Assignment One

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Direction:

Please answer all the questions below and hand in your answers before the due day. All work, must be handed in **on time**.

Due day:

April. 12, 2021

Please hand it in by the class time.

Questions:

1. For each of the following functions, indicate how much the function's value will change if its argument is increased fourfold.

   a. $log_2 n$ b. $\sqrt{n}$ c. n d. n^2 e. n^3 f. 2^n

1. a. $\log_2 n$ answer: let $f(n) = \log_2 n$, then $f(4n) = \log_2(4n) = 2 + \log_2 n$

b. $\sqrt{n}$ answer. let $f(n) = \sqrt{n}$, then $f(4n) = \sqrt{4n} = 2\sqrt{n}$

c. n answer: let $f(n) = n$, then $f(4n) = 4n$

d. n^2 answer: let $f(n) = n^2$, then $f(4n) = (4n)^2 = 16n^2$

e. n^3 answer: let $f(n) = n^3$, then $f(4n) = (4n)^3 = 64n^3$

f. 2^n answer: let $f(n) = 2^n$, then $f(4n) = 2^{(4n)} = 16 \cdot 2^n$

2. Prove (by using the definitions of the notations involved) or disprove (by giving a specific counterexample) the following assertions.

   a. If $t(n) \in O(g(n))$, then $g(n) \in \Omega(t(n))$.

   b. $\Theta(\alpha g(n)) = \Theta(g(n))$, where $\alpha > 0$.

   c. $\Theta(g(n)) = O(g(n)) \cap \Omega(g(n))$.

   d. For any two nonnegative functions $t(n)$ and $g(n)$ defined on the set of nonnegative integers, either $t(n) \in O(g(n))$, or $t(n) \in \Omega(g(n))$, or both.

2. a. if $t(n) \in O(g(n))$, then $g(n) \in \Omega(t(n))$

proof: $t(n) \in O(g(n)) \Leftrightarrow \exists c_1 > 0, \forall n \geq n_0, \ t(n) \leq c_1 g(n)$

$\Leftrightarrow \exists c_1 > 0, \forall n \geq n_0, \ g(n) \geq \frac{1}{c_1} t(n)$

let $c_2 = \frac{1}{c_1}$, $\exists c_2 > 0, \forall n \geq n_0, \ g(n) \geq c_2 t(n)$

$\Leftrightarrow g(n) \in \Omega(t(n))$

b. $\Theta(\alpha g(n)) = \Theta(g(n))$, where $\alpha > 0$

proof:

$\lim_{n \to \infty} \frac{\alpha g(n)}{g(n)} = \alpha > 0$, then $\alpha g(n) \in \Theta(g(n))$

$\lim_{n \to \infty} \frac{g(n)}{\alpha g(n)} = \frac{1}{\alpha} > 0$. then $g(n) \in \Theta(\alpha g(n))$

$\therefore \Theta(g(n)) = \Theta(\alpha g(n))$

c. $\Theta(g(n)) = O(g(n)) \cap \Omega(g(n))$

proof: let $t(n) \in O(g(n)) \cap \Omega(g(n))$, then $\exists c_1, c_2 > 0, \forall n \geq n_0$,

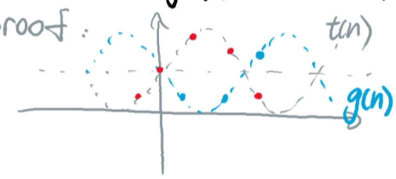
$c_1 g(n) \leq t(n) \leq c_2 g(n)$

$\therefore t(n) \in \Theta(g(n))$

$\therefore \Theta(g(n)) = O(g(n)) \cap \Omega(g(n))$

d. $t(n) \geq 0$, $g(n) \geq 0$, $n \in N^*$, either $t(n) \in O(g(n))$, or $t(n) \in \Omega(g(n))$, or both.

disproof:



let $t(n) = \sin x + 1$, $g(n) = -\sin x + 1$

then $t(n) \notin O(g(n))$

$t(n) \notin \Omega(g(n))$

$t(n) \notin O(g(n)) \cap \Omega(g(n))$

3. Solve the following recurrence relations.

a. $x(n) = 3x(n-1)$ for $n > 1$, $x(1) = 4$

b. $x(n) = x(n-1) + n$ for $n > 0$, $x(0) = 0$

c. $x(n) = x(n/2) + n$ for $n > 1$, $x(1) = 1$ (solve for $n = 2^k$)

3. a. $x(n) = 3x(n-1)$, for $n > 1$, $x(1) = 4$

$\frac{x(n)}{x(n-1)} = 3 \Rightarrow \frac{x(n)}{\cancel{x(n-1)}} \cdot \frac{\cancel{x(n-1)}}{\cancel{x(n-2)}} \cdots \frac{\cancel{x(2)}}{x(1)} = 3^{n-1}$

$\therefore x(n) = 4 \cdot 3^{n-1}$

b. $x(n) = x(n-1) + n$ for $n > 0$, $x(0) = 0$

$$\left.\begin{aligned} x(n) - x(n-1) &= n \\ x(n-1) - x(n-2) &= n-1 \\ &\vdots \\ x(2) - x(1) &= 2 \\ x(1) - x(0) &= 1 \end{aligned}\right\} \Rightarrow x(n) - x(0) = \frac{(1+n)n}{2}$$

$\therefore x(n) = \frac{n(n+1)}{2}$

c. $x(n) = x(n/2) + n$ for $n > 1$, $x(1) = 1$ (solve for $n = 2^k$)

$$x(n) = x\left(\frac{n}{2}\right) + n$$

$$x(2^k) = x(2^{k-1}) + 2^k \quad (k \in Z^+) \quad . \quad x(1) = x(2^0) = 1$$

$$\left.\begin{aligned} x(2^k) - x(2^{k-1}) &= 2^k \\ x(2^{k-1}) - x(2^{k-2}) &= 2^{k-1} \\ &\vdots \\ x(2) - x(2^0) &= 2 \end{aligned}\right\} \Rightarrow x(2^k) - x(2^0) = \frac{2(1-2^k)}{1-2}$$

$$\Rightarrow x(n) = x(2^k) = 2^{k+1} - 1$$