Sequence Definition of Uniform Continuity
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Let (X, d) and (Y, ϱ) be metric spaces and $f : X \to Y$.

Show that the following are equivalent:

(1)

$$\forall \epsilon > 0 \;\; \exists \delta(\epsilon) > 0 \;\; \forall x \in X \;\; \forall y \in X \;\; [d(x, y) < \delta] \Rightarrow [\varrho(f(x), f(y)) < \epsilon]$$

(2)

$$\text{For arbitrary sequences } (x_n)_{n=1}^{\infty} \text{ and } (y_n)_{n=1}^{\infty} \text{ in } X \text{ with } \lim_{n\to\infty} d(x_n, y_n) = 0$$
$$\text{then } \lim_{n\to\infty} \varrho(f(x_n), f(y_n)) = 0$$

Proof.
($\Rightarrow$) Fix arbitrary sequences (x_n) and (y_n) with $d(x_n, y_n) \to 0$. We have that:

$$\forall \gamma > 0 \;\; \exists N_0(\gamma) \;\; \forall n \geq N_0 \;\; d(x_n, y_n) < \gamma$$

Let ϵ be given and show:

$$\exists G_0(\epsilon) > 0 \;\; \forall g \geq G_0 \;\; \varrho(f(x_n), f(y_n)) < \epsilon$$

Pick, using notation found in **(1)**, $G_0 := N_0(\delta(\epsilon))$. I claim that this satisfies the above. Pick $g \geq G_0$ and by the above we have that $d(x_g, y_g) < \delta(\epsilon)$ and using **(1)**: $\varrho(f(x_g), f(y_g)) < \epsilon$.

The above is equivalent to:

$$\lim_{n\to\infty} \varrho(f(x_n), f(y_n)) = 0$$

($\Leftarrow$) We will prove this implication using contraposition. Take $\neg$**(1)**:

$$\exists \epsilon > 0 \;\; \forall \delta > 0 \;\; \exists x(\delta) \in X \;\; \exists y(\delta) \in X \;\; d(x, y) < \delta \text{ and } \varrho(f(x), f(y)) \geq \epsilon$$

I write $x(\delta)$ and $y(\delta)$ to denote the dependance of x and y on the choice of arbitrary δ.

Construct sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ as follows using the above notation:

$$\text{for every } n \in \mathbb{N} \text{ let } \delta_n := \frac{1}{n} \text{ and } x_n := x(\delta_n),\; y_n := y(\delta_n)$$

We immedietly have that: $d(x_n, y_n) \to 0$. Since for any ϵ we can find n_0 with $\delta_{n_0} = n_0^{-1} < \epsilon$. And by our defintion of (x_n) and (y_n) we have: $d(x_{n_0}, y_{n_0}) < \delta_{n_0} < \epsilon$. And since δ_n is decreasing we are done.

Also by the definitons of the sequences we have that for any n $\varrho(f(x_n), f(y_n)) \geq \epsilon$. So if we pick $\frac{\epsilon}{2}$ we cannot find n with $\varrho(f(x_n), f(y_n)) < \frac{\epsilon}{2}$. Therefore: $\varrho(f(x_n), f(y_n)) \not\to 0$. We have exhibited sequences which converge to each other but do not converge when composed with f. This is $\neg$**(2)** and we are done. $\square$