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HW 2

1. Let ω be a sample space and $\mathbb{P}$ be a probability measure. Prove that there cannot exist events E, F that satisfy

$\mathbb{P}(E/F) = \frac{2}{5}, \mathbb{P}(E \bigcup F) = \frac{1}{2}, and\ \mathbb{P}((E \cap F)^c) = \frac{3}{4}$

$\mathbb{P}((E \cap F)^c) = \frac{3}{4}$

$1 - \mathbb{P}(E \cap F) = \frac{3}{4}$

$\mathbb{P}(E \cap F) = \frac{1}{4}$

$\mathbb{P}(E \backslash F) = \mathbb{P}(E) - \mathbb{P}(E \cap F) = \frac{2}{5}$

$\mathbb{P}(E) = \frac{2}{5} + \frac{1}{4} = \frac{13}{20}$

For $E \subseteq (E \bigcup F)$ to be true, $\mathbb{P}(E) \leq \mathbb{P}(E \bigcup F)$

But $\frac{13}{20} \leq \frac{1}{2}$ is not true, so the events E and F will not be able to satisfy the conditions.

2. Given a sample space ω and a probability measure $\mathbb{P}$, two events $A \subseteq \omega$ and $B \subseteq \omega$ are said to be independent if $\mathbb{P} = \mathbb{P}(A)\mathbb{P}(B)$. Assume that the events E_1 *and* E_2 are independent.

(a) Prove that the events E_1^c and E_2^c are also independent.

(b) If, in addition, $\mathbb{P}(E_1) = \frac{1}{2}$ and $\mathbb{P}(E_2) = \frac{1}{3}$, prove that $\mathbb{P}(E_1 \bigcup E_2) = \frac{2}{3}$

(c) Let E_3 be a third event such that $\mathbb{P}(E_3) = \frac{1}{4}$, satisfying in addition that E_1 and E_3 are independent and also that E_2 and E_3 are independent. Prove that
$\frac{17}{24} \leq \mathbb{P}(E_1 \bigcup E_2 \bigcup E_3) \leq \frac{19}{24}$

(a) $\mathbb{P}(E_1^c \cap E_2^c) = \mathbb{P}((E_1 \bigcup E_2)^c)$
$= 1 - \mathbb{P}(E_1 \bigcup E_2)$
$= 1 - (\mathbb{P}(E_1) + \mathbb{P}(E_2) - \mathbb{P}(E_1 \cap E_2))$
$= 1 - (\mathbb{P}(E_1) + \mathbb{P}(E_2) - \mathbb{P}(E_1)\mathbb{P}(E_2))$
$= (1 - \mathbb{P}(E_1))(1 - \mathbb{P}(E_2))$
$= \mathbb{P}(E_1^c)\mathbb{P}(E_2^c))$

(b) $\mathbb{P}(E_1 \bigcup E_2) = \mathbb{P}(E_1) + \mathbb{P}(E_2) - \mathbb{P}(E_1 \cap E_2)$
$= \mathbb{P}(E_1) + \mathbb{P}(E_2) - \mathbb{P}(E_1)\mathbb{P}(E_2)$
$= \frac{1}{2} + \frac{1}{3} - \frac{1}{6}$
$= \frac{4}{6} = \frac{2}{3}$

(c) $\mathbb{P}(E_1 \bigcup E_2 \bigcup E_3) = \mathbb{P}(E_1) + \mathbb{P}(E_2) + \mathbb{P}(E_3) - \mathbb{P}(E_1 \cap E_2) - \mathbb{P}(E_1 \cap E_3) - \mathbb{P}(E_2 \cap E_3) + \mathbb{P}(E_1 \cap E_2 \cap E_3)$
$\mathbb{P}(E_1) + \mathbb{P}(E_2) + \mathbb{P}(E_3) - \mathbb{P}(E_1) \times \mathbb{P}(E_2) - \mathbb{P}(E_1) \times \mathbb{P}(E_3) - \mathbb{P}(E_2) \times \mathbb{P}(E_3) + \mathbb{P}(E_1 \cap E_2 \cap E_3) = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{6} - \frac{1}{12} - \frac{1}{8} + \mathbb{P}(E_1 \cap E_2 \cap E_3)$
$= \frac{17}{24} + \mathbb{P}(E_1 \cap E_2 \cap E_3)$
Here we get $\frac{17}{24} \leq \mathbb{P}(E_1 \bigcup E_2 \bigcup E_3)$

