Compactness

Many of these problems are from a collection made by Behnam Esmayli, who also references a collection created by Cezar Lupu.

Common Lemmas:

(1) If $f : X \to Y$ is a continuious function between metric spaces (X, d) and (Y, ϱ) then $f(X)$ is bounded.

Proof. Assume towards a contradiction that $f(X)$ is not bounded. Therefore, for a fixed $y \in f(X)$ we have that for every R there is some $f(x)$ such that $\varrho(y, f(x)) > R$. Therefore we can construct some sequence $(x_n)_{n=1}^{\infty}$ such that $\varrho(f(x_n), y) > n$. Since X is compact there must be some $(x_{n_k})_{k=1}^{\infty}$ with $x_{n_k} \to x_0\,(\in X)$. Since f is continuous, we have that $f(x_{n_k}) \to f(x_0)\,(\in Y)$. Since these sequences are covergent, it must be that $\varrho(f(x_{n_k}), y) \to \varrho(f(x_0), y)$. However, $\varrho(x_{n_k}, y) > n_k > k$ for all k. This is a contradiction and therefore $f(X)$ is bounded. $\square$

(2) If $f : X \to \mathbb{R}$ with X compact, then f must achieve a maximum and minimum value.

Proof. It suffices to show only f achieving a maximum since the cases are the same. Applying the first lemma we know that $f(X)$ is a bounded subset of $\mathbb{R}$. Therefore $\alpha := \sup f(X)$ exists. Since α is the supremum of the range of f for every ϵ there must be some x such that $\alpha - \epsilon < f(x) \leq \alpha$. Letting $\epsilon_n := \frac{1}{n}$ we can construct a sequence $(x_n)_{n=1}^{\infty}$ with $f(x_n) \to \alpha$. Also since X is compact there must be some $(x_{n_k})_{k=1}^{\infty}$ with $x_{n_k} \to x_0\,(\in X)$. Since $f(x_n)$ is convergent, its subsequences must also be convergent to the same limit. And by the continuity of f we have:

$$f(x_0) \leftarrow f(x_{n_k}) \to \alpha$$

By the uniqueness of limits $f(x_0) = \alpha$ is a maximum. $\square$

Problems:

(1) For $f : X \to \mathbb{R}$ with X compact and $\forall x \in X \quad f(x) > 0$, then there exists $\delta > 0$ such that for all $x \in X \quad f(x) \geq \delta$

Proof. By applying lemma 2 we know that f must achieve a minimum for some $x_0 \in X$ and by hypothesis we know that $f(x_0) > 0$. Therefore we can pick $\delta := f(x_0)$ and we know that for all $x \in X \quad f(x) \geq f(x_0) = \delta$. $\square$

(2) For a set compact subset K and F a closed subset of some metric space (X, d). Show that if $K \cap F = \emptyset$ then there exists some $\delta > 0$ such that:

$$\forall p \in K \quad \forall q \in F \quad d(p, q) \geq \delta$$

Proof. Assume towards a contradiction that for all $\delta > 0$ there exists some $p \in K$ and $q \in F$ such that $d(p, q) < \delta$. Letting $\delta_n := \frac{1}{n}$ for any n we can construct a sequences $(p_n)_{n=1}^{\infty}$ in K and $(q_n)_{n=1}^{\infty}$ in F such that:

$$d(p_n, q_n) \to 0$$

Since p_n is a sequence in a compact set there must exist some subsequence $(p_{n_k})_{k=1}^{\infty}$ that is convergent to some $p_0 \in K$. I claim that the sequence given by $(q_{n_k})_{k=1}^{\infty}$ is convergent to p_0.

$$d(q_{n_k}, p_0) \leq d(q_{n_k}, p_{n_k}) + d(p_{n_k}, p_0)$$

and we can fix some n_0 where for all $n \geq n_0$:

$$d(q_{n_k}, p_0) \leq d(q_{n_k}, p_{n_k}) + d(p_{n_k}, p_0) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Since the sequence q_{n_k} is convergent sequence in a closed set, its limit point is necessarily included in the set. Therefore, $p_0 \in F$. We can conlude that $\{p_0\} \subset F \cap K$ which is a contradiction. $\square$

I am confused here because I seem to have solved this problem without employing the strategy of **(1)**. My idea is that you could fix an element of F and define a function the is the distance from the points in K to the fixed element of F. There is necessarily a positive mimimum of this function. For each element of F there is a fixed minimum value, and we could define another funcion on F that ties these elements to their minimum distanced elements in F and I would claim that this set has a minimum positive value in which case the question would follow but I don't kow how to prove the last step!

