Limits and continuous functions

Textbook Reading : [JL] Sections 3.1, 3.2 (Lebl)

***Exercise* 3.1.1:** *Find the limit (and prove it of course) or prove that the limit does not exist*

a) $\lim_{x\to c} \sqrt{x}$, *for* $c \geq 0$ *b)* $\lim_{x\to c} x^2 + x + 1$, *for* $c \in \mathbb{R}$ *c)* $\lim_{x\to 0} x^2\cos(1/x)$
d) $\lim_{x\to 0} \sin(1/x)\cos(1/x)$ *e)* $\lim_{x\to 0} \sin(x)\cos(1/x)$

— **Solution 3.1.1** :

— a) For any sequence of positive reals $x_n \to c$, we know that $f(x_n) = \sqrt{x_n} \to \sqrt{c}$, thus $\lim_{x\to c}\sqrt{x} = \sqrt{c}$.

— b) For any sequence of reals $x_n \to c$, we know that $x_n^2 \to c^2$, thus $f(x_n) = x_n^2 + x_n + 1 \to c^2 + c + 1$, hence

$$\lim_{x\to c} x^2 + x + 1 = c^2 + c + 1.$$

— c) For any sequence of reals $x_n \to 0$, we have $x_n^2 \to 0$, and $\left|\cos\frac{1}{x_n}\right| \leqslant 1$, then

$$-x_n^2 \leqslant x_n^2 \cos\frac{1}{x_n} \leqslant x_n^2.$$

By the squeeze theorem, we deduce by passing to the limit that $f(x_n) = x_n^2\cos\frac{1}{x_n} \to 0$, hence

$$\lim_{x\to 0} f(x) = 0.$$

— d) We have

$$f(x) = \sin\frac{1}{x}\cos\frac{1}{x} = \frac{1}{2}\sin\frac{2}{x}.$$

Let us show that the limit of $f(x)$ as $x \to 0$ does not exist ! Let $x_n = \dfrac{4}{(2n+1)\pi} \to 0$. By contradiction, if $\lim_{x\to 0} f(x) = l$ exists then one must have $f(x_n) \to l$. We have

$$f(x_n) = \frac{1}{2}\sin\frac{(2n+1)\pi}{2} = (-1)^n$$

which is a divergent sequence, henceforth : $\lim_{x\to 0} f(x)$ does not exist.

— e) We use here the inequality $|\sin x| \leqslant x$ for all $x \in \mathbb{R}$. We have : For any sequence of reals $x_n \to 0$, we have $|\sin x| \leqslant x$, and $\left|\cos\frac{1}{x_n}\right| \leqslant 1$, then

$$0 \leqslant \left|\sin x_n \cos\frac{1}{x_n}\right| = |\sin x_n| \left|\cos\frac{1}{x_n}\right| \leqslant |x_n|.$$

$x_n \to 0 \implies |x_n| \to 0$, and by the squeeze theorem, we deduce by passing to the limit that $f(x_n) = \left|\sin x_n \cos\frac{1}{x_n}\right| \to 0$, hence

$$\lim_{x\to 0} f(x) = 0.$$

***Exercise* 3.1.8:** *Find example functions f and g such that the limit of neither $f(x)$ nor $g(x)$ exists as $x \to 0$, but such that the limit of $f(x) + g(x)$ exists as $x \to 0$.*

— **Solution 3.1.8** : Think to $g = -f$, with for example $f(x) = \sin\frac{1}{x}$ and $c = 0$. We have $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} g(x)$ do not exist but

$$\lim_{x\to 0}(f(x) + g(x)) = \lim_{x\to 0}(f(x) - f(x)) = 0 \text{ exists.}$$

— Prove the following :

Corollary 3.1.11. *Let $S \subset \mathbb{R}$ and let c be a cluster point of S. Suppose $f: S \to \mathbb{R}$, $g: S \to \mathbb{R}$, and $h: S \to \mathbb{R}$ are functions such that*

$$f(x) \leq g(x) \leq h(x) \qquad \text{for all } x \in S \setminus \{c\}.$$

Suppose the limits of $f(x)$ and $h(x)$ as x goes to c both exist, and

$$\lim_{x \to c} f(x) = \lim_{x \to c} h(x).$$

Then the limit of $g(x)$ as x goes to c exists and

$$\lim_{x \to c} g(x) = \lim_{x \to c} f(x) = \lim_{x \to c} h(x).$$

—

— **Solution Cor3.1.11** : Suppose $\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = l$ exists, then for any sequence $x_n \to c$, we have $f(x_n) \to l$ and $h(x_n) \to l$. On the other hand, one has for all n :

$$f(x_n) \leqslant g(x_n) \leqslant h(x_n),$$

hence by the squeeze theorem, $\{g(x_n)\}_n$ is a convergent sequence and $g(x_n) \to l$. Therefore, $\lim_{x \to c} g(x) = l$. The proof the corollary is finished.

