Sets, Set Operations, and Mathematical Induction (corrections)

Reading : [JL] Section 0.3

Exercises

1. Exercise 0.3.6

a) Let's pove that

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \quad (1)$$

By definition

$$\begin{aligned} A \cap (B \cup C) &= \{x | x \in A \text{ and } x \in B \cup C\} \\ &= \{x | x \in A \text{ and } (x \in B \text{ or } x \in C)\} \\ &= \{x | \mathcal{P}(x)\} \end{aligned}$$

$$\begin{aligned} (A \cap B) \cup (A \cap C) &= \{x | x \in A \cap B \text{ or } x \in A \cap C\} \\ &= \{x | (x \in A \text{ and } x \in B \text{ or }) \text{ or } (x \in A \text{ and } x \in C)\} \\ &= \{x | \mathcal{Q}(x)\} \end{aligned}$$

Since, in mathematical logic, we have equivalence between assertions $\mathcal{P}(x)$ and $\mathcal{Q}(x)$ then equality (1) holds true. Recall De Morgan's law in mathematical logic can be proved as afollows :

p	q	r	(p **and** q)	(p **and** r)	(p **and** q) **or** (p **and** r)	(q **or** r)	p **and** (q **or** r)
1	**1**	**1**	**1**	**1**	**1**	**1**	**1**
1	**1**	**0**	**1**	**0**	**1**	**1**	**1**
1	**0**	**1**	**0**	**1**	**1**	**1**	**1**
1	**0**	**0**	**0**	**0**	**0**	**0**	**0**
0	**1**	**1**	**0**	**0**	**0**	**1**	**0**
0	**1**	**0**	**0**	**0**	**0**	**1**	**0**
0	**0**	**1**	**0**	**0**	**0**	**1**	**0**
0	**0**	**0**	**0**	**0**	**0**	**0**	**0**

b) We have to pove that

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \quad (2)$$

It follows similarly from the following De Morgan's law in mathematical logic :

p	q	r	(p **or** q)	(p **or** r)	(p **or** q) **and** (p **or** r)	(q **and** r)	p **or** (q **and** r)
1	**1**	**1**	**1**	**1**	**1**	**1**	**1**
1	**1**	**0**	**1**	**1**	**1**	**1**	**1**
1	**0**	**1**	**1**	**1**	**1**	**1**	**1**
1	**0**	**0**	**1**	**1**	**1**	**0**	**1**
0	**1**	**1**	**1**	**1**	**1**	**1**	**1**
0	**1**	**0**	**1**	**0**	**1**	**1**	**1**
0	**0**	**1**	**0**	**1**	**1**	**1**	**1**
0	**0**	**0**	**0**	**0**	**0**	**0**	**0**

2. Exercise 0.3.11 Denote $P(n)$ the assertion $n < 2^n$. One has to prove by induction that $P(n)$ is true for all n.

base case : for $n = 0$, $0 < 2^0$ is true.

Induction step : assume $P(n)$ true and let us show that $P(n+1)$ is true so. We have, for all $n \in \mathbb{N}$, $1 \leqslant 2^n$ hence,

$$n \leqslant 2^n \implies n + 1 \leqslant 2^n + 1 < 2^n + 2^n = 2.2^n = 2^{n+1},$$

thus $P(n) \implies P(n+1)$.

Conclusion By induction, we deduce that $P(n)$ is true for all $n \in \mathbb{N}$.

3. Exercise 0.3.12 Let A any finite set of cardinality $n = |A|$ and $\mathcal{P}(A)$ the set of parts of A. Denote $P(n)$ the assertion

$$|\mathcal{P}(A)| = 2^n.$$

One has to prove by induction that $P(n)$ is true for all $n \geqslant 0$.

base case : for $n = 0$, $A = \emptyset$ and $\mathcal{P}(A) = \{\emptyset\}$, hence $|\mathcal{P}(A)| = 1 = 2^0$, is true.

Induction step : assume $P(n)$ true and let us show that $P(n+1)$ is true so. Let $n \geqslant 0$, and $|A| = n+1 \geqslant 1$. Then A contains at least one element x. Hence, the cardinality of $\mathcal{P}(A)$ equals $C_x + N_x$ the number of parts of A that contain x (denoted by C_x) plus the number of parts of A that do not contain x (denoted by N_x). Let the set $B = A \setminus \{x\}$, then $|\mathcal{P}(B)| = n$. One has by induction

$$N_x = |\mathcal{P}(B)| = 2^n.$$

On the other hand, any subset of A that do not contain x can be written as the union of $\{x\}$ and a subset of B, we have as many subsets which contain x as those which do not contain it. Henceforth,

$$C_x = |\mathcal{P}(B)| = 2^n.$$

Thus,

$$|\mathcal{P}(A)| = C_x + N_x = 2^n + 2^n = 2^{n+1}.$$

Therefore, $P(n) \implies P(n+1)$.

Conclusion By induction, we deduce that $P(n)$ is true for all $n \in \mathbb{N}$.

4. Exercise 0.3.15 Denote $P(n)$ the assertion $n^3 + 5n$ is divisible by 6 and let us prove by induction that $P(n)$ is true for all $n \in \mathbb{N}$.

base case : for $n = 0$, 0 is divisible by 6 : is true.

