Differentiable functions

Textbook Reading : [JL] Sections 4.1, 4.2 (Lebl)

Solution of 4.1.1

$$\frac{f(x)g(x)-f(c)g(c)}{x-c} = f(x)\frac{(g(x)-g(c))}{x-c} + \frac{f(x)-f(c)}{x-c}g(c)$$

Passing to the limit as $x \to c$, $f(x) \to f(c)$ by continuity of f , and we obtain

$$(fg)'(c) = f(c)g'(c) + f'(c)g(c).$$

Solution of 4.1.3 : Let us show that

$$(x^n)' = \begin{cases} nx^{n-1} & \text{if } n > 0 \text{ or } (n < 0 \text{ and } x \neq 0) \\ 0 & \text{if } n = 0 \end{cases},$$

First case $n = 0$ then $x^0 = 1$ and $(x^0)' = 0$,

If $n \geqslant 1$, then the proof follows by induction.

For $n = 1$,$(x)' = 1 = 1x^0$: True

Induction step : Assume $(x^n)' = nx^{n-1}$, then by the product rule :

$$(x^{n+1})' = (x^n x)' = nx^{n-1}x + x^n 1 = (n+1)x^n.$$

Conclusion step : $(x^n)' = nx^{n-1}$ for all $n \geqslant 1$ and all $x \in \mathbb{R}$.

If $n \leqslant -1$, then $m = -n \geqslant 1$ and we write $x^n = 1/x^m$ and by the differential rule $(1/f)' = -f'/f^2$ at all x such that $f(x) \neq 0$, we obtain from the previous result that for all $x \neq 0$,

$$(x^n)' = (1/x^m)' = \frac{-mx^{m-1}}{x^{2m}} = -mx^{-m-1} = nx^{n-1}.$$

Solution of 4.1.4 : A polynomial of degree n writes for some coefficients ($a_0, a_1, ..., a_n$ constants) :

$$P_n(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

then by the result of 4.1.3, and the linearity of the differentiation, P_n is differentiable and

$$P_n'(x) = a_1 + 2a_2x + 3a_3x^2 + \cdots + na_nx^{n-1}$$

Remark that P_n' is a polynomial of degree $n-1$.

Solution of 4.1.5 : $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$$

We have :

$$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0}\frac{f(x)}{x} = l$$

Let us show that $l = 0$. For all $\varepsilon > 0$, let $x \neq 0$ then

$$\left|\frac{f(x)}{x}\right| = \begin{cases} |x| & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$$

then $\exists \delta > 0$ ($\delta = \varepsilon$) , such that

$$0 < |x| < \delta \implies \begin{cases} \left|\frac{f(x)}{x}\right| = |x| < \varepsilon & \text{if } x \in \mathbb{Q} \\ \left|\frac{f(x)}{x}\right| = 0 < \varepsilon & \text{otherwise} \end{cases} \implies \left|\frac{f(x)}{x}\right| < \varepsilon$$

hence $l = 0$. We conclude that f is differentiable at $x = 0$ and that $f'(0) = 0$.

Since f is differentiable at 0 then f is continuous at 0.

Let us show that f is discontinuous at any other $x \neq 0$. By contradiction, assume there exists $c \neq 0$ such that f is continuous at c. Then we know by density of rational numbers (and irrational numbers) in $\mathbb{R}$ that there exist two sequences $\{r_n\}_n$ of rationals and $\{i_n\}_n$ of irrationals such that $r_n \to c$ and $i_n \to c$, hence, by continuity assumption on f,

$$f(r_n) = r_n^2 \to c^2 \neq 0 \leftarrow 0 = f(i_n)$$

which leads to a contradiction. Thus f is discontinuous at any $c \neq 0$.

Solution of 4.1.6 We assume $|x - \sin x| \leqslant x^2$ for all x real.

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\sin x - x + x}{x} = 1$$

since $0 \leqslant \left|\frac{\sin x - x}{x}\right| \leqslant x \to 0$ and $x/x \to 1$. Thus f is differentiable at $x = 0$ and $f'(0) = 1$.

Solution of 4.1.7 : Let $f(x) = \sin x$ and let us use Trigonometric rule

$$\sin a - \sin b = 2\cos\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$$

then

$$\frac{f(x) - f(c)}{x - c} = \frac{\sin x - \sin c}{x - c} = \cos\left(\frac{x+c}{2}\right)\frac{\sin\left(\frac{x-c}{2}\right)}{\frac{x-c}{2}}$$

then by the change of variable $\frac{x-c}{2} = X \to 0$ as $x \to c$ and the the previous result $\lim_{X\to 0} \frac{\sin X}{X} = 1$ and the continuity of Cosine function, one obtains

$$\lim_{x\to 0} \frac{f(x) - f(c)}{x - c} = \lim_{x\to 0} \cos\left(\frac{x+c}{2}\right) \lim_{X\to 0} \frac{\sin(X)}{X} = \cos c$$

then f is differentiable at any real c and $f'(c) = \cos c$.