Taking the events E_2 and E_3 because they have the smallest probabilities:
$E_1 \cap E_2 \cap E_3 \subseteq E_2 \cap E_3$
$\mathbb{P}(E_1 \cap E_2 \cap E_3) \leq \mathbb{P}(E_2 \cap E_3) = \frac{1}{12}$
Substituting from before:
$\frac{17}{24} + \mathbb{P}(E_1 \cap E_2 \cap E_3) = \frac{17}{24} + \frac{1}{12} = \frac{19}{24}$
Hence proving that $\frac{17}{24} \leq \mathbb{P}(E_1 \bigcup E_2 \bigcup E_3) \leq \frac{19}{24}$

3. Eight rooks are placed randomly on a chess board. What is the probability that none of the rooks can capture any of the other rooks? (In non-chess terms: Randomly pick 8 unit squares from an 8×8 square grid. What is the probability that no two squares share a row or a column?)

There are 8! ways of placing 8 rooks such that none of the rooks can capture any of the other rooks. Total ways of placing eight rooks is $\binom{64}{8}$

The probability would be $\frac{8!}{\binom{64}{8}} = 9.11 \times 10^{-6}$

4. We roll two fair six-sided dice. Consider the events

E: The sum of the outcomes is even.

F: At least one outcome is 6.

Calculate the conditional probabilities $\mathbb{P}(E|F)$ and $\mathbb{P}(F|E)$.

$\mathbb{P}(E|F) = \frac{\mathbb{P}(E \cap F)}{\mathbb{P}(F)}$

$(E \cap F) = 5 \quad \{(2,6),(4,6),(6,6),(6,2),(6,4)\}$

$(F) = 11 \quad \{(1,6),(2,6),(3,6),(4,6),(5,6),(6,6),(6,1),(6,2),(6,3),(6,4),(6,5)\}$

$\mathbb{P}(E|F) = \frac{5}{11} = 0.45$

$\mathbb{P}(F|E) = \frac{\mathbb{P}(F \cap E)}{\mathbb{P}(E)}$

$(F \cap E) = \mathbb{P}(E \cap F)$

$(E) = 18$, where the other half has odd sums

$\mathbb{P}(F|E) = \frac{5}{18} = 0.28$

5. A fair six-sided die is rolled repeatedly.

(a) Give an expression for the probability that the first five rolls give a four at most two times.

(b) Calculate the probability that the first two does not appear before the fifth roll.

(c) Calculate the probability that the first six appears before the twentieth roll, but not before the fifth roll.

(a) This will be a binomial r.v., with 5 independent trials and a probability of success of $\frac{1}{6}$. $X \sim Bin(5, \frac{1}{6}) \rightarrow P(X=0) + P(X=1) + P(X=2) = \binom{5}{0}\frac{1}{6}^0(1-\frac{1}{6})^{5-0} + \binom{5}{1}\frac{1}{6}^1(1-\frac{1}{6})^{5-1} + \binom{5}{2}\frac{1}{6}^2(1-\frac{1}{6})^{5-2} = 0.96$

(b) By independence, $P(X=4) = \frac{5}{6}^4 = 0.48$. This is almost geometric, but we do not care if the fifth one is actually a success or not.

(c) Let Event A be where no 6 is rolled before the fifth roll, and Event B where a 6 is rolled somewhere between the 5^{th} and 19^{th} roll. We want to get $P(A \cap B) = P(A) \times P(B)$. To get $P(B)$, we get its complement as it is similar to how the probability for Event A was obtained: $P(B^c) = (\frac{5}{6})^{15}$. Hence, $P(B) = (1 - (\frac{5}{6})^{15})$.
Combining both probabilities, we get:
$P(A) \times P(B) = (\frac{5}{6})^4 \times (1 - (\frac{5}{6})^{15}) = 0.45$

6. The statement "some days are snowy" has 16 letters (treating different appearances of the same letter as distinct). Pick one of them uniformly at random (i.e. each with equal probability 1/16). Let X be the length of the word to which the letter which was chosen belongs. Determine the possible values that X may attain, and the probability mass function of X.

X can either be 3 (are), 4 (some days) or 5 (snowy)

The probability mass function of X will depend on the number of letters the word has. So:

$\mathbb{P}(X=3) = \frac{3}{16} = 0.1875$

$\mathbb{P}(X=4) = \frac{8}{16} = 0.5$

$\mathbb{P}(X=5) = \frac{5}{16} = 0.3125$

For the case where X = 4, although s was repeated, as they are considered distinct, the total number of letters that belong to 4-letter words will still be 8.