2.3 1D Plane Waves

Saturday, October 14, 2023 7:33 PM

EM Wave Equations — Applications

$$\frac{\partial}{\partial z}\frac{\partial E_x}{\partial z} = -\mu_0 \boxed{\frac{\partial}{\partial z}\frac{\partial H_y}{\partial t}}$$

$$\boxed{\frac{\partial}{\partial t}\frac{\partial H_y}{\partial z}} = -\varepsilon_0 \frac{\partial}{\partial t}\frac{\partial E_x}{\partial t}$$

$V \leftrightarrow E_x$ $\quad Z_0 = \sqrt{\frac{L}{C}}$ $\quad \eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}$

$I \leftrightarrow H_y$ $\quad v = \frac{1}{\sqrt{LC}}$ $\quad v = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$

$$\frac{\partial^2 E_x}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 E_x}{\partial t^2} \qquad E_x(z,t) = E_{xs}e^{j\omega t}$$

$$\frac{d^2 E_{xs}}{dz^2} = \frac{-\omega^2}{v_0^2}E_{xs} = -k_0^2 E_{xs}$$

$$E_{xs} = E_{x0}^{+}e^{-jk_0 z} + E_{x0}^{-}e^{jk_0 z}$$

incorporates forward and backward waves

$$I_s = \frac{V^+}{Z_0}e^{-j\beta z} - \frac{V^-}{Z_0}e^{j\beta z}$$

$$H_{ys} = \frac{E_{x0}^{+}}{\eta_0}e^{-jk_0 z} - \frac{E_{x0}^{-}}{\eta_0}e^{jk_0 z}$$

Q1 What is the unit of k_0 as used above? $\frac{1}{m}$

Q2 What is the physical interpretation of k_0? Spatial density of wave crests

Intrinsic Impedance of Conductive Material

ε_0 is boring! (We are not in a vacuum)

$\varepsilon_0 \to \varepsilon$

$$\underbrace{\vec{\nabla} \times \vec{H}}_{\frac{1}{m}\cdot\frac{A}{m}} = \sigma\vec{E} + \varepsilon\frac{\partial \vec{E}}{\partial t} \qquad \to \vec{J} = \sigma\vec{E} \quad (I,\ V)$$

now, in conductive materials, $\sigma \neq 0$

$$\vec{\nabla} \times \vec{H}_s e^{j\omega t} = \sigma\vec{E}_s + j\omega\varepsilon\vec{E}_s$$

$$= \underbrace{(j\omega\varepsilon + \sigma)}_{j\omega\varepsilon_{effective}} E_s$$

$$j\omega\varepsilon_{eff} = j\omega\varepsilon + \sigma$$

$$\varepsilon_{eff} = \varepsilon + \frac{\sigma}{j\omega}$$

$$= \varepsilon - j\frac{\sigma}{\omega}$$

$$\eta = \sqrt{\frac{\mu_0}{\varepsilon - j\frac{\sigma}{\omega}}}$$

$\sqrt{\varepsilon - j\frac{\sigma}{\omega}}$

Q1 In a non-conductive material with $\sigma = 0$, ε_{eff} is: Purely real

Q2 Continuing with the same material, is it lossy to EM waves? No

Q3 In a conductive material with $\sigma > 0$, ε_{eff} is: Complex

Q4 Continuing with the same material, is it lossy to EM waves? Yes

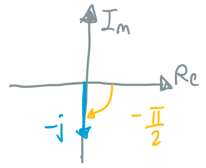
Wavenumber (k) in Conductive Material

Vacuum: $\frac{d^2 E_{xs}}{dz^2} = -k_o^2 E_{xs}$ $\qquad jk_o = j\omega\sqrt{\mu_o \varepsilon_o}$

