Ph.D. Preliminary Examination (Analysis)

August, 2011

INSTRUCTIONS: *Do all six problems. In order to receive maximum credit, solutions to problems must be clearly and carefully presented and should contain the necessary details. All problems are worth the same number of points.*

1. Let $\beta > 0$ and $\{u_n\}$ be a sequence of positive real numbers such that $\dfrac{u_{n+1}}{u_n} \le \beta$ for every $n \in \mathbf{N}$. Prove that

$$\limsup_{n\to\infty} \sqrt[n]{u_n} \le \limsup_{n\to\infty} \left(\frac{u_{n+1}}{u_n}\right).$$

2. Let $f : [0,1] \to \mathbf{R}$ be continuously differentiable on $[0,1]$ and satisfy $f(1) = 0$. Show that

$$\int_0^1 |f(x)|^2 dx \le 4 \int_0^1 x^2 |f'(x)|^2 dx.$$

3. Let (M,d) be a compact metric space and $f : M \to \mathbf{R}$ be a continuous function on M. For any given $\varepsilon > 0$, show that there exists an $\omega = \omega(\varepsilon, f) > 0$ such that

$$|f(x) - f(y)| \le \omega d(x,y) + \varepsilon$$

holds for all $x, y \in M$.

4. Let $n, m \in \mathbf{N}$ and $f : \mathbf{R}^n \to \mathbf{R}^m$ be C^1 on $\mathbf{R}^n$. Suppose that $Df(0)$ is one-to-one. Prove that there exists an open neighborhood U of the orgin in $\mathbf{R}^n$ such that f is one-to-one on U.

5. Let $f : [0,1]^2 \to [0,\infty)$ be Riemann integrable over $[0,1]^2$. Suppose that

$$\int_{[0,1]^2} f = 0.$$

Prove that $\{(x,y) \in [0,1]^2 : f(x,y) > 0\}$ is a set of measure zero.

6. Let $v : \mathbf{R}^2 \to \mathbf{R}$ be a C^2 function such that v is nonconstant on $\Omega = \{(x,y) : x^2 + y^2 \le 1\}$ and $v(x,y) = 0$ for all $(x,y) \in \partial\Omega$. Suppose that for some $\lambda \in \mathbf{R}$ we have

$$\Delta v(x,y) = \lambda v(x,y)$$

for every $(x,y) \in \Omega$. Prove that $\lambda < 0$.