CS 5100 – PS5

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Problem 1: Quantifying Uncertainty

1. **Show from first principles that P (a | b ∧ a) = 1. In other words, use identities from basic Bayesian probability theory to show that these are equivalent.**

   According to conditional probability $P(A|B) = \frac{P(A \wedge B)}{P(B)}\ if\ P(B) > 0$, we have:

   $$P(a|b \wedge a) = \frac{P(a \wedge (b \wedge a))}{P(b \wedge a)}$$

   Since $a \wedge a = a$, the equation will be $P(a|b \wedge a) = \frac{P(a \wedge (b \wedge a))}{P(b \wedge a)} = \frac{P(b \wedge a)}{P(b \wedge a)} = 1$

   Therefore, $P(a|b \wedge a) = 1$

2. **For each of the following statements, either prove it is true or give a counterexample.**
   a. **If P (a | b, c) = P (b | a, c), then P (a | c) = P (b | c)**

      Answer: True

      According to conditional probability $P(A|B) = \frac{P(A,B)}{P(B)}\ if\ P(B) > 0$ and Product rule $P(A|B) = P(A \wedge B) = P(A \mid B)P(B)$, we have:

      $$P(a|b,c) = \frac{P(a,b,c)}{P(b,c)} \text{ and } P(b|a,c) = \frac{P(b,a\ ,c)}{P(a,c)}$$

      Since P (a | b, c) = P (b | a, c), we could know that P (b, c) = P (a, c). Since P (b, c) = P(b | c)P(c) and P (a, c) = P(a | c)P(c), we can say they P(b | c) = P(a | c).

   b. **If P (a | b, c) = P (a), then P (b | c) = P (b)**

      Answer: False

      P(a | b, c) = P(a) only indicates that a is independent of b and c, it does not say anything regarding the dependence of b and c. For example, a and b record the results of two independent coin flips, and c = b.

   c. **If P (a | b) = P (a), then P (a | b, c) = P (a | c)**

      Answer: False

      P(a | b) = P(a) implies that a is independent of b, but it does not say that a is independent of b given c. For example, a and b record the results of two independent coin flips, and c equals the xor of a and b.

3. **Given the full joint distribution shown in Figure 1, calculate the following:**
   a. P (toothache) = 0.108 + 0.012 + 0.016 + 0.064 = 0.2

b. P (toothache | cavity) = $\frac{P(toothache) \wedge \mathrm{P(cavity)}}{P(cavity)} = \frac{0.108 + 0.012}{0.108 + 0.012 + 0.072 + 0.008} = \frac{0.12}{0.2} = 0.6$

c. P (toothache | cavity $\vee$ catch) = $\frac{P(toothache) \wedge \mathrm{P(cavity \vee catch)}}{P(cavity \vee \mathrm{catch})} =$
$\frac{0.108 + 0.012 + 0.016}{0.108 + 0.012 + 0.016 + 0.072 + 0.008 + 0.144} = \frac{0.136}{0.36} = 0.3778$

Problem 2: The St. Petersburg Paradox

a) Show that the expected monetary value of this game is infinite

In this game, if the coin lands heads on the first flip I will win \$2, if it lands heads on the second flip, I will win \$4, and so on. The probability of the outcomes are $\frac{1}{2}, \frac{1}{2} \times \frac{1}{2}, \ldots$ Therefore, the expected monetary value would be: $\frac{1}{2} \times 2 + (\frac{1}{2})^2 \times 2^2 + + (\frac{1}{2})^3 \times 2^3 + \ldots$ $= \sum_{n=1}^{\infty} (\frac{1}{2})^n \times 2^n = \sum_{n=1}^{\infty} 1 = \infty$ which is infinite.

b) How much would you, personally, pay to play the game? Give an informal explanation.

I would like to spend at most \$100, since more pay more gain, and each game the probability of heads is 0.5. However, I don't want to spend too much time and money on this game.

c) What is the expected utility of the game under this assumption? Assume that you start off with \$0 and that playing the game is free.

The probability of outcomes are same, and if we use the logarithmic scale $U(S_n) = alog_2 n + b$ where a and b are some constants since we start off with \$0, the expected monetary would be: $\sum_{n=1}^{\infty} (\frac{1}{2})^n \times (alog_2(0 + 2^n) + b) = \sum_{n=1}^{\infty} (\frac{1}{2})^n \times (an + b)$

d) What is the expected utility if your initial wealth is \$k and you had to pay \$c to play the game?

The expected monetary would be: $\sum_{n=1}^{\infty} (\frac{1}{2})^n \times (alog_2(k - c + 2^n) + b)$

e) Assuming the same conditions as in part (d), what is the maximum amount that it would be rational to pay to play the game?

It would only be rational to play the game if we can increase our expected utility; otherwise. We should not play. If we do not play, the utility is $U(\mathrm{S_k}) = a\log_2 k + b$. So we should only play the game if:

$a\left[\sum_{n=1}^{\infty} \frac{\log_2(\text{k-c}+2^{\text{n}})}{2^{\text{n}}}\right] + \text{b} \geq \text{alog}_2\text{k} + \text{b}$

$$= \mathrm{U(S_k)} \leftrightarrow \sum_{n=1}^{\infty} \frac{\log_2(\text{k-c} + 2^{\text{n}})}{2^{\text{n}}} \geq \log_2 \text{k}$$

The maximum amount of c that one should pay to play the game if the maximum amount that still leads to the above inequality being true. If we assume that we must pay all our wealth to play the game, i.e., k = c, then the inequality comes:

$$\sum_{n=1}^{\infty} \frac{\log_2(\text{c-c} + 2^{\text{n}})}{2^{\text{n}}} \geq \log_2 \text{c}$$

$$\leftrightarrow \sum_{n=1}^{\infty} \frac{\text{n}}{2^{\text{n}}} \geq \log_2 \text{c}$$

$$\leftrightarrow \text{c} \leq 2^2 = 4$$