

## Compactness

Many of these problems are from a collection made by Behnam Esmayli, who also references a collection created by Cezar Lupu.

### Common Lemmas:

(1) If  $f : X \rightarrow Y$  is a continuous function between metric spaces  $(X, d)$  and  $(Y, \rho)$  then  $f(X)$  is bounded.

*Proof.* Assume towards a contradiction that  $f(X)$  is not bounded. Therefore, for a fixed  $y \in f(X)$  we have that for every  $R$  there is some  $f(x)$  such that  $\rho(y, f(x)) > R$ . Therefore we can construct some sequence  $(x_n)_{n=1}^{\infty}$  such that  $\rho(f(x_n), y) > n$ . Since  $X$  is compact there must be some  $(x_{n_k})_{k=1}^{\infty}$  with  $x_{n_k} \rightarrow x_0 (\in X)$ . Since  $f$  is continuous, we have that  $f(x_{n_k}) \rightarrow f(x_0) (\in Y)$ . Since these sequences are convergent, it must be that  $\rho(f(x_{n_k}), y) \rightarrow \rho(f(x_0), y)$ . However,  $\rho(x_{n_k}, y) > n_k > k$  for all  $k$ . This is a contradiction and therefore  $f(X)$  is bounded.  $\square$

(2) If  $f : X \rightarrow \mathbb{R}$  with  $X$  compact, then  $f$  must achieve a maximum and minimum value.

*Proof.* It suffices to show only  $f$  achieving a maximum since the cases are the same. Applying the first lemma we know that  $f(X)$  is a bounded subset of  $\mathbb{R}$ . Therefore  $\alpha := \sup f(X)$  exists. Since  $\alpha$  is the supremum of the range of  $f$  for every  $\epsilon$  there must be some  $x$  such that  $\alpha - \epsilon < f(x) \leq \alpha$ . Letting  $\epsilon_n := \frac{1}{n}$  we can construct a sequence  $(x_n)_{n=1}^{\infty}$  with  $f(x_n) \rightarrow \alpha$ . Also since  $X$  is compact there must be some  $(x_{n_k})_{k=1}^{\infty}$  with  $x_{n_k} \rightarrow x_0 (\in X)$ . Since  $f(x_n)$  is convergent, its subsequences must also be convergent to the same limit. And by the continuity of  $f$  we have:

$$f(x_0) \leftarrow f(x_{n_k}) \rightarrow \alpha$$

By the uniqueness of limits  $f(x_0) = \alpha$  is a maximum.  $\square$

(3) If  $f : X \rightarrow Y$  is continuous with  $X$  compact and  $Y$  a metric space then the image of  $f$  is compact.

*Proof.* Take an arbitrary sequence  $(f(x_n))_{n=1}^{\infty}$  in  $\text{Im} f$ . There must exist some convergent subsequence  $x_{n_k}$  of  $x_n$  with  $x_{n_k} \rightarrow x_0 (\in X)$ . By the continuity of  $f$  we know that  $f(x_{n_k}) \rightarrow f(x_0)$  which proves the compactness of  $\text{Im} f$ .  $\square$

(4) A sequence  $(x_n)_{n=1}^{\infty}$  being convergent to a point  $x_0$  is equivalent to every subsequence  $(x_{n_k})$  of  $(x_n)$  having a convergent subsequence  $x_{n_{k_\ell}} \rightarrow x_0$ .

*Proof.* This first way is easy since if  $(x_n)$  is convergent to  $x_0$  then its subsequences are as well, and subsequences of subsequences, are, of course, subsequences so they all converge to  $x_0$  as well. To show the other way, I will use contraposition. Consider the negation of  $x_n \rightarrow x_0$ :

$$\exists \epsilon > 0 \quad \forall N_0 \quad \exists n \geq N_0 \quad d(x_n, x_0) \geq \epsilon$$

We can use the above rule to inductively define a subsequence  $(x_{n_k})$  of  $(x_n)$  such that for all  $k$ ,  $d(x_{n_k}, x_0) \geq \epsilon$ . This clearly cannot have a subsequence convergent to  $x_0$  since all of its elements are of a minimum distance of  $\epsilon$  from  $x_0$ . We have shown the negation of the second statement which yields the desired equivalence.  $\square$

**Problems:**

(1) For  $f : X \rightarrow \mathbb{R}$  with  $X$  compact and  $\forall x \in X \quad f(x) > 0$ , then there exists  $\delta > 0$  such that for all  $x \in X \quad f(x) \geq \delta$

