Properties of continuous functions

Textbook Reading : [JL] Sections 3.1, 3.2, 3.3, 3.4, 3.5 (Lebl)

Challenging (exercise 1) : Solution :

$\Longrightarrow$) Let $f : \mathbb{R} \to \mathbb{R}$ continuous, and let U open in $\mathbb{R}$. By definition,

$$f^{-1}(U) = \{x \in \mathbb{R},\ f(x) \in U\}.$$

For all $c \in f^{-1}(U)$, $c \in \mathbb{R}$, such that $f(c) \in U$. Since U is open then $\exists \varepsilon > 0$, such that $(f(c)-\varepsilon, f(c)+\varepsilon) \subset U$. Hence,

$$\forall y \in \mathbb{R},\ |y - f(c)| < \varepsilon \implies y \in U$$

By continuity of f, for any $c \in f^{-1}(U)$, $\exists \delta > 0$, such that $\forall x \in \mathbb{R}$:

$$|x - c| < \delta \implies |f(x) - f(c)| < \varepsilon \implies f(x) \in U \implies x \in f^{-1}(U)$$

which means exactly that $(c - \delta, c + \delta) \subset f^{-1}(U)$, hence $f^{-1}(U)$ is open.

$\Longleftarrow$) Let f be such that for all open set U in $\mathbb{R}$, $f^{-1}(U)$ is an open set of $\mathbb{R}$. Then, for all $c \in \mathbb{R}$ and all $\varepsilon > 0$, $(f(c) - \varepsilon, f(c) + \varepsilon) = U$ is an open set of $\mathbb{R}$,thus

$$f^{-1}(U) = \{x \in \mathbb{R},\ f(x) \in U\} = \{x \in \mathbb{R},\ |f(x) - f(c)| < \varepsilon\}$$

is open and it contains c since $|f(c) - f(c)| = 0 < \varepsilon$, it follows that $\exists \delta > 0$, such that $(c - \delta, c + \delta) \subset f^{-1}(U)$, henceforth, $\forall x \in \mathbb{R}$,

$$|x - c| < \delta \implies f(x) \in U \implies |f(x) - f(c)| < \varepsilon.$$

Hence, f is continuous at any $c \in \mathbb{R}$ thus it is continous on all $\mathbb{R}$.

Exercise 3.3.1 There are many examples! A Solution : Let $f : [0, 1] \to \mathbb{R}$, discontinuous at $x = 1/2$,

$$f(x) = \begin{cases} x & \text{if } x \neq \dfrac{1}{2} \\ 0 & \text{if } x = \dfrac{1}{2} \end{cases}$$

f never achieves the value $y = 1/2$, that means $\nexists x \in [0, 1]$,such that $f(x) = 1/2$.

Exercise 3.3.2 (Seen in course session) $f : [0, 1] \to \mathbb{R}$, bounded discontinuous at $x = 0$ and $x = 1$,

$$f(x) = \begin{cases} x & \text{if } x \neq 0 \text{ and } x \neq 1 \\ \dfrac{1}{2} & \text{else} \end{cases}$$

Use a graph to visualize.

Exercise 3.3.3 Let $f : (0, 1) \to \mathbb{R}$, continuous such that

$$\lim_{x\to 0} f(x) = 0 = \lim_{x\to 1} f(x).$$

Let $\tilde{f}$ the extension of f defined by

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \neq 0 \text{ and } x \neq 1 \\ 0 & \text{if } x = 0 \text{ or } x = 1 \end{cases}$$

then $\tilde{f} : [0, 1] \to \mathbb{R}$ is a continuous function, thus by the Min/Max Bolzano's theorem, there exist $c, d \in [0, 1]$, such that

$$\tilde{f}(c) = \min_{x\in[0,1]} \tilde{f}(x) \text{ and } \tilde{f}(d) = \max_{x\in[0,1]} \tilde{f}(x).$$

