conclude that $\gamma \leq m'$. Therefore the set $\Xi \cup \{m'\}$ is a chain. Since $m' \notin \Xi$ we have Ξ as a proper subset of $\Xi \cup \{m'\}$ which contradicts the maximality of Ξ.

It follows that m is maximal and this section of the proof is complete.

[Zorn's Lemma] $\Rightarrow$ (3)
Let $(X, \leq)$ be a partial ordering where every chain in X has a least upper bound. If each chain has a least upper bound then it has an upper bound. Zorn's Lemma applies and X has a maximal element and we are done.

$\square$