“Optimization”

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Homework #1

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1)

knowing that convexity is preserved under intersection, if S is a convex set $\xrightarrow{\text{"a line is convex"}}$ The intersection of S with a line is Convex (I)

Now, taking two points x_1 and x_2 from S, and supposing that intersection of S with any line is Covex, then, Considering Covex Combination of x_1 and x_2 as L $\Longrightarrow$

$L = \theta x_1 + (1-\theta)x_2$ for $0 \leq \theta \leq 1$ and $L \in S$

$\Longrightarrow$ for each $(x_1, x_2) \in S$ with $0 \leq \theta \leq 1$

$\theta x_1 + (1-\theta)x_2 \in S \Longrightarrow S$ is covex (II)

(I), (II) $\Longrightarrow$ a set C is convex $\Longleftrightarrow$ its intersection with any line is convex

2)

Considering distances equal to Euclidean norm (norm-2);

Therefore, we have to prove the following statement:

$S = \{x \mid \|x-a\|_2 \leq \|x-b\|_2\} \Longrightarrow$ S is a half space.

knowing that norms are non-negative, by taking the inequality above to the power of 2 we have:

$$\|x-a\|_2^2 \leq \|x-b\|_2^2 \Longrightarrow (x-a)^T(x-a) \leq (x-b)^T(x-b)$$

$$\Longrightarrow x^Tx - 2a^Tx + a^Ta \leq x^Tx - 2b^Tx + b^Tb$$

$$\Longrightarrow \underbrace{2(b-a)^T}_{(a')^T} x \leq \underbrace{b^Tb - a^Ta}_{b'} \Longrightarrow (a')^T x \leq b' \quad (\text{I})$$

$\Longrightarrow$ Therefore, the set 'S' is a halfspace according to the obtained inequality (I).

3)

a) According to the first problem statement that a set is convex if and only if its intersection with any line is convex. Therefore, by assuming an arbitrary line in the vector form of the following:

$L = \{\hat{x} + dy\}$ where $\hat{x} \in \mathbb{R}^n$, $d \in \mathbb{R}$, $y \in \mathbb{R}^n$

Then showing the intersection of line 'L' with the set 'C' by substituting the points of 'L' with points of 'C' we have:

$$(\hat{x}+dy)^T A(\hat{x}+dy) + b^T(\hat{x}+dy) + c = ed^2 + fd + g$$

where:

$\underline{e = y^T A y}$, $f = b^T y + 2\hat{x}^T A y$, $g = c + b^T\hat{x} + \hat{x}^T A\hat{x}$

Therefore, the intersection of 'C' and 'L' is as follows:

$$\{\hat{x} + dy \mid \overbrace{ed^2 + fd + g \leq 0}^{\mathrm{I}}\}$$

Generally, we need to prove that 'L' intersects 'C' as a <u>continuous line segment</u> where d is continuous. Since I is non-positive, 'e' parameter should be positive for the intersection to be continuous.

$\Rightarrow e \geq 0 \Rightarrow y^T A y \geq 0 \Rightarrow \boxed{A \succeq 0}$ ✓

b)

Let $H = \{x \mid g^T x + h = 0\}$. We define e, f, and g as in the solution of previous part and, in addition,

$$h = g^T v, \quad i = g^T \hat{x} + h$$

Now, we can assume that $\hat{x} \in H$, i.e., $i = 0$. The intersection of $C \cap H$ with the line defined by $\hat{x}$ and v is:

$$\{\hat{x} + dv \mid ed^2 + fd + g \leq 0, \; hd = 0\}$$

If $h = g^T v \neq 0$, the intersection is the singleton $\{\hat{x}\}$, if $g \leq 0$, or it is empty. In either case it is a convex set. If $h = g^T v = 0$, the set reduces to

$$\{\hat{x} + dv \mid ed^2 + fd + g \leq 0\}$$

which is convex if $e \geq 0$. Therefore, $C \cap H$ is convex if $g^T v = 0 \Rightarrow v^T A v \geq 0$

This is true if there exists λ such that $A + \lambda g g^T \succeq 0$; then,

$$v^T A v = v^T (A + \lambda g g^T) v \geq 0$$

$\boxed{\text{for all } v \text{ satisfying } g^T v = 0.}$ ✓

C) No, the convers is not true considering the following Counter example:

Considering the special condition where $x, A \in \mathbb{R}$ and $A=-2$ and $c=-2 \Rightarrow -2x^2-2 \leq 0$ for all $x \in R$

$\Rightarrow \boxed{'C' = \mathbb{R}}$ which is convex. Therefore, we showed that in special cases, A may be negative while 'C' is convex. ✓

Hence, the inverse of the statement is not ~~t~~ true. ✓

4)

The following proof is based on the definition of convex sets.

