Linear Algebra Lecture 21

Eigenvalues - Eigenvectors

$\det[A-\lambda I]=0$.

TRACE $=\lambda_1+\lambda_2+\cdots+\lambda_n$

in goes a "vector x". out goes a vector

Ax <a kind of function>

also must for square matrices!

Eigenvectors

Ax parallel to x } can find a λ

$\boxed{Ax=\lambda x}$

eigenvalue

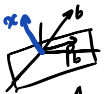
when $\lambda=0$. $Ax=\lambda x=0$. x is in the null space

If A is singular, $\lambda=0$ is egenvalue [we set "x" is non-zero]

there's "non-zero x" matches $Ax=0\cdot x$.

- Both λ and x are unknown. we need a good and new way to find out all x's and the λ's matching them.

Now we think about it on condition $A=P$ (projection matrix):



What are x's and λ's for projection matrix?

Any x in plane: $Px=x$. $\lambda=1$.

Any $x \perp$ plane: $Px=0$. $\lambda=0$.

$A=\begin{bmatrix}0&1\\1&0\end{bmatrix}$ $x=\begin{bmatrix}1\\1\end{bmatrix}$ $Ax=\begin{bmatrix}1\\1\end{bmatrix}$ $\lambda=1$ $Ax=x$

$x=\begin{bmatrix}-1\\1\end{bmatrix}$ $Ax=\begin{bmatrix}1\\-1\end{bmatrix}$. $\lambda=-1$ $Ax=-x$

Fact:

Sum of λ's $=$

$a_{11}+a_{22}+\cdots+a_{nn}$ (trace)

How to solve $Ax=\lambda x$:

Rewrite $(A-\lambda I)x=0$

Singular (remove) minus λI and become sigular

KEY Euation (Characteristic Eq polynomial uation)

$\rightarrow$ $\det(A-\lambda I)=0$. FIND λ first. λ squared n at most! $\rightarrow$ there is n λ at most. may different, may same.

Symmetric, real ? anti-symmetric

$A=\begin{bmatrix}3&1\\1&3\end{bmatrix}$ $\det(A-\lambda I)=\begin{vmatrix}3-\lambda&1\\1&3-\lambda\end{vmatrix}=(3-\lambda)^2-1$
$=\lambda^2-6\lambda+8$ $\xrightarrow{\text{factor into}}$ $(\lambda-2)(\lambda-4)$. $\lambda_1=4$, $\lambda_2=2$.

trace det$A$

(linear coefficient)

$A-4I=\begin{bmatrix}-1&1\\1&-1\end{bmatrix}$. $x_1=\begin{bmatrix}1\\1\end{bmatrix}$ x's $\Rightarrow$ in null space of $(A-\lambda I)$ and isn't zero.

$\begin{bmatrix}1&-1\\0&0\end{bmatrix}$ independent eigenvectors $\begin{bmatrix}0&1\\1&0\end{bmatrix}\xrightarrow{+3I}\begin{bmatrix}3&1\\1&3\end{bmatrix}$

$A-2I=\begin{bmatrix}1&1\\1&1\end{bmatrix}$ $x_2=\begin{bmatrix}-1\\1\end{bmatrix}$

eigenvalues are 3 bigger

eigenvectors don't change

If $Ax=\lambda x$. Just the scalar that should exist change, but because it's arbitrary x's match them won't change. one-one matching.

then $(A+3I)x=\lambda x+3x=(\lambda+3)x$

NOT SO GREAT $A+B$. AB (because A and B are uncertain to share the same eigenvectors

If $Ax=\lambda x$, B has eigenvalues α_1.

$B\cancel{x}=\alpha\cancel{x}$

~~$(A+B)x=(\lambda+\alpha)x$~~

Example (rotation!~)

$Q=\begin{bmatrix}0&-1\\1&0\end{bmatrix}$ ($\cos\frac{\pi}{2}$, $-\sin\frac{\pi}{2}$, $\sin\frac{\pi}{2}$, $\cos\frac{\pi}{2}$) trace: $0+0=\lambda_1+\lambda_2$

90° rotation det $=1=\lambda_1\lambda_2$

$\det(Q-\lambda I)=\begin{bmatrix}-\lambda&-1\\1&-\lambda\end{bmatrix}=\lambda^2+1=0$. $\lambda_1=i$ $\lambda_2=-i$

Complex numbers have to enter 18.06 at this moment! Boo..! ($\rightarrow$ n λ's must exist)

whole $\rightarrow$ present the structure of the matrix

(eigenvalues and eigenvectors are both very special on one matrix's own — especially eigenvalues)

$A=\begin{bmatrix}3&1\\0&3\end{bmatrix}$. $\det(A-\lambda I)=\begin{vmatrix}3-\lambda&1\\0&3-\lambda\end{vmatrix}=(3-\lambda)(3-\lambda)$

triangular! $\lambda_1=3$, $\lambda_2=3$

We can read the eigenvalues off and they're right on the diagonal.

