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Computational Methods for Quantitative Finance

Finite Element Methods for Derivative Pricing

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ISSN 1616-0533 ISSN 2195-0687 (electronic)
Springer Finance
ISBN 978-3-642-35400-7 ISBN 978-3-642-35401-4 (eBook)
DOI 10.1007/978-3-642-35401-4
Springer Heidelberg New York Dordrecht London

Library of Congress Control Number: 2013932229

Mathematics Subject Classification: 60J75, 60J25, 60J35, 60J60, 65N06, 65K15, 65N12, 65N30

JEL Classification: C63, C16, G12, G13

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Preface

The subject of mathematical finance has undergone rapid development in recent years, with mathematical descriptions of financial markets evolving both in volume and technical sophistication. Pivotal in this development have been *quantitative models* and *computational methods* for calibrating mathematical models to market data, and for obtaining option prices of concrete products from the calibrated models.

In this development, two broad classes of computational methods have emerged: statistical sampling approaches and grid-based methods. They correspond, roughly speaking, to the characterization of arbitrage-free prices as conditional expectations over all sample paths of a stochastic process model of the market behavior, or to the characterization of prices as solutions (in a suitable sense) of the corresponding Kolmogorov forward and/or backward partial differential equations, or PDEs for short, the canonical example being the Black–Scholes equation and its extensions.

Sampling methods contain, for example, Monte-Carlo and Quasi-Monte-Carlo Methods, whereas grid-based methods contain, for example, Finite Difference, Finite Element, Spectral and Fourier transformation methods (which, by the use of the Fast Fourier Transform, require approximate evaluation of Fourier integrals on grids). The present text discusses the analysis and implementation of grid-based methods.

The importance of numerical methods for the efficient valuation of derivative contracts cannot be overstated: often, the selection of mathematical models for the valuation of derivative contracts is determined by the ease and efficiency of their *numerical evaluation* to the extent that computational efficiency takes priority over mathematical sophistication and general applicability.

Having said this, we hasten to add that the computational methods presented in these notes approximate the (forward and backward) pricing partial (integro) differential equations and inequalities by finite dimensional *discretizations* of these equations which are amenable to numerical solution on a computer. The methods incur, therefore, naturally an error due to this replacement of the forward pricing equation by a discretization, the so-called *discretization error*. One main message to be conveyed by these notes is that, using numerical analysis and advanced solution methods, efficient discretizations of the pricing equations for a wide range of market models and term sheets are available, and there is no obvious necessity to confine financial modeling to processes which entail "exactly solvable" PIDEs.

We caution the reader, however, that this reasoning implies that the error estimates presented in these notes are *bounds on the discretization error*, i.e. the error in the computed solution with respect to the *exact solution of one particular market model under consideration*. An equally important theme is the quantitative analysis of the error inherent in the financial models themselves, i.e. the so-called *modeling errors*. Such errors are due to assumptions on the markets which were (explicitly or implicitly) used in their derivations, and which may or may not be valid in the situations where the models are used. It is our view that a unified, numerical pricing methodology that accommodates a wide range of market models can facilitate quantitative verification of dependence of prices on various assumptions implicit in particular classes of market models.

Thus, to give "non-experts" in computational methods and in numerical analysis an introduction to grid-based numerical solution methods for option pricing problems is one purpose of the present volume. Another purpose is to acquaint numerical analysts and computational mathematicians with formulation and numerical analysis of typical initial-boundary value problems for partial integro-differential equations (PIDEs) that arise in models of financial markets with jumps. Financial contracts with early exercise features lead to optimal stopping problems which, in turn, lead to unilateral boundary value problems for the corresponding PIDEs. Efficient numerical solution methods for such problems have been developed over many years in solvers for contact problems in mechanics. Contrary to the differential operators which arise with obstacle problems in mechanics, however, the PIDEs in financial models with jumps are, as a rule, *nonsymmetric* (due to the presence of a drift term which, in turn, is mandated by no-arbitrage conditions in the pricing of derivative contracts). The numerical analysis of the corresponding algorithms in financial applications cannot rely, therefore, on energy minimization arguments so that many well-established algorithms are ruled out.

