Linear Algebra Lecture 5.

Chapter 3

$PA = LU$ $[\,\cdot\,][\,\cdot\,]$ factorization.

Vector Space and subspace

Permutations P: execute row exchanges

↳ identity matrix with reordered rows

n rows → $n! = n(n-1)\cdots(3)(2)(1)$

Counts reorderings

→ counts all $n \times n$ permutations

$\boxed{A = LU \text{ becomes } PA = LU}$

$P^{-1} = P^{T} \Leftrightarrow P^{T}P = I$

Transpose.

$$\begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}^{T} = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 3 & 1 \end{bmatrix}$$

R^T R (for rectangular matrix)

$$(A^T)_{ij} = A_{ji}.$$

Symmetric matrices. $A^T = A$ Example $\begin{bmatrix} 3 & 1 & 7 \\ 1 & 2 & 9 \\ 7 & 9 & 4 \end{bmatrix}$

R^TR is always symmetric WHY?

(RR^T)

$$\begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 4 \\ 3 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 11 & 7 \\ 11 & 13 & 11 \\ 7 & 11 & 17 \end{bmatrix}$$

TAKE TRANSPOSE!

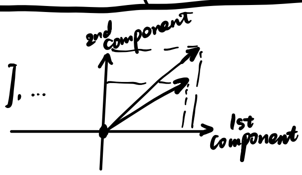
$(R^TR)^T = R^TR^{TT} = R^TR$. So, it's symmetric.

(reverse and transpose.)

Vector spaces

Examples: R^2 = all 2-dim real vectors. $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} \pi \\ e \end{bmatrix}$, ...

= "x-y plane"



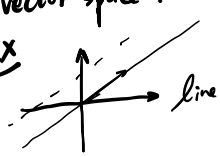
R^3 = all vectors with 3 components $\begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$

R^n: all column vectors with n real components

(number of component.)

a vector space inside R^2 — subspace of R^2.

EX



line in R^2. through the zero vector.

List Subspaces of R^2.

① all of R^2. plane

② any line through $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ L

③ zero vector only Z

R^3

R^3

plane th.

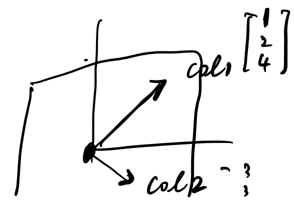
line th.

Z.

$A = \begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}$ columns in R^3

all there linear combinations form a subspace of R^3.

called column space $C(A)$



Linear Algebra Lecture 6.

Vector spaces and subspaces

Column space of A : Solving $Ax=b$

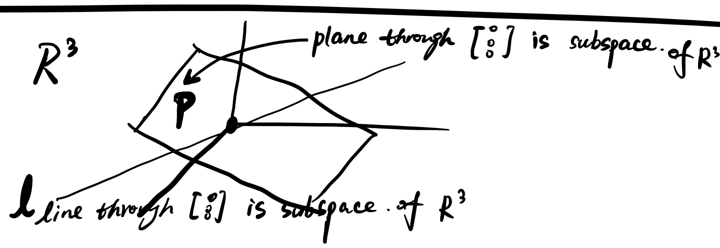
Nullspace of A.

2 ways to build up subspaces.

Vector space requirements

$v+w$ and cv are in the space.

⇒ all combs. $\boxed{cv+dw}$ are in the space



2 subspaces: P and L

$P \cup L$ (union) = all vectors in P or L (or both.)

This ~~is~~ (is not) a subspace

$P \cap L$ (intersect) : all vectors in both P and L.

Subspaces S and T.

intersection $S \cap T$. is a subspace.

Column Space of A. is subspace of R^4 $C(A)$

$A=\begin{bmatrix}1&1&2\\2&1&3\\3&1&4\\4&1&5\end{bmatrix}$ \\ all linear combs. of columns of A.

Does $Ax=b$ have a solution for every b? NO

4 equations, 3 unknowns.

$$Ax: \begin{bmatrix}1&1&2\\2&1&3\\3&1&4\\4&1&5\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}b_1\\b_2\\b_3\\b_4\end{bmatrix}$$

Which b's allow this system to be solved??? $m \times n$

Can solve $Ax=b$ exactly when b is in $C(A)$ in R^4 m

2-D subspace of R^4.

