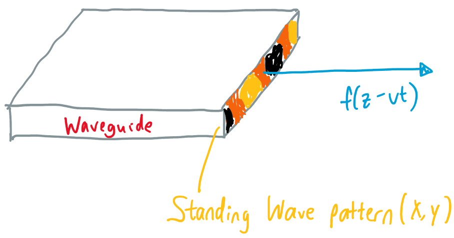
3.2-3.4 Waveguides

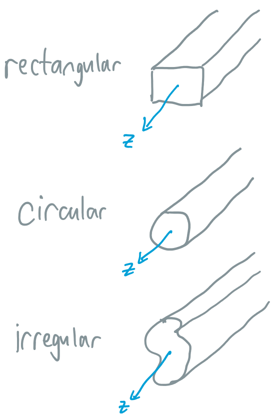
Monday, November 20, 2023 10:04 PM

Fields Formulation

- Waves in a waveguide: cross-section components & travelling component
- Cross-section components are standing waves, denoted as Standing wave pattern (x,y)
- Travelling component travelling down $+\hat{z}$, denoted as $f(z-vt)$



Building a Waveguide



$$\vec{E} = \vec{E}_\perp + \hat{z}E_z, \quad \vec{H} = \vec{H}_\perp + \hat{z}H_z$$

$$\vec{\nabla} = \vec{\nabla}_\perp + \hat{z}\frac{\partial}{\partial z}$$

transverse $= -j\omega\mu H_\perp$

normal

$$\vec{\nabla} \times \vec{E} = \left(\vec{\nabla}_\perp + \hat{z}\frac{\partial}{\partial z}\right) \times \left(\vec{E}_\perp + \hat{z}E_z\right) = -j\omega\mu\left(H_\perp + \hat{z}H_z\right) \quad (A)$$

$$\vec{\nabla} \times \vec{H} = \left(\vec{\nabla}_\perp + \hat{z}\frac{\partial}{\partial z}\right) \times \left(\vec{H}_\perp + \hat{z}H_z\right) = j\omega\varepsilon\left(\vec{E}_\perp + \hat{z}E_z\right) \quad (B)$$

$$\left(\vec{\nabla}_\perp \times \hat{z}E_z\right) + \left(\hat{z}\frac{\partial}{\partial z} \times \vec{E}_\perp\right) = -j\omega\mu H_\perp \quad (C)$$

$$\left(\vec{\nabla}_\perp \times \hat{z}H_z\right) + \left(\hat{z}\frac{\partial}{\partial z} \times \vec{H}_\perp\right) = j\omega\varepsilon \vec{E}_\perp \quad (D)$$

Sub (D) to (C) Looking at traverse — no z direction

$$\left(\vec{\nabla}_\perp \times \hat{z}E_z\right) + \left(\hat{z}\frac{\partial}{\partial z} \times \frac{(\vec{\nabla}_\perp \times \hat{z}H_z) + (\hat{z}\frac{\partial}{\partial z} \times \vec{H}_\perp)}{j\omega\varepsilon}\right) = -j\omega\mu\vec{H}_\perp$$

$$j\omega\varepsilon\left(\vec{\nabla}_\perp \times \hat{z}E_z\right) + \frac{\partial}{\partial z}\hat{z} \times \left(\vec{\nabla}_\perp \times \hat{z}H_z + \frac{\partial}{\partial z}\hat{z} \times \vec{H}_\perp\right) = \omega^2\mu\varepsilon\vec{H}_\perp$$

replace with jk_z

BAC-CAB

$$\hat{z} \times \left(\vec{\nabla}_\perp \times \hat{z}\right) = \vec{\nabla}_\perp\overbrace{(\hat{z}\cdot\hat{z})}^{1} - \hat{z}\overbrace{(\vec{\nabla}_\perp\cdot\hat{z})}^{0} = \vec{\nabla}_\perp$$

$$\hat{z} \times \left(\hat{z} \times \vec{H}_\perp\right) = \hat{z}\underbrace{(\hat{z}\cdot\vec{H}_\perp)}_{0} - \vec{H}_\perp\underbrace{(\hat{z}\cdot\hat{z})}_{1} = -\vec{H}_\perp$$

Equating z components from (A) and (B)

Equating z components from Ⓐ and Ⓑ

$\vec{\nabla}_\perp \times \vec{E}_\perp = -j\omega\mu H_z \hat{z}$ Ⓔ

$\vec{\nabla}_\perp \times \vec{H}_\perp = j\omega\varepsilon E_z \hat{z}$ Ⓕ

Sub $\vec{E}_\perp + \vec{H}_\perp$ into Ⓔ and Ⓕ to get: (Helmholtz Equation)

