18.06 Linear Algebra Lec 1. Chapter 1.

n linear equations n unknowns

Row picture

* Column picture.

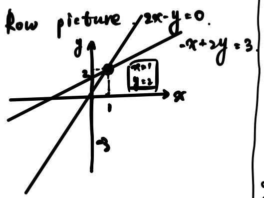
matrix.

$2x - y = 0$

$-x + 2y = 3$

$$\underset{A}{\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}}\underset{X}{\begin{bmatrix}x\\ y\end{bmatrix}} = \underset{b}{\begin{bmatrix}0\\ 3\end{bmatrix}}$$

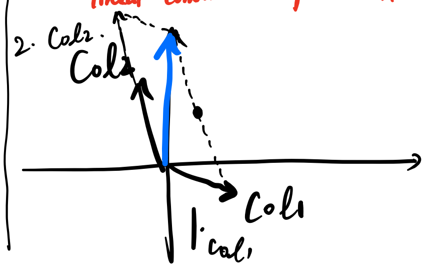
Row picture



Column picture.

$$x\begin{bmatrix}2\\ -1\end{bmatrix} + y\begin{bmatrix}-1\\ 2\end{bmatrix} = \begin{bmatrix}0\\ 3\end{bmatrix}$$

linear combination of columns



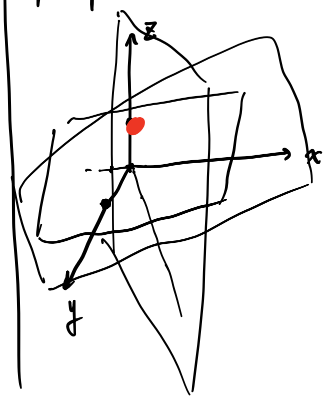
$2x - y = 0$

$-x + 2y - z = -1$

$-3y + 4z = 4$

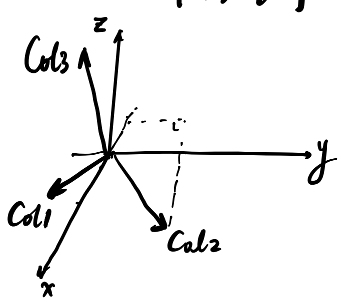
$$A = \begin{bmatrix}2 & -1 & 0\\ -1 & 2 & -1\\ 0 & -3 & 4\end{bmatrix} \quad b = \begin{bmatrix}0\\ -1\\ 4\end{bmatrix}$$

Row picture



Column picture.

$$x\begin{bmatrix}2\\ -1\\ 0\end{bmatrix} + y\begin{bmatrix}-1\\ 2\\ -3\end{bmatrix} + z\begin{bmatrix}0\\ -1\\ 4\end{bmatrix} = \begin{bmatrix}0\\ -1\\ 4\end{bmatrix}$$



$x=0.\ y=0.\ z=1.$

Can I solve $Ax = b$ for every b ?

Do the linear Combinations of the colums fill 3-D space?

— For this matrix, the answer is YES.

dimension.

$Ax = b$. ① still the linear combination of columns.

$\boxed{Ax \text{ is the comb. of columns of } A.}$

$$\begin{bmatrix}2 & 5\\ 1 & 3\end{bmatrix}\begin{bmatrix}1\\ 2\end{bmatrix} = 1\begin{bmatrix}2\\ 1\end{bmatrix} + 2\begin{bmatrix}5\\ 3\end{bmatrix} = \begin{bmatrix}12\\ 7\end{bmatrix}$$

② row picture.

Linear Algebra Lecture 2.

Elimination { Success / Failure.

Back-Substitution

Elimination matrices

Matrix multiplication

$x + 2y + z = 2$

$3x + 8y + z = 12$

$4y + z = 2.$

$Ax = b.$

$x + 2y + z = 2.$

$2y - 2z = 6$

$5z = -10.$

$x = 2$

$y = 1$

$z = -2$

$Ux = c.$

$$\begin{bmatrix} \boxed{1} & 2 & 1 & 2 \\ 3 & 8 & 1 & 12 \\ 0 & 4 & 1 & 2 \end{bmatrix} \xrightarrow{(2,1)} \begin{bmatrix} \boxed{1} & 2 & 1 & 2 \\ 0 & \boxed{2} & -2 & 6 \\ 0 & 4 & 1 & 2 \end{bmatrix} \xrightarrow{(3,2)} \begin{bmatrix} \boxed{1} & 2 & 1 & 2 \\ 0 & \boxed{2} & -2 & 6 \\ 0 & 0 & \boxed{5} & -10 \end{bmatrix}$$

(3.1)

1st pivot

2nd pivot

A .. Augmented matrix.

tack on the extra column

U (upper triangular) c

temperare failure. → 0 at pivot place but there is non-0. below it, and we do the row exchanging to let non-0 at that pivot place

complete failure → 0 at pivot place. but there is no row for us the do the exchanging which to put a non-0 number at the pivot place.

(Such as $\begin{matrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & -4 \end{matrix}$)

But pivot can't be 0, ← the elimination failures.

Matrices

step 1.