*Proof.* By applying lemma 2 we know that  $f$  must achieve a minimum for some  $x_0 \in X$  and by hypothesis we know that  $f(x_0) > 0$ . Therefore we can pick  $\delta := f(x_0)$  and we know that for all  $x \in X \quad f(x) \geq f(x_0) = \delta$ .  $\square$

(2) For a set compact subset  $K$  and  $F$  a closed subset of some metric space  $(X, d)$ . Show that if  $K \cap F = \emptyset$  then there exists some  $\delta > 0$  such that:

$$\forall p \in K \quad \forall q \in F \quad d(p, q) \geq \delta$$

*Proof.* Assume towards a contradiction that for all  $\delta > 0$  there exists some  $p \in K$  and  $q \in F$  such that  $d(p, q) < \delta$ . Letting  $\delta_n := \frac{1}{n}$  for any  $n$  we can construct a sequences  $(p_n)_{n=1}^\infty$  in  $K$  and  $(q_n)_{n=1}^\infty$  in  $F$  such that:

$$d(p_n, q_n) \rightarrow 0$$

Since  $p_n$  is a sequence in a compact set there must exist some subsequence  $(p_{n_k})_{k=1}^\infty$  that is convergent to some  $p_0 \in K$ . I claim that the sequence given by  $(q_{n_k})_{k=1}^\infty$  is convergent to  $p_0$ .

$$d(q_{n_k}, p_0) \leq d(q_{n_k}, p_{n_k}) + d(p_{n_k}, p_0)$$

and we can fix some  $n_0$  where for all  $n \geq n_0$ :

$$d(q_{n_k}, p_0) \leq d(q_{n_k}, p_{n_k}) + d(p_{n_k}, p_0) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Since the sequence  $q_{n_k}$  is convergent sequence in a closed set, its limit point is necessarily included in the set. Therefore,  $p_0 \in F$ . We can conclude that  $\{p_0\} \subset F \cap K$  which is a contradiction.  $\square$

I am confused here because I seem to have solved this problem without employing the strategy of (1). My idea is that you could fix an element of  $F$  and define a function that is the distance from the points in  $K$  to the fixed element of  $F$ . There is necessarily a positive minimum of this function. For each element of  $F$  there is a fixed minimum value, and we could define another function on  $F$  that ties these elements to their minimum distanced elements in  $F$  and I would claim that this set has a minimum positive value in which case the question would follow but I don't know how to prove the last step!

(3) Let  $A$  be a non singular linear map  $\mathbb{R}^n \rightarrow \mathbb{R}^n$  (an  $n \times n$  matrix) show that there exists some  $\delta > 0$  such that:

$$\|A(x)\| = \|Ax\| \geq \delta \|x\| \quad \forall x \in \mathbb{R}^n$$

*Proof.* Note that since  $A$  is nonsingular:  $\|Ax\| = 0 \iff \|x\| = 0$  since the only vector mapped to the 0 vector is the 0 vector. Therefore we can fix the sphere of radius 1, denoted  $S$ . This is compact since it is a closed and bounded subset of  $\mathbb{R}^n$ . Since  $\|x\|$  and  $\|Ax\|$  are nonzero for  $x \in S$ . We can then define the function  $f : S \rightarrow \mathbb{R}^n$  as follows:

$$f(x) := \frac{\|Ax\|}{\|x\|}$$

This function is defined over a compact set and is positive for all  $x \in S$ . Therefore, problem (1) applies and there must exist some  $\delta > 0$  such that:

$$f(x) = \frac{\|Ax\|}{\|x\|} \geq \delta \iff \|Ax\| \geq \delta \|x\| \quad (x \in S)$$

It remains to show the above for the rest of  $\mathbb{R}^n$ . Since this inequality holds trivially for the zero vector, fix some arbitrary non zero vector,  $x$ , in  $\mathbb{R}^n$ .  $x$  can be represented as  $t\hat{x}$  with  $\hat{x}$  a unit vector and  $t \in \mathbb{R}$  nonzero. Since  $\hat{x}$  is a unit vector  $\hat{x} \in S$  and we can write:

$$\begin{aligned} \|A(x)\| &= \|A(t\hat{x})\| = t\|A(\hat{x})\| \geq t\delta\|\hat{x}\| = \delta\|x\| \\ &\iff \|Ax\| \geq \delta\|x\| \end{aligned}$$

