Compactness

Many of these problems are from a collection made by Behnam Esmayli, who also references a collection created by Cezar Lupu.

Common Lemmas:

(1) If $f : X \to Y$ is a continuious function between metric spaces (X, d) and (Y, ϱ) then $f(X)$ is bounded.

Proof. Assume towards a contradiction that $f(X)$ is not bounded. Therefore, for a fixed $y \in f(X)$ we have that for every R there is some $f(x)$ such that $\varrho(y, f(x)) > R$. Therefore we can construct some sequence $(x_n)_{n=1}^{\infty}$ such that $\varrho(f(x_n), y) > n$. Since X is compact there must be some $(x_{n_k})_{k=1}^{\infty}$ with $x_{n_k} \to x_0\,(\in X)$. Since f is continuous, we have that $f(x_{n_k}) \to f(x_0)\,(\in Y)$. Since these sequences are covergent, it must be that $\varrho(f(x_{n_k}), y) \to \varrho(f(x_0), y)$. However, $\varrho(x_{n_k}, y) > n_k > k$ for all k. This is a contradiction and therefore $f(X)$ is bounded. $\square$

(2) If $f : X \to \mathbb{R}$ with X compact, then f must achieve a maximum and minimum value.

Proof. It suffices to show only f achieving a maximum since the cases are the same. Applying the first lemma we know that $f(X)$ is a bounded subset of $\mathbb{R}$. Therefore $\alpha := \sup f(X)$ exists. Since α is the supremum of the range of f for every ϵ there must be some x such that $\alpha - \epsilon < f(x) \le \alpha$. Letting $\epsilon_n := \frac{1}{n}$ we can construct a sequence $(x_n)_{n=1}^{\infty}$ with $f(x_n) \to \alpha$. Also since X is compact there must be some $(x_{n_k})_{k=1}^{\infty}$ with $x_{n_k} \to x_0\,(\in X)$. Since $f(x_n)$ is convergent, its subsequences must also be convergent to the same limit. And by the continuity of f we have:

$$f(x_0) \leftarrow f(x_{n_k}) \to \alpha$$

By the uniqueness of limits $f(x_0) = \alpha$ is a maximum. $\square$

(3) If $f : X \to Y$ is continuous with X compact and Y a metric space then the image of f is compact.

Proof. Take an arbitrary sequence $(f(x_n))_{n=1}^{\infty}$ in $\mathrm{Im} f$. There must exist some convergent subsequence x_{n_k} of x_n with $x_{n_k} \to x_0 (\in X)$. By the continuity of f we know that $f(x_{n_k}) \to f(x_0)$ which proves the compactness of $\mathrm{Im} f$.

$\square$

(4) A sequence $(x_n)_{n=1}^{\infty}$ being convergent to a point x_0 is equivalent to every subsequence (x_{n_k}) of (x_n) having a convergent subsequence $x_{n_{k_\ell}} \to x_0$.

Proof. This first way is easy since if (x_n) is convergent to x_0 then its subsequences are as well, and subsequences of subsequences, are, of course, subsequences so they all converge to x_0 as well. To show the other way, I will use contraposition. Consider the negation of $x_n \to x_0$:

$$\exists \epsilon > 0 \quad \forall N_0 \quad \exists n \ge N_0 \quad d(x_n, x_0) \ge \epsilon$$

We can use the above rule to inductively define a subsequence (x_{n_k}) of (x_n) such that for all k, $d(x_{n_k}, x_0) \ge \epsilon$. This clearly cannot have a subsequence convergent to x_0 since all of its elements are of a minimum distance of ϵ from x_0. We have shown the negation of the second statement which yields the desired equivalance. $\square$

Problems:
(1) For $f : X \to \mathbb{R}$ with X compact and $\forall x \in X \quad f(x) > 0$, then there exists $\delta > 0$ such that for all $x \in X \quad f(x) \geq \delta$

Proof. By applying lemma 2 we know that f must achieve a minimum for some $x_0 \in X$ and by hypothesis we know that $f(x_0) > 0$. Therefore we can pick $\delta := f(x_0)$ and we know that for all $x \in X \quad f(x) \geq f(x_0) = \delta$. $\square$

