Orthogonal Projections
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Definition: For S a subspace of $\mathbb{R}^n$ with a given basis: $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ and a point $\mathbf{g} \in \mathbb{R}^n$ then $\mathbf{a} \in S$ is the orthogogonal projection of $\mathbf{g}$ onto S if:

$$\forall i \leq k \quad (\mathbf{g} - \mathbf{a}) \cdot \mathbf{u}_i = 0$$

Lemma 1: For a vector sub space S of $\mathbb{R}^n$ with a given basis: $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ if some vector $\mathbf{v} \in S$ is such that for all $i \leq k$ $\mathbf{v} \cdot \mathbf{u}_i = 0$ then $\mathbf{v} = \mathbf{0}$

Proof. Since $\mathbf{v} \in S$ we can write:

$$\mathbf{v} = t_1\mathbf{u}_1 + t_2\mathbf{u}_2 + \cdots + t_k\mathbf{u}_k$$

It is sufficient to show that the magnitude of $\mathbf{v}$ is 0:

$$|\mathbf{v}|^2 = \mathbf{v} \cdot \mathbf{v} = \mathbf{v} \cdot (t_1\mathbf{u}_1 + t_2\mathbf{u}_2 + \cdots + t_k\mathbf{u}_k) = t_1\mathbf{u}_1\mathbf{v} + t_2\mathbf{u}_2\mathbf{v} + \cdots + t_k\mathbf{u}_k\mathbf{v} = 0$$

$$\iff |\mathbf{v}| = 0$$

$$\iff \mathbf{v} = 0$$

$\square$

Proposition 1: If an orthogonal projection of $\mathbf{g}$ onto S exists it is unique

Proof. Let $\mathbf{a}$ and $\mathbf{b}$ be orthogonal projections of $\mathbf{g}$ onto S

By defintion for an arbitrary $i \leq k$ we have:

$$0 = (\mathbf{g} - \mathbf{a}) \cdot \mathbf{u}_i = (\mathbf{g} - \mathbf{a} + \mathbf{b} - \mathbf{b}) \cdot \mathbf{u}_i = (\mathbf{g} - \mathbf{b} - (\mathbf{a} - \mathbf{b})) \cdot \mathbf{u}_i$$

$$= (\mathbf{g} - \mathbf{b}) \cdot \mathbf{u}_i - (\mathbf{a} - \mathbf{b}) \cdot \mathbf{u}_i = -(\mathbf{a} - \mathbf{b}) \cdot \mathbf{u}_i$$

$$\iff (\mathbf{a} - \mathbf{b}) \cdot \mathbf{u}_i = 0$$

Since $(\mathbf{a} - \mathbf{b}) \in S$ we can apply lemma 1:

$$\Rightarrow (\mathbf{a} - \mathbf{b}) = 0$$

$$\iff \mathbf{a} = \mathbf{b}$$

$\square$

Proposition 2: If $\mathbf{a} \in S$ is an orthogonal projection of $\mathbf{g}$ relative to a basis: $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ then it is also orthogonal relative to any other basis

Proof. suppose that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is also a basis of S and take an arbitrary $\mathbf{v}_i$

$$(\mathbf{g} - \mathbf{a}) \cdot \mathbf{v}_i = (\mathbf{g} - \mathbf{a}) \cdot (t_1\mathbf{u}_1 + t_2\mathbf{u}_2 + \cdots + t_k\mathbf{u}_k) \quad \because \ \mathbf{v}_i \in S$$

$$= t_1(\mathbf{g} - \mathbf{a}) \cdot \mathbf{u}_1 + t_2(\mathbf{g} - \mathbf{a}) \cdot \mathbf{u}_2 + \cdots + t_k(\mathbf{g} - \mathbf{a}) \cdot \mathbf{u}_k = 0$$

$$\Rightarrow (\mathbf{g} - \mathbf{a}) \cdot \mathbf{v}_i = 0$$

$\square$

Proposition 3: The given definition is equivalent to saying that the orthogonal projection of $\mathbf{a}$ onto S (denoted $\mathbf{g}$) then $(\mathbf{g}-\mathbf{a})$ is orthogonal to every vector in S.

Proof. ($\Rightarrow$) Fixing an arbitrary $\mathbf{v} \in S$ we know that $\mathbf{v}$ can be written as $\sum_{i=1}^{k} a_i \mathbf{u}_i$ for each $\mathbf{u}_i$ being a basis vector of S. Therefore:

$$(\mathbf{g}-\mathbf{a}) \cdot \mathbf{v} = (\mathbf{g}-\mathbf{a}) \cdot \sum_{i=1}^{k} a_i \mathbf{u}_i = \sum_{i=1}^{k} (\mathbf{g}-\mathbf{a}) \cdot a_i \mathbf{u}_i = 0$$

($\Leftarrow$) This is trivial because if $(\mathbf{g}-\mathbf{a})$ is orthogonal to all vectors in S then in particular it is perpendicular to basis vectors of S. $\square$

Given the results of propositions 1 and 2, orthogonal projections are independant of choice of basis vectors and are unique. Therefore, the following notation will be introduced for $\mathbf{a}$ being $\mathbf{g}$'s orthogonal projection onto S:

$$P_S(\mathbf{g}) = \mathbf{a}$$

Proposition 4: If a point $\mathbf{g}$ is included in S then its projection onto S is itself.

Proof. The proof is trivial since if a point exists which satisfies the first definition then the point is unique. If we consider $\mathbf{g}$ itself then:

$$(\mathbf{g}-\mathbf{g}) = \mathbf{0}$$

$$\Rightarrow \forall i \leq k \quad \mathbf{u}_i \cdot (\mathbf{g}-\mathbf{g}) = \mathbf{0}$$

And $\mathbf{g}$ is its own projection. This also works in reverse in a sense because, if a projection of a point is calculated to be in exactly the same location as the point itself then the point must have been in S in the first place! I use this result to determine whether certain points are included in a subspace in the program that accompanies this.

