$\mathcal{N}$-type : current player wins

$\mathcal{P}$-type : other player wins

notation recap.

T-shirt material

Meanwhile in the game tree :

1. If G is a game then $G+G$ is $\mathcal{P}$

by mimicking

2. $(*n) + (*m)$ is $\mathcal{P}$ iff $n = m$.

$\Leftarrow$ (follows from Fact (1))

$\Rightarrow$ (by contradiction)

S'pose

If $(*n)+(*m)$ is P

then $n \neq m$.

but

WLOG $n > m$

$(*n)+(*m) \rightsquigarrow (*m)+(*m)$ $\mathcal{P}$

3. If A is $\mathbb{P}$ then $o(G) = o(G+A)$ $\forall$ game G

$\mathbb{P} + \mathbb{N} = \mathbb{N}$

$\mathbb{P} + \mathbb{P} = \mathbb{P}$

ⓐ S'pose $o(G) = \mathbb{P}$

The other player pursues their winning strategy in both games.

ⓑ S'pose $o(G) = \mathbb{N}$

$$\underset{\mathbb{N}}{G} \rightsquigarrow \underset{\mathbb{P}}{H} \qquad (\underbrace{G+A}_{\mathbb{N}}) \rightsquigarrow (\underbrace{H+A}_{\mathbb{P}})$$

★ $\boxed{\textbf{Defn.} \quad G \equiv H \text{ if } \forall J, \quad o(G+J) = o(H+J)}$

this is an equivalence relation.

Note:

$n > m$

$(*n)$ and $(*m)$ are equivalent iff $n = m$.

$\underbrace{+(*m)}_{J}$ $\quad$ $\underbrace{+(*m)}_{J}$

N $\quad$ P

Claim. $G \equiv H$ iff $o(G+H) = \mathcal{P}$

ⓐ If $G \equiv H$ then $o(G+H) = P$

(the desired conclusion.)

$$\underbrace{o(G+G)}_{P} = \overbrace{o(H+G)}^{P}$$

by mimicking

by assumption

ⓑ if $o(G+H) = P$ then $G \equiv H$.

WTS. $o(G+J) = o(H+J) \ \forall J$

$$o(G+J) = o(G+J+\underbrace{H+H}_{P})$$

$$= o(\underbrace{G+H}_{P} + H+J)$$

$$= o(H+J)$$

Thm. Any game is equivalent a game of the form $(*n)$.

N iff $n>0$.

Pf. Label all the P-States $(*0)$.

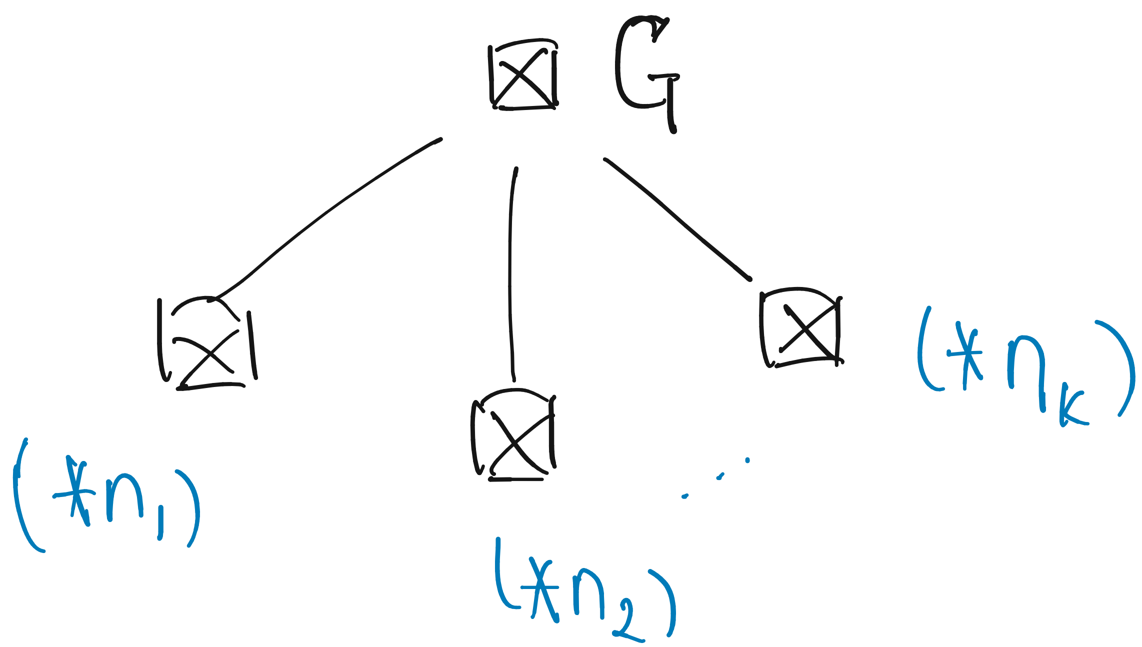
Note!

if all $(*n_i)$'s are

s.t. $n_i > 0$

then

$$f(s) = (*0)$$



$$f(s) = \underbrace{\text{mex}}_{\text{minimum excludant.}}(n_1, \dots, n_k) = m$$

Claim. $o(G + *m) = P$

$m' < m$

Ⓐ if the current player moves $(*m) \rightsquigarrow (*m')$

then the opponent moves $G \to H$

where $H \approx *m'$. } this move is available

because $m = \text{mex}(n_1, \ldots, n_k)$

The resultant game is $(*m') + (*m')$

& therefore $\forall\, m' < n$,

& therefore in P.

$\exists$ a child H of G s.t. $\underbrace{H \approx (*m')}_{\text{IH}.}$

(B) if the current player plays in G:

(B.1) $G \rightsquigarrow H$ where $H \approx (*m')$ where $m' < m$.

The other player moves $(*m) \rightsquigarrow (*m')$

to make the game $(*m') + (*m') \rightsquigarrow P$

(B.2) $G \rightsquigarrow H$ where $H \approx (*m')$ where $m' > m$.

The other player moves H to H' where

$H' \approx (*m)$ [H' exists by the mex mechanism.]