# Characterization of real numbers (corrections)

**Textbook Reading : [JL] Sections 0.3, 1.1 and 1.2**

**Exercises**

**1. Exercise 1.1.1** Let $F$ be an ordered field and $x, y, z \in F$. We use the fact that $<$ is compatible with $+$ and $.$ in the field $F$ :

$$\begin{cases} z > y \implies z + (-y) > y + (-y) = 0 \implies z - y > 0 \\ \text{and} \\ x < 0 \implies x - x < -x \implies 0 < -x \end{cases} \implies (-x).(z-y) > 0$$

hence, by operations properties in the field $F$,

$$x.(y-z) = x.y - x.z > 0$$

and by compatibility, adding $x.y$ in both sides, we obtain

$$x.y - x.z + x.z > x.z,$$

thus $x.y > x.z$.

**2. Exercise 1.1.2** Let $S$ be an ordered set and $A \subset S$ a finite subset, say $A = \{x_1, x_2, ..., x_n\}$, that is $|A| = n \geqslant 1$. Let us show that in a first step that $A$ is upper bounded and that $\max A \in A$.

**Proof** by induction :

**base** case : if $n = 1$, then $A = \{x_1\}$ and we have $\max A = x_1 \in A$.

**Induction** step : Let $P(n)$ the property that for every finite (non empty) subset $A = \{x_1, x_2, ..., x_n\} \subset S$, of size $n$, there exists $j \in \{1, 2, ..., n\}$ such that $\max A = x_j \in A$. Then $P(n+1)$ is the property that for every finite (non empty) subset $B = \{y_1, y_2, ..., y_n, y_{n+1}\} \subset S$, of size $n+1$, there exists $i \in \{1, 2, ..., n, n+1\}$ such that $\max B = y_i \in B$.

**Assume** $P(n)$ true and let $B = \{y_1, y_2, ..., y_n, y_{n+1}\} \subset S$. Then, $B = B' \cup \{y_{n+1}\}$ where $B' = \{y_1, y_2, ..., y_n\} \subset S$ and $|B'| = n$, so there exists $j \in \{1, 2, ..., n\}$ such that $\max B' = y_j \in B'$. Since $S$ is ordered, then one has : either $(i) : y_j \leqslant y_{n+1}$ or $(ii) : y_{n+1} \leqslant y_j$. Hence,

**case** (i) $\forall y \in B' : y \leqslant \max B' = y_j \leqslant y_{n+1}$ thus $\forall y \in B' \cup \{y_{n+1}\} : y \leqslant y_{n+1}$ which implies that $y_{n+1} = \max B \in B$.

**case** (ii) $\forall y \in B' \cup \{y_{n+1}\} = B : y \leqslant y_j$ which implies that $y_j = \max B \in B$.

**In** both cases, $\max B$ exists and belongs to $B$. Thus $P(n+1)$ is true.

**Conclusion** step : It follows by induction that $P(n)$ is true for all $n \geqslant 1$.

**The** proof of lower bound and existence of a minimum is similar and let to the reader.

**3. Exercise 1.1.5** $S$ an ordered set. $A \subset S$ and $b$ is an upper bound for $A$ such that $b \in A$. Let us show that $b = \sup A$.

**Proof** Since $b \in A$ then $b \leqslant \sup A$, but $b$ is an upper bound for $A$ and $\sup A$ is the least upper bound for $A$, then $\sup A \leqslant b$. Henceforth,

$$\sup A \leqslant b \leqslant \sup A \implies b = \sup A.$$

**4. Exercise 1.1.6** $S$ an ordered set. $\emptyset \neq A \subset S$ such that $A$ is bounded above. Assume $\sup A$ exists and $\sup A \notin A$. Let us show that $A$ contains a countably infinite subset.

**Proof** $A \neq \emptyset$ implies that there exists $x_1 \in A$. Thus $x_1 < \sup A$ since $\sup A \notin A$. If one assume that $A = \{x_1\}$ then $\sup A = x_1$ which is in contradiction with $x_1 < \sup A$ , i.e., $\{x_1\} \subsetneq A$ (this means that $\{x_1\}$ is a **proper** subset of $A$), hence there exists $x_2 \neq x_1$ such that $\{x_1, x_2\} \subset A$. But, by the same argument $\sup A \notin A$ we conclude that $x_2 < \sup A$ and that $\{x_1, x_2\} \subsetneq A$ because if-not then either $x_1$ or $x_2$ is equal $\sup A$ which leads to a contradiction. hence, following the same arguments, we can can find, for all $n \geqslant 1$, a subset $B_n = \{x_1, x_2, ..., x_n\} \subsetneq A$. This means that $A$ contains a countably infinite subset $B = \{x_1, x_2, ..., x_n, ...\}$, which ends the proof.

**Remark :** To be more rigorous, the statement "$\forall n \geqslant 1$, there exists a finite subset $B_n = \{x_1, x_2, ..., x_n\} \subsetneq A$" must be proved by induction. This is an easy exercise left to the reader.