(2) For a set compact subset K and F a closed subset of some metric space (X, d). Show that if $K \cap F = \emptyset$ then there exists some $\delta > 0$ such that:

$$\forall p \in K \quad \forall q \in F \quad d(p, q) \geq \delta$$

Proof. Assume towards a contradiction that for all $\delta > 0$ there exists some $p \in K$ and $q \in F$ such that $d(p, q) < \delta$. Letting $\delta_n := \frac{1}{n}$ for any n we can construct a sequences $(p_n)_{n=1}^{\infty}$ in K and $(q_n)_{n=1}^{\infty}$ in F such that:

$$d(p_n, q_n) \to 0$$

Since p_n is a sequence in a compact set there must exist some subsequence $(p_{n_k})_{k=1}^{\infty}$ that is convergent to some $p_0 \in K$. I claim that the sequence given by $(q_{n_k})_{k=1}^{\infty}$ is convergent to p_0.

$$d(q_{n_k}, p_0) \leq d(q_{n_k}, p_{n_k}) + d(p_{n_k}, p_0)$$

and we can fix some n_0 where for all $n \geq n_0$:

$$d(q_{n_k}, p_0) \leq d(q_{n_k}, p_{n_k}) + d(p_{n_k}, p_0) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Since the sequence q_{n_k} is convergent sequence in a closed set, its limit point is necessarily included in the set. Therefore, $p_0 \in F$. We can conlude that $\{p_0\} \subset F \cap K$ which is a contradiction. $\square$

I am confused here because I seem to have solved this problem without employing the strategy of **(1)**. My idea is that you could fix an element of F and define a function the is the distance from the points in K to the fixed element of F. There is necessarily a positive mimimum of this function. For each element of F there is a fixed minimum value, and we could define another funcion on F that ties these elements to their minimum distanced elements in F and I would claim that this set has a minimum positive value in which case the question would follow but I don't kow how to prove the last step!

(3) Let A be a non singular linear map $\mathbb{R}^n \to \mathbb{R}^n$ (an $n \times n$ matrix) show that there exists some $\delta > 0$ such that:

$$||A(x)|| = ||Ax|| \geq \delta ||x|| \quad \forall x \in \mathbb{R}^n$$

Proof. Note that since A is nonsingular: $||Ax|| = 0 \iff ||x|| = 0$ since the only vector mapped to the 0 vector is the 0 vector. Therefore we can fix the sphere of radius 1, denoted S. This is compact since it is a closed and bounded susbset of $\mathbb{R}^n$. Since $||x||$ and $||Ax||$ are nonzero for $x \in S$. We can then define the function $f : S \to \mathbb{R}^n$ as follows:

$$f(x) := \frac{||Ax||}{||x||}$$

This function is defined over a compact set and is positive for all $x \in S$. Therefore, problem **(1)** applies and there must exist some $\delta > 0$ such that:

$$f(x) = \frac{||Ax||}{||x||} \geq \delta \iff ||Ax|| \geq \delta ||x|| \quad (x \in S)$$

It remains to show the above for the rest of $\mathbb{R}^n$. Since this inequality holds trivially for the zero vector, fix some arbitrary non zero vector, x, in $\mathbb{R}^n$. x can be represented as $t\hat{x}$ with $\hat{x}$ a unit vector and $t \in \mathbb{R}$ nonzero. Since $\hat{x}$ is a unit vector $\hat{x} \in S$ and we can write:

$$||A(x)|| = ||A(t\hat{x})|| = t||A(\hat{x})|| \geq t\delta||\hat{x}|| = \delta||x||$$

$$\iff ||Ax|| \geq \delta||x||$$

Therefore the delta found for S is sufficient for every $x \in \mathbb{R}^n$. □

(4) Let K and F be compact and disjoint subsets of $\mathbb{R}^n$. Show that there exists open sets U and V that contain K and F, respectively, such that $U \cap V = \emptyset$

Proof. Since K and F are both compact sets we can apply the results of problem **(2)** and find δ such that forall $k \in K$ and $f \in F$ it will be that $d(k, f) \geq \delta$. By comactness we can also find finite $\delta/2$ dense sets in both K and F. More specifically, there exists:

$$\{k_0, k_1, \dots, k_n\} \subset K \text{ and } \{f_0, f_1, \dots, f_\ell\} \subset F$$

With the properties:

$$\forall k \in K \;\; \exists i \leq n \;\; d(k, k_i) < \frac{\delta}{2} \text{ and } \forall f \in F \;\; \exists j \leq \ell \;\; d(f, f_j) < \frac{\delta}{2}$$