Therefore the delta found for  $S$  is sufficient for every  $x \in \mathbb{R}^n$ .  $\square$

**(4)** Let  $K$  and  $F$  be compact and disjoint subsets of  $\mathbb{R}^n$ . Show that there exists open sets  $U$  and  $V$  that contain  $K$  and  $F$ , respectively, such that  $U \cap V = \emptyset$

*Proof.* Since  $K$  and  $F$  are both compact sets we can apply the results of problem **(2)** and find  $\delta$  such that for all  $k \in K$  and  $f \in F$  it will be that  $d(k, f) \geq \delta$ . By compactness we can also find finite  $\delta/2$  dense sets in both  $K$  and  $F$ . More specifically, there exists:

$$\{k_0, k_1, \dots, k_n\} \subset K \quad \text{and} \quad \{f_0, f_1, \dots, f_\ell\} \subset F$$

With the properties:

$$\forall k \in K \quad \exists i \leq n \quad d(k, k_i) < \frac{\delta}{2} \quad \text{and} \quad \forall f \in F \quad \exists j \leq \ell \quad d(f, f_j) < \frac{\delta}{2}$$

For each element of the  $\delta/2$  coverings define the open sets:

$$U_{k_i} := B\left(k_i, \frac{\delta}{2}\right) \quad U_{f_j} := B\left(f_j, \frac{\delta}{2}\right)$$

Now define the open coverings of  $K$  and  $F$  as:

$$U_K := \bigcup_{i=0}^n U_{k_i} \quad \text{and} \quad U_F := \bigcup_{j=0}^{\ell} U_{f_j}$$

Clearly  $K$  and  $F$  are subsets of  $U_K$  and  $U_F$ , respectively. And  $U_K$  and  $U_F$  are both open as they are finite unions of open sets. We must also have that  $U_K \cap U_F = \emptyset$ . Otherwise there would exist some  $x$  with  $x \in U_K$  and  $x \in U_F$  which would imply the existence of  $k_i \in K$  and  $f_j \in F$  such that:

$$\begin{aligned} d(k_i, x) &< \frac{\delta}{2} \quad \text{and} \quad d(f_j, x) < \frac{\delta}{2} \\ \Rightarrow d(k_i, f_j) &\leq d(k_i, x) + d(x, f_j) < \delta \end{aligned}$$

Which is a contradiction with  $\delta$  being chosen such that every element in  $K$  and  $F$  are  $\delta$  distance apart.  $\square$

The next theme that is described in the collection is that taking limits gives items inside of compact sets!

**(5)** If  $K \subset \mathbb{R}^n$  is compact, then there is a farthest point of  $K$  to the origin.

*Proof.* Begin by defining the set  $A$  as follows:

$$A := \{x \in \mathbb{R} : \exists k \in K \quad d(\mathbf{0}, k) = x\}$$

Since  $K$  is compact it is bounded,  $A$  is also bounded above because of this fact. Therefore we can define  $\alpha := \sup A$  ( $\in \mathbb{R}$ ). We can also define a sequence  $(x_n)_{n=1}^{\infty}$  in  $A$  with  $x_n \rightarrow \alpha$ . This sequence yields a sequence  $(k_n)_{n=1}^{\infty}$  in  $K$  such that  $d(k_n, \mathbf{0}) \rightarrow \alpha$ . Since  $k_n$  is a sequence in a compact set there exists a further subsequence  $(k_{n_k})_{k=1}^{\infty}$  such that  $k_{n_k} \rightarrow k_0$  ( $\in K$ ). Since the distance metric

is a continuous function and because of the properties of the sequence that  $k_{n_k}$  is a subsequence of we have:

$$d(k_0, \mathbf{0}) \leftarrow d(k_{n_k}, \mathbf{0}) \rightarrow \alpha$$

By the uniqueness of limits,  $d(k_0, \mathbf{0}) = \alpha$  and this is a maximal distance from the origin!  $\square$

**(6)** For  $K \subset \mathbb{R}^n$  a compact set that is disjoint from a closed set  $F$ , then there is a closest point of  $K$  to the set  $F$ .

