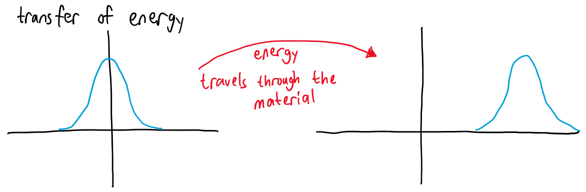
1. Transmission Lines

Tuesday, September 5, 2023 9:18 PM

1.1) Wave Properties and Mathematical Formulation

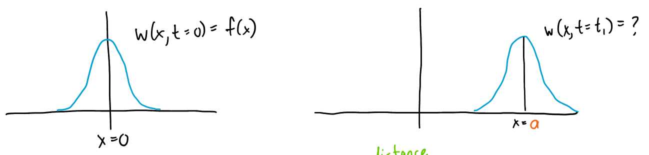
What is a Wave?

- A transfer of energy



- ex. acoustic waves
  - water "
  - elastic "
  - EM "

Basic Idea: Translation



$f(x - vt)$ — characteristic

- How is a related to t_1?

$a = t_1 v$ (distance; time; uniform movement)

Q1 In the video describing wave translation, the wave pulse, $f(x)$, has a peak at $x=0$ when $t=0$. The pulse moves uniformly with velocity v to a new position where the peak is now at $x=a$ when $t=t_1$ (ie. $a = vt_1$)

$f(x+a) = f(x+vt_1)$

$f(x)$

> $f(x-a) = f(x-vt_1)$ at $x=a$, $f(x-a) = f(a-a) = f(0)$ is the peak of the pulse

Q2 We have just found that a wave traveling at uniform speed, v, will cover a distance d in time t given by the formula $d = vt$. let's apply this to where we live in Vancouver, an earthquake-prone zone. An earthquake is an elastic wave giving rise to two waves: a compressional wave and a shear wave

Parameters we will need:

1) Distance from Vancouver to the epicenter, d, is 420 km
2) Compression wave velocity, V_p, is 7 km/s
3) The shear wave velocity, V_s, is 3 km/s

What is the time difference (in seconds) between the arrival of the compression and shear waves in Vancouver?

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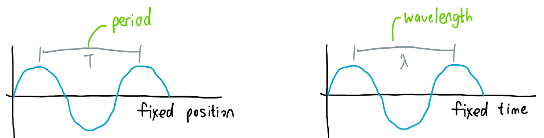
Basic Wave Properties — Waves and Phase Velocity

Sine waves (two snapshots)



T — angular frequency (rad/s)

$\omega = \frac{2\pi}{T}$ density of wave

λ — (rad/m)

$\nu = \frac{2\pi}{\lambda}$ density of wave

$$T \qquad \omega = \frac{2\pi}{T}$$ angular frequency (rad/s); density of wave per distance

$$\lambda \qquad K = \frac{2\pi}{\lambda}$$ (rad/m); density of wave per time

$$\boxed{v = f\lambda}$$ (phase velocity)

$$= (2\pi f)\frac{\lambda}{2\pi}$$

$$= \omega \frac{1}{\frac{2\pi}{\lambda}} \quad (k)$$

$$\boxed{v = \frac{\omega}{k} \quad \frac{rad/s}{rad/m}}$$ velocity of a point on a wave

Q We have previously shown that a wave traveling to the right along the x-axis, in the direction of increasing positive x, can be described by the function $f(x-vt)$ where v is the wavespeed or phase velocity in one dimension, x is the spatial wave location, t is the time, and f is the pulse shape. Let's assume that our wave has the shape of a cosine (periodic). We also know for a periodic wave that $V_{phase} = \frac{\omega}{k}$. Making these substitutions, our wave becomes $\cos(kx - \omega t)$.

The phase velocity of this wave is:

$V_{phase} = \frac{k}{\omega}$

Indeterminate

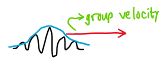
> $V_{phase} = \frac{\omega}{k}$

$V_{phase} = \frac{\omega^2}{k^2}$

Basic Wave Properties – Group Velocity

- attenuation — wave amplitude decreases as the wave goes through the material

- reflection & refraction
- dispersion & group velocity — a bunch of frequencies mixed together in a packet and look at how the speed of the packet moves
  - different frequencies of radiation have different speeds

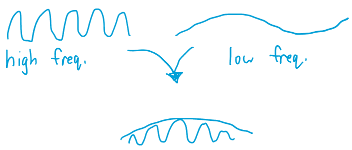


- diffraction — enhanced form of light scattering and bending. Has to do with shadows

- Group Velocity

$$A(x,t) = \cos\left[(K+\delta K)x - (\omega + \delta\omega)t\right] + \cos\left[(K-\delta K)x - (\omega - \delta\omega)t\right] \qquad \delta\omega \ll \omega$$

$$= 2\cos(Kx - \omega t)\cos(\delta K x - \delta\omega t)$$



Simple Case

$\omega = V_p K$

$\frac{d\omega}{dK} = V_p$ — phase

$$V_p = \frac{c}{n(\omega)} = \frac{\omega}{K}$$

prism; speed of light; index of refraction

$\frac{d\omega}{dK}$?

$K = \frac{\omega n(\omega)}{c}$ — If we can calculate this, we can easily get $\frac{d\omega}{dK}$: $\frac{dK}{d\omega}$

$$\frac{d\omega}{dk} = \left(\frac{dk}{d\omega}\right)^{-1} = \left(\frac{1}{c}\left[n(\omega) + \omega\frac{dn}{d\omega}\right]\right)^{-1} = \frac{c}{\boxed{n(\omega) + \omega\frac{dn}{d\omega}}}$$ group velocity of light waves going to the prism

in some books, known as $\frac{c}{\boxed{n_g}}$

Q1 Suppose we have two waves of the cosine type, with frequencies $\omega + \delta\omega$ and $\omega - \delta\omega$ where $\delta\omega \ll \omega$. The corresponding wave numbers will be

Q1 Suppose we have two waves of the cosine type, with frequencies $\omega+\delta\omega$ and $\omega-\delta\omega$ where $\delta\omega \ll \omega$. The corresponding wave numbers will be $k+\delta k$ and $k-\delta k$. The sum of our two waves, $A(x,t)$ is given by $A(x,t) = \cos[(k+\delta k)x-(\omega+\delta\omega)t] + \cos[(k-\delta k)x-(\omega-\delta\omega)t]$

$A(x,t)$ can be written as:

- $2\cos(kx+\omega t)\cos(\delta kx+\delta\omega t)$
- $2\cos(kx-\omega t)\cos(\delta kx+\delta\omega t)$
- > $2\cos(kx-\omega t)\cos(\delta kx-\delta\omega t)$ Using trig identity $\cos(C)+\cos(D) = 2\cos\left(\frac{(C+D)}{2}\right)\cos\left(\frac{(C-D)}{2}\right)$
- $2\cos(kx+\omega t)\cos(\delta kx-\delta\omega t)$

Q2 Conditions: $\delta k \ll k$ and $\delta\omega \ll \omega$

In the interpretation of the results of the previous question on "close" sinusoids, the final results may be interpreted as a

- > rapidly oscillating wave with a slowly varying modulation envelope The rapid oscillation will have frequency ω while the slowly varying oscillation will have frequency $\delta\omega$
- rapidly oscillating wave with a rapidly varying modulation envelope
- slowly oscillating wave with a slowly varying modulation envelope
- slowly oscillating wave with a rapidly varying modulation envelope

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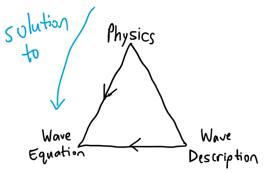
1.2) Modelling Lossless Transmission Lines

Motivation for Differentiation of the Wave Description

$f(x-vt) \leftarrow$ Model for Waves



Differentiating the Wave Description

Forward waves $\rightarrow f(x-vt)$

Physics: $V(x,t) = V^+ f(x-vt)$ Voltage

$I(x,t) = I^+ f(x-vt)$ Current

$= \frac{V^+}{Z} f(x-vt)$

$V(x,t) = V^+ f(x-vt) = V^+ f(\eta(x,t))$ $\quad \eta(x,t) = x - \underset{\text{velocity}}{v}t$

