Sequencies of real numbers (corrections)

Textbook Reading : [JL] Sections 2.1, 2.2

Exercises

1. We say a set $F \subset \mathbb{R}$ is open is closed if its complement F^c is open (see assignment 3). It follows that $\emptyset$, and $\mathbb{R}$ are closed.
   (a) Let $a, b \in \mathbb{R}$ with $a < b$. We have $[a,b]^c = (-\infty, a) \cap (b, +\infty)$ which is a finite intersection of open sets (see assignment 3) then it is open.
   (b) We have $\mathbb{Z}^c = \bigcup_{n \in Z}(n, n+1)$ which is a union of open sets $(n, n+1)$ (see assignment 3) then it is open. Hence, $\mathbb{Z}$ is closed in $\mathbb{R}$.
   (c) By contradiction, assume that $\mathbb{Q}$ is closed, then the set of irrationals $\mathbb{Q}^c$ is open, i.e.
$$\forall i \in \mathbb{Q}^c, \exists \varepsilon > 0 : (x - \varepsilon, x + \varepsilon) \subset \mathbb{Q}^c,$$
   but by the density of $\mathbb{Q}$ in $\mathbb{R}$, $x - \varepsilon < x + \varepsilon \implies \exists r \in \mathbb{Q} : x - \varepsilon < r < x + \varepsilon$, which is a contradiction with $(x - \varepsilon, x + \varepsilon) \subset \mathbb{Q}^c$. Thus $\mathbb{Q}$ is not closed.
2. The proof is based on the De Moivre identity :
   (a) Let us prove that if F_λ is closed for all $\lambda \in \Lambda$ then the following (intersection) set is closed :
$$\bigcap_{\lambda \in \Lambda} F_\lambda := \{x \in \mathbb{R} : \forall \lambda \in \Lambda; x \in F_\lambda\}.$$
   It follows directly from the De Moivre identity
$$\left(\bigcap_{\lambda \in \Lambda} F_\lambda\right)^c = \bigcup_{\lambda \in \Lambda} F_\lambda^c$$
   and the fact that F_λ^c are open sets (see assignment 3).
   (b) Let $n \geqslant 1$ an integer, and let $F_1, ..., F_n \subset \mathbb{R}$. Let us prove that if $F_1, ..., F_n$ are closed sets then the following (finite union) set is closed :
$$\bigcup_{k=1}^{n} F_k := F_1 \cup F_2 \cup ... \cup F_n.$$
   Similarly, it follows directly from the De Moivre identity
$$\left(\bigcup_{k=1}^{n} F_\lambda\right)^c = \bigcap_{k=1}^{n} F_k^c$$
   and the fact that F_k^c are open sets (see assignment 3).
3. In this question, we have to show that a closed set contains the limits of its convegent sequencies. Let $F \subset \mathbb{R}$ closed and $\{x_n\}$ a sequence of elements of F that converges to $x \in \mathbb{R}$. Let us show that $x \in F$. By contradiction, assume that $x \notin F$ then $x \in F^c$ which is an open set since F is closed. Then, there exists $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subset F^c$. However, since $x_n \to x$ then for this $\varepsilon > 0$,
$$\exists M \in \mathbb{N}, \forall n : n \geqslant M \implies |x_n - x| < \varepsilon,$$
   then $x_M \in F$ and $|x_M - x| < \varepsilon$ implies $x_M \in (x - \varepsilon, x + \varepsilon) \subset F^c$ which gives a contradiction. The proof is finished.

4. **Exercise 2.2.3 in textbook :** By induction, we have to show that if $\{x_n\}$ is a convergent sequence, and $k \in \mathbb{N}^*$, then

$$\lim_{n\to+\infty} x_n^k = \left(\lim_{n\to+\infty} x_n\right)^k.$$

   (a) **base step :** for $k = 1$, the identity is evident.

   (b) **induction step :** Assume the assertion is true for a power k, i.e., if $x_n \to x$ then $x_n^k \to x^k$ and let us show that it is also true for the power $k+1$. Let $x_n \to x$, and let us write (as in the proof of convegence of a product pf convergent sequencies)

$$x_n^{k+1} - x^{k+1} = \left(x_n^k - x^k\right) x_n + (x_n - x)\, x^k,$$

   thus, since x_n is convergent then it is bounded, say $\exists B > 0$ such that $|x_n| \leqslant B$ for all n, and we have

$$\begin{aligned}\left|x_n^{k+1} - x^{k+1}\right| &\leqslant \left|x_n^k - x^k\right| |x_n| + |x_n - x|\, |x|^k, \\ &\leqslant \left|x_n^k - x^k\right| B + |x_n - x|\, |x|^k, \\ &\leqslant C\left(\left|x_n^k - x^k\right| + |x_n - x|\right),\end{aligned}$$

   where $C = \max(B, |x|^k) > 0$. Since we have $x_n \to x$ and $x_n^k \to x^k$ then for all $\varepsilon > 0$,

$$\exists M_1 \in \mathbb{N}, \forall n : n \geqslant M_1 \implies |x_n - x| < \frac{\varepsilon}{2C}, \text{ and } \exists M_k \in \mathbb{N}, \forall n : n \geqslant M_k \implies |x_n^k - x^k| < \frac{\varepsilon}{2C},$$

   hence, let $M_{k+1} = \max(M_1, M_k) \in \mathbb{N}$,then

$$\forall n : n \geqslant M_{k+1} \implies |x_n^{k+1} - x^{k+1}| < C\left(\frac{\varepsilon}{2C} + \frac{\varepsilon}{2C}\right) = \varepsilon,$$

   which implies that $x_n^{k+1} \to x^{k+1}$.