*Proof.* By applying the result of problem **(2)** we know that there is some  $\delta > 0$  such that for all  $k \in K$  and  $f \in F$   $d(k, f) \geq \delta$ . We can then define the set:

$$A := \{x \in \mathbb{R} : \exists k \in K \exists f \in F \ d(f, k) = x\}$$

By the above, we know that  $A$  is bounded below by some positive  $\delta$ . Therefore  $\beta := \inf A$  exists and is positive as well. As before, we can fix sequences  $(k_n)_{n=1}^\infty$  and  $(f_n)_{n=1}^\infty$  such that  $d(k_n, f_n) \rightarrow \beta$ . Since  $(k_n)$  is a sequence in a compact set it omits a sequence  $(k_{n_\ell})_{\ell=1}^\infty$  which is convergent to some  $k_0 \in K$ . We know that  $d(k_{n_\ell}, f_{n_\ell}) \rightarrow \beta$ . It is also true that  $d(k_0, f_{n_\ell}) \rightarrow \beta$  since:

$$d(k_0, f_{n_\ell}) - \beta \leq d(k_0, k_{n_\ell}) + d(k_{n_\ell}, f_{n_\ell}) - \beta < \frac{\epsilon}{2} + \frac{\epsilon}{2} + \beta - \beta = \epsilon$$

and:

$$\begin{aligned} \beta - d(k_0, f_{n_\ell}) &= \beta - (k_0, f_{n_\ell}) + d(k_0, k_{n_\ell}) - d(k_0, k_{n_\ell}) \\ &= \beta + d(k_0, k_{n_\ell}) - ((k_0, f_{n_\ell}) + d(k_0, k_{n_\ell})) \leq \beta + d(k_0, k_{n_\ell}) - d(k_{n_\ell}, f_{n_\ell}) \\ &< \beta + \frac{\epsilon}{2} - (\beta - \frac{\epsilon}{2}) = \epsilon \end{aligned}$$

Since  $(f_{n_\ell})$  is within  $\beta$  of  $k_0$  in its limit then for some  $N_0$  with  $\ell \geq N_0$  the sequence  $(f_{n_\ell})_{\ell \geq N_0}^\infty$  must be contained in the closed ball,  $B$ , of radius  $2\beta$  around  $k_0$ . Since  $B$  is closed and bounded in  $\mathbb{R}^n$  then it is compact and therefore  $(f_{n_\ell})_{\ell \geq N_0}^\infty$  must have a convergent subsequence  $f_{n_{\ell_g}} \rightarrow f_0$  but since this is also a sequence in  $F$  which is closed  $f_0$  must be contained in  $F$ . I will now show that  $d(k_0, f_0) = \beta$ :

$$d(k_0, f_0) - \beta \leq d(k_0, f_{n_{\ell_g}}) + d(f_{n_{\ell_g}}, f_0) - \beta$$

Since we know that  $d(k_0, f_{n_\ell}) \rightarrow \beta$  we can write:

$$\leq \beta + \frac{\epsilon}{2} + \frac{\epsilon}{2} - \beta = \epsilon$$

Also:

$$\begin{aligned} \beta - d(k_0, f_0) &\leq \beta - d(k_0, f_0) - d(f_0, f_{n_{\ell_g}}) + d(f_0, f_{n_{\ell_g}}) \\ &= \beta + d(f_0, f_{n_{\ell_g}}) - (d(k_0, f_0) + d(f_0, f_{n_{\ell_g}})) \leq \beta + d(f_0, f_{n_{\ell_g}}) - d(k_0, f_{n_{\ell_g}}) \\ &< \beta + \frac{\epsilon}{2} - (\beta - \frac{\epsilon}{2}) = \epsilon \\ |\beta - d(k_0, f_0)| &< \epsilon \quad \text{for all } \epsilon > 0 \\ &\iff \beta = d(k_0, f_0) \end{aligned}$$

We have found points in  $F$  and  $K$  which are at a distance apart equal to the infimum of all the distances between  $K$  and  $F$  which is positive!  $\square$

**(7)** Let  $X$  be a metric space and  $f : X \rightarrow X$  be a contraction mapping. Suppose that  $K \subset X$  is a compact and nonempty set that satisfies  $f(K) = K$ . Prove that  $K$  contains one single point.

*Proof.* Note that since compact sets are bounded they have a finite diameter which is defined to be the supremum of all of the distances of all of the pairs of points in the set. For a compact set  $C$  this is denoted as  $\text{diam } C$ . Also note that for a given compact set  $C$  there exist points in  $C$  which achieve this diameter. Therefore if we assume that  $K$  has more than one point there exists  $x^*, y^* \in K$  such that  $d(x^*, y^*) = \text{diam } K > 0$ . If we were to consider  $f(K)$  we can easily show that this is compact.

