

## Orthogonal Projections

**Definition:** For  $S$  a subspace of  $\mathbb{R}^n$  with a given basis:  $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k\}$  and a point  $\mathbf{g} \in \mathbb{R}^n$  then  $\mathbf{a} \in S$  is the orthogonal projection of  $\mathbf{g}$  onto  $S$  if:

$$\forall i \leq k \quad (\mathbf{g} - \mathbf{a}) \cdot \mathbf{u}_i = 0$$

**Lemma 1:** For a vector sub space  $S$  of  $\mathbb{R}^n$  with a given basis:  $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k\}$  if some vector  $\mathbf{v} \in S$  is such that for all  $i \leq k$   $\mathbf{v} \cdot \mathbf{u}_i = 0$  then  $\mathbf{v} = \mathbf{0}$

*Proof.* Since  $\mathbf{v} \in S$  we can write:

$$\mathbf{v} = t_1 \mathbf{u}_1 + t_2 \mathbf{u}_2 + \dots + t_k \mathbf{u}_k$$

It is sufficient to show that the magnitude of  $\mathbf{v}$  is 0:

$$\begin{aligned} |\mathbf{v}|^2 &= \mathbf{v} \cdot \mathbf{v} = \mathbf{v} \cdot (t_1 \mathbf{u}_1 + t_2 \mathbf{u}_2 + \dots + t_k \mathbf{u}_k) = t_1 \mathbf{u}_1 \mathbf{v} + t_2 \mathbf{u}_2 \mathbf{v} + \dots + t_k \mathbf{u}_k \mathbf{v} = 0 \\ &\iff |\mathbf{v}| = 0 \\ &\iff \mathbf{v} = \mathbf{0} \end{aligned}$$

□

**Proposition 1:** If an orthogonal projection of  $\mathbf{g}$  onto  $S$  exists it is unique

*Proof.* Let  $\mathbf{a}$  and  $\mathbf{b}$  be orthogonal projections of  $\mathbf{g}$  onto  $S$

By definition for an arbitrary  $i \leq k$  we have:

$$\begin{aligned} 0 &= (\mathbf{g} - \mathbf{a}) \cdot \mathbf{u}_i = (\mathbf{g} - \mathbf{a} + \mathbf{b} - \mathbf{b}) \cdot \mathbf{u}_i = (\mathbf{g} - \mathbf{b} - (\mathbf{a} - \mathbf{b})) \cdot \mathbf{u}_i \\ &= (\mathbf{g} - \mathbf{b}) \cdot \mathbf{u}_i - (\mathbf{a} - \mathbf{b}) \cdot \mathbf{u}_i = -(\mathbf{a} - \mathbf{b}) \cdot \mathbf{u}_i \\ &\iff (\mathbf{a} - \mathbf{b}) \cdot \mathbf{u}_i = 0 \end{aligned}$$

Since  $(\mathbf{a} - \mathbf{b}) \in S$  we can apply lemma 1:

$$\begin{aligned} &\Rightarrow (\mathbf{a} - \mathbf{b}) = \mathbf{0} \\ &\iff \mathbf{a} = \mathbf{b} \end{aligned}$$

□

**Proposition 2:** If  $\mathbf{a} \in S$  is an orthogonal projection of  $\mathbf{g}$  relative to a basis:  $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k\}$  then it is also orthogonal relative to any other basis

*Proof.* suppose that  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$  is also a basis of  $S$  and take an arbitrary  $\mathbf{v}_i$

$$\begin{aligned} (\mathbf{g} - \mathbf{a}) \cdot \mathbf{v}_i &= (\mathbf{g} - \mathbf{a}) \cdot (t_1 \mathbf{u}_1 + t_2 \mathbf{u}_2 + \dots + t_k \mathbf{u}_k) \quad \because \mathbf{v}_i \in S \\ &= t_1 (\mathbf{g} - \mathbf{a}) \cdot \mathbf{u}_1 + t_2 (\mathbf{g} - \mathbf{a}) \cdot \mathbf{u}_2 + \dots + t_k (\mathbf{g} - \mathbf{a}) \cdot \mathbf{u}_k = 0 \\ &\Rightarrow (\mathbf{g} - \mathbf{a}) \cdot \mathbf{v}_i = 0 \end{aligned}$$

Given the results of propositions 1 and 2, orthogonal projections are independant of choice

of basis vectors and are unique. Therefore, the following notation will be introduced for  $\mathbf{a}$  being  $\mathbf{g}$ 's orthogonal projection onto  $S$ :

$$P_S(\mathbf{g}) = \mathbf{a}$$

□