Chapters and exercises given with a numbering are from *Basic Analysis: Introduction to Real Analysis (Vol I)* by J. Lebl.

Reading Section 0.3

Exercises

1. Exercise 0.3.6
2. Exercise 0.3.11
3. Exercise 0.3.12
4. Exercise 0.3.15
5. Exercise 0.3.19
6. In this exercise, you will prove that

$$|\{q \in \mathbb{Q} : q > 0\}| = |\mathbb{N}|.$$

In what follows, we will use the following theorem without proof:

Theorem. *Let $q \in \mathbb{Q}$ with $q > 0$. Then*

1) if $q \in \mathbb{N}$ and $q \neq 1$, then there exists unique prime numbers $p_1 < p_2 < \cdots < p_N$ and unique exponents $r_1, \ldots, r_N \in \mathbb{N}$ such that

$$q = p_1^{r_1} p_2^{r_2} \cdots p_N^{r_N}, \tag{$\dagger$}$$

2) if $q \notin \mathbb{N}$, then there exist unique prime numbers $p_1 < p_2 < \ldots < p_N$, $q_1 < q_2 < \cdots < q_M$ with $p_i \neq q_j$ for all $i \in \{1, \ldots, N\}$, $j \in \{1, \ldots M\}$, and unique exponents $r_1, \ldots, r_N, s_1, \ldots s_M \in \mathbb{N}$ such that

$$q = \frac{p_1^{r_1} p_2^{r_2} \cdots p_N^{r_N}}{q_1^{s_1} q_2^{s_2} \cdots q_M^{s_M}}.. \tag{$\ddagger$}$$

Define $f : \{q \in \mathbb{Q} : q > 0\} \to \mathbb{N}$ as follows: $f(1) = 1$, if $q \in \mathbb{N}\backslash\{1\}$ is given by ($\dagger$), then

$$f(q) = p_1^{2r_1} \cdots p_N^{2r_N},$$

and if $q \in \mathbb{Q}\backslash\mathbb{N}$ is given by ($\ddagger$), then

$$f(q) = p_1^{2r_1} \cdots p_N^{2r_N} q_1^{2s_1 - 1} \cdots q_M^{2s_M - 1}.$$

(a) Compute $f(4/15)$. Find q such that $f(q) = 108$.

(b) Use the **Theorem** to prove that f is a bijection.

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Exercise* 0.3.6*: *Prove:*

a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

b) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Exercise* 0.3.11*: *Prove by induction that* $n < 2^n$ *for all* $n \in \mathbb{N}$.

Exercise* 0.3.12*: *Show that for a finite set* A *of cardinality* n*, the cardinality of* $\mathcal{P}(A)$ *is* 2^n.

Exercise* 0.3.15*: *Prove that* $n^3 + 5n$ *is divisible by* 6 *for all* $n \in \mathbb{N}$.

Exercise* 0.3.19*: *Give an example of a countably infinite collection of finite sets* $A_1, A_2, \ldots$, *whose union is not a finite set.*