Assuming $v, y \in C'$ where $C' = \{x \in R_+^2 \mid x_1 x_2 \geq 1\}$

$\Rightarrow v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ where $v_1, v_2, y_1, y_2 \geq 0$, $v_1 v_2 \geq 1$, $y_1 y_2 \geq 1$

Considering $z = \theta v + (1-\theta) y \Rightarrow z_1 = \theta v_1 + (1-\theta) y_1$ $\quad 0 \leq \theta \leq 1$

$z_2 = \theta v_2 + (1-\theta) y_2$

Therefore, there are following possible conditions:

(I) $v \geq y \Rightarrow z = \theta v + (1-\theta) y = y + \underbrace{(v-y)\theta}_{\text{Positive}} \Rightarrow z \geq y$

we had: $y_1 y_2 \geq 1$, $y_1, y_2 \geq 0$ $\quad z_1, z_2 \geq y_1, y_2 \Rightarrow z_1 z_2 \geq 1$, $z_1, z_2 \geq 0$ $\Rightarrow z \in C'$

$\Rightarrow \boxed{C' \text{ is convex}}$

(II) $y \geq v \Rightarrow$ exactly the same as (I) $\Rightarrow \boxed{C' \text{ is convex.}}$

(III) $y \ngeq v$ & $v \ngeq y$ (ie. $(v_1 - y_1)(v_2 - y_2) \leq 0$)

In this case we have: $z_1 z_2 = (\theta v_1 + (1-\theta) y_1)(\theta v_2 + (1-\theta) y_2)$

$= \theta^2 v_1 v_2 + (1-\theta)^2 y_1 y_2 + \theta(1-\theta) v_1 y_2 + \theta(1-\theta) v_2 y_1$

$= \underbrace{\theta v_1 v_2 + (1-\theta) y_1 y_2}_{\geq 1} \underbrace{- \theta(1-\theta) \underbrace{(y_1 - v_1)(y_2 - v_1)}_{\ominus}}_{\oplus} \Rightarrow z_1 z_2 \geq 1$

$\Rightarrow \boxed{C' \text{ is convex}}$

(I)(II)(III): Considering all three possible conditions above, C' is convex.

5)

a)

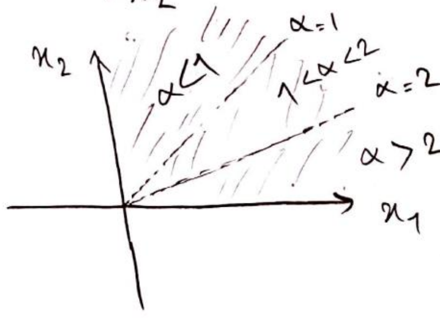
The Hessian Matrix of f is as the following:

$$\nabla^2 f(x) = \begin{bmatrix} 0 & -1/x_2^2 \\ -1/x_2^2 & 2x_1/x_2^3 \end{bmatrix}$$ $\Rightarrow$ Hessian Matrix is not positive or negative semidefinite.

$\Rightarrow$ $\boxed{\text{f is not convex or concave.}}$

To evaluate if f is quasi convex or quasiconcave following graph will be considered:

$x_1/x_2 = \alpha \quad x_1, x_2 > 0 \Rightarrow x_1 = \alpha x_2$



Therefore, both super level sets and sub level sets of the 'f' are convex sets $\Rightarrow$

$\boxed{\text{'f' is both quasiconvex and quasi concave.}}$

b)

Again, by computing Hessian Matrix of 'f' we have:

$$\nabla^2 f(x) = \frac{1}{x_1 x_2}\begin{bmatrix} 2/x_1^2 & 1/x_1x_2 \\ 1/x_1x_2 & 2/x_2^2 \end{bmatrix}$$

$\Rightarrow \nabla^2 f(x) \succeq 0$, since, all minors of the matrix are non-negative. Hence, it's positive semidefinite.

