Compactness

Many of these problems are from a collection made by Behnam Esmayli, who also references a collection created by Cezar Lupu.

Common Lemmas:

(1) If $f : X \to Y$ is a continuious function between metric spaces (X, d) and (Y, ϱ) then $f(X)$ is bounded.

Proof. Assume towards a contradiction that $f(X)$ is not bounded. Therefore, for a fixed $y \in f(X)$ we have that for every R there is some $f(x)$ such that $\varrho(y, f(x)) > R$. Therefore we can construct some sequence $(x_n)_{n=1}^{\infty}$ such that $\varrho(f(x_n), y) > n$. Since X is compact there must be some $(x_{n_k})_{k=1}^{\infty}$ with $x_{n_k} \to x_0 \, (\in X)$. Since f is continuous, we have that $f(x_{n_k}) \to f(x_0) \, (\in Y)$. Since these sequences are covergent, it must be that $\varrho(f(x_{n_k}), y) \to \varrho(f(x_0), y)$. However, $\varrho(x_{n_k}, y) > n_k > k$ for all k. This is a contradiction and therefore $f(X)$ is bounded. $\square$

(2) If $f : X \to \mathbb{R}$ with X compact, then f must achieve a maximum and minimum value.

Proof. It suffices to show only f achieving a maximum since the cases are the same. Applying the first lemma we know that $f(X)$ is a bounded subset of $\mathbb{R}$. Therefore $\alpha := \sup f(X)$ exists. Since α is the supremum of the range of f for every ϵ there must be some x such that $\alpha - \epsilon < f(x) \leq \alpha$. Letting $\epsilon_n := \frac{1}{n}$ we can construct a sequence $(x_n)_{n=1}^{\infty}$ with $f(x_n) \to \alpha$. Also since X is compact there must be some $(x_{n_k})_{k=1}^{\infty}$ with $x_{n_k} \to x_0 \, (\in X)$. Since $f(x_n)$ is convergent, its subsequences must also be convergent to the same limit. And by the continuity of f we have:

$$f(x_0) \leftarrow f(x_{n_k}) \to \alpha$$

By the uniqueness of limits $f(x_0) = \alpha$ is a maximum. $\square$

Problems:

(1) For $f : X \to \mathbb{R}$ with X compact and $\forall x \in X \quad f(x) > 0$, then there exists $\delta > 0$ such that for all $x \in X \quad f(x) \geq \delta$

Proof. By applying lemma 2 we know that f must achieve a minimum for some $x_0 \in X$ and by hypothesis we know that $f(x_0) > 0$. Therefore we can pick $\delta := f(x_0)$ and we know that for all $x \in X \quad f(x) \geq f(x_0) = \delta$. $\square$