Rather than trying to cover all possible numerical approaches for the computational solution of pricing equations, we decided to focus on Finite Difference and on Finite Element Methods. Finite Element Methods (FEM for short) are based on particularly general, so-called *weak, or variational formulations* of the pricing equation. This is, on the one hand, the natural setting for FEM; on the other hand, as we will try to show in these notes, the variational formulation of the forward and backward equations (in price or in log-price space) on which the FEM is based has a very natural correspondence on the "stochastic side", namely the so-called *Dirichlet form* of the stochastic process model for the dynamics of the risky asset(s) underlying the derivative contracts of interest. As we show here, FEM based numerical solution methods allow for a unified numerical treatment of rather *general classes of market models*, including local and stochastic volatility models, square root driving processes, jump processes which are either stationary (such as Lévy processes) or nonstationary (such as affine and polynomial processes or processes which are additive in the sense of Sato), for which transform based numerical schemes are not immediately applicable due to lack of stationarity.

In return for this restriction in the types of methods which are presented here, we tried to accommodate within a single mathematical solution framework a wide range of mathematical models, as well as a reasonably large number of term sheet features in the contracts to be valued.

The presentation of the material is structured in two parts: Part I "Basic Methods", and Part II "Advanced Methods". The material in the first part of these notes has evolved over several years, in graduate courses which were taught to students in the joint ETH and Uni Zürich MSc programme in quantitative finance, whereas Part II is based on PhD research projects in computational finance.

This distinction between Parts I and II is certainly subjective, and we have seen it evolve over time, in line with the development of the field. In the formulation of the methods and in their analysis, we have tried to maintain mathematical rigor whenever possible, without compromising ease of understanding of the computational methods per se. This has, in particular in Part I, lead to an engineering style of method presentation and analysis in many places. In Part II, fewer such compromises have been made. The formulation of forward and backward equations for rather large classes of jump processes has entailed a somewhat heavy machinery of Sobolev spaces of fractional and variable, state dependent order, of Dirichlet forms, etc. There is a close correspondence of many notions to objects on the stochastic side where the stochastic processes in market models are studied through their Dirichlet forms.

We are convinced that many of the numerical methods presented in these notes have applications beyond the immediate area of computational finance, as Kolmogorov forward and backward equations for stochastic models with jumps arise naturally in many contexts in engineering and in the sciences. We hope that this broader scope will justify to the readers the analytical apparatus for numerical solution methods in particular in Part II.

The present material owes much in style of presentation to discussions of the authors with students in the UZH and ETH MSc quantitative finance and in the ETH MSc Computational Science and Engineering programmes who, during the courses given by us during the past years, have shaped the notes through their questions, comments and feedback. We express our appreciation to them. Also, our thanks go to Springer Verlag for their swift and easy handling of all nonmathematical aspects at the various stages during the preparation of this manuscript.

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Part I
Basic Techniques and Models

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Chapter 1
Notions of Mathematical Finance

The present notes deal with topics of computational finance, with focus on the analysis and implementation of numerical schemes for pricing derivative contracts. There are two broad groups of numerical schemes for pricing: stochastic (Monte Carlo) type methods and deterministic methods based on the numerical solution of the Fokker–Planck (or Kolmogorov) partial integro-differential equations for the price process. Here, we focus on the latter class of methods and address finite difference and finite element methods for the most basic types of contracts for a number of stochastic models for the log returns of risky assets. We cover both, models with (almost surely) continuous sample paths as well as models which are based on price processes with jumps. Even though emphasis will be placed on the (partial integro)differential equation approach, some background information on the market models and on the derivation of these models will be useful particularly for readers with a background in numerical analysis.

Accordingly, we collect synoptically terminology, definitions and facts about models in finance. We emphasise that this is a *collection* of terms, and it can, of course, in no sense claim to be even a short survey over mathematical modelling in finance. Readers who wish to obtain a perspective on mathematical modelling principles for finance are referred to the monographs of Mao [120], Øksendal [131], Gihman and Skorohod [71–73], Lamberton and Lapeyre [109], Shiryaev [152], as well as Jacod and Shiryaev [97].

1.1 Financial Modelling

Stocks Stocks are shares in a company which provide partial ownership in the company, proportional with the investment in the company. They are issued by a company to raise funds. Their value reflects both the company's real assets as well as the estimated or imagined company's earning power. *Stock* is the generic term for assets held in the form of shares. For publicly quoted companies, stocks are quoted and traded on a stock exchange. An *index* tracks the value of a *basket of stocks*.