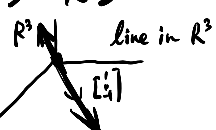
throw one of a few columns, stay the same.

"pivot column" the vector space just

Nullspace of A. $N(A)$ how we look at the weights/coefficients in the combinations

\\ all solutions x's $x=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$ to $Ax=0$. n in R^3

$Ax=\begin{bmatrix}1&1&2\\2&1&3\\3&1&4\\4&1&5\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$ contains $\begin{bmatrix}0\\0\\0\end{bmatrix}$, $\begin{bmatrix}1\\1\\-1\end{bmatrix}$.



Check that solutions to $Ax=0$ always give a subspace. (when $b \neq \vec{0}$)

If $Av=0$ and $Aw=0$ ① then $A(v+w) = Av+Aw = 0$. ② then $A(cv)=0$. ✓

Only if $b=\begin{bmatrix}0\\0\\0\end{bmatrix}$. the solutions pass through the origin! and it's necessary to be a vector space.

Linear Algebra. Lecture 7

Computing the nullspace ($Ax=0$)

Pivot variables – free variables

Special Solutions – rref(A) = R

during elimination, (in rectangles) the solutions to the system stay the same. because I'm doing the legitimate operations on the equations

So the null space stay the same.

$$A=\begin{bmatrix}1&2&2&2\\2&4&6&8\\3&6&8&10\end{bmatrix}$$

① elimination $Ax=0 \to Ux=0$.

each one free variable matching a kind of linear combination using its previous columns (have the same level with previous pivot) ⇒ matching one basis.

$$\to\begin{bmatrix}1&2&2&2\\0&0&2&4\\0&0&2&4\end{bmatrix}$$

the 2nd column is dependent on the earlier column

$$\to\begin{bmatrix}1&2&2&2\\0&0&2&4\\0&0&0&0\end{bmatrix}=U$$

echelon form [like staircase form]

rank of A = # of pivots

r = 2.

now we can solve $Ux=0$ instead of $Ax=0$.

dim C(A) = dim R(A)

number of pivot variables ⇒ r equations here. really independent

pivot rows

• pivot columns → pivot variables

FREE VARIABLES: we can assign any number freely to variables

choose freely x_2 and x_4 (the ones multiply col2, col4)

free columns → "free" means that

n variables in total → $n-r$ = 4 - 2 as number of free variables

Then we can solve the equations for x_1 and x_3.

special solutions

$$x=c\begin{bmatrix}-2\\1\\0\\0\end{bmatrix}+d\begin{bmatrix}2\\0\\-2\\1\end{bmatrix}$$

a line.

$x_1+2x_2+2x_3+2x_4=0.$

$2x_3+4x_4=0.$

② "special numbers I give to the free variables" give one of the free variables one and the others 0. (give them those special zero one values)

→ a good way to be sure I got everybody.

R = reduced row echelon form and zeros above + below pivots = 1

do elimination upwards

notice $\begin{bmatrix}1&0\\0&1\end{bmatrix}=I$ in pivot rows/cols

$$\begin{bmatrix}1&2&2&2\\0&0&2&4\\0&0&0&0\end{bmatrix}\to\begin{bmatrix}1&2&0&-2\\0&0&2&4\\0&0&0&0\end{bmatrix}\to\begin{bmatrix}1&2&0&-2\\0&0&1&2\\0&0&0&0\end{bmatrix}=R=\text{rref}(A)$$

get all the information as clear as it can be.

the identity part of the matrix at the same time

$I_{r\times r}$ So: $\begin{bmatrix}1&0\\0&1\end{bmatrix}$ pivot cols; $\begin{bmatrix}2&-2\\0&2\end{bmatrix}$ free cols; $F_{r\times(n-r)}$ free part of the matrix

"weight matrix" of pivot variables. "weight matrix" of free variables.

$x_1+2x_2-2x_4=0$

$x_3+2x_4=0.$

$Ax=0 \to Ux=0$ → $\boxed{Rx=0.}$ all the same

rref form. change to

$$R=\begin{bmatrix}I&F\\0&0\end{bmatrix}$$

r pivot rows (rank.)

r pivot cols, $n-r$ free cols

$Rx=0.$

$$\begin{bmatrix}-2&2\\1&0\\0&-2\\0&1\end{bmatrix}$$

So $RN=0$

nullspace matrix N (Columns = special solutions)

$$N=\begin{bmatrix}-F\\I\end{bmatrix}\begin{matrix}r\\n-r\end{matrix}$$

this I isn't same as the I in R. It refers to the choose of one zero numbers for free variables.

