Completness of real numbers (corrections)

Textbook Reading : [JL] Sections 2.2, 2.3, 2.4, 2.5 (Lebl)

Exercises

1. **(Exercise 2.3.5)** is omitted.
2. **(Exercise 2.3.5)**
   (a) $x_n = \dfrac{(-1)^n}{n},\ n > 0$: We have, by the Squeeze-Theorem
   $$0 \leqslant \left|\frac{(-1)^n}{n}\right| \leqslant \frac{1}{n} \implies x_n \to 0 \implies \limsup x_n = \liminf x_n = 0.$$
   (b) $x_n = \dfrac{(n-1)(-1)^n}{n},\ n > 0$: Since $\forall n > 0;\ -1 < x_n < 1$ then we have
   $$-1 \leqslant \liminf x_n \leqslant \limsup x_n \leqslant 1.$$
   On the other hand $\left\{\dfrac{(n-1)}{n}\right\}_{n>0} = \left\{1 - \dfrac{1}{n}\right\}_{n>0}$, then
   $$x_{2p} = \frac{2p-1}{2p} \to 1 \implies 1 \leqslant \limsup x_n$$
   thus $1 \leqslant \limsup x_n \leqslant 1 \implies \boldsymbol{\limsup x_n = 1}$, and similarly
   $$x_{2p+1} = -\frac{2p}{2p+1} \to -1 \implies 1 \geqslant \liminf x_n$$
   thus $-1 \leqslant \liminf x_n \leqslant -1 \implies \boldsymbol{\liminf x_n = 1}$.
3. **(Exercise 2.3.6)** Let $\{x_n\}$ and $\{y_n\}$ bounded such that $x_n \leqslant y_n$, for all n. We have
   $$\limsup x_n = \lim_{n\to+\infty} a_n^x \text{ and } \limsup y_n = \lim_{n\to+\infty} a_n^y$$
   where
   $$a_n^x = \sup\{x_k,\ k \geqslant n\} \text{ and } a_n^y = \sup\{y_k,\ k \geqslant n\}$$
   and we have, for all $k \geqslant n$, $x_k \leqslant y_k \leqslant a_n^y$ thus a_n^y is an upper bound of $\{x_k,\ k \geqslant n\}$ and since $a_n^x = \sup\{x_k,\ k \geqslant n\}$ is the least upper bound then $a_n^x \leqslant a_n^y$ and this holds for all n, then, passing to the limit, we obtain
   $$\lim_{n\to+\infty} a_n^x \leqslant \lim_{n\to+\infty} a_n^y \implies \limsup x_n \leqslant \limsup y_n.$$
   Similar reasoning using the definition of the infimum will conduct us to $\liminf x_n \geqslant \liminf y_n$.
4. **(Exercise 2.3.7)** Let $\{x_n\}$ and $\{y_n\}$ bounded.
   (a) $\{x_n\}$ and $\{y_n\}$ bounded then there exist $C_1 > 0$ and $C_2 > 0$ such that, for all n,
   $$|x_n| \leqslant C_1 \text{ and } |y_n| \leqslant C_2,$$
   then
   $$|x_n + y_n| \leqslant |x_n| + |y_n| \leqslant C_1 + C_2 = C > 0$$
   hence, $\{x_n + y_n\}_n$ is bounded.

(b) Let us show that

$$\liminf\{x_n\} + \liminf\{y_n\} \leqslant \liminf\{x_n + y_n\}.$$

From (a), $\{x_n + y_n\}_n$ is bounded then one can extract a convergent subsequence to $\liminf\{x_n + y_n\}$, say

$$x_{n_k} + y_{n_k} \to \liminf\{x_n + y_n\}, \text{ as } k \to +\infty,$$

on the other hand, we have

$$\liminf x_n = \inf\{x_k, k \geqslant n\} \leqslant x_n, \ \forall n, \implies \liminf x_n \leqslant x_{n_k}$$

$$\liminf y_n = \inf\{y_k, k \geqslant n\} \leqslant y_n, \ \forall n, \implies \liminf y_n \leqslant y_{n_k}$$

thus

$$\liminf x_n + \liminf y_n \leqslant x_{n_k} + y_{n_k}$$

then, passing to the limit as $k \to +\infty$, one obtains

$$\liminf x_n + \liminf y_n \leqslant \liminf\{x_n + y_n\}.$$

(c) Let pick up a counter example when this last inequality is strict. Let $x_n = (-1)^n$ and $y_n = (-1)^{n+1}$ then

$$\liminf x_n = \liminf y_n = -1 - 1 = -2$$

and, since $x_n + y_n = 0$, for all n,

$$\liminf\{x_n + y_n\} = 0 > -2.$$

5. **(Exercise)** Let $\{x_n\}$ be a bounded sequence of real numbers. Let us show that

$$x_n \to 0 \iff \limsup |x_n| = 0?$$

We know that

$$x_n \to 0 \iff |x_n| \to 0,$$

$\Longleftarrow$) we have

$$0 = \limsup |x_n| = \lim_{n\to+\infty} a_n$$

where

$$a_n = \sup\{|x_k|, \ k \geqslant n\}$$

and for all n,

$$0 \leqslant |x_n| \leqslant a_n$$

thus, passing to the limit, we obtain by the Squeeze-theorem

$$\lim_{n\to+\infty} |x_n| = 0$$

$\Longrightarrow$) Assume $\lim_{n\to+\infty} |x_n| = 0$ then any subsequence of $\{|x_n|\}$ converges to 0. However, we know that there exists a subsequence $\{|x_{n_k}|\}$ of $\{|x_n|\}$ that converges to $\limsup |x_n|$, hence

$$\limsup |x_n| = \lim_{k\to+\infty} |x_{n_k}| = \lim_{n\to+\infty} |x_n| = 0.$$

6. **(Exercise)** By contradiction, assume there exists a sequence $\{x_n\}$ such that

$$\liminf x_n = -1, \ \lim_{n\to+\infty} x_n = 0, \ \limsup x_n = 1$$

Since $\lim_{n\to+\infty} x_n = 0$ then $\{x_n\}$ is bounded and in this case we know that $\{x_n\}$ converges if and only if

$$\liminf x_n = \limsup x_n$$

which is in contradiction with hypothesis. Thus, no such a sequence exists.

7. **(Exercise 2.4.8)** Let $\{x_n\}$ Cauchy sequence, then

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N},\ \forall n,\ n+1 > n \geqslant N \implies |x_{n+1} - x_n| < \varepsilon.$$

This means that the sequence $y_n = x_{n+1} - x_n \to 0$. However, the given assertion

$$\exists M \in \mathbb{N},\ \forall n \geqslant M : |x_{n+1} - x_n| \leqslant |x_n - x_{n-1}|$$

means that the sequence $\{y_n\}$ is monotone decreasing after a rank M. We can thus seek for a counter example of a Cauchy sequence $\{x_n\}$ such that $\{y_n\}$ is not monotone! Let us put $x_n = \dfrac{(-1)^n}{n}$ then we know that $x_n \to 0$ thus $\{x_n\}$ is Cauchy. However,

$$y_n = x_{n+1} - x_n = \frac{(-1)^{n+1}}{n+1} - \frac{(-1)^n}{n} = (-1)^{n+1}\left(\frac{1}{n} + \frac{1}{n+1}\right)$$

has an alternating sign thus it is not monotone, in fact observe that

$$y_{2p} < 0,\ \ y_{2p+1} > 0, y_{2p+2} < 0.$$

Hence, the given assertion is false.