[theorem]Question Theorem **??**

Name: Sherwin Adrien Tiu **Student Number:** 35443258
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HW 6

1. Suppose that X has moment generating function

$$M_X(t) = \tfrac{1}{4}e^{-3t} + \tfrac{1}{2} + \tfrac{1}{4}e^{t}.$$

(a) Find the mean and variance of X by differentiating the m.g.f. above.

(b) Find the p.m.f. of X. Use your expression for the p.m.f. to check your answers from part (a).

(a)

$$M_X'(t) = -\frac{3}{4}e^{-3t} + \frac{1}{4}e^{t}$$

$$M_x''(t) = \frac{9}{4}e^{-3t} + \frac{1}{4}e^{t}$$

$$\mu = \mathbb{E}(x) = M_x'(0) = \frac{-3}{4} + \frac{1}{4} = -\frac{1}{2}$$

$$\sigma^2 = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = M_x''(0) - (M_x'(0))^2 = (\frac{9}{4} + \frac{1}{4}) - (-\frac{1}{2})^2 = \frac{9}{4}$$

(b)

$$M_x'(t) = \sum_k e^{tk}\mathbb{P}(X = k)$$

$$X = -3 \to \mathbb{P}(-3) = \frac{1}{4}$$

$$X = 0 \to \mathbb{P}(0) = \frac{1}{2}$$

$$X = 1 \to \mathbb{P}(1) = \frac{1}{4}$$

$$\mathbb{E}(X) = -3(\frac{1}{4}) + 0(\frac{1}{2}) + 1(\frac{1}{4}) = -\frac{1}{2}$$

$$\mathbb{E}(X^2) = -3^2(\frac{1}{4}) + 0^2(\frac{1}{2}) + 1^2(\frac{1}{4}) = \frac{10}{4}$$

$$\sigma^2 = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \frac{9}{4}$$

2. You have two dice, one with three sides labeled $0, 1, 2$ (perhaps this problem takes place in a two-dimensional world) and one with 4 sides, labeled $0, 1, 2, 3$. Let X_1 be the outcome of rolling the first die, and X_2 the outcome of rolling the second. Denote the joint p.m.f. of X_1 and X_2 by p_{X_1,X_2}. That is, $p_{X_1,X_2}(m,n) := \mathbb{P}[X_1 = m, X_2 = n]$ for all $(m,n) \in \{0,1,2\} \times \{0,1,2,3\}$. The rolls are independent.

   (a) What is the joint p.m.f. of (X_1, X_2)?

   $p(x,y) = (\frac{1}{3})(\frac{1}{4}) = \frac{1}{12}; x \in 0,1,2, y \in 0,1,2,3$

   (b) Let $Y_1 = X_1 \cdot X_2$ and $Y_2 = \max\{X_1, X_2\}$. Make a table for the joint p.m.f. of (Y_1, Y_2).

$y_1 y_2$	0	1	2	3	p_{y_i}
0	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{3}{6}$
1	0	$\frac{1}{12}$	0	0	$\frac{1}{12}$
2	0	0	$\frac{1}{6}$	0	$\frac{1}{6}$
3	0	0	0	$\frac{1}{12}$	$\frac{1}{12}$
4	0	0	$\frac{1}{12}$	0	$\frac{1}{12}$
5	0	0	0	0	0
6	0	0	0	$\frac{1}{12}$	$\frac{1}{12}$
p_{y_2}	$\frac{1}{12}$	$\frac{3}{12}$	$\frac{5}{12}$	$\frac{3}{12}$	

   (c) Are Y_1 and Y_2 independent?

   $\mathbb{P}(Y_1, Y_2) \stackrel{?}{=} \mathbb{P}(Y_1)\mathbb{P}(Y_2)$

   Take $Y_1 = 2$, $Y_2 = 1$

   $\mathbb{P}(Y_1, Y_2) = 0$

   $\mathbb{P}(Y_1)\mathbb{P}(Y_2) = \frac{2}{12} \times \frac{4}{12} \neq 0$

   Not Independent.