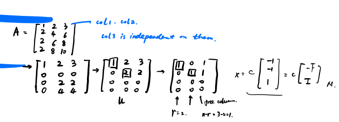
$Rx=0.$

$$[I\ F]\begin{bmatrix}x_{pivot}\\x_{free}\end{bmatrix}=0.$$

$\boxed{x_{pivot}=-F\,x_{free}}$

$r\times(n-r)$, $(n-r)\times 1$. ⇒ $r\times 1$

→ There are "$n-r$" kinds of x_{free} and matched x_{pivot}.



Linear Algebra lecture 8

Complete solution of $\boxed{Ax=b}$

Rank r

$r=m$: Solution exists $r=n$: Solution is unique.

$x_1+2x_2+2x_3+2x_4=b_1$

$2x_1+4x_2+6x_3+8x_4=b_2$

$3x_1+6x_2+8x_3+10x_4=b_3$

$$\left[\begin{array}{cccc|c} 1&2&2&2&b_1\\ 2&4&6&8&b_2\\ 3&6&8&10&b_3 \end{array}\right] \to \left[\begin{array}{cccc|c} 1&2&2&2&b_1\\ 0&0&2&4&b_2-2b_1\\ 0&0&0&0&b_3-b_2-b_1 \end{array}\right]$$

x_2, x_4 free variables

ok let $b=\begin{bmatrix}1\\5\\6\end{bmatrix}$

$0=b_3-b_1-b_2$

pivot columns

Augmented matrix $=[A\ b]$

$\boxed{\text{Solvability}}$ Condition on b.

$Ax=b$ solvable when b is in $C(A)$ — column way

row way: If a comb. of rows of A gives zero row, then same comb. of entries of b $\boxed{\text{must give } 0.}$ get $0=0$ from the process of b.

To find complete solutions to $Ax=b$.

① $x_{particular}$: set all free variables to zero $x_2=0,\ x_4=0$.

Solve $Ax=b$ for pivot variables.

$x_1+2x_3=1$ $x_1=-2$.

$2x_3=3$. $x_3=3/2$. (halves)

$x_p=\begin{bmatrix}-2\\0\\3/2\\0\end{bmatrix}$

② $x_{nullspace}$.

③ $x=x_p+x_n$ complete!

all vector out of $x_{nullspace}$.

$x_{complete}=\begin{bmatrix}-2\\0\\3/2\\0\end{bmatrix}+c_1\begin{bmatrix}-2\\1\\0\\0\end{bmatrix}+c_2\begin{bmatrix}2\\0\\-2\\1\end{bmatrix}$ 2 dimension subspace

x_n: all combinations of the special solutions.

go through this guy instead of the original. subspace.

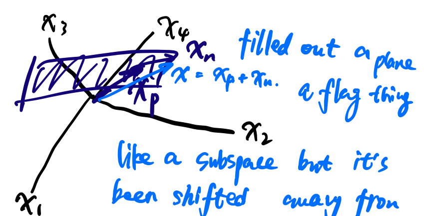
$Ax_p=b$. → find one particular solution.

plus.

$Ax_n=0$.

$A(x_p+x_n)=b$.

Plot all solutions x in R^4



filled out a plane, a flag thing, like a subspace but it's been shifted away from origin. (doesn't contain zero).

m by n matrix A of rank r. (know $r\le m$, $r\le n$.)

<"case of 'full rank'"> get here! And the # of solution turns to 0 or 1.

$\boxed{\text{Full column rank}}$ means $\boxed{r=n}$: No free variables.

$N(A)=\{\text{zero vector}\}$ Solution to $Ax=b$: $x=x_p$ (if exists) unique solution if it exists (0 or 1 solution)

$A=\begin{bmatrix}1&3\\2&1\\6&1\\5&1\end{bmatrix}\to\begin{bmatrix}1&3\\0&-5\\0&-17\\0&-14\end{bmatrix}\to\begin{bmatrix}1&0\\0&-5\\0&0\\0&0\end{bmatrix}\to\begin{bmatrix}1&0\\0&1\\0&0\\0&0\end{bmatrix}$ $R=\begin{bmatrix}I\\0\end{bmatrix}$

Depending on the n, r, m and the relationship between b and columns of A

get here. # of solution still could be infinity ($n-m$ free variables left, just is the condition of) just no zero rows here.