N. Hilber et al., *Computational Methods for Quantitative Finance*, Springer Finance,
DOI 10.1007/978-3-642-35401-4_1,

Assets for which future prices are not known with certainty are called *risky assets*, while assets for which the future prices are known are called *risk free*.

Price Process The price at which a stock can be bought or sold at any given time t on a stock exchange is called *spot price* and we shall denote it by S_t. All possible future prices S_t as functions of t (together with probabilistic information on the likelihood of a particular price history) constitute the *price process* $S = \{S_t : t \geq 0\}$ of the asset. It is mathematically modelled by a *stochastic process* to be defined below.

Derivative Securities A *derivative security*, derivative for short, is a security whose value depends on the value of one or several *underlying assets* and the decisions of the investor. It is also called *contingent claim*. It is a financial contract whose value at *expiration time* (or *time of maturity*) T is determined by the price process of the underlying assets up to time T. After choosing a price process for the asset(s) under consideration, the task is to determine a price for the derivative security on the asset. There are several types of derivatives: options, forwards, futures and swaps. We focus exemplary on the pricing of options, since pricing other assets leads to closely related problems.

Options An *option* is a derivative which gives its holder the *right, but not the obligation* to make a specified transaction at or by a specified date at a specified price. Options are sold by one party, the *writer* of the option, to another, the *holder*, of the option. If the holder chooses to make the transaction, he *exercises* the option. There are many conditions under which an option can be exercised, giving rise to different types of options. We list the main ones: *Call options* give the right (but not the obligation) to buy, *put options* give the right (but not an obligation) to sell the underlying at a specified price, the so-called *strike price* K. The simplest options are the *European call and put* options. They give the holder the right to buy (resp., sell) exactly at *maturity* T. Since they are described by very simple rules, they are also called *plain vanilla* options. Options with more sophisticated rules than those for plain vanillas are called *exotic options*. A particular type of exotic options are *American options* which give the holder the right (but not the obligation) to buy (resp., sell) the underlying at any time t at or before maturity T. For European options the price does not depend on the path of the underlying, but only on the realisation at maturity T. There are also so-called *path dependent* options, like Asian, lookback or barrier contracts. The value of *Asian options* depends on the average price of the option's underlying over a period, *lookback options* depend on the maximum or minimum asset price over a period, and *barrier options* depend on particular price level(s) being attained over a period.

Payoff The *payoff* of an option is its value at the time of exercise T. For a European call with strike price K, the payoff g is

$$g(S_T) = (S_T - K)_+ = \begin{cases} S_T - K & \text{if } S_T > K, \\ 0 & \text{else.} \end{cases}$$

At time $t \le T$ the option is said to be *in the money*, if $S_t > K$, the option is *out of the money*, if $S_t < K$, and the option is said to be *at the money*, if $S_t \approx K$.

Modelling Assumptions Most market models for stocks assume the existence of a *riskless bank account* with *riskless interest rate* $r \ge 0$. We will also consider stochastic interest rate models where this is not the case. However, unless explicitly stated otherwise, we assume that money can be deposited and borrowed from this bank account with continuously compounded, known interest rate r. Therefore, 1 currency unit in this account at $t = 0$ will give e^{rt} currency units at time t, and if 1 currency unit is borrowed at time $t = 0$, we will have to pay back e^{rt} currency units at time t. We also assume a *frictionless market*, i.e. there are no transaction costs, and we assume further that there is no default risk, all market participants are rational, and the market is efficient, i.e. there is no arbitrage.

1.2 Stochastic Processes

We refer to the texts Mao [120] and Øksendal [131] for an introduction to stochastic processes and stochastic differential equations. Much more general stochastic processes in the Markovian and non-Markovian setup are treated in the monographs Gihman and Skorohod [71–73] as well as Jacod and Shiryaev [97].

Prices of the so-called risky assets can be modelled by stochastic processes in continuous time $t \in [0, T]$ where the maturity $T > 0$ is the *time horizon*. To describe stochastic price processes, we require a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Here, Ω is the set of elementary events, $\mathcal{F}$ is a σ-algebra which contains all events (i.e. subsets of Ω) of interest and $\mathbb{P} : \mathcal{F} \to [0, 1]$ assigns a probability of any event $A \in \mathcal{F}$.

