Friday, December 9, 2022 3:04 PM

Method of Undetermined Coefficients

$y_g = y_h + y_p$ $\qquad y(0)=y_0$

$ay'' + by' + c = f(t)$ $\qquad y'(0)=v_0$

Basic idea:

$f(t)$ has a family of derivatives

$f'(t), f''(t), \ldots$

e.g. $f(t)=t^3$
$f'(t)=3t^2$
$f''(t)=6t$
$f'''(t)=6$

$\{t^3, t^2, t, 1\}$

$y_p = At^3 + Bt^2 + Ct + D$

$f(t)=\sin(t)$
$f'(t)=\cos(t)$
$f''(t)=-\sin(t)$

$\{\sin(t), \cos(t)\}$

$y_p = A\sin t + B\cos t$

$y''+3y = t^2 = f(t)$

$y_h = \cos(\sqrt{3}t) + \sin(\sqrt{3}t)$

$y_p = At^2 + Bt + C$

$y_p' = 2At + B$

$y_p'' = 2A$

$2A + 3(At^2 + Bt + C) = t^2$

True for all t

t^2: $3A=1$, $\boxed{A=\frac{1}{3}}$

t: $\boxed{B=0}$

1: $2A+3C=0$

$\frac{2}{3}+3C=0$

$\boxed{C=-\frac{2}{9}}$

$y''+3y=\sin(\sqrt{3}t)$

$y_h = C_1\cos(\sqrt{3}t) + C_2\sin(\sqrt{3}t)$

$y_p = At\cos(\sqrt{3}t) + Bt\sin(\sqrt{3}t)$

both part of homogenous solution

$y_p' = (A\cos(\sqrt{3}t) + B\sin(\sqrt{3}t)) + (-\sqrt{3}A\sin(\sqrt{3}t) + \sqrt{3}B\cos(\sqrt{3}t))\,t$

from derivatives ...

$y_p'' = 2(-\sqrt{3}A\sin(\sqrt{3}t) + \sqrt{3}B\cos(\sqrt{3}t)) - 3(A\cos(\sqrt{3}t) + B\sin(\sqrt{3}t))\,t$

$\to 2(-\sqrt{3}A\sin(\sqrt{3}t) + \sqrt{3}B\cos(\sqrt{3}t)) - 3(A\cos(\sqrt{3}t) + B\sin(\sqrt{3}t))t + 3(At\cos(\sqrt{3}t) + Bt\sin(\sqrt{3}t)) = \sin(\sqrt{3}t)$

$\sin$: $-2\sqrt{3}A = 1$

$\cos$: $2\sqrt{3}B = 0$

Systems

Homog: $\frac{d}{dt}\vec{x} = A\vec{x}$ $\qquad A$ = matrix, $\vec{x}$ = vector

1st order ODE

$x' - at = 0 \leftrightarrow x' = ax$

Inhomog: $\frac{d}{dt}\vec{x} = A\vec{x} + b(t)$

Ansatz Method

$\vec{x}(t) = e^{rt}\vec{q}$ — constant vector; time dependent

$\frac{d}{dt}\vec{x}(t) = re^{rt}\vec{q} = r\vec{x} \longrightarrow r\vec{x} = A\vec{x}$

$\to r$ is an eigenvalue of A

$\to q$ is the eigenvector associated with r

If $\vec{x}$ is an n-dimensional vector

$\to A = M_{n\times n}$ ($n\times n$ matrix)

There are at most n eigenvectors, n eigenvalues

Call them $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$

$\vec{x}_i = e^{r_i t}\vec{q}_i$

2x2 Matrix

1) Two real, distinct eigenvalues

$X(t) = C_1e^{r_1t}\vec{q}_1 + C_2e^{r_2t}\vec{q}_2$

2) One real, repeated root

$X(t) = e^{rt}\vec{q} + e^{rt}t\vec{p}$

$(A-rI)\vec{p} = \vec{q}$ → Generalized Eigenvector

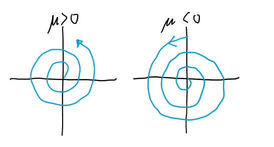
3) Complex conjugates

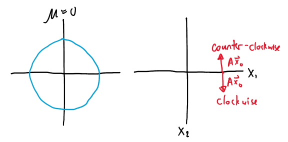
$r = \mu \pm i\omega$

$\vec{q} = \vec{a} \pm i\vec{b}$

$\vec{x}_r = e^{\mu t}(\cos(\omega t)\vec{a} - \sin(\omega t)\vec{b})$

$\vec{x}_i = e^{\mu t}(\sin(\omega t)\vec{a} + \cos(\omega t)\vec{b})$





How to find direction of spiral?