3. (**Not marked**) In this guided exercise you are asked to prove the version of the law of unconscious statistician from Section 6.1, and then to use it in order to give a short and easy proof of (part (b)) linearity of expectation and (part (c)) of the fact that expectation of a product of independent random variables is the product of the expectations:

   Let X_1 and X_2 be two discrete random variables. In parts (b) and (c) suppose that either $\mathbb{P}[X \geq 0, Y \geq 0] = 1$ or that $\mathbb{E}[|X|]$ and $\mathbb{E}[|Y|]$ are both finite (or both). Prove the following claims from lecture:

   (a) If $g : \mathbb{R}^2 \to \mathbb{R}$ is a function, then

   $$\mathbb{E}[\, g(X_1, X_2)] = \sum_{k_1, k_2} g(k_1, k_2) \cdot \mathbb{P}(X_1 = k_1, X_2 = k_2).$$

   Hint: Let $A_\ell := \{(k_1, k_2) : g(k_1, k_2) = \ell \text{ and } \mathbb{P}(X_1 = k_1, X_2 = k_2) > 0\}$. Observe that $\{g(X_1, X_2) = \ell\} = \{(X_1, X_2) \in A_\ell\}$. Remember that the left hand side is by definition

   $$\mathbb{E}\left[g(X_1, X_2)\right] = \sum_{\ell\colon |A_\ell| > 0} \ell \cdot \mathbb{P}(g(X_1, X_2) = \ell) = \sum_{\ell\colon |A_\ell| > 0} \ell \cdot \mathbb{P}((X_1, X_2) \in A_\ell).$$

   Finally, write $\ell \cdot \mathbb{P}((X_1, X_2) \in A_\ell) = \sum_{(k_1, k_2) \in A_\ell} g(k_1, k_2) \mathbb{P}[X_1 = k_1, X_2 = k_2]$.

   (b) $\mathbb{E}[X_1 + X_2] = \mathbb{E}\, X_1 + \mathbb{E}\, X_2$.
   Hint: Express $X_1 + X_2$ as $g(X_1, X_2)$ for an appropriate choice of g and apply part (a).[1]

   (c) Show that if X_1 and X_2 are independent then

   $$\mathbb{E}[X_1 X_2] = \mathbb{E}[X_1] \mathbb{E}[X_2].$$

   Hint: Express $\mathbb{E}[XY]$ as $\mathbb{E}[g(X, Y)]$ for an appropriate choice of $g : \mathbb{R}^2 \to \mathbb{R}$. Use part (a) to conclude.

[1] Recall that a sum with non-negative summands can be split into two sums and can also be rearranged in an arbitrary fashion. The same is true for an absolutely convergent sum. E.g.,

$$\sum_{k_1, k_2} (k_1 + k_2) p_{X_1, X_2}(k_1, k_2) = \sum_{k_1} \sum_{k_2} k_1 p_{X_1, X_2}(k_1, k_2) + \sum_{k_2} \sum_{k_1} k_2 p_{X_1, X_2}(k_1, k_2),$$

and $\sum_{k_1, k_2} a_{k_1} b_{k_2} = \sum_{k_1} a_{k_1} \sum_{k_2} b_{k_2}$.

4. Compute the moment generating functions of

(i) $X \sim \text{Geom}(p)$.

$$\begin{aligned}
M(t) &= \sum_{k=1}^{\text{inf}} e^{kt} p(1-p)^{k-1} \\
&= p(1-p)^{-1} \sum_{k=1}^{\text{inf}} ((1-p)e^t)^k \\
&= p(1-p)^{-1} \times \frac{(1-p)e^t}{1-qe^t} \\
&= \frac{p(1-p)e^t}{(1-p)(1-(1-p)e^t)} \\
&= \frac{pe^t}{1-(1-p)e^t} \\
\text{Bounds:} (1-p)e^t &< 1 \\
e^t &< \frac{1}{1-p} \\
t &< ln(\frac{1}{1-p}) \\
t &< -ln(1-p)
\end{aligned}$$

(ii) (**Not marked**) $Y \sim \text{Exp}(\lambda)$ and of $Z \sim \text{Poisson}(\mu)$.