$\mu_o \to \mu$, $\varepsilon_o \to \varepsilon_{eff} \to \varepsilon - j\frac{\sigma}{\omega}$

$$
\begin{aligned}
jk = j\omega\sqrt{\mu\,\varepsilon_{eff}} &= j\omega\sqrt{\mu}\sqrt{\varepsilon - j\tfrac{\sigma}{\omega}} \\
&= j\omega\sqrt{\mu}\sqrt{\varepsilon}\sqrt{\cancel{1} - \tfrac{j\sigma}{\omega\varepsilon}} \\
&= j\omega\sqrt{\mu(-j)\tfrac{\sigma}{\omega}} \\
&= j\sqrt{(-j)\mu\sigma\omega} \\
&= j\left(\tfrac{1}{\sqrt{2}} - j\tfrac{1}{\sqrt{2}}\right)\sqrt{\mu\sigma\omega} \\
&= (1+j)\sqrt{\tfrac{\mu\sigma\omega}{2}}
\end{aligned}
$$

very good conductor, σ is very large — ignore the 1



$\sqrt{-j} = \sqrt{e^{-j\frac{\pi}{2}}} = e^{-j\frac{\pi}{4}}$

$= \cos\left(\frac{\pi}{4}\right) + j\sin\left(-\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$

Q1 What is $\sqrt{j}$? $\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}$

Q2–4

Non-conductive Material Properties ($\sigma = 0$)

- Attenuation Coefficient (α) $= 0$
- Phase Constant (β) > 0
- Lossy to EM wave? No

Q5-7

Conductive Material Properties ($\sigma > 0$)

- Attenuation Coefficient (α) > 0
- Phase Constant (β) > 0
- Lossy to EM wave? Yes

Skin Depth and Power

$jk = (1+j)\sqrt{\frac{\omega\mu\sigma}{2}} = \alpha + j\beta$ $\qquad e^{-jkz} = e^{-(\alpha + j\beta)z}$

$\to \alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}}$ $\qquad E_{xs} = E_{xo}e^{-(\alpha + j\beta)z} = E_{xo}e^{-\alpha z}e^{-j\beta z}$

$jk = (1+j)\sqrt{\quad 2} = \alpha + j\beta \qquad e \quad = e$

$$\rightarrow \alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}} \qquad E_{xs} = E_{x0}\, e^{-(\alpha + j\beta)z} = E_{x0}\, e^{-\alpha z} e^{-j\beta z}$$

$$\alpha z = 1 \rightarrow z = \frac{1}{\alpha} = \sqrt{\frac{2}{\omega\mu\sigma}} = \overset{\text{delta}}{\underline{\delta}} \text{ Skin Depth}$$

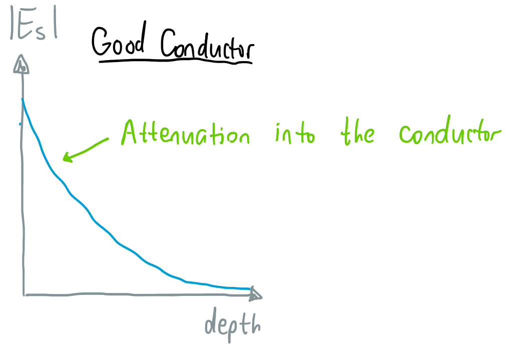
$$\delta_{Cu} = \frac{0.066}{\sqrt{f}}\ \text{m} \qquad f = 60\,\text{Hz} \rightarrow \delta = 9\,\text{mm} \qquad H_{ys} = \frac{E_{x0}}{\eta} e^{-\alpha z} e^{-j\beta z} \qquad \eta = \sqrt{\frac{\mu}{\varepsilon_0}} = \sqrt{\frac{\mu}{-j\frac{\sigma}{\omega}}}$$