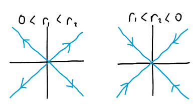
1) Pick a point $\vec{x}_0$ (e.g. $(1,0)$ or $(0,1)$)

2) Evaluate $A\vec{x}_0$

1) Distinct Real Roots

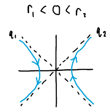
(1.1) r_1 and r_2 have the same sign

$0<r_1<r_2$ $\qquad r_1<r_2<0$



(1.2) r_1 and r_2 have opposite signs

$r_1 < 0 < r_2$



Inhomogeneous System

1) $\vec{b}(t) = \vec{b}$ = constant

$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$

$\vec{x}_p = \vec{w}$ = constant

$\vec{0} = A\vec{w} + \vec{b}$

$A\vec{w} = -\vec{b} \leftrightarrow w = -A^{-1}\vec{b}$

ex. $\frac{d}{dt}\vec{x} = \begin{bmatrix}-1 & -4\\ 1 & -1\end{bmatrix}\vec{x} + \begin{bmatrix}\sin(t)\\ -e^{-3t}\end{bmatrix}$

$\vec{x} = \vec{x}_h + \vec{x}_p$

Fourier Applied to PDEs (Homogenous)

PDE	Heat: $u_t = \alpha u_{xx}$	Wave: $u_{tt} = c^2 u_{xx}$
BCs: Dirichlet	$u(0,t) = u(L,t) = 0$	$u(0,t) = u(L,t) = 0$
Neumann	$u_x(0,t) = u_x(L,t) = 0$	$u_x(0,t) = u_x(L,t) = 0$
ICs:	$u(x,0) = u_0(x)$	$u(x,0) = f(x)$, $u_t(x,0) = g(x)$

Solution Method

Separation of Variables

$u(x,t) = X(x)\,T(t)$

Heat

$T'(t)X(x) = \alpha X''T$

$\frac{T'}{\alpha T} = \frac{X''}{X} = -\lambda$

$X(x) = C_1\cos(\sqrt{\lambda}x) + C_2\sin(\sqrt{\lambda}x)$

$T(t) = e^{-\lambda\alpha t}$

Fourier Series only used for space

$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$ $\qquad \lambda = \left(\frac{n\pi}{L}\right)^2$

$u_n(x,t) = e^{-\lambda\alpha t}(a_n\cos(\sqrt{\lambda}x) + b_n\sin(\sqrt{\lambda}x))$

$u(x,t) = \sum_{n=1}^{\infty} e^{-\frac{n^2\pi^2}{L^2}\alpha t}(a_n\cos(\sqrt{\lambda}x) + b_n\sin(\sqrt{\lambda}x))$

Dirichlet:
$u(0,t) = u(L,t) = 0 \longrightarrow a_n = 0$

Neumann:
$u_x(0,t) = u_x(L,t) = 0 \longrightarrow b_n = 0$

Initial Condition:
$u(x,0) = u_0(x) = \sum_{n=0}^{\infty} a_n\cos\left(\frac{n\pi}{L}x\right)$ Neumann

OR

$\sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi}{L}x\right)$ Dirichlet

Wave

$T''X = c^2X''T$

$\frac{T''}{c^2T} = \frac{X''}{X} = -\lambda$

$X(x) = C_1\cos(\sqrt{\lambda}x) + C_2\sin(\sqrt{\lambda}x)$

$T(t) = C_3\cos(\sqrt{\lambda}t) + C_4\sin(\sqrt{\lambda}t)$

Fourier series used for both

$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$

$u_n = \left[b_n\frac{L}{n\pi c}\sin\left(\frac{n\pi c}{L}t\right) + c_n\cos\left(\frac{n\pi c}{L}t\right)\right]\sin\left(\frac{n\pi}{L}x\right)$

Dirichlet:
$u(0,t) = u(L,t) = 0$

Neumann:
$u_x(0,t) = u_x(L,t) = 0$

Initial Condition:
$u(x,0) = f(x) = \sum C_n\sin\left(\frac{n\pi x}{L}\right)$

$u_t(x,0) = g(x) = \sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi x}{L}\right)$ — Dirichlet

$\to C_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$

> Flip to cos for Neumann ICs.

