Compactness

Many of these problems are from a collection made by Behnam Esmayli, who also references a collection created by Cezar Lupu.

Common Lemmas:
If $f : X \to Y$ is a continuious function between metric spaces (X, d) and (Y, ϱ) then $f(X)$ is bounded.

Proof. Assume towards a contradiction that $f(X)$ is not bounded. Therefore, for a fixed $y \in f(X)$ we have that for every R there is some $f(x)$ such that $\varrho(y, f(x)) > R$. Therefore we can construct some sequence $(x_n)_{n=1}^{\infty}$ such that $\varrho(f(x_n), y) > n$. Since X is compact there must be some $(x_{n_k})_{k=1}^{\infty}$ with $x_{n_k} \to x_0\,(\in X)$. Since f is continuous, we have that $f(x_{n_k}) \to f(x_0)\,(\in Y)$. Since these sequences are covergent, it must be that $\varrho(f(x_{n_k}), y) \to \varrho(f(x_0), y)$. However, $\varrho(x_{n_k}, y) > n_k > k$ for all k. This is a contradiction and therefore $f(X)$ is bounded. $\square$