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HW 5

1. Problem 1

(a) For f(x) to be a p.d.f., it must satisfy:
$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^1 (3x - b)\,dx = 1$
$[\frac{3}{2}x^2 - bx]_0^1 = 1$
$\frac{3}{2} - b = 1$
$b = \frac{1}{2}$
Another condition is that f(x) ≥ 0 for $x \in [0, 1]$
$3x - \frac{1}{2} \geq 0$
$x \geq \frac{1}{6}$
So, for $x \in [0, \frac{1}{6})$, there is no value of b for which this is the p.d.f. of some random variable x.

(b) f(x) must satisfy
$\int_{\infty}^{\infty} f(x)\,dx = \int_{-b}^{b} (\frac{1}{2}cos(x))\,dx = 1$
$\frac{1}{2}(sin(b) - sin(-b)) = 1$
$sin(b) = 1$
$b = \frac{\pi}{2}$

2. Problem 2

PDF of X:

f(x) $= \begin{cases} \frac{1}{c} & 0 \leq x \leq c \\ 0 & otherwise \end{cases}$

CDF of X:

F(x) $= \int_0^x \frac{1}{c}dt = \frac{x}{c}$

$$F_x(x) = \begin{cases} 0 & x < 0 \\ \frac{x}{c} & 0 \leq x \leq c \\ 1 & x > c \end{cases}$$

Bounds of y:

$0 \leq xc$

$-c \leq x - c0$

$c \leq c - x0$

$c \leq Y0$

CDF of Y:

$$\begin{aligned} F_y(y) &= \mathbb{P}(Y \le Y) \\ &= \mathbb{P}(c - x \le Y) \\ &= \mathbb{P}(x \ge c - Y) \\ &= 1 - \mathbb{P}(x < c - Y) \\ &= 1 - F_x(c - Y) \\ &= 1 - \frac{c - Y}{c} \\ &= \frac{Y}{c}, 0 \le Y \le c \end{aligned}$$

$F_y(Y) = \begin{cases} 0 & Y < 0 \\ \frac{Y}{c} & 0 \le Y \le c \\ 1 & Y > c \end{cases}$

From above, x and Y have the same CDF.

$$\begin{aligned} f(y) &= \frac{d}{dy} F_y(y) \\ &= \frac{d}{dy}\frac{Y}{c} \\ &= \frac{1}{c}, 0 \le y \le c \end{aligned}$$

PDF of Y:

f(y) $= \begin{cases} \frac{1}{c} & 0 \le y \le c \\ 0 & otherwise \end{cases}$ And x and Y also has the same pdf.

3. Problem 3

(a) $\int_2^\infty cx^{-3}\, dx = 1$
$[-\frac{1}{2}cx^2]_2^\infty = 1$
$-\frac{1}{2}c[0 - \frac{1}{4}] = 1$ $\frac{1}{8}c = 1$ $c = 8$

(b) $F(x) = \int_2^x 8t^{-3} dt = -4x^{-2} + 1$
F(x) $= \begin{cases} 0 & x \le 2 \\ -4x^{-2} + 1 & x > 2 \end{cases}$

(c) $\mathbb{P}(x > 3|x < 5) = \frac{\mathbb{P}(3<x<5)}{\mathbb{P}(x<5)} = \frac{\mathbb{P}(X<5) - \mathbb{P}(x<3)}{\mathbb{P}(x<5)} = \frac{(1-\frac{4}{25})-(1-\frac{4}{9})}{(1-\frac{4}{25})} = 0.3386$

(d) $F_x(x) = 1 : median = 0.5$
$-4x^{-2} + 1 = 0.5$
$x^{-2} = \frac{1}{8}$
$x = 2\sqrt{2}$

(e) $\mathbb{E}[\sqrt{x}] = \int_2^\infty \sqrt(x) f(x)\,dx = \int_2^\infty \sqrt(x) 8x^{-3}\,dx = [-\frac{16}{3}x^{-\frac{3}{2}}]_2^\infty = \frac{16}{3}\frac{1}{2^{\frac{3}{2}}} = 1.886$

4. Problem 4

(a) f(x) $= \begin{cases} \frac{1}{\ell} & 0 < x < \ell \\ 0 & otherwise \end{cases}$

$$\begin{aligned} \mathbb{P}(Y \le b) &= 1 - \mathbb{P}(Y > b) \\ &= 1 - \mathbb{P}(x > b, \ell - x > b) \\ &= 1 - \mathbb{P}(x > b, x < \ell - b) \\ &= 1 - \mathbb{P}(b < x < \ell - b) \\ &= 1 - \int_b^{\ell - B} \frac{1}{\ell}\,dx \\ &= 1 - \frac{1}{\ell}(\ell - b - b) \\ &= \frac{2b}{\ell} \end{aligned}$$