Induction step : assume $P(n)$ true and let us show that $P(n+1)$ is true so. We have, for all $n \in \mathbb{N}$, ther exists $k \in \mathbb{N}$ such that

$$n^3 + 5n = 6k.$$

thus

$$\begin{aligned}(n+1)^3 + 5(n+1) &= n^3 + 3n^2 + 3n + 1 + 5n + 5 \\ &= n^3 + 5n + 3n^2 + 3n + 6 \\ &= n^3 + 5n + 3n(n+1) + 6 \\ &= 6k + 3n(n+1) + 6.\end{aligned}$$

Since, $n(n+1)$ is even number, there exists $k' \in \mathbb{N}$ such that $n(n+1) = 2k'$. Hence

$$(n+1)^3 + 5(n+1) = 6k + 6k' + 6$$

is divisible by 6. Therefore, $P(n) \implies P(n+1)$.

Conclusion By induction, we deduce that $P(n)$ is true for all $n \in \mathbb{N}$.

5. Exercise 0.3.19 One may take $\mathbb{N}$ as an example : $\mathbb{N}$ is countably infinite, put $A_n = \{n\}$ for $n \in \mathbb{N}$, then

$$\mathbb{N} = \bigcup_{n\in\mathbb{N}} A_n = \{0\} \cup \{1\} \cup \{2\} \cup \ldots$$

5. Using the theorem (that essentially gives existence and unicity of prime number decomposition), one has to prove that

$$|\{q \in \mathbb{Q} | q > 0\}| = |\mathbb{N}|.$$

a) We have $q = 4/15 \in \mathbb{Q} \setminus \mathbb{N}$,

$$q = \frac{2^2}{3^1 \times 5^1} \implies f(q) = 2^{2(2)} \times 3^{2(1)-1} \times 5^{2(1)-1} = 240.$$

Let us find now q such that $f(q) = 108$, we have the unique prime numbers decomposition

$$108 = 2^2 \times 3^3,$$

thus

$$f(q) = 108 = 2^{2(1)} \times 3^{2(2)-1},$$

hence

$$q = \frac{2^1}{3^2} = \frac{2}{9}.$$

b) $f : \{q \in \mathbb{Q} | q > 0\} \to \mathbb{N}$ is bijective if it is 1-1 and onto.

Proof of 1-1 ? Let q and q' such that $f(q) = f(q')$, then :

case if $f(q) = f(q') = 1$ here we obtain $q = q' = 1$,

case when $f(q) = f(q') = F$ and $F \neq 1$, then $F \in \mathbb{N} \setminus \{1\}$ admits a unique prime number decomposition (ordered by odd and even powers) as follows

$$F = F_1^{2t_1} F_2^{2t_2} ... F_k^{2t_k} F_{k+1}^{2t_{k+1}-1} F_{k+2}^{2t_{k+2}-1} ... F_m^{2t_m-1}$$

where $t_1, t_2, ..., t_m$ are positive integers. Thus, if all the powers are even then

$$F = F_1^{2t_1} F_2^{2t_2} ... F_m^{2t_m} \implies q = q' = F_1^{t_1} F_2^{t_2} ... F_m^{t_m}$$

by unicity of prime factor decomposition. On the other hand, if there are even and odd powers, then

$$F = F_1^{2t_1} F_2^{2t_2} ... F_k^{2t_k} F_{k+1}^{2t_{k+1}-1} F_{k+2}^{2t_{k+2}-1} ... F_m^{2t_m-1} \implies q = q' = \frac{F_1^{t_1} F_2^{t_2} ... F_m^{t_m}}{F_{k+1}^{t_{k+1}} F_{k+2}^{t_{k+2}} ... F_m^{t_m}}$$

also by unicity of prime number decomposition.

Conclusion : f is 1-1 .

Proof of onto ? Let $F \in \mathbb{N}$, then :

case if $F = 1$ then thers exists $q = 1$, such that $f(q) = f(1) = 1$.

else if $F \in \mathbb{N} \setminus \{1\}$ then it admits a unique prime number decomposition (ordered by odd and even powers) as follows

$$F = F_1^{2t_1} F_2^{2t_2} ... F_k^{2t_k} F_{k+1}^{2t_{k+1}-1} F_{k+2}^{2t_{k+2}-1} ... F_m^{2t_m-1}$$

where $t_1, t_2, ..., t_m$ are positive integers. Thus, if all the powers are even then

$$F = F_1^{2t_1} F_2^{2t_2} ... F_m^{2t_m} \implies q = F_1^{t_1} F_2^{t_2} ... F_m^{t_m}$$

thus ther exists q such that $f(q) = F$. On the other hand, if there are both even and odd powers, then

$$F = F_1^{2t_1} F_2^{2t_2} ... F_k^{2t_k} F_{k+1}^{2t_{k+1}-1} F_{k+2}^{2t_{k+2}-1} ... F_m^{2t_m-1} \implies q = \frac{F_1^{t_1} F_2^{t_2} ... F_m^{t_m}}{F_{k+1}^{t_{k+1}} F_{k+2}^{t_{k+2}} ... F_m^{t_m}}$$

thus ther exists q such that $f(q) = F$.

Conclusion : f is onto.

Therefore $f : \{q \in \mathbb{Q} | q > 0\} \to \mathbb{N}$ is a bijection.

Conclusion :

$$|\{q \in \mathbb{Q} | q > 0\}| = |\mathbb{N}|.$$