To do this fix some arbitrary sequence in  $(y_n)_{n=1}^\infty \subset f(K)$ . This induces some  $(x_n)_{n=1}^\infty$  in  $K$ . This has a convergent subsequence  $x_{n_k} \rightarrow x_0 \in K$ . Since  $f$  is a contraction mapping we know:

$$\begin{aligned} f(f(x_0), f(x_{n_k})) &\leq kd(x_0, x_{n_k}) \rightarrow 0 \quad (k \in [0, 1)) \\ \Rightarrow y_{n_k} &= f(x_{n_k}) \rightarrow f(x_0) \in f(K) \end{aligned}$$

Since  $f(K)$  is also compact we can conclude that there must also be points  $x^{**}, y^{**} \in f(K)$  such that  $d(x^{**}, y^{**}) = \text{diam } f(K)$ . Also there must exist  $x_0, y_0 \in K$  for which  $x^{**} = f(x_0)$  and  $y^{**} = f(y_0)$

$$\begin{aligned} d(x^{**}, y^{**}) &= d(f(x_0), f(y_0)) < d(x_0, y_0) \leq \text{diam } K \\ \Rightarrow \text{diam } f(K) &< \text{diam } K \Rightarrow K \neq f(K) \end{aligned}$$

Therefore,  $K$  cannot have multiple points! □

(8) Let  $r : [a, b] \rightarrow \mathbb{R}^n$  be a continuous curve (the path of a particle). Suppose that  $r(a) = (0, 0, \dots, 0)$  and  $r(b) = (1, 1, \dots, 1)$ . Prove that there exists a last time the particle touches the surface of the unit sphere.

*Proof.* Since  $r$  is continuous with a point inside the unit sphere and a point outside of the unit sphere we know that there exists at least one point where the particle is on the unit sphere. Therefore we can define the set:

$$A := \{x \in \mathbb{R}^n : d(r(x), \mathbf{0}) = 1\}$$

This is nonempty and bounded above and below by  $b$ , and  $a$  respectively! Therefore the supremum and infimums of  $A$  exist and are themselves defined as inputs of  $r$ . Let  $\alpha := \inf A$  and  $\beta := \sup A$ . The notion that  $f(\alpha)$  is on the unit sphere is intuitively obvious and not the question so I will only prove that  $f(\beta)$  is on the unit sphere. The proofs are also exactly the same!

Assume towards a contradiction that  $f(\beta)$  is not on the unit sphere  $\Rightarrow d(f(\beta), \mathbf{0}) \neq 1$ . Therefore we are left with two cases:

[Case 1]  $d(f(\beta), \mathbf{0}) < 1$

Because of the above, there must exist an  $\epsilon$  such that  $d(f(x), \mathbf{0}) < 1 - \epsilon$ . We can then consider the ball  $B$  of radius  $\epsilon$  centered at  $f(\beta)$ . Elements of this ball are also within the unit ball and therefore of distance less than 1 of the origin. Because for  $\gamma \in B(f(\beta), \epsilon)$

$$d(\mathbf{0}, \gamma) \leq d(\mathbf{0}, f(\beta)) + d(f(\beta), \gamma) < 1 - \epsilon + \epsilon = 1$$

By the continuity of  $r$  there must exist some  $\delta$  where if we consider  $\xi$  with  $|x - \xi| < \delta$ . And if we let  $\delta'$  be the minimum of  $\delta$  and  $b - \beta$  we can find some  $\xi < \beta$  such that  $d(f(\xi), \mathbf{0}) < 1$ . The existence of  $\xi$  contradicts  $\beta$  being the supremum of  $A$ .

[Case 2:]  $d(f(\beta), \mathbf{0}) > 1$

The argument is exactly the same.

Therefore  $\beta$  must be on the unit sphere and is the "last" time the particle is on the unit sphere. □

The next theme is about every sequence having a convergent subsequence which converges in the set!

(9) For  $A \subset \mathbb{R}$  with  $A$  compact and  $x \in A$ . Suppose that every convergent subsequence of  $(x_n)_{n \in \mathbb{N}}$

converges to  $x$ . Show that the entire sequence converges to  $x$ . This was kind of confusing since it did not state that the sequence was in the compact set, but eh whatever.