($m\ge n$. → may the solution doesn't exist.)

$\boxed{\text{Full row rank}}$ means $\boxed{r=m}$

only can — What's important is ~~the solution always exists~~! for every b (because $r=m<n$.

($n<m$. impossible full column rank!)

the process to get zero row essentially

we do elimination and we don't get any zero rows which means there is no combination of rows to get zero. So there's no requirement for b

Can solve $Ax=b$ for every b.

must Exists for ∞ solutions

non-zero / column space is R^m. $b\in R^m$ must in it. must have a kind of l.c.

~~when F=0~~: F doesn't exist because it could be turn into "all zero" from not all zero. So if all zero, initially should be all zero, it means that variable doesn't even exist. so it will turn into more condition

Left with $\underset{n-m}{\overset{\shortparallel}{n-r}}$ free variables.

eg $A=\begin{bmatrix}1&2&6&5\\3&1&1&1\end{bmatrix}$ $r=2=2$.

$R=\begin{bmatrix}1&0&F\\0&1&F\end{bmatrix}$ $R=[I\ F]$ (F could be partly mixed into I)

$\boxed{r=m=n}$ just full rank. $R=I$.

$A=\begin{bmatrix}1&2\\3&1\end{bmatrix}$

→ invertible matrices.

So for every b, the solution exists and is unique. is $\begin{bmatrix}b_1\\b_2\\\vdots\\b_n\end{bmatrix}$

no "F" → nullspace is {zero vector}

no free variables

1 solution

$r<m$. $r<n$.

$R=\begin{bmatrix}I&F\\0&0\end{bmatrix}$ 0 or ∞ solutions.

Linear Algebra Lecture 9

Linear independence

Spanning a Space.

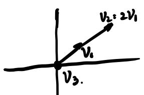
BASIS and dimension.

Suppose A is m by n. with $m<n$.
(more unknown x's than equations)

Then there are some non-zero solutions to $Ax=0$ (there is something in nullspace.)

Reason: There will be free variables! (r couldn't up to n. because of the limination of m)

$\boxed{\text{Independence}}$ Vector $x_1, x_2, \dots, x_n$ are independent if.
no combination gives zero vector (except The zero comb. call $c_i=0$)
$c_1x_1 + c_2x_2 + \dots + c_nx_n \neq 0$



$2v_1 - v_2 = 0$.
$0v_1 + cv_3 = 0$.
If one vector is zero, the independence dead.

$A = \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} x_1 & x_3 & x_5 \\ x_2 & x_4 & x_6 \end{bmatrix} \end{array}$, and then $Ax=0$ have non-zero solutions.
$m<n$. free variables always exist.

When we have a matrix, we're interested in whether its columns are dependent. or independent.

Repeat: when $v_1, \dots, v_n$ are columns of A. → combination ~
"put in directly"

They are independent if nullspace of A is {zero vector} rank $=n$ (no free column/variable) $N(A)=\{0\}$.

They are dependent if $Ac=0$, for some non-zero c. rank $<n$ Yes free variable
something in the nullspace

Vectors $v_1, \dots, v_l$ $\boxed{\text{span}}$ a space
means: The space consists of all comb. of those vector

$\boxed{\text{Basis}}$ for a space is a sequence of vectors $v_1, v_2, \dots, v_d$ with 2 properties.

1. They are Independent. (not more)
2. They span the space. (not less).

R^n. n vectors give basis if the $n \times n$ matrix with those cols is invertible.

Example:

Space is R^3.
One basis is $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$. Another basis is $\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}2\\2\\5\end{bmatrix}$ $\cancel{\begin{bmatrix}3\\?\\8\end{bmatrix}}$

row space!! $\begin{bmatrix}0\\0\\1\end{bmatrix}$

Given a space: cols
Every basis for the space has the same number of vectors.

Def: Dimension of the space

Space is $C(A)$.
$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix}$

$2=$ the rank(A) = the number of pivot columns = dimension of $C(A)$.