(3) Let A be a non singular linear map $\mathbb{R}^n \to \mathbb{R}^n$ (an $n \times n$ matrix) show that there exists some $\delta > 0$ such that:

$$||A(x)|| = ||Ax|| \geq \delta ||x|| \ \ \forall x \in \mathbb{R}^n$$

Proof. Note that since A is nonsingular: $||Ax|| = 0 \iff ||x|| = 0$ since the only vector mapped to the 0 vector is the 0 vector. Therefore we can fix the sphere of radius 1, denoted S. This is compact since it is a closed and bounded susbset of $\mathbb{R}^n$. Since $||x||$ and $||Ax||$ are nonzero for $x \in S$. We can then define the function $f : S \to \mathbb{R}^n$ as follows:

$$f(x) := \frac{||Ax||}{||x||}$$

This function is defined over a compact set and is positive for all $x \in S$. Therefore, problem **(1)** applies and there must exist some $\delta > 0$ such that:

$$f(x) = \frac{||Ax||}{||x||} \geq \delta \iff ||Ax|| \geq \delta ||x|| \quad (x \in S)$$

It remains to show the above for the rest of $\mathbb{R}^n$. Since this inequality holds trivially for the zero vector, fix some arbitrary non zero vector, x, in $\mathbb{R}^n$. x can be represented as $t\hat{x}$ with $\hat{x}$ a unit vector and $t \in \mathbb{R}$ nonzero. Since $\hat{x}$ is a unit vector $\hat{x} \in S$ and we can write:

$$||A(x)|| = ||A(t\hat{x})|| = t||A(\hat{x})|| \geq t\delta||\hat{x}|| = \delta||x||$$

$$\iff ||Ax|| \geq \delta||x||$$

Therefore the delta found for S is sufficient for every $x \in \mathbb{R}^n$. $\square$

(4) Let K and F be compact and disjoint subsets of $\mathbb{R}^n$. Show that there exists open sets U and V that contain K and F, respectively, such that $U \cap V = \emptyset$

Proof. Since K and F are both compact sets we can apply the results of problem **(2)** and find δ such that forall $k \in K$ and $f \in F$ it will be that $d(k, f) \geq \delta$. By comactness we can also find finite $\delta/2$ dense sets in both K and F. More specifically, there exists:

$$\{k_0, k_1, \ldots, k_n\} \subset K \ \text{ and } \ \{f_0, f_1, \ldots, f_\ell\} \subset F$$

With the properties:

$$\forall k \in K \ \ \exists i \leq n \ \ d(k, k_i) < \frac{\delta}{2} \ \text{ and } \ \forall f \in F \ \ \exists j \leq \ell \ \ d(f, f_j) < \frac{\delta}{2}$$

For each element of the $\delta/2$ coverings define the open sets:

$$U_{k_i} := B\left(k_i, \frac{\delta}{2}\right) \ \ U_{f_j} := B\left(f_j, \frac{\delta}{2}\right)$$

Now define the open coverings of K and F as:

$$U_K := \bigcup_{i=0}^{n} U_{k_i} \quad \text{and} \quad U_F := \bigcup_{j=0}^{\ell} U_{f_j}$$

Clearly K and F are subsets of U_K and U_F, respectively. And U_K and U_F are both open as they are finite unions of open sets. We must also have that $U_K \cap U_F = \emptyset$. Otherwise there would exists some x with $x \in U_K$ and $x \in U_F$ which would imply the existance of $k_i \in K$ and $f_j \in F$ such that:

$$d(k_i, x) < \frac{\delta}{2} \quad \text{and} \quad d(f_j, x) < \frac{\delta}{2}$$
$$\Rightarrow d(k_i, f_j) \leq d(k_i, x) + d(x, f_j) < \delta$$

Which is a contradiction with δ being chosen such that every element in K and F are δ distance apart. $\square$

The next theme that is described in the collection is that taking limits gives items inside of compact sets!

(5) If $K \subset \mathbb{R}^n$ is compact, then there is a farthest point of K to the origin.

Proof. Begin by defining the set A as follows:

$$A := \{x \in \mathbb{R} : \exists k \in K \ \ d(\mathbf{0}, k) = x\}$$

Since K is compact it is bounded, A is also bounded above because of this fact. Therefore we can define $\alpha := \sup A \, (\in \mathbb{R})$. We can also define a sequence $(x_n)_{n=1}^{\infty}$ in A with $x_n \to \alpha$. This sequence yields a sequence $(k_n)_{n=1}^{\infty}$ in K such that $d(k_n, \mathbf{0}) \to \alpha$. Since k_n is a sequence in a compact set there exists a futher subsequence $(k_{n_k})_{k=1}^{\infty}$ such that $k_{n_k} \to k_0 \, (\in K)$. Since the distance metric is a continuous function and because of the properties of the sequence that k_{n_k} is a subsequence of we have:

$$d(k_0, \mathbf{0}) \leftarrow d(k_{n_k}, \mathbf{0}) \to \alpha$$

By the uniqueness of limits, $d(k_0, \mathbf{0}) = \alpha$ and this is a maximal distance from the origin! $\square$

(6) For $K \subset \mathbb{R}^n$ a compact set that is disjoint from a closed set F, then there is a closest point of K to the set F.

