Characterization of real numbers (corrections)

Textbook Reading : [JL] Sections 1.2, 1.3, 1.4, 1.5, 2.1

Exercises

1. Let $x < y$ reals, then by the density of $\mathbb{Q}$ in $\mathbb{R}$, there exits $r \in \mathbb{Q}$ such that $x < r < y$. By the same argument ofthe density of $\mathbb{Q}$ in $\mathbb{R}$, there exits $r' \in \mathbb{Q}$ such that $x < r < r' < y$. Let $i = r + \dfrac{r'-r}{\sqrt{2}}$: Thus, $x < i < y$. Let us show that i is not rational. By contradiction, assuming $r + \dfrac{r'-r}{\sqrt{2}} = i \in \mathbb{Q}$ leads to $\sqrt{2} = \dfrac{i-r}{r'-r} \in \mathbb{Q}$ which is false. Hence $i = r + \dfrac{r'-r}{\sqrt{2}}$ is irrational and satisfy $x < i < y$.

2. Let $E \subset (0,1)$ be the set

$$E = \{x \in (0,1) : \forall j \in \mathbb{N}^*, \exists d_{-j} \in \{1,2\} : x = 0.d_{-1}d_{-2}....\}.$$

Let us consider the function

$$\begin{aligned} f :& E \to \mathcal{P}(\mathbb{N}^*) \\ & x = 0.d_{-1}d_{-2}.... \mapsto f(x) = \{j \in \mathbb{N}^* : d_{-j} = 2\}. \end{aligned}$$

One can show that f is a bijection. In fact, let us denote

$$\overline{f(x)} = \mathbb{N}^* \diagdown f(x) = f(x) = \{j \in \mathbb{N}^* : d_{-j} = 1\}$$

since there are only two digits 1 or 2 defining elements of E.

(a) **Let us show that f is 1-1 :** let $x = 0.d_{-1}d_{-2}....$ and $y = 0.e_{-1}e_{-2}....$ such that $f(x) = f(y)$, and $\overline{f(x)} = \overline{f(y)}$, i.e.

$$\{j \in \mathbb{N}^* : d_{-j} = 2\} = \{j \in \mathbb{N}^* : e_{-j} = 2\} \text{ and } \{j \in \mathbb{N}^* : d_{-j} = 1\} = \{j \in \mathbb{N}^* : e_{-j} = 1\},$$

then $x = y$. Thus, f is 1-1.

(b) **Let us show that f is onto :** If A is a subset of $\mathbb{N}^*$ then :

i. If $A = \emptyset$ then $A = f(x)$ where $x = 0.111.....$

ii. If $A \neq \emptyset$, then for any $j \in A$, we put $d_{-j} = 2$ and for any $j \in \overline{A} = \mathbb{N}^* \diagdown A$, we put $d_{-j} = 1$. Hence, for this sequence d_{-j}, for all $j \in \mathbb{N}^*$, we have $f(x) = A$.

(c) Conclusion : $f : E \to \mathcal{P}(\mathbb{N}^*)$ is a bijection which implies that $|E| = |\mathcal{P}(\mathbb{N}^*)|$.

3. Recall that a set S is countably infinite if $|S| = |\mathbb{N}|$, i.e. there exists a bijection $f : S \to \mathbb{N}$.

(a) We know that $|2\mathbb{N}| = |2\mathbb{N}+1|$, and if A and B are two disjoint countably infinite sets then, there exists two bijections $h : A \to 2\mathbb{N}$ and $g : A \to 2\mathbb{N}+1$. Let us put $f : A \cup B \to \mathbb{N}$ defined by :

$$f(x) = \begin{cases} h(x) & \text{if } x \in A \\ g(x) & \text{if } x \in B \end{cases}$$

We can show that f is a bijection and conclude that $|A \cup B| = |\mathbb{N}|$.

i. It is clear that f is onto because for any $n \in \mathbb{N}$, if $n \in 2\mathbb{N}$ there exists $x \in A$ such that $n = h(x) = f(x)$ (since $h : A \to 2\mathbb{N}$ is bijective), and if $n \in 2\mathbb{N}+1$ there exists $y \in B$ such that $n = h(y) = f(y)$ (since $h : A \to 2\mathbb{N}+1$ is bijective), hence in both cases there exists z in A or B such that $n = f(z)$, then f is onto.

