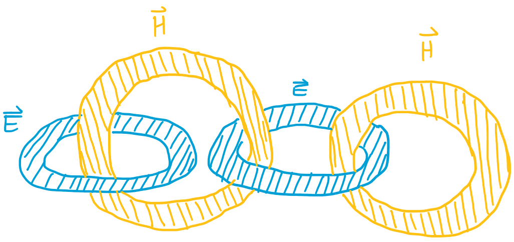
2.2 EM Waves

Tuesday, October 10, 2023 10:28 PM

Maxwell's Rotations

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\mu \frac{\partial \vec{H}}{\partial t} \qquad \vec{\nabla} \times \vec{H} = \varepsilon \frac{\partial \vec{E}}{\partial t}$$



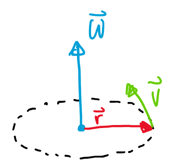
$$\frac{1}{2} \vec{\nabla} \times \vec{E} = \overrightarrow{rotation}_{\vec{E}} = -\frac{1}{2} \mu \frac{\partial \vec{H}}{\partial t}$$

$$\frac{1}{2} \vec{\nabla} \times \vec{H} = \overrightarrow{rotation}_{\vec{H}} = \frac{1}{2} \varepsilon \frac{\partial \vec{E}}{\partial t}$$

Taking $\vec{\nabla} \times \vec{\nabla} \times \vec{E}$ puts us in the $\vec{E}$ plane

$\rightarrow$ wave equation in $\vec{E}$

- A Mechanical Analogy
  > Consider a rigid rotating mass on a string, the tangential velocity of the mass is v. The rotation can be expressed by a constant angular velocity, $\vec{\omega}$, expressed as: $\vec{v} = \vec{\omega} \times \vec{r}$ where $\vec{r}$ is the position of the mass. In general, $\vec{r} = \langle x, y, z \rangle$



Q A relationship between the tangential velocity $\vec{v}$ and the angular velocity $\vec{\omega}$ can be established by taking the curl on both sides: $\vec{\nabla} \times \vec{v} = \vec{\nabla} \times (\vec{\omega} \times \vec{r})$.

Using the identity $\vec{\nabla} \times (\vec{a} \times \vec{b}) = a(\vec{\nabla} \cdot \vec{b}) + (\vec{b} \cdot \vec{\nabla})\vec{a} - \vec{b}(\vec{\nabla} \cdot \vec{a}) - (\vec{a} \cdot \nabla)\vec{b}$, which of the following is a valid relationship between $\vec{v}$ and $\vec{\omega}$?

$\vec{v} = \vec{\omega}$

$\vec{\nabla} \times \vec{v} = \vec{\omega}$

> $\underline{\vec{\nabla} \times \vec{v} = 2\vec{\omega}}$

$\vec{v} = \vec{\nabla} \times \vec{\omega}$

Maxwell's Phasor Notation

$$\vec{\nabla}\cdot\vec{E}=0 \qquad \vec{\nabla}\cdot\vec{B}=0$$
$$\vec{\nabla}\times\vec{E}=-\mu\frac{\partial\vec{H}}{\partial t} \qquad \vec{\nabla}\times\vec{H}=\varepsilon\boxed{\frac{\partial\vec{E}}{\partial t}}$$

non-phasor
$\vec{E}=\vec{E}(x,y,z,t)$
$\vec{H}=\vec{H}(x,y,z,t)$

Phasor $\rightarrow$ time dependence $e^{j\omega t}$

$$\vec{E}(x,y,z,t)=\boxed{\vec{E}_s(x,y,z,\omega)\,e^{j\omega t}}$$
$$\vec{H}(x,y,z,t)=\vec{H}_s(x,y,z,\omega)\,e^{j\omega t}$$

$$\cancel{e^{j\omega t}}\,\vec{\nabla}\times\vec{H}_s(x,y,z,\omega)=\varepsilon\boxed{\vec{E}_s(x,y,z,\omega)\,j\omega\,\cancel{e^{j\omega t}}}$$
$$\vec{\nabla}\times\vec{H}_s=j\omega\varepsilon\vec{E}_s$$

Application — Wave Equation

$$\nabla^2\vec{E}=\frac{1}{v^2}\frac{\partial^2\vec{E}}{\partial t^2}\quad\left(\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)$$
$$\nabla^2\left[\vec{E}_s(x,y,z,\omega)\,e^{j\omega t}\right]=\frac{1}{v^2}\frac{\partial^2}{\partial t^2}\left[\vec{E}_s(x,y,z,\omega)\,e^{j\omega t}\right]$$

