Linear Algebra Lecture 18

Determinants $\det A = |A|$

Properties 1, 2, 3, 4-10

± signs

square.

→ vertical bars, rather than brackets

① $\det I = 1$

② Exchange rows reverse sign of det. $\Rightarrow \det P = \begin{cases} 1 & \text{even times} \\ -1 & \text{odd times} \end{cases}$

③a $\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ (the factor "t" comes out) LINEAR FOR EACH ROW

③b $\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$ ⇒ We're only after linearity in each row (separately) rather than "every rows"

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① $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$

② $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$

$\left(\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \right)$

$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ 0 & d - \frac{c}{a} b \end{vmatrix} = ad - bc$

④ 2 equal rows → det = 0

$\begin{vmatrix} a & b \\ c & d \\ a & b \end{vmatrix}$, $\cancel{\begin{vmatrix} 0 & 0 \\ c & d \\ a & b \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \\ 0 & 0 \end{vmatrix} = 0}$ exchange those rows → same matrix !! rather than single row's linearity (sign → 0)

⑤ substract $l \times$ row i from row k, DET Doesn't change.

$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \to \begin{vmatrix} a & b \\ c-la & d-lb \end{vmatrix} \overset{③b}{=} \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a & b \\ -la & -lb \end{vmatrix} \overset{③a}{=} \begin{vmatrix} a & b \\ c & d \end{vmatrix} + (-l)\underline{\begin{vmatrix} a & b \\ a & b \end{vmatrix}} \overset{④}{=} \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

⑥ Row of zeros → $\det A = 0$. by ③b / ③a $t=0$

⑦ $\det U = \begin{vmatrix} d_1 & * & * & * \\ 0 & d_2 & * & * \\ 0 & 0 & \ddots & \\ 0 & 0 & 0 & d_n \end{vmatrix} = (d_1)(d_2)\cdots(d_n)$ product of pivots

"the pivot formula have all complicated mess (horrible stuff) already built in quite efficiently (during elimination)"

suppose d's are not 0. and we do elimination up (⑤) (kill off-diagonal terms). and factor out d's (③a), and get I ($\det I = 1$) (①)

the elimination could guarantee this !!!

⑧ $\det A = 0$ when A is singular (det. is a fair test for invertibility of non-invertibility)

$\det A \neq 0$ when A is invertible $A \to U \to D \to d_1 d_2 \cdots d_n$ (get down to) $= (R)$ (D is as the special and good form of R)

⑨ $\det AB = (\det A)(\det B)$ [Matrix didn't have the (whole) linear property (eg. adding), but has this multiplying property] !!! AMAZING

// $\det A^{-1} = 1/\det A$

$A^{-1}A = I$. "take determinants of both sides" $(\det A^{-1})(\det A) = 1$

eg $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ $A^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix}$; $\underline{\det A^2 = (\det A)^2}$ volume

$\underline{\det 2A = 2^n \det A}$

⑩ $\underline{\det A^T = \det A}$

$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix}$ ✓ there's nothing special about rows/cols and the "x" row

Prove #10 using 1-9

$|A^T| = |A|$

$\Leftarrow |U^T L^T| = |LU|$

$\Leftarrow |U^T||L^T| = |L||U|$

+ permutation { odd / even.

Linear Algebra lecture 19

Formula for det A ($n!$ terms)

Cofactor formula

Tridiagonal matrices

① det I = 1

② SIGN REVERSE WITH ROW Exchange.

③ det is linear in each row separately

① $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$ ② $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$ ③ $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix}$

"METHOD~!"

3×3: 1→3→9→27 !

$ad-bc$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \\ a_{31} & 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 0 \end{vmatrix}$$

slip this up like craze !

row numbers are (1,2,3)

col numbers are some permutation of 1,2,3

How many survivors ?

When do we get a survivor? —The survivor have one entry from each row and each column

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

BIG FORMULA get all the horrible stuff spread out

$$\det A = \sum_{n!\text{ terms}} \pm a_{1\alpha}a_{2\beta}a_{3\gamma}\cdots a_{n\omega}$$

where come from?

$(\alpha,\beta,\gamma,\dots,\omega) = \text{perm of } (1,2,\dots,n)$

Example

$$\begin{vmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{vmatrix}$$

$(4,3,2,1) \to +1$ $(3,2,1,4) \to -1$

"Ok, we're executing a determinant formula here."