*Proof.* An equivalence to the definition of convergence is that, for a sequence  $(x_n)_{n \in \mathbb{N}}$  if there is some  $x_0$  where every sequence  $(x_{n_k})_{k \in \mathbb{N}}$  has a further subsequence  $(x_{n_{k_\ell}})_{\ell \in \mathbb{N}}$  with  $x_{n_{k_\ell}} \rightarrow x_0$ . Then we can conclude  $x_n \rightarrow x_0$ . These facts are equivalent, I'll include it as a lemma (4 probably). To apply this fact to solve this problem fix an arbitrary subsequence  $(x_{n_k})$  of  $(x_n)$ . Since  $(x_{n_k})$  is a sequence in  $A$  which is compact there is a convergent subsequence of it  $(x_{n_{k_\ell}})$ . Since  $(x_{n_{k_\ell}})$  is itself a subsequence of  $(x_n)$ , by hypothesis we know that  $x_{n_{k_\ell}} \rightarrow x$ . Since every sequence has a further subsequence convergent to the same point  $(x_n)$  is convergent!  $\square$

(10) Let  $X$  be a compact metric space and  $f : X \rightarrow X$  be a contraction map:

$$\forall x, y \quad d(f(x), f(y)) < d(x, y)$$

Show that  $f$  has a unique fixed point

*Proof.* Let the set  $\Psi$  be defined as:

$$\Psi := \{\gamma \in \mathbb{R} : \exists x \quad \gamma = d(x, f(x))\}$$

We know that  $A$  is bounded below by 0 therefore  $\alpha := \inf \Psi$  exists. I will show that there exists a point  $x$  such that  $d(x, f(x)) = \alpha$ . Because  $\alpha$  is an infimum we can find a sequence  $(x_n)$  with  $d(x_n, f(x_n)) \rightarrow \alpha$ . Since  $x_n$  is a sequence in  $X$ , which is compact we have  $x_{n_k} \rightarrow x_0$  and  $d(x_0, f(x_0)) = \alpha$ . With this information we must conclude that  $\alpha = 0$ . If  $\alpha > 0$  then  $x_0 \neq f(x_0)$  and since  $x_0, f(x_0) \in X$  we can calculate  $f(x_0), f(f(x_0)) \in X$ . By the contraction property of  $f$  we know that  $d(f(x_0), f(f(x_0))) < d(x_0, f(x_0)) = \alpha$  but this cannot be since  $\alpha$  is a lower bound. We must conclude that  $\alpha = 0$  and therefore  $d(x_0, f(x_0)) = 0 \iff f(x_0) = x_0$ .

To show uniqueness, assume that two points  $x_0, y_0$  satisfy this property. Construct the set  $\{x_0, y_0\}$  which is a compact set. We should have that  $f(\{x_0, y_0\}) = \{x_0, y_0\}$  but this is a contradiction with the results of problem (7).  $\square$

The next theme is about continuous functions on compact sets are automatically uniformly continuous! Ie for  $f : X \rightarrow Y$  with  $X$  compact, then  $f$  is uniformly continuous.

*Proof.* Assume the negation of uniform continuity:

$$\exists \epsilon > 0 \quad \forall \delta > 0 \quad \exists x, y \in X \quad d(x, y) < \delta \quad \text{and} \quad d(f(x), f(y)) \geq \epsilon$$

Using this rule we can then let  $\delta_n := \frac{1}{n}$  and find sequences  $(x_n)$  and  $(y_n)$  with  $d(x_n, y_n) \rightarrow 0$  and  $d(f(x_n), f(y_n)) \geq \epsilon$ . By compactness of  $X$  there exists some  $(x_{n_k}) \rightarrow x_0$  and we have that  $d(x_0, y_{n_k}) \rightarrow 0$  which means that  $y_{n_k} \rightarrow x_0$ . However  $d(f(x_{n_k}), f(y_{n_k})) \geq \epsilon$  for all  $k$  and this is a contradiction.  $\square$