$[\nabla_\perp^2 + (k^2 - k_z^2)] H_z = 0$ TE Waves

$[\nabla_\perp^2 + (k^2 - k_z^2)] E_z = 0$ TM Waves

$k_x^2 + k_y^2 + k_z^2 = k^2 = \omega^2 \mu\varepsilon$

$k^2 - k_z^2 = k_x^2 + k_y^2$ (see in-class 11-29-2023 ⓞ)

TE: $H_z = e^{-jk_z z} f(x,y)$

$f(x,y) = (A\cos k_x x + B\sin k_x x)\cdot(C\cos k_y y + D\sin k_y y)$
(see in-class 11-29-2023 ⊛)

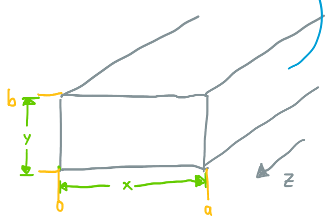
along (tangential to) the side wall

@ $x=0$, $x=a$, $E_y = 0 \rightarrow E_y \propto \frac{\partial H_z}{\partial x} = 0$
See in-class 11-29-2023 △

@ $x=0$ B must be 0

@ $x=a$ $-A\sin k_x a = 0$

Sides are perfect conductors by Coulomb's Law



$E_x = \frac{\partial H_z}{\partial y} = 0$

@ $y=0$, $D=0$

@ $y=b$, $k_y = \frac{n\pi}{b}$, $n \in \mathbb{Z}$

(see in-class 12-06-2023 ☺)

> If A is 0, everything becomes 0. So

$A \neq 0$, $k_x = \frac{m\pi}{a}$, $k_y = \frac{n\pi}{b}$, $m, n \in \mathbb{Z}$

TE: $H_z = G\cos k_x x \cos k_y y e^{-jk_z z}$

$k_x = \frac{m\pi}{a}$ $k_y = \frac{n\pi}{b}$

$k_x^2 + k_y^2 + k_z^2 = ? = k^2 = \omega^2\mu\varepsilon \rightarrow$ a sphere

k_z depends on geometry. But there is a square root, so k_z can be imaginary. So we won't have energy going down the waveguide. Cutoff

Q1 BAC-CAB Rule is a way of decoupling rotations. True

Q2 Components of $\vec{E}$ tangential to the surface of the waveguide must vanish. True

Cutoff

Cutoff

$$k_x^2 + k_y^2 + k_z^2 = k^2 = \omega^2 \mu \varepsilon = \frac{\omega^2}{v^2} \quad , \quad v = \frac{c}{n}$$

$$k_z = \sqrt{\frac{\omega^2 n^2}{c^2} - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

At $k_z = 0$

no propagation

CUT-OFF

$$\omega_{c,mn} = \frac{c}{n}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

$$\lambda_{c,mn} = \frac{v}{f_{c,mn}} = \frac{c}{n}\,\frac{2\pi}{\omega_{c,mn}}$$

comes from the source

Interpretation of Cutoff

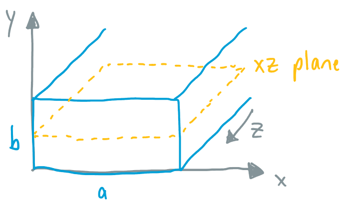
Example: TE_{10} $[m=1,\ n=0]$ dominant mode $\left(k_z = \sqrt{\underbrace{\omega^2\mu\varepsilon}_{k^2} - k_x^2} \text{ where } k_x = \frac{m\pi}{a}\right)$

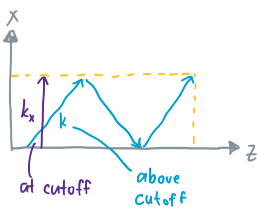
Along guide, $\lambda_g = \dfrac{2\pi}{k_z} = \dfrac{2\pi}{\sqrt{\left(\frac{2\pi f}{c}\right)^2 - \left(\frac{m\pi}{a}\right)^2}}$ $\longrightarrow$ $m=1,\ v=c,\ \frac{v}{f} = \lambda_0$

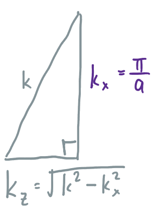
$$= \frac{2\pi}{2\pi\sqrt{\frac{1}{\lambda_0^2} - \left(\frac{1}{2a}\right)^2}}$$