$(A-\lambda I)x=\begin{bmatrix}0&1\\0&0\end{bmatrix}\begin{bmatrix}\ \\ \ \end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$ $x_1=\begin{bmatrix}1\\0\end{bmatrix}$ $x_2=$ No 2nd INDEP x ! matrices where eigenvectors are don't give the complete story (only 1 independent eigenvector) [<2]

This is a degenerate matrix

Linear Algebra Lecture 22

Diagonalizing a matrix $S^{-1}AS = \Lambda$ [diagonalization]

Powers of A / equation $u_{k+1} = Au_k$ ✓✓✓

Suppose n independent eigenvectors of A. → make sure we're able to invert S

Put them in columns of S [the eigenvector matrix]

diagonal eigenvalue matrix Λ (capital lambda)

$AS = A\begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} = \begin{bmatrix} \lambda_1 x_1 & \cdots & \lambda_n x_n \end{bmatrix} = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} = S\Lambda$

Back again!

separate out those eigenvalues from the x's (multiplying numbers)

(But for this equation, we can repeating the x_i and λ_i and make the equation okey)

– when $m < n$ ($r < n$), the indep. eigenvectors can't form a matrix and can't invert.

$S^{-1}AS = \Lambda$ There's a small number of matrices that don't have n independent eigenvectors. But the most matrices that we can diagonalize! and if can do it

→ $A = S\Lambda S^{-1}$ "the new factorization!" [replacement for LU from elimination or QR from Gram-Schmidt] [give a great way to understand the powers of a matrix]

If $Ax = \lambda x$.

$A^2x = \lambda Ax = \lambda^2 x$ ⇒ eigenvectors are the same. eigenvalues are λ^2.

$A^2 = S\Lambda S^{-1}S\Lambda S^{-1} = S\Lambda^2 S^{-1}$

$A^k = S\Lambda^k S^{-1}$ $A^{-1} = S\Lambda^{-1}S^{-1}$ (1 over eigenvalues of A)

Theorem

$A^k \to 0$ as $k \to \infty$

if all $|\lambda_i| < 1$

"Which matrices are diagonalizable?"

A is sure to have n indep. eigenvectors, (and be diagonalizable)

if all the λ's are different ✓

(no repeated λ's) [every λ have at least 1 eigenvector]

Repeated eigenvalues // may or may not have n indep. eigenvectors

→ identity matrix I → eigenvalues: $\underbrace{1, \ldots, 1}_{n}$, but there is no shortage of eigenvectors for I. (every vectors in $\mathbb{R}^n$ is eigenvector)

($S\Lambda S^{-1} = \Lambda$, if A is already diagonal, Λ is the same as A)

→ Suppose A is triangular matrix.

$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ $\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix}$ how often the eigenvalues are repeated.

$\lambda = 2, 2$ ⇒ algebraic multiplicity

$A - 2I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. we don't have enough eigenvectors

The root of the polynomial (Sum of λ's)

geometric multiplicity

looks for eigenvectors (in null space of $(A - \lambda I)$)

Equation $u_{k+1} = Au_k$ (one step equation)

It solves difference equation (at first order. first-order system) ← vectors and a matrix; ↑ it only goes up one level

Start with given vector u_0

$u_1 = Au_0$. $u_2 = A^2u_0$ $\boxed{u_k = A^k u_0}$

To really solve: write

substitute $A^p x$ with $\lambda^p x$ directly

"certain amount of.."

$u_0 = c_1x_1 + c_2x_2 + \cdots + c_nx_n$ expand the initial vector into eigenvectors.

$u_{100} = A^{100}u_0 = c_1\lambda_1^{100}x_1 + c_2\lambda_2^{100}x_2 + \cdots + c_n\lambda_n^{100}x_n$

$= \underset{n\times n}{\Lambda^{100}}\underset{n\times n}{S}\underset{n\times 1}{C}$ $(= S\Lambda^{100}C)$

$Au_0 = S\Lambda S^{-1}SC = S\Lambda C$

$AAu_0 = S\Lambda S^{-1}S\Lambda C$

$= S\Lambda^2 C$

$A^n u_0 = S\Lambda^n C$

follow each one seperately! every A brings in λ's goes into

and let each eigenvector go its own way, multiplying by λ_i each step. therefore by λ to the hundredth power after 100 steps

this also requires a complete set of eigenvectors, otherwise, we might not be able to expand u_0 in the eigenvectors and we couldn't get started.

Fibonacci example: 0, 1, 1, 2, 3, 5, ... $F_{100} = ?$

F_0 F_1 F_2

TRICK $u_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$ $u_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$u_{k+1} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}u_k$

$\begin{vmatrix} -\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = 0$. $\lambda^2 - \lambda - 1 = 0$. $\lambda = \frac{1 \pm \sqrt{5}}{2}$ $\lambda_1 = \frac{1+\sqrt{5}}{2}$ $\lambda_2 = \frac{1-\sqrt{5}}{2}$ $\begin{bmatrix} \frac{-1-\sqrt{5}}{2} & 1 \\ 1 & \frac{1-\sqrt{5}}{2} \end{bmatrix} \to \begin{bmatrix} \frac{-1-\sqrt{5}}{2} \\ 0 \end{bmatrix}$

$F_{k+2} = F_{k+1} + F_k$. (second-order single equation)

(Like having a second-order differential equation with second derivatives)

$F_{k+1} = F_{k+1}$ ↘ to be a system

One-step equation $u_{k+1} = \overset{A}{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}\begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$

$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - \lambda - 1 = 0$. $F_{k+2} - F_{k+1} - F_k = 0$ $\lambda_1 = \frac{1+\sqrt{5}}{2} \approx 1.618\ldots$ $\lambda_2 = \frac{1-\sqrt{5}}{2} \approx -.618\ldots$

eigenvalues are controlling the growth. crucial numbers!