$u_{tt} + bu_t = c^2u_{xx}$

$u = XT$

$XT'' + bT'X = c^2X''T$

$\frac{T'' + bT'}{c^2T} = \frac{X''}{X} = -\lambda$

$T'' + bT' = -c^2\lambda T$

$T'' + bT' + c^2\lambda T = 0$

Guess $t = e^{rt}$

$r^2 + br + c^2\lambda = 0$

$r = \frac{-b \pm \sqrt{b^2 - 4c^2\lambda}}{2}$

D'Alembert

$u_{tt} = c^2u_{xx}$ $\qquad x\in[0,L]$ $\qquad$ IC: $u(x,0) = f(x)$, $u_t(x,0) = g(x)$

$u(x,t) = A(x-ct) + B(x+ct)$

$A = \frac{1}{2}\left[F(x) - \frac{1}{c}\int_0^{x-ct} G(s)\,ds\right]$

$B = \frac{1}{2}\left[F(x) + \frac{1}{c}\int_0^{x+ct} G(s)\,ds\right]$

$x = x - ct$, S = original x

BCs:
$u(0,t) = u(L,t) = 0$ $\qquad (u_x(0,t) = u_x(L,t) = 0)$

$F(z)$ = odd periodic extension of $f(x)$ (even)

$G(z)$ = odd periodic extension of $g(x)$ (even)

example

$u_{tt} = 25u_{xx}$ $\qquad 0<x<3$

$u(x,0) = x^3(1-x) = f(x)$

$u_t(x,0) = 0$ $\qquad u_x(0,t) = u_x(3,t) = 0$

Find $u(x,t)$ for $x=1$, $t=1000$

$u(x,t) = \frac{1}{2}[F(x-5t) + F(x+5t)]$ $\qquad$ BC: $2\cdot3=6$

$F(x) = \begin{cases} x^3(1-x) & 0<x<3 \\ -x^3(1+x) & -3<x<0 \end{cases}$ $\qquad F(x+n6) = F(x)$, $\forall x\in\mathbb{N}$

$u(1,1000) = \frac{1}{2}[F(1-5000) + F(1+5000)]$ $\qquad (-4999)$, (5001)

-4999 and 5001 are not between -3 and $+3$

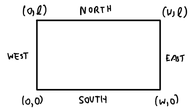
Remapping: $\exists\, n_1, n_2$ such that

$-3 < -4999 - n_16 < 3$ $\qquad -3 < 5001 + n_26 < 3$

$4996 < n_16 < 5002$ $\qquad -5004 < n_26 < -4998$

Laplacian PDE

$\Delta u = 0 \leftrightarrow u_{xx} + u_{yy} = 0$



North: $u(x,h) = f(x)$
South: $u(x,0) = g(x)$
East: $u(w,y) = h(y)$
West: $u(0,y) = m(y)$

$u(x,y) = u_N + u_S + u_E + u_W$

East

$u(x,y) = X(x)\,Y(y)$

$X''Y + XY'' = 0$

$X''Y = -XY''$

$-\frac{Y''}{Y} = \frac{X''}{X} = -\lambda$

BC: $y(0) = y(h) = 0$

$y'' - \lambda y = 0$

$\vdots$

$\lambda = -\left(\frac{n\pi}{h}\right)^2$

$\to Y(y) = C_1\sin\left(\frac{n\pi}{h}y\right)$

$X'' + \left(\frac{n\pi}{h}\right)^2X = 0$ $\qquad$ Guess $X = e^{rx}$

$r^2e^{rx} + \left(\frac{n\pi}{h}\right)^2e^{rx} = 0$

$r = \pm\frac{n\pi}{h}$

$X(x) = Ae^{\frac{n\pi}{h}x} + Be^{-\frac{n\pi}{h}x}$

BC $X(0) = 0 \to X(0) = A + B = 0$
$\to A = -B$

$X(x) = B\left(e^{-\frac{n\pi}{h}x} - e^{\frac{n\pi}{h}x}\right) = B\sinh\left(\frac{n\pi}{h}x\right)$