(11) Let  $X$  be a compact subset of  $\mathbb{R}^n$ , and  $f : X \rightarrow \mathbb{R}$  be continuous. Prove that any given  $\epsilon > 0$  there exists some  $M$  such that for all  $x, y$ :

$$|f(x) - f(y)| \leq M|x - y| + \epsilon$$

*Proof.* We know that since  $f$  is defined over a compact space and is continuous it must be uniformly continuous. Letting  $\epsilon$  be given we can find some  $\delta_\epsilon > 0$  where for all  $x, y$ :

$$d(x, y) < \delta_\epsilon \Rightarrow d(f(x), f(y)) < \epsilon$$

For those  $x, y$  with  $d(x, y) \geq \delta_\epsilon$   $X$  and  $\text{Im } f$  are compact we have the inequalities:

$$\delta \leq |x - y| \leq \text{diam } X \quad \text{and} \quad d(f(x), f(y)) < \text{diam } \text{Im } f$$

If we let  $M$  be such that  $M\delta_\epsilon > \text{diam Im } f$  then for arbitrary  $x, y$ :  
 If  $d(x, y) < \delta_\epsilon$ :

$$\varrho(f(x), f(y)) < \epsilon \leq Md(x, y) + \epsilon$$

If  $d(x, y) \geq \delta_\epsilon$ :

$$\varrho(f(x), f(y)) < \text{diam Im } f < M\delta_\epsilon \leq Md(x, y) + \epsilon$$

□

Slightly confused here, was it meant that  $M$  is dependant on  $\epsilon$ ? Also I did not utilize any of the properties from  $\mathbb{R}$  or  $\mathbb{R}^n$  and this works for continuous functions defined over any compact spaces.

**(12)** Suppose that  $X$  is a compact metric space,  $Y$  is a metric space and  $f : X \rightarrow Y$  is one-to-one, onto, and continuous. Show that  $f^{-1} : Y \rightarrow X$  is continuous. Is the compactness of  $X$  necessary?

*Proof.* Since we know that  $f^{-1}$  exists we can fix some arbitrary sequence  $(y_n)_{n=1}^\infty$  convergent to  $y_0$  in  $\text{Im } f$ .  $f^{-1}$  is continuous iff:

$$f^{-1}(y_n) \rightarrow f^{-1}(y_0)$$

We know that for all  $n$ ,  $y_n = f(x_n)$  and  $y_0 = f(x_0)$ . So,  $y_n$  omits a sequence  $(x_n)$  in  $X$ . Furthermore we can equivalently state the above as:

$$f^{-1}(y_n) \rightarrow f^{-1}(y_0) \iff x_n \rightarrow x_0$$

Suppose that  $x_n \not\rightarrow x_0$ . This implies the existance of some subsequence  $(x_{n_k})$  such that for all  $k$ ,  $d(x_{n_k}, x_0) \geq \epsilon$ . There must be a subsequence  $(x_{n_{k_\ell}})$  that is convergent to  $x_1 \neq x_0$  because  $X$  is compact. Since  $y_n \rightarrow y_0 \iff f(x_n) \rightarrow f(x_0)$  it must be that  $f(x_{n_{k_\ell}}) \rightarrow f(x_0)$  but also by the continuity of  $f$  we have that  $f(x_{n_{k_\ell}}) \rightarrow f(x_1)$  which are not the same point by the one to one property of  $f$ . This is a contradiction and we have that:

$$x_n \rightarrow x_0 \iff f^{-1}(y_n) \rightarrow f^{-1}(y_0)$$

Which shows the continuity of  $f^{-1}$ .

□

**(12)** This counter example shows why the above requires compactness. Take the interval  $[0, 1)$  and wrap it around the unit circle  $\mathbb{S}^1$ . Letting  $f : [0, 1) \rightarrow \mathbb{S}^1$  be this continuous mapping with  $\lim_{x \rightarrow 1^-} f(x) = f(0)$ .

*Proof.* Such a mapping can be parameterized by the following:

$$f : [0, 1) \rightarrow \mathbb{S}^1 \quad f(t) := (\cos(2\pi t), \sin(2\pi t))$$

The surjectivity and injectivity properties of  $f$  follow from this, as does continuity. However, if we consider  $f^{-1} : \mathbb{S}^1 \rightarrow [0, 1)$  and in particular at the point  $(1, 0)$ . For any gap  $\delta$  around  $(0, 1)$  we can find points on  $[0, 1)$  close to 0 that map within  $\delta$  of  $(0, 1)$  and points close to 1 which map to within  $\delta$  of  $(0, 1)$ . If it was required that  $[0, 1)$  had to be compact then to preserve the injectivity of  $f$  the endpoints of the function would have to map to different locations and the above contradiction would not be obtained.

□