Limits and continuous functions

Textbook Reading : [JL] Sections 3.1, 3.2 (Lebl)

***Exercise* 3.1.1:** *Find the limit (and prove it of course) or prove that the limit does not exist*

a) $\lim_{x\to c}\sqrt{x}$, *for* $c \geq 0$ *b)* $\lim_{x\to c} x^2 + x + 1$, *for* $c \in \mathbb{R}$ *c)* $\lim_{x\to 0} x^2\cos(1/x)$
d) $\lim_{x\to 0}\sin(1/x)\cos(1/x)$ *e)* $\lim_{x\to 0}\sin(x)\cos(1/x)$

***Exercise* 3.1.8:** *Find example functions f and g such that the limit of neither $f(x)$ nor $g(x)$ exists as $x \to 0$, but such that the limit of $f(x) + g(x)$ exists as $x \to 0$.*

— Prove the following :

Corollary 3.1.11. *Let $S \subset \mathbb{R}$ and let c be a cluster point of S. Suppose $f : S \to \mathbb{R}$, $g : S \to \mathbb{R}$, and $h : S \to \mathbb{R}$ are functions such that*

$$f(x) \leq g(x) \leq h(x) \qquad \text{for all } x \in S \setminus \{c\}.$$

Suppose the limits of $f(x)$ and $h(x)$ as x goes to c both exist, and

$$\lim_{x\to c} f(x) = \lim_{x\to c} h(x).$$

Then the limit of $g(x)$ as x goes to c exists and

$$\lim_{x\to c} g(x) = \lim_{x\to c} f(x) = \lim_{x\to c} h(x).$$

***Exercise* 3.1.10:** *Suppose that $f : \mathbb{R} \to \mathbb{R}$ be a function such that for every sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathbb{R}$, the sequence $\{f(x_n)\}_{n=1}^{\infty}$ converges. Prove that f is constant, that is, $f(x) = f(y)$ for all $x, y \in \mathbb{R}$.*

***Exercise* 3.2.1:** *Using the definition of continuity directly prove that $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) := x^2$ is continuous.*

***Exercise* 3.2.2:** *Using the definition of continuity directly prove that $f : (0, \infty) \to \mathbb{R}$ defined by $f(x) := 1/x$ is continuous.*

***Exercise* 3.2.3:** *Define $f : \mathbb{R} \to \mathbb{R}$ by*

$$f(x) := \begin{cases} x & \text{if } x \text{ is rational,} \\ x^2 & \text{if } x \text{ is irrational.} \end{cases}$$

Using the definition of continuity directly prove that f is continuous at 1 and discontinuous at 2.

***Exercise* 3.2.4:** *Define* $f: \mathbb{R} \to \mathbb{R}$ *by*

$$f(x) := \begin{cases} \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Is f *continuous? Prove your assertion.*

***Exercise* 3.2.5:** *Define* $f: \mathbb{R} \to \mathbb{R}$ *by*

$$f(x) := \begin{cases} x\sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Is f *continuous? Prove your assertion.*

***Exercise* 3.2.9:** *Give an example of functions* $f: \mathbb{R} \to \mathbb{R}$ *and* $g: \mathbb{R} \to \mathbb{R}$ *such that the function* h*, defined by* $h(x) := f(x) + g(x)$*, is continuous, but* f *and* g *are not continuous. Can you find* f *and* g *that are nowhere continuous, but* h *is a continuous function?*

***Exercise* 3.2.10:** *Let* $f: \mathbb{R} \to \mathbb{R}$ *and* $g: \mathbb{R} \to \mathbb{R}$ *be continuous functions. Suppose that* $f(r) = g(r)$ *for all* $r \in \mathbb{Q}$*. Show that* $f(x) = g(x)$ *for all* $x \in \mathbb{R}$*.*

***Exercise* 3.2.11:** *Let* $f: \mathbb{R} \to \mathbb{R}$ *be continuous. Suppose* $f(c) > 0$*. Show that there exists an* $\alpha > 0$ *such that for all* $x \in (c - \alpha, c + \alpha)$*, we have* $f(x) > 0$*.*