$\frac{\partial\eta}{\partial x} = 1 \quad \frac{\partial\eta}{\partial t} = -v$

$$\frac{\partial V}{\partial x} = V^+ \frac{df}{d\eta}\frac{d\eta}{dx} = \boxed{V^+ \frac{df}{d\eta}} \cdot 1$$

$$\frac{\partial V}{\partial t} = V^+ \frac{df}{d\eta}\frac{d\eta}{dt} = \boxed{V^+ \frac{df}{d\eta}}(-v) = \frac{\partial V}{\partial x}(-v)$$

$$\frac{\partial V}{\partial t} = -v\frac{\partial V}{\partial x} \rightarrow \boxed{\frac{\partial V}{\partial x} + \frac{1}{v}\frac{\partial V}{\partial t} = 0}$$ Wave travelling to the right in 1D

$-$ Wave travelling to the left

Q1 Consider a wave described by $f(x+vt)$ where v is a positive number. As t increases, the wave travels in the direction of

- > decreasing x
- increasing x

> decreasing x

increasing x

x = 1

both increasing and decreasing x

Q2 Let us denote the derivative of $f(x+vt)$ with respect to the argument $x+vt$ by f' (i.e. $f' = \frac{df}{d(x+vt)}$). When we differentiate $f(x+vt)$ with respect to t, we obtain the result

> vf'

$-vf'$

$-v\,f(x+vt)$

$v(f')'$

Q3 Suppose $V(z,t) = V^+ g\left(\alpha \cdot \left(t - \frac{z}{v}\right)\right)$ where g = some function of wave, α = constant, v = velocity, z = distance that the wave travels down the transmission line, and V^+ = voltage amplitude constant

$V(z,t)$ satisfies the equation given by:

$\frac{\partial V}{\partial z} + \frac{\alpha}{v}\frac{\partial V}{\partial t} = 0$

> $\frac{\partial V}{\partial z} + \frac{1}{v}\frac{\partial V}{\partial t} = 0$

$\frac{\partial V}{\partial z}\frac{\partial z}{\partial g} + \frac{\alpha}{v}\frac{\partial V}{\partial t}\frac{\partial t}{\partial g} = 0$

$\frac{\partial V}{\partial t} + \frac{1}{v}\frac{\partial V}{\partial z} = 0$

$\eta(x,t) = \alpha t - \frac{\alpha x}{v}$

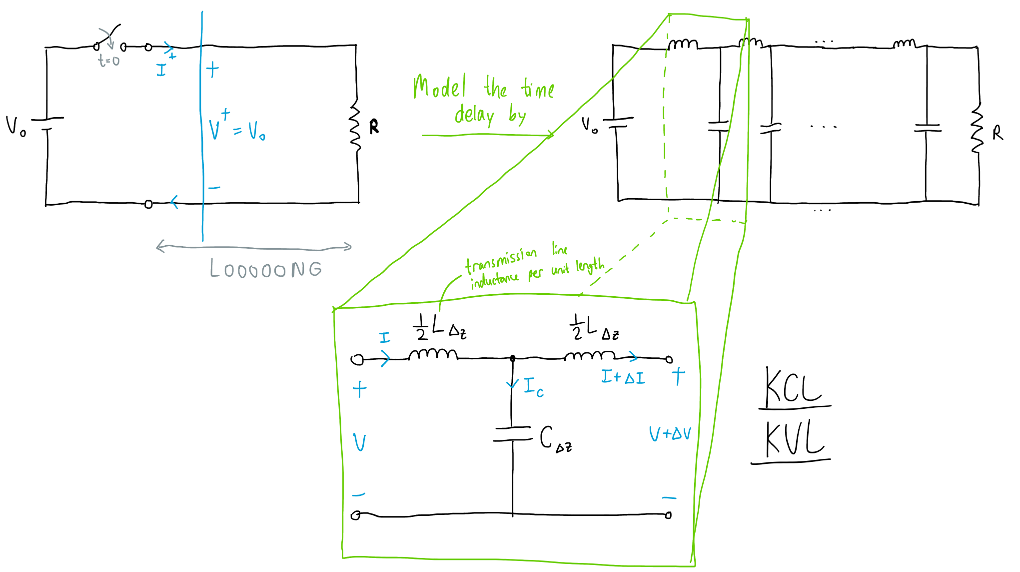
$\frac{\partial \eta}{\partial x} = \frac{-\alpha}{v} \quad \frac{\partial \eta}{\partial t} = \alpha$

$\frac{\partial V}{\partial x} = V^+ g \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial x} = V^+ g \frac{\partial f}{\partial \eta}\frac{-\alpha}{v}$

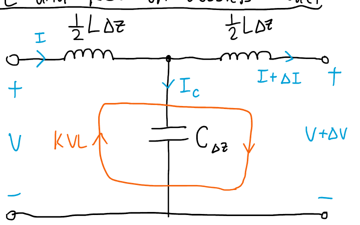
$\frac{\partial V}{\partial t} = V^+ g \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial t} = V^+ g \frac{\partial f}{\partial \eta}\alpha$

$\rightarrow \frac{\partial V}{\partial x} = \frac{\partial V}{\partial t}\cdot\frac{-1}{v} \rightarrow \frac{\partial V}{\partial x} + \frac{1}{v}\frac{\partial V}{\partial t} = 0$

Lossless Transmission Line Model



KVL and KCL on Lossless Model



KVL: $\cancel{V} - \frac{1}{2}L\Delta z\frac{\partial I}{\partial t} - \frac{1}{2}L\Delta z\frac{\partial}{\partial t}(I+\Delta I) - (\cancel{V}+\Delta V) = 0$

$L\Delta z\frac{\partial I}{\partial t}(I + \cancel{\Delta I}^{\,0}) = -\Delta V$

KVL: $\cancel{V} - \frac{1}{2}L\Delta z \frac{\partial I}{\partial t} - \frac{1}{2}L\Delta z \frac{\partial}{\partial t}(I+\Delta I) - (\cancel{V}+\Delta V) = 0$

$L\Delta z \frac{\partial I}{\partial t}(1+\cancel{\Delta I}^{\,0}) = -\Delta V$

$\frac{\Delta V}{\Delta z} = -L\frac{\partial I}{\partial t}$

$\frac{\partial V}{\partial z} = -L\frac{\partial I}{\partial t}$

KCL: $I = I_c + I + \Delta I$

$\cancel{I} = C\Delta z \frac{\partial V}{\partial t} + \cancel{I} + \Delta I$

$\frac{\Delta I}{\Delta z} = -C\frac{\partial V}{\partial t}$

$\frac{\partial I}{\partial z} = -C\frac{\partial V}{\partial t}$

Q1 Which of the following is true?

> Waves traveling down a transmission line have a maximum speed limit

The transmission line model presented in this video with lumped L and C elements dissipate power as voltage and current waves pass through

A transmission line described by the lumped element model only supports waves traveling in one direction

Q2 Is the statement true: "A wave is launched down a transmission line when a battery is switched on."

It depends on the battery voltage, yes if it is higher than 5V

> Yes

No

Q3 Under which of the following scenario(s) is/are the lossless transmission line model applicable for modeling the transmission line?

