Corrigé du Td 2 corps des nombres complexes
2023/2024

Exercice1. Forme algèbrique
$z_1 = (2-i)\,\overline{(3+8i)} = 14-13i$
$z_2 = (1+i)\,\overline{(1+i)} = 2$
$z_3 = (1+i)^3 = 2i-2$
$z_4 = \frac{1}{1+i} = \frac{1-i}{(1+i)(1-i)} = \frac{1}{2} - \frac{i}{2}$
$z_5 = \frac{1-2i}{3+i} = \frac{(1-2i)(3-i)}{10} = \frac{1}{10} - \frac{7}{10}i.$
Exercice2. Forme trigonométrique: $z = |z|(\cos\theta + i\sin\theta)$
$z_1 = 1+i\sqrt{3} \Rightarrow |z_1| = \sqrt{1^2+\left(\sqrt{3}\right)^2} = 2$ et $\left\{\begin{array}{l}\cos\theta = \frac{1}{2}\\ \sin\theta = \frac{\sqrt{3}}{2}\end{array}\right. \Rightarrow \theta = \frac{\pi}{3} + 2k\pi, k\in\mathbb{Z}.$

$$z_1 = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

$z_2 = 1-i \Rightarrow |z_2| = \sqrt{2}$ et $\left\{\begin{array}{l}\cos\theta = \frac{1}{\sqrt{2}}\\ \sin\theta = \frac{-1}{\sqrt{2}}\end{array}\right. \Rightarrow \theta = -\frac{\pi}{4} + 2k\pi, k\in\mathbb{Z}.$

$$z_1 = \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$$

$z_3 = \frac{1+i}{-1-i\sqrt{3}} \Rightarrow |z_2| = \frac{|1+i|}{\left|-1-i\sqrt{3}\right|} = \frac{\sqrt{2}}{2}$ et $\arg z_3 = \arg(1+i) - \arg\left(-1-i\sqrt{3}\right) = \theta_1 - \theta_2$
$\left\{\begin{array}{l}\cos\theta_1 = \frac{1}{\sqrt{2}}\\ \sin\theta_1 = \frac{1}{\sqrt{2}}\end{array}\right. \Rightarrow \theta_1 = \frac{\pi}{4} + 2k\pi, k\in\mathbb{Z}.$ $\left\{\begin{array}{l}\cos\theta_2 = \frac{-1}{2}\\ \sin\theta_2 = \frac{-\sqrt{3}}{2}\end{array}\right. \Rightarrow \theta_1 = \pi + \frac{\pi}{3} + 2k\pi = \frac{4\pi}{3} + 2k\pi, k\in\mathbb{Z}.$

$$\begin{aligned} z_3 &= \frac{\sqrt{2}}{2}\left(\cos\left(\frac{\pi}{4} - \frac{4\pi}{3}\right) + i\sin\left(-\frac{\pi}{4} - \frac{4\pi}{3}\right)\right) \\ &= \frac{\sqrt{2}}{2}\left(\cos\left(-\frac{13\pi}{12}\right) + i\sin\left(-\frac{13\pi}{12}\right)\right). \end{aligned}$$

$z_4 = (1+i)\left(-1-i\sqrt{3}\right) \Rightarrow |1+i|\left|-1-i\sqrt{3}\right| = 2\sqrt{2}.$
$\arg z_4 = \arg(1+i) + \arg\left(-1-i\sqrt{3}\right) = \frac{\pi}{4} + \frac{4\pi}{3} = \frac{19\pi}{12}.$

$$z_4 = 2\sqrt{2}\left(\cos\left(\frac{19\pi}{12}\right) + i\sin\left(\frac{19\pi}{12}\right)\right)$$

2. $z = \frac{4+4i}{1-i\sqrt{3}}$
1. Forme algèbrique $z = \frac{4+4i}{1-i\sqrt{3}} = \left(1-\sqrt{3}\right) + i\left(1+\sqrt{3}\right).$
2. Forme trigonométrique . $|z| = \frac{|4+4i|}{\left|1-i\sqrt{3}\right|} = 2\sqrt{2}, \arg z = \underbrace{\arg(4+4i)}_{\theta_1} - \underbrace{\arg\left(1-i\sqrt{3}\right)}_{\theta_2}.$

