Properties of continuous functions

Textbook Reading : [JL] Sections 3.1, 3.2, 3.3, 3.4, 3.5 (Lebl)

Challenging (exercicse 1) :

Let $f : \mathbb{R} \to \mathbb{R}$. Recall that if $U \subset \mathbb{R}$, the *inverse image* of U is the set

$$f^{-1}(U) := \{x \in \mathbb{R} : f(x) \in U\}.$$

Prove that f is continuous if and only if for every open set $U \subset \mathbb{R}$, $f^{-1}(U)$ is open.

***Exercise* 3.3.1:** *Find an example of a discontinuous function $f: [0,1] \to \mathbb{R}$ where the conclusion of the intermediate value theorem fails.*

***Exercise* 3.3.2:** *Find an example of a* bounded *discontinuous function $f: [0,1] \to \mathbb{R}$ that has neither an absolute minimum nor an absolute maximum.*

***Exercise* 3.3.3:** *Let $f: (0,1) \to \mathbb{R}$ be a continuous function such that $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 1} f(x) = 0$. Show that f achieves either an absolute minimum or an absolute maximum on $(0,1)$ (but perhaps not both).*

***Exercise* 3.3.4:** *Let*

$$f(x) := \begin{cases} \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Show that f has the intermediate value property. That is, whenever $a < b$, if there exists a y such that $f(a) < y < f(b)$ or $f(a) > y > f(b)$, then there exists a $c \in (a,b)$ such that $f(c) = y$.

***Exercise* 3.3.14:** *Suppose $f: [0,1] \to (0,1)$ is a bijection. Prove that f is not continuous.*

***Exercise* 3.3.15:** *Suppose $f: \mathbb{R} \to \mathbb{R}$ is continuous.*

a) Prove that if there is a c such that $f(c)f(-c) < 0$, then there is a $d \in \mathbb{R}$ such that $f(d) = 0$.

b) Find a continuous function f such that $f(\mathbb{R}) = \mathbb{R}$, but $f(x)f(-x) \geq 0$ for all $x \in \mathbb{R}$.

***Exercise* 3.4.11:** *Prove:*

a) If $f: S \to \mathbb{R}$ and $g: S \to \mathbb{R}$ are uniformly continuous, then $h: S \to \mathbb{R}$ given by $h(x) := f(x) + g(x)$ is uniformly continuous.

b) If $f: S \to \mathbb{R}$ is uniformly continuous and $a \in \mathbb{R}$, then $h: S \to \mathbb{R}$ given by $h(x) := af(x)$ is uniformly continuous.

Challenging (exercicse 2) :

Let $S \subset \mathbb{R}$. We say that $f : S \to \mathbb{R}$ is *Lipschitz continuous* on S if there exists $L \geq 0$ such that for all $x, y \in S$,

$$|f(x) - f(y)| \leq L|x - y|.$$

Prove that if $f : S \to \mathbb{R}$ is Lipschitz continuous on S then f is uniformly continuous on S.

(a) Prove that $f(x) = \cos x$ is Lipschitz continuous on $\mathbb{R}$.

(b) Prove that $f(x) = x^{1/3}$ is uniformly continuous on $[0,1]$ and is not Lipschitz continuous on $[0,1]$.