***Exercise* 3.1.10:** *Suppose that $f: \mathbb{R} \to \mathbb{R}$ be a function such that for every sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathbb{R}$, the sequence $\{f(x_n)\}_{n=1}^{\infty}$ converges. Prove that f is constant, that is, $f(x) = f(y)$ for all $x, y \in \mathbb{R}$.*

—

— **Solution 3.1.10** : By contradiction, assume that for every sequence $\{z_n\}_n$ (not necessarily convergent), the sequence $\{f(z_n)\}_n$ converges, and that there exist $x \neq y$ reals such that $f(x) \neq f(y)$. Let the sequence $\{z_n\}_n$ defined by

$$z_n = \begin{cases} x & \text{if } n \text{ is odd} \\ y & \text{if } n \text{ is even} \end{cases}$$

then the sequence $\{f(z_n)\}_n$ converges (say to a limit l), where by definition

$$f(z_n) = \begin{cases} f(x) & \text{if } n \text{ is odd} \\ f(y) & \text{if } n \text{ is even} \end{cases}$$

henceforth, any subsequence of $\{f(z_n)\}_n$ converges too to the same limit l. However, one has

$$f(z_{2k+1}) = f(x) \to f(x)$$

and

$$f(z_{2k}) = f(y) \to f(y)$$

which is a contradiction, since $f(x) \neq f(y)$.

***Exercise* 3.2.1:** *Using the definition of continuity directly prove that $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) := x^2$ is continuous.*

***Exercise* 3.2.2:** *Using the definition of continuity directly prove that $f: (0, \infty) \to \mathbb{R}$ defined by $f(x) := 1/x$ is continuous.*

***Exercise* 3.2.3:** *Define $f: \mathbb{R} \to \mathbb{R}$ by*

$$f(x) := \begin{cases} x & \text{if } x \text{ is rational,} \\ x^2 & \text{if } x \text{ is irrational.} \end{cases}$$

Using the definition of continuity directly prove that f is continuous at 1 and discontinuous at 2.

—

— **Solution of 3.2.1** : Let us show that f is continuous at any point $x_0 \in \mathbb{R}$, i.e.

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall x \in \mathbb{R} : |x - x_0| < \delta \implies |x^2 - x_0^2| < \varepsilon$$

We have

$$|x^2 - x_0^2| = |x - x_0||x + x_0| \leqslant |x - x_0| \left(|x| + |x_0|\right)$$

One has

$$|x-x_0|\,(|x|+|x_0|) = |x-x_0|\,(|x-x_0+x_0|+|x_0|)$$
$$\leqslant |x-x_0|\,(|x-x_0|+2|x_0|) = \alpha\,(\alpha+2|x_0|)$$

and we search $\alpha>0$ such that $\alpha\,(\alpha+2|x_0|)$. We study the sign of the following second order polynomial in α :

$$\alpha^2+2\alpha|x_0|-\varepsilon.$$
$$\Delta' = x_0^2+\alpha\varepsilon^2>0$$

then there exist two roots

$$\alpha^{\pm} = \alpha\pm\sqrt{\Delta'} = |x_0|\pm\sqrt{x_0^2+\alpha\varepsilon^2}$$

such that $\alpha^+>0$ and $\alpha^-<0$. Then $\alpha^2-2\alpha|x_0|-\varepsilon<0$.for all $\alpha\in(\alpha^-,\alpha^+)$. Let $\delta\in(0,\alpha^+)$, then $\delta>0$ and $\forall x\in\mathbb{R}:|x-x_0|<\delta \implies$

$$|x^2-x_0^2|<\delta\,(\delta+2|x_0|)<\varepsilon.$$

Hence f is continuous at any $x_0\in\mathbb{R}$, so f is continuous on $\mathbb{R}$.