   (c) **conclusion step :** if $\{x_n\}$ is a convergent sequence, and $k \in \mathbb{N}^*$, then

$$\lim_{n\to+\infty} x_n^k = \left(\lim_{n\to+\infty} x_n\right)^k.$$

5. **Exercise 2.2.5 :** Let $x_n = \dfrac{n - \cos(n)}{n}$, $n \in \mathbb{N}^*$. Using the squeez theorem, let us show that $\{x_n\}$ converges and find the limit. We have, $-1 \leqslant \cos n \leqslant 1$,

$$1 - \frac{1}{n} \leqslant x_n = 1 - \frac{\cos(n)}{n} \leqslant 1 + \frac{1}{n}$$

   and $(1 - 1/n) \to 1$ and $(1 + 1/n) \to 1$ then $x_n \to 1$.

6. The proof is direct by the hint and the squeez theorem, since $a_0 - 1/n \to a_0$ and the constant sequence $\{a_0\}$ converges also to a_0.

7. Let $E \subset \mathbb{R}$ nonempty. We say $x \in \mathbb{R}$ is a **cluster** point of E if for every $\varepsilon > 0$, $(x - \varepsilon, x + \varepsilon) \cap E \setminus \{x\} \neq \emptyset$.

   (a) We shall prove that x is a cluster point of E if and only if there exists a sequence $\{x_n\}$ of elements of $E \setminus \{x\}$ such that $x_n \to x$.

      i. right implication $\implies$) : let $x \in \mathbb{R}$ a **cluster** point of E then by definition for every $n \in \mathbb{N}^*$,put $\varepsilon = 1/n > 0$, there exists $x_n \neq x$, such that

$$x - \frac{1}{n} < x_n < x + \frac{1}{n},$$

      thus, by the squeez theorem, the sequence $\{x_n\}$ of element of $E \setminus \{x\}$ converges to x.

      ii. left implication $\impliedby$) : assume that there exists a sequence $\{x_n\}$ of elements of $E \setminus \{x\}$ such that $x_n \to x$, then for any $\varepsilon > 0$,

$$\exists M \in \mathbb{N}, \forall n : n \geqslant M \implies |x_n - x| < \varepsilon,$$

      hence there exists $x_M \in E \setminus \{x\}$ such that $|x_M - x| < \varepsilon$, which means that $x_M \in (x - \varepsilon, x + \varepsilon)$, thus $(x - \varepsilon, x + \varepsilon) \cap E \setminus \{x\} \neq \emptyset$, then x is a cluster point of E, which ends the proof.

(b) Let A the set of all cluster points of E. Consider its complement A^c and let us show that is is open. For all $x \in A^c$, x is not a cluster point of E, then by definition,

$$\exists \varepsilon > 0 : (x - \varepsilon, x + \varepsilon) \cap E \diagdown \{x\} = \emptyset,$$

which implies that

$$(x - \varepsilon, x) \cap E = \emptyset \text{ and } (x, x + \varepsilon) \cap E = \emptyset.$$

Let us show that for any $y \in (x - \varepsilon, x + \varepsilon)$, y is not a cluster point of E. In fact, we have :

i. Case 1 : If $y = x$ then there is nothing to show since x is not a cluster point by assumption.

ii. Case 2 : If $y \in (x - \varepsilon, x)$ then putting $\varepsilon' = \min(x - y, y - (x - \varepsilon)) > 0$, we obtain $(y - \varepsilon', y + \varepsilon') \cap E = \emptyset$ which gives also

$$(y - \varepsilon', y + \varepsilon') \cap E \diagdown \{y\} = \emptyset,$$

and implies that y is not a cluster point of E.

iii. Case 3 : If $y \in (x, x + \varepsilon)$ then putting $\varepsilon'' = \min(y - x, (x + \varepsilon) - y) > 0$, we obtain $(y - \varepsilon'', y + \varepsilon'') \cap E = \emptyset$ which gives also

$$(y - \varepsilon'', y + \varepsilon'') \cap E \diagdown \{y\} = \emptyset,$$

and implies that y is not a cluster point of E.

Conclusion : $\exists \varepsilon > 0 : (x - \varepsilon, x + \varepsilon) \subset A^c$ which implies that A^c is open and therefore A is closed.