**5. Exercise 1.2.7**

**Proof** Direct consequence of the positivity of $x, y$ and the inequality

$$0 \leqslant (\sqrt{x} - \sqrt{y})^2 = x + y - 2\sqrt{xy}.$$

**6. Exercise 1.2.9** $A, B$ nonempty and bounded subsets of reals.

$$C = A + B = \{a + b |\, a \in A \text{ and } b \in B\}.$$

**Proof** for sup : For all $c = a + b \in C$ :

$$a \leqslant \sup A \text{ and } b \leqslant \sup B \implies a + b = c \leqslant \sup A + \sup B$$

thus $\sup A + \sup B$ is an upper bound of $C$, hence $\sup C \leqslant \sup A + \sup B$ since it is the least one.

**Let** us show now that $\sup C \geqslant \sup A + \sup B$. We have

$$a + b = c \leqslant \sup C : \forall a \in A \text{ and } \forall b \in B$$
$$\implies a \leqslant \sup C - b : \forall a \in A \text{ and } \forall b \in B$$

thus, $\forall b \in B$, $\sup C - b$ is an upper bound of $A$ and hence $\sup A \leqslant \sup C - b$ since it is the least one. Then,

$$\forall b \in B : \sup A + b \leqslant \sup C \implies \forall b \in B : b \leqslant \sup C - \sup A,$$

hencforth, $\sup C - \sup A$ is an upper bound of $B$ which yields $\sup B \leqslant \sup C - \sup A$, since it is the least one. So, we obtain $\sup A + \sup B \leqslant \sup C$. Conclusion :

$$\sup C = \sup A + \sup B$$

**Proof** for inf : Use the same reasoning with lower bounds.

**Remark :** This exercise can be solved very easily using the definition of sup (or inf) with $\varepsilon$ which will be seen in worksheet 3.

**7.** Let $E = \{x \in \mathbb{R} : x > 0 \text{ and } x^3 < 2\}$. In our proof bellow, we assume that we do not know anything on the cubic root $\sqrt[3]{2}$!! and also on the continuity of the funcion $x \mapsto x^3$ which implies directly that $E$ is the open interval $(0, \sqrt[3]{2})$ whose supremum is simply equal $\sqrt[3]{2}$.

**Proof of (a)** By contradiction, assume that $E$ **is** **not** **bounded above**, then

$$\forall M > 0, \exists x \in E \,/\, x > M$$

Thus

$$0 < M < x \implies 0 < M^3 < x^3 < 2.$$

We conclude that

$$\forall M > 0 : M^3 < 2,$$

which is not true for $M = 2$ for example. We deduce, by contradiction, that $E$ **is bounded above.**

**Proof of (b)** Let $r = \sup E$ (which exists by part (a)). **Let us prove that** $r > 0$ **and** $r^3 = 2$.

**Since** $r = \sup E$ then $\forall x \in E$, $0 < x \leqslant r$ thus $r > 0$.

**Let us show now** that $r^3 \leqslant 2$ and $r^3 \geqslant 2$ in order to deduce that $r^3 = 2$.

**By contradiction**, assume that $r^3 < 2$, then we can find an $0 < h \leqslant 1$, such that $r^3 < (r+h)^3 < 2$. In fact, using the identity $a^3 - b^3 = (a-b)\left(a^2 + ab + b^2\right)$, we must have

$$0 < (r+h)^3 - r^3 = h\left((r+h)^2 + h(r+h) + h^2\right) < 2 - r^3 \tag{1}$$

Since $1 \in E$ and $r = \sup E$ then $r \geqslant 1$, thus $r + h > 1$, then

$$h < r + h < (r+h)^2 < (r+h)^3 \implies$$

$$(r+h)^2 + h(r+h) + h^2 < 3(r+h)^3$$

Hence, inequality (1) holds if we have $3h(r+h)^3 < 2 - r^3$, i.e, $3h(r+1)^3 < 2 - r^3$ since $h$ is expected in $(0,1]$, thus the sufficient condition for inequality (1) is

$$0 < h < \frac{2 - r^3}{3(r+1)^3} \text{ and } h \leqslant 1.$$

We can thus choose for example

$$h = \min\left(1, \frac{2 - r^3}{3(r+1)^3}\right)$$

which satisfies $0 < h \leqslant 1$ and ensures the inequality (1).

**In summary**, we can find $h \in (0,1]$ such that $r^3 < (r+h)^3 < 2$ which implies that $r + h \in E$ and results in contradiction with $r = \sup E \geqslant r + h$, because $h > 0$.

**We conclude by contradiction** that $r^3 \geqslant 2$.

**Similarly,** and by contradiction method, if we assume that $r^3 > 2$ then one can find $h \in (0,1]$ such that $r^3 > (r-h)^3 > 2$ which means that

$$\forall x \in E : x \geqslant 0 \text{ and } x^3 < 2 < (r-h)^3 < r^3$$

which implies

$$\forall x \in E : x < (r-h) < r$$

hence $r - h$ becomes an upper bound of $E$ but this lead to a contradiction since $r$ is the least upper bound of $E$. Hence, one must have $r^3 \leqslant 2$.

**Final conclusion** : $r > 0$, $r^3 \leqslant 2$ and $r^3 \geqslant 2$ then $r^3 = 2$.

**Remark :** Others proofs are welcomed.