(rank of matrix ; dimension of column subspace of A)

$\boxed{\dim C(A) = r}$

$\begin{bmatrix}1\\-1\\1\\0\end{bmatrix} \begin{bmatrix}-1\\0\\0\\1\end{bmatrix}$

$\boxed{\begin{aligned} \dim N(A) &= \#\text{ free variables} \\ &= n-r \end{aligned}}$

Linear Algebra Lecture 10

① Correct error in Lect. 9 !!
② Four Fundamental subspaces (for matrix A)

4 Subspaces

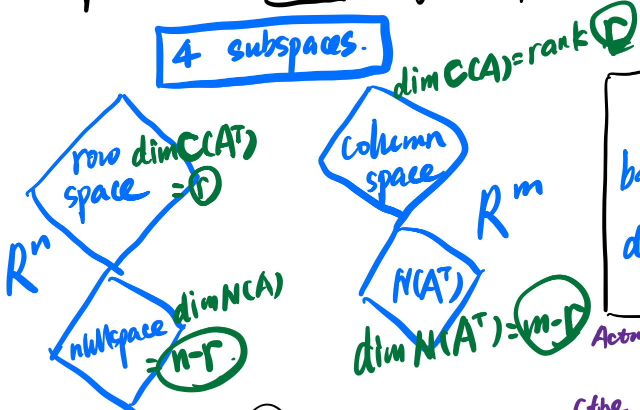
column space $C(A)$ in $\mathbb{R}^m$

nullspace $N(A)$ in $\mathbb{R}^n$

row space = all combinations of rows of A = all combinations of columns of A^T = $C(A^T)$ in $\mathbb{R}^n$

as good as A. Just happens to be n by m.

nullspace of A^T = $N(A^T)$ = left nullspace of A. in $\mathbb{R}^m$



	$C(A)$	$N(A)$	$C(A^T)$	$N(A^T)$
basis ?	pivot cols of A (not of A!)	Special Solutions	first r rows of R (not of A)	last m-r rows of E ↓ the track of "$A \to R$"
dimension?	r	n-r	r	m-r

Actually, they only shares the zero vector. (of $\mathbb{R}^m$)

(the null space is perpendicular to the row space ?!)

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} \xrightarrow{\dots} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C(R) \neq C(A)$ different column space

row operations preserve the row space. but they influence the column space (rank doesn't change, but the space change)

→ still in the space

same row space

Basis for the row space of A or of R is the first r rows of R

"sitting there in R." (Sometimes true for "of A". but not all.

$A \to R$ do the row exchanges and get $\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$)

4th space: $N(A^T)$

$$A^T y = 0. \to y^T A^{TT} = 0^T \to y^T A = 0^T$$

$$\begin{bmatrix} \quad \end{bmatrix}\begin{bmatrix} \, \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix} \qquad \begin{bmatrix} \; y^T \; \end{bmatrix}\begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} \quad 0 \quad \end{bmatrix}$$

⇓

"left nullspace of A"

how $A \to R$?

we do the reduced row echelon form of the matrix: (not n×n as Gauss-Jordan but we can still do that) keep track of E.

$$\text{rref}: \underbrace{[A_{m\times n} \; I_{m\times m}]}_{E} \to \underbrace{[R_{m\times n} \; E_{m\times m}]}_{=}$$

▶ $EA = R$

(rref of a nice invertible square matrix is the identity, in the notes before, R is just I then E is A inverse)

and last m-r rows of E 's transpose combinate the rows of A and get zero vector in the last m-r rows of R

New vector spaces !

All 3x3 matrices !! M

stretch the idea from $\mathbb{R}^n$ to $\mathbb{R}^{n\times n}$. But we can still add and multiply by numbers.

every 3x3 matrix is one of my "vectors"

→ entitled to call them vectors because they could obey the proper operations $A+B$. cA (not AB for now) just ignore we can do this for now

"subspaces" of M: eg. upper triangular matrices

eg. symmetric matrices

"∩"

eg. diagonal matrices — dim of this subspace is 3

"smaller ?"
"contained in ?"
precisely!
compute dimension?! count basis!