Solution of 4.1.8

Already proved in 4.1.3 for the particular case $f(x) = x$, now the chain rule $(g \circ f)'(x) = (g'(f(x))f'(x)$ the result follows immediately by the change of variable $X = f(x)$.

Solution of 4.1.11

Let f bounded then $\exists M \geqslant 0 : \forall x \in I, |f(x)| \leqslant M$, then, since $g(c) = 0$,

$$\frac{h(x) - h(c)}{x - c} = \frac{f(x)g(x) - f(c)g(c)}{x - c} = f(x)\frac{g(x) - g(c)}{x - c}$$

and since $g'(c) = 0$,

$$0 \leqslant \left|\frac{h(x) - h(c)}{x - c}\right| \leqslant |f(x)| \left|\frac{g(x) - g(c)}{x - c}\right| \underset{x\to c}{\to} |f(x)|\,|g'(c)| = 0$$

then

$$\frac{h(x) - h(c)}{x - c} \underset{x\to c}{\to} = 0$$

which implies that h is differentiable at c and $h'(c) = 0$.

Solution of 4.1.15

Using $f(c) = g(c) = 0$, and $g'(c) \neq 0$

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f(x) - f(c)}{g(x) - g(c)} = \lim_{x\to c} \frac{\frac{f(x) - f(c)}{x - c}}{\frac{g(x) - g(c)}{x - c}} = \frac{f'(c)}{g'(c)}$$

and since f' and g' are continuous then f'/g' is continuous at any $x \in (a, b)$ such that $g'(x) \neq 0$, in particular at $x = c$, and we obtain

$$\lim_{x\to c} \frac{f'(x)}{g'(x)} = \frac{f'(c)}{g'(c)}$$

which achieves the proof.

Solution of 4.2.5 :

For all $c \in \mathbb{R}$, one has for all $x \neq c$, and by the assumption on f,

$$0 \leqslant \left| \frac{f(x) - f(c)}{x - c} \right| \leqslant |x - c| \underset{x \to c}{\to} = 0$$

which implies that f is differentiable at c and $f'(c) = 0$. Then f is differentiable on $\mathbb{R}$. Hence, for any two reals a, b, $(a < b)$, f is continuous on $[a, b]$ and differentiable on (a, b) and by the MVT theorem we deduce that $\exists c \in (a, b)$, such that

$$\frac{f(b) - f(a)}{b - a} = f'(c) = 0$$

therefore $f(a) = f(b)$ and this is for whatever a, b reals then f is constant.

Remark : The conclusion can be obtained directly by the MVT theorem on the interval $\mathbb{R} = (-\infty, +\infty)$ but we missed the proof for this in the course.

Solution of 4.2.7 : **Remark : we miss the continuity of f' in this exercise so add it to the question.**

By contradiction, assume $\exists x \in (a, b)$, $x \neq c$, such that $f'(x) \leqslant 0$, then $f'(x) < 0$ since by hypothesis $f'(x) \neq 0$, then by the continuity of f' and the IVT theorem there exists d between c and x, then $b \in (a, b)$ such that $f'(d) = 0$ which is in contradiction with the hypothesis $f'(x) \neq 0$ for any x in (a, b). Hence the proof follows.

Solution of 4.2.8 :

Let $h = f - g$ is differentiable on (a, b) interval and $h'(x) = f'(x) - g'(x) = 0$ for all $x \in (a, b)$ then $h(x) = C$ constant hence $f(x) = g(x) + C$.

Solution of 4.2.9 : Comparing with previous exercise 4.1.15, we miss the continuity of f' and g' and we deal with an interval instead of a set I of $\mathbb{R}$. Let us prove first that $g(x) \neq 0$ for any $x \neq c$. In fact, by contradiction, assume that there exists $d \neq c$ (for example $c < d$) such that $g(d) = 0$, then since g is continuous on $[c, d]$ and differentiable on (c, d) then Rolle's theorem implies that there exists $e \in (c, d)$ such that $g'(e) = 0$ which is in contradiction with the hypothesis $g'(x) \neq 0$ for any $x \neq c$. Hence, we deduce that $g(x) \neq 0$ for any $x \neq c$. Now, it follows from exercise 4.1.15 that

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)}$$

Moreover, by the MVT applied to f and g on (c, x) for all $x \neq c$, recall that f and g are both continuous on $[c, x]$ and differentiable on (c, x), then there exist d in (c, x) by the Cauchy's MVT (or the generalized MVT theorem, **see Th. 4.2.5 in Real analysis 1 page 165**) such that

