Linear Algebra Lecture 30

LINEAR TRANSFORMATIONS T

input $\xrightarrow{\text{map}}$ output

without coordinates : no matrix

find matrix behind T.

with coordinate $\Rightarrow$ MATRIX

$T(v+w) = T(v) + T(w)$

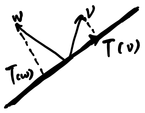
$T(cv) = c\,T(v)$ $\to T(0) = 0$

$T(cv+dw) = c\,T(v) + d\,T(w)$

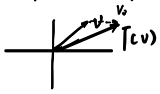
Example 1: Projection

a mapping

$T: \mathbb{R}^2 \to \mathbb{R}^2$



Example 2: Shift whole plane by v_0



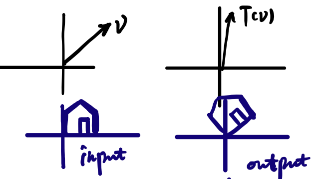
This transformation is not linear.

NON Example.

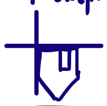
$T(v) = \|v\|$ $T: \mathbb{R}^3 \to \mathbb{R}^1$ Not linear.

Example 3 Rotation by 45°

$T: \mathbb{R}^2 \to \mathbb{R}^2$



$A: \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$



Matrix A.

$T(v) = Av$

linear. Transformation.

$A(v+w) = Av + Aw$ ✓ $A(cv) = cAv$ ✓

Start $T: \mathbb{R}^3 \to \mathbb{R}^2$ (Linear)

Example: $T(v) = Av$ ← 2 by 3 matrix.

output in $\mathbb{R}^2$ input in $\mathbb{R}^3$

Information needed to know $T(v)$ for all inputs "the space of inputs"

$T(v_1)$, $T(v_2)$ given, ..., $T(v_n)$ for any input basis $v_1, \dots, v_n$.

If I know what T does to every vector in a basis. we know everything of T.

Every $v = c_1v_1 + \dots + c_nv_n$.

Then I know what $T(v) = c_1T(v_1) + \dots + c_nT(v_n)$

Coordinates come from a basis chosen

of $v = c_1v_1 + \dots + c_nv_n$

amount of each basis

Construct matrix A that represents lin tr T:

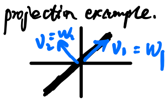
$T: \mathbb{R}^n \to \mathbb{R}^m$

Choose basis $v_1, \dots, v_n$ for inputs $\mathbb{R}^n$

—— $w_1, \dots, w_m$ for outputs $\mathbb{R}^m$

Want matrix A

projection example.



$v = c_1v_1 + c_2v_2$ (c_1, c_2)

$T(v) = c_1v_1$ $(c_1, 0)$

• same basis → input, output coords.

eigenvector basis

leads to diagonal matrix Λ

Proj. onto 45° line

use standard basis $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = w_1$, $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = w_2$

matrix $P = \dfrac{aa^T}{a^Ta} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$

Just standard basis, not best.

$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1 \\ 0 \end{bmatrix}$

A input coords $\Rightarrow$ output coords

Rule to find A ← **Given basis** $v_1 - v_n$, $w_1 - w_m$

1st column of A: Write $T(v_1) = a_{11}w_1 + a_{21}w_2 + \dots + a_{m1}w_m$

connection between $T(v)$ and w

2nd column of A: $T(v_2) = a_{12}w_1 + \dots + a_{m2}w_m$.

Everything Okay

$A\left(\begin{smallmatrix}\text{input}\\\text{coords}\end{smallmatrix}\right) = \left(\begin{smallmatrix}\text{output}\\\text{coords}\end{smallmatrix}\right)$

matrix A's coords

connection between C_{out} and C_{in}

$T = \dfrac{d}{dx}$ Input: $c_1 + c_2x + c_3x^2$ basis: $1, x, x^2$

linear!

Output deriv. : $c_2 + 2c_3x$ basis:

$A\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} c_2 \\ 2c_3 \end{bmatrix}$ $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

matrix multiplication really come from linear transformation!

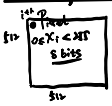
Linear Algebra Lecture 31

Change of Basis

Compression of Images

Transformation $\Leftrightarrow$ Matrix — the matrix is the coordinate-based description of the linear transformation

---



pixels $x \in \mathbb{R}^n$, $n = (512)^2$

JPEG
joint photographic experts group

Standard basis $\begin{bmatrix}1\\0\\ \vdots\\0\end{bmatrix} \begin{bmatrix}0\\1\\ \vdots\\0\end{bmatrix} \cdots \begin{bmatrix}0\\0\\ \vdots\\1\end{bmatrix}$ | Better basis | Fourier basis 8×8 $\begin{bmatrix}1\\w\\w^2\\ \vdots\\w^{63}\end{bmatrix}$

signal x P

$\downarrow$ change basis, loss less

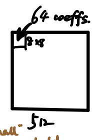
coefficients c

lossy $\downarrow$ compression, throw away "small", set some threshold — thresholding

$\hat{c}$ (may zeros)

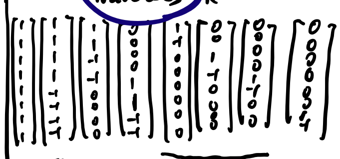
$\downarrow$

$\hat{x} = \sum \hat{c}_i v_i$



Video
sequence of images
correlated

Wavelets $\mathbb{R}^8$



Orthogonal.

$\downarrow$

$W^{-1} = W^T$.