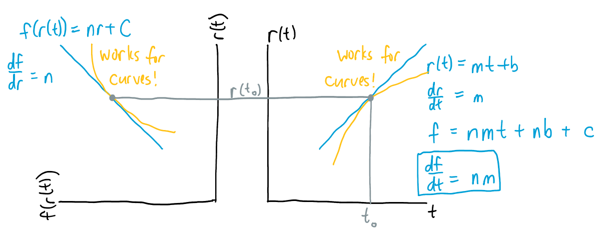
> Modeling a 100m long lossless transmission line terminated by a 50Ω resistive load

> Modeling a 100m long lossless transmission line terminated by an open circuit

> Modeling a 100m long lossless transmission line terminated by a short circuit

> Modeling an infinitely long lossless transmission line

The Chain Rule



$$\frac{df}{dt} = \frac{df}{dr}\frac{dr}{dt}$$

1.3) Simple Waves on Transmission Lines

Intuition for 1st and 2nd Order Wave Equations

1st order right-travelling wave: $\frac{\partial V}{\partial x} + \frac{1}{v}\frac{\partial V}{\partial t} = 0$ ←

left-travelling wave: $\frac{\partial V}{\partial x} - \frac{1}{v}\frac{\partial V}{\partial t} = 0$ ←

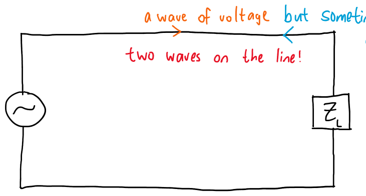
Intuition for 1st and 2nd Order Wave Equations

1st order right-travelling wave: $\frac{\partial v}{\partial x} + \frac{1}{v}\frac{\partial v}{\partial t} = 0$ ←

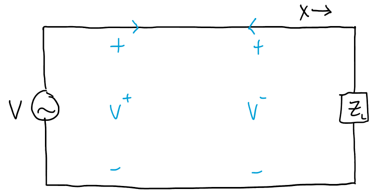
left-travelling wave: $\frac{\partial v}{\partial x} - \frac{1}{v}\frac{\partial v}{\partial t} = 0$ ←

2nd order: $\frac{\partial^2 v}{\partial t^2} - \frac{1}{v^2}\frac{\partial^2 v}{\partial t^2} = 0$

a wave of voltage but sometimes, when the wave hits the load, the load appears "stiff" to the wave. So we get a reflection back.



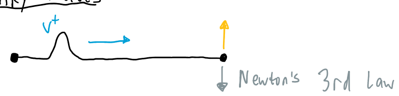
Since we have two first-order equations here, it will lead us to that 2nd order equation previously.

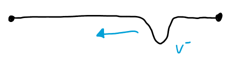


$$V(x,t) = V^+ f(x-vt) + V^- f(x+vt)$$

$$\frac{\partial^2 V}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 V}{\partial t^2} = 0$$

Slinky Waves





Simple Waves on Transmission Lines — Voltage Waves

KVL: $\frac{\partial V}{\partial z} = -L\frac{\partial I}{\partial t}$ ⟶ $\frac{\partial^2 V}{\partial z^2} = -L\boxed{\frac{\partial^2 I}{\partial t \partial z}}$

KCL: $\frac{\partial I}{\partial z} = -C\frac{\partial V}{\partial t}$ ⟶ $\boxed{\frac{\partial^2 I}{\partial t \partial z}} = -C\frac{\partial^2 V}{\partial t^2}$

$$\boxed{\frac{\partial^2 V}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 V}{\partial t^2}}$$

$\frac{\partial^2 V}{\partial z^2} = LC\frac{\partial^2 V}{\partial t^2}$ → $\frac{1}{v^2}$ → velocity

$\left[\frac{V}{m^2}\right] = [LC]\left[\frac{V}{t^2}\right]$ Volts

$= \left[\frac{1}{\frac{m^2}{t^2}}\right]\left[\frac{V}{t^2}\right]$ time

$[LC] = \frac{HF}{m^2}$

$v = \frac{1}{\sqrt{LC}}$

Simple Voltage and Current Waves on Lossless Lines

$\frac{\partial^2 V}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 V}{\partial t^2}$ ⟶ $V(z,t) = f_1\left(t - \frac{z}{v}\right) + f_2\left(t + \frac{z}{v}\right)$

forward V^+ backward V^-

KVL

$\frac{\partial I}{\partial t} = -\frac{1}{L}\frac{\partial V}{\partial z}$

$\frac{\partial I}{\partial t} = -\frac{1}{L}\frac{\partial}{\partial z}\left[f_1\left(t-\frac{z}{v}\right) + f_2\left(t+\frac{z}{v}\right)\right]$

$I(z,t) = \frac{1}{Lv}\left[f_1\left(t-\frac{z}{v}\right) - f_2\left(t+\frac{z}{v}\right)\right]$

$\eta_1(z,t) = t - \frac{z}{v}$ $\quad \frac{\partial \eta_1}{\partial z} = -\frac{1}{v}$ $\quad \frac{\partial \eta_1}{\partial t} = 1$

$\eta_2(z,t) = t + \frac{z}{v}$ $\quad \frac{\partial \eta_2}{\partial z} = \frac{1}{v}$ $\quad \frac{\partial \eta_2}{\partial t} = 1$

→ $\frac{\partial I}{\partial t} = -\frac{1}{L}\left(\frac{\partial f}{\partial \eta_1}\frac{\partial \eta_1}{\partial z} + \frac{\partial f}{\partial \eta_2}\frac{\partial \eta_2}{\partial z}\right)$

$= -\frac{1}{L}\left(\frac{\partial f}{\partial \eta_1}\left(-\frac{1}{v}\right) + \frac{\partial f}{\partial \eta_2}\right)$

$\int \frac{\partial I}{\partial t} = \int \left(\frac{1}{Lv}\left(\frac{\partial f}{\partial \eta_1} - \frac{\partial f}{\partial \eta_2}\right)\right)$

$I(z,t) = \frac{1}{Lv}\left(f_1\left(t-\frac{z}{v}\right) - f_2\left(t+\frac{z}{v}\right)\right) = I^+ + I^-$

$\boxed{\frac{\partial f}{\partial z}} + \boxed{\frac{1}{v}\frac{\partial f}{\partial t}} = 0$

$L^+ f = 0$

$L^+ = \frac{\partial}{\partial z} + \frac{1}{v}\frac{\partial}{\partial t}$

$L^+ V = \underline{L^+ f(z-vt)} + L^+ g(z+vt)$ one-wave wave equation

$L^- n = 0$

$$I(z,t) = \frac{1}{LV}\left(f_1\left(t-\frac{z}{v}\right) - f_2\left(t+\frac{z}{v}\right)\right) = I^+ + I^-$$

$$L^+V = L^+\cancel{f(z-vt)}^{\,0} + L^+ g(z+vt)$$ one-wave wave equation

$$L^- g = 0$$

$$L^- = \frac{\partial}{\partial z} - \frac{1}{v}\frac{\partial}{\partial t}$$

$$L^-L^+V = 0 + L^-L^+ g(z+vt)$$

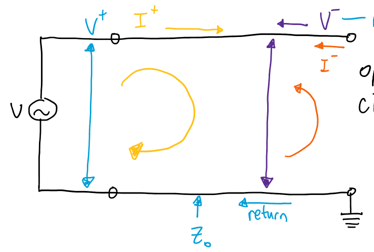
$$L^-L^+V = 0 + L^+\cancel{L^- g(z+vt)}^{\,0}$$

$$L^-L^+V = 0$$

$$L^-L^+V = \left[\frac{\partial^2}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right]V = 0$$

$$L^-L^+ = \frac{\partial^2}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}$$

Intuition of Backward Current Sign Flip



opposite happens for short circuit

$I^+ \rightarrow I_0$

$I^- \rightarrow -I_0$ → cancel out because current can't flow anymore — reflected current

Q On the transmission line,

> ~~the polarity of the backward current, I^-, is the reverse of the polarity of the backward voltage, V^-~~

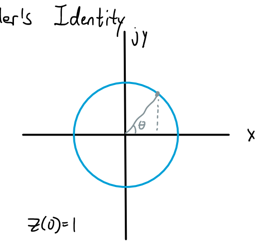
the polarity of the backward current, I^-, is the same of the polarity of the forward voltage, V^+

the polarity of the backward current, I^-, is the same of the polarity of the forward current, I^+

the polarity of the forward current, I^+, is the reverse of the polarity of the forward voltage, V^+

Voltage Functions and Notations

Euler's Identity



$x + jy = z$

$$\boxed{z = \cos\theta + j\sin\theta}$$

$$\frac{dz}{d\theta} = -\sin\theta + j\cos\theta$$

$$= j(\underbrace{j\sin\theta + \cos\theta}_{z})$$

$$\frac{dz}{d\theta} = jz$$

$$\frac{dz}{z} = j\,d\theta \rightarrow \ln z = j\theta + c$$

$$e^{\ln z} = z = e^{j\theta + c} = \underbrace{e^{c}}_{1} e^{j\theta}$$

$$z(0) = 1 = e^{c}$$

$$\boxed{z = e^{j\theta}}$$

Voltage Function & Notations

$$V^+ = f_1\left(t - \frac{z}{v}\right)$$

f = cosine exp.

