# Correction of TEST 1

1. **(1pt)** We have $B \subset A$ and $\forall x \in A$, $\exists y \in B$ such that $x \leqslant y$.
   (a) Since $B \subset A$ then $\forall y \in B$, $y \in A \implies y \leqslant \sup A$ then $B$ is upper bounded in $\mathbb{R}$ hence $\sup B$ exists.
   (b) From (a), $\sup A$ is an upper bound of $B$ then $\sup B \leqslant \sup A$ since $\sup B$ is the least upper bound. Assume by contradiction that $\sup B < \sup A$, then by definition, there exists $x \in A$ such that
   $$\sup B < x \leqslant \sup A$$
   and by hypothesis on $B$, there exists $y \in B$ such that
   $$\sup B < x \leqslant y$$
   which yields a contradiction since $y \in B \implies y \leqslant \sup B$. Conclusion : $\sup B = \sup A$.

2. **(1pt)** Let $x, y \in \mathbb{R}$, show that :
   (a) One has
   $$\begin{aligned} |x| = |x-y+y| \leqslant |x-y| + |y| &\implies |x| - |y| \leqslant |x-y| \\ |y| = |y-x+x| \leqslant |x-y| + |x| &\implies |y| - |x| \leqslant |x-y| \\ &\implies ||x|-|y|| \leqslant |x-y|. \end{aligned}$$
   (b) One has
   $$\max\{x,y\} = \frac{x+y+|x-y|}{2} = \begin{cases} \dfrac{x+y+(x-y)}{2} = x & \text{if } x \geqslant y \\ \dfrac{x+y-(x-y)}{2} = y & \text{if } x < y \end{cases} : \text{True}$$
   $$\min\{x,y\} = \frac{x+y-|x-y|}{2} = \begin{cases} \dfrac{x+y-(x-y)}{2} = y & \text{if } x \geqslant y \\ \dfrac{x+y+(x-y)}{2} = x & \text{if } x < y \end{cases} : \text{True}$$
   (c) $\implies$) Evident. let us show that $x = y \impliedby |x-y| < \varepsilon$, $\forall \varepsilon > 0$. By contrapositive reasoning,
   $$\exists \varepsilon > 0 : |x-y| \geqslant \varepsilon \implies |x-y| > 0 \implies x \neq y$$
   thus the indirect implication is true too.

3. **(1pt)** A set $I \subset \mathbb{R}$ is an **interval** if and only if $I$ contains at least 2 points and for all $a, c \in I$ and $b \in \mathbb{R}$ such that $a < b < c$, we have $b \in I$. Say (and justify) if the following sets are or are not intervals :
   $$(0,1],\ \{0,1\},\ (0,1) \cap \mathbb{Q},\ \mathbb{R} \setminus \mathbb{Q}.$$
   (a) $(0,1]$ contains$1/2$ and $1$ and for all $a, c \in (0,1]$ such that $a < c$, $(a,c) \subset (0,1]$ thus $(0,1]$ is an interval of $\mathbb{R}$.
   (b) $\{0,1\}$ is not an interval since it contains $0$ and $1$ but $0 < 1/2 < 1$ and $1/2 \notin \{0,1\}$.
   (c) $(0,1)$ contains$1/4$ and $1/2$ and for all $a, c \in (0,1)$ such that $a < c$, $(a,c) \subset (0,1)$ thus $(0,1)$ is an interval of $\mathbb{R}$.
   (d) $(0,1) \cap \mathbb{Q}$ is not an interval since it contains $1/4$ and $1/2$ but $1/4 < \sqrt{2}-1 < 1/2$ and $\sqrt{2}-1 \notin (0,1) \cap \mathbb{Q}$.
   (e) $\mathbb{R} \setminus \mathbb{Q}$ is not an interval since it contains $-\pi$ and $\pi$ but $-\pi < 0 < \pi$ and $0 \notin \mathbb{R} \setminus \mathbb{Q}$.

4. **(1pt)** For $a < b$ reals, let $f : (a,b] \to (0,1]$ defined by $f(x) = \alpha x + \beta$ such that $f(x)$ tends to $0$ as $x \to a$ and $f(b) = 1$. Theses two relations yield
   $$\alpha a + \beta = 0 \text{ and } \alpha b + \beta = 1 \implies \alpha = \frac{1}{b-a} \text{ and } \beta = 1 - \alpha b = \frac{-a}{b-a}.$$
   $f$ is a bijection since for all $y \in (0,1]$, there exists a unique $x \in (a,b]$ such that $f(x) = y$, in fact
   $$x = \frac{y-\beta}{\alpha} = (b-a)\left(y + \frac{a}{b-a}\right) = a + y(b-a)$$
   We have $x > a$ since $y > 0$ and $b - a > 0$, and on the other hand $x \leqslant b$ since $y \leqslant 1$ and $b - a > 0$, thus $x \in (a,b]$ is uniquely determined in $(a,b]$.

