Orthogonal Projections

Defintion: For S a subspace of $\mathbb{R}^n$ with basis vectors $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ and a point $\mathbf{g} \in \mathbb{R}^n$ then $\mathbf{a} \in S$ is the orthogogonal projection of $\mathbf{g}$ onto S if:

$$\forall i \leq k \quad (\mathbf{g} - \mathbf{a}) \cdot \mathbf{u}_i = 0$$

Proposition 1: If an orthogonal projection of $\mathbf{g}$ onto S exists it is unique

Proof. Let $\mathbf{a}$ and $\mathbf{b}$ be orthogonal projections of $\mathbf{g}$ onto S and consider:

$$\Rightarrow \mathbf{a} = a_1\mathbf{u}_1 + \cdots + a_k\mathbf{u}_k \text{ and } \mathbf{b} = b_1\mathbf{u}_1 + \cdots + b_k\mathbf{u}_k$$

Consider:

$$\mathbf{a} - \mathbf{b} = (a_1 - b_1)\mathbf{u}_1 + \cdots + (a_k - b_k)\mathbf{u}_k$$

For every i let $z_i := a_i - b_i$ and we can write:

$$\mathbf{a} - \mathbf{b} = z_1\mathbf{u}_1 + \cdots + z_k\mathbf{u}_k$$

By defintion we have:

$$\begin{aligned} 0 = (\mathbf{g} - \mathbf{a}) \cdot \mathbf{u}_k = (\mathbf{g} - \mathbf{a} + \mathbf{b} - \mathbf{b}) \cdot \mathbf{u}_k = (\mathbf{g} - \mathbf{b} - (\mathbf{a} - \mathbf{b})) \cdot \mathbf{u}_k \\ = (\mathbf{g} - \mathbf{b}) \cdot \mathbf{u}_k - (\mathbf{a} - \mathbf{b}) \cdot \mathbf{u}_k = -(\mathbf{a} - \mathbf{b}) \cdot \mathbf{u}_k \\ \iff (\mathbf{a} - \mathbf{b}) \cdot \mathbf{u}_k = 0 \end{aligned}$$

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