$$\underset{\text{instantaneous}}{V(z,t)} = \frac{1}{2}\underbrace{\left(\overbrace{Nd\,e^{j\theta}}^{V_0}\right) e^{\pm j\beta z} e^{j\omega t}}_{B} + B^*$$

$$V_c(z,t) = V_0 e^{\pm j\beta z} e^{j\omega t}$$

$$V_s(z,t) = V_0\, e^{\mp j\beta z}$$

Q1 $\cos(x) =$

$\frac{e^{jx} + e^{-jx}}{2j}$

$\frac{e^{jx} - e^{-jx}}{2j}$

$e^{jx} - e^{-jx}$

$2j$

$\frac{e^{jx}-e^{-jx}}{2j}$

$\frac{e^{jx}-e^{-jx}}{2}$

> $\frac{e^{jx}+e^{-jx}}{2}$

Q2 In the complex voltage representation $V_c(z,t) = V_0 e^{\pm j\beta z} e^{j\omega t}$, what are the units of:

ω
- > rad/s
- rad/m
- V

β
- rad/s
- > rad/m
- V

1.4) Lossy Transmission Lines and Phasor Representation

Energy Flow in Transmission Line via TEM Waves

TEM for T (transmission) Lines

transverse electromagnetic waves

$$V(z,t) = \underbrace{f_1\left(t-\frac{z}{v}\right)}_{V^+} + \cancel{f_2\left(t+\frac{z}{v}\right)}$$ to make things simple

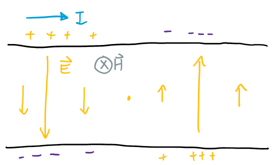
$$I(z,t) = \frac{1}{Z_0}\left\{\underbrace{f_1\left(t-\frac{z}{v}\right)}_{\text{forward},\ I^+} - \underbrace{\cancel{f_2\left(t+\frac{z}{v}\right)}}_{\text{backward}}\right\}$$

voltage

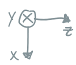
velocity $v = \frac{1}{\sqrt{LC}}$

$Z_0 = \sqrt{\frac{L}{C}} = Lv$ characteristic impedance

We want to visualize this.



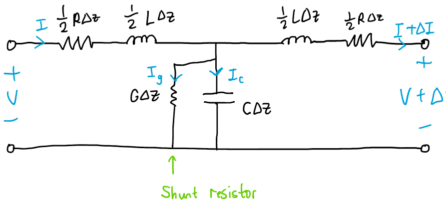
Each line in the xy plane is perpendicular/transverse to the direction of the current and transverse to the direction of the energy flow.



$[\vec{E}] = \frac{V}{m}$

$[\vec{H}] = \frac{A}{m}$

Full Transmission Line Model



KVL: $\frac{\partial v}{\partial z} = -\left(IR + L\frac{\partial I}{\partial t}\right)$

KCL: $\frac{\partial I}{\partial z} = -\left(GV + C\frac{\partial V}{\partial t}\right)$

Q The following losses are modeled by the series resistance R:
- > electrical resistance in the conductors
- current leakage through the dielectric between conductors (dielectric loss)
- capacitive charging loss
- > magnetic field switch losses (often insignificant)

Phasor Solutions

- Assume that all time variation is of the solution dependent on $e^{j\omega t}$.

So all first time derivatives will be replaced by $j\omega e^{j\omega t}$ and second time derivatives will be replaced by $-\omega^2 e^{j\omega t}$

- Assume that all time variation is of the solution dependent on $e^{j\omega t}$.
So all first time derivatives will be replaced by $j\omega e^{j\omega t}$ and second time derivatives will be replaced by $-\omega^2 e^{j\omega t}$
- Previously:
$$\frac{\partial V}{\partial z} = -\left(RI + L\frac{\partial I}{\partial t}\right)$$
$$\frac{\partial I}{\partial z} = -\left(GV + C\frac{\partial V}{\partial t}\right)$$
- Combining this we get one voltage equation:
$$\frac{\partial^2 V}{\partial z^2} = LC\frac{\partial^2 V}{\partial t^2} + \underbrace{(LG+RC)\frac{\partial V}{\partial t} + RGV}$$
If there was no damping, resistance, conductance in the system, then last two terms would disappear

- Assuming the phasor time dependence we end up with the phasor expression
$$\frac{d^2 V_s}{dz^2} = -\omega^2 LCV_s + j\omega(LG+RC)V_s + RGV_s$$

- We can regroup the RHS of the previous equation to get
$$\frac{d^2 V_s}{dz^2} = \underbrace{(R+j\omega L)}_{Z}\underbrace{(G+j\omega C)}_{Y}V_s = \gamma^2 V_s$$

- Here $\gamma = \sqrt{(R+j\omega L)(G+j\omega C)} = \sqrt{ZY} = \alpha + j\beta$ (→ wave number: β)
if there is no resistance R or conductance G, then α_0 and β_0 is k (as seen at the very top of the page)

whose solution is $V_s = \underbrace{V_0^+ e^{-\gamma z}}_{\text{Wave going towards the wall}} + \underbrace{V_0^- e^{\gamma z}}_{\text{Wave reflected back}}$

and for current, we can follow analogous procedures to get
$$I_s = \underbrace{I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z}}_{\text{Same as for } V_s}$$

- We can apply similar methods to the first equations that we obtained using KVL & KCL, obtaining
$$\frac{dV_s}{dz} = -(R+j\omega L)I_s = -ZI_s \quad eq$$
and
$$\frac{dI_s}{dz} = -(G+j\omega C)V_s = -YV_s$$

- Now substitute voltage and current solutions into eq to get
$$-\gamma V_0^+ e^{-\gamma z} + \gamma V_0^- e^{\gamma z} + \gamma V_0^- e^{\gamma z} = -Z(I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z})$$
$$-\gamma V_s^+ = -ZI_0^+$$
$$\frac{V_0^+}{I_0^+} = \frac{Z}{\gamma} \longrightarrow Z_0 \quad \text{Characteristic Impedance}$$

- We look at the forward & backward waves and match and we get the characteristic impedance as

Z_0 = transmission line voltage to current ratio at same position
$$= \frac{V_0^+}{I_0^+} = -\frac{V_0^-}{I_0^-} = \frac{Z}{\gamma} = \frac{Z}{\sqrt{ZY}} = \sqrt{\frac{Z}{Y}}$$

- Using the previous results for Z and Y we have the explicit expression
$$Z_0 = \sqrt{\frac{(R+j\omega L)}{(G+j\omega C)}} = |Z_0|\, e^{j\theta}$$

From the video, we have obtained $\gamma = \sqrt{(R+j\omega L)(G+j\omega C)} = \sqrt{ZY} = \alpha + j\beta$, $V_s = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}$
Considering the forward component of the phasor voltage, we have: $V_s^+ = V_0^+ e^{-(\alpha+j\beta)z} = V_0^+ e^{-\alpha z} e^{-j\beta z}$

Q1 From the mathematical intuition, $e^{-\alpha z}$ is ...

the oscillation (rotation of the phasor)

> the loss (exponential decay)

Q2 And $e^{-j\beta z}$ is ...

