Linear Algebra Lecture 29

final and best factorization of a matrix!

Singular Value Decomposition : SVD

$A = U\Sigma V^T$ // Σ (sigma) diagonal → all positive !

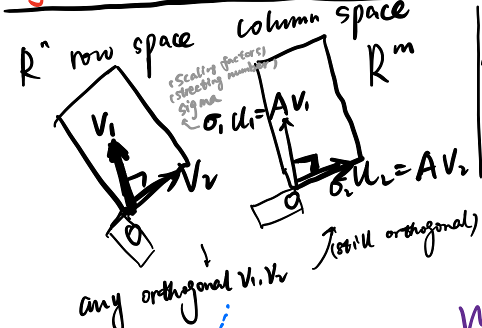
U, V ~~orthogonal~~ orthonormal

any A

symmetric pos. def

$A = Q\Lambda Q^T$

~~$A = S\Lambda S^{-1}$~~



$A[v_1\, v_2 \cdots v_r]$ row $\rightarrow (Av_i = \sigma_i u_i)$

$= [u_1 \cdots u_r]\begin{bmatrix}\sigma_1 & & 0\\ & \sigma_2 & \\ 0 & & \ddots\ \sigma_r\end{bmatrix}$

$v_{r+1} \cdots v_n$, $0 \cdots 0$ using bases of null space to complete

Goal: $AV = U\Sigma$ orthonormal

$A = \begin{bmatrix}4 & 4\\ -3 & 3\end{bmatrix}$ v_1, v_2 in rowspace R^2 (V^T)

u_1, u_2 in colspace R^2 (U)

$\sigma_1 > 0,\ \sigma_2 > 0$

$Av_1 = \sigma_1 u_1$

$Av_2 = \sigma_2 u_2$

We imagine that we can really use "$v_1 \cdots v_r$" and "$\sigma_1 \cdots \sigma_r$"

$AV = U\Sigma$

$A = U\Sigma V^{-1} = U\Sigma V^T$

two unknown vectors. But we want to know them both at once.

→ Cook up some expression. that will make the U's disappear.

$A^TA = V\Sigma \cancel{U^TU}\Sigma V^T$ (I, collapse)

$= V\begin{bmatrix}\sigma_1^2 & & \\ & \sigma_2^2 & \\ & & \ddots\end{bmatrix}V^T \Rightarrow V$

share the same eigenvalues (pos. def.!) (AB's evalues are same as BA's)

$\begin{bmatrix}4 & -3\\ 4 & 3\end{bmatrix}\begin{bmatrix}4 & 4\\ -3 & 3\end{bmatrix} = \begin{bmatrix}25 & 7\\ 7 & 25\end{bmatrix}$.

$\begin{bmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\end{bmatrix} = 32\begin{bmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\end{bmatrix}$

eigenvectors are easier to find out!

$\begin{bmatrix}1/\sqrt{2}\\ -1/\sqrt{2}\end{bmatrix} = 18\begin{bmatrix}1/\sqrt{2}\\ -1/\sqrt{2}\end{bmatrix}$

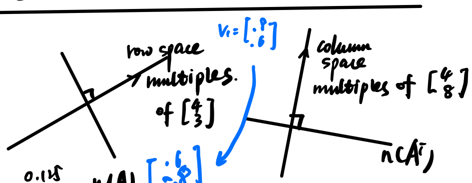
$AA^T = U\Sigma \cancel{V^TV}\Sigma U^T$

$= U\begin{bmatrix}\sigma_1^2 & & \\ & \sigma_2^2 & \\ & & \ddots\end{bmatrix}U^T \Rightarrow U$

$$\begin{bmatrix}4 & 4\\ -3 & 3\end{bmatrix} = \underset{U}{\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}}\underset{\Sigma}{\begin{bmatrix}\sqrt{32} & \\ & \sqrt{18}\end{bmatrix}}\underset{V^T}{\begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2}\\ 1/\sqrt{2} & -1/\sqrt{2}\end{bmatrix}}\ \triangle$$

Example 2

$A = \begin{bmatrix}4 & 3\\ 8 & 6\end{bmatrix}$



$A^TA = \begin{bmatrix}80 & 60\\ 60 & 45\end{bmatrix}$ $\overset{0.125}{=\!=}$

$$\begin{bmatrix}4 & 3\\ 8 & 6\end{bmatrix} = \underset{U}{\tfrac{1}{\sqrt{5}}\begin{bmatrix}1 & 2\\ 2 & -1\end{bmatrix}}\begin{bmatrix}\sqrt{125} & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}.8 & .6\\ .6 & -.8\end{bmatrix}$$

Essense:

"We're choosing the right basis for the four subspaces of linear algebra"

$v_1, \ldots, v_r$: orthonormal basis for row space

$u_1, \ldots, u_r$: — — — column space

$v_{r+1}, \ldots, v_n$ — — — null space

$u_{r+1} \cdots u_m$ — — $N(A^T)$

and

$Av_i = \sigma_i u_i$