2.6 3D Wave Solutions

Monday, October 30, 2023 11:11 PM

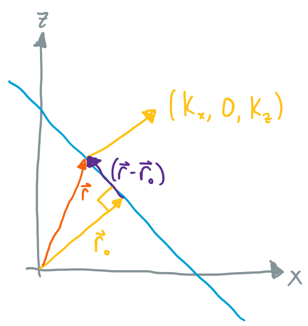
Wave Solutions in 3D — Plane Waves

How? 1D: $f(t \pm \frac{z}{v}) + f(\alpha(t \pm \frac{z}{v})) \rightarrow e^{\pm j\omega t \pm \beta z} = e^{j\omega(t \pm \frac{\beta}{\omega} z)} = e^{j\omega(t \pm \frac{1}{\frac{\omega}{\beta}} z)} = e^{j\omega(t + \frac{z}{v})}$

Guess? $f(t \pm (\frac{x}{v} + \frac{y}{v} + \frac{z}{v}))$ $(\beta z = (0, 0, \beta) \cdot (0, 0, z))$

3D: Replace 0s : $(m, n, \beta) \cdot (x, y, z) =$ constant

2D



$$(\vec{r} - \vec{r}_0) \cdot \overbrace{(k_x, 0, k_z)}^{\vec{k}} = 0$$

$$\vec{r} \cdot \vec{k} - \vec{r}_0 \cdot \vec{k} = 0$$

$$\vec{r} \cdot \vec{k} = \text{constant}$$

$$r_x k_x + r_z k_z = C$$

$$X k_z + Z k_z = C$$

Interpretation:

$$\beta = \frac{2\pi}{\lambda} \longrightarrow \vec{k} = (k_x, k_y, k_z)$$

$$|\vec{k}| = \sqrt{k_x^2 + k_y^2 + k_z^2} = \frac{2\pi}{\lambda} = \frac{2\pi f}{v} = \frac{\omega}{v}$$

$$v = |\vec{v}| \qquad |\vec{k}| = \frac{\omega}{v} = \omega\sqrt{\mu\varepsilon}$$

$$\vec{E} = \vec{E}_0 \, e^{j(\omega t - \vec{k} \cdot \vec{r})}$$

$$\vec{H} = \vec{H}_0 \, e^{j(\omega t - \vec{k} \cdot \vec{r})}$$

Q1 Consider a plane wave described by the following electric field:

$$\vec{E}(x, y, z, t) = \vec{E}_0 \, e^{j\omega t} e^{-j\vec{k} \cdot \vec{r}}$$

Which of the following vectors have the same direction as the wave's direction of travel?

$\vec{r}$ $\quad$ $\underline{\vec{k}}$ $\quad$ $\vec{E}$ $\quad$ $\vec{E}_0$

Q2 Continuing with the same expression, which of the following is perpendicular to the (wavefront) plane?

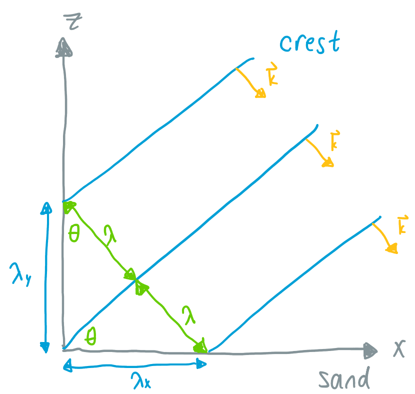
$\vec{r}$ $\quad$ $\underline{\vec{k}}$ $\quad$ $\vec{E}$ $\quad$ $\vec{E}_0$

Q3 Given the following plane description: $3x + 2y + z = 0$. What is a vector normal to this plane?

$\underline{(3,2,1)}$ $\quad$ $(1,1,1)$ $\quad$ $(0,1,2)$ $\quad$ $(1,2,3)$

Counting Wave Crests ~~at the Beach~~ in 2D

"(Done in a beach but with a lightboard)"



$$\lambda_x = \frac{\lambda}{\sin\theta} \qquad \lambda_z = \frac{\lambda}{\cos\theta}$$

$$K_x = \frac{2\pi}{\lambda_x} = \frac{2\pi\sin\theta}{\lambda}$$

$$K_z = \frac{2\pi}{\lambda_z} = \frac{2\pi\cos\theta}{\lambda}$$

$$|\vec{k}| = \sqrt{K_x^2 + K_z^2} = \sqrt{\left(\frac{2\pi}{\lambda}\sin\theta\right)^2 + \left(\frac{2\pi}{\lambda}\cos\theta\right)^2} = \frac{2\pi}{\lambda}$$

Application of Plane Waves to Maxwell's Equations

3D Maxwell-Faraday $\rightarrow$ $\vec{\nabla} \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$

$\vec{E} = \vec{E}_0 e^{j\phi}$ $\qquad$ $\phi = \omega t - K_x X - K_y Y - K_z z$

$\vec{H} = \vec{H}_0 e^{j\phi}$

CURL

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial x & \partial y & \partial z \\ E_x & E_y & E_z \end{vmatrix}$$

$E_x = E_{0x} e^{j\phi}$ $\qquad$ $E_y = E_{0y} e^{j\phi}$ $\qquad$ $E_z = E_{0z} e^{j\phi}$

z component of $\vec{\nabla} \times \vec{E}$: $\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}$

$E_{..} = E_{0..} e^{j\phi}$

$$E_x = E_{0x} e^{j\phi}$$
$$E_y = E_{0y} e^{j\phi}$$
$$\frac{\partial}{\partial x}(e^{j\phi}) = -jK_x e^{j\phi}$$
$$\frac{\partial}{\partial y}(e^{j\phi}) = -jK_y e^{j\phi}$$

$$\frac{\partial E_y}{\partial x} = E_{0y} \frac{\partial}{\partial x} e^{j\phi}$$
$$\frac{\partial E_x}{\partial y} = E_{0x} \frac{\partial}{\partial y} e^{j\phi}$$

$$\rightarrow E_{0y}(-jK_x)e^{j\phi} - E_{0x}(-jK_y)e^{j\phi}$$
$$\rightarrow e^{j\phi}[(-jK_x)E_{0y} - (-jK_y)E_{0x}]$$

$$\underbrace{\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial x & \partial y & \partial z \\ E_x & E_y & E_z \end{vmatrix}}_{\vec{\nabla} \times \vec{E}} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -jK_x & -jK_y & -jK_z \\ E_x & E_y & E_z \end{vmatrix} = -\mu \frac{\partial}{\partial t} \vec{H}_0 e^{j\phi} = -\mu j\omega \vec{H}_0 e^{j\phi}$$

from derivative of phase

$$\phi = \omega t - K_x x - K_y y - K_z z$$
$$= \omega t - \underbrace{\vec{K} \cdot \vec{r}}_{\text{constant}}$$

$$\downarrow$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -jK_x & -jK_y & -jK_z \\ E_{0x}e^{j\phi} & E_{0y}e^{j\phi} & E_{0z}e^{j\phi} \end{vmatrix}$$

$$-j\vec{K}_x \vec{E}_0 e^{j\phi} = -\mu j\omega \vec{H}_0 e^{j\phi}$$
$$\vec{K}_x \vec{E}_0 = \mu \omega \vec{H}_0$$

$\boxed{\text{Conclusion}}$

$$\vec{K} \times \vec{E}_0 = \mu\omega \vec{H}_0$$
$$|\vec{K}| = \frac{\omega}{c} = \omega\sqrt{\mu\varepsilon}$$
$$\vec{K} = \omega\sqrt{\mu\varepsilon}\,\hat{n}$$

plane of constant phase ($\vec{E}$ vibrating in this plane)

$\hat{n}$

$$\cancel{\omega}\sqrt{\mu\varepsilon}\,\hat{n} \times \vec{E}_0 = \mu \cancel{\omega} \vec{H}_0$$
$$\hat{n} \times \vec{E}_0 = \sqrt{\frac{\mu}{\varepsilon}}\, \vec{H}_0$$

$\rightarrow$ intrinsic impedance

$$\vec{E}_0 \perp \vec{H}_0$$

$$\vec{\nabla} \cdot \vec{E} = 0 \rightarrow \frac{\partial}{\partial x} E_x + \frac{\partial}{\partial y} E_y + \frac{\partial}{\partial z} E_z = 0$$
$$-jK_x E_{0x} e^{j\phi} - jK_y E_{0y} e^{j\phi} - jK_z E_{0z} e^{j\phi} = 0$$
$$\cancel{-je^{j\phi}}(K_x E_{0x} + K_y E_{0y} + K_z E_{0z}) = 0$$

to other side

$\cancel{j e} \; (k_x E_{0x} + k_y E_{0y} + k_z E_{0z}) = 0$

to other side

$$\vec{K} \cdot \vec{E_0} = 0 \longrightarrow \underline{\vec{K} \perp \vec{E_0}}$$

Q1 From $\hat{n} \times \vec{E_0} = \sqrt{\frac{\mu}{\varepsilon}} \vec{H_0}$, what does $\frac{\mu}{\varepsilon}$ represent?

Impedance squared per length squared $\left(\frac{\Omega^2}{m^2}\right)$

Impedance per length $\left(\frac{\Omega}{m}\right)$

Impedance (Ω)

<u>Impedance squared (Ω^2)</u>

Q2 The lesson went through the process of simplifying Faraday's law to acquire plane wave relationships. If the same process is applied to Ampere's law, what is the result?

1) Start with Ampere's law: $\vec{\nabla} \times \vec{H} = j\omega\varepsilon\vec{E}$

2) Substitute: $\vec{H} = \vec{H_0} e^{j\phi}$, $\vec{E} = \vec{E_0} e^{j\phi}$ where $\phi = \omega t - k_x x - k_y y - k_z z$ (in other words, $\phi = \omega t - \vec{k} \cdot \vec{r}$)

3) Take the curl similar to how it was done in the lesson and simplify.

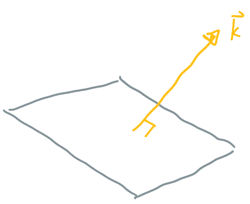
This will lead to: ($\hat{n}$ is a unit vector in the direction of $\vec{k}$)

$$\hat{n} \times \vec{H_0} = -\sqrt{\frac{\varepsilon}{\mu}} \vec{E_0}$$

<u>In-Class 11/01/23</u>

$$\vec{E} = \vec{E_0} e^{j(\omega t - \vec{k} \cdot \vec{r})} \longrightarrow f(\omega t - \vec{k}\vec{r}) = f\left(t - \frac{\vec{k}}{\omega}\vec{r}\right)$$

$$f\left(t - \frac{z}{v}\right)$$



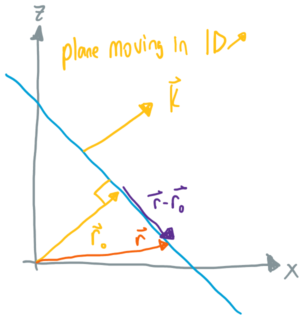
Let $\phi = \omega t - \vec{k}\vec{r}$ and choose $t = 0$

$$\phi = -\vec{k}\vec{r}$$

Constant $\phi \rightarrow -\vec{k} \cdot \vec{r} = \alpha$

a constant

$k_x + k_y + k_z = \text{Constant} \rightarrow$ a plane in 3D



$$\vec{K}\cdot(\vec{r}-\vec{r}_0)=0$$

$$\vec{K}\cdot\vec{r} = \vec{K}\cdot\vec{r}_0 = \text{constant}$$

$$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

·Preamble:

$$\frac{\partial}{\partial x} e^{-jk_x x} = -jk_x e^{-jk_x x}$$

$$\vec{\nabla}\left(e^{-j\vec{k}\cdot\vec{r}}\right) = \vec{\nabla}\left(e^{-jk_x x}\cdot e^{-jk_y y}\cdot e^{-jk_z z}\right)$$

$$\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \rightarrow (-jk_x, -jk_y, -jk_z)\, e^{-j\vec{k}\cdot\vec{r}}$$

$$\vec{E} = \vec{E}_0 e^{-j\vec{k}\cdot\vec{r}}$$

(Short-circuited Algebra version)

$$\vec{\nabla}\cdot\vec{E} = 0 \rightarrow -j(k_x, k_y, k_z)\cdot\vec{E}$$

$$= -j(k_x, k_y, k_z)\cdot\vec{E}_0 e^{-j\vec{k}\cdot\vec{r}}$$

$$= -j\vec{K}\cdot\vec{E} = 0$$

$\left.\begin{matrix}\vec{K}\perp\vec{E} \\ \vdots \\ \vec{K}\perp\vec{H}\end{matrix}\right\}$ $\vec{E}$ and $\vec{H}$ has to be in a plane

(Long version)

$$\vec{\nabla}\cdot\vec{E} = 0 \rightarrow \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\cdot(E_{ox}, E_{oy}, E_{oz})\, e^{-j\vec{k}\cdot\vec{r}}$$

$$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\cdot\left(E_{ox} e^{-j\vec{k}\cdot\vec{r}}, E_{oy} e^{-j\vec{k}\cdot\vec{r}}, E_{oz} e^{-j\vec{k}\cdot\vec{r}}\right)$$

$$= \frac{\partial}{\partial x} e^{-j\vec{k}\cdot\vec{r}} E_{ox} + \frac{\partial}{\partial y} e^{-j\vec{k}\cdot\vec{r}} E_{oy} + \frac{\partial}{\partial z} e^{-j\vec{k}\cdot\vec{r}} E_{oz}$$

$$= (-jk_x, -jk_y, -jk_z)\cdot(E_{ox}, E_{oy}, E_{oz})\, e^{-j\vec{k}\cdot\vec{r}}$$

$$= 0 \rightarrow \vec{K}_0\cdot\vec{E} = 0 \rightarrow \vec{K}\cdot\vec{E}$$

$$= \vec{K}\cdot\vec{E}_0 = 0$$