The smaller segment can only go from 0 to $\frac{\ell}{2}$. So
$\mathbb{P}(Y \le b) = \frac{2b}{\ell}, 0 < b < \frac{\ell}{2}$

(b) PDF:

f(b) $= \begin{cases} \frac{2}{\ell} & 0 \le b \le \ell \\ 0 & otherwise \end{cases}$

This is equivqalent to $Unif[0, \frac{\ell}{2}]$

5. Problem 5

$f_x(x) = 2e^{-2x}$
$F_x(x) = -e^{-2x}$
$\mathbb{P}(x \in [0,1]) = F_x(1) - F_x(0) = -e^{-2} + 1$
$\mathbb{P}(x \in [a,2]) = F_x(2) - F_x(a) = -e^{-4} + e^{-2a}$
For a $>$ 1, events are disjoint (not independent). So assume $0 \le a \le 1$.
$\mathbb{P}(x \in [0,1], x \in [a,2]) = \mathbb{P}(x \in [a,1]) = F_x(1) - F_x(a) = -e^{-2} + e^{-2a}$
By independence:
$-e^{-2} + e^{-2a} = (-e^{-2} + 1)(-e^{-4} + e^{-2a})$
$a = 0.062$

6. Problem 6 (Challenge)

7. Problem 7

(a) $\mathbb{P}(X < 6) = \Phi(\frac{6-2}{2}) = \Phi(2) = 0.97725$

(b) $\mathbb{P}(X \leq 6) = \mathbb{P}(X < 6) = 0.97725$

(c)

$$\begin{aligned}
\mathbb{P}(X < 1 | X > -1) &= \frac{\mathbb{P}(X < 1, X > -1)}{\mathbb{P}(X > -1)} \\
&= \frac{\mathbb{P}(-1 < x < 1)}{1 - \mathbb{P}(X \leq -1)} \\
&= \frac{\mathbb{P}(\frac{-1-2}{2} < \frac{x-2}{2} < \frac{1-2}{2})}{1 - \Phi(-\frac{3}{2})} \\
&= \frac{\mathbb{P}(-\frac{3}{2} < Z < -\frac{1}{2})}{1 - 0.06681} \\
&= \frac{\mathbb{P}(Z < -\frac{1}{2}) - \mathbb{P}(Z < -\frac{3}{2})}{1 - 0.06681} \\
&= \frac{\mathbb{P}(Z \geq \frac{1}{2}) - \mathbb{P}(Z \geq \frac{3}{2})}{1 - 0.06681} \\
&= \frac{(1 - \mathbb{P}(Z < \frac{1}{2})) - (1 - \mathbb{P}(Z < \frac{3}{2}))}{1 - 0.06681} \\
&= \frac{(1 - 0.69146) - (1 - 0.93319)}{1 - 0.06681} \\
&= 0.25904
\end{aligned}$$

(d) $\mathbb{E} = Var(X) + (\mathbb{E}(X))^2 = 4 + 2^2 = 8$

(e) $\mathbb{P}(X > c) = 1 - \mathbb{P}(X \leq c) = \frac{1}{3}$

$\mathbb{P}(X \leq c) = 0.667$

$\Phi(\frac{c-2}{2}) = \Phi(0.43)$ from the table

$c = 2.86$

8. Problem 8 (Confidence Interval)

9. Problem 9 (Confidence Interval)

10. Problem 10

$\mu = np = 10000 \times \frac{1}{36} = 277.78$

$\sigma = \sqrt{np(1-p)} = \sqrt{10000(\frac{1}{36})(\frac{35}{36})} = 16.434$

$$\begin{aligned}
\mathbb{P}(280 < X < 300) &= \mathbb{P}(\frac{280 - \mu}{\sigma} < \frac{X - \mu}{\sigma} < \frac{300 - \mu}{\sigma}) \\
&= \mathbb{P}(0.135 < Z < 1.352) \\
&= \mathbb{P}(Z < 1.35) - \mathbb{P}(Z < 0.14) \\
&= 0.91149 - 0.55567 \\
&= 0.35582
\end{aligned}$$