Cheaper and close

standard basis

$$\begin{bmatrix}P_1\\P_2\\ \vdots\\P_8\end{bmatrix} = c_1\begin{bmatrix}\ \end{bmatrix} + c_2\begin{bmatrix}\ \end{bmatrix} + \cdots +$$

$= c_1 w_1 + \cdots + c_8 w_8$

$$P = \underbrace{\begin{bmatrix} \ \end{bmatrix}}_{W}\begin{bmatrix}c_1\\c_2\\ \vdots\\c_8\end{bmatrix}$$

$P = Wc$.

$c = W^{-1}P$

Good Basis:

① Fast FFT. FWT

② FEW IS ENOUGH

Change of basis

Columns of W = new basis vectors

$$\begin{bmatrix}x\end{bmatrix}_{\text{old basis}} \longrightarrow \begin{bmatrix}c\end{bmatrix}_{\text{in new basis}} \quad x = Wc$$

x as "a whole", so it stays the same.

What had changed are "basis" and "x's coords under the basis"

T with respect to $v_1, \dots, v_8$ it has matrix A

with respect to $w_1 \dots w_8$ it has matrix B

SIMILAR $\boxed{B = M^{-1}AM}$ — change of basis matrix

What is A? Using basis $v_1 \dots v_8$

Know T completely from $T(v_1)$. $T(v_2) \cdots T(v_8)$
"what T does to v_i"

Because every $x = c_1 v_1 + \cdots + c_8 v_8$

Then $T(x) = c_1 T(v_1) + \cdots + c_8 T(v_8)$

Write $T(v_1) = a_{11}v_1 + a_{21}v_2 + \cdots + a_{81}v_8$

$T(v_2) = a_{12}v_1 + \text{———} + a_{82}v_8$

$$[A] = \begin{bmatrix} a_{11} & a_{12} & \\ \vdots & \vdots & \\ a_{81} & a_{82} & \end{bmatrix} \quad \begin{bmatrix}v_1\\ \vdots\\ v_8\end{bmatrix}$$

Eigenvector basis — very fast but more calculation

$T(v_i) = \lambda_i v_i$

What is A?

$$\begin{bmatrix}\lambda_1 & & 0\\ & \lambda_2 & \\ 0 & & \ddots\end{bmatrix}$$

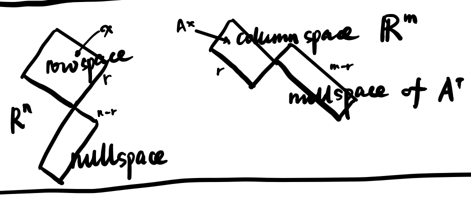
Linear Algebra Lecture 33 (32 is quiz review)

4 subspaces

Left - inverses

Right - inverses

Pseudo - inverses



2-sided inverse.

$$AA^{-1} = I = A^{-1}A$$

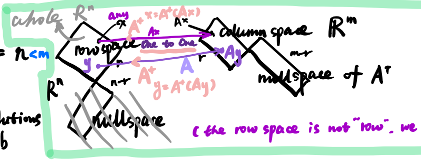
$r = m = n$ full rank

left inverse

full column rank $r = n < m$

nullspace $= \{0\}$

indep. cols. / 0 or 1 solutions to $Ax = b$



(the row space is not "row". we can just ignore where it comes from)

A^TA invertible. $\underline{(A^TA)^{-1}A^T}\, A = I$

$\underset{n\times m}{A^{-1}_{left}}\ \underset{m\times n}{A} = I_{n\times n}$

$AA^{-1}_{left} = A(A^TA)^{-1}A^T = P$

projection matrix (onto column space of matrix A)

⇓

"Trying to be identify matrix."

right inverse

full row rank $r = m < n$

nullspace of $A^T = \{0\}$

indep. rows. / $Ax=b$ always have solutions; $(n-m)$ free variables; ∞ solutions to $Ax=b$

AA^T invertible. $A\,\underline{A^T(AA^T)^{-1}} = I$

$AA^{-1}_{right} = I$

$A^{-1}_{right}A = A^T(AA^T)^{-1}A$

projection onto row space

It's those null spaces that are screwing up inverses, because if a matrix takes a vector to zero, there's no way an inverse can bring it back to life.

the matrix A got the nullspace hanging around, where it's knocking vectors to zero and then it got almost all vectors in between → almost all vectors have a row space component and a null space component, and A kills the nullspace component

But if we look at the vector that are in the row space, with no null space component, just in the row space.

then, they all go into the column space.

If $x \neq y$ both in row space, then $Ax \neq Ay$ (in col space). (Limit A in row space and column space. it's inverse mapping! —— pseudo-inverse)

Proof. Suppose $Ax = Ay$

$A(x-y) = 0$ → $x-y=0$ $x=y$

$x-y$ IN NULL SPACE but also in the ROW SPACE

Find the pseudo inverse A^+

① Start from SVD : $A = U\Sigma V^T$

"put all the problems into the diagonal matrix"

$\Sigma = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_r \\ & & & 0 \end{bmatrix}_{m\times n}$ rank r, m rows, n cols

$\Sigma^+ = \begin{bmatrix} 1/\sigma_1 & & 0 \\ & \ddots & \\ & & 1/\sigma_r \\ 0 & & & 0 \end{bmatrix}_{n\times m}$

$\Sigma\Sigma^+ = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ & & 1 \\ 0 & & & 0 \end{bmatrix}_{m\times m}$ → projection matrix onto column space

brings you into the two good spaces

$\Sigma^+\Sigma = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ & & 1 \\ 0 & & & 0 \end{bmatrix}_{n\times n}$

projection matrix onto row space

$$A^+ = V\Sigma^+U^T$$