We shall always assume the probability space to be complete, i.e. if $B \subset A$ with $A \in \mathcal{F}$ and $\mathbb{P}[A] = 0$, then $B \in \mathcal{F}$. We equip $(\Omega, \mathcal{F}, \mathbb{P})$ with a *filtration*, i.e. a family $\mathbb{F} = \{\mathcal{F}_t : 0 \le t \le T\}$ of σ-algebras which are monotonic with respect to t in the sense that for $0 \le s \le t \le T$ holds that $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}_T \subset \mathcal{F}$. In financial modelling, the σ-algebra $\mathcal{F}_t \in \mathbb{F}$ represents the information available in the model up to time t. We assume that the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ satisfies the *usual assumptions*, i.e.

(i) $\mathcal{F}$ is $\mathbb{P}$-complete,
(ii) $\mathcal{F}_0$ contains all $\mathbb{P}$-null subsets of Ω and
(iii) The filtration $\mathbb{F}$ is right-continuous: $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$.

Definition 1.2.1 (Stochastic processes) A *stochastic process* $X = \{X_t : 0 \le t \le T\}$ is a family of random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$, parametrised by the time variable t. For $\omega \in \Omega$, the function $X_t(\omega)$ of t is called *a sample path* of X. The process is $\mathbb{F}$-*adapted* if X_t is $\mathcal{F}_t$ measurable (denoted by $X_t \in \mathcal{F}_t$) for each t.

To model asset prices by stochastic processes, knowledge about past events up to time t should be incorporated into the model. This is done by the concept of *filtration*.

Definition 1.2.2 (Natural filtration) We call $\mathbb{F}^X = \{\mathcal{F}_t^X : 0 \le t \le T\}$ the *natural filtration for* X if it is the completion with respect to $\mathbb{P}$ of the filtration $\widetilde{\mathbb{F}}^X = \{\widetilde{\mathcal{F}}_t^X : 0 \le t \le T\}$, where for each $0 \le t \le T$, $\widetilde{\mathcal{F}}_t^X = \sigma(X_r : r \le s)$.

A stochastic process is called *càdlàg* (from French 'continue à droite avec des limités à gauche') if it has càdlàg sample paths, and a mapping $f : [0, T] \to \mathbb{R}$ is said to be càdlàg if for all $t \in [0, T]$ it has a left limit at t and is right-continuous at t. A stochastic process is called *predictable* if it is measurable with respect to the σ-algebra $\widehat{\mathcal{F}}$, where $\widehat{\mathcal{F}}$ is the smallest σ-algebra generated by all adapted càdlàg processes on $[0, T] \times \Omega$.

Asset prices are often modelled by *Markov processes*. In this class of stochastic processes, the stochastic behaviour of X after time t depends on the past only through the current state X_t.

Definition 1.2.3 (Markov property) A stochastic process $X = \{X_t : 0 \le t \le T\}$ is *Markov* with respect to $\mathbb{F}$ if

$$\mathbb{E}[f(X_s)|\mathcal{F}_t] = \mathbb{E}[f(X_s)|X_t],$$

for any bounded Borel function f and $s \ge t$.

No arbitrage considerations require discounted log price processes to be martingales, i.e. the best prediction of X_s based on the information at time t contained in $\mathcal{F}_t$ is the value X_t. In particular, the expected value of a martingale at any finite time T based on the information at time 0 equals the initial value X_0, $\mathbb{E}[X_T|\mathcal{F}_0] = X_0$.

Definition 1.2.4 (Martingale) A stochastic process $X = \{X_t : 0 \le t \le T\}$ is a *martingale* with respect to $(\mathbb{P}, \mathbb{F})$ if

(i) X is $\mathbb{F}$ adapted,
(ii) $\mathbb{E}[|X_t|] < \infty$ for all $t \ge 0$,
(iii) $\mathbb{E}[X_s|\mathcal{F}_t] = X_t$ $\mathbb{P}$-a.s. for $s \ge t \ge 0$.