> the oscillation (rotation of the phasor)

the loss (exponential decay)

Q3 Here, we have introduced the characteristic impedance $Z_o = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$. Is the following statement true:

"When R is 0 and $G = 0$, Z_o is always imaginary

True

> False When $R = 0, G = 0$, $Z_o = \sqrt{\frac{0 + j\omega L}{0 + j\omega C}} = \sqrt{\frac{L}{C}}$ which is purely real

Intuition of Characteristic Impedance

$$V = V_o^+ e^{-\gamma z}$$

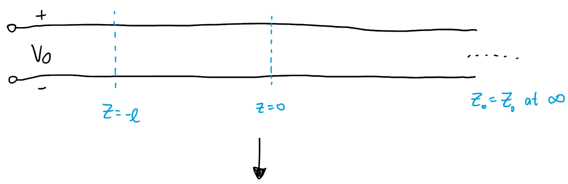
$$I = I_o^+ e^{-\gamma z} = \frac{V_o^+}{Z_o^+} e^{-\gamma z} = \frac{V_o^+}{LV} e^{-\gamma z} = \frac{V_o^+}{L \cdot \frac{1}{LC}} e^{-\gamma z} = \frac{V_o}{\sqrt{\frac{L}{C}}} e^{-\gamma z}$$

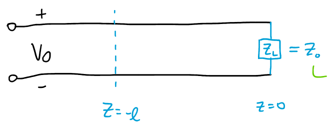
Lossless line

$\gamma = \alpha + j\beta \longrightarrow \alpha = 0, \ \gamma = j\beta$

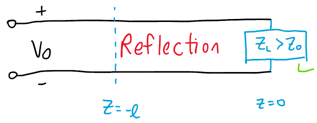
(for the case)

Ratio of $\frac{V}{I} = \frac{V_o^+ \cancel{e^{-j\beta z}}}{\frac{V_o}{Z_o} \cancel{e^{-j\beta z}}} = Z_o$ → constant along the line

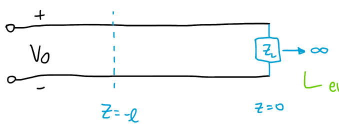




└ think of this as the electrical stiffness of the line



└ eg. when you push and the object is very stiff, there is some pushback



open circuit

└ everything (wave) gets reflected because the O.C. does not consume any of the incoming power

Q Given that a voltage wave $V_s = (5V)\, e^{-j\beta z}$ enters an infinitely long transmission line with characteristic

Q Given that a voltage wave $V_s = (5V)\, e^{-j\beta z}$ enters an infinitely long transmission line with characteristic impedance $Z_0 = 50\,\Omega$, what is the corresponding current amplitude I_0 in $I_s = I_0 e^{-j\beta z}$?

$I_0 =$ 0.1 A $Z_0 = 50 = \frac{V_0}{I_0} = \frac{5}{I_0}$

Summary

Q1 Consider a TEM wave travelling down a transmission line in the $+z$ direction. If $\vec{E}$ is in the $+x$ direction, what is the direction of $\vec{H}$?

- $+\hat{x}$
- $-\hat{x}$
- $-\hat{z}$
- $+\hat{z}$
- > $+\hat{y}$ Right Hand Rule
- $-\hat{y}$

Q2 In a lossless transmission line, the characteristic impedance (Z_0) must be 0

- True
- > False

1.5 Power Loss in Transmission Lines

Low Loss Propagation

$R \ll \omega L$ and $G \ll \omega C$. Low Loss Model where G and R control Damping (Friction in Mechanics)

BASIC BEHAVIOUR:

- Substituting $\gamma = \alpha + j\beta$, our solution for the voltage, assuming real amplitudes $V_0^{\pm}$, becomes

$$V_s = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}$$
$$= V_0^+ e^{-\alpha z} e^{-j\beta z} + V_0^- e^{\alpha z} e^{j\beta z} \qquad \text{(HB8, (48))}$$

- We multiply the phasor solution of (HB8, (48)) by $e^{j\omega t}$ and take the real part to get the real instantaneous voltage (assuming that the amplitudes, V_0^+ and V_0^- are real) as

$$V(z,t) = \underbrace{V_0^+ \underbrace{e^{-\alpha z}}_{\text{damping}} \underbrace{\cos(\omega t - \beta z)}_{\text{travelling}}}_{\text{forward wave}} + \underbrace{V_0^- e^{\alpha z} \cos(\omega t + \beta z)}_{\text{reflected wave}} \qquad \text{(HB8, (49))}$$

$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)}$ (take the real part)

- Main idea: damped (attenuated) travelling wave in forward and backward directions. Simplifications $R \ll \omega L$ and $G \ll \omega C$ for α, β and Z_0:
  1. Use binomial expansion on γ to obtain simplified values for α and β
  2. Use binomial expansion and the $\frac{1}{1+x}$ result for small x to get the final approximation for Z_0
- With our first Transmission Line model completed, we move to power transmission!

Power Loss and dB

Wave: $V_s(z) = V_0 e^{-\alpha z} e^{-j\beta z}$ HB8, 61

$I_s(z) = I_0 e^{-\alpha z} e^{-j\beta z} = \frac{V_0}{Z_0} e^{-\alpha z} e^{-j\beta z}$ $Z_0 = |Z_0| e^{j\theta}$ HB8, 62

$\langle P \rangle = \frac{1}{2} \operatorname{Re}\left(V_s(z)\, I_s^*(z)\right)$ HB8, 63

$$\langle P\rangle = \frac{1}{2}\,Re\left(V_s(z)\,I_s^*(z)\right) \qquad HB8,63$$

$$\langle P(z)\rangle = \frac{1}{2}\,Re\left[\frac{V_0 V_0^*}{|z_0|}e^{-2\alpha z}e^{j\theta}\right]$$

$$= \frac{1}{2}\frac{|V_0|^2}{|z_0|}e^{-2\alpha z}\cos\theta \qquad HB8,64$$

$$\langle P(z)\rangle = \langle P(0)\rangle e^{-2\alpha z} \qquad HB8,65 \qquad \langle P(0)\rangle = \frac{1}{2}\frac{|V_0|^2}{|z_0|}\cos\theta$$

$$\frac{\langle P(z)\rangle}{\langle P(0)\rangle} = e^{-2\alpha z} = 10^{-K\alpha z} \qquad HB8,66$$

set $\alpha z = 1 \longrightarrow e^{-2} = 10^{-K} = K = \log_{10}(e^2) = 0.869$ HB8, 67

$$\text{Power Loss (dB)} = 10\log_{10}\left[\frac{\langle P(0)\rangle}{\langle P(z)\rangle}\right] = 8.69\,\alpha z \qquad HB8,68$$

$$\| $$

$$8.69\,\frac{dB}{\cancel{Np}}\cdot\frac{1\,\cancel{Np}}{\cancel{m}}\cdot\cancel{m}$$

Q1 Given a voltage wave travelling down a transmission line: $V_s = V_0 e^{-\alpha z}e^{-j\beta z}$.
If $V_0 = 5V$, $\alpha = 0.1/m$, what is the voltage amplitude at the location 5m?

$$V_s = (5V)\,e^{-(0.1)(5)} = 3.03\,V$$

Q2 Consider the following voltage wave on a transmission line: $V_s = (5V)\,e^{-\alpha z}e^{-j\beta z}$ with $\alpha = 0.001/m$ and $Z_0 = 50\Omega$.
After the wave travels for 100m, what is the time averaged power dissipated?

$\langle P_{loss}\rangle = 0.0453\ W$

$$\langle P_i\rangle = \frac{1}{2}\frac{V_s^2}{Z_0} = \frac{1}{2}(5)\,e^{-0.1(100)}/(50) = 0.2047 \qquad P\langle 0\rangle = \frac{V_s^2}{R} = \frac{5^2}{50} = 0.5$$

$$\langle P_o\rangle = P\langle 0\rangle - P\langle 100\rangle$$

$$= 0.5 -$$

Nepers vs dB

Amplitude ratios: $20\log_{10}\left(\frac{A_2}{A_1}\right)$ dB $\qquad u = \log_{10}\left(\frac{A_2}{A_1}\right),\ 10^u = \left(\frac{A_2}{A_1}\right),\ \ln 10^u = u\ln 10 = \ln\left(\frac{A_2}{A_1}\right)$