For each element of the $\delta/2$ coverings define the open sets:

$$U_{k_i} := B\left(k_i, \frac{\delta}{2}\right) \;\; U_{f_j} := B\left(f_j, \frac{\delta}{2}\right)$$

Now define the open coverings of K and F as:

$$U_K := \bigcup_{i=0}^{n} U_{k_i} \text{ and } U_F := \bigcup_{j=0}^{\ell} U_{f_j}$$

Clearly K and F are subsets of U_K and U_F, respectively. And U_K and U_F are both open as they are finite unions of open sets. We must also have that $U_K \cap U_F = \emptyset$. Otherwise there would exists some x with $x \in U_K$ and $x \in U_F$ which would imply the existance of $k_i \in K$ and $f_j \in F$ such that:

$$d(k_i, x) < \frac{\delta}{2} \text{ and } d(f_j, x) < \frac{\delta}{2}$$

$$\Rightarrow d(k_i, f_j) \leq d(k_i, x) + d(x, f_j) < \delta$$

Which is a contradiction with δ being chosen such that every element in K and F are δ distance apart. □

The next theme that is described in the collection is that taking limits gives items inside of compact sets!

(5) If $K \subset \mathbb{R}^n$ is compact, then there is a farthest point of K to the origin.

Proof. Begin by defining the set A as follows:

$$A := \{x \in \mathbb{R} : \exists k \in K \;\; d(\mathbf{0}, k) = x\}$$

Since K is compact it is bounded, A is also bounded above because of this fact. Therefore we can define $\alpha := \sup A \, (\in \mathbb{R})$. We can also define a sequence $(x_n)_{n=1}^{\infty}$ in A with $x_n \to \alpha$. This sequence yields a sequence $(k_n)_{n=1}^{\infty}$ in K such that $d(k_n, \mathbf{0}) \to \alpha$. Since k_n is a sequence in a compact set there exists a futher subsequence $(k_{n_k})_{k=1}^{\infty}$ such that $k_{n_k} \to k_0 \, (\in K)$. Since the distance metric is a continuous function and because of the properties of the sequence that k_{n_k} is a subsequence of we have:

$$d(k_0, \mathbf{0}) \leftarrow d(k_{n_k}, \mathbf{0}) \rightarrow \alpha$$

By the uniqueness of limits, $d(k_0, \mathbf{0}) = \alpha$ and this is a maximal distance from the origin! $\square$

(6) For $K \subset \mathbb{R}^n$ a compact set that is disjoint from a closed set F, then there is a closest point of K to the set F.

Proof. By applying the result of problem **(2)** we know that there is some $\delta > 0$ such that for all $k \in K$ and $f \in F$ $d(k, f) \geq \delta$. We can then define the set:

$$A := \{x \in \mathbb{R} : \exists k \in K \;\; \exists f \in F \;\; d(f, k) = x\}$$

By the above, we know that A is bounded below by some positive δ. Therefore $\beta := \inf A$ exists and is positive as well. As before, we can fix sequences $(k_n)_{n=1}^{\infty}$ and $(f_n)_{n=1}^{\infty}$ such that $d(k_n, f_n) \rightarrow \beta$. Since (k_n) is a sequence in a compact set it omits a sequence $(k_{n_\ell})_{\ell=1}^{\infty}$ which is convergent to some $k_0 \in K$. We know that $d(k_{n_\ell}, f_{n_\ell}) \rightarrow \beta$. It is also true that $d(k_0, f_{n_\ell}) \rightarrow \beta$ since:

$$d(k_0, f_{n_\ell}) - \beta \leq d(k_0, k_{n_\ell}) + d(k_{n_\ell}, f_{n_\ell}) - \beta < \frac{\epsilon}{2} + \frac{\epsilon}{2} + \beta - \beta = \epsilon$$

and:

$$\beta - d(k_0, f_{n_\ell}) = \beta - (k_0, f_{n_\ell}) + d(k_0, k_{n_\ell}) - d(k_0, k_{n_\ell})$$
$$= \beta + d(k_0, k_{n_\ell}) - ((k_0, f_{n_\ell}) + d(k_0, k_{n_\ell})) \leq \beta + d(k_0, k_{n_\ell}) - d(k_{n_\ell}, f_{n_\ell})$$
$$< \beta + \frac{\epsilon}{2} - (\beta - \frac{\epsilon}{2}) = \epsilon$$