1 is ignored, $\sigma \gg 1$

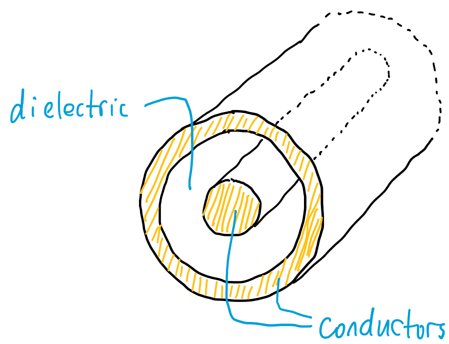
$$\langle \vec{S} \rangle = \frac{1}{2} \mathrm{Re}\left(E_{xs} \cdot H_{ys}^{*} \right)$$

Skin Depth Visualization

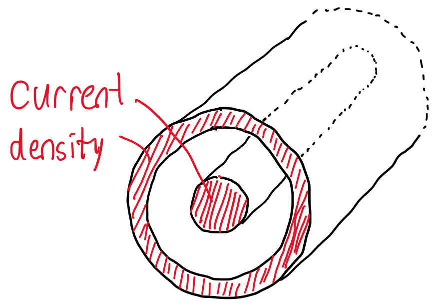
- In a simplistic 1D picture, as an EM wave travels into a conductor, power is dissipated as heat and the oscillation amplitude of the E and H fields decay exponentially with depth



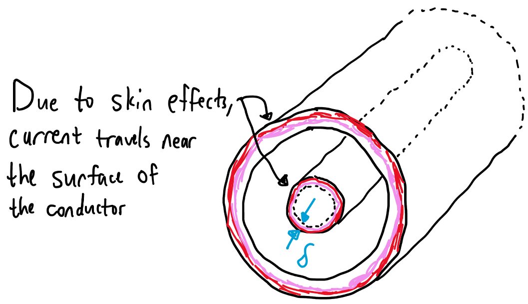
- When considering more complicated geometries such as a coaxial cable, this frequency-dependent decay also plays a role. Consider the following coaxial geometry with a center conductor, an outer conductor (grounded), and dielectric between the conductors:



- When no skin is present, one would expect the entire cross-section of the cable to carry current:



- However, due to skin effects, the current can only travel near the surface of the conductor when the frequency is sufficiently high:



Which means the effective cross-section for current flow is reduced under the effect of skin effect.

Resistance

- Resistance of a conductor is defined by $R = \frac{\rho L}{A}$ where ρ is the resistivity (inverse of conductivity), L is the conductor length, and A is the conductor cross-section area. Resistance per length can be expressed as:

$$\frac{R}{L} = \frac{1}{\sigma A}$$

Q1 An EM wave with a frequency of 28.8 kHz is traveling in a copper conductor with the following properties:

with the following properties:

$\sigma = 5.96 \times 10^{7}$ S/m

$\mu = \mu_0 = 4\pi \times 10^{-7}$ H/m

$$\delta = \sqrt{\frac{2}{\omega\mu\sigma}} = \sqrt{\frac{2}{2\pi f \mu_0 \sigma}} = \sqrt{\frac{2}{2\pi(28800)(4\pi\times10^{-7})(5.96\times10^{7})}}$$

What is the skin depth of this copper conductor in cm? 3.84×10^{-2} cm

Q2 When considering current traveling on the center conductor with radius r under noticeable skin effect, the conduction cross-section area can be expressed as...

> $A = \pi r^2 - \pi (r-\delta)^2$

$A = \pi \delta^2$

$A = \pi (r-\delta)^2$

$A = \pi r^2$

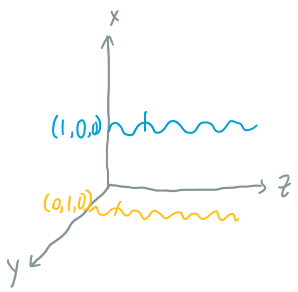
Q3 If the skin effect on the center conductor is so prominent that $\delta \ll r$, the conduction area can be approximated by...