New ratios: $N_p \rightarrow \ln\left(\frac{A_2}{A_1}\right)N_p$ (waves $\propto e^{-\alpha z}$)

$$\ln\left(\frac{A_2}{A_1}\right)N_p = 20\log_{10}\left(\frac{A_2}{A_1}\right)dB$$

$$1\,N_p = \frac{20\log_{10}\left(\frac{A_2}{A_1}\right)dB}{\ln\left(\frac{A_2}{A_1}\right)} = \frac{20u}{u\ln 10}\,dB \approx 8.69\,dB$$

$$\boxed{\#dB = 20\log_{10}\left(\frac{A_{source}}{A_{receiver}}\right)} \qquad e^{\#\,Nepers} = \frac{A_{source}}{A_{receiver}}$$

$$\boxed{Nepers = \ln\left(\frac{A_{source}}{A_{receiver}}\right)} \qquad 20\log_{10}e^{\#Neper} = 20\log_{10}\left(\frac{A_{source}}{A_{receiver}}\right)$$

$$\#dB = 20\log_{10}e^{1}$$

$$\#dB = 20\log_{10}(3) \approx 10$$

---

16 Reflection and Standing Waves

Reflection Generation

> Basic equations for an incident wave and reflected wave given by

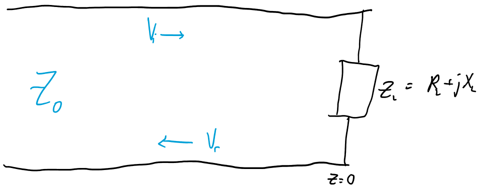
incident reflected

> Basic equations for an incident wave and reflected wave given by

$$V_s = \overset{\text{incident}}{V_i(z)} + \overset{\text{reflected}}{V_r(z)}$$
$$= V_{o_i} e^{-\alpha z} e^{-j\beta z} + V_{o_r} e^{\alpha z} e^{j\beta z} \quad \text{HB 8, 70a, 70b}$$

> The reflection will be caused by a load, Z_L, located at $z=0$ at the end of the line

• Consider the figure below



• At $z=0$, at the load, Z_L, the voltage and current are respectively:

$$V_L = V_{o_i} + V_{o_r}$$
$$I_L = I_{o_i} + I_{o_r}$$
$$= \frac{1}{Z_o}[V_{o_i} - V_{o_r}] = \frac{V_L}{Z_L} = \frac{[V_{o_i} + V_{o_r}]}{Z_L} \quad \text{HB 8, 71, 72}$$

• Now solve for the ratio $\frac{V_{o_r}}{V_i}$

$$I_L = \frac{V_L}{Z_L} = \frac{[V_{oi} + V_{or}]}{Z_L} = \frac{1}{Z_o}[V_{oi} - V_r]$$
$$Z_o(V_{oi} + V_{or}) = Z \quad V_{oi} - V_{or}]$$
$$Z_o\left(1 + \frac{V_{or}}{V_{oi}}\right) = Z_L\left(\frac{V_{or}}{V_{oi}}\right)$$
$$Z_o(1+\Gamma) = Z_L(1-\Gamma)$$
$$Z_o - Z_L = -Z_o\Gamma - Z_L\Gamma = (Z_o + Z_L)\Gamma$$
$$\Gamma = (Z_L - Z_o)(Z_L + Z_o)$$

• Continuing,

$$\Gamma = \frac{V_{or}}{V_{oi}} = \frac{Z_L - Z_o}{Z_L + Z_o} = |\Gamma| e^{j\theta}$$

• The voltage at the load is the sum of the incident and reflected voltages at $z=0$. That ratio is the transmission coefficient:

$$\tau = \frac{V_L}{V_{oi}} = \frac{[V_{oi} + V_{or}]}{V_{oi}} = \frac{[V_{oi} + \Gamma V_{oi}]}{V_{oi}} = 1 + \Gamma \quad \text{HB 8, (74, 75)}$$

Q1 From this video, $\Gamma = |\Gamma| e^{j\theta}$.
When $|\Gamma| \neq 0$ and $\theta \neq \pi m$ where $m \in \mathbb{Z}$:

Γ has an imaginary component which indicates loss in the form of heat dissipation at the boundary

> Γ has an imaginary component which introduces a phase shift of the reflected wave

Video Recap: The reflection coefficient between a transmission line and the load, Γ, is defined as

$$\Gamma = \frac{V_{or}}{V_{oi}} = \frac{Z_L - Z_o}{Z_L + Z_o}$$

$$\Gamma = \frac{V_{or}}{V_{oi}} = \frac{Z_L - Z_0}{Z_L + Z_0}$$

It is clear from the 1st part of the equation that Γ is a ratio between the amplitudes of the reflection and the incident voltage waves ($V_{or} = \Gamma V_{oi}$)

Q2 Given that $V_{oi} = 5\,V$ and $\Gamma = 0$, what is V_{or}?

5 V

> 0 V

2.5 V

$V_{or} = 0 \cdot 5$
$= 0$

Q3 What is the physical interpretation when $\Gamma = 0$?

> There is no reflection

The voltage wave is partially reflected

The voltage wave is fully reflected

Q4 Given that $V_{oi} = 5\,V$ and $\Gamma = -0.5$, what is V_{or}?

> -2.5 V

2.5 V

5 V

Q5 What is the physical interpretation when $\Gamma = -0.5$?

The voltage wave is partially reflected with no polarity change at the boundary

> The voltage wave is partially reflected with reversed polarity at the boundary

The voltage wave is fully reflected

Power Reflection and Transmission

- Using the Reflection Coefficient to Find the Time-Averaged Power Transfer

$$\Gamma = \frac{V_{or}}{V_{oi}} = \frac{Z_L - Z_0}{Z_L + Z_0} = |\Gamma| e^{j\theta}$$

$$\tau = \frac{V_L}{V_{oi}} = \frac{[V_{oi} + V_{or}]}{V_{oi}} = \frac{[V_{oi} + \Gamma V_{oi}]}{V_{oi}} = 1 + \Gamma$$

- The application of these coefficients should be straightforward with V and I waves (e.g. reflected voltage V_{or} is simply ΓV_{oi}). But what about the reflection and transmission of power?
- Let's examine the incident and reflected power at the boundary between a lossless transmission line and a load (or between two lossless transmission lines with different characteristic impedances). The incident time-averaged power can be expressed by:

$$\langle P_i \rangle = \frac{1}{2} \Re \left\{ \frac{V_0 V_0^*}{|Z_0|} e^{j\theta} \right\} = \frac{1}{2} \frac{|V_0|^2}{|Z_0|} \cos\theta$$

and the reflected time-averaged power by (note the voltage amplitude is simply substituted with the reflected voltage amplitude):

and the reflected time-averaged power by (note the voltage amplitude is simply substituted with the reflected voltage amplitude):

$$\langle P_r \rangle = \frac{1}{2} \mathcal{R}\left\{ \frac{(\Gamma V_0)(\Gamma^* V_0^*)}{|Z_0|} e^{j\theta} \right\} = \frac{1}{2} \frac{|\Gamma|^2 |V_0|^2}{|Z_0|} \cos\theta$$

- The ratio between reflected & incident power is

$$\frac{\langle P_r \rangle}{\langle P_i \rangle} = |\Gamma|^2$$

everything else cancels out!

- Through similar procedures, one can conclude that the transmission coefficient can be expressed as

$$\frac{\langle P_t \rangle}{\langle P_i \rangle} = 1 - |\Gamma|^2$$

Transmission lines A and B with different characteristic impedances, Z_{A0} and Z_{B0}, are connected in series. The reflection coefficient at the connection is given by $\Gamma = 0.2$. For an incident time-averaged power of 10W, find:

Q1 $\langle P_r \rangle$ = 0.4 W $\quad P_i = 10\,W$, $\frac{P_r}{P_i} = 0.2^2 = 0.04$, $P_r = 0.4$

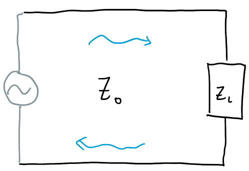
Q2 $\langle P_t \rangle$ = 9.6 W $\quad \frac{P_t}{P_i} = 0.96$, $P_t = 9.6$

Consider the same setup as above but this time $\Gamma = 0.1 + j0.1$. Incident power remains as 10W.

Q3 $\langle P_r \rangle$ = 0.2 W $\quad |\Gamma|^2 = 0.02$, $\frac{\langle P_r \rangle}{\langle P_i \rangle} = 0.02$, $\langle P_r \rangle = 0.2$

Q4 $\langle P_t \rangle$ = 9.8 W $\quad \frac{\langle P_t \rangle}{\langle P_i \rangle} = 0.98$, $\langle P_t \rangle = 9.8$

VSWR – Engineering Setting



Z_0 = characteristic imp.

