Exercises given with a numbering are from *Basic Analysis: Introduction to Real Analysis (Vol I)* by J. Lebl.

Reading Sections 2.2, 2.3, 2.4, 2.5 (**Lebl**)

Exercises

1. ~~Exercise 2.2.9~~
2. Exercise 2.3.5
3. Exercise 2.3.6
4. Exercise 2.3.7

See p 2

5. Let $\{x_n\}$ be a bounded sequence of real numbers. Prove

$$\lim_{n\to\infty} x_n = 0,$$

if and only if

$$\limsup_{n\to\infty} |x_n| = 0.$$

6. Does there exist a sequence $\{x_n\}$ such that

$$\liminf_{n\to\infty} x_n = -1, \quad \lim_{n\to\infty} x_n = 0, \quad \limsup_{n\to\infty} x_n = 1?$$

Either give an example, or explain why no such example exists.

7. Exercise 2.4.8

See p 2

Exercise* 2.3.5*:

a) *Let* $x_n := \frac{(-1)^n}{n}$*. Find* $\limsup_{n\to\infty} x_n$ *and* $\liminf_{n\to\infty} x_n$*.*

b) *Let* $x_n := \frac{(n-1)(-1)^n}{n}$*. Find* $\limsup_{n\to\infty} x_n$ *and* $\liminf_{n\to\infty} x_n$*.*

Exercise* 2.3.6*: *Let* $\{x_n\}_{n=1}^\infty$ *and* $\{y_n\}_{n=1}^\infty$ *be bounded sequences such that* $x_n \leq y_n$ *for all* n*. Show*

$$\limsup_{n\to\infty} x_n \leq \limsup_{n\to\infty} y_n \qquad \text{and} \qquad \liminf_{n\to\infty} x_n \leq \liminf_{n\to\infty} y_n.$$

Exercise* 2.3.7*: *Let* $\{x_n\}_{n=1}^\infty$ *and* $\{y_n\}_{n=1}^\infty$ *be bounded sequences.*

a) *Show that* $\{x_n + y_n\}_{n=1}^\infty$ *is bounded.*

b) *Show that*

$$\left(\liminf_{n\to\infty} x_n\right) + \left(\liminf_{n\to\infty} y_n\right) \leq \liminf_{n\to\infty} (x_n + y_n).$$

Hint: One proof is to find a subsequence $\{x_{n_m} + y_{n_m}\}_{m=1}^\infty$ *of* $\{x_n + y_n\}_{n=1}^\infty$ *that converges. Then find a subsequence* $\{x_{n_{m_i}}\}_{i=1}^\infty$ *of* $\{x_{n_m}\}_{m=1}^\infty$ *that converges.*

c) *Find an explicit* $\{x_n\}_{n=1}^\infty$ *and* $\{y_n\}_{n=1}^\infty$ *such that*

$$\left(\liminf_{n\to\infty} x_n\right) + \left(\liminf_{n\to\infty} y_n\right) < \liminf_{n\to\infty} (x_n + y_n).$$

Hint: Look for examples that do not have a limit.

Exercise* 2.4.8*: *True or false, prove or find a counterexample: If* $\{x_n\}_{n=1}^\infty$ *is a Cauchy sequence, then there exists an* M *such that for all* $n \geq M$*, we have* $|x_{n+1} - x_n| \leq |x_n - x_{n-1}|$*.*