Therefore, for all $x \in [0, 1]$, $\tilde{f}(c) \leqslant \tilde{f}(x) \leqslant \tilde{f}(d)$. It follows that, $\forall x \in (0, 1)$,

$$\tilde{f}(c) \leqslant \tilde{f}(x) = f(x) \leqslant \tilde{f}(d)$$

Assume that neither c neither d is in $(0,1)$, thus $\forall x \in (0,1)$,

$$0 = \tilde{f}(c) \leqslant f(x) \leqslant \tilde{f}(d) = 0$$

then $f \equiv 0$ is constant on $(0,1)$. Hence, if on chooses $f \equiv c \neq 0$ any other non zero constant, one obtains a contradiction. It follows that, when f in not constant, at least c or d lies in $(0,1)$, say c for example. Then, $\forall x \in (0,1)$,

$$f(c) = \tilde{f}(c) \leqslant f(x) \leqslant \tilde{f}(d) = 0$$

which implies that c is an absolute minimum of f. Remark that not necessarily both c and d lie in $(0,1)$ as shown in the following example :

$$f(x) = \begin{cases} x & \text{if } 0 < x \leqslant \dfrac{1}{2} \\ 1-x & \text{if } \dfrac{1}{2} < x < 1 \end{cases}$$

f is continuous on $(0,1)$ and has limit 0 at 0 and 1. Use a graph to visualize that $d = 1/2$ is an absolute maximum but there is no absolute minimum (here $c = 0$ or $c = 1$ realizes the minimum of $\tilde{f}$ but not of f)

Exercise 3.3.4 Let $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} \sin\dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Remark that f is discontinuous at 0 since $\lim_{x\to 0} f(x) = \lim_{x\to 0} \sin\dfrac{1}{x}$ does not exist. However, for any $a < b$, if there is y such that $f(a) < y < f(b)$ then :

1st case : if $0 < a < b$ or $a < b < 0$, then the restriction $f_{|[a,b]} : [a,b] \to \mathbb{R}$ is continuous thus it has the I.V.T property, i.e., $\exists c \in (a,b)$, such that $f(c) = y$.

2nd case : if $a \leqslant 0 < b$, then : since

$$-1 \leqslant f(a) < y < f(b) \leqslant 1$$

$\exists \alpha$ real, such that $\sin(\alpha + 2n\pi) = y$ for all integer n, i.e., $f(\dfrac{1}{\alpha + 2n\pi}) = y$. But the sequence $\dfrac{1}{\alpha + 2n\pi} \searrow 0$ decreasing then there exists n_0 integer such that

$$0 < c_0 = \frac{1}{\alpha + 2n_0\pi} < \frac{b}{2}$$

hence we have $a \leqslant c_0 < b$ and $f(c_0) = \sin(\alpha + 2n_0\pi) = y$.

3rd case : if $a < 0 \leqslant b$, then : since

$$-1 \leqslant f(a) < y < f(b) \leqslant 1$$

$\exists \beta$ real, such that $\sin(\beta - 2n\pi) = y$ for all integer n, i.e., $f(\dfrac{1}{\beta - 2n\pi}) = y$. But the sequence $\dfrac{1}{\beta - 2n\pi} \nearrow 0$ is increasing then there exists n_1 integer such that

$$\frac{a}{2} < c_1 = \frac{1}{\beta - 2n_1\pi} < b$$

hence we have $a < c_1 \leqslant b$ and $f(c_1) = \sin(\beta - 2n_1\pi) = y$.

Exercise 3.3.14 Let $f : [0,1] \to (0,1)$, a bijection, then $f([0,1]) = (0,1)$. By contradiction, if we assume f is continuous then by the Min/Max theorem, f achieves at $c \in [0,1]$ a minimum value and at $d \in [0,1]$ a maximum value, and

$$f([0,1]) = [f(c), f(d)]$$

which is a contradiction with $f([0,1]) = (0,1)$. Then, f is discontinuous.

Exercise 3.3.15 Let $f : \mathbb{R} \to \mathbb{R}$ continuous.

a) One can assume $c > 0$. The restriction $f_{|[-c,c]} : [-c,c] \to \mathbb{R}$ is continuous then it admits the I.V.T property, i.e., $f(c)f(-c) < 0 \implies \exists d \in (-c,c),\ f(d) = 0$.

b) For example, let $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x \sin x$, then f is continuous on $\mathbb{R}$ and satisfy

$$f(x)f(-x) = x^2 \sin^2 x \geqslant 0, \ \forall x \in \mathbb{R}.$$

We have for all integer n, $f(\pi/2+2n\pi) = \pi/2+2n\pi$ is an unbounded increasing sequence of reals with n and $f(\pi/2-2n\pi) = \pi/2-2n\pi$ is an unbounded decreasing sequence of reals with n, then for all $y \in \mathbb{R}$, there exists an integer n_0 such that

$$f(a) = \pi/2 - 2n_0\pi < y < \pi/2 + 2n_0\pi = f(b)$$

where $a = \pi/2 - 2n_0\pi$ and $b = \pi/2 + 2n_0\pi$. Now, $a < b$, and $f : [a, b] \to \mathbb{R}$ is continuous and admits the I.V.T property then there exists $c \in (a, b)$ such that $f(c) = y$, which mean that $f : \mathbb{R} \to \mathbb{R}$ is onto then $f(\mathbb{R}) = \mathbb{R}$.