$\begin{cases} \cos\theta_1 = \frac{1}{\sqrt{2}} \\ \sin\theta_1 = \frac{1}{\sqrt{2}} \end{cases} \Rightarrow \theta_1 = \frac{\pi}{4} + 2k\pi, k \in \mathbb{Z}$. $\begin{cases} \cos\theta_2 = \frac{1}{2} \\ \sin\theta_2 = \frac{-\sqrt{3}}{2} \end{cases} \Rightarrow \theta_1 = -\frac{\pi}{3} + 2k\pi, k \in \mathbb{Z}$.

$$z = 2\sqrt{2}\left(\cos\left(\frac{7\pi}{12}\right) + i\sin\left(\frac{7\pi}{12}\right)\right)$$

$$\frac{1}{z} = \frac{1}{2\sqrt{2}}\left(\cos\left(-\frac{7\pi}{12}\right) + i\sin\left(-\frac{7\pi}{12}\right)\right),\ \text{car}\ \left|\frac{1}{z}\right| = \frac{1}{|z|}\ \text{et}\ \arg\frac{1}{z} = -\arg z.$$

. $z^{2009} = \left(2\sqrt{2}\right)^{2009}\left(\cos\left(\frac{2009.7}{12}\right) + i\sin\left(\frac{2009.7}{12}\right)\right) = 2^{3013.5}\left(\cos\left(\left(1171 + \frac{11}{12}\right)\pi\right) + i\sin\left(\left(1171 + \frac{11}{12}\right)\pi\right)\right) =$
$= 2^{3013.5}\left(\cos\left(\left(\frac{23}{12}\right)\pi\right) + i\sin\left(\left(\frac{23}{12}\right)\pi\right)\right) = 2^{3013.5}\left(\cos\left(\left(\frac{23}{12}\right)\pi\right) + i\sin\left(\left(\frac{23}{12}\right)\pi\right)\right).$
. $\overline{z} = |z|\left(\cos(-\arg z) + i\sin(-\arg z)\right) = z = 2\sqrt{2}\left(\cos\left(-\frac{7\pi}{12}\right) + i\sin\left(-\frac{7\pi}{12}\right)\right).$
3. Forme exponentielle:
$z_1 = 2 - 2i = 2\sqrt{2}e^{-i\frac{\pi}{4}}, z_2 = 3\sqrt{3} - 3i = 6e^{-i\frac{\pi}{6}}, z_3 = \frac{5}{4}i = \frac{5}{4}e^{i\frac{\pi}{2}}, z_4 = -1 = e^{i\pi}$.

Exercice 3. $z_1 = 1 + i\sqrt{3}, z_2 = 1 + i, z_3 = \frac{z_1}{z_2}$.
1. $z_3 = \frac{1-\sqrt{3}}{2} + i\frac{1+\sqrt{3}}{2}$.
2. $|z_3| = \frac{|z_1|}{|z_2|} = \frac{2}{\sqrt{2}} = \sqrt{2}, \arg z_3 = \arg z_1 - \arg z_2 = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$.

$$z_3 = \sqrt{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right).$$

3. On a $\begin{cases} z_3 = \frac{1-\sqrt{3}}{2} + i\frac{1+\sqrt{3}}{2} \\ z_3 = \sqrt{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right) \end{cases}$
Par identificationde la forme algèbrique avec la forme trigonométrique
$\begin{cases} \frac{1-\sqrt{3}}{2} = \sqrt{2}\cos\frac{\pi}{12} \\ \frac{1+\sqrt{3}}{2} = \sqrt{2}\sin\frac{\pi}{12} \end{cases} \Rightarrow \begin{cases} \cos\frac{\pi}{12} = \frac{1-\sqrt{3}}{2\sqrt{2}} \\ \sin\frac{\pi}{12} = \frac{1+\sqrt{3}}{2\sqrt{2}} \end{cases}.$

Exercice 4.
1. Racine carrée de $1, i, 3 + 4i$.
Pour $w = 1$, cherchons un $z = x + iy$ tel que $z^2 = w$

$$\begin{aligned} z^2 &= w = 1 \\ x^2 - y^2 + 2ixy &= 1,\ \text{et}\ |z| = 1, \end{aligned}$$

d'où

$$\begin{cases} x^2 - y^2 = 1 \\ 2xy = 0 \\ x^2 + y^2 = 1 \end{cases} \Rightarrow \begin{cases} x = 1 \Rightarrow y = 0 \\ x = -1 \Rightarrow y = 0 \end{cases} \Rightarrow z_1 = 1 \text{ ou } z_2 = -1$$

