18.100A: Complete Lecture Notes

Lecture 16

Weierstrass's Example of a Continuous and Nowhere Differentiable Function

Theorem 1

If $f : I \to \mathbb{R}$ is differentiable at $c \in I$, then f is continuous at c.

Proof: Since every point of I is a cluster point of I, f is continuous at $c \in I \iff \lim_{x\to c} f(x) = f(c)$. Now,

$$\begin{aligned}
\lim_{x\to c} f(x) &= \lim_{x\to c}(f(x) - f(c) + f(c)) \\
&= \lim_{x\to c}\left((x-c)\frac{f(x)-f(c)}{x-c} + f(c)\right) \\
&= 0 \cdot f'(c) + f(c) = f(c).
\end{aligned}$$

□

Question 2. *Is the converse true? Does f being continuous imply that f is differentiable?*

The answer, is **no**.

Example 3

Let $f(x) = |x|$. Then, f is not differentiable at 0.

Proof: We find a sequence $x_n \to 0$ such that

$$\lim_{n\to\infty} \frac{f(x_n) - f(0)}{x_n - 0} \text{ does not exist.}$$

Let $x_n = \frac{(-1)^n}{n}$. Then, $\lim_{n\to\infty} x_n = 0$. However,

$$\frac{f(x_n) - f(0)}{x_n - 0} = \frac{|(-1)^n/n|}{(-1)^n/n} = (-1)^n,$$

and $\lim_{n\to\infty}(-1)^n$ does not exist. ■

Question 4. *If $f : \mathbb{R} \to \mathbb{R}$ is continuous, then does there exist a $c \in \mathbb{R}$ such that f is differentiable at c?*

The answer is again **no**! This was shown by Weierstrass, aka the Godfather.

The basic idea is to build a continuous function that is a sum of highly oscillating functions.

Remark 5. *Note that we number the upcoming theorems so we may reference them a bit later in this lecture.*

Theorem 6 (Theorem I)
We will show the following

1. $\forall x, y \in \mathbb{R}$, $|\cos x - \cos y| \leq |x - y|$.
2. Let $c \in \mathbb{R}$. Then, for all $K \in \mathbb{N}$, $\exists y \in (c + \pi/K, c + 3\pi/K)$ such that

$$|\cos(Kc) - \cos(Ky)| \geq 1.$$

Proof:

1. In the proof of continuity of $\sin x$, we showed that $\forall x, y \in \mathbb{R}$, $|\sin x - \sin y| \leq |x - y|$. Thus,

$$|\cos x - \cos y| = |\sin(x + \pi/2) - \sin(y + \pi/2)| \leq |x - y|.$$

2. The function $f(x) = \cos(Kx)$ is a $\frac{2\pi}{K}$-periodic function. In particular, $([-1, 1] \setminus \cos(Kc)) \subset f(c + \pi/K, c + 3\pi/K)$.

   If $\cos Kc \geq 0$, then we choose y such that $\cos(Ky) = -1$. If $\cos(Kc) < 0$, then we choose y such that $\cos(Ky) = 1$. This completes the proof

$\square$

Theorem 7 (Theorem II)
For all $a, b, c \in \mathbb{R}$,

$$|a + b + c| \geq |a| - |b| - |c|.$$

Proof: We apply the Triangle Inequality twice:

$$|a| = |a + b + b + (-b) + (-c)| \leq |a + b + b| + |b + c| \leq |a + b + c| + |b| + |c|.$$

$\square$

Theorem 8 (Theorem III)
We will show the following:

1. $\forall x \in \mathbb{R}$, $\sum_{k=0}^{\infty} \frac{\cos(160^k x)}{4^k}$ is absolutely convergent.
2. The function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \sum_{k=0}^{\infty} \frac{\cos(160^k x)}{4^k}$ is bounded and continuous.

Proof:

1. $\forall k$, $\left|\frac{\cos(160^k x)}{4^k}\right| \leq 4^{-k}$. Hence, by the Comparison Test,

$$\sum_{k=0}^{\infty} \left|\frac{\cos(160^k x)}{4^k}\right| \quad \text{converges.}$$

2. For all $x \in \mathbb{R}$, $|f(x)| \leq \sum_{k=0}^{\infty} \frac{|\cos(160^k x)|}{4^k} \leq \sum 4^{-k} = \frac{4}{3}$. Therefore, f is bounded.