when things are evolving in time by a first-order system

$F_{100} \approx c_1 \frac{1}{2}\left(\frac{1+\sqrt{5}}{2}\right)^{100} (+ 0)$

$A - \lambda I = \begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix}, \begin{bmatrix} \lambda \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ so, $x_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}$ $x_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$

Linear Algebra Lecture 23 (parallel lecture compared to powers of a matrix)

The key idea is the solutions to constant coefficient linear equations are exponentials

Differential Equations $\frac{du}{dt} = Au$ ✓

Exponential e^{At} of a matrix ✓

Example

$\frac{du_1}{dt} = -u_1 + 2u_2$

$u(0) = \begin{bmatrix}1\\0\end{bmatrix}$

$\frac{du_2}{dt} = u_1 - 2u_2$

$\boxed{\frac{du}{dt} = Au}$

A itself is singular. of course $\lambda = 0, -3$

$\rightarrow A = \begin{bmatrix}-1 & 2\\ 1 & -2\end{bmatrix}$

$|A - \lambda I| = \begin{vmatrix}-1-\lambda & 2\\ 1 & -2-\lambda\end{vmatrix} = \lambda^2 + 3\lambda = 0.$ $\lambda(\lambda+3)$

$Ax_1 = 0x_1$

$\lambda_1 = 0$. $x_1 = \begin{bmatrix}2\\1\end{bmatrix}$

$\lambda_2 = -3$

$A-(-3)I$

$= \begin{bmatrix}2 & 2\\1 & 1\end{bmatrix} x_2 = \begin{bmatrix}1\\-1\end{bmatrix}$ $Ax_2 = -3x_2$

why the number of eigenvectors $\leq n$? ✓ ⇒ Because "square" ($\in R^n$) indepen

why different λ leads to different x? — Different $A-\lambda I$. different null space?

the x_1 from $N(A-\lambda_1 I)$. x_2 from $N(A-\lambda_2 I)$.

x_1, x_2 must be indep. because (can't use one the represent the other) $Ax_1 = \lambda_1 x_1$, $Ax_2 = \lambda_2 x_2$ and one eigenvector can only match one eigenvalue

$A-\lambda I \rightarrow 2$ indep x's ⇒ with another λ's x. dim may not be 3? may be must

Solution: "two eigenvalues, two special solutions — two pure exponential solutions"

$u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2$

analogue of pure powers last time

analog: $c_1 \lambda_1^k x_1 + c_2 \lambda_2^k x_2$ $[u_{k+1} = Au_k]$

Check: $\frac{du}{dt} = Au$. Plug in $e^{\lambda_1 t}x_1$: $\lambda_1 \cancel{e^{\lambda_1 t}} x_1 = A e^{\lambda_1 t} x_1$

(on the left-hand side, we take the time derivative)

$= c_1 \cdot 1 \cdot \begin{bmatrix}2\\1\end{bmatrix} + c_2 e^{-3t}\begin{bmatrix}1\\-1\end{bmatrix}$

"use the initial condition"

Use $u(0) = \begin{bmatrix}1\\0\end{bmatrix}$: At $t=0$. $c_1\begin{bmatrix}2\\1\end{bmatrix} + c_2\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$

$S\,c = u(0)$

$\begin{bmatrix}2 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$

All stay right forever — each pure exponential goes on its own way once you start it.

By getting it right at the start, we know how much of each pure exponential is in the solution

Solution: $c_1 = \frac{1}{3}$. $c_2 = \frac{1}{3}$

$\rightarrow \frac{1}{3}\begin{bmatrix}2\\1\end{bmatrix} + e^{-3t}\begin{bmatrix}1\\-1\end{bmatrix}$

steady state. $u(\infty) = \frac{1}{3}\begin{bmatrix}2\\1\end{bmatrix}$

stability picture

① Stability / need $e^{At} \to 0$ / $\mathrm{Re}\,\lambda < 0$ goes to 0.

$|e^{(-3+6i)t}| = e^{-3t}$ ⇒ It's the real part that matters!

$|e^{6it}| = 1$ $(\cos 6t + i\sin 6t)$

this complex number runs around the unit circle (oscillate between two components)

② Steady state: $\lambda_1 = 0$ and other $\mathrm{Re}\,\lambda < 0$.

③ Blow up! if any $\mathrm{Re}\,\lambda > 0$

2x2 stability $\mathrm{Re}\,\lambda_1 < 0$. $\mathrm{Re}\,\lambda_2 < 0$

$A = \begin{bmatrix}a & b\\c & d\end{bmatrix}$; trace $a+d = \lambda_1 + \lambda_2 < 0$ (the complex parts will be conjugates of each other)

det > 0 $(\lambda_1\lambda_2)$

trace < 0 still blow up $\begin{bmatrix}-2 & 0\\0 & 1\end{bmatrix}$

$\frac{du}{dt} = Au$ ★Set $\boxed{u = Sv}$ constant eigenvector matrix $u \to Sv$

"Now this matrix A couples "u" and t.

The whole point of eigenvectors is to "uncouple." (diagonalize)

calculate $\frac{dv}{dt}$. and get Λ!!!

