Differentiable functions

Textbook Reading : [JL] Sections 4.1, 4.2 (Lebl)

***Exercise* 4.1.1:** *Prove the product rule. Hint: Prove and use* $f(x)g(x) - f(c)g(c) = f(x)\big(g(x) - g(c)\big) + \big(f(x) - f(c)\big)g(c)$.

***Exercise* 4.1.3:** *For* $n \in \mathbb{Z}$, *prove that* x^n *is differentiable and find the derivative, unless, of course,* $n < 0$ *and* $x = 0$. *Hint: Use the product rule.*

***Exercise* 4.1.4:** *Prove that a polynomial is differentiable and find the derivative. Hint: Use the previous exercise.*

***Exercise* 4.1.5:** *Define* $f : \mathbb{R} \to \mathbb{R}$ *by*

$$f(x) := \begin{cases} x^2 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{otherwise.} \end{cases}$$

Prove that f *is differentiable at* 0, *but discontinuous at all points except* 0.

***Exercise* 4.1.6:** *Assume the inequality* $|x - \sin(x)| \le x^2$. *Prove that* $\sin$ *is differentiable at* 0, *and find the derivative at* 0.

***Exercise* 4.1.7:** *Using the previous exercise, prove that* $\sin$ *is differentiable at all* x *and that the derivative is* $\cos(x)$. *Hint: Use the sum-to-product trigonometric identity as we did before.*

***Exercise* 4.1.8:** *Let* $f : I \to \mathbb{R}$ *be differentiable. For* $n \in \mathbb{Z}$, *let* f^n *be the function defined by* $f^n(x) := \big(f(x)\big)^n$. *If* $n < 0$, *assume* $f(x) \neq 0$ *for all* $x \in I$. *Prove that* $(f^n)'(x) = n\big(f(x)\big)^{n-1} f'(x)$.

***Exercise* 4.1.11:** *Suppose* $f : I \to \mathbb{R}$ *is bounded,* $g : I \to \mathbb{R}$ *is differentiable at* $c \in I$, *and* $g(c) = g'(c) = 0$. *Show that* $h(x) := f(x)g(x)$ *is differentiable at* c. *Hint: You cannot apply the product rule.*

***Exercise* 4.1.15:** *Prove the following simple version of L'Hôpital's rule. Suppose* $f : (a,b) \to \mathbb{R}$ *and* $g : (a,b) \to \mathbb{R}$ *are differentiable functions whose derivatives* f' *and* g' *are continuous functions. Suppose that at* $c \in (a,b)$, $f(c) = 0$, $g(c) = 0$, $g'(x) \neq 0$ *for all* $x \in (a,b)$, *and* $g(x) \neq 0$ *whenever* $x \neq c$. *Note that the limit of* $f'(x)/g'(x)$ *as* x *goes to* c *exists. Show that*

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}.$$

***Exercise* 4.2.5:** *Suppose* $f : \mathbb{R} \to \mathbb{R}$ *is a function such that* $|f(x) - f(y)| \le |x - y|^2$ *for all* x *and* y. *Show that* $f(x) = C$ *for some constant* C. *Hint: Show that* f *is differentiable at all points and compute the derivative.*

***Exercise* 4.2.7:** *Suppose $f\colon (a,b) \to \mathbb{R}$ is a differentiable function such that $f'(x) \neq 0$ for all $x \in (a,b)$. Suppose there exists a point $c \in (a,b)$ such that $f'(c) > 0$. Prove $f'(x) > 0$ for all $x \in (a,b)$.*

***Exercise* 4.2.8:** *Suppose $f\colon (a,b) \to \mathbb{R}$ and $g\colon (a,b) \to \mathbb{R}$ are differentiable functions such that $f'(x) = g'(x)$ for all $x \in (a,b)$, then show that there exists a constant C such that $f(x) = g(x) + C$.*

***Exercise* 4.2.9:** *Prove the following version of L'Hôpital's rule. Suppose $f\colon (a,b) \to \mathbb{R}$ and $g\colon (a,b) \to \mathbb{R}$ are differentiable functions and $c \in (a,b)$. Suppose that $f(c) = 0$, $g(c) = 0$, $g'(x) \neq 0$ when $x \neq c$, and that the limit of ${f'(x)}/{g'(x)}$ as x goes to c exists. Show that*

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}.$$

Compare to Exercise 4.1.15. Note: Before you do anything else, prove that $g(x) \neq 0$ when $x \neq c$.

***Exercise* 4.2.11:** *Prove the theorem Rolle actually proved in 1691:* If f is a polynomial, $f'(a) = f'(b) = 0$ for some $a < b$, and there is no $c \in (a,b)$ such that $f'(c) = 0$, then there is at most one root of f in (a,b), that is at most one $x \in (a,b)$ such that $f(x) = 0$. *In other words, between any two consecutive roots of f' is at most one root of f. Hint: Suppose there are two roots and see what happens.*

***Exercise* 4.2.12:** *Suppose $a,b \in \mathbb{R}$ and $f\colon \mathbb{R} \to \mathbb{R}$ is differentiable, $f'(x) = a$ for all x, and $f(0) = b$. Find f and prove that it is the unique differentiable function with this property.*

Exercises

1. Prove that the polynomial equation $\frac{x^{1121}}{1121} + \frac{x^{2021}}{2021} + x + 1 = 0$ has exactly one real root.
2. Compute the fourth Taylor polynomial for:
   (a) $f(x) = \sin x$ at $x = 0$.
   (b) $f(x) = \frac{1}{1-x}$ at $x = -1$.
3. Compute:
   (a)
   $$\lim_{x\to 0} \frac{x - \sin x}{x^3}$$
   (b)
   $$\lim_{x\to \frac{\pi}{2}} \frac{1 - \sin x}{\left(x - \frac{\pi}{2}\right)^2}$$
4. Suppose that $f : (a,b) \to \mathbb{R}$ is three times continuously differentiable, $c \in (a,b)$, $f'(c) = f''(c) = 0$ and $f'''(c) > 0$. Prove that f has a neither a local maximum nor a local minimum at c.