   We now show that f is continuous over $\mathbb{R}$. Suppose $c \in \mathbb{R}$ and $x_n \to c$. Note that $\{|f(x_n) - f(c)|\}_n$ is bounded, and thus

$$\lim_{n\to\infty} |f(x_n) - f(c)| = 0 \iff \limsup_{n\to\infty} |f(x_n) - f(c)| = 0.$$

We claim that for all $\epsilon > 0$, $\limsup_{n\to\infty} |f(x_n)-f(c)| \leq \epsilon$. Let $\epsilon > 0$. Choose M_0 such that $\sum_{k=M_0+1}^{\infty} 4^{-k} < \frac{\epsilon}{2}$. Then,

$$\begin{aligned}
\limsup_{n\to\infty} |f(x_n) - f(c)| &= \limsup_n \left| \sum_{k=0}^{M_0} \frac{\cos(160^k x_n)}{4^k} - \frac{\cos(160^k c)}{4^k} + \sum_{k=M_0+1}^{\infty} \frac{\cos(160^k x_n)}{4^k} - \frac{\cos(160^k c)}{4^k} \right| \\
&\leq \limsup_n \sum_{k=0}^{M_0} 4^{-k} |\cos(160^k x_n) - \cos(160^k c)| + \sum_{k=M_0+1}^{\infty} 4^{-k}(|\cos(160^k x_n)| + |\cos(160^k c)|) \\
&\leq \limsup_n \left(\sum_{k=0}^{M_0} 40^k \right) |x_n - c| + \epsilon = \epsilon.
\end{aligned}$$

□

Theorem 9 (Weierstrass)
The function $f(x) = \sum_{k=0}^{\infty} \frac{\cos(160^k x)}{4^k}$ is nowhere differentiable.

Proof: Let $c \in \mathbb{R}$. We will construct a sequence $x_n \to c$ such that $\left\{ \frac{f(x_n)-f(c)}{x_n - c} \right\}_n$ is unbounded. By Theorem I 2), $\forall n \in \mathbb{N}$ there exists an x_n such that **a)** $\frac{\pi}{160^n} < x_n - c < \frac{3\pi}{160^n}$ and **b)** $|\cos(160^n c) - \cos(160^n x_n)| \geq 1$.

By **a)**, $x_n \neq 0 \forall n$ and $|x_n - c| \leq \frac{3\pi}{160^n} \to 0$. Let $f_k(x) = \frac{\cos(160^k x)}{4^k}$ so $f(x) = \sum f_k(x)$. Let $n \in \mathbb{N}$. Thus, denote

$$\begin{aligned}
f(c) - f(x_n) &= f_n(c) - f_n(x_n) + \sum_{k=0}^{n-1}(f_k(c) - f_k(x_n)) + \sum_{k=n}^{\infty}(f_k(c) - f_k(x_n)) \\
&:= a_n + b_n + c_n.
\end{aligned}$$

Therefore, by Theorem II,

$$|f(c) - f(x_n)| \geq |a_n| - |b_n| - |c_n|.$$

By **b)**, $|a_n| = 4^{-n}|\cos(160^k x_n) - \cos(160^k c)| \geq 4^{-n}$. Furthermore, we have

$$|b_n| \leq \sum_{k=0}^{n-1} 4^{-k} |\cos(160^k c) - \cos(160^k x_n)| \leq \sum_{k=0}^{n-1} 4^{-k} \cdot 160^k |x_n - c| \leq \frac{3\pi}{160^n} \sum_{k=0}^{n-1} 40^k = \frac{3\pi}{160^n} \cdot \frac{40^n - 1}{39} \leq \frac{4^{-n+1}}{13}.$$

Finally, we have

$$|c_n| \leq \sum_{k=n+1}^{\infty} 4^{-k}(|\cos(160^k c)| + |\cos(160^k x_n)|) \leq 2 \sum_{k=n+1}^{\infty} 4^{-k} = 2 \cdot 4^{-n-1} \cdot \frac{4}{3} = 4^{-n} \frac{2}{3}.$$

Therefore, by the above inequalities, we have

$$|f(c) - f(x_n)| \geq 4^{-n} \left(1 - \frac{4}{13} - \frac{2}{3} \right) = 4^{-n} \cdot \frac{1}{39}.$$

Therefore,

$$\frac{|f(c) - f(x_n)|}{|c - x_n|} \geq \frac{160^n}{3\pi} \cdot 4^{-n} \cdot \frac{1}{39} = \frac{40^n}{117\pi}.$$

Thus, $\left\{ \frac{f(x_n)-f(c)}{x_n - c} \right\}_n$ is unbounded. □

Remark 10. *In other words, this proof by Weierstrass shows that there exists a continuous function that is nowhere differentiable!*