

Orthogonal Projections

Definition: For S a subspace of \mathbb{R}^n with a given basis: $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k\}$ and a point $\mathbf{g} \in \mathbb{R}^n$ then $\mathbf{a} \in S$ is the orthogonal projection of \mathbf{g} onto S if:

$$\forall i \leq k \quad (\mathbf{g} - \mathbf{a}) \cdot \mathbf{u}_i = 0$$

Lemma 1: For a vector sub space S of \mathbb{R}^n with a given basis: $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k\}$ if some vector $\mathbf{v} \in S$ is such that for all $i \leq k$ $\mathbf{v} \cdot \mathbf{u}_i = 0$ then $\mathbf{v} = \mathbf{0}$

Proof. Since $\mathbf{v} \in S$ we can write:

$$\mathbf{v} = t_1 \mathbf{u}_1 + t_2 \mathbf{u}_2 + \dots + t_k \mathbf{u}_k$$

It is sufficient to show that the magnitude of \mathbf{v} is 0:

$$\begin{aligned} |\mathbf{v}|^2 &= \mathbf{v} \cdot \mathbf{v} = \mathbf{v} \cdot (t_1 \mathbf{u}_1 + t_2 \mathbf{u}_2 + \dots + t_k \mathbf{u}_k) = t_1 \mathbf{u}_1 \mathbf{v} + t_2 \mathbf{u}_2 \mathbf{v} + \dots + t_k \mathbf{u}_k \mathbf{v} = 0 \\ &\iff |\mathbf{v}| = 0 \\ &\iff \mathbf{v} = \mathbf{0} \end{aligned}$$

□

Proposition 1: If an orthogonal projection of \mathbf{g} onto S exists it is unique

Proof. Let \mathbf{a} and \mathbf{b} be orthogonal projections of \mathbf{g} onto S

By definition for an arbitrary $i \leq k$ we have:

$$\begin{aligned} 0 &= (\mathbf{g} - \mathbf{a}) \cdot \mathbf{u}_i = (\mathbf{g} - \mathbf{a} + \mathbf{b} - \mathbf{b}) \cdot \mathbf{u}_i = (\mathbf{g} - \mathbf{b} - (\mathbf{a} - \mathbf{b})) \cdot \mathbf{u}_i \\ &= (\mathbf{g} - \mathbf{b}) \cdot \mathbf{u}_i - (\mathbf{a} - \mathbf{b}) \cdot \mathbf{u}_i = -(\mathbf{a} - \mathbf{b}) \cdot \mathbf{u}_i \\ &\iff (\mathbf{a} - \mathbf{b}) \cdot \mathbf{u}_i = 0 \end{aligned}$$

Since $(\mathbf{a} - \mathbf{b}) \in S$ we can apply lemma 1:

$$\begin{aligned} &\Rightarrow (\mathbf{a} - \mathbf{b}) = \mathbf{0} \\ &\iff \mathbf{a} = \mathbf{b} \end{aligned}$$

□

Proposition 2: If $\mathbf{a} \in S$ is an orthogonal projection of \mathbf{g} relative to a basis: $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k\}$ then it is also orthogonal relative to any other basis

Proof. suppose that $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is also a basis of S and take an arbitrary \mathbf{v}_i

$$\begin{aligned} (\mathbf{g} - \mathbf{a}) \cdot \mathbf{v}_i &= (\mathbf{g} - \mathbf{a}) \cdot (t_1 \mathbf{u}_1 + t_2 \mathbf{u}_2 + \dots + t_k \mathbf{u}_k) \quad \because \mathbf{v}_i \in S \\ &= t_1 (\mathbf{g} - \mathbf{a}) \cdot \mathbf{u}_1 + t_2 (\mathbf{g} - \mathbf{a}) \cdot \mathbf{u}_2 + \dots + t_k (\mathbf{g} - \mathbf{a}) \cdot \mathbf{u}_k = 0 \\ &\Rightarrow (\mathbf{g} - \mathbf{a}) \cdot \mathbf{v}_i = 0 \end{aligned}$$

Given the results of propositions 1 and 2, orthogonal projections are independant of choice

of basis vectors and are unique. Therefore, the following notation will be introduced for \mathbf{a} being \mathbf{g} 's orthogonal projection onto S :

$$P_S(\mathbf{g}) = \mathbf{a}$$

□