Orthogonal Projections

Definition: For S a subspace of $\mathbb{R}^n$ with a given basis: $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ and a point $\mathbf{g} \in \mathbb{R}^n$ then $\mathbf{a} \in S$ is the orthogogonal projection of $\mathbf{g}$ onto S if:

$$\forall i \leq k \quad (\mathbf{g} - \mathbf{a}) \cdot \mathbf{u}_i = 0$$

Lemma 1: For a vector sub space S of $\mathbb{R}^n$ with a given basis: $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ if some vector $\mathbf{v} \in S$ is such that for all $i \leq k$ $\mathbf{v} \cdot \mathbf{u}_i = 0$ then $\mathbf{v} = \mathbf{0}$

Proof. Since $\mathbf{v} \in S$ we can write:

$$\mathbf{v} = t_1\mathbf{u}_1 + t_2\mathbf{u}_2 + \cdots + t_k\mathbf{u}_k$$

It is sufficient to show that the magnitude of $\mathbf{v}$ is 0:

$$|\mathbf{v}|^2 = \mathbf{v} \cdot \mathbf{v} = \mathbf{v} \cdot (t_1\mathbf{u}_1 + t_2\mathbf{u}_2 + \cdots + t_k\mathbf{u}_k) = t_1\mathbf{u}_1\mathbf{v} + t_2\mathbf{u}_2\mathbf{v} + \cdots + t_k\mathbf{u}_k\mathbf{v} = 0$$
$$\iff |\mathbf{v}| = 0$$
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□

Proposition 1: If an orthogonal projection of $\mathbf{g}$ onto S exists it is unique

Proof. Let $\mathbf{a}$ and $\mathbf{b}$ be orthogonal projections of $\mathbf{g}$ onto S

$$\Rightarrow \mathbf{a} = a_1\mathbf{u}_1 + \cdots + a_k\mathbf{u}_k \text{ and } \mathbf{b} = b_1\mathbf{u}_1 + \cdots + b_k\mathbf{u}_k$$

Consider:

$$\mathbf{a} - \mathbf{b} = (a_1 - b_1)\mathbf{u}_1 + \cdots + (a_k - b_k)\mathbf{u}_k$$

For every i let $z_i := a_i - b_i$ and we can write:

$$\mathbf{a} - \mathbf{b} = z_1\mathbf{u}_1 + \cdots + z_k\mathbf{u}_k$$

By defintion for an arbitrary $i \leq k$ we have:

$$0 = (\mathbf{g} - \mathbf{a}) \cdot \mathbf{u}_i = (\mathbf{g} - \mathbf{a} + \mathbf{b} - \mathbf{b}) \cdot \mathbf{u}_i = (\mathbf{g} - \mathbf{b} - (\mathbf{a} - \mathbf{b})) \cdot \mathbf{u}_i$$
$$= (\mathbf{g} - \mathbf{b}) \cdot \mathbf{u}_i - (\mathbf{a} - \mathbf{b}) \cdot \mathbf{u}_i = -(\mathbf{a} - \mathbf{b}) \cdot \mathbf{u}_k$$
$$\iff (\mathbf{a} - \mathbf{b}) \cdot \mathbf{u}_i = 0$$

Since $(\mathbf{a} - \mathbf{b}) \in S$ we can apply lemma 1, meaning that: $(\mathbf{a} - \mathbf{b}) = \mathbf{0}$ □