Pour $w = i$ cherchons un $z = x + iy$ telle que $z^2 = w$

$$\begin{aligned} z^2 &= w = i \\ x^2 - y^2 + 2ixy &= i,\ \text{et}\ |z| = 1, \end{aligned}$$

d'où

$$\left\{\begin{array}{c} x^2-y^2=0 \\ 2xy=1 \\ x^2+y^2=1 \end{array}\right. \Rightarrow \left\{\begin{array}{l} x=\frac{1}{\sqrt{2}} \Rightarrow y=\frac{1}{2\sqrt{2}} \\ x=\frac{-1}{\sqrt{2}} \Rightarrow y=\frac{-1}{2\sqrt{2}} \end{array}\right. \Rightarrow z_1=\frac{1}{\sqrt{2}}+i\frac{1}{2\sqrt{2}} \text{ ou } z_2=\frac{-1}{\sqrt{2}}-i\frac{1}{2\sqrt{2}}.$$

Pour $w=3+4i$ cherchons un $z=x+iy$ tel que $z^2=w$

$$\begin{aligned} z^2 &= w=3+4i \\ x^2-y^2+2ixy &= 3+4i, \text{ et } |z|=5, \end{aligned}$$

d'où

$$\left\{\begin{array}{c} x^2-y^2=3 \\ 2xy=4 \\ x^2+y^2=5 \end{array}\right. \Rightarrow \left\{\begin{array}{c} x=2 \Rightarrow y=1 \\ x=-2 \Rightarrow y=-1 \end{array}\right. \Rightarrow z_1=2+i \text{ ou } z_2=-2-i.$$

2. Résoudre les équations:

$z^2+z+1=0, \Delta=-3=i^2 3 \Rightarrow \left\{\begin{array}{c} z_1=\frac{-1+i\sqrt{3}}{2}, \\ z_2=\frac{-1-i\sqrt{3}}{2} \end{array}\right.$.

$z^2-(1+2i)\,z+i=0, \Delta=-1=i^2 \Rightarrow \left\{\begin{array}{c} z_1=\frac{1+3i}{2} \\ z_2=\frac{1+i}{2} \end{array}\right.$.

3. Racine cubique de $2-2i$.

$w=2-2i=2\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right)$, cherchons un $z=r(\cos\theta+i\sin\theta)$ tel que

$$\begin{aligned} z^3 &= 2\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right) \\ r^3(\cos 3\theta+i\sin 3\theta) &= 2\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right) \end{aligned}$$

$$\left\{\begin{array}{c} r^3=2\sqrt{2} \\ 3\theta=-\frac{\pi}{4}+2k\pi, k=0,1,2. \end{array}\right.$$

$$\left\{\begin{array}{c} r=\sqrt[3]{2\sqrt{2}} \\ \theta=-\frac{\pi}{12}+\frac{2k\pi}{3}, k=0,1,2. \end{array}\right.$$

$$\begin{aligned} k &= 0 \Rightarrow \theta=\frac{-\pi}{12} \Rightarrow z_1=\sqrt[3]{2\sqrt{2}}\left(\cos\frac{-\pi}{12}+i\sin\left(\frac{-\pi}{12}\right)\right) \\ k &= 1 \Rightarrow \theta=\frac{7\pi}{12} \Rightarrow z_2=\sqrt[3]{2\sqrt{2}}\left(\cos\frac{-7\pi}{12}+i\sin\left(\frac{-7\pi}{12}\right)\right) \\ k &= 2 \Rightarrow \theta=\frac{15\pi}{12} \Rightarrow z_2=\sqrt[3]{2\sqrt{2}}\left(\cos\frac{15\pi}{12}+i\sin\left(\frac{15\pi}{12}\right)\right). \end{aligned}$$

Exercice 5: $a,b\in\mathbb{N}$

1. $\left(a+b\sqrt{3}\right)=4+2\sqrt{3} \Leftrightarrow a^2+3b^2+2ab\sqrt{3}=4+2\sqrt{3}$.