Linear Algebra Lecture 11

Bases of new vector spaces → M = all 3 by 3 matrices ($\dim M = 9$) → Symmetric 3x3 ($\dim S = 6$). upper triangular 3x3 ($\dim U = 6$)

Rank one matrices

Small world graphs

Basis for M = all 3x3's

$\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}, \dots, \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}$

Intersection

$S \cap U$ = symm and upper triangular = diagonal 3x3's

$\dim(S \cap U) = 3.$

$S \cup U = ?$

$\dim S = 6.\ \dim U = 6$

S and U head in difference direction, such as W_1: $\langle 0,1 \rangle$, W_2: $\langle 1,0 \rangle$. the union of W_1 and W_2 is the union of x axis and y axis. It means little... rather that the union of entries. we really need to focus on the "combination" in "space" → use the operation of vector spaces to combine them

$(R^2 = (x,y) = x(1,0) + y(0,1))$

Sum

combinations of things in S and things in U

$S+U$ = any element of S + any element of U = all 3x3's. $\dim(S+U) = 9.$

$\Rightarrow \dim S + \dim U = \dim(S \cap U) + \dim(S+U)$

$\frac{d^2y}{dx^2} + y = 0.\ y = \underbrace{\cos x}, \underbrace{\sin x}$, 解 (null space)

BASIS

Complete Solution: $y = c_1 \cos x + c_2 \sin x$

$\dim \begin{pmatrix}\text{Solution}\\\text{space}\end{pmatrix} = 2.$

⇒ Differential equations (微分方程)

↳ an equation the relates one of more unknown functions and their dirivatives.

generally represent physical quantities

... their rates of change

Ordinary D.E. (ODE) [containing an unknown function of one real or complex variable x and its derivatives and some given functions of]

"常......"

partial D.E. (PDE)

often represented by a variable (often denoted y)

and Linear differential equations are the differential equations that are linear in the unknown function and its derivatives.

$A = \begin{bmatrix}1&4&5\\2&8&10\end{bmatrix}_{2\times3}$ $r=1$

$\dim C(A) = \text{rank} = \dim C(A^T)$

Rank 1 matrix

$A = UV^T$

(U. V are all column vector of A)

$= \begin{bmatrix}1\\2\end{bmatrix}_{1\times2} \begin{bmatrix}1&4&5\end{bmatrix}_{1\times3}$

M = all 5x17 matrices.

subset of rank 4 matrices

not a subspace.

Example: In R^4. $v = \begin{bmatrix}v_1\\v_2\\v_3\\v_4\end{bmatrix}$ S = all vectors v in R^4 with $v_1+v_2+v_3+v_4 = 0$.

$Av = 0$

S = null space of $A = [\boxed{1}\ 1\ 1\ 1]$ ($n=4$) ★ rank $= 1 = r$.

$\dim N(A) = n - r = 3$ ✓

Or see it directly.

Basis for S [special vectors]

$\begin{bmatrix}-1\\1\\0\\0\end{bmatrix} \begin{bmatrix}-1\\0\\1\\0\end{bmatrix} \begin{bmatrix}-1\\0\\0\\1\end{bmatrix}$

$N = \begin{bmatrix}-1&-1&-1\\1&0&0\\0&1&0\\0&0&1\end{bmatrix}$ $\begin{matrix}-F_{r\times(n-r)}\\I_{(n-r)\times(n-r)}\end{matrix}$

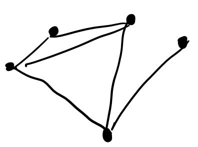
$C(A) = R^1.\ N(A^T) = \{0\}.$

$C(A^T) = c\begin{bmatrix}1\\1\\1\\1\end{bmatrix}.$

$3 + 1 = 4 = n$

$1 + 0 = m$

Graph = {nodes, edges}



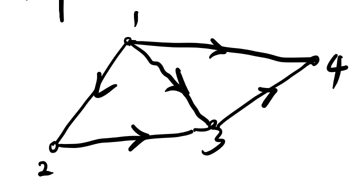
Linear Algebra Lecture 12.

Graphs + Networks.

Incidence Matrices

Kirchhoff's Laws

Graph: Nodes. Edges



$n=4$ nodes.

$m=5$ edges.

Incidence Matrix

$$A=\begin{array}{c}\begin{array}{cccc}\text{node}1 & 2 & 3 & 4\end{array}\\ \begin{bmatrix}-1 & 1 & 0 & 0\\ 0 & -1 & 1 & 0\\ -1 & 0 & 1 & 0\\ -1 & 0 & 0 & 1\\ 0 & 0 & -1 & 1\end{bmatrix}\end{array}\begin{array}{l}\text{edge}1\\2\\3\\4\\5\end{array}$$

subgraph. loop inside node 1.2.3

loop correspond to dependent (linearly dependent rows)

properties of graphs lead to the properties of its matrices.

