

## Compactness

Many of these problems are from a collection made by Behnam Esmayli, who also references a collection created by Cezar Lupu.

### Common Lemmas:

(1) If  $f : X \rightarrow Y$  is a continuous function between metric spaces  $(X, d)$  and  $(Y, \varrho)$  then  $f(X)$  is bounded.

*Proof.* Assume towards a contradiction that  $f(X)$  is not bounded. Therefore, for a fixed  $y \in f(X)$  we have that for every  $R$  there is some  $f(x)$  such that  $\varrho(y, f(x)) > R$ . Therefore we can construct some sequence  $(x_n)_{n=1}^{\infty}$  such that  $\varrho(f(x_n), y) > n$ . Since  $X$  is compact there must be some  $(x_{n_k})_{k=1}^{\infty}$  with  $x_{n_k} \rightarrow x_0 (\in X)$ . Since  $f$  is continuous, we have that  $f(x_{n_k}) \rightarrow f(x_0) (\in Y)$ . Since these sequences are convergent, it must be that  $\varrho(f(x_{n_k}), y) \rightarrow \varrho(f(x_0), y)$ . However,  $\varrho(x_{n_k}, y) > n_k > k$  for all  $k$ . This is a contradiction and therefore  $f(X)$  is bounded.  $\square$

(2) If  $f : X \rightarrow \mathbb{R}$  with  $X$  compact, then  $f$  must achieve a maximum and minimum value.

*Proof.* It suffices to show only  $f$  achieving a maximum since the cases are the same. Applying the first lemma we know that  $f(X)$  is a bounded subset of  $\mathbb{R}$ . Therefore  $\alpha := \sup f(X)$  exists. Since  $\alpha$  is the supremum of the range of  $f$  for every  $\epsilon$  there must be some  $x$  such that  $\alpha - \epsilon < f(x) \leq \alpha$ . Letting  $\epsilon_n := \frac{1}{n}$  we can construct a sequence  $(x_n)_{n=1}^{\infty}$  with  $f(x_n) \rightarrow \alpha$ . Also since  $X$  is compact there must be some  $(x_{n_k})_{k=1}^{\infty}$  with  $x_{n_k} \rightarrow x_0 (\in X)$ . Since  $f(x_n)$  is convergent, its subsequences must also be convergent to the same limit. And by the continuity of  $f$  we have:

$$f(x_0) \leftarrow f(x_{n_k}) \rightarrow \alpha$$

By the uniqueness of limits  $f(x_0) = \alpha$  is a maximum.  $\square$

### Problems:

(1) For  $f : X \rightarrow \mathbb{R}$  with  $X$  compact and  $\forall x \in X \quad f(x) > 0$ , then there exists  $\delta > 0$  such that for all  $x \in X \quad f(x) \geq \delta$

*Proof.* By applying lemma 2 we know that  $f$  must achieve a minimum for some  $x_0 \in X$  and by hypothesis we know that  $f(x_0) > 0$ . Therefore we can pick  $\delta := f(x_0)$  and we know that for all  $x \in X \quad f(x) \geq f(x_0) = \delta$ .  $\square$

(2) For a set compact subset  $K$  and  $F$  a closed subset of some metric space  $(X, d)$ . Show that if  $K \cap F = \emptyset$  then there exists some  $\delta > 0$  such that:

$$\forall p \in K \quad \forall q \in F \quad d(p, q) \geq \delta$$

*Proof.* Assume towards a contradiction that for all  $\delta > 0$  there exists some  $p \in K$  and  $q \in F$  such that  $d(p, q) < \delta$ . Letting  $\delta_n := \frac{1}{n}$  for any  $n$  we can construct a sequences  $(p_n)_{n=1}^{\infty}$  in  $K$  and  $(q_n)_{n=1}^{\infty}$  in  $F$  such that:

$$d(p_n, q_n) \rightarrow 0$$

Since  $p_n$  is a sequence in a compact set there must exist some subsequence  $(p_{n_k})_{k=1}^{\infty}$  that is convergent to some  $p_0 \in K$ . I claim that the sequence given by  $(q_{n_k})_{k=1}^{\infty}$  is convergent to  $p_0$ .

$$d(q_{n_k}, p_0) \leq d(q_{n_k}, p_{n_k}) + d(p_{n_k}, p_0)$$

and we can fix some  $n_0$  where for all  $n \geq n_0$ :

$$d(q_{n_k}, p_0) \leq d(q_{n_k}, p_{n_k}) + d(p_{n_k}, p_0) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Since the sequence  $q_{n_k}$  is convergent sequence in a closed set, its limit point is necessarily included in the set. Therefore,  $p_0 \in F$ . We can conclude that  $\{p_0\} \subset F \cap K$  which is a contradiction.  $\square$

I am confused here because I seem to have solved this problem without employing the strategy of (1). My idea is that you could fix an element of  $F$  and define a function the is the distance from the points in  $K$  to the fixed element of  $F$ . There is necessarily a positive minimum of this function. For each element of  $F$  there is a fixed minimum value, and we could define another function on  $F$  that ties these elements to their minimum distanced elements in  $F$  and I would claim that this set has a minimum positive value in which case the question would follow but I don't know how to prove the last step!