$\rightarrow S\frac{dv}{dt} = ASv$ $\boxed{\frac{dv}{dt}} = \boxed{S^{-1}ASv} = \boxed{\Lambda v}$ $\left(\frac{dv_1}{dt} = \lambda_1 v_1, \cdots\right)$

If we use the eigenvectors as basis, then our system of equation is just diagonal (no coupling any more)

⇒ Pure exponentials matrix: $v(t) = e^{\Lambda t} v(0)$ $u(t)$ could be write as this simply one

the answer is $e^{At}u(0)$ for sure (directly got by $\frac{du}{dt} = Au$)

$\rightarrow \boxed{u(t) = Se^{\Lambda t}S^{-1}u(0)}$ ⇒ $e^{At} = Se^{\Lambda t}S^{-1}$? why right

? What's going on with this matrix in the exponential

So we get the solution in the form of matrix

this eries is better, because it always converges ("divide by $n!$") so whatever matrix A and however large t is, the series adds up to a finite sum (terms go to 0)

Matrix exponential $e^{At} = I + At + \frac{(At)^2}{2} + \frac{(At)^3}{6} + \cdots + \frac{(At)^n}{n!} + \cdots$ $\left(e^x = \sum_0^\infty \frac{x^n}{n!}\right)$ [e to the e^{At} is completely defined]

There's a power series for exponentials! the one beautiful Taylor series

$\frac{1}{1-x} = \sum_0^\infty x^n$ [the geometric series, the nicest power series of all]

$(I - At)^{-1} = I + At + (At)^2 + (At)^3 + \cdots$

$|\lambda(At)| < 1 \rightarrow$ converge

do the same thing for matrices that we do for ordinary functions

$e^{At} = I + At + \frac{(At)^2}{2} + \frac{(At)^3}{6} + \cdots + \frac{(At)^n}{n!} + \cdots$

$= \underset{SS^{-1}}{I} + S\Lambda S^{-1}t + \frac{S\Lambda^2 S^{-1}}{2}t^2 + \cdots = Se^{\Lambda t}S^{-1}$ (based on the condition that A can be diagonalized)

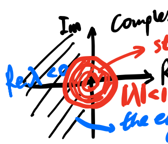
Our whole point is that to take the exponential of a diagonal matrix

↳ be completely decoupled!

$e^{\Lambda t} = \begin{bmatrix}e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t}\end{bmatrix}$ $\Lambda = \begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix}$ (Λ's are all diagonal, and add up — getting "$e^x = \ldots$" ordinary functions to get $e^{\lambda_i t}$ in entries)

S and S^{-1} aren't moving. the key is $e^{\Lambda t} = [e^{\lambda_1 t} \cdots e^{\lambda_n t}]$.

$\rightarrow$ So only when $\mathrm{Re}\,\lambda_i < 0$ (for $\forall i \geq 1$) the e^{At} get smaller and smaller as t increases (each $e^{\lambda_i t}$ goes to 0)



stability rigion for powers of A $|\lambda| < 1$

the eigenvalues should be be this side for stability in differential equations

$y'' + by' + ky = 0$ (one equation; second order)

$\rightarrow$ 2x2 1st order

$u = \begin{bmatrix}y'\\y\end{bmatrix}$ $u' = \begin{bmatrix}y''\\y'\end{bmatrix} = \begin{bmatrix}-b & -k\\1 & 0\end{bmatrix}\begin{bmatrix}y'\\y\end{bmatrix}$

$y' = y'$

coefficients of the equation

trivial equations

5^{th} order $\rightarrow$ 5x5 1st order

Linear Algebra Lecture 24

Markov matrices ✓
steady state
Fourier Series ✓ & Projections

$A = \begin{bmatrix} .1 & .01 & .3 \\ .2 & .99 & .3 \\ .7 & 0 & .4 \end{bmatrix}$

① All entries ≥ 0
② all columns add to 1.
(after squaring, true again, true always)

steady state : $\lambda = 1$
correspond, totally connected with an eigenvalue of one be

1. $\lambda = 1$ is an eigenvalue. / eigenvector $x_1 > 0$
2. All other $|\lambda_i| < 1$ $(i \neq 1)$ → go to 0

$u_k = A^k u_0 = c_1 \lambda_1^k x_1 + c_2 \lambda_2^k x_2 + \dots = c_1 x_1 > 0$
▷ steady state! $\boxed{x_1 \text{ part of } u_0}$

$A - 1I = \begin{bmatrix} -.9 & .01 & .3 \\ .2 & -.01 & .3 \\ .7 & 0 & -.6 \end{bmatrix} \begin{bmatrix} .6 \\ 33 \\ .7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ because rows are dependent.

shift it by 1

$[(1,1,1) \text{ is in } N(A^T)]$ ✓
↳ What vector is in $N(A)$? (x_1)

Naturely, All columns of $A-I$ add to zero → $A-I$ is singular

wow Fact: eigenvalues of A = eigenvalues of A^T are The Same!

$\det(A - \lambda I) = 0$ $N(A-\lambda I)$
$\Downarrow$ *
$\det(A^T - \lambda I) = 0$ $N((A-\lambda I)^T)$

Application of Markov matrices

$u_{k+1} = A u_k$. A is Markov $\begin{bmatrix} u_{cal} \\ u_{mass} \end{bmatrix}_0 = \begin{bmatrix} 0 \\ 1000 \end{bmatrix}$

$\begin{bmatrix} u_{cal} \\ u_{mass} \end{bmatrix}_{t=k+1} = \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix} \begin{bmatrix} u_{cal} \\ u_{mass} \end{bmatrix}_k$

$(\begin{bmatrix} .9 \\ .1 \end{bmatrix} u_{cal} + [\ ] u_{mass})$

$\begin{bmatrix} u_{cal} \\ u_{mass} \end{bmatrix}_1 = \begin{bmatrix} 200 \\ 800 \end{bmatrix}$

a severe limitation on the example
(the same Markov matrix/probability act at every time)