Exercise 3.4.11

a) f and g uniformly continuous on S : implies, $\forall \varepsilon > 0$, $\forall c \in S$, there exist $\delta_1 = \delta_1(\varepsilon) > 0$ and $\delta_2 = \delta_2(\varepsilon) > 0$ such that $\forall x \in S$:

$$\begin{cases} |x-c| < \delta_1 \implies |f(x) - f(c)| < \dfrac{\varepsilon}{2} \\ |x-c| < \delta_2 \implies |g(x) - g(c)| < \dfrac{\varepsilon}{2} \end{cases}$$

thus, for $\delta = \min(\delta_1, \delta_2) > 0$, one has for all $x \in S$ such that $|x - c| < \delta$,

$$\begin{aligned} |f(x) + g(x) - f(c) - g(c)| &= |f(x) - f(c) + g(x) - f(c)| \\ &\leqslant |f(x) - f(c)| + |g(x) - f(c)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

Then $f + g$ is uniformly continuous on S.

b) f uniformly continuous on S : implies, $\forall \varepsilon > 0$, $\forall c \in S$, there exist $\delta = \delta(\varepsilon) > 0$ such that $\forall x \in S$:

$$|x - c| < \delta \implies |f(x) - f(c)| < \frac{\varepsilon}{1 + |a|}$$

thus, for this $\delta > 0$, one has for all $x \in S$ such that $|x - c| < \delta$,

$$\begin{aligned} |h(x) - h(c)| &= |a||f(x) - f(c)| \\ &\leqslant \frac{|a|}{1 + |a|}\varepsilon < \varepsilon \end{aligned}$$

Then h is uniformly continuous on S.

Challenging (exercise 2) :

Let f Lipschitz continuous with Lipschitz constant $L \geqslant 0$,

i) If $L = 0$ then f is constant then uniformly continuous.

ii) If $L > 0$, then $\forall \varepsilon > 0$, $\forall c \in S$, there exist

$$\delta = \delta(\varepsilon) = \frac{\varepsilon}{L} > 0$$

such that $\forall x \in S$:

$$|x - c| < \delta \implies |f(x) - f(c)| \leqslant L|x - c| < L\delta = \varepsilon$$

then f is uniformly continuous.

a) $f(x) = \cos x$, then $\forall x, y \in \mathbb{R}$, we know that $|\sin \alpha| \leqslant |\alpha|$ and that

$$\cos x - \cos y = -2 \sin\left(\frac{x-y}{2}\right) \sin\left(\frac{x+y}{2}\right)$$

thus

$$\begin{aligned} |\cos x - \cos y| &= 2\left|\sin\left(\frac{x-y}{2}\right)\right| \left|\sin\left(\frac{x+y}{2}\right)\right| \\ &\leqslant 2\left|\sin\left(\frac{x-y}{2}\right)\right| \leqslant |x - y| \end{aligned}$$

then f is Lipschitz continuous with Lipschitz constant $L = 1$.

b) Let $f(x) = x^{\frac{1}{3}}$, since $f : [0,1] \to \mathbb{R}$ is a continuous function on a closed interval $[0,1]$ then f is uniformly continuous. Let us show now that $f$is not Lipschitz on $[0,1]$. By contradiction, if f is Lipschitz on $[0,1]$ with Lipschitz constant $L > 0$ ($L \neq 0$ since f is not constant) then : $\forall x, y \in [0,1]$, $|f(x) - f(y)| \leqslant L|x - y|$. Hence for $y = 0$, one obtains that for all $n \in \mathbb{N}^*$, $x_n = 1/n \in (0,1)$,

$$x_n^{\frac{1}{3}} \leqslant L x_n \implies L \geqslant \frac{1}{x_n^{\frac{2}{3}}} = n^{\frac{2}{3}} \to +\infty$$

which leads to a contradiction since $L \in \mathbb{R}$. Henceforth, f can not be Lipschitz continuous.