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(c)}{g(x) - g(c)} = \frac{f'(d)}{g'(d)},$$

hence, the fact that when $x \to c : d \to c$ since $d \in (c, x)$, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(d)}{g'(d)} = \lim_{d \to c} \frac{f'(d)}{g'(d)} = \lim_{x \to c} \frac{f'(x)}{g'(x)} \ (\exists).$$

Solution of 4.2.11

We know that a polynomial f is infinitely continuously differentiable on $\mathbb{R}$. Let $a < b$ such that $f'(a) = f'(b) = 0$ and there is no $c \in (a, b)$ such that $f'(c) = 0$. Assume, by contradiction, that there are at least two roots distinct roots $c_1 \neq c_2$ of f in (a, b), that is $f(c_1) = f(c_2)$, then by the MVT we deduce that there exists $c_3 \in (c_1, c_2)$ such that $f'(c_3) = 0$ and c_3 lies in (a, b) which is in contradiction with the fact that there is no $c \in (a, b)$ such that $f'(c) = 0$.

Solution of 4.2.12

Let $f : \mathbb{R} \to \mathbb{R}$ differentiable such that $f'(x) = a$ (constant) for all x, and $f(0) = b$ for all x, hence f is continuously on any closed interval $[0, x]$ (here we assumed without restriction $x > 0$), and differentiable on $(0, x)$, then by the MVT : there exists $c \in (0, x)$ such that

$$f(x) - f(0) = f'(c)(x - 0) \implies f(x) = f'(c)x + f(0) = ax + b$$

Now, if there is others constants c, d such that $f(x) = cx + d$ then $f(0) = d = b$ and $f'(x) = c = a$ thus a and b are uniquely determined.

Solutions of exercises :

Ex1 A polynomial is infinitely continuously differentiable on $\mathbb{R}$.

$$P(x)=\frac{x^{1121}}{1121}+\frac{x^{2021}}{2021}+x+1$$

$$P(0)=1>0 \text{ and } P(-1)=\frac{-1}{1121}+\frac{-1}{2021}<0$$

hence, by the IVT we conclude that there exists at least one root $c\in(-1,0)$ of P, i.e. $P(c)=0$. Assume by contradiction that there are two real roots $c_1\neq c_2$ in $(-1,0)$, i.e. $P(c_1)=P(c_2)=0$ then by the Rolle's theorem, there exists $d\in(c_1,c_2)$ such that $P'(d)=0$. But, since 1120 and 2020 are even numbers, then for all $x\in\mathbb{R}$,

$$P'(x)=x^{1120}+x^{2020}+1>0$$

which gives the contradiction $P'(d)=0$. Therefore, P has exactly one real root.

Ex2 We have,

a) f is infinitely continuously differentiable on $\mathbb{R}$: $f(x)=\sin(x)$, $f'(x)=\cos x$, $f''(x)=-\sin(x)$, $f'''(x)=-\cos x$, $f^{(4)}(x)=\sin x$, $f^{(5)}(x)=\cos x$, thus, for all $x\in\mathbb{R}$, $\exists c\in(0,x)$:

$$\begin{aligned} f(x)&=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\frac{f^{(4)}(0)}{4!}x^4+\frac{f^{(5)}(c)}{5!}x^5\\ &=x-\frac{x^3}{6}+\frac{\cos c}{120}x^5\end{aligned}$$

Remark : Recall that $c=c(x)$ depends on x. Observe also that we can write $f(x)=x-\dfrac{x^3}{6}+O(x^5)=x-\dfrac{x^3}{6}+o(x^4)$ since $\cos c$ is a bounded function and $\dfrac{\cos c}{120}x=\varepsilon(x)\to 0$ as $x\to 0$.

b) Observe first that f is a rational function thus it is infinitely continuously differentiable on $\mathbb{R}\setminus\{1\}$. We can thus apply a Taylor expansion of f on any interval $(a,b)\subset(-\infty,1)$ or $(a,b)\subset(1,+\infty)$.