Like in between of pivot formula and big formula,

It's got easy stuff times horrible stuff

Cofactors 3×3 IN PARENTHESIS

$$\det = a_{11}(a_{22}a_{33} - a_{23}a_{32})$$
$$+ a_{12}(\quad)$$
$$+ a_{13}(\quad)$$

$$\begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & a_{23} \\ a_{31} & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ - & - & 0 \\ - & - & 0 \end{vmatrix}$$

$$a_{12}(-a_{21}a_{33} + a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Cofactor formula (along row 1) / col j

$$\det A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

Cofactor of $a_{ij} = C_{ij}$

$\pm \det \begin{pmatrix} n-1 \text{ matrix} \\ \text{with row } i \text{, col } j \text{ erased} \end{pmatrix}$ minor

$i+j$ even, $i+j$ odd [Built into the cofactors] sign

"$i+j$ rule" (Of the "newest / constant" matrix)

EX "tri-diagonal matrix of ones"

$$A_4 \begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{vmatrix}$$

$|A_1| = 1,\ |A_2| = 0,\ |A_3| = -1$

$|A_4| = 1\cdot|A_3| - 1|A_2|$

$\underline{|A_n| = 1|A_{n-1}| - |A_{n-2}|}$

(These matrices have period 6)

$|A_4| = -1$ $|A_5| = 0$

$|A_6| = 1$

$|A_7| = 1$

Linear Algebra Lecture 20.

Applications:

1. Formula for A^{-1}
2. Cramers Rule for $x = A^{-1}b$
3. $|\det A|$ = Volume of box

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ $\boxed{A^{-1} = \frac{1}{\det A} C^T}$ ← products n-1 entries

(the matrix of cofactors)

products of n entries

Check $AC^T = (\det A) I$

$$\begin{bmatrix} a_{11} \cdots a_{1n} \\ \\ a_{n1} \cdots a_{nn} \end{bmatrix}\begin{bmatrix} C_{11} & C_{21} \cdots C_{n1} \\ C_{1n} & \\ C_{1n} & C_{2n} \cdots C_{nn} \end{bmatrix} = \begin{bmatrix} \det A & & 0 \\ & \det A & \\ 0 & & \det A \end{bmatrix}$$

One row i times the (cofactors from a different row j)^T = 0 :

$A_s = \begin{bmatrix} a & b \\ a & b \end{bmatrix}$ "A screwed up"

"When we do this multiplication, what we're doing is using the cofactor formula for [Think from the meaning!] the determinant which row i = row j!

we exchange row i and row j. then Erase the origin row j and put just row i there. then the multiplying above. — of course "0" (minus exists too) (make cofactors of row j is matched with above)

number

numerically

CRAMER'S RULE

$Ax = b$

$x = A^{-1}b = \frac{1}{\det A} C^T b$

multiplying cofactors by the entries of B

$x_1 = \frac{\det B_1}{\det A}$ Cramer realize "C^Tb" have

"But actually, Cramer's Rule is a disastrous way to go.

$x_j = \frac{\det B_j}{\det A}$ just n+1 dets ?....

$B_1 = \left[\, b \;\; \begin{matrix} n-1 \\ \text{columns} \\ \text{of } A \end{matrix} \,\right]$ = A with column 1 replaced by b

use cofactor formula down column 1

→ B_j = A with column j replaced by b

$x_1 = C_{11}b_1 + C_{21}b_2 + \cdots + C_{n1}b_n$

$(x_i = C_{1i}b_1 + C_{2i}b_2 + \cdots + C_{ni}b_n)$ [could use "row way"]

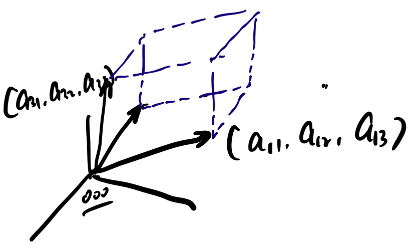
3×3

$|\det A|$ = volume of box [parallelepiped]

(a formula for the area of a parallelogram)

" what we know is the coordinates of the corners instead of the base and height.

sign: tell us whether these three is a right-handed box or left-handed box



when A is I, the box is a unit cube. (± 1)

when $A = Q$ (orthog. matrix) $(Q^TQ = I)$, the box is a unit cube again, and it's just rotated (± 1)

1 . 2 . 3a . 3b

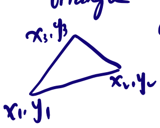
I ± t

(also satisfying 3a)

area = $\det\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ = ad−bc for parallelogram

$\frac{1}{2}$(ad−bc) for triangles

triangle



area = $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

row operations are equivalent to moving the triangle to start at the origin.