There is a one-to-one correspondence between models that satisfy the *no free lunch with vanishing risk* condition and the existence of a so-called *equivalent local martingale measure* (ELMM). We refer to [54, 55] for details. The most widely used price process is a *Brownian motion* or *Wiener process*. Its use in modelling log returns in prices of risky assets goes back to Bachelier [4]. Recall that the *normal distribution* $\mathcal{N}(\mu, \sigma^2)$ with mean $\mu \in \mathbb{R}$ and variance σ^2 with $\sigma > 0$ has the density

$$f_{\mathcal{N}}(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/(2\sigma^2)},$$

and it is symmetric around μ. Normality assumptions in models of log returns of risky assets' prices imply the assumption that upward and downward moves of prices occur symmetrically.

Definition 1.2.5 (Wiener process) A stochastic process $X = \{X_t : t \geq 0\}$ is a *Wiener process* on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ if (i) $X_0 = 0$ $\mathbb{P}$-a.s., (ii) X has independent increments, i.e. for $s \leq t$, $X_t - X_s$ is independent of $\mathcal{F}_s = \sigma(X_u, u \leq s)$, (iii) $X_{t+s} - X_t$ is normally distributed with mean 0 and variance $s > 0$, i.e. $X_{t+s} - X_t \sim \mathcal{N}(0, s)$, and (iv) X has $\mathbb{P}$-a.s. continuous sample paths. We shall denote this process by W for N. Wiener.

In the Black–Scholes stock price model, the price process S of the risky asset is modelled by assuming that the return due to price change in the time interval $\Delta t > 0$ is

$$\frac{S_{t+\Delta t} - S_t}{S_t} = \frac{\Delta S_t}{S_t} = r\,\Delta t + \sigma\,\Delta W_t,$$

in the limit $\Delta t \to 0$, i.e. that it consists of a deterministic part $r\,\Delta t$ and a random part $\sigma(W_{t+\Delta t} - W_t)$. In the limit $\Delta t \to 0$, we obtain the stochastic differential equation (SDE)

$$\mathrm{d}S_t = r\,S_t\,\mathrm{d}t + \sigma\,S_t\,\mathrm{d}W_t, \qquad S_0 > 0. \tag{1.1}$$

The above SDE admits the unique solution

$$S_t = S_0 e^{(r-\sigma^2/2)t+\sigma W_t}.$$

This exponential of a Brownian motion is called the *geometric Brownian motion*. The stochastic differential equation (1.1) for the geometric Brownian motion is a special case of the more general SDE

$$\mathrm{d}X_t = b(t, X_t)\,\mathrm{d}t + \sigma(t, X_t)\,\mathrm{d}W_t, \qquad X_0 = Z, \tag{1.2}$$

for which we give an existence and uniqueness result.

Theorem 1.2.6 *We consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\mathbb{F}$ and a Brownian motion W on $(\Omega, \mathcal{F}, \mathbb{P})$ adapted to $\mathbb{F}$. Assume there exists $C > 0$ such that $b, \sigma : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ in (1.2) satisfy*

$$|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq C|x-y|, \qquad x, y \in \mathbb{R},\ t \in \mathbb{R}_+, \tag{1.3}$$

$$|b(t,x)| + |\sigma(t,x)| \leq C(1+|x|), \qquad x \in \mathbb{R},\ t \in \mathbb{R}_+. \tag{1.4}$$

Assume further $X_0 = Z$ for a random variable which is $\mathcal{F}_0$-measurable and satisfies $\mathbb{E}[|Z|^2] < \infty$. Then, for any $T \geq 0$, (1.2) admits a $\mathbb{P}$-a.s. unique solution in $[0, T]$ satisfying

$$\mathbb{E}\Big[\sup_{0 \leq t \leq T} |X_t|^2\Big] < \infty. \tag{1.5}$$

We refer to [131, Theorem 5.2.1] or [120, Theorem 2.3.1] for a proof of this statement. Note that the Lipschitz continuity (1.3) implies the linear growth condition (1.4) for time-independent coefficients $\sigma(x)$ and $b(x)$. For any $t \geq 0$ one has $\int_0^t |b(s, X_s)| \, \mathrm{d}s < \infty$, $\int_0^t |\sigma(s, X_s)|^2 \, \mathrm{d}s < \infty$, $\mathbb{P}$-a.s., i.e. the solution process X is a particular case of a so-called *Itô process*. Equation (1.2) is formally the differential form of the equation

$$X_t = Z + \int_0^t b(s, X_s) \, \mathrm{d}s + \int_0^t \sigma(s, X_s) \, \mathrm{d}W_s,$$

for $t \in [0, T]$. In the derivation of pricing equations, it will become important to check under which conditions the integrals with respect to W, i.e. $\int_0^t \phi_s \, \mathrm{d}W_s$, are martingales. The notion of stochastic integrals is discussed in detail in [120, Sect. 1.5].