An octagon with the sides taking the length of r, $A \approx 2(1+\sqrt{2})r^2$

> A long rectangular strip, $A \approx 2\pi r \delta$

A circle, $A \approx \pi r^2$

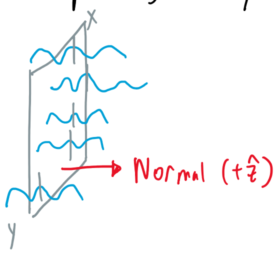
In-Class 10-18-23 (Covering 2.2)



How do you eliminate this functional dependence?

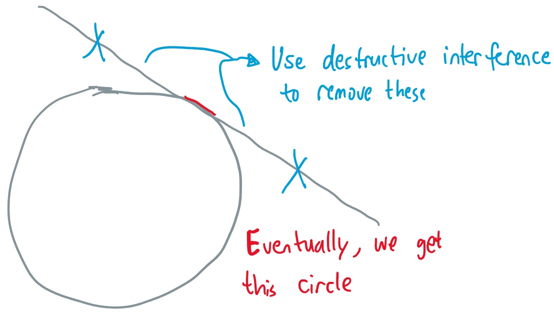
- infinite # of strings, in-phase, travelling in the +z direction

- Look at all the second peaks, and you see a plane! (At that snapshot)

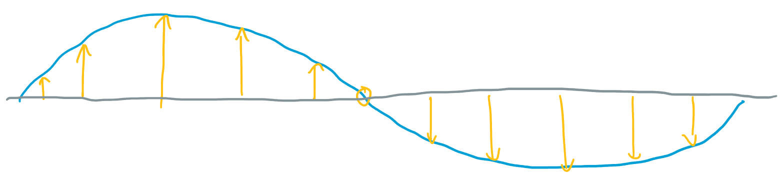


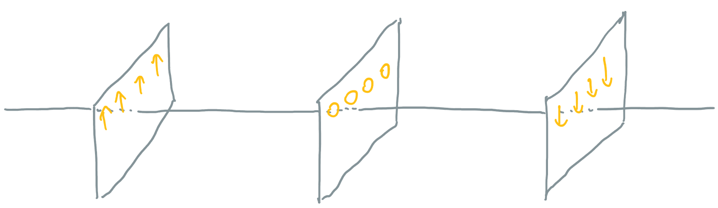
- The plane carries information of the wave in the z-direction
  - We can model in 3D using vibrating strings

- Charges are points, and they are spherical
  When spheres get very very large, we can model this as a plane

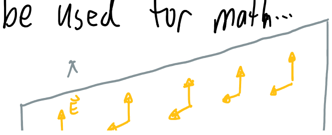


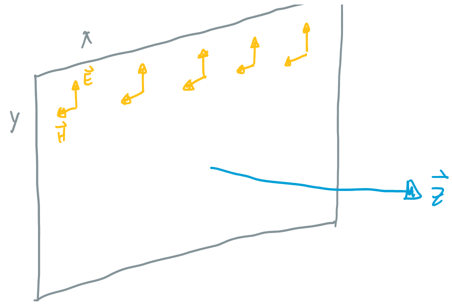
- But don't forget vectors!





- If your position is fixed, you'll see the change in time
  - Waves of different phases will pass
- If you stay on the plane, the phase will be constant, "freeze" the coordinates. The phase has a space & time component which will be used for math...





- **The Math**

1D: $\sin(\overbrace{\omega t - kz}^{\text{wave densities in space — scalar}})\ e^{j(\omega t - kz)}$ $\quad t\uparrow,\ z\uparrow$ by $z = vt$, phase remains constant

$\phi = \omega t - kz = \text{constant}$

Big step: need to count waves, but also need to add direction

↳ in 3D $\quad k$ becomes a vector!

but want this to be true

3D: $\phi = \underbrace{\omega t - \vec{k} z}_{} \neq \text{constant}$

$\rightarrow \omega t - \vec{k} \cdot \vec{z} = \text{constant}$

normals to the plane

At $t = 0$: $\quad$ 2 spaces

$\phi = -\vec{k} \cdot \vec{z} = \text{constant} \rightarrow \underbrace{ax + by + cz}_{\text{a plane}} = \text{constant}$

0

wavenumber

Collection of normals in the plane.

Dimension: $\frac{1}{m}$

GEOMETRIC OPTICS

- e.g. Snell's law, ...

In-class 10-20-23

1D $\quad f(z - vt) \rightarrow f(\underbrace{\omega t - \beta z}_{}) \rightarrow e^{j\omega t} e^{-j\beta z}$

↳ constant

sitting at a point on a wave

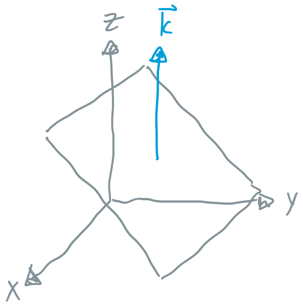
In 3D: sitting on a plane

3D $\quad \omega t - \vec{k}\vec{z} = \text{Constant}$

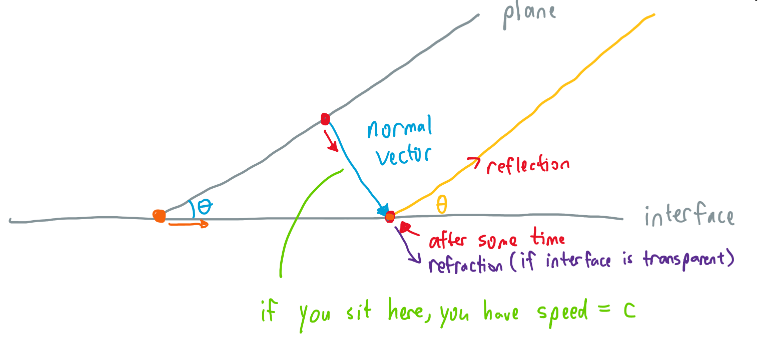
$\vec{k} \cdot \vec{z} = \text{constant} \qquad ax + by + cz = \text{constant}$

$\vec{k}\cdot\vec{z} = \text{constant}$ $\qquad ax + by + cz = \text{constant}$

$\vec{k}\cdot\vec{r} = \text{constant}$ $\qquad \vec{k} = (k_x, k_y, k_z)$, $\vec{r} = (x, y, z)$



- We can describe our magnetic fields as a combination of planes



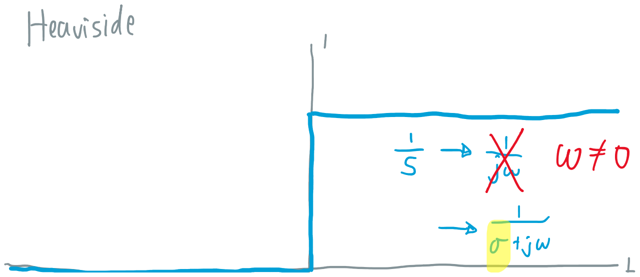
> Energy is carried by normal of the plane

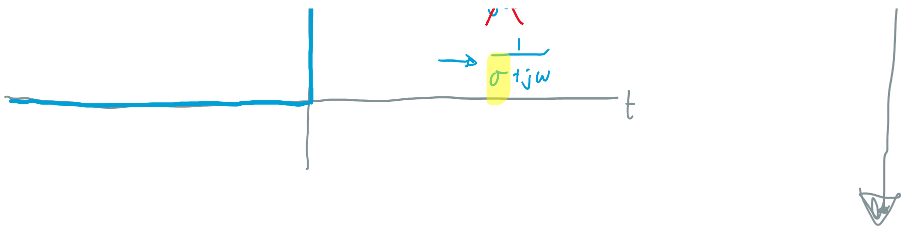
> Components of k are not physical

> Energy associated with a point is ≈ 0
> > Area of a point = 0
> > Energy needs an area (Recall $V = \frac{V}{m}$, $I = \frac{A}{m}$)

> Points don't matter
> > No step functions
> > No (true) DC (DC doesn't exist)

Heaviside





- The interface is where the incident, reflected and transmitted waves all coexist (at an instant)

- $\vec{E} = \vec{E}_0 e^{j(\omega t - \vec{k}\cdot\vec{r})}$

  $\vec{E}_0$ → amplitude, associated with the same phase

  → constant for ELEC 311

  $\vec{E} = \vec{E}^0 e^{j\omega t} e^{-jk_x x} e^{-jk_y y} e^{-jk_z z}$

- Ohmic Current

Ampere's $\vec{\nabla} \times \vec{H} = \varepsilon \frac{\partial \vec{E}}{\partial t} + \sigma \vec{E}$

σ: S/m, $\vec{E}$: V/m, $\frac{V}{R}$

$\frac{\partial \vec{E}}{\partial t}$: displacement current → $\frac{A}{m^2}$

$\vec{\nabla} \times \vec{H}$ → $\frac{\partial}{\partial x} \times \frac{A}{m}$ → $\frac{A}{m^2}$

In phasor:

$$\vec{\nabla} \times \vec{H} = \varepsilon j\omega \vec{E} + \sigma \vec{E}$$

$$= (j\omega \varepsilon + \sigma)\vec{E}$$

$$= j\omega\left(\varepsilon + \frac{\sigma}{j\omega}\right)\vec{E}$$

$$\vec{\nabla} \times \vec{H} = j\omega \varepsilon_{eff} \vec{E} \qquad \varepsilon_{eff} = \varepsilon + \frac{\sigma}{j\omega}$$

TL: $Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$ H $L \to \mu$ H/m, F $C \to \varepsilon$ F/m, $R = G = 0 \to Z_0 = \sqrt{\frac{L}{C}}$

EM: $\eta = \sqrt{\frac{\mu}{\varepsilon}}$

$= \sqrt{\frac{\mu}{\varepsilon_{eff}}}$

TL: $K = \omega\sqrt{LC}$ replace with the magic number $\quad K/\omega = \sqrt{LC} \to v^{-1} = \sqrt{LC}$

TL: $K = \omega\sqrt{LC}$ replace magic nu $\qquad K/\omega = \sqrt{LC} \rightarrow v^{-1} = \sqrt{LC}$

EM: $K = \omega\sqrt{\mu\varepsilon_{eff}}$ Now complex!

$$= \omega\sqrt{\mu\left(\varepsilon + \frac{\sigma}{j\omega}\right)}$$

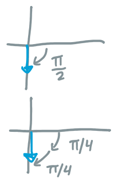
Suppose σ is very big (and we're looking at metals, so ε is small)

$$= \omega\sqrt{\frac{\mu\sigma}{j\omega}} \qquad \left(\frac{1}{\sqrt{j}} \cdot \frac{\sqrt{-j}}{\sqrt{-j}}\right)$$

$$= \sqrt{\omega}\sqrt{\mu\sigma}\sqrt{-j}$$

$$= \sqrt{\omega}\sqrt{\mu\sigma}\sqrt{e^{-j\frac{\pi}{2}}}$$

$$= \sqrt{\omega}\sqrt{\mu\sigma}\, e^{-j\frac{\pi}{4}}$$



$$K = \sqrt{\omega\mu\sigma}\left(\frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}}\right)$$

$$e^{-jkz} = e^{-j\sqrt{\frac{\omega\mu\sigma}{2}}z}\, e^{-\sqrt{\frac{\omega\mu\sigma}{2}}z}$$

skin depth is Inverse of this
> frequency ↑, skin depth ↓
> cross-section ↓, resistance ↑

Quiz 2.3 10/25/23

Skin depth of perfect conductor ($\sigma = \infty$) $= 0$

This is analogous to a short circuit in transmission lines

?