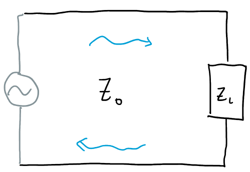
Z_L = load

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = |\Gamma| e^{j\phi}$$

① $Z_L = Z_0$ $\quad \Gamma = 0$ — no reflection

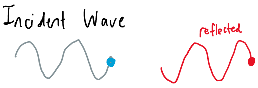
② $Z_L \neq Z_0$

VSWR – Physics Setting



① $Z_L = 0$ short $\Gamma = -1$ from $\left(\frac{-Z_0}{Z_0}\right)$

Incident Wave



② $Z_L \neq 0$

???

VSWR – Math Intuition

VSWR - Math Intuition

$$V_{ST}(z) = V_0 e^{-j\beta z} + \Gamma V_0 e^{j\beta z} = V_0 e^{-j\beta z}\left[1 + \Gamma e^{2j\beta z}\right]$$

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = |\Gamma|\, e^{j\phi}$$

$$V_{ST}(z) = V_0 e^{-j\beta z}\left[1 + |\Gamma|\, e^{j(2\beta z + \phi)}\right]$$

Example: Open Circuit Load:

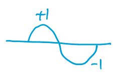
$|\Gamma| = 1$ and $\phi = 0$ $Z_L \to \infty$

$$V_{ST} = V_0 e^{-j\beta z} + 1 V_0 e^{j\beta z} = V_0 \cdot 2\cos(\beta z)$$

Conclusion: $V(z,t) = \mathcal{R}\{V_{ST}\, e^{j\omega t}\} = 2V_0 \cos(\beta z)\cos(\omega t)$

VSWR - Engineering Application

$$V_{ST}(z) = V_0 e^{-j\beta z}\left[1 + |\Gamma|\, e^{j(2\beta z + \phi)}\right]$$



When is the amplitude of V_{ST} max?

$V_0(1 + |\Gamma|)$

$V_0(1 - |\Gamma|)$

Implication: Gives us access to the phase of $\Gamma \to \phi$

VSWR: Voltage Standing Wave Ratio

- We can find our load impedance by measuring the VSWR and the phase of the reflection coefficient

$$VSWR = \frac{1 + |\Gamma|}{1 - |\Gamma|}$$

Q1 VSWR is

> Infinite if all incident waves are reflected

> 1 if there is no reflected wave

Q2 Assuming that you can measure the voltage at any point of a transmission line, how can the amplitude of the reflection coefficient, $|\Gamma|$, be acquired if Z_0 and Z_L are both unknown?

Noticing that VSWR is the ratio between max and min voltage amplitudes, you measure the voltages at multiple points and find the maximum & minimum oscillation amplitudes. You think $|\Gamma| = \frac{S+1}{S-1}$ where S is the VSWR and you successfully calculate $|\Gamma|$ and live happily ever after.

> Noticing that VSWR is the ratio between max and min voltage amplitudes, you measure the voltages at multiple

$\frac{S-1}{\ }$

work and you successfully calculate |Γ| and live happily ever after.

> Noticing that VSWR is the ratio between max and min voltage amplitudes, you measure the voltages at multiple points and find the maximum & minimum oscillation amplitudes. You think $|\Gamma| = \frac{S-1}{S+1}$ where S is the VSWR and you successfully calculate $|\Gamma|$ and live happily ever after.

$S = \frac{1+|\Gamma|}{1-|\Gamma|}$

Don't know

42

Q3 Consider a transmission line that is terminated by a mismatched load such that reflection exist within the line ($Z_L \neq Z_0$). The maximum voltage amplitude has been measured to be 85V, and the minimum voltage amplitude has been measured to be 17V. What is the VSWR?

$\frac{85V}{17V} = 5$

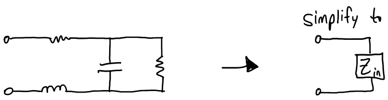
Q4 For a system where VSWR is known to be 7.1, what is the magnitude of the reflection current $|\Gamma|$?

$|\Gamma| = \frac{7.1-1}{7.1+1} = \frac{6.1}{8.1} = 0.753$

1.7 Black Magic Design: the Smith Chart

Input Impedance

Normal Circuits:

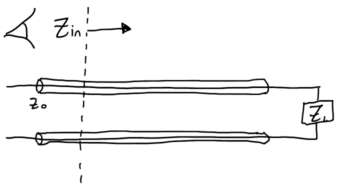


T-Lines:

depends on where you are on the line!

$$Z_{in} = \frac{V}{I} = \frac{V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}}{\frac{V_0^+}{Z_0} e^{-\gamma z} - \frac{V_0^-}{Z_0} e^{\gamma z}} \qquad \gamma = j\beta$$

Input impedance looking into an arbitrary point of a transmission line system looking towards the load



Q1 In the video, we've ended with the input impedance expression

$$Z_{in} = \frac{V}{I} = \frac{V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}}{\frac{V_0^+}{Z_0} e^{-\gamma z} - \frac{V_0^-}{Z_0} e^{\gamma z}}$$

where in this case $\gamma = j\beta$ as we're considering a lossless line, and $\beta = \frac{2\pi}{\lambda}$. Substituting $\beta = \frac{2\pi}{\lambda}$ and $V_0^- = \Gamma V_0^+$ into the expression, which of the following expressions do we acquire?

$Z_{in} = \frac{1}{Z_0} \frac{e^{-j\frac{2\pi}{\lambda}z} + \Gamma e^{j\frac{2\pi}{\lambda}z}}{e^{-j\frac{2\pi}{\lambda}z} - \Gamma e^{j\frac{2\pi}{\lambda}z}}$

$Z_{in} = Z_0 \frac{1+\Gamma}{1-\Gamma}$

> $Z_{in} = Z_0 \frac{e^{-j\frac{2\pi}{\lambda}z} + \Gamma e^{j\frac{2\pi}{\lambda}z}}{e^{-j\frac{2\pi}{\lambda}z} - \Gamma e^{j\frac{2\pi}{\lambda}z}}$

$Z_{in} = Z_0 \frac{(1+\Gamma)e^{j\frac{2\pi}{\lambda}z}}{(1-\Gamma)e^{-j\frac{2\pi}{\lambda}z}}$

$z_{in} = z_0 \frac{\ldots}{(1-\Gamma)e^{-j\beta z}}$

Q2 Consider the Z_{in} expression from the previous answer. When substituting z by $z+\frac{\lambda}{2}$, how does input impedance change?

Hint: what can $e^{j\pi}$ and $e^{-j\pi}$ be simplified to?

> Not at all

The magnitude of the input impedance increases

AC transmission becomes DC

Loss (α) gets introduced

Q3 Assume that the Z_{in} is known at $z=\frac{3}{8}$ for a line that is 1λ long. What is the next location where the input impedance is the same?

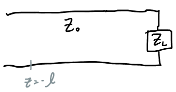
$z=\frac{6}{8}\lambda$

$z=\frac{4}{8}\lambda$

> $z=\frac{5}{8}\lambda$

$z=\frac{7}{8}\lambda$

Constructing the Smith Chart



$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0}$ $\rightarrow |\Gamma| e^{j\phi}$

normalize

① $z_L = \frac{Z_L}{Z_0}$

② $\Gamma = \frac{z_L - 1}{z_L + 1}$

$z_L = \frac{1+\Gamma}{1-\Gamma}$

$V_s(z) = V_0 e^{-j\beta z} + \Gamma V_0 e^{j\beta z}$

$= V_0 e^{-j\beta z}(1 + \Gamma e^{2j\beta z})$

$V_s(z=-\ell) = V_0^+ e^{j\beta\ell}(1 + \Gamma e^{-2j\beta\ell})$

$I_s(z=-\ell) = \frac{V_0^+ e^{j\beta\ell}}{Z_0}(1 - \Gamma e^{-2j\beta\ell})$

wave impedance

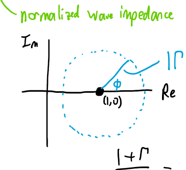
Z_w (at $z=0$) $= Z_0 \cdot \frac{1+\Gamma}{1-\Gamma}$

z_{in} (at $z=0$) $= \frac{1+\Gamma}{1-\Gamma}$

normalized wave impedance

$z_{in} = \frac{V_s(z=-\ell)}{I_s(z=-\ell)} \frac{1}{Z_0} = r + jx$

$= \frac{[1+\Gamma e^{-2j\beta\ell}]}{[1-\Gamma e^{-2j\beta\ell}]} = \frac{[1+|\Gamma| e^{j\phi} e^{-2j\beta\ell}]}{[1-|\Gamma| e^{j\phi} e^{-2j\beta\ell}]}$



real, imaginary

$r + jx = \frac{1+\Gamma}{1-\Gamma} = \frac{1+\Gamma_r+\Gamma_i}{1-\Gamma_r-\Gamma_i}$