$\Rightarrow$ $\boxed{f(x) \text{ is convex}} \Rightarrow \boxed{f(x) \text{ is quasi convex}}$

$\Rightarrow$ $f(x)$ is neither concave nor quasi concave since its not a line.

C)

The mentioned function is a pointwise maximization operation which preserves convexity. Therefore, it is sufficient for convexity of $f(x)$ to prove convexity of each of the $\underbrace{|A^{(i)}x - b^{(i)}|}_{f_i(x)}$. Now, since each $f_i(x)$ is a composition of Norm function and an Affine function, $f_i(x)$ is convex; Hence, $\boxed{f(x) \text{ is convex}}$

Therefore, $\boxed{f(x) \text{ is quasi convex}}$

$\Rightarrow$ $\boxed{f(x) \text{ is neither concave nor quasi concave}}$

6)

The determinant of the mentioned matrix is as follows:

$$\boxed{y f(z) - z f(y) - x f(z) + z f(x) + x f(y) - y f(x) \geq 0}$$ ✓

$$\Leftrightarrow y f(z) + z f(x) + x f(y) \geq z f(y) + y f(x) + x f(z)$$

$$\Leftrightarrow (y-x) f(z) + (z-y) f(x) \geq (z-x) f(y) \quad \text{(I)}$$

We have: dom $f(x)$ is convex, $x < y < z$

$$\Leftrightarrow \exists \theta : \; y = \theta x + (1-\theta) z \quad \text{(II)}$$

$$\text{(I)} \xrightarrow{\div (z-x)} \frac{y-x}{z-x} f(z) + \frac{z-y}{z-x} f(x) \geq f(y)$$

$$\xrightarrow{\text{(II)}} \left(\frac{y-x}{z-x}\right) f(z) + \left(\frac{z-y}{z-x}\right) f(x) \geq f(\theta x + (1-\theta) z)$$

now: $\frac{z-y}{z-x} \overset{\text{(II)}}{=} \frac{z - (\theta x + (1-\theta) z)}{z-x} = \theta$ and $\frac{y-x}{z-x} \overset{\text{(II)}}{=} 1-\theta$

$$\Leftrightarrow (1-\theta) f(z) + \theta f(x) \geq f(\theta x + (1-\theta) z)$$ ✓

$$\Leftrightarrow \boxed{\text{function 'f' is convex}}$$ ✓

Also, considering that all of the mentioned steps are true in both ways, the exactly inverse procedure is also true. ✓

7)

Considering the definition of convex functions, we have to prove the following:

$$\text{I}: \frac{(f(\theta x+(1-\theta)y))^2}{g(\theta x+(1-\theta)y)} \leq \alpha \frac{(f(x))^2}{g(x)} + (1-\alpha)\frac{(f(y))^2}{g(y)} \Bigg\} \text{III}$$

for all $x, y \in \mathbb{R}^n$ and $0 \leq \theta \leq 1$.

Now, from the convexity and non-negativity of 'f' and concavity and positivity of 'g' we conclude that:

$$\text{II}: \frac{(f(\theta x+(1-\theta)y))^2}{g(\theta x+(1-\theta)y)} \leq \frac{(\theta f(x)+(1-\theta)f(y))^2}{\theta g(x)+(1-\theta)g(y)} \Bigg\} \text{IV}$$

Now, we have to compare the right hand-side of the II with the right hand-side of the I:

(for f^2/g to be convex, IV should be smaller than III)

$$\Rightarrow \frac{(f(\theta x+(1-\theta)y))^2}{g(\theta x+(1-\theta)y)} \leq \theta\frac{(f(x))^2}{g(x)} + (1-\theta)\frac{(f(y))^2}{g(y)}$$

Now, by canceling out equal terms of the both sides of the inequality above, following inequality will be obtained:

$$0 \leq (f(x)g(y) - f(y)g(x))^2$$

which is always true; hence, $\frac{f^2(x)}{g(x)}$ is convex. ✓

8)

In order to evaluate quasiconvexity of the 'f' function, we have to determine the condition of its sub-level sets. Therefore, for $f(x) \leq L$

$$\Rightarrow \frac{a^T x + b}{c^T x + d} \leq L \xrightarrow{c^T x + d > 0} a^T x + b - L(c^T x + d) \leq 0$$

$$\Rightarrow \underbrace{(a^t - Lc^t)}_{a'^t} x + \underbrace{(b-d)}_{b'} \leq 0 \Rightarrow \underbrace{a'^t x + b' \leq 0}_{\text{halfspace} \Rightarrow \text{Convex}}$$

Hence, for $f(x)$, all of the sub-level sets are convex. Since domain of $f(x)$ is also convex ($\{x \mid c^T x + d > 0\}$),

$\Rightarrow$ $\boxed{\text{the } f(x) \text{ itself is always quasiconvex!}}$

Now, to evaluate convexity, the Hessian matrix of $f(x)$ will be computed:

$$\nabla^2 f(x) = -(c^T x + d)^{-2} (ac^T + ca^T) + (a^T x + b)(c^T x + d)^{-3} cc^T$$

Now, assuming there is x_0 that $c^T x_0 + d = 1$ and $ax_0 + b$ equal to any desired value ($ax_0 + b = V$) we have:

$$\nabla^2 f(x) = -ac^T + ca^T + \underbrace{V cc^T}_{(\mathrm{I})}$$

Considering the term (I) in the equation above, if we set $(ax_0 + b)$ so that 'V' becomes large and negative, it is obvious that $\nabla^2 f(x)$ is not Positive Semi Definite.

Therefore, we have to seek for trivial Solution.

Assuming that $a = kc$ $(k \in \mathbb{R})$ we have:

$$f(x) = \frac{kc^T x + b}{c^T x + d} = k + \frac{b - kd}{c^T x + d}$$

Now, we know that $(c^T x + d)$ is convex since its affine function. considering: $c^T x + d = g(x)$

$$\Rightarrow f(x) = k + \frac{b - kd}{g(x)}$$

Now, as k (constant) is convex, $(b - kd)$ should be non-negative for $\frac{b-kd}{g(x)}$ to be convex, Subsequently for $f(x)$ to be convex. $\Longrightarrow$ Reminding that $k = \frac{a}{c}$,

$\boxed{f(x) \text{ is convex if } b \geq kd}$

hence, $\boxed{b \geq \frac{a}{c} d}$

Another trivial solution is that assuming $\boxed{c = 0}$
this way, $f(x)$ reduces to affine function of $f(x) = a'^T x + b'$ which is convex.

$$\Longrightarrow \boxed{f(x) \text{ is convex if } \begin{cases} 1)\ b \geq \frac{a}{c} d & \text{where } \vec{a} = k\vec{c},\ c \neq 0 \\ 2)\ c = 0 \end{cases}}$$

✓

Implementation Problem

9.

There are multiple mathematical optimization tools and software such as CVX, CVXPY, CVXOPT, CVXR, PICOS, DCCP, DMCP, NCVX and so on. The main differences between these tools are based on kind of ‘solvers’, ‘programming languages’, ‘APIs’, and ‘libraries’ they implement. Each one of these tools or applications are suitable for some specific optimization problems and may have better performance at some specific tasks.

The mentioned problem is solved by CVXOPT:

$$\begin{aligned} min \quad & f(x) = x_1 + 3x_2 \\ s.t. \quad & -x_1 + x_2 \leq 2 \\ & x_1 + x_2 \geq 2 \\ & x_2 \geq 0 \\ & 2x_1 - 3x_2 \leq 5 \end{aligned}$$

In [5]:

```
from cvxopt import matrix, solvers
```

In [10]:

```
A = matrix([ [-1.0, -1.0, 0.0, 2.0], [1.0, -1.0, -1.0, -3.0] ])
b = matrix([ 2.0, -2.0, 0.0, 5.0 ])
c = matrix([ 1.0, 3.0 ])

sol = solvers.lp(c,A,b)
print(sol['x'])
```

```
     pcost       dcost       gap    pres   dres   k/t
 0:  1.0000e+00  1.0000e+00  4e+00  6e-01  3e-16  1e+00
 1:  1.6732e+00  1.6745e+00  4e-01  5e-02  5e-16  9e-02
 2:  1.9479e+00  1.9535e+00  2e-01  3e-02  7e-16  5e-02
 3:  1.9994e+00  1.9995e+00  2e-03  3e-04  2e-16  6e-04
 4:  2.0000e+00  2.0000e+00  2e-05  3e-06  9e-17  6e-06
 5:  2.0000e+00  2.0000e+00  2e-07  3e-08  2e-16  6e-08
Optimal solution found.
[ 2.00e+00]
[-2.57e-08]
```