5. **(2pt)** The sets $(-1,1)$ and $\mathbb{R}$ have the same cardinalty if one finds a bijection from $\mathbb{R}$ to $(-1,1)$. Many examples can given : for example $f:\mathbb{R}\to(-1,1)$

$$f(x)=\frac{x}{1+|x|}$$

$f$ is a bijection, in fact : for all $y\in(-1,1)$,

(a) if $y\in[0,1)$ then

$$\frac{x}{1+|x|}=y\geqslant 0 \implies x\geqslant 0$$
$$\frac{x}{1+x}=y \implies x=\frac{y}{1-y}\in\mathbb{R}$$

(b) if $y\in(-1,0)$ then

$$\frac{x}{1+|x|}=y\geqslant 0 \implies x<0$$
$$\frac{x}{1-x}=y \implies x=\frac{y}{1+y}\in\mathbb{R}$$

In both cases, $x=\dfrac{y}{1-|y|}$ exists and is uniquely determined such that $f(x)=y$, hence $f$ is a bijection.
Conclusion : $|(-1,1)|=|\mathbb{R}|$.

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6. **(1pt)** $\left\{\dfrac{(-1)^n}{n}\right\}_{n\geqslant 1}$ is convergent and the limit is 0 by the **ST** theorem, since

$$0\leftarrow -\frac{1}{n}\leqslant\frac{(-1)^n}{n}\leqslant\frac{1}{n}\to 0$$

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7. **(1pt)** $\left\{x_n:=\dfrac{n}{n+1}\right\}_{n\geqslant 0}$ is monotone strictly increasing because $\forall n\in\mathbb{N}$,

$$x_{n+1}-x_n=\frac{n+1}{n+2}-\frac{n}{n+1}=\frac{1}{n+2}>0$$

and is bounded since, $\forall n\in\mathbb{N}$,

$$0<x_n=\frac{n}{n+1}<1$$

Since $\{x_n\}_{n\geqslant 0}$ is increasing then

$$\inf\{x_n : n\geqslant 0\}=\min\{x_n : n\geqslant 0\}=x_0=0,$$

and since $\{x_n\}_{n\geqslant 0}$ is bounded then by **Bolzano-Weirestrass th.**

$$\sup\{x_n : n\geqslant 0\}=\lim_{n\to+\infty}x_n=1.$$

Finally, since $n\neq n+1$ for all integer $n$ , then $\max\{x_n : n\geqslant 0\}$ does not exists since for any integer $n$, $x_n\neq 1$.

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8. **(1pt)** The sequence $\left\{a_n:=\dfrac{(-1)^n\, n}{n+1}\right\}_{n\geqslant 0}$ is not convergent because, there are two subsequences convergent to different limits :

$$\begin{cases} a_{2n}=\dfrac{2n}{2n+1}\to 1 \\ a_{2n+1}=-\dfrac{2n+1}{2n+3}\to -1 \quad \neq 1 \end{cases}$$

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9. **(1pt)** Let $\{x_n\}_n$ a sequence of reals :

(a) True since $|x_n-0|=|x_n|=||x_n|-0|$, in other words :

$$|x_n-0|<\varepsilon \iff ||x_n|-0|<\varepsilon$$

(b) $x_n=(-1)^n$, $\lim_{n\to+\infty}|x_n|=1$ but $\lim_{n\to+\infty}x_n$ does not exist.

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10. **(2pt)** Let $\{x_n\}_{n\geqslant 0}$ a sequence of non zero reals (i.e. $x_n\neq 0$, $\forall n$) such that the limit

$$L:=\lim_{n\to+\infty}\frac{x_{n+1}}{x_n},\ \text{exists.}$$

(a) By contra positive reasoning, assume $x_n \to l \neq 0$. Then, $x_{n+1} \to l$ and since $x_n \neq 0$, $\forall n$, and $l \neq 0$ then $\dfrac{x_{n+1}}{x_n} \to 1$ thus $L = 1$ is not less then 1.

(b) Assume $L > 1$ then and by contradiction suppose $\{x_n\}_{n \geqslant 0}$ is bounded. By assumption,

$$L = \lim_{n \to +\infty} \left| \frac{x_{n+1}}{x_n} \right|$$

Hence, for $\varepsilon = L - 1 > 0$, there exists an integer $M$ such that for all $n \geqslant M$ :

$$L - \varepsilon < \left| \frac{x_{n+1}}{x_n} \right| < L + \varepsilon$$

that gives for all $n \geqslant M$ :

$$\left| \frac{x_{n+1}}{x_n} \right| > 1 \implies |x_{n+1}| > |x_n|$$

thus $\{|x_n|\}_{n \geqslant M}$ is monotone increasing. Since $\{|x_n|\}_{n \geqslant M}$ is bounded then there exists a convergent (monotone increasing) subsequence such that $|x_{n_k}| \to l = \sup_{k \geqslant M} |x_{n_k}| \geqslant 0$ and we have, since $x_n \neq 0$, $\forall n$,

$$l = \sup_{k \geqslant M} |x_{n_k}| \geqslant |x_{n_M}| > 0.$$

On the other hand, one has $L = \lim_{k \to +\infty} \left| \dfrac{x_{n_k+1}}{x_{n_k}} \right|$ exists, then by contra-positive reasoning as in (a) :

$$l \neq 0 \implies L = 1$$

which give a contradiction with the hypothesis $L > 1$.