Since (f_{n_ℓ}) is within β of k_0 in its limit then for some N_0 with $\ell \geq N_0$ the sequence $(f_{n_\ell})_{\ell \geq N_0}^{\infty}$ must be contained in the closed ball, B, of radius 2β around k_0. Since B is closed and bounded in $\mathbb{R}^n$ then it is compact and therefore $(f_{n_\ell})_{\ell \geq N_0}^{\infty}$ must have a convergent subsequence $f_{n_{\ell_g}} \rightarrow f_0$ but since this is also a sequence in F which is closed f_0 must be contained in F. I will now show that $d(k_0, f_0) = \beta$:

$$d(k_0, f_0) - \beta \leq d(k_0, f_{n_{\ell_g}}) + d(f_{n_{\ell_g}}, f_0) - \beta$$

Since we know that $d(k_0, f_{n_\ell}) \rightarrow \beta$ we can write:

$$\leq \beta + \frac{\epsilon}{2} + \frac{\epsilon}{2} - \beta = \epsilon$$

Also:

$$\beta - d(k_0, f_0) \leq \beta - d(k_0, f_0) - d(f_0, f_{n_{\ell_g}}) + d(f_0, f_{n_{\ell_g}})$$
$$= \beta + d(f_0, f_{n_{\ell_g}}) - (d(k_0, f_0) + d(f_0, f_{n_{\ell_g}}) \leq \beta + d(f_0, f_{n_{\ell_g}}) - d(k_0, f_{n_{\ell_g}})$$
$$< \beta + \frac{\epsilon}{2} - (\beta - \frac{\epsilon}{2}) = \epsilon$$
$$|\beta - d(k_0, f_0)| < \epsilon \qquad \text{for all } \epsilon > 0$$
$$\iff \beta = d(k_0, f_0)$$

We have found points in F and K which are at a distance apart equal to the infimum of all the distances between K and F which is positive! $\square$

(7) Let X be a metric space and $f : X \rightarrow X$ be a contraction mapping. Suppose that $K \subset X$ is a compact and nonempty set that satisfies $f(K) = K$. Prove that K contains one single point.

Proof. Note that since compact sets are bounded they have a finite diameter which is defined to be the supremum of all of the distances of all of the pairs of points in the set. For a compact set C this is denoted as $diam\, C$. Also note that for a given compact set C there exist points in C which acieve this diamater. Therefore if we assume that K has more than one point there exists $x^*, y^* \in K$ such that $d(x^*, y^*) = diam\, K > 0$. If we were to consider $f(K)$ we can easily show that this is compact.
To do this fix some arbitrary sequence in $(y_n)_{n=1}^{\infty}$ n $f(K)$. This induces some $(x_n)_{n=1}^{\infty}$ in K. This has a convergent subsequence $x_{n_k} \to x_0 \in K$. Since f is a contraction mapping we know:

$$f(f(x_0), f(x_{n_k})) \leq kd(x_0, x_{n_k}) \to 0 \quad (k \in [0,1))$$

$$\Rightarrow y_{n_k} = f(x_{n_k}) \to f(x_0) \in f(K)$$

Since $f(K)$ is also compact we can conclude that there must also be points $x^{**}, y^{**} \in f(K)$ such that $d(x^{**}, y^{**}) = diam\, f(K)$. Also there must exist $x_0, y_0 \in K$ for whice $x^{**} = f(x_0)$ and $y^{**} = f(y_0)$

$$d(x^{**}, y^{**}) = d(f(x_0), f(y_0)) < d(x_0, y_0) \leq diam\, K$$

$$\Rightarrow diam\, f(K) < diam\, K \Rightarrow K \neq f(K)$$

Therefore, K cannot have multiple points! □

(8) Let $r : [a, b] \to \mathbb{R}^n$ be a conitinuous curve (the path of a particle). Suppose that $r(a) = (0, 0, \ldots, 0)$ and $r(b) = (1, 1, \ldots, 1)$. Prove that there exists a last time the particle touches the surface of the unit sphere.