Subtract 3×row1 from row2

row operation using left matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix}$$

E_{21}

$$\begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = \begin{matrix} 3 \times col\,1 \\ + \\ 4 \times col\,2 \\ + \\ 5 \times col\,3 \end{matrix}$$

matrix × column = column

$$\begin{bmatrix} 1 & 2 & 7 \end{bmatrix} \begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix} = \begin{matrix} 1 \times row1 \\ + \\ 2 \times row2 \\ + \\ 7 \times row3 \end{matrix}$$

1×3 3×3

row × matrix Combining the rows

= row

How matrix × column and row × matrix look like?

$A \times [\;]$ $[\;] \times A$.

"repeat". Just put together

So matrix × matrix is the combinig of rows operations and columns operations

$E_{(index)}$ the elementary matrices so there are the ways the realize the mun. of matrixs.

step 2. Subtract 2×row2 from row3.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix}$$

E_{32}.

~~Gauss given order~~

$E_{32}(E_{21}A) = U$

$\underbrace{(E_{32}E_{21})}_{E}A = U$

→ whole job.

We can move the parentheses. and the outing won't change. the associative low

$$\underset{P}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & a \\ d & c \end{bmatrix}$$

permutation matrix → we use it to exchange two rows

Instead of doing the mul. we think about the reversing steps: how to get A from U?

Inverses $\underset{E^{-1}}{\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{E}{\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} = \underset{I}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$ identity matrix.

undoes = invert.

Linear Algebra lecture 3. — Page 1

Matrix multiplication (4 ways!)

Inverse of A. AB A^T → "flip across the diagonal"

Gauss-Jordan | find A^{-1}

① regular way

row 3, col 4

$A_{m\times n}$ $B_{n\times p}$ $C = AB$, $m\times p$

$C_{34} = (\text{row 3 of } A)\cdot(\text{column 4 of } B)$

$= a_{31}b_{14} + a_{32}b_{24} + \dots + a_{3n}b_{n4}$

accumulating the sums

$= \sum_{k=1}^{n} a_{3k}b_{k4}$

by

② column way

col 1, A(col 1)

$A_{m\times n}$ $B_{n\times p}$ $C_{m\times p}$.

columns of C are combination of $\boxed{\text{columns of } A}$

columns "rest by each other" side by side multiplying by A and getting the columns of the answer

come from

columns that are used the combinating the columns of A.

tell us "what the combination is?"

③ row way

A B C

rows of C are the combinating of rows of B

also as the "set" of columns waiting to be combinated

should "1 to 1" explain ★ all rows / all lines. realize the importance of "scalar"

④ 4th way $AB = \text{sum of } ((\text{cols of } A) \times (\text{rows of } B))$ ← of matched pairs! "k"

column of A × row of B.

$m\times 1$ $1\times p$

A tells us "what the combination of rows is?" → and if we want to get a concrete entry, we focus of the column numbers of rows of B, so it equals to $\sum a_{ik}b_{kj}$

B tells us "What the combination of columns is?"

also as the "set" of rows waiting to be combinated coefficient.

$$\begin{bmatrix}2\\3\\4\end{bmatrix}\begin{bmatrix}1 & 6\end{bmatrix} = \begin{bmatrix}2 & 12\\3 & 18\\4 & 24\end{bmatrix}$$

→ a special matrix! Becausing all rows line on the direction of vector (1, 6)

As big picture of 1st way! column index of A equals to row index of B.

fix the "i" and "j" and get every entries' compenents and repeat for "n" rounds, finishing accumulating all of the things eventually.

⑤ Block

$$\begin{bmatrix}A_1 & A_2\\A_3 & A_4\end{bmatrix}\begin{bmatrix}B_1 & B_2\\B_3 & B_4\end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix}$$

A B

$A_1B_1 + A_2B_3$

Lecture 3 — Page 2.

Inverses (square matrices)

$A^{-1}A = I = AA^{-1}$

If it exists. invertible, nonsingular

(non-invertible)

Singular Case.

No inverse — this dopey matrix :) spells disaster for an inverse.

$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$

"column space" "row space"

Because: You can find a non-zero (not equals to 0) vector x with $Ax = 0$.

Cool! It means there are several columns of A can be picked to be combinated and turn out "0"!

It's easy to be proved — If A have an inverse. $Ax = 0$

$A^{-1}Ax = A^{-1}0 = 0$.

$\Rightarrow x = 0$. but x is not 0. ▫

can never escape from zero.

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

A I

column way ⇒

$A \times$ column j of A^{-1} = column j of I.

Gauss-Jordan (solve ~~2~~ n equations at once)

$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{array}\right] \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

minus

A I   I A^{-1}

Start with this long matrix — double-length A.I.

eliminate untill the original "A" part down to I.

and another part. must be A Inverse.

overall matrix

E

write together and operate together.

$E_1E_2 \cdots E_n [A \mid I] = [E_1E_2 \cdots E_n A \mid E_1E_2 \cdots E_n I] = [I \mid E_1E_2 \cdots E_n I]$

E's

$E_1E_2 \cdots E_n A$ is just A^{-1}

smartly tuck on the identity

$E_1E_2 \cdots E_n I = E_1E_2 \cdots E_n$ is just A^{-1}

$[I \; A^{-1}]$

$EA = I$ tells us $E = A^{-1}$.