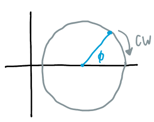
When $\ell = 0 \rightarrow$ starting point

---

at Load: $Z_{in} = \frac{1+\Gamma}{1-\Gamma}$

at any z: $Z_{in} = \frac{1+|\Gamma| e^{j(\phi-2\beta\ell)}}{1-|\Gamma| e^{j(\phi-2\beta\ell)}}$ $\rightarrow \Gamma_r + \Gamma_i$

$\Gamma = |\Gamma| e^{j\phi}$



conjugate

$r + jx = \frac{1+\Gamma_r+j\Gamma_i}{1-\Gamma_r+j\Gamma_i} \times \frac{1-\Gamma_r+j\Gamma_i}{1-\Gamma_r+j\Gamma_i} = \frac{1-\Gamma_r+j\Gamma_i+\Gamma_r-\Gamma_r^2+\Gamma_r \times j\Gamma_i+j\Gamma_i-j\Gamma_i\Gamma_r-\Gamma_i^2}{(1+\Gamma_r)^2+\Gamma_i^2}$

$= \frac{1+2j\Gamma_i-\Gamma_r^2-\Gamma_i^2}{(1+\Gamma_r)^2+\Gamma_i^2}$

$r = \frac{1-\Gamma_r^2-\Gamma_i^2}{1-2\Gamma_r+\Gamma_r^2+\Gamma_i^2}$ $x = \frac{2\Gamma_i}{1-2\Gamma_r+\Gamma_r^2+\Gamma_i^2}$

$r\Gamma_r^2 + \Gamma_r^2 - 2r\Gamma_r + r\Gamma_i^2 = 1$

$\Gamma_r^2(r+1) - 2r\Gamma_r + \Gamma_i^2(r+1) = 1-r$

$\Gamma_r^2 - \frac{2r\Gamma_r}{r+1} + \Gamma_i^2 = \frac{1-r}{r+1}$

$\Gamma_r^2 - \frac{2r}{r+1}\Gamma_r + \left(\frac{r}{r+1}\right)^2 + \Gamma_i^2 = \frac{(1-r)}{(1+r)} + \frac{r^2}{(r+1)^2} = \frac{\overbrace{(1-r)(1+r)}^{1-r^2} + r^2}{(1+r)^2} = \frac{1}{(r+1)^2}$

$$\Gamma_r^2 - \frac{2r\Gamma_r}{r+1} + \Gamma_i^2 = \frac{1-r}{r+1}$$

$$\Gamma_r^2 - \frac{2r}{r+1}\Gamma_r + \left(\frac{r}{r+1}\right)^2 + \Gamma_i^2 = \frac{(1-r)}{(1+r)} + \frac{r^2}{(r+1)^2} = \frac{\overbrace{(1-r)(1+r)}^{1-r^2} + r^2}{(1+r)^2} = \frac{1}{(r+1)^2}$$

$$\left(\Gamma_r - \frac{r}{r+1}\right)^2 + \Gamma_i^2 = \frac{1}{(r+1)^2}$$

two families of circles!

Using the Smith Chart

$Z_L = 50 + j100\,\Omega$, $Z_0 = 50\,\Omega$ | $Z_L = 25 - j50\,\Omega$, $Z_0 = 50\,\Omega$

$z_L = 1.0 + j2.0$ | $z_L = 0.5 - j1.0$

$\boxed{A}$

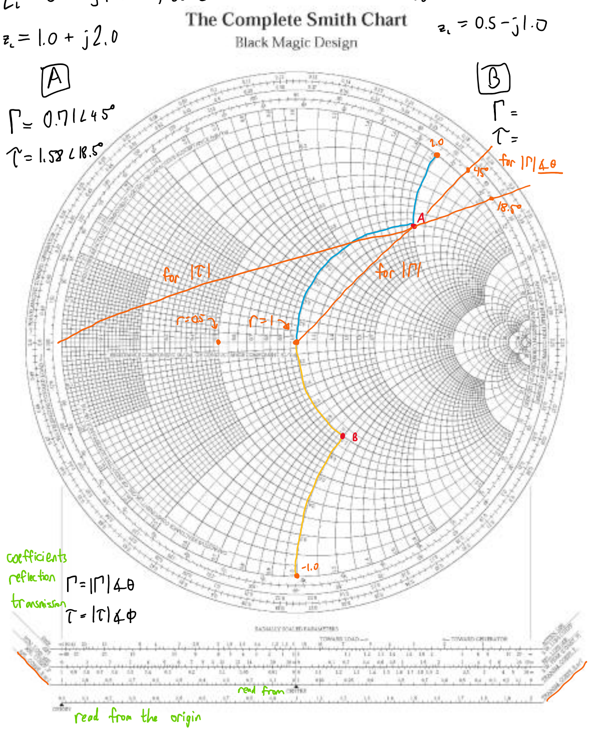
$\Gamma = 0.71\angle 45°$

$\tau = 1.58\angle 18.5°$

$\boxed{B}$

$\Gamma =$

$\tau =$



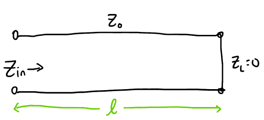
Admittances on the Smith Chart, Necessary for Analysis

Admittance $Y = \frac{1}{Z}$

$Y = G + jB$ (G: conductance, B: susceptance)

$Z = R + jX$ (R: resistance, X: reactance)

Short Circuit Load on T-Line — Analysis



$$z_L = \frac{Z_L}{Z_0} = 0 \qquad y_L = \frac{1}{z_L} = \infty$$

$$|\Gamma| = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{z_L - 1}{z_L + 1} = -1$$

$$z_{in} = \frac{1 + \overset{=-1}{\Gamma} e^{-2j\beta\ell}}{1 - \underset{=-1}{\Gamma} e^{-2j\beta\ell}} = \frac{e^{j\beta\ell} - e^{-j\beta\ell}}{e^{j\beta\ell} + e^{-j\beta\ell}} = \frac{2j\sin\beta\ell}{2\cos\beta\ell} = j\tan\beta\ell$$

keep cot in "same quadrant"

$$y_{in} = \frac{1}{z_{in}} = \frac{1}{j\tan\beta\ell} = -j\cot\beta\ell \qquad 0 < \ell < \frac{\lambda}{4} \qquad \ell = 0,\ z_{in} = 0,\ y_{in} = \infty!$$

Stub Matching

Max V $\quad V = V_0^+ e^{-j\beta z} + \Gamma V_0^+ e^{j\beta z}$

Min I $\quad I = \dfrac{V_0^+ e^{-j\beta z} - \Gamma V_0^+ e^{j\beta z}}{Z_0}$

Get ratio

$$V_{max} = V_0^+ e^{-j\beta z}\left(1 + \Gamma e^{2j\beta z}\right) \longrightarrow V_0^+ \left(1 + |\Gamma|\right)$$

$$I_{min} = \frac{V_0^+ e^{-j\beta z}\left(1 - \Gamma e^{2j\beta z}\right)}{Z_0} \longrightarrow \frac{V_0^+ \left(1 - |\Gamma|\right)}{Z_0}$$

$$Z_{max} = \frac{V_{max}}{I_{min}} = \frac{1+|\Gamma|}{1-|\Gamma|}\left(Z_0\right)$$

$$z_{max} = \frac{Z_{max}}{Z_0} = \frac{1+|\Gamma|}{1-|\Gamma|}$$ normalized