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Section: MATH 302 102

HW 3

1. (a) To get probabilities of selecting two cakes of the same type for each bakery, we get the number of different types of cakes present, multiply it to the number of ways to get two of a certain type of cake, and divide that by the total number of ways to choose two cakes regardless of type.

   Bakery 1 -> 2 types -> P(selecting two cakes of same type Bakery 1) $= \frac{2\times\binom{6}{2}}{\binom{12}{2}} = \frac{5}{11}$

   Bakery 2 -> 3 types -> P(selecting two cakes of same type Bakery 2) $= \frac{3\times\binom{4}{2}}{\binom{12}{2}} = \frac{3}{11}$

   Bakery 3 -> 4 types -> P(selecting two cakes of same type Bakery 2) $= \frac{4\times\binom{3}{2}}{\binom{12}{2}} = \frac{2}{11}$

   Randomly walk into any of the three bakeries mean equal probability of choosing a bakery. So, we have $\frac{1}{3}$ chance of entering a certain bakery.

   Combining everything, we get $\frac{1}{3} \times \frac{5}{11} + \frac{1}{3} \times \frac{3}{11} + \frac{1}{3} \times \frac{2}{11} = \frac{10}{33}$

   (b) Simplifying the question, we get: P(went to bakery 2 | bought two different kinds of cakes), which is in the P(A | B) format. We get $\frac{P(A \bigcup B)}{P(B)}$.

   $P(A \bigcup B) =$ P(bakery 2) $\times$ P(bought two different kinds of cakes) $=$ P(bakery 2) $\times$ (1 - P(bought two cakes of similar type))

   $P(A \bigcup B) = \frac{1}{3} \times (1 - \frac{3}{11})$

   For condition B, we can turn it into 1 - P(bought two cakes of similar type) $= 1 - \frac{10}{33}$

   Combiniing both together, we get $\frac{\frac{1}{3}\times(1-\frac{3}{11})}{1-\frac{10}{33}} = \frac{8}{23}$

2. The following equation gives us the total probability that the assembly line will identify products as faulty

P(faulty in both tests) $=$ P(actually working) P(faulty in both tests | actually working) $+$ P(actually faulty) P(faulty in both tests | actually faulty)

   (a) P(faulty in both tests) $=$ (1 - 0.01)(1 - 0.95)(1 - 0.95) $+$ (0.01)(0.98)(0.98) $=$ 0.012

   As the tests are independent, we multiply the two conditional probabilities above twice to represent the two tests.

   So, P(actually faulty | faulty in both tests) $= \frac{P(actuallyfaulty)P(faulty\ in\ both\ tests\ |\ actually\ faulty)}{P(faultyinbothtests)} = \frac{0.01\times0.98\times0.98}{0.012} = 0.795$

   (b) This time, this is only for one test.

   P(faulty in one test) $=$ (1 - 0.01)(1 - 0.95) $+$ (0.01)(0.98) $=$ 0.0593

   So, P(actually faulty | faulty in one test) $= \frac{P(actuallyfaulty)P(faulty\ in\ one\ test\ |\ actually\ faulty)}{P(faultyinonetest)} = \frac{0.01\times0.98}{0.0593} = 0.1653$

3. To assess independence, multiply two probabilities and see if the products match what we want as combined result.

(a) P(even) $= \frac{50}{100}$, there are 50 even numbers between 1 and 100 inclusive.
P(divisible by 5) $= \frac{20}{100}$, there are 20 numbers between 1 and 100 inclusive that are divisible by 5 (e.g. 5, 10, 15, ... , 95, 100)
P(even and divisible by 5) = P(divisible by 10) $= \frac{10}{100}$, there are 10 numbers between 1 and 100 inclusive that are divisible by 10 (e.g. 10, 20, ... , 90, 100)
P(even) $\times$ P(divisible by 5) $= \frac{50}{100} \times \frac{20}{100} = \frac{10}{100}$ =P(even and divisible by 5)
So events E and F are independent.

(b) P(prime) $= \frac{25}{100}$
P(one of the digits is 2) $= \frac{19}{100}$
P(prime and one of the digits is 2) $= \frac{3}{100}$, the numbers are 2, 23, and 29
Multiplying events A and B, we get $\frac{25}{100} \times \frac{19}{100} \neq \frac{3}{100}$
So events A and B are dependent.

(c) Let n = 7.
P(even) $= \frac{3}{7}$, where the even numbers are 2, 4, and 6.
P(divisible by 5) $= \frac{1}{7}$, where the number divisible by 5 is 5 itself.
P(even and divisible by 5) $= \frac{0}{7}$ since the smallest positive number to satisfy this condition is 10.
Multiplying the probabilities of the two events, we have $\frac{3}{7} \times \frac{1}{7} \neq \frac{0}{7}$
So after changing n from 100 to 7, the two events became dependent.

4. Proof of the memoryless property:

$$P(X = n+m \mid X > n)$$

$$\downarrow$$

$$P(X = m+1 \mid X > 1) = P(X = m)$$

$$\rightarrow \frac{P(X = m+1 \cap X > 1)}{P(X > 1)}$$

$$= \frac{P(X = m+1)}{P(X>1)}$$

$$= \frac{P(X = m+1)}{1 - P(X=1)}$$

Let $p = P(X=1)$

$$P(X=m) = \frac{P(X=m+1)}{1-p}$$

$$P(X = m+1) = (1-p)\, P(X=m)$$

Given: $P(X=1) = p(1-p)^{1-1} = p$

$m=1$: $P(X = 1+1) = (1-p)\,P(X=1) = (1-p)\,p$

$m=2$: $P(X = 2+1) = (1-p)\,P(X=2) = (1-p)^2\,p$

$m=3$: $P(X = 3+1) = (1-p)\,P(X=3) = (1-p)^3\,p$

$m=k$: $P(X=m) = (1-p)^{m-1}\,p$ for $m > 1$

■ $X \sim \text{Geom}(p)$

Proof of the geometric random variable given the memoryless property

$$P(X=n+m \mid X>n) = P(X=m)$$

$$= \frac{P(X=n+m \cap X>n)}{P(X>n)}$$

$$= \frac{P(X=n+m)}{P(X>n)}$$

$$= \frac{(1-p)^{(n+m-1)}\, p}{P(X>n)}$$

$$P(X>n) = \sum_{m=n+1}^{\infty} P(X=m) = \sum_{m=n+1}^{\infty} (1-p)^{m-1}\, p$$

$$= p(1-p)^n \sum_{m=0}^{\infty} (1-p)^m$$

$$= p(1-p)^n \frac{1}{1-(1-p)} = p(1-p)^n \frac{1}{p}$$

$$= (1-p)^n$$

$$= \frac{(1-p)^{(n+m-1)}\, p}{(1-p)^n}$$

$$= (1-p)^{(m-1)}\, p$$

$$= P(X=m)$$

■ $P(X=n+m \mid X>n) = P(X=m)$; $n, m \in \mathbb{N}$

5. (a) It is a binomial distribution since the probabilities are independent and the number of games is finite.

   So the solution is to get the number of ways to choose a suit, $\binom{4}{1}$, getting all the 13 cards in the suit, 1, all divided by the total ways of getting 13 cards, $\binom{52}{13}$.

   This gives p $= \frac{\binom{4}{1}\times 1}{\binom{52}{13}} = 6.3 \times 10^{-12}$

   n is 50

   (b) It is a geometric distribution since there are infinite games possible for this.

   p = P(first time to get at least an ace) = 1 - P(not getting an ace) $= 1 - \frac{\binom{48}{13}}{\binom{52}{13}} = 0.6962$.

   (c) This is a binomial distribution with n $= 50$ and p $= \frac{\binom{4}{3}\times\binom{48}{10}}{\binom{52}{13}} = 0.0412$

6.

7. (a) $P(X \leq 3) = \frac{1}{7} + \frac{1}{14} + \frac{3}{14} = \frac{3}{7}$

(b) $P(X < 3) = \frac{1}{7} + \frac{1}{14} = \frac{3}{14}$

(c) $P(X < 4.12 | X > 1.6) = \frac{1}{7} + \frac{1}{14} + \frac{3}{14} + \frac{2}{7} = \frac{4}{7}$