$\square$

Theorem 1 If a point $\mathbf{a} \in S$ minimizes the distance between $\mathbf{g}$ and the space S then $P_S(\mathbf{g}) = \mathbf{a}$

Proof. We can represent vectors in S as parameterized in k variables where k is the dimension of the basis of S. Therefore distances between S and $\mathbf{g}$ can be written as:

$$d(t_1, t_2, \ldots, t_k) = ||t_1\mathbf{u}_1 + t_2\mathbf{u}_2 + \cdots + t_k\mathbf{u}_k - \mathbf{g}||$$

Since d is always positive it suffices to consider d^2 when talking about local extrema. Which can be written as:

$$(t_1\mathbf{u}_1 + t_2\mathbf{u}_2 + \cdots + t_k\mathbf{u}_k - \mathbf{g}) \cdot (t_1\mathbf{u}_1 + t_2\mathbf{u}_2 + \cdots + t_k\mathbf{u}_k - \mathbf{g})$$

Since the product rule holds for the dot product or inner product we can write the derivate of d^2 with respect to the variable t_i as:

$$2\mathbf{u}_i \cdot (t_1\mathbf{u}_1 + t_2\mathbf{u}_2 + \cdots + t_k\mathbf{u}_k - \mathbf{g})$$

To have a minimum we must have that:

$$\forall i \leq k \quad \frac{\partial\, d^2}{\partial\, t_i} = 0$$

$$\forall i \leq k \quad 2\mathbf{u}_i \cdot (t_1\mathbf{u}_1 + t_2\mathbf{u}_2 + \dots + t_k\mathbf{u}_k - \mathbf{g}) = 0$$
$$\iff \forall i \leq k \quad \mathbf{u}_i \cdot (t_1\mathbf{u}_1 + t_2\mathbf{u}_2 + \dots + t_k\mathbf{u}_k - \mathbf{g}) = 0$$

Therefore **a** satisfies this then it is **g**'s orthogonal projection by my original definition. $\square$

Notice that if if we allow points in S to be written as linear combinations of S's basis $\mathbf{u}_i$ we can calculate the projection point by solving the system of equations:

$$\mathbf{u}_1 \cdot (t_1\mathbf{u}_1 + t_2\mathbf{u}_2 + \dots + t_k\mathbf{u}_k - \mathbf{a}) = 0$$
$$\dots$$
$$\mathbf{u}_k \cdot (t_1\mathbf{u}_1 + t_2\mathbf{u}_2 + \dots + t_k\mathbf{u}_k - \mathbf{a}) = 0$$

Which is equivalent to:

$$\mathbf{u}_1 \cdot t_1\mathbf{u}_1 + \mathbf{u}_1 \cdot t_2\mathbf{u}_2 + \dots + \mathbf{u}_1 \cdot t_k\mathbf{u}_k = \mathbf{u}_1 \cdot \mathbf{a}$$
$$\dots$$
$$\mathbf{u}_k \cdot t_1\mathbf{u}_1 + \mathbf{u}_k \cdot t_2\mathbf{u}_2 + \dots + \mathbf{u}_k \cdot t_k\mathbf{u}_k = \mathbf{u}_k \cdot \mathbf{a}$$

Which is again equivalent to solving the matrix equation:

$$\begin{bmatrix} \mathbf{u}_1 \cdot \mathbf{u}_1 & \mathbf{u}_1 \cdot \mathbf{u}_2 & \dots & \mathbf{u}_1 \cdot \mathbf{u}_k \\ \mathbf{u}_2 \cdot \mathbf{u}_1 & \mathbf{u}_2 \cdot \mathbf{u}_2 & \dots & \mathbf{u}_2 \cdot \mathbf{u}_k \\ \dots & \dots & \dots & \dots \\ \mathbf{u}_k \cdot \mathbf{u}_1 & \mathbf{u}_k \cdot \mathbf{u}_2 & \dots & \mathbf{u}_k \cdot \mathbf{u}_k \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ \dots \\ t_k \end{bmatrix} = \begin{bmatrix} \mathbf{u}_1 \cdot \mathbf{a} \\ \mathbf{u}_2 \cdot \mathbf{a} \\ \dots \\ \mathbf{u}_k \cdot \mathbf{a} \end{bmatrix}$$

Taking for granted that this matrix is invertable we can calculate $P_S(\mathbf{g})$'s coordinates with respect to the chosen basis as:

$$\begin{bmatrix} t_1 \\ t_2 \\ \dots \\ t_k \end{bmatrix} = \begin{bmatrix} \mathbf{u}_1 \cdot \mathbf{u}_1 & \mathbf{u}_1 \cdot \mathbf{u}_2 & \dots & \mathbf{u}_1 \cdot \mathbf{u}_k \\ \mathbf{u}_2 \cdot \mathbf{u}_1 & \mathbf{u}_2 \cdot \mathbf{u}_2 & \dots & \mathbf{u}_2 \cdot \mathbf{u}_k \\ \dots & \dots & \dots & \dots \\ \mathbf{u}_k \cdot \mathbf{u}_1 & \mathbf{u}_k \cdot \mathbf{u}_2 & \dots & \mathbf{u}_k \cdot \mathbf{u}_k \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{u}_1 \cdot \mathbf{a} \\ \mathbf{u}_2 \cdot \mathbf{a} \\ \dots \\ \mathbf{u}_k \cdot \mathbf{a} \end{bmatrix}$$

This is the method employed by my program to calculate projections!