— **Solution of 3.2.2** : Let us show that f is continuous at any point $x_0\in(0,+\infty)$, i.e.

$$\forall\varepsilon>0,\ \exists\delta>0,\ \forall x\in(0,+\infty):|x-x_0|<\delta \implies |\frac{1}{x}-\frac{1}{x_0}|<\varepsilon$$

We have

$$|\frac{1}{x}-\frac{1}{x_0}| = \frac{|x-x_0|}{xx_0}$$

If we let $|x-x_0|<x_0$ then

$$x = |x_0-(x-x_0)|>x_0-|x-x_0|$$

and we obtain

$$|\frac{1}{x}-\frac{1}{x_0}|\leqslant\frac{|x-x_0|}{x_0\,(x_0-|x-x_0|)}$$

Let $\alpha=|x-x_0|$, then we must have

$$\frac{\alpha}{x_0\,(x_0-\alpha)}<\varepsilon?$$

then

$$\alpha<\varepsilon x_0\,(x_0-\alpha) \implies \alpha(1+\varepsilon x_0)<\varepsilon x_0^2 \implies \alpha<\frac{\varepsilon x_0^2}{1+\varepsilon x_0}$$

We finally put $\delta=\min(x_0,\dfrac{\varepsilon x_0^2}{1+\varepsilon x_0})>0$, then $\forall x\in(0,+\infty):|x-x_0|<\delta \implies$

$$|\frac{1}{x}-\frac{1}{x_0}|\leqslant\frac{\alpha}{x_0\,(x_0-\alpha)}<\varepsilon.$$

Hence f is continuous at any $x_0\in(0,+\infty)$, so f is continuous on $(0,+\infty)$.

— **Solution of 3.2.3** :

— Let us show that f is continuous at 1, i.e.

$$\forall\varepsilon>0,\ \exists\delta>0,\ \forall x\in\mathbb{R}:|x-1|<\delta \implies |f(x)-f(1)|<\varepsilon$$

Thus, $f(1)=1$,and one has

$$|f(x)-f(1)| = \begin{cases}|x-1| & \text{if } x \text{ rational}\\ |x^2-1| = |x-1||x+1| & \text{if } x \text{ irrational}\end{cases}$$

Thus, if x is rational, one put $\delta=\delta_1=\varepsilon$ and obtains $|f(x)-f(1)|<\varepsilon$. In the other case, if x is irrational, then we put

$$\delta=\delta_2=\frac{\varepsilon}{1+|x+1|}>0,$$

such that to have

$$|x-1|<\delta \implies |x-1||x+1|<\frac{|x+1|\varepsilon}{1+|x+1|}<\varepsilon.$$

Finally, one puts

$$\delta=\min(\delta_1,\delta_2)=\delta_2=\frac{\varepsilon}{1+|x+1|}>0$$

and obtains $\forall x\in\mathbb{R}:|x-1|<\delta \implies |f(x)-f(1)|<\varepsilon$. We conclude that f is continuous at 1.

— Now, let us show that f is not continuous at 2, i.e.

$$\exists \varepsilon > 0,\ \forall \delta > 0,\ \exists x \in \mathbb{R} : |x-2| < \delta \text{ and } |f(x) - f(2)| \geqslant \varepsilon$$

We have $f(2) = 2$, and

$$|f(x) - f(2)| = \begin{cases} |x-2| & \text{if } x \text{ rational} \\ |x^2 - 2| & \text{if } x \text{ irrational} \end{cases}$$

We have $2 + \dfrac{1}{\sqrt{n}} \to 2$, hence for all $\delta > 0,\ \exists M \in \mathbb{N}^*,\ \forall n \geqslant M$:

$$2 - \delta < 2 + \frac{1}{\sqrt{n}} < 2 + \delta$$

and since $2 + \dfrac{1}{\sqrt{n}} \in \mathbb{Q}^c$, then

$$|f(2 + \frac{1}{\sqrt{n}}) - f(2)| = |(2 + \frac{1}{\sqrt{n}})^2 - 2| = 2 + \frac{1}{n} + \frac{4}{\sqrt{n}} > 2$$

Thus, for $\varepsilon = 2$, let $x = 2 + \dfrac{1}{\sqrt{M}}$

$$\forall \delta > 0,\ \exists x_M \in \mathbb{R} : |x-2| < \delta \text{ and } |f(x) - f(2)| = 2 + \frac{1}{M} + \frac{4}{\sqrt{M}} > 2.$$

Hence, f is discontinous at 2.