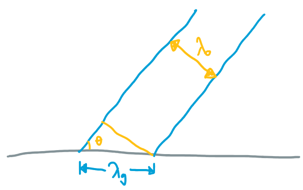
$$= \frac{\lambda_0}{\sqrt{1 - \left(\frac{\lambda_0}{2a}\right)^2}}$$

Recall — Counting wave crests at the beach (2.6)









$$k_z = \sqrt{\omega^2\mu\varepsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

$$\lambda_g = \frac{2\pi}{k_z}$$

Apps 1 – TE

Phase Velocity : $v_p = \dfrac{\omega}{k_z} = \dfrac{\omega}{\sqrt{\omega^2\mu\varepsilon - k_\perp^2}} \cdot \dfrac{\frac{1}{\omega\sqrt{\mu\varepsilon}}}{\frac{1}{\omega\sqrt{\mu\varepsilon}}}$ $\quad k_\perp^2 = k_x^2 + k_y^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$

$$V_p = \frac{1}{\sqrt{\mu\varepsilon}} \cdot \frac{1}{\underbrace{\sqrt{1-\frac{k_z^2}{\omega^2\mu\varepsilon}}}_{\text{less than 1!}}} > \frac{1}{\sqrt{\mu\varepsilon}}$$

in air: $\frac{1}{\sqrt{\mu_0\varepsilon_0}} = c$

phase velocity $> c$? Math is correct, and not violating Einstein's relation since a point has no energy

Group velocity: $V_g = \frac{d\omega}{dk_z} = \left(\frac{dk_z}{d\omega}\right)^{-1} \rightarrow V_p \cdot V_g = \frac{1}{\mu\varepsilon} = v^2$

In a vacuum, c^2.
For waves, it will be the group speed
$\rightarrow V_g \neq c$

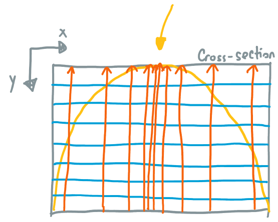
Transverse Impedance: $Z^{TE} = \frac{\omega\mu}{k_z} = \frac{E_x}{H_y} = \frac{-E_y}{H_x}$

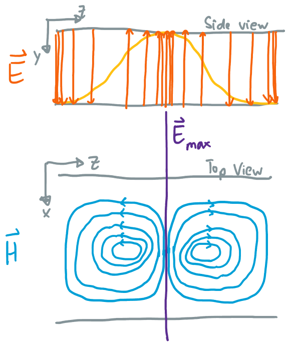
Apps 2 — TE10 Shape

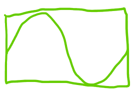
$$E_{ys} = E_0 \sin\left(\frac{m\pi x}{a}\right) e^{-jk_z z}$$

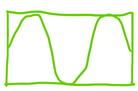
1

$$k_z = \sqrt{\omega^2\mu\varepsilon - \left(\frac{m\pi}{a}\right)^2}$$

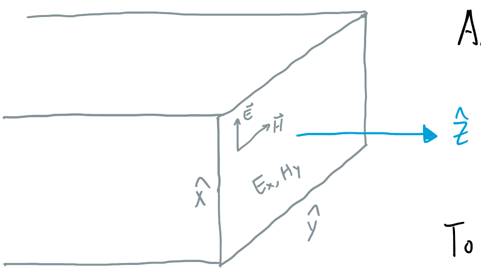




TE20: 

TE30: 

In-class 11-22-2023



$E_z = 0$
by definition
of transverse
in TEM

Ampère: $\vec{\nabla}\times\vec{H} = \overbrace{\cancel{\sigma\vec{E}}}^{\text{ohmic}} + \overbrace{\varepsilon\frac{\partial\vec{E}}{\partial t}}^{\text{displacement}}$

conductivity in middle of waveguide
= conductivity of air
= 0

To generate $\vec{H}$, we need a current. (By RHR (curling hand))

If is an $\vec{H}$ into plane, there should be a current, in $\hat{z}$ direction.

But there's no E_z, so there's no current. $\rightarrow \varepsilon\frac{\partial\vec{E}}{\partial t} = 0 = \vec{\nabla}\times\vec{H}$

We assumed $\vec{\nabla} \times \vec{H} \neq 0$ because there's a current, but there's no current. So we can't have TEM (plane) waves down the guide.

How do we get energy down the guide?

> Plane waves bounce off sides of waveguide (ping pong), so we get transverse standing waves.

From TE part above:

$$\vec{k} \cdot \vec{k} = \omega^2 \mu \varepsilon$$

$$k_x^2 + k_y^2 + k_z^2 = \omega^2 \mu \varepsilon$$

$$k_z = \pm \sqrt{\omega^2 \mu \varepsilon - \underbrace{k_x^2 - k_y^2}_{\text{depends on boundary conditions of the side of the waveguide}}}$$

When frequency is low enough, k_z will be imaginary $\longrightarrow$ Cut off

In-class 11-29-2023

$$H_z = A(x,y)\, \underbrace{e^{j\omega t - jk_z z}}_{\text{one-dimensional}} \qquad ⓪$$

$$k_x^2 + k_y^2 + k_z^2 = k^2 = \omega^2 \mu \varepsilon$$

$$k_z^2 = \underbrace{k^2 - k_x^2 - k_y^2}_{> 0}$$

We don't want this to be imaginary or we get exponential decay

$$k_z^2 = \omega^2 \mu \varepsilon - \overbrace{k_x^2 - k_y^2}^{\text{fixed by geometry}}$$

$$\omega^2 \mu \varepsilon = -k_x^2 - k_y^2$$

$\omega \rightarrow$ cut off frequency

$$\frac{\partial^2}{\partial x^2} H_z + \cancel{\frac{\partial^2}{\partial y^2}} H_z + (k_x^2 + \cancel{k_y^2}) H_z = 0 \qquad ⊛$$

Assume no y dependence

$$\frac{\partial^2}{\partial x^2} H_z + \cancel{\frac{\partial^2}{\partial y^2}} H_z + (k_x^2 + \cancel{k_y^2}) H_z = 0$$

Assume no y dependence

$(*)$

$$\frac{\partial^2 H_z}{\partial x^2} + k_x^2 H_z = 0$$

$$\frac{\partial^2 H_z}{\partial x^2} = -k_x^2 H_z$$

$$H_z = C \sin k_x x + D \cos k_x x$$

$$\vec{\nabla} \times \vec{H} = j\omega\varepsilon\vec{E} \quad ; \quad H_x, H_y = 0$$

(Δ)

$$\begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial x & \partial y & \partial z \\ 0 & 0 & H_z \end{bmatrix} \longrightarrow E_y \propto \frac{\partial H_z}{\partial x}$$

In-class 12-01-2023

TE_{10} lowest frequency, highest energy

$\frac{n\pi}{b}$

$\frac{m\pi}{a}$

How does a wave get transmitted down the waveguide?

> It bounces off the walls

For TEM waves, none gets transmitted. No energy goes to the end of the waveguide. $k_z = 0$

$$V_p V_g = \frac{\omega}{k_z} \frac{d\omega}{dk_z} = \sqrt{\frac{\omega^2}{\omega^2\mu\varepsilon - k_\perp}}$$

$$V_g = \frac{1}{\sqrt{\mu\varepsilon}} \sqrt{1 - \frac{k_\perp^2}{\omega^2\mu\varepsilon}}$$

$$V_g = \frac{d\omega}{dk_z} = \left(\frac{dk_z}{d\omega}\right)^{-1}$$

$$= \left(\frac{d}{d\omega}\left(\omega\sqrt{\frac{1}{\omega^2\mu\varepsilon - k_\perp}}\right)\right)^{-1}$$

$$= \left(\omega\left(-\left(\frac{1}{\omega^2\mu\varepsilon - k_\perp}\right)\left(2\omega\mu\varepsilon - k_\perp dk_\perp\right)\right) - \sqrt{\frac{1}{\omega^2\mu\varepsilon - k_\perp}}\right)$$

$(\omega (\; (\omega^2 \mu \varepsilon - k_\perp)(\omega \mu \varepsilon - k_\perp a k_\perp)) \quad \sqrt{\omega^2 \mu \varepsilon - k_\perp})$

In-class 12-06-23 LAST LECTURE

$$f(x,y) = (A\cos(k_x x) + B\sin(k_x x)) \cdot (C\cos(k_y y) + D\sin(k_y y))$$

$$\frac{\partial f}{\partial y} = -k_y C \sin k_y y + k_y D \cos k_y y$$

If $k_y = 0$, get a wave with infinite wavelength

→ DC, will not propagate

@ $y = 0$

$$\frac{\partial f}{\partial y} = -k_y(0) + k_y D(1)$$

D should be 0

@ $y = b$

C cannot be 0 (see why A can't be 0)

$\sin k_y b \rightarrow k_y b = n\pi$