[theorem]Question Theorem **??**

Name: Sherwin Adrien Tiu **Student Number:** 35443258
Section: MATH 302 102

HW 7

1. Problem 1

   $f_x(x) = 2e^{-2x}, x \geq 0$

   $f_y(y) = \frac{1}{2}, 1 \leq y \leq 3$

   $y - x \geq \frac{1}{2} \rightarrow x \leq y - \frac{1}{2}$

$$
\begin{aligned}
\mathbb{P}(Y - X \geq \frac{1}{2}) &= \int_1^3 \int_0^{y-\frac{1}{2}} f_x(x) f_y(y) dx dy \\
&= \int_1^3 \int_0^{y-\frac{1}{2}} \frac{1}{2} \times 2e^{-2x} dx dy \\
&= \int_1^3 -\frac{1}{2}[e^{-2x}]_0^{y-\frac{1}{2}} dy \\
&= \int_1^3 -\frac{1}{2}(e^{-2y+1} - 1) dy \\
&= -\frac{1}{2}[-\frac{1}{2}e^{-2y+1} - y]_1^3 \\
&= 0.9097
\end{aligned}
$$

2. Problem 2

(a)

$$\int_0^\infty \int_0^\infty C\frac{e^{-x}-e^{-x-2y}}{e^y-1}dxdy = 1$$
$$C\int_0^\infty \int_0^\infty e^{-x}(e^{-y}+e^{-2y})dxdy = 1$$
$$C[-e^{-x}]_0^\infty[-e^{-y}-\frac{1}{2}e^{-2y}]_0^\infty = 1$$
$$C(1)(\frac{3}{2}) = 1$$
$$C = \frac{2}{3}$$

(b) $f_x(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_{-\infty}^{\infty} \frac{2}{3}\frac{e^{-x}-e^{-x-2y}}{e^y-1}dy = \frac{2}{3}e^{-x}\int_{-\infty}^{\infty}(e^{-y}+e^{-2y})dy = \frac{2}{3}e^{-x}(\frac{3}{2}) = e^{-x}$

For X and Y to be independent, we would have $f_y(y) = \frac{f(x,y)}{f_x(x)} = \frac{\frac{2}{3}\frac{e^{-x}-e^{-x-2y}}{e^y-1}}{e^{-x}}$

$f_y(y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_{-\infty}^{\infty} \frac{2}{3}\frac{e^{-x}-e^{-x-2y}}{e^y-1}dx = \frac{2}{3}(e^{-y}+e^{-2y})\int_{-\infty}^{\infty} e^{-x}dx = \frac{2}{3}(e^{-y}+e^{-2y})$

The product is satisfied, so they are independent.

(c)

$$\begin{aligned}
\mathbb{P}(X<Y) &= \int_0^\infty \int_0^y \frac{2}{3}\frac{e^{-x}-e^{-x-2y}}{e^y-1}dxdy \\
&= \frac{2}{3}\int_0^\infty [-e^{-x}]_0^y(e^{-y}+e^{-2y})dy \\
&= \frac{2}{3}\int_0^\infty -(e^y-1)(e^{-y}+e^{-2y})dy \\
&= \frac{2}{3}\int_0^\infty (e^{-y}-e^{-3y})dy \\
&= \frac{2}{3}[-e^{-y}+\frac{1}{3}e^{-3y}]_0^\infty \\
&= (\frac{2}{3})(\frac{2}{3}) \\
&= \frac{4}{9}
\end{aligned}$$

3. Problem 3

   $X \sim Exp(\mu)$

   $Y \sim Exp(\lambda)$

   For the case $\mu \neq \lambda$:

$$
\begin{aligned}
f_{x+y}(z) &= \int_{-\infty}^{\infty} f_x(x) f_y(z-x) dx \\
&= \int_0^z f_x(x) f_y(z-x) dx \\
&= \int_0^z \mu e^{-\mu x} \lambda e^{-\lambda(z-x)} dx \\
&= \mu\lambda \int_0^z e^{-\mu x - \lambda(z-x)} dx \\
&= \frac{\mu\lambda}{-\mu+\lambda} [e^{-\mu x - \lambda(z-x)}]_0^z \\
&= \frac{\mu\lambda}{-\mu+\lambda} (e^{-\mu z} - e^{-\lambda z})
\end{aligned}
$$

   For the case $\mu = \lambda$:

$$
\begin{aligned}
&= \int_0^z \lambda e^{-\lambda x} \lambda e^{-\lambda(z-x)} dx \\
&= \lambda^2 \int_0^z e^{-\lambda x - \lambda(z-x)} dx \\
&= \lambda^2 \int_0^z e^{-\lambda z} dx \\
&= \lambda^2 z e^{-\lambda z}
\end{aligned}
$$

   When $z < 0$, p.d.f. of X+Y $= 0$

4. Problem 4

   $\mathbb{P}(z \leq Z \leq W \leq w), 0 \leq z \leq w \leq 1$
   $= \mathbb{P}(Z \leq X_1 \leq W, Z \leq X_2 \leq W, ..., Z \leq X_n \leq W)$
   $= \mathbb{P}(Z \leq X_1 \leq W)\mathbb{P}(Z \leq X_2 \leq W)...\mathbb{P}(Z \leq X_n \leq W)$
   $= (w - z)^n$

   $\mathbb{P}(z \leq Z \leq W \leq w)$
   $= \mathbb{P}(W \leq w) - \mathbb{P}(Z \leq z, W \leq w)$
   $= \mathbb{P}(W \leq w) - (w - z)^n$
   $= F_{z,w}(z, w)$

   $f_{z,w}(z, w) = \frac{\partial^2}{\partial z \partial w}\mathbb{P}(W \leq w) - (w - z)^n$
   $= n(n - 1)(w - z)^{n-2}$

   $$f_{z,w}(z, w) = \begin{cases} n(n-1)(w-z)^{n-2}, & 0 \leq z \leq w \leq 1 \\ 0, & otherwise \end{cases}$$

5. Problem 5

   6.34

   Area of quadrilateral: $\frac{3}{2}$

   (a) $f_{x,y}(x, y) = \begin{cases} \frac{2}{3}, & \text{for } (x, y) \in D \\ 0, & \text{for } (x, y) \notin D \end{cases}$

      Marginal density functions:
      $x \leq 0, x \geq 2 \rightarrow f_x(x) = 0$
      $0 < x \leq 1 \rightarrow f_x(x) = \int_0^1 \frac{2}{3} dy = \frac{2}{3}$
      $1 < x < 2 \rightarrow f_x(x) = \int_0^{2-x} \frac{2}{3} dy = \frac{4}{3} - \frac{2}{3}x$

      $y \leq 0, y \geq 1 \rightarrow f_y(y) = 0$
      $0 < y < 1 \rightarrow f_y(y) = \int_0^{2-y} \frac{2}{3} dx = \frac{4}{3} - \frac{2}{3}y$

   (b) $\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_x(x) dx = \int_0^1 \frac{2}{3} x dx + \int_1^2 (\frac{4}{3} - \frac{2}{3}x) x dx = \frac{7}{9}$
      $\mathbb{E}[Y] = \int_{-\infty}^{\infty} y f_y(y) dy = \int_0^1 (\frac{4}{3} - \frac{2}{3}y) y dy = \frac{4}{9}$

   (c) Not independent. The joint density is not a product of the marginal densities

6.36

(a)

$$
\begin{aligned}
f(x,y) &= ce^{-\frac{x^2}{2}-\frac{(x-y)^2}{2}} \\
1 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} ce^{-\frac{x^2}{2}-\frac{(x-y)^2}{2}}dydx \\
&= c\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{2}}dydx \\
&= c\int_{-\infty}^{\infty} \sqrt{2\pi}e^{-\frac{x^2}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}e^{-\frac{(x-y)^2}{2}}dydx \\
&= c\int_{0}^{\infty} \sqrt{2\pi}e^{-\frac{x^2}{2}} \text{ By p.d.f. of a N(x,1) variable} \\
&= C2\pi \\
C &= \frac{1}{2\pi}
\end{aligned}
$$

(b)

$$
\begin{aligned}
f_x(x) &= \int_{-\infty}^{\infty} \frac{1}{2\pi}e^{-\frac{x^2}{2}-\frac{(x-y)^2}{2}}dy \\
&= \frac{1}{2\pi}\sqrt{(2\pi)}e^{-\frac{x^2}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{(2\pi)}}e^{-\frac{(x-y)^2}{2}}dy \\
&= \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}
\end{aligned}
$$

$$
\begin{aligned}
f_y(y) &= \int_{-\infty}^{\infty} \frac{1}{2\pi}e^{-\frac{x^2}{2}-\frac{(x-y)^2}{2}}dx \\
&= \int_{-\infty}^{\infty} \frac{1}{2\pi}e^{\frac{-2x^2+2xy-y^2}{2}}dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}e^{\frac{-2[(x-\frac{y}{2})^2-\frac{y^2}{4}]}{2}}dx \\
&= \frac{1}{\sqrt{4\pi}}e^{-\frac{y^2}{4}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{pi}}e^{(x-\frac{y}{2})^2}dx \\
&= \frac{1}{\sqrt{4\pi}}e^{-\frac{y^2}{4}} \text{ By p.d.f. of a N}(\frac{y}{2}, 1)\text{ variable}
\end{aligned}
$$

(c) Not independent. The joint density is not a product of the marginal densities

7.18

$$f_X(X) = \begin{cases} 2e^{-2x}, & \text{for } x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$f_Y(Y) = \begin{cases} 4xe^{-2x}, & \text{for } x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{aligned} f_{X+Y}(z) &= \int_{-\infty}^{\infty} f_x(x) f_y(z-x) dx \\ &= \int_0^z 2e^{-2x}(4(z-x)e^{-2(z-x)})dx \\ &= 8\int_0^z e^{-2z}(z-x)dx \\ &= 8e^{-2z}\int_0^z (z-x)dx \\ &= 8e^{-2z}(\frac{z^2}{2}) \\ &= 4z^2e^{-2z} \end{aligned}$$

$$f_X + Y(z) = \begin{cases} 4z^2e^{-2z}, & \text{for } z \geq 0 \\ 0, & \text{otherwise} \end{cases}$$