***Exercise* 3.2.4:** *Define $f\colon \mathbb{R} \to \mathbb{R}$ by*

$$f(x) := \begin{cases} \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Is f continuous? Prove your assertion.

***Exercise* 3.2.5:** *Define $f\colon \mathbb{R} \to \mathbb{R}$ by*

$$f(x) := \begin{cases} x\sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Is f continuous? Prove your assertion.

— **Solution of 3.2.4** : f is continuous on $\mathbb{R}^*$ since it is a composition of $x \mapsto 1/x$ continuous from $\mathbb{R}^*$to $\mathbb{R}$ and $y \mapsto \sin y$ continuous on $\mathbb{R}$. It remains to study the continuity at 0 : Since $\lim_{x\to 0} f(x) = \lim_{x\to 0} \sin\dfrac{1}{x}$ does not exist then f is not continuous at 0.

— **Solution of 3.2.5** : f is continuous on $\mathbb{R}^*$ since it is a product of $x \mapsto x$ continuous on $\mathbb{R}$ and the composition of $x \mapsto 1/x$ continuous from $\mathbb{R}^*$to $\mathbb{R}$ and $y \mapsto \sin y$ continuous on $\mathbb{R}$. It remains to study the continuity at 0 : Since

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} x \sin\frac{1}{x} = 0 = f(0)$$

(limit 0 is obtained by the fact that $|x \sin\dfrac{1}{x}| \leqslant |x|$) then f continuous at 0. Hence, $f : \mathbb{R} \to \mathbb{R}$ is a continuous function.

***Exercise* 3.2.9:** *Give an example of functions $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ such that the function h, defined by $h(x) := f(x) + g(x)$, is continuous, but f and g are not continuous. Can you find f and g that are nowhere continuous, but h is a continuous function?*

***Exercise* 3.2.10:** *Let $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ be continuous functions. Suppose that $f(r) = g(r)$ for all $r \in \mathbb{Q}$. Show that $f(x) = g(x)$ for all $x \in \mathbb{R}$.*

***Exercise* 3.2.11:** *Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous. Suppose $f(c) > 0$. Show that there exists an $\alpha > 0$ such that for all $x \in (c - \alpha, c + \alpha)$, we have $f(x) > 0$.*

- **Solution of 3.2.9** : Let us take the example of the Dirichlet function in order to give directly function that are nowhere continuous : Let
$$f : \mathbb{R} \to \mathbb{R} : f(x) = \begin{cases} 1 & \text{if } x \text{ rational} \\ 0 & \text{if } x \text{ irrational} \end{cases}$$
Then f is nowhere continuous, in fact, for any $x_0 \in \mathbb{R}$,
  - if $x_0 \in \mathbb{Q}$ then there exists by density a sequence of irrationals $i_n \to x_0$, then for all $n : f(i_n) = 0$ and
$$\lim_{n\to+\infty} f(i_n) = 0 \neq f(x_0) = 1$$
.
  - if $x_0 \in \mathbb{Q}^c$ then there exists also by density a sequence of rationals $r_n \to x_0$, then for all $n : f(r_n) = 1$ and
$$\lim_{n\to+\infty} f(r_n) = 1 \neq f(x_0) = 0.$$
Let $g = -f$, then f and g are nowhere continuous but $f + g = f - f = 0$ is continuous elsewhere.
- **Solution of 3.2.10** : For all x reals, there exists by density a sequence of rationals $r_n \to x$, then $g(r_n) = f(r_n)$ converge to the same limit l. Since g and f are continuous, then $l = g(x) = f(x)$.
- **Solution of 3.2.11** : Let $f : \mathbb{R} \to \mathbb{R}$ be continuous, and $f(c) > 0$. Using the continuity of f at c, we have
$$\forall \varepsilon > 0, \exists \alpha > 0, \forall x \in \mathbb{R} : |x - c| < \alpha \implies |f(x) - f(c)| < \varepsilon$$
Thus, $\forall x \in \mathbb{R}$, such that $c - \alpha < x < c + \alpha$, one has
$$f(c) - \varepsilon < f(x) < f(c) + \varepsilon,$$
Hence, taking $\varepsilon = f(c) > 0$,we obtain $0 < f(x) < 2f(c)$, which ends the proof.