What about the null space of A?

!!! SHIT. x is exactly the weights of potentials

Ax is the total potential differences of each node!

$$Ax=\begin{bmatrix}x_2-x_1\\x_3-x_2\\x_3-x_1\\x_4-x_1\\x_4-x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}$$

combinations of columns of A to get outing potential differences of each node

$x=x_1, x_2, x_3, x_4$ 电势 potentials at the nodes

x's meaning is: how to pick the potentials from every part of tiny edge that are out or in the node (choose "if pick" and "the times")

how to combinate the A
x is the choose of potentials.

用A作用于x后

x_2-x_1, etc. ↳ 电势差.

potential differences, across the edges

We want the outing to be "all zero". and we find we must mul each column the same scalar!!! (pass all for the same time)

rank = 3 = (node − 1)

$$x=c\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$$

"null space means all the potentials vectors that make potential differences to be zero vector.

→ x must be constant vector, as for what is x, determained by "c" which is the given base.

$\dim N(A)=1$. Absolutely. if we set any one variable of x to be 0. the x should be 0. and dimension of $N(A)=0$, and this means that:

any "node−1" columns will be independent actually

means we can't change nothing "to the potentials of each nodes in the condition that we don't pass one of the nodes.

Null space of A transpose $N(A^T)$ $\dim N(A^T)=m-r = 5-3 = 2$

$\boxed{A^Ty=0}$ the most fundamental equation of applied mathematics

y is the real choose of "if walk" and "what times": (every column is the real "edge")

the only way to change nothy is to "don't move"

We want to find the ways to walk and be back to the origion!

$\dim N(A^T)=m-r$

$$\begin{array}{ccccc}\downarrow 1 & \downarrow 2 & & \downarrow 3 & \\ \end{array}$$
$$\begin{bmatrix}-1 & 0 & -1 & -1 & 0\\ 1 & -1 & 0 & 0 & 0\\ 0 & 1 & 1 & 0 & -1\\ 0 & 0 & 0 & 1 & 1\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\\y_4\\y_5\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

<final step>

$\boxed{A^Ty=0}$ ⇒ "Kirchoff's Current Law" (KCL)

add current sources → $A^Ty=f$

$x = x_1 . x_2 . x_3 . x_4$

to definite the potentials at nodes

$A \downarrow \boxed{e = (Ax)}$ add batties in edges

x_2-x_1, etc. ⓔ

get the potential differences at nodes

same size as A. to give the conductance of the edges = $\frac{1}{\text{resistance}}$ (电导5阻) (电阻)

C $\boxed{y=Ce}$

$\xrightarrow{C(Ax)}$ currents $y_1 . y_2 . y_3 . y_4 . y_5$ on edges

↑ A^T!!! (Graphs in vision of edges matching the columns)

avaliable changes in potential

physical constands in "how much current happens" OHM's Law (欧姆定律)

make some currents happen.

In KCL:

(as a balanced equation — a conservation law)

asking for five currents that satisfy KCL

$$\begin{cases}-y_1-y_3-y_4=0\\ y_1-y_2=0\\ y_2+y_3-y_5=0\\ y_4+y_5=0\end{cases}$$

"net flow is zero" (in equals to out)

charge doesn't accumulate at the nodes, and it travels around.

redraw the graph:

[just graph level!]

Basis for $N(A^T)$

$$\begin{bmatrix}1\\1\\-1\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\\-1\\1\end{bmatrix}$$

two independent solutions to $A^Ty=0$ (loops)

$\boxed{C(A^T)}$ (row space of A) ⇒ correspond to edges

col1. col2. col4 [pivot clolumns] ↳ independent

in graph, those there edges have no loop.

Use formular for dimension: (rank = n−1)

$\dim N(A^T) = m-r$ = #loops ⇒ of independent

#loops = #edges − (#nodes − 1)

#nodes − #edges + #loops = 1 (0 dim, 1 dim, 2 dim)

works for every graph Euler's Formula (topology fact about any graph)

the "loops" match "dependencies"; they're in the graph.

Dependencies came from loops, they were the things in the $N(A^T)$

A graph with no loops is called a tree.

$\boxed{A^TCAx=f}$ (the unknown x)