(3) Let  $A$  be a non singular linear map  $\mathbb{R}^n \rightarrow \mathbb{R}^n$  (an  $n \times n$  matrix) show that there exists some  $\delta > 0$  such that:

$$\|A(x)\| = \|Ax\| \geq \delta\|x\| \quad \forall x \in \mathbb{R}^n$$

*Proof.* Note that since  $A$  is nonsingular:  $\|Ax\| = 0 \iff \|x\| = 0$  since the only vector mapped to the 0 vector is the 0 vector. Therefore we can fix the sphere of radius 1, denoted  $S$ . This is compact since it is a closed and bounded subset of  $\mathbb{R}^n$ . Since  $\|x\|$  and  $\|Ax\|$  are nonzero for  $x \in S$ . We can then define the function  $f : S \rightarrow \mathbb{R}$  as follows:

$$f(x) := \frac{\|Ax\|}{\|x\|}$$

This function is defined over a compact set and is positive for all  $x \in S$ . Therefore, problem (1) applies and there must exist some  $\delta > 0$  such that:

$$f(x) = \frac{\|Ax\|}{\|x\|} \geq \delta \iff \|Ax\| \geq \delta\|x\| \quad (x \in S)$$

It remains to show the above for the rest of  $\mathbb{R}^n$ . Since this inequality holds trivially for the zero vector, fix some arbitrary non zero vector,  $x$ , in  $\mathbb{R}^n$ .  $x$  can be represented as  $t\hat{x}$  with  $\hat{x}$  a unit vector and  $t \in \mathbb{R}$  nonzero. Since  $\hat{x}$  is a unit vector  $\hat{x} \in S$  and we can write:

$$\begin{aligned} \|A(x)\| &= \|A(t\hat{x})\| = t\|A(\hat{x})\| \geq t\delta\|\hat{x}\| = \delta\|x\| \\ &\iff \|Ax\| \geq \delta\|x\| \end{aligned}$$

Therefore the delta found for  $S$  is sufficient for every  $x \in \mathbb{R}^n$ .  $\square$

(4) Let  $K$  and  $F$  be compact and disjoint subsets of  $\mathbb{R}^n$ . Show that there exists open sets  $U$  and  $V$  that contain  $K$  and  $F$ , respectively, such that  $U \cap V = \emptyset$

*Proof.* Since  $K$  and  $F$  are both compact sets we can apply the results of problem (2) and find  $\delta$  such that for all  $k \in K$  and  $f \in F$  it will be that  $d(k, f) \geq \delta$ . By compactness we can also find finite  $\delta/2$  dense sets in both  $K$  and  $F$ . More specifically, there exists:

$$\{k_0, k_1, \dots, k_n\} \subset K \quad \text{and} \quad \{f_0, f_1, \dots, f_\ell\} \subset F$$

With the properties:

$$\forall k \in K \quad \exists i \leq n \quad d(k, k_i) < \frac{\delta}{2} \quad \text{and} \quad \forall f \in F \quad \exists j \leq \ell \quad d(f, f_j) < \frac{\delta}{2}$$

For each element of the  $\delta/2$  coverings define the open sets:

$$U_{k_i} := B\left(k_i, \frac{\delta}{2}\right) \quad U_{f_j} := B\left(f_j, \frac{\delta}{2}\right)$$

Now define the open coverings of  $K$  and  $F$  as:

$$U_K := \bigcup_{i=0}^n U_{k_i} \quad \text{and} \quad U_F := \bigcup_{j=0}^{\ell} U_{f_j}$$

Clearly  $K$  and  $F$  are subsets of  $U_K$  and  $U_F$ , respectively. And  $U_K$  and  $U_F$  are both open as they are finite unions of open sets. We must also have that  $U_K \cap U_F = \emptyset$ . Otherwise there would exist some  $x$  with  $x \in U_K$  and  $x \in U_F$  which would imply the existence of  $k_i \in K$  and  $f_j \in F$  such that:

$$\begin{aligned} d(k_i, x) &< \frac{\delta}{2} \quad \text{and} \quad d(f_j, x) < \frac{\delta}{2} \\ \Rightarrow d(k_i, f_j) &\leq d(k_i, x) + d(x, f_j) < \delta \end{aligned}$$

Which is a contradiction with  $\delta$  being chosen such that every element in  $K$  and  $F$  are  $\delta$  distance apart.  $\square$

The next theme that is described in the collection is that taking limits gives items inside of compact sets!

**(5)** If  $K \subset \mathbb{R}^n$  is compact, then there is a farthest point of  $K$  to the origin.

*Proof.* Begin by defining the set  $A$  as follows:

$$A := \{x \in \mathbb{R} : \exists k \in K \quad d(\mathbf{0}, k) = x\}$$

Since  $K$  is compact it is bounded,  $A$  is also bounded above because of this fact. Therefore we can define  $\alpha := \sup A$  ( $\in \mathbb{R}$ ). We can also define a sequence  $(x_n)_{n=1}^{\infty}$  in  $A$  with  $x_n \rightarrow \alpha$ . This sequence yields a sequence  $(k_n)_{n=1}^{\infty}$  in  $K$  such that  $d(k_n, \mathbf{0}) \rightarrow \alpha$ . Since  $k_n$  is a sequence in a compact set there exists a further subsequence  $(k_{n_k})_{k=1}^{\infty}$  such that  $k_{n_k} \rightarrow k_0$  ( $\in K$ ). Since the distance metric is a continuous function and because of the properties of the sequence that  $k_{n_k}$  is a subsequence of we have:

$$d(k_0, \mathbf{0}) \leftarrow d(k_{n_k}, \mathbf{0}) \rightarrow \alpha$$

By the uniqueness of limits,  $d(k_0, \mathbf{0}) = \alpha$  and this is a maximal distance from the origin!  $\square$