Q The phasor voltage can be expressed as $\vec{E}_s(x,y,z,\omega)=\mathrm{Re}\{\vec{E}_s\}+j\,\mathrm{Im}\{\vec{E}_s\}$. The measurable instantaneous electric field is $\vec{\mathcal{E}}(x,y,z,t)=\mathrm{Re}\{\vec{E}_s(x,y,z,\omega)\,e^{j\omega t}\}$. For an EM wave travelling in the $+\hat{z}$ direction where the E field oscillates in the $\hat{x}$ axis with real amplitude E_{x0} and the H field oscillates in the $\hat{y}$ axis, the instantaneous E field, $\vec{\mathcal{E}}$, can be further expressed as ...

> $\vec{\mathcal{E}}=E_{x0}\cos(\omega t-\beta z)\,\hat{x}$

$\vec{\mathcal{E}}=E_{x0}\sin(\omega t-\beta z)\,\hat{x}$

$\vec{\mathcal{E}}=E_{x0}\sin(\omega t-\beta z)\,\hat{z}$

$\vec{\mathcal{E}}=E_{x0}\cos(\omega t-\beta z)\,\hat{z}$

Maxwell's Equations in Plane Wave (3D)

$$\vec{\nabla}\cdot\vec{E}=0 \qquad \vec{\nabla}\cdot\vec{B}=0$$
$$\boxed{\vec{\nabla}\times\vec{E}=-\mu\frac{\partial\vec{H}}{\partial t}} \qquad \vec{\nabla}\times\vec{H}=\varepsilon\frac{\partial\vec{E}}{\partial t}$$

gradient divergence Laplacian curl

gradient divergence Laplacian curl

$$\vec\nabla\times(\vec\nabla\times\vec E)=\vec\nabla(\vec\nabla\cdot\vec E)-\vec\nabla^2\vec E=\nabla\times\left(-\mu\frac{\partial\vec H}{\partial t}\right)=-\mu\frac{\partial}{\partial t}(\vec\nabla\times\vec H)$$

0 LHS | RHS $\left(\varepsilon\frac{\partial\vec E}{\partial t}\right)$

$$-\vec\nabla^2\vec E=-\mu\varepsilon\frac{\partial^2\vec E}{\partial t^2}$$

$$\frac{\partial^2\vec E}{\partial x^2}+\frac{\partial^2\vec E}{\partial y^2}+\frac{\partial^2\vec E}{\partial z^2}=\mu\varepsilon\frac{\partial^2\vec E}{\partial t^2}$$

Q Given the general 3D wave equation for electric fields: $\vec\nabla^2\vec E=\mu\varepsilon\frac{\partial^2\vec E}{\partial t^2}$. If an E field oscillating in the $\hat x$ axis is substituted into the wave equation: $\vec E=(E_x,0,0)=\left(E_0e^{j(\omega t-\beta z)},0,0\right)$, which of the following simplification is valid?

> $\frac{\partial^2\vec E}{\partial z^2}=\mu\varepsilon\frac{\partial^2E_x}{\partial t^2}$

$\frac{\partial^2E_x}{\partial x^2}=\mu\varepsilon\frac{\partial^2E_x}{\partial t^2}$

$\frac{\partial^2E_x}{\partial z^2}=-\mu\varepsilon\frac{\partial^2E_x}{\partial t^2}$

$\frac{\partial^2E_x}{\partial y^2}=\mu\varepsilon\frac{\partial^2E_x}{\partial t^2}$

In-class

$$\vec A\times(\vec B\times\vec C)=\alpha\vec B+\beta\vec C\longrightarrow\vec B(\vec A\cdot\vec C)-\vec C(\vec A\cdot\vec B)$$

BAC-CAB rule

α, β: a number

$\vec B\times\vec C$: $\perp$ BC plane (R)

$\vec A\times(\vec B\times\vec C)$: $\perp$ R plane $\longrightarrow$ BC plane!

How many ways to reorganize this?

$\vec B(\vec A\cdot\vec C)\leftrightarrow(\vec A\cdot\vec C)\vec B$

$\vec B(\vec C\cdot\vec A)\leftrightarrow(\vec C\cdot\vec A)\vec B$

$4\times4=16$

Where does the constraint come from?

e.g. $\vec A$ is an operator

$\vec B$ is an operator

We can have $\vec B(\vec A\cdot\vec C)$

but not $\vec B(\vec C\cdot\vec A)$?

Order matters!