5. (**Only parts (c) and (d) will be marked**)

(a) Show that if X and Y are independent random variables, and $f, g : \mathbb{R} \to \mathbb{R}$ then $f(X)$ and $g(Y)$ are also independent. Hint: By the definition of independence, it suffices to prove that for all $B, B' \subset \mathbb{R}$ we have that

$$\mathbb{P}[f(X) \in B, g(Y) \in B'] = \mathbb{P}[f(X) \in B]\mathbb{P}[g(Y) \in B'].$$

Note that $\{f(X) \in B\} = \{X \in \{x : f(x) \in B\}\}$ and similarly, $\{g(Y) \in B\} = \{Y \in \{y : g(y) \in B'\}\}$.

(b) Show that if X and Y are independent then the moment generating functions of X, Y and $X + Y$ satisfy the following relation for all t:

$$M_{X+Y}(t) = M_X(t)M_Y(t).$$

Hint: Express $M_{X+Y}(t)$ as $\mathbb{E}[f(X)f(Y)]$ for an appropriate $f : \mathbb{R} \to \mathbb{R}_+$ and use part (a) of this question together with part (c) of Question 3 with $g(x, y) = f(x)f(y)$.

(c) Let X and Y be two independent Poisson r.v.'s with parameters μ, λ. Compute the m.g.f. of $X+Y$ and compare it with that of $W \sim \text{Poisson}(\mu+\lambda)$. Conclude that $X+Y \sim \text{Poisson}(\mu+\lambda)$.

$$\begin{aligned}
\mathbb{P}(x) &= e^{-\mu}\frac{\mu}{k!} \\
M_x(t) &= \sum_{k\geq 0} e^{tk}e^{-\mu}\frac{\mu}{k!} \\
&= e^{-\mu}e^{e^t}\mu \\
&= e^{-\mu(e^t-1)} \\
M_y(t) &= \sum_{k\geq 0} e^{tk}e^{-\lambda}\frac{\lambda}{k!} \\
&= e^{-\lambda}e^{e^t}\lambda \\
&= e^{-\lambda(e^t-1)} \\
M_{x+y}(t) &= M_x(t)M_y(t) \\
&= e^{-\mu(e^t-1)}e^{-\lambda(e^t-1)} \\
&= e^{(\mu+\lambda)(e^t-1)} \\
&= M_w(t)
\end{aligned}$$

(d) Let $X \sim \mathcal{N}(a, b)$ and $Y \sim \mathcal{N}(c, d)$ be independent. Compute the m.g.f. of $X+Y$ and compare it with that of $W \sim \mathcal{N}(a+c, b+d)$. Conclude that $X + Y \sim \mathcal{N}(a+c, b+d)$.

Let Z $\sim N(0,1)$

$$\begin{aligned}
f_z(x) &= \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} \\
M_z(t) &= \int_{-\inf}^{\inf} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} \times e^{tx} dx \\
&= \int_{-\inf}^{\inf} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2} + \frac{2tx}{2} + \frac{t^2}{2} - \frac{t^2}{2}} dx \\
&= e^{\frac{t^2}{2}} \int_{-\inf}^{\inf} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-t)^2}{2}} dx \\
&= e^{\frac{(t)^2}{2}} \times 1 \rightarrow p.d.f.\ of\ N(t,1) \\
&= e^{\frac{(t)^2}{2}}
\end{aligned}$$

Let V $\sim N(\mu, \sigma^2)$

$$\begin{aligned}
M_v(t) &= \mathbb{E}[e^{t(\sigma T + \mu)}] \\
&= e^{t\mu} \mathbb{E}[e^{t(\sigma T)}] \\
&= e^{t\mu} M_z(t\sigma) \\
&= e^{t\mu} e^{\frac{(t\sigma)^2}{2}}
\end{aligned}$$

$$\begin{aligned}
M_{X+Y}(t) &= M_x(t) M_y(t) \\
&= e^{ta + \frac{(tb)^2}{2}} e^{tc + \frac{(td)^2}{2}} \\
&= e^{ta + tc + \frac{t^2(b^2+d^2)}{2}} \\
&= e^{t(a+c) + \frac{t^2(b^2+d^2)}{2}} \\
&= M_w(t)
\end{aligned}$$

Remarks: We will later be able to give alternative proofs for the conclusions of parts (c) and (d) using the convolution formula.