Proposition 1.2.7 *Let the process ϕ be predictable and let ϕ satisfy, for $T \geq 0$,*

$$\mathbb{E}\left[\int_0^T |\phi_t|^2 \, \mathrm{d}t\right] < \infty. \tag{1.6}$$

Then, the process $M = \{M_t : t \geq 0\}$, $M_t := \int_0^t \phi_s \, \mathrm{d}W_s$ is a martingale.

For a proof of this statement, we refer to [131, Theorem 3.2.1]. In mathematical finance, we are interested in the dynamics of $f(t, X_t)$, e.g. where $f(t, X_t)$ denotes the option price process. Here, the Itô formula plays an important role.

Theorem 1.2.8 (Itô formula) *Let X be given by the Itô process (1.2), and let $f(t, x) \in C^2([0, \infty) \times \mathbb{R})$, i.e. f is twice continuously differentiable on $[0, \infty) \times \mathbb{R}$. Then, for $Y_t = f(t, X_t)$ we obtain*

$$\mathrm{d}Y_t = \frac{\partial f}{\partial t}(t, X_t) \, \mathrm{d}t + \frac{\partial f}{\partial x}(t, X_t) \, \mathrm{d}X_t + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(t, X_t) \cdot (\mathrm{d}X_t)^2, \tag{1.7}$$

where $(\mathrm{d}X_t)^2 = (\mathrm{d}X_t) \cdot (\mathrm{d}X_t)$ is computed according to the rules

$$\mathrm{d}t \cdot \mathrm{d}t = \mathrm{d}t \cdot \mathrm{d}W_t = \mathrm{d}W_t \cdot \mathrm{d}t = 0, \quad \mathrm{d}W_t \cdot \mathrm{d}W_t = \mathrm{d}t.$$

We refer to [120, Theorem 1.6.2] for a proof of the Itô formula. A sketch of the proof is given in [131, Theorem 4.1.2]. We note in passing that the smoothness requirements on the function f in Theorem 1.2.8 can be substantially weakened. We refer to [132, Sects. II.7 and II.8] and [40, Sect. 8.3] for general versions of the Itô formula for Lévy processes and semimartingales.

1.3 Further Reading

An introduction to financial modelling and option pricing can be found in Wilmott et al. [161] and the corresponding student version [162]. More details on risk-neutral pricing, absence of arbitrage and equivalent martingale measures are given in Delbaen and Schachermayer [53]. For a general introduction to stochastic differential equations, see Øksendal [131] and Mao [120], Protter [132] and the references therein.

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Chapter 2
Elements of Numerical Methods for PDEs

In this chapter, we present some elements of numerical methods for partial differential equations (PDEs). The PDEs are classified into elliptic, parabolic and hyperbolic equations, and we indicate the corresponding type of problems that they model. PDEs arising in option pricing problems in finance are mostly parabolic. Occasionally, however, elliptic PDEs arise in connection with so-called "infinite horizon problems", and hyperbolic PDEs may appear in certain pure jump models with dominating drift.

Therefore, we consider in particular the heat equation and show how to solve it numerically using finite differences or finite elements. Finite difference methods (FDM) consist of finding an approximate solution on a grid by replacing the derivatives in the differential equation by difference quotients. Finite element methods (FEM) are based instead on variational formulations of the differential equations and determine approximate solutions that are usually piecewise polynomials on some partition of the (log) price domain. We start with recapitulating some function spaces as well as the classification of PDEs.

