Axiom of Choice in Analysis

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Exersize 1: A compact set E has a countable and dense subset

Proof. Since E is compact it is totally bounded, meaning that for all ϵ there is $\{x_1, x_2, \ldots, x_n\}$ such that $E \subset \bigcup_{i=1}^{n} B(x_i, \epsilon)$. Let $\epsilon_n := \frac{1}{n}$ and denote the set given by the bounding of ϵ_n as T_n. For every n, T_n is finite. We can form the union $T := \bigcup_{n=1}^{\infty} T_n$ which is countable. Fixing some arbitrary $x \in E$ and $\delta > 0$ we can find j such that $\frac{1}{j} < \delta \Rightarrow T_j \subset T$. Therefore there exists some x_i in T_j for which $d(x_i, x) < \frac{1}{j} < \delta$. This implies that T is dense in E. □

Exersize 2: For $D := \{x_1, x_2, x_3, \ldots\}$, dense in some compact set E then for every $x \in E$ and $\delta > 0$ there exists i and k with $i \leq k$ such that $d(x_i, x) < \delta$

Proof. Since E is compact it is totally bounded. Let $T = z_1, z_2, \ldots, z_j$ be the collection of points given by bounding E by $\delta/2$. Define the function:

$$g : T \to \mathcal{P}(E) \setminus \{\emptyset\} \quad f(z_i) \mapsto \{\gamma \in D : d(\gamma, z_i) < \delta/2\}$$

For all z_i, $g(z_i)$ is nonempty since D is dense in E. Let $f : \mathcal{P}(E) \setminus \{\emptyset\} \to E$ be a choice function on E (guarunteed by the axiom of choice). We can then form the composition:

$$f \circ g : T \to D$$

which selects exactly one element from each $g(z_i)$. $R := \text{Range}(f \circ g) \subset D$ is finite. Fixing some arbitrary $x \in E$ there must be some $z_j \in T$ such that $d(x_j, x) < \delta/2$ there must also be some $x_\ell \in R$ with $d(x_\ell, z_i) < \delta/2$. Therefore we can calculate:

$$d(x, x_\ell) \leq d(x_\ell, z_j) + d(x, z_j) < \delta$$

□