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11. **(1pt)** Let $c > 0$, prove that : (Hint : question 10.)

(a) if $c < 1$ then

$$L = \lim_{n \to +\infty} \left| \frac{x_{n+1}}{x_n} \right| = \lim_{n \to +\infty} \left| \frac{c^{n+1}}{c^n} \right| = c < 1$$

then $\lim_{n \to +\infty} c^n = 0$.

(b) if $c > 1$ then

$$L = \lim_{n \to +\infty} \left| \frac{x_{n+1}}{x_n} \right| = \lim_{n \to +\infty} \left| \frac{c^{n+1}}{c^n} \right| = c > 1$$

hence the sequence $\{c^n\}_{n \geqslant 0}$ is unbounded.

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12. **(1pt)** Let $n! := n(n-1)...3.2.1$ and $0! = 1$. (Use question 10.)

$$L = \lim_{n \to +\infty} \left| \frac{\dfrac{2^{n+1}}{(n+1)!}}{\dfrac{2^n}{n!}} \right| = \lim_{n \to +\infty} \frac{2}{n+1} = 0 < 1.$$

then

$$\lim_{n \to +\infty} \frac{2^n}{n!} = 0$$

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13. **(1pt)** Let $x \geqslant 0$ and $n \in \mathbb{N}$ : By induction :

(a) Initialization : for $n = 0$, $(1+x)^0 = 1 \geqslant 1 + 0x$ , true.

(b) Induction step : Assume the property true for $n$ and let us show it for $n+1$.

$$\begin{aligned}(1+x)^{n+1} &= (1+x)^n(1+x) \geqslant (1+nx)(1+x) \\ &\geqslant 1 + (n+1)x + nx^2 \geqslant 1 + (n+1)x.\end{aligned}$$

(c) Conclusion step : for all $x \geqslant 0$ and $n \in \mathbb{N}$ : $(1+x)^n \geqslant 1 + nx$.

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14. **(1pt)** Let $p > 0$. For any $\varepsilon > 0$, let $d = \varepsilon^{-1/p}$ then for all $n \geqslant d$ :

$$\left|n^{-p}\right| \leqslant d^{-p} = \varepsilon$$

hence the inequality is true for all $n \geqslant M = [d] + 1$ and we conclude that

$$\lim_{n\to+\infty} n^{-p} = 0$$

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15. **(2pt)** Let $p > 1$ : then $p^{\frac{1}{n}} > 1^{\frac{1}{n}} \implies p^{\frac{1}{n}} - 1 > 0$ for all $n \in \mathbb{N}^*$. On the other hand, one has by question 13 :

$$p = \left(1 + \left(p^{\frac{1}{n}} - 1\right)\right)^n \geqslant 1 + n\left(p^{\frac{1}{n}} - 1\right)$$

hence

$$0 < p^{\frac{1}{n}} - 1 \leqslant \frac{p-1}{n},$$

and we deduce by the ST theorem that

$$0 < p^{\frac{1}{n}} - 1 \leqslant \frac{p-1}{n} \to 0 \implies p^{\frac{1}{n}} - 1 \to 0$$

$$\implies \lim_{n\to+\infty} p^{\frac{1}{n}} = 1$$

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16. **(2pt)** Let $\{x_n\}_n, \{y_n\}_n$ two convergent sequences of reals such that $\lim_{n\to+\infty} |x_n - y_n| = 0$. Let $\lim_{n\to+\infty} x_n = x$ and $\lim_{n\to+\infty} y_n = y$. Then for all $\varepsilon > 0$ , there exists $M_1, M_2, M_3$ integers such that :

$$\forall n \geqslant M_1 : |x - x_n| < \frac{\varepsilon}{3}$$

$$\forall n \geqslant M_2 : |x_n - y_n| < \frac{\varepsilon}{3}$$

$$\forall n \geqslant M_3 : |y_n - y| < \frac{\varepsilon}{3}$$

hence, for $n = \max(M_1, M_2, M_3)$,

$$\begin{aligned} 0 \leqslant |x - y| &= |x - x_n + x_n - y_n + y_n - y| \\ &\leqslant |x - x_n| + |x_n - y_n| + |y_n - y| \leqslant \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

henceforth

$$(\forall \varepsilon > 0 : |x - y| < \varepsilon) \implies |x - y| = 0 \implies x = y.$$