$\begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix}$. $\lambda_1 = 1$. $\begin{bmatrix} -.1 & .2 \\ .1 & -.2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ ↑x_1

$\lambda_2 = .7$ $\begin{bmatrix} .2 & .2 \\ .1 & .1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ ↑x_2

$u_k = c_1 1^k \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 (.7)^k \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$u_0 = \begin{bmatrix} 0 \\ 1000 \end{bmatrix} = \underset{\frac{1000}{3}}{c_1} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \underset{\frac{2000}{3}}{c_2} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

projections with orthonomal basis

expansion
$q_1 \cdots q_n$. (in R^n)

$\begin{bmatrix} q_1 & \cdots & q_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = v$
$Q \quad x = v$

$x = Q^{-1} v = Q^T v$
$x_i = q_i^T v$

Any $v = x_1 q_1 + x_2 q_2 + \dots + x_n q_n$ (expanding the vector into the basis)
take the dot product (inner) q_1

$q_1^T v = x_1 + 0 + \_\_ + 0$

FOURIER SERIES: $f(x) = f(x+2\pi)$ PERIODIC FCN

"function space" Leave finite dimensional vector spaces and go to infinite dimensional vector spaces

$f(x) = a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \dots$

We have an "orthogonal" infinite basis for function space, and all we want to do is express a function in that basis. (get an expansion in an orthonormal basis)

For vectors: $v^T w = v_1 w_1 + \cdots v_n w_n$.
↓continuum
For functions: $f^T g = \int_0^{2\pi} f(x) g(x) dx$ the dot (inner) product of two functions will be the integral of those functions. dx
integration

$\int_0^{2\pi} \sin x \cos x \, dx = \frac{1}{2} (\sin x)^2 \Big|_0^{\pi} = 0$

a_0: average of $f(x)$

the first Fourier coefficient a_1: take the inner product of everything with $\cos x$

$\int_0^{2\pi} f(x) \cos x \, dx = a_1 \underbrace{\int_0^{2\pi} (\cos x)^2 dx}_{\pi} + 0 + \dots$

$\therefore a_1 = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos x \, dx$ formula for coefficients in a Fourier Series

Quiz Review

$\boxed{\text{algebraic multiplicity} \geq \text{geometric multiplicity}}$

$\frac{1}{9} \begin{bmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{bmatrix}$

$\lambda = 0, 0, 1$

For eigenvalue $\lambda = 0$ 8

geometric multiplicity (of 0's eigenvectors) $= \boxed{\dim N(A)} = n - r(A) \boxed{\leq}$

$\boxed{\text{algebraic multiplicity}}$ (of 0) $= n -$ algebraic multiplicity of non-zero eigenvalues

$\Rightarrow$ algebraic multiplicity of non-zero eigenvalues $\leq n + r(A) - n = r(A)$

Invertible $\Leftrightarrow$ no zero evalues

trace of $(A + I) = (\lambda_1 + 1) + \dots + (\lambda_n + 1) = \underbrace{\lambda_1 + \dots + \lambda_n}_{\text{trace of } A \text{ "eigenvalues'"}} + n$.

$\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$

$\lambda^2 - \lambda + 1 = 0$

$\lambda = \frac{1 \pm \sqrt{3}}{2} = \frac{1 + \sqrt{3} i}{2}, \frac{1 - \sqrt{3} i}{2}$

squares of them add to 1. → on the unit circle
↓
the need of $i\theta$

$\cos\theta + i \sin\theta$

$e^{i\pi/3}, e^{-i\pi/3}$
"e to the π over 3" $(e^{\pm 2\pi i} = 1)$

$\Rightarrow$ "6 powers" back to one!

↳ "A to the 6." 's evalues are 1 and 1.
$(\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix})^6 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix})$

$\Downarrow$
definated by Ruler Falmula: $e^{i\theta} = \cos\theta + i \sin\theta$

$e^{x+ij} = e^x e^{ij}$

Linear Algebra Lecture 25.

Symmetric matrices. (must be square) real

their Eigenvalues / Eigenvectors

(Start: Positive Definite Matrices)

① The eigenvalues are REAL

② The eigenvectors are PERPENDICULAR

Can be chosen.

BASE! There's always enough eigenvectors.

"have a complete set of eigenvectors

Usual $A = S\Lambda S^{-1}$

Symmetric case $A = A^T$

$A = Q\Lambda Q^{-1}$ — have orthonormal eigenvectors as columns of Q

$\boxed{A = Q\Lambda Q^T}$ ⇒ matrix is "broken down into" pure eigenvalues and eigenvectors.

Spectral theorem (in mechanics, it's often called the principle axis theorem.)

Δ (spectrum: is the set of eigenvalues of the matrices) it somehow comes from the idea of the spectrum of light — as a combination of pure things

Why can be chosen to be perpendicular?

①
$Ax_1 = \lambda_1 x_1$. $\quad x_1^T A x_2 = x_1^T \lambda_1^T x_2$

$Ax_2 = \lambda_2 x_2 \quad x_2^T A x_1 = x_2^T \lambda_2^T x_1$

$\lambda_1 \neq \lambda_2$. So $x_1^T x_2$ and $x_2^T x_1$ must be 0

So x_1 is perpendicular to x_2.

Conclusion: It means that different eigenvalues's eigenvectors (not only independent!) must be perpendicular mutually for symmetric matrices and, one eigenvalue's eigenvectors can be transformed to be orthonormal.

---

② why real eigenvalues? $(\overline{a+ib} = a - ib)$ $\quad \overline{ab} = \bar{a}\bar{b}$, $\quad \overline{a+b} = \bar{a} + \bar{b}$

e^x can be transformed to sum of real part and imaginary part

$Ax = \lambda x \xrightarrow{\text{always}} A\bar{x} = \bar{\lambda}\bar{x}$ (real matrix)

pairs? same?

$\bar{x}^T A^T = \bar{x}^T \bar{\lambda}^T$

$\bar{x}^T A = \bar{x}^T \bar{\lambda}^T$

$\bar{x}^T A x = \lambda \bar{x}^T x \qquad \bar{x}^T A x = \bar{x}^T \bar{\lambda}^T x$

$\lambda \bar{x}^T x = \bar{\lambda}\, \bar{x}^T x$ — (length)² of complex numbers

$[\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \bar{x}_1 x_1 + \bar{x}_2 x_2 + \dots$

$(a - ib)(a + ib) = a^2 + b^2$

$\lambda = \bar{\lambda}$ λ is real!

But when A is complex.

$\bar{A}\bar{x} = \bar{\lambda}\bar{x} \Rightarrow \bar{x}^T \bar{A}^T = \bar{x}^T \bar{\lambda}^T$

"Good matrices"

Real λ's.

Perpendicular x's.

$A = \bar{A}^T$ — if A is complex!

$A = A^T$ (A is real in 99% cases)

$A = Q\Lambda Q^T$

$= \begin{bmatrix} q_1 & q_2 & \cdots \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \end{bmatrix} \begin{bmatrix} - q_1^T - \\ - q_2^T - \\ \vdots \end{bmatrix} = \lambda_1 q_1 q_1^T + \lambda_2 q_2 q_2^T + \dots$

4th way (the sum of columns × rows)

"Every symmetric matrix breaks up into these pieces"

$A(A^TA)^{-1}A^T \to \dfrac{aa^T}{a^Ta}$. q's are unit vectors. $q^Tq = 1$ so we don't need to divide by a^Ta

So $q_i q_i^T$'s are projection matrices

$\left(\begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix} [p_1, p_2, \dots, p_n] = \begin{bmatrix} p_1p_1 & p_1p_2 & \cdots \\ \vdots & \ddots & \\ & & 1 \end{bmatrix} \text{(symmetric!)} \right)$

$((q_i q_i^T)^T = q_i q_i^T)$

→ So, Every Symm matrix is a comb. of perpendicular projection matrix.

---

real → positive? negative?

$A = A^T$.

Signs of pivots same as signs of λ's

$\boxed{\#\text{ positive pivots} = \#\text{ positive } \lambda\text{'s}}$

Δ For every matrix:
Product of pivots = product of eigenvalues = determinant

To prove this, we should use the "inertia theorem of Sylvester":
It means, for a symmetric matrix A, nonsingular matrix W
A and WAW^T have the same number of positive, negative, and zero eigenvalues
are said to be congruent
So: $A = Q\Lambda Q^T$ is as a congruence relation
We can then inspect the signs of the diagonal entries of D to determine the inertia of A (its pivots)

Positive definite symmetric matrix

- all eigenvalues are positive
- all pivots are positive
- all subdeterminants are positive.

---

anti-symmetric matrix (skew-symmetric matrix) → evectors orthogonal too.

→ evalues: pure imaginary

$\begin{bmatrix} -\lambda & -1 & 0 \\ 1 & -\lambda & -1 \\ 0 & 1 & -\lambda \end{bmatrix} = -\lambda^3 - 2\lambda = 0$

$\lambda^3 + 2\lambda = 0$

$\lambda(\lambda^2 + 2) = 0$.

$\lambda = \sqrt{2}i, -\sqrt{2}i$

Orthogonal evectors:

$AA^T = A^TA$

- symmetric matrices $= A^2$
- anti-symmetric matrices $= -A^2$
- orthogonal matrices $= I$?

Linear Algebra Lecture 26.

COMPLEX vectors matrices — inner product — Fourier matrix F_n

DISCRETE Fourier

FAST Fourier Transform = FFT

$[\ ]_{n\times n} \times [\ ]_{n\times n}$ $\cancel{n^2}$ multiplications $\to \frac{n}{2}\log_2 n$

---

length? $\bar{z}^T z$ is good, bcz it's length square $[1\ -i]\begin{bmatrix}1\\-i\end{bmatrix} = 1+1=2$. length is $\sqrt{2}$

$(\bar{z}_1 z_1 = |z_1|^2)$

$z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$ in complex n dimensional space C^n

$z^H z$ "z Hermitian z" $= |z_1|^2 + \cdots + |z_n|^2$

use symbol "H" to express transpose with taking conjugates "Hermite"

And for inner product in C^n: $\bar{y}^T x = y^H x$

"Symmetric" $A^T = A$ no good if A complex.

→ take $\bar{A}^T = A$. "complex vision of symmetry"

Hermitian matrices $(A^H = A)$ $\begin{bmatrix} 2 & 3+i \\ 3-i & 5 \end{bmatrix}$

Perpendicular unit "q"

$q_1, q_2, \ldots, q_n$ $\quad Q = \begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix}$

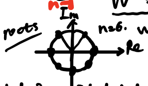
$\bar{q}_i^T q_j = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$ $\quad \bar{Q}^T Q = I$ $(Q^H Q)$ orthogonal matrix → unitary

---

$F_n = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & w & w^2 & \cdots & w^{n-1} \\ 1 & w^2 & w^4 & \cdots & w^{2(n-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & w^{n-1} & w^{2(n-1)} & \cdots & w^{(n-1)^2} \end{bmatrix}$ $(w \in C)$ $(F_n)_{ij} = w^{ij}$, $i,j = 0, \ldots, n-1$

$w^0\ w^1 \cdots w^{n-1}$ (0 … n-1)

$W^n = 1$ $\quad w = e^{i2\pi/n} = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$



$n=6.\ w = e^{i2\pi/6}$

$n=4.\ w = e^{i2\pi/4} = i$ $\quad i, i^2=-1, i^3=-i, i^4=1$

$F_4 = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}$ cols orthogonal, divide by 2. → cols orthonomal

$F_4^H F_4 = I$

$F_n^{-1} = F_n^H$

$(W_{64})^2 = W_{32}$.

P (permutation)

$[F_{64}] = \begin{bmatrix} I & D \\ I & -D \end{bmatrix}\begin{bmatrix} F_{32} & 0 \\ 0 & F_{32} \end{bmatrix}\begin{bmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{bmatrix}$ → it takes the even numbered components first and the ~~the~~ odd (in 0, …, n-1 indexs)

$64^2 \to 2(32)^2 + \text{fix}$ 32 fix up cost

$D = \begin{bmatrix} 1 & & & \\ & w & & \\ & & \ddots & \\ & & & w^{31} \end{bmatrix}$

$\to 2[2(16)^2 + 16] + 32$

"recursion" ⇒ $F_{64} \to F_{32} \to F_{16} \to \cdots \to 6 \times 32$ factoring the matrix properly

$[F_{64}]\ \begin{bmatrix} F_{32} & 0 \\ 0 & F_{32} \end{bmatrix}\ \begin{bmatrix} F_{16} & & & \\ & F_{16} & & \\ & & F_{16} & \\ & & & F_{16} \end{bmatrix}$ $\quad \log_2 64 \cdot n/2$ $\quad \boxed{\frac{1}{2} n \log_2 n}$

$\frac{n}{\times n} \to \frac{\frac{1}{2}\log_2 n}{\times n}$

Linear Algebra Lecture 27

Positive Definite Matrix (Tests)

Tests for Minumum of ($x^TAx > 0$) (except at $x=0$)

Ellipsoids in R^n. (geometry)

$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$

① $\lambda_1 > 0, \lambda_2 > 0$ [evalues]

② $a > 0$, $ac - b^2 > 0$ (leading submatrices) [det.]

③ pivots $a > 0$, $\frac{ac-b^2}{a} > 0$ [pivots]

④ $x^TAx > 0$

function of x_1, x_2 quadratic form

$$\underbrace{[x_1\ x_2]}_{x^T}\begin{bmatrix} 2 & 6 \\ 6 & 18 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [x_1\ x_2]\begin{bmatrix} 2x_1 + 6x_2 \\ 6x_1 + 18x_2 \end{bmatrix} = 2x_1^2 + 12x_1x_2 + 18x_2^2.$$

$ax^2 \quad 2bxy \quad cy^2$. > 0

$2(x_1+3x_2)^2 > 0$? $x_1 = 1, x_2 = -1$.

Examples

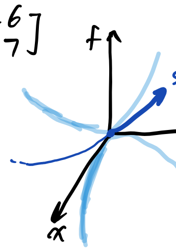
$\begin{bmatrix} 2 & 6 \\ 6 & 18 \end{bmatrix}$ pivots: 2. positive semidefinite $\lambda = 0, 20$.

borderline $\exists \vec{x} \neq 0$

Graphs of $f(x,y) = \vec{x}^TA\vec{x}$

$= ax^2 + 2bxy + cy^2$

$\begin{bmatrix} 2 & 6 \\ 6 & 7 \end{bmatrix}$



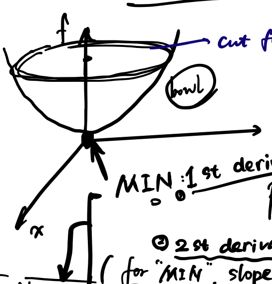
saddle points (at the maximum in some directions; at the minimum in some other directions)

$2x^2 + 12xy + 7y^2 = 2(x+3y)^2 - 11y^2$

$\begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix}$ $x^TAx = 2x_1^2 + 12x_1x_2 + 20x_2^2$ underlying pure squares.

> 0 except at $x = 0$ $\quad = 2(x_1 + 3x_2)^2 + 2x_2^2$ multiplier inside

pivots outside



cut following $f = C$ → get "ellipse" cut through a saddle point → get hyperbola "slice through"

MIN: ① 1st derivatives $= 0$ pure quadratic → sits at the origin.

② 2nd derivatives > 0 (for "MIN". slope have to be increasing. curvature have to go upwards)

"MIN"

Calculus

1st DERIV. $= 0$

MIN ~ $\frac{d^2u}{dx^2} > 0$

$\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$ $f_{xx} > 0$, $f_{yy} > 0$ — and should be big enough to overcome the cross-derivative for minimum

18.06 MIN ~ MATRIX OF 2nd DERNS IS POS DEF Hessian Matrix

$f(x_1, x_2, \ldots, x_n)$

$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \\ \vdots & \end{bmatrix}$ $\quad H_{ij} = \frac{\partial^2 f}{\partial x_i \cdot \partial x_j}$

$\begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix} \to \begin{bmatrix} 2 & 6 \\ 0 & 2 \end{bmatrix}$

$L = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$

3x3 example

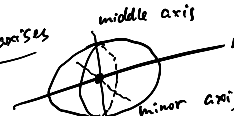
$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ dets 2. 3. 4 pivots $2, \frac{3}{2}, \frac{4}{3}$

eigenvalues $2-\sqrt{2}$, 2, $2+\sqrt{2}$

f $x^TAx = 2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3 > 0$

$= 1$ → football / ellipsoid

determine the length of axises



$Q\Lambda Q^T$ diagonalization for a symmetric matrix

principal axis theorem

Linear Algebra Lecture 28

A^TA is positive definite!

SIMILAR MATRICES A, B / JORDAN FORM

$B = M^{-1}AM$

If A, B are pos def. $x^TAx > 0$, $x^TBx > 0$

$x^T(A+B)x > 0$ ∴ So is $A+B$.

with a pos. def. matrix, you never have to do row exchanges. You never run into unsuitably small numbers or zeroes in the pivot position —— Great matrix to compute with!

Now suppose A m by n $\boxed{\text{rank } A = n}$ (full rank) → pos. def.

A^TA square, symmetric, pos def

$x^T(A^TA)x = (Ax)^TAx = \|Ax\|^2 > 0$

$n \times n$ matrices

A and B are similar means: for some some M

$B = M^{-1}AM$

Example: special A suppose full eigenvectors.

"matrices family" similar → have connection between each other → the outstanding member of the family is the diagonal guy (may not exists) simplest, neatest

$S^{-1}AS = \Lambda$ → A is similar to Λ

$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, $\Lambda = \begin{bmatrix} 3 & \\ & 1 \end{bmatrix}$ 3, 1

$\overset{M^{-1}}{\begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \overset{M}{\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}} = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 9 \\ 1 & 6 \end{bmatrix} = \overset{B}{\begin{bmatrix} -2 & -15 \\ 1 & 6 \end{bmatrix}}$ 3, 1

They have what things in common?

$\begin{cases} x + y = 4 \\ xy = 3 \end{cases}$ $x(4-x) = 3$, $x^2 - 4x + 3 = 0$ ✓

Similar matrices have the same eigenvalues!! And the same # eigenvectors

Why? $Ax = \lambda x$, $B = M^{-1}AM$. B's evalues = ?

(another direction: A, B share the same evalues, "M" exists.) ↘ $B = S_1\Lambda S_1^{-1}$, $A = S_2\Lambda S_2^{-1}$.

Based on S_1, S_2 are both invertible.

let $S_1 = M^{-1}S_2$, $M^{-1} = S_1S_2^{-1}$, $M = S_2S_1^{-1}$

$M^{-1}AM = S_1S_2^{-1}S_2\Lambda S_2^{-1}S_2S_1^{-1} = S_1\Lambda S_1^{-1} = B$.

$M^{-1}AMM^{-1}x = \lambda M^{-1}x$

⇒ $BM^{-1}x = \lambda M^{-1}x$ ⇒ Eigenvector of B is M^{-1} (evector of A)

So B times some vectors $= \lambda$ times them.

The eigenvectors didn't stay the same!

BAD CASE $\lambda_1 = \lambda_2 = 4$. has two families!

1. One small family has $\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$ $4I$ → only itself is similar to it.
$M^{-1}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}M = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$ (because we can't even reverse the pivots..)

2. Big family includes $\begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$ → somehow the best one in this family... won't be diagonalizable (if can, → similar to $\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$, but not)
(all other matrices that have evalues 4, 4)
↳ Jordan form climax
(in each family, he picked out the nicest — the most diagonal one, but not completely diagonal.) [cover all matrices!]
(There isn't a diagonal matrix in this family)
In some way, Jordan completed the diagonalization by coming as near as he could Which is his Jordan form

more members of family

$\begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$, $\begin{bmatrix} 5 & 1 \\ -1 & 3 \end{bmatrix}$, $\begin{bmatrix} 4 & 0 \\ 17 & 4 \end{bmatrix}$ $\begin{bmatrix} a & \bullet \\ \bullet & 8-a \end{bmatrix}$ $\begin{cases} \text{trace} = 8 \\ \text{det} = 16 \end{cases}$

$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ little blocks 2 independent evectors (dim $N(A) = 2$)

$\lambda = 0, 0, 0, 0$.

Jordan form —— we have a one above the diagonal for every missing eigenvector 2 missing.

$\begin{bmatrix} 0 & 1 & 7 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ → not be similar to each other

$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ Jordan blocks ["as a kind of view"] [as a kind of classification] and must have.

$J_i = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{bmatrix}$ ⇒ $\begin{bmatrix} 0 & 1 & & 0 \\ & 0 & \ddots & \\ & & \ddots & 1 \\ 0 & \cdots & & 0 \end{bmatrix}$ a Jordan block has one eigenvector only

(easiest form: $[c]$)

Every square A is similar to a Jordan matrix J

$J = \begin{bmatrix} \boxed{J_1} & & \\ & \boxed{J_2} & \\ & & \ddots & \\ & & & \boxed{J_d} \end{bmatrix}$

(Jordan matrix becomes a diagonal matrix)

blocks = # eigenvalues.

Start with any A. is the evalues are distinct, then A is similar to Λ Good: J is Λ