Proof. By applying the result of problem **(2)** we know that there is some $\delta > 0$ such that for all $k \in K$ and $f \in F$ $\ d(k, f) \geq \delta$. We can then define the set:

$$A := \{x \in \mathbb{R} : \exists k \in K \ \ \exists f \in F \ \ d(f, k) = x\}$$

By the above, we know that A is bounded below by some positive δ. Therefore $\beta := \inf A$ exists and is positive as well. As before, we can fix sequences $(k_n)_{n=1}^{\infty}$ and $(f_n)_{n=1}^{\infty}$ such that $d(k_n, f_n) \to \beta$. Since (k_n) is a sequence in a compact set it omits a sequence $(k_{n_\ell})_{\ell=1}^{\infty}$ which is convergent to some $k_0 \in K$. We know that $d(k_{n_\ell}, f_{n_\ell}) \to \beta$. It is also true that $d(k_0, f_{n_\ell}) \to \beta$ since:

$$d(k_0, f_{n_\ell}) - \beta \leq d(k_0, k_{n_\ell}) + d(k_{n_\ell}, f_{n_\ell}) - \beta < \frac{\epsilon}{2} + \frac{\epsilon}{2} + \beta - \beta = \epsilon$$

and:

$$\beta - d(k_0, f_{n_\ell}) = \beta - (k_0, f_{n_\ell}) + d(k_0, k_{n_\ell}) - d(k_0, k_{n_\ell})$$
$$= \beta + d(k_0, k_{n_\ell}) - ((k_0, f_{n_\ell}) + d(k_0, k_{n_\ell})) \leq \beta + d(k_0, k_{n_\ell}) - d(k_{n_\ell}, f_{n_\ell})$$
$$< \beta + \frac{\epsilon}{2} - (\beta - \frac{\epsilon}{2}) = \epsilon$$

Since (f_{n_ℓ}) is within β of k_0 in its limit then for some N_0 with $\ell \geq N_0$ the sequence $(f_{n_\ell})_{\ell \geq N_0}^{\infty}$ must be contained in the closed ball, B, of radius 2β around k_0. Since B is closed and bounded in $\mathbb{R}^n$ then it is compact and therefore $(f_{n_\ell})_{\ell\geq N_0}^{\infty}$ must have a convergent subsequence $f_{n_{\ell_g}} \to f_0$ but since this is also a sequence in F which is closed f_0 must be contained in F. I will now show that $d(k_0, f_0) = \beta$:

$$d(k_0, f_0) - \beta \leq d(k_0, f_{n_{\ell_g}}) + d(f_{n_{\ell_g}}, f_0) - \beta$$

Since we know that $d(k_0, f_{n_\ell}) \to \beta$ we can write:

$$\leq \beta + \frac{\epsilon}{2} + \frac{\epsilon}{2} - \beta = \epsilon$$

Also:

$$\beta - d(k_0, f_0) \leq \beta - d(k_0, f_0) - d(f_0, f_{n_{\ell_g}}) + d(f_0, f_{n_{\ell_g}})$$
$$= \beta + d(f_0, f_{n_{\ell_g}}) - (d(k_0, f_0) + d(f_0, f_{n_{\ell_g}}) \leq \beta + d(f_0, f_{n_{\ell_g}}) - d(k_0, f_{n_{\ell_g}})$$
$$< \beta + \frac{\epsilon}{2} - (\beta - \frac{\epsilon}{2}) = \epsilon$$
$$|\beta - d(k_0, f_0)| < \epsilon \qquad \text{for all } \epsilon > 0$$
$$\iff \beta = d(k_0, f_0)$$

We have found points in F and K which are at a distance apart equal to the infimum of all the distances between K and F which is positive! □

(7) Let X be a metric space and $f : X \to X$ be a contraction mapping. Suppose that $K \subset X$ is a compact and nonempty set that satisfies $f(K) = K$. Prove that K contains one single point.

Proof. Note that since compact sets are bounded they have a finite diameter which is defined to be the supremum of all of the distances of all of the pairs of points in the set. For a compact set C this is denoted as $diam\, C$. Also note that for a given compact set C there exist points in C which acieve this diamater. Therefore if we assume that K has more than one point there exists $x^*, y^* \in K$ such that $d(x^*, y^*) = diam\, K > 0$. If we were to consider $f(K)$ we can easily show that this is compact.
To do this fix some arbitrary sequence in $(y_n)_{n=1}^{\infty}$ n $f(K)$. This induces some $(x_n)_{n=1}^{\infty}$ in K. This has a convergent subsequence $x_{n_k} \to x_0 \in K$. Since f is a contraction mapping we know:

$$f(f(x_0), f(x_{n_k})) \leq kd(x_0, x_{n_k}) \to 0$$
$$\Rightarrow y_{n_k} = f(x_{n_k}) \to f(x_0) \in f(K)$$

Since $f(K)$ is also compact we can conclude that there must also be points $x^{**}, y^{**} \in f(K)$ such that $d(x^{**}, y^{**}) = diam\, f(K)$.

$$d(x^{**}, y^{**}) <$$

□