$u_n(x,y) = a_n\sin\left(\frac{n\pi}{h}y\right)\sinh\left(\frac{n\pi}{h}x\right)$

$u_E(x,y) = \sum_{n=1}^{\infty} a_n\sin\left(\frac{n\pi}{h}y\right)\sinh\left(\frac{n\pi}{h}x\right)$

Neumann → cosh, cos

$u_E(w,y) = \ell(y) = \sum_{n=1}^{\infty} \underbrace{a_n\sinh\left(\frac{n\pi}{h}w\right)}_{b_n}\sin\left(\frac{n\pi}{h}y\right)$ — Fourier sine series

$b_n = \frac{2}{h}\int_0^h \ell(y)\sin\left(\frac{n\pi}{h}y\right)dy$

$a_n\sinh\left(\frac{n\pi}{h}w\right) = b_n$

$a_n = \frac{b_n}{\sinh\left(\frac{n\pi}{h}w\right)}$

ex. $\frac{d}{dt}\vec{x} = \begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix}\vec{x} + \begin{bmatrix} \sin(t) \\ -e^{-3t} \end{bmatrix}$

$\vec{x} = \vec{x}_h + \vec{x}_p$

$\begin{vmatrix} -1-\lambda & -4 \\ 1 & -1-\lambda \end{vmatrix} = (-1-\lambda)^2 + 4 = 0$

$1 + 2\lambda + \lambda^2 + 4 = 0$

$\lambda^2 + 2\lambda + 5 = 0$

$\lambda = \frac{-2 \pm \sqrt{4-4(5)}}{2}$

$\lambda = -1 \pm 2i$

$\lambda = -1+2i$

$(A - \lambda i)\vec{q} = \vec{0}$

$\left[\begin{array}{cc|c} -1-(-1+2i) & -4 & 0 \\ 1 & -1-(-1+2i) & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 2i & -4 & 0 \\ 1 & -2i & 0 \end{array}\right]$

$x - 2iy = 0$

$x = 2iy$

$\vec{q} = \begin{bmatrix} 2i \\ 1 \end{bmatrix}$

$\vec{q}_{1,2} = \vec{a} \pm i\vec{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \pm i \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$\vec{x}_h = c_1 e^{-t}\left(\cos(2t)\begin{bmatrix} 0 \\ 1 \end{bmatrix} - \sin(2t)\begin{bmatrix} 2 \\ 0 \end{bmatrix}\right)$
$+ c_2 e^{-t}\left(\sin(2t)\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \cos(2t)\begin{bmatrix} 2 \\ 0 \end{bmatrix}\right)$

$\vec{x}_p = \sin(t)\vec{u} + \cos(t)\vec{w} + e^{-3t}\vec{p}$

$\vec{x}_p' = \cos(t)\vec{u} - \sin(t)\vec{w} - 3e^{-3t}\vec{p}$

matrix

$= A[\sin(t)\vec{u} + \cos(t)\vec{w} + e^{-3t}\vec{p}]$
$+ \sin(t)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + e^{-3t}\begin{bmatrix} 0 \\ -1 \end{bmatrix}$

$\sin(t)$: $\vec{w} = A\vec{u} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\cos(t)$: $\vec{u} = A\vec{w}$

e^{-3t}: $-3\vec{p} = A\vec{p} + \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

Resonance — See lecture 19
Slides 6-8 Theory
Slides 9-10 Example

$\exists\, n_1, n_2$ such that

$-3 < -4999 - n_1 6 < 3$ | $-3 < 5001 + n_1 6 < 3$

$4996 < n_1 6 < 5002$ | $-5004 < n_2 6 < -4998$

$\frac{4996}{6} < n_1 < \frac{5002}{6}$ | $\frac{-5004}{6} < n_2 < \frac{-4998}{6}$

$832.66 < n_1 < 833.66$ | $-834 < n_2 < -833$

$n_1 = 833$ | $n_2 = -833$ or -844

$Z_1 = -4999 + 833 \cdot 6$
$= -1$
$-3 < -1 < 3$ ✓

$Z_2 = 5001 - 833 \cdot 6$
$= 3$
$-3 \le 3 \le 3$ ✓

$u(1,1000) = \frac{1}{2}[F(-1) + F(3)]$
$= \frac{1}{2}[0 + -54]$
$= \boxed{-27}$