6. Suppose that $X \sim \text{Exp}(\mu)$. Find the probability density function of $Y = \ln(X)$.

$$
\begin{aligned}
\mathbb{P}(X \leq x) &= 1 - e^{-\lambda x} \\
\mathbb{P}(X \leq e^y) &= 1 - e^{-\lambda e^y} \\
F_y(Y) = \mathbb{P}(Y \leq y) = \mathbb{P}(ln(X) \leq y) &= \mathbb{P}(X \leq e^y) \\
&= 1 - e^{-\lambda e^y} \\
f_y(Y) = \lambda e^y e^{-\lambda e^y} &= \lambda e^{y - \lambda e^y}
\end{aligned}
$$

7. Let $X \sim \text{Bin}(n, p)$. Find the moment generating function $M_X(t)$ of X. Use your answer to calculate $\mathbb{E}[X]$ and $\text{Var}(X)$.

   Hint: Consider $Y = \sum_{i=1}^n \xi_i$ where $\xi_1, \ldots, \xi_n$ are i.i.d. Bernoulli(p) random variables. Then X and Y have the same distribution and so $M_X = M_Y$, $\mathbb{E}[X^n] = \mathbb{E}[Y^n]$ for all n and $\text{Var}(X) = \text{Var}(Y)$. You may use the following generalization of 5(b): $M_Y(t) = M_{\sum_{i=1}^n \xi_i}(t) = \prod_{j=1}^n M_{\xi_j}(t)$ (by independence of $\xi_1, \ldots, \xi_n$).

$$
\begin{aligned}
M(t) &= \sum_k e^{kt} \binom{n}{k} p^k (1-p)^{n-k} \\
&= \sum_k \binom{n}{k} (pe^t)^k (1-p)^{n-k} \\
&= (pe^t + (1-p))^n \\
M'(t) &= n(pe^t + (1-p))^{n-1}(pe^t) \\
M''(t) &= n(n-1)(pe^t + (1-p))^{n-2}(pe^t)^2 + n(pe^t + (1-p))^{n-1}(pe^t)
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}[X] &= M'(0) = np \\
Var(X) &= M''(0) - (M'(0))^2 \\
&= n(n-1)(p-p)^{n-2}p^2 + n(p-p)^{n-1}p - (np)^2 \\
&= np(1 + (n-1)p) - (np)^2 \\
&= np(1 + (n-1)p - np) \\
&= np(1 + np - p - np) \\
&= np(1-p)
\end{aligned}
$$

8. (**Not marked**) Let X be a continuous random variable. Denote its PDF by f. Suppose that (i) $f(x) = 0$ for all $x < 0$, (ii) $f(x) > 0$ for all $x \geq 0$, (iii) f is right-continuous at 0 and continuous elsewhere, and (iv) for all $s, t \geq 0$

$$\mathbb{P}[X > t + s \mid X > s] = \mathbb{P}[X > t] \quad \text{(memoryless property for a continuous r.v.).}$$

Show that $X \sim \operatorname{Exp}(\lambda)$ where $\lambda := f(0)$.

Hint: Use the memoryless property in order to derive a differential equation for $G = 1 - F_X$ and solve it.

Remark: We proved in lecture that if $X \sim \operatorname{Exp}(\lambda)$ then the above holds. This exercise shows that among random variables satisfying (i)-(iii), the memoryless property characterizes the Exponential family of distributions.

9. **Challenge (not marked):** Let X and Y be continuous random variables defined on the same probability space. Denote their joint cumulative distribution function by

$$F_{X,Y}(x, y) := \mathbb{P}[X \leq x, Y \leq y].$$

Let $S \subseteq \mathbb{R}^2$ be an open set such that the partial derivatives $\partial_1 F_{X,Y}, \partial_2 F_{X,Y}, \partial_1^2 F_{X,Y}, \partial_2^2 F_{X,Y}, \partial_1\partial_2 F_{X,Y}$ and $\partial_2\partial_1 F_{X,Y}$ exist and are continuous on S. Show that for all $(s, t) \in S$ and all $\varepsilon, \delta > 0$ such that $B := (s, s + \varepsilon] \times (t, t + \delta] \subseteq S$ we have that

$$\mathbb{P}[(X, Y) \in B] = \int_B \partial_1\partial_2 F_{X,Y}(x, y)\mathrm{d}x\mathrm{d}y.$$

Remark: It follows that if $\mathbb{P}[(X, Y) \in S] = 1$, then X, Y have a joint density function

$$f_{X,Y}(x, y) := \begin{cases} \partial_1\partial_2 F_{X,Y}(x, y) = \partial_2\partial_1 F_{X,Y}(x, y) & (x, y) \in S \\ 0 & (x, y) \notin S \end{cases}.$$

Hint: Use the relation from lecture: For all $s, t \in \mathbb{R}$ and all $\varepsilon, \delta \geq 0$ we have that

$$\mathbb{P}[X \in (s, s+\varepsilon], Y \in (t, t+\delta]] = F_{X,Y}(s+\varepsilon, t+\delta) + F_{X,Y}(s, t) - F_{X,Y}(s+\varepsilon, t) - F_{X,Y}(s, t+\delta).$$

10. **Challenge (not marked):** Let $X_1, \ldots, X_n$ be continuous random variables defined on the same probability space. Denote the probability density function of X_i by f_i. Assume that for all closed sets $B \subset \mathbb{R}^n$ we have that

$$\mathbb{P}[(X_1, \ldots, X_n) \in B] = \int_B f_1(x_1)f_2(x_2)\ldots f_n(x_n)\mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_n.$$

Show that $X_1, \ldots, X_n$ are independent.

Hint: By a fact stated in lecture, $X_1, \ldots, X_n$ are independent if and only if for all $t_1, \ldots, t_n \in \mathbb{R}$ we have that $\{X_1 \leq t_1\}, \{X_2 \leq t_2\} \ldots$ and $\{X_n \leq t_n\}$ are independent.

11. **Challenge (not marked):** Let $X_1, \ldots, X_n$ be continuous random variables defined on the same probability space.

(a) Show that for all $i \in [n] := \{1, 2, \ldots, n\}$ and $s \in \mathbb{R}$

$$\lim_{t\to\infty} \mathbb{P}[\{X_i \le s\} \cap \{X_j \le t \text{ for all } j \in [n] \setminus \{i\}\}] = \mathbb{P}[X_i \le s] \quad \text{and that}$$

$$\lim_{t\to-\infty} \mathbb{P}[\{X_i > s\} \cap \{X_j > t \text{ for all } j \in [n] \setminus \{i\}\}] = \mathbb{P}[X_i > s].$$

Hint: Use the hint from challenge question 12 in HW assignment 5. Namely, that for every nested sequence of events $B_1 \subset B_2 \subset \cdots$ we have that $\mathbb{P}(\cup_{m=1}^{\infty} B_m) = \lim_{m\to\infty} \mathbb{P}(B_m)$.

(b) Assume that there are some $G_1, \ldots, G_n : \mathbb{R} \to \mathbb{R}_+$ such that: (i) $\lim_{t\to-\infty} G_j(t) = 1$ for all $j \in [n]$, and (ii) for all $t_1, \ldots, t_n \in \mathbb{R}$ we have that

$$\mathbb{P}[X_1 > t_1, X_2 > t_2 \ldots, X_n > t_n] = G_1(t_1)G_2(t_2)\cdots G_n(t_n).$$

Show that $X_1, \ldots, X_n$ are independent and that $\mathbb{P}[X_i > t] = G_i(t)$ for all $1 \le i \le n$ and $t \in \mathbb{R}$.

Hint: By a fact stated in lecture, $X_1, \ldots, X_n$ are independent if and only if for all $t_1, \ldots, t_n \in \mathbb{R}$ we have that $\{X_1 \le t_1\}, \{X_2 \le t_2\} \ldots$ and $\{X_n \le t_n\}$ are independent. By another fact from lecture, for every n independent events $A_1, \ldots, A_n$ and every $i \in [n]$ we have that $A_1^c, \ldots, A_i^c, A_{i+1}, \ldots, A_n$ are independent. Apply this with $i = n$ for some appropriate events.

12. **Challenge (not marked):** Let X and Y be two independent random variables. Denote their CDFs by F_X and F_Y, respectively. Assume that F_X and F_Y are continuous functions. Show that

$$\mathbb{P}[X = Y] = 0.$$

13. Let X_1, X_2 be independent $\text{Exp}(\lambda)$ random variables. Let $W_1 := \min\{X_1, X_2\}$ and $W_2 := \max\{X_1, X_2\} - \min\{X_1, X_2\}$. Show that $W_1 \sim \text{Exp}(2\lambda), W_2 \sim \text{Exp}(\lambda)$ and that W_1 and W_2 are independent.

Hint: You may use the following without further justification, beyond what is stated below. By Question 11 (b), it suffices to show that

$$\forall s, t \ge 0, \quad \mathbb{P}[W_1 > s, W_2 > t] = \mathbb{P}[\text{Exp}(2\lambda) > s]\mathbb{P}[\text{Exp}(\lambda) > t] = e^{-2\lambda s}e^{-\lambda t}.$$

By Question 12 $\mathbb{P}[W_2 > 0] = \mathbb{P}[X_1 \ne X_2] = 1$. This can be used to verify the last display when $t = 0$ (using a theorem from lecture concerning the distribution of W_1), and that for $t > 0$ this implies that

$$\mathbb{P}[W_1 > s, W_2 > t] = \mathbb{P}[W_1 > s, W_2 > t, X_1 < X_2] + \mathbb{P}[W_1 > s, W_2 > t, X_2 < X_1]$$
$$= \mathbb{P}[X_1 > s, X_2 - X_1 > t, X_1 < X_2] + \mathbb{P}[X_2 > s, X_1 - X_2 > t, X_2 < X_1]$$
$$= \mathbb{P}[X_1 > s, X_2 > t + X_1] + \mathbb{P}[X_2 > s, X_1 > t + X_2]$$

By symmetry, the two probabilities are equal to one another. Consider

$$B := \{(x, y) \in \mathbb{R}^2 : x > s \quad \text{and} \quad x + t < y < \infty\},$$

and observe that $\{X_1 > s, X_2 > t + X_1\} = \{(X_1, X_2) \in B\}$. Hence, using the definition of a joint probability density function and Fubini's theorem

$$\mathbb{P}[X_1 > s, X_2 > t+X_1] = \mathbb{P}[(X_1, X_2) \in B] = \iint_B f_{X_1,X_2}(x, y)\mathrm{d}x\mathrm{d}y = \int_s^\infty \int_{t+x}^\infty f_{X_1,X_2}(x, y)\mathrm{d}y\mathrm{d}x,$$

where f_{X_1,X_2} is the joint probability density function of X_1, X_2. Finally, by independence and a fact from Section 6.3 $f_{X_1,X_2}(x, y) = f_{X_1}(x)f_{X_2}(y) = \lambda e^{-\lambda x} \cdot \lambda e^{-\lambda y}$.

$$\begin{aligned}
X_1, X_2 > 0 &\Longrightarrow W_1 > 0 \\
\mathbb{P}(W_1 > s) &= \mathbb{P}(min(X_1, X_2) > s) \\
&= \mathbb{P}(X_1 > s)\mathbb{P}(X_2 > s) \\
&= e^{-\lambda s}e^{-\lambda s} \\
&= e^{-2\lambda s} \\
F_{w_1}(s) &= \mathbb{P}(W_1 \leq s) = \mathbb{P}(W_1 > s) \\
&= 1 - e^{-2\lambda s} \\
f_{w_1}(s) &= 2\lambda e^{-2\lambda s} \\
&\sim Exp(2\lambda)
\end{aligned}$$

$$\begin{aligned}
f_{W_2} &= \int_0^\infty f_{X_1}(X_2 + W_2)f_{X_2}(X_2)dX_2 + \int_t^\infty f_{X_1}(X_2 - t)f_{X_2}(X_2)dX_2 \\
&= \int_0^\infty \lambda e^{-\lambda X_2}dX_2 + \int_t^\infty \lambda e^{-\lambda X_2}dX_2 \\
&= [-e^{-\lambda y}]_0^\infty [-e^{-\lambda y}]_t^\infty \\
&= e^{-\lambda t} \\
&\sim Exp(\lambda)
\end{aligned}$$

Showing Independence,

$$\begin{aligned}
\mathbb{P}[W_1 > s, W_2 > t] &= \mathbb{P}[\mathrm{Exp}(2\lambda) > s]\mathbb{P}[\mathrm{Exp}(\lambda) > t] \\
&= \mathbb{P}[W_1 > s]\mathbb{P}[W_2 > t] \\
&= e^{-2\lambda s}e^{-\lambda t}
\end{aligned}$$