2.1 Function Spaces

The variational formulation and the analysis of the finite element method require tools from functional analysis, in particular Hilbert spaces (see Appendix A). Let G be a non-empty open subset of $\mathbb{R}^d$. If a function $u : G \to \mathbb{R}$ is sufficiently smooth, we denote the partial derivatives of u by

$$D^{\mathbf{n}}u(x) := \frac{\partial^{|\mathbf{n}|}u(x)}{\partial x_1^{n_1} \cdots \partial x_d^{n_d}} = \partial_{x_1}^{n_1} \cdots \partial_{x_d}^{n_d} u(x), \quad x = (x_1, \ldots, x_d) \in G, \tag{2.1}$$

where $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{N}_0^d$ is a multi-index. The order of the partial derivative is given by $|\mathbf{n}| = \sum_{i=1}^d n_i$. For any integer $n \in \mathbb{N}_0$, we define

$$C^n(G) = \{u : D^{\mathbf{n}}u \text{ exists and is continuous on } G \text{ for } |\mathbf{n}| \leq n\},$$

N. Hilber et al., *Computational Methods for Quantitative Finance*, Springer Finance,
DOI 10.1007/978-3-642-35401-4_2,

and set $C^\infty(G) = \bigcap_{n\geq 0} C^n(G)$. The support of u is denoted by $\operatorname{supp} u$, and we define $C_0^n(G)$, $C_0^\infty(G)$ consisting of all functions $u \in C^n(G)$, $C^\infty(G)$ with compact support $\operatorname{supp} u \Subset G$.

We denote by $L^p(G)$, $1 \leq p \leq \infty$ the usual space which consists of all Lebesgue measurable functions $u : G \to \mathbb{R}$ with finite L^p-norm,

$$\|u\|_{L^p(G)} := \begin{cases} (\int_G |u(x)|^p \, \mathrm{d}x)^{1/p} & \text{if } 1 \leq p < \infty, \\ \operatorname{ess\,sup}_G |u(x)| & \text{if } p = \infty, \end{cases}$$

where ess sup means the *essential supremum* disregarding values on nullsets. The case $p = 2$ is of particular interest. The space $L^2(G)$ is a Hilbert space with respect to the inner product $(u, v) = \int_G u(x)v(x)\, \mathrm{d}x$.

Let $\mathcal{H}$ be a Hilbert space with the inner product $(\cdot,\cdot)_\mathcal{H}$ and norm $\|u\|_\mathcal{H} := (u, u)_\mathcal{H}^{1/2}$. We denote by $\mathcal{H}^*$ the dual space of $\mathcal{H}$ which consists of all bounded linear functionals $u^* : \mathcal{H} \to \mathbb{R}$ on $\mathcal{H}$. $\mathcal{H}^*$ can be identified with $\mathcal{H}$ by the Riesz representation theorem.

Theorem 2.1.1 (Riesz representation theorem) *For each $u^* \in \mathcal{H}^*$ there exists a unique element $u \in \mathcal{H}$ such that*

$$\langle u^*, v\rangle_{\mathcal{H}^*,\mathcal{H}} = (u, v)_\mathcal{H} \quad \forall v \in \mathcal{H}.$$

The mapping $u^* \mapsto u$ is a linear isomorphism of $\mathcal{H}^*$ onto $\mathcal{H}$.

The theory of parabolic partial differential equations requires the introduction of Hilbert space-valued L^p-functions. As above, let $\mathcal{H}$ be a Hilbert space with the norm $\|\cdot\|_\mathcal{H}$. Denote by J the interval $J := (0, T)$ with $T > 0$, and let $1 \leq p \leq \infty$. The space $L^p(J; \mathcal{H})$ is defined by

$$L^p(J; \mathcal{H}) := \{u : \overline{J} \to \mathcal{H} \text{ measurable} : \|u\|_{L^p(J;\mathcal{H})} < \infty\},$$

with the norm

$$\|u\|_{L^p(J;\mathcal{H})} := \begin{cases} (\int_J \|u(t)\|_\mathcal{H}^p \, \mathrm{d}t)^{1/p} & \text{if } 1 \leq p < \infty, \\ \operatorname{ess\,sup}_J \|u(t)\|_\mathcal{H} & \text{if } p = \infty. \end{cases}$$

Furthermore, for $n \in \mathbb{N}_0$ let $C^n(J; \mathcal{H})$ be the space of $\mathcal{H}$-valued functions that are of the class C^n with respect to t.