ii. Let us show that f is 1-1 : assume by contradiction that there exist $x \neq y$ in $A \cup B$ such that $f(x) = f(y)$.There must be three cases as follows :

A. $x, y \in A \implies h(x) = h(y) \implies x = y$ because h is 1-1, $\implies$ contradiction.

B. $x, y \in B \implies g(x) = g(y) \implies x = y$ because g is 1-1, $\implies$ contradiction.

C. $(x \in A$ and $y \in B) \implies h(x) = g(y), \implies$ contradiction, since $h(x) \in \mathbb{N}$ and $g(y) \in 2\mathbb{N}$ and $2\mathbb{N} \cap (2\mathbb{N}+1) = \emptyset$.

4. Using the previous result, we have $\mathbb{R} = \mathbb{Q} \cup (\mathbb{R} \setminus \mathbb{Q})$ and $\mathbb{R} \setminus \mathbb{Q}$ and $\mathbb{Q}$ are disjoints sets. By contradiction, if one assumes that $\mathbb{R} \setminus \mathbb{Q}$ is countably (infinite) then $\mathbb{R}$ becomes countably infinite, which is a contradiction.

5. Let A a bounded subset of $\mathbb{R}$. We have to show that

$$a_0 = \sup A \iff \forall \varepsilon > 0, \exists a \in A : a_0 - \varepsilon < a.$$

(a) $\implies$) By contradiction, assume that $a_0 = \sup A$ and :

$$\exists \varepsilon > 0, \forall a \in A : a \leqslant a_0 - \varepsilon,$$

in this case, $\forall a \in A : a \leqslant a_0 - \varepsilon < a_0$,implies that $a_0 - \varepsilon$ is an upper bound of A and since a_0 is the least upper bound then $a_0 \leqslant a_0 - \varepsilon$ which leads to a contradiction because $\varepsilon > 0$. Thus,

$$a_0 = \sup A \implies \forall \varepsilon > 0, \exists a \in A : a_0 - \varepsilon < a.$$

(b) $\impliedby$) By contrapositive reasoning, the indirect implication is evident, in fact : let us show

$$\exists \varepsilon > 0, \forall a \in A : a \leqslant a_0 - \varepsilon \implies a_0 \neq \sup A.$$

Since $\forall a \in A : a \leqslant a_0 - \varepsilon$ then $a_0 - \varepsilon$ is an upper bound of A, but $a_0 - \varepsilon < a_0$ because $\varepsilon > 0$, hence a_0 can not be the least upper bound, i.e. $a_0 \neq \sup A$.

6. We say a set $U \subset \mathbb{R}$ is open if for every $x \in U$ there exists $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subset U$. Since the definition is vacuous for $U = \emptyset$, it follows that the empty set is open. It is also clear from the definition that $U = \mathbb{R}$ is open. Such notions are found in topology in second year L2 in third semester.

(a) Let $a, b \in \mathbb{R}$ with $a < b$.

i. Let us prove that the set $(-\infty, a) = \{x \in \mathbb{R} : x < a\}$ is open : Let $x \in (-\infty, a)$ and let $\varepsilon = a - x > 0$,

$$\forall y \in \mathbb{R} : x - \varepsilon < y < x + \varepsilon \implies y < a \implies y \in (-\infty, a),$$

thus $(x - \varepsilon, x + \varepsilon) \subset (-\infty, a)$ which implies that $(-\infty, a)$ is open

ii. Similarly, let us prove that the set $(b, +\infty) = \{x \in \mathbb{R} : b < x\}$ is open : Let $x \in (-\infty, a)$ and let $\varepsilon = x - b > 0$,

$$\forall y \in \mathbb{R} : x - \varepsilon < y < x + \varepsilon \implies b < y \implies y \in (b, +\infty),$$

thus $(x - \varepsilon, x + \varepsilon) \subset (b, +\infty)$ which implies that $(b, +\infty)$ is open.

iii. Let us now remark that $(a, b) = (-\infty, b) \cap (a, +\infty)$. From i. and ii. , $(-\infty, b)$ and $(a, +\infty)$ are open sets, thus

$$\forall x \in (a, b) : x \in (-\infty, b) \text{ and } (a, +\infty)$$

hence there exists $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that $(x - \varepsilon_1, x + \varepsilon_1) \subset (-\infty, b)$ and $(x - \varepsilon_2, x + \varepsilon_2) \subset (a, +\infty)$, therefore

$$(x - \varepsilon_1, x + \varepsilon_1) \cap (x - \varepsilon_2, x + \varepsilon_2) \subset (-\infty, b) \cap (a, +\infty).$$

If we put $\varepsilon = \min(\varepsilon_1, \varepsilon_2) > 0$, then

$$(x - \varepsilon_1, x + \varepsilon_1) \cap (x - \varepsilon_2, x + \varepsilon_2) = (x - \varepsilon, x + \varepsilon) \subset (a, b),$$

which ends the proof.

(b) Let us prove that if U_λ is open for all $\lambda \in \Lambda$ then the following (union) set is open :

$$\bigcup_{\lambda \in \Lambda} U_\lambda := \{x \in \mathbb{R} : \exists \lambda \in \Lambda; x \in U_\lambda\}.$$

If $x \in \bigcup_{\lambda \in \Lambda} U_\lambda$ then by definition, $\lambda \in \Lambda; x \in U_\lambda$, and since U_λ is open then there exists $\varepsilon > 0$ such that

$$(x - \varepsilon, x + \varepsilon) \subset U_\lambda \subset \bigcup_{\lambda \in \Lambda} U_\lambda,$$

hence $\bigcup_{\lambda \in \Lambda} U_\lambda$ is open.

(c) Let $n \geqslant 1$ an integer, and let $U_1, ..., U_n \subset \mathbb{R}$. Let us prove that if $U_1, ..., U_n$ are open sets then the following (finite intersection) set is open :

$$\bigcap_{k=1}^{n} U_k := U_1 \cap U_2 \cap ... \cap U_n.$$

Let $x \in \bigcap_{k=1}^{n} U_k$, then for all $k \in \{1, 2, ..., n\}$, $x \in U_k$ which is open, then there exists $\varepsilon_k > 0$ such that $(x - \varepsilon_k, x + \varepsilon_k) \subset U_k$. It follows that

$$\bigcap_{k=1}^{n} (x - \varepsilon_k, x + \varepsilon_k) \subset \bigcap_{k=1}^{n} U_k.$$

If we put $\varepsilon = \min(\varepsilon_1, \varepsilon_2, ..., \varepsilon_k) > 0$, then

$$\bigcap_{k=1}^{n} (x - \varepsilon_k, x + \varepsilon_k) = (x - \varepsilon, x + \varepsilon) \subset \bigcap_{k=1}^{n} U_k,$$

which ends the proof.

(d) No the set of rationals $\mathbb{Q}$ is not open in $\mathbb{R}$. Let us prove this by contradiction : assume $\mathbb{Q}$ is open in $\mathbb{R}$, then :

$$\forall r \in \mathbb{Q}, \exists \varepsilon > 0 : (x - \varepsilon, x + \varepsilon) \subset \mathbb{Q},$$

which is a contradiction since the interval $(x - \varepsilon, x + \varepsilon)$ contains an infinite number of irrationals. Hence, $\mathbb{Q}$ is not open in $\mathbb{R}$.

7. Let us prove that

$$\lim_{n \to +\infty} \frac{1}{n^2 + 20n + 2020} = 0.$$

On must show that :

$$\forall \varepsilon > 0, \exists M \in \mathbb{N} : \forall n \geqslant M, \left| \frac{1}{n^2 + 20n + 2020} - 0 \right| < \varepsilon. \qquad (1)$$

We have, for all integer n,

$$\left| \frac{1}{n^2 + 20n + 2020} \right| = \frac{1}{n^2 + 20n + 2020} < \frac{1}{20n},$$

thus, if $\frac{1}{20n} < \varepsilon$, i.e., if $n > \frac{1}{20\varepsilon}$ then the inequality in (1) is satisfied. Henceforth, for all $\varepsilon > 0$, if we chose the integer

$$M = [\frac{1}{20\varepsilon}] + 1 > \frac{1}{20\varepsilon}$$

then

$$\forall n \geqslant M, \left| \frac{1}{n^2 + 20n + 2020} - 0 \right| < \varepsilon,$$

which ends the proof.