We have $f(x)=\dfrac{1}{1-x}$,$f'(x)=\dfrac{1}{(1-x)^2}$, $f''(x)=\dfrac{2}{(1-x)^3}$, $f'''(x)=\dfrac{6}{(1-x)^4}$, $f^{(4)}(x)=\dfrac{24}{(1-x)^5}$, $f^{(5)}(x)=\dfrac{120}{(1-x)^6}$. Since $-1\in(-\infty,1)$ then, for all $x\in(-\infty,1)$, f is 5 times continuously differentiable on $[x,-1]$ (if $x<-1$) or on $[-1,x]$ if $1>x>-1$, then by Taylor expansion at order 5, there exists c between -1 and x such that :

$$\begin{aligned} f(x)&=f(-1)+\frac{f'(-1)}{1!}(x+1)+\frac{f''(-1)}{2!}(x+1)^2+\frac{f'''(-1)}{3!}(x+1)^3+\frac{f^{(4)}(-1)}{4!}(x+1)^4+\frac{f^{(5)}(c)}{5!}(x+1)^5\\ &=\frac{1}{2}+\frac{1}{4}(x+1)+\frac{1}{8}(x+1)^2+\frac{1}{16}(x+1)^3+\frac{1}{32}(x+1)^4+\frac{1}{(1-c)^6}(x+1)^5\end{aligned}$$

Ex3 Let us use Taylor expansion to compute the limits :

a) From previous ex1 (a), we have

$$\begin{aligned}\lim_{x\to0}\frac{x-\sin(x)}{x^3}&=\lim_{x\to0}\frac{x-\left(x-\dfrac{x^3}{6}+\dfrac{\cos c}{120}x^5\right)}{x^3}\\ &=\lim_{x\to0}\left(\frac{1}{6}-\frac{\cos c}{120}x^3\right)=\frac{1}{6}\end{aligned}$$

since $\dfrac{\cos c}{120}x^3\to 0$ as $x\to 0$ (bounded$\times(\to 0)$=0).

b) In this case, let us use the change of variable $X=x-\dfrac{\pi}{2}$ then $x=X+\dfrac{\pi}{2}$ and we have, using the rule $\sin(X+\dfrac{\pi}{2})=\cos X$,

$$\lim_{x\to\frac{\pi}{2}}\frac{1-\sin(x)}{\left(x-\frac{\pi}{2}\right)}=\lim_{X\to0}\frac{1-\cos(X)}{X}$$

Now, the Taylor expansion of $\cos X$ of order 2 around $X_0 = 0$ gives (recall that O means a bounded function around X_0)

$$\cos X = 1 - \frac{X^2}{2} + O(X^2)$$

then

$$\begin{aligned}
\lim_{x\to\frac{\pi}{2}} \frac{1-\sin(x)}{\left(x-\frac{\pi}{2}\right)} &= \lim_{X\to 0} \frac{1-\cos(X)}{X} \\
&= \lim_{X\to 0} \frac{\frac{X^2}{2} + O(X^2)}{X} = \lim_{X\to 0} \left(\frac{X}{2} + O(X)\right) = 0
\end{aligned}$$

Ex4 Let $f : (a,b) \to \mathbb{R}$ in $\mathcal{C}^3(a,b)$ (means three times continuously differentiable on (a,b)) such that $f'(c) = f''(c) = 0$ and $f'''(c) > 0$ for a c in (a,b).

Let us prove first that there exists $\delta > 0$ such that $\forall x \in (a,b)$, $x \in (c-\delta, c+\delta) \implies f'''(x) > 0$. By definition of continuity of f''' at c, for $\varepsilon = f'''(c) > 0$, $\exists \delta > 0$, $\forall x \in (a,b)$,

$$\begin{aligned}
x \in (c-\delta, c+\delta) \quad &\implies \quad -\varepsilon < f'''(x) - f'''(c) < \varepsilon \\
&\implies \quad -f'''(c) < f'''(x) - f'''(c) < f'''(c) \\
&\implies \quad 0 < f'''(x) < 2f'''(c)
\end{aligned}$$

By contradiction, assume that f has a local maximum at c, then there exists $\delta' > 0$ such

$$\forall x \in (a,b), x \in (c-\delta', c+\delta') \implies f(c) \geqslant f(x).$$

Let $\delta'' = \min(\delta, \delta') > 0$, then we have

$$\forall x \in (a,b), x \in (c-\delta'', c+\delta'') \implies f(c) \geqslant f(x) \text{ and } f'''(x) > 0.$$

Now by a Taylor expansion of f of order 2 at c, we have for a d between c and $x \in (c-\delta'', c+\delta'')$, and since $f'(c) = f''(c) = 0$,

$$\begin{aligned}
f(x) &= f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(d)}{3!}(x-c)^3 \\
&= f(c) + \frac{f'''(d)}{3!}(x-c)^3 \\
\implies f(x) - f(c) &= \frac{f'''(d)}{3!}(x-c)^3
\end{aligned}$$

which changes sign from $x < c$ to $x > c$ and gives us a the contradiction. In fact, $d \in (c-\delta'', c+\delta'')$ and if $x = c + \delta''/2 \implies f(x) - f(c) \geqslant 0$ since $f'''(d) > 0$ hence the contradiction.

The reasoning for local minimum is similar.