Proof. Since r is continuous with a point inside the unit sphere and a point outside of the unit sphere we know that there exists at least one point where the particle is on the unit sphere. Therefore we can define the set:

$$A := \{x \in \mathbb{R}^n : d(r(x), \mathbf{0}) = 1\}$$

This is nonempty and bounded above and below by b, and a respectively! Therefore the supremum and infimums of A exist and are themselves defined as inputs of r. Let $\alpha := \inf A$ and $\beta := \sup A$. The notion that $f(\alpha)$ is on the unit sphere is intuitively obvious and not the question so I will only prove that $f(\beta)$ is on the unit sphere. The proofs are also exactly the same!
Assume towards a contradiction that $f(\beta)$ is not on the unit sphere $\Rightarrow d(f(\beta), \mathbf{0}) \neq 1$. Therefore we are left with two cases:
[Case 1] $d(f(\beta), \mathbf{0}) < 1$
Because of the above, there must exist an ϵ such that $d(f(x), \mathbf{0}) < 1 - \epsilon$. We can then consider the ball B of radius ϵ centered at $f(\beta)$. Elements of this ball are also within the unit ball and therefore of distance less than 1 of the origin. Becauase for $\gamma \in B(f(\beta), \epsilon)$

$$d(\mathbf{0}, \gamma) \leq d(\mathbf{0}, f(\beta)) + d(f(\beta), \mathbf{0}) < 1 - \epsilon + \epsilon = 1$$

By the continuity of r there must exist some δ where if we consider ξ with $|x - \xi| < \delta$. And if we let δ' be the minimum of δ and $b - \beta$ we can find some $\xi < \beta$ such that $d(f(\xi), \mathbf{0}) < 1$. The existance of ξ contradetics β being the supremum of A.
[Case 2:] $d(f(\beta), \mathbf{0}) > 1$
The argument is exactly the same.
Therefore β must be on the unit sphere and is the "last" time the particle is on the unit sphere. □

The next theme is about every sequence having a convergent subsequence which converges in the set!
(9) For $A \subset \mathbb{R}$ with A compact and $x \in A$. Suppose that every convergent subsequence of $(x_n)_{n \in \mathbb{N}}$

converges to x. Show that the entire sequence converges to x. This was kind of confusing since it did not state that the sequence was in the compact set, but eh whatever.

Proof. An equivalence to the definition of convergence is that, for a sequence $(x_n)_{n\in\mathbb{N}}$ if there is some x_0 where every sequence $(x_{n_k})_{k\in\mathbb{N}}$ has a further subsequence $(x_{n_{k_\ell}})_{\ell\in\mathbb{N}}$ with $x_{n_{k_\ell}} \to x_0$. Then we can conclude $x_n \to x_0$. These facts are equivalent, I'll include it as a lemma (4 probably). To apply this fact to solve this problem fix an arbitrary subsequence (x_{n_k}) of (x_n). Since (x_{n_k}) is a sequence in A which is compact there is a convergent subsequence of it $(x_{n_{k_\ell}})$. Since $(x_{n_{k_\ell}})$ is itself a subsequence of (x_n), by hypothesis we know that $x_{n_{k_\ell}} \to x$. Since every sequence has a futher subsequence convergent to the same point (x_n) is convergent! $\square$

(10) Let X be a compact metric space and $f : X \to X$ be a contraction map:

$$\forall x, y \quad f(f(x), f(y)) < d(x, y)$$

Show that f has a unique fixed point

Proof. Let the set Ψ be defined as:

$$\Psi := \{\gamma \in \mathbb{R} : \exists x \quad \gamma = d(x, f(x))\}$$

We know that A is bounded below by 0 therefore $\alpha := \inf \Psi$ exists. I will show that there exists a point x such that $d(x, f(x)) = \alpha$. Because α is an infimum we can find a sequence (x_n) with $d(x_n, f(x_n)) \to \alpha$. Since x_n is a sequence in X, which is compact we have $x_{n_k} \to x_0$ and $d(x_0, f(x_0)) = \alpha$. With this information we must conclude that $\alpha = 0$. If $\alpha > 0$ then $x_0 \neq f(y_0)$ and since $x_0, f(x_0) \in X$ we can calculate $f(x_0), f(f(x_0)) \in X$. By the contraction property of f we know that $d(f(x_0), f(f(x_0))) < d(x_0, f(x_0)) = \alpha$ but this cannot be since α is a lower bound. We must conclude that $\alpha = 0$ and therefore $d(x_0, f(x_0)) = 0 \iff f(x_0) = x_0$.
To show uniqueness, assume that two points x_0, y_0 satisfy this property. Construct the set $\{x_0, y_0\}$ which is a compact set. We should have that $f(\{x_0, y_0\}) = \{x_0, y_0\}$ but this is a contradiction with the results of problem **(7)**. $\square$