$\left\{\begin{array}{c} a^2+3b^2=4 \\ 2ab=2 \end{array}\right. \Rightarrow \left\{\begin{array}{c} a^2+3b^2=4 \\ ab=1 \end{array}\right. \Rightarrow \{a=1 \text{ et } b=1 \text{ d'où } \left(1+\sqrt{3}\right)^2 = 4+2\sqrt{3}$.

2. $z^2-\left(2+\left(1-\sqrt{3}\right)\right)z+1+\sqrt{3}+\left(1-\sqrt{3}\right)i=0$.

$\Delta=-4-2\sqrt{3}=-\left(4+2\sqrt{3}\right)=i^2\left(1+\sqrt{3}\right)^2 \Rightarrow \left\{\begin{array}{c} z_1=1+i \\ z_2=1-i\sqrt{3} \end{array}\right.$.

$|z_1|=\sqrt{2}, |z_2|=2$.

3. $\frac{z_1}{z_2}=\frac{1+i}{1-i\sqrt{3}}=\frac{1-\sqrt{3}}{4}+i\frac{1+\sqrt{3}}{4}$.

$\arg\frac{z_1}{z_2}=\arg z_1-\arg z_2=\frac{7\pi}{12}, \left|\frac{z_1}{z_2}\right|=\frac{\sqrt{2}}{2}$.

$$\frac{z_1}{z_2}=\frac{\sqrt{2}}{2}\left(\cos\left(\frac{7\pi}{12}\right)+i\sin\left(\frac{7\pi}{12}\right)\right).$$

Par identification de la forme algèbrique et la forme trigonométrique.

$\left\{\begin{array}{c} \frac{\sqrt{2}}{2}\cos\left(\frac{7\pi}{12}\right)=\frac{1-\sqrt{3}}{4} \\ \frac{\sqrt{2}}{2}\sin\left(\frac{7\pi}{12}\right)=\frac{1+\sqrt{3}}{4} \end{array}\right. \Rightarrow \left\{\begin{array}{c} \cos\left(\frac{7\pi}{12}\right)=\frac{1-\sqrt{3}}{2\sqrt{2}} \\ \sin\left(\frac{7\pi}{12}\right)=\frac{1+\sqrt{3}}{2\sqrt{2}} \end{array}\right.$.

Exercice 6:

1. $z_A=2e^{i\frac{\pi}{6}}, z_B=2e^{-i\frac{\pi}{6}}, z_E=-\sqrt{3}-i, z_F=2i$.

2. a) $\left(\frac{z_A}{2}\right)^{2013}=e^{i\frac{\pi}{6}2013}=e^{i\pi\left(335+\frac{1}{2}\right)}=\underbrace{e^{i\pi 334}}_{=1}e^{i\frac{3}{2}\pi}=-i$.

$\left(i\frac{z_E}{2}\right)^{2013}=(-1)^{2013}\left(e^{i\pi\frac{2}{3}2013}\right)=(-1)\underbrace{\left(e^{i2\pi 671}\right)}_{=1}=-1$.

D'où $\left(\frac{z_A}{2}\right)^{2013}+\left(\frac{iz_E}{2}\right)^{2013}=-i-1$.

$2\alpha=\left(-1+\sqrt{3}\right)+i\left(1+\sqrt{3}\right)$.

$z_D=\alpha^2=-\sqrt{3}+i=2e^{i\frac{5}{6}\pi}$.

b) $\frac{z_D}{z_E}=-ie^{i\frac{\pi}{6}}=\underbrace{e^{i\pi}}_{=-1}\underbrace{e^{i\frac{\pi}{2}}}_{=i}e^{i\frac{\pi}{6}}=e^{i\frac{5\pi}{3}}$.

$$\begin{aligned}\left(\frac{z_D}{z_E}\right)^n &= e^{i\frac{5n\pi}{3}}=e^{in\left(\frac{5\pi}{3}-2\pi\right)}=e^{-in\frac{\pi}{3}} \\ &= \cos\left(-n\frac{\pi}{3}\right)+i\sin\left(-n\frac{\pi}{3}\right)\end{aligned}$$

$$\begin{aligned}\left(\frac{z_D}{z_E}\right)^n &\in \mathbb{R} \Leftrightarrow \sin\left(-n\frac{\pi}{3}\right)=0 \Rightarrow -n\frac{\pi}{3}=k\pi, k\in\mathbb{Z}_- \\ n &= -3k, k\in\mathbb{Z}_-.\end{aligned}$$