2.2 Partial Differential Equations

For $k \in \mathbb{N}$ we let

$$D^k u(x) := \{D^\mathbf{n} u(x) : |\mathbf{n}| = k\}$$

be the set of all partial derivatives of order k. If $k = 1$, we regard the elements of $D^1u(x) =: Du(x)$ as being arranged in a row vector

$$Du = (\partial_{x_1} u, \dots, \partial_{x_d} u).$$

If $k = 2$, we regard the elements of $D^2u(x)$ as being arranged in a matrix

$$D^2u = \begin{pmatrix} \partial_{x_1}\partial_{x_1} u & \dots & \partial_{x_1}\partial_{x_d} u \\ \vdots & \ddots & \vdots \\ \partial_{x_d}\partial_{x_1} u & \dots & \partial_{x_d}\partial_{x_d} u \end{pmatrix}.$$

Hence, the Laplacian Δu of u can be written as

$$\Delta u := \sum_{i=1}^{d} \partial_{x_i}\partial_{x_i} u = \mathrm{tr}[D^2u], \tag{2.2}$$

where $\mathrm{tr} : \mathbb{R}^{d\times d} \to \mathbb{R}$, $\mathbf{B} \mapsto \mathrm{tr}[\mathbf{B}] = \sum_{i=1}^{d} \mathbf{B}_{ii}$ is the trace of a $d \times d$-matrix $\mathbf{B}$. In the following, we write $\partial_{x_i x_j}$ instead of $\partial_{x_i}\partial_{x_j}$ to simplify the notation.

A *partial differential equation* is an equation involving an unknown function of two or more variables and certain of its derivatives. Let $G \subset \mathbb{R}^d$ be open, $x = (x_1, \dots, x_d) \in G$, and $k \in \mathbb{N}$.

Definition 2.2.1 An expression of the form

$$F(D^k u(x), D^{k-1}u(x), \dots, Du(x), u(x), x) = 0, \quad x \in G,$$

is called a kth order partial differential equation, where the function

$$F : \mathbb{R}^{d^k} \times \mathbb{R}^{d^{k-1}} \times \dots \times \mathbb{R}^d \times \mathbb{R} \times G \to \mathbb{R}$$

is given and the function $u : G \to \mathbb{R}$ is the unknown.

Let $a_{ij}(x), b_i(x), c(x)$ and $f(x)$ be given functions. For a *linear first order PDE* in $d + 1$ variables, F has the form

$$F(Du, u, x) = \sum_{i=0}^{d} b_i(x)\partial_{x_i} u + c(x)u - f(x).$$

Setting $x_0 = t$, $x_1 = x$, $b(x) = (b_1(x), b_2(x))^\top = (1, b)^\top$, $b \in \mathbb{R}_+$, and $c = 0$, for example, we obtain the (hyperbolic) transport equation with constant speed b of propagation $\partial_t u + b\partial_x = f(t, x)$.

For a *linear second order PDE* in $d + 1$ variables, F takes the form

$$F(D^2u, Du, u, x) = -\sum_{i,j=0}^{d} a_{ij}(x)\partial_{x_i x_j} u + \sum_{i=0}^{d} b_i(x)\partial_{x_i} u + c(x)u - f(x).$$

Let $b(x) = (b_0(x), \ldots, b_d(x))$ and assume that the matrix $A(x) = \{a_{ij}(x)\}_{i,j=0}^{d}$ is symmetric with real eigenvalues $\lambda_0(x) \le \lambda_1(x) \le \cdots \le \lambda_d(x)$. We can use the eigenvalues to distinguish three types of PDEs: elliptic, parabolic and hyperbolic.

Definition 2.2.2 Let $\mathcal{I} = \{0, \ldots, d\}$. At $x \in \mathbb{R}^{d+1}$, a PDE is called

(i) Elliptic $\Leftrightarrow \lambda_i(x) \neq 0, \forall i \wedge \operatorname{sign}(\lambda_0(x)) = \cdots = \operatorname{sign}(\lambda_d(x))$,
(ii) Parabolic $\Leftrightarrow \exists! j \in \mathcal{I} : \lambda_j(x) = 0 \wedge \operatorname{rank}(A(x), b(x)) = d + 1$,
(iii) Hyperbolic $\Leftrightarrow (\lambda_i(x) \neq 0,\ \forall i) \wedge \exists! j \in \mathcal{I} : \operatorname{sign} \lambda_j(x) \neq \operatorname{sign} \lambda_k(x),\ k \in \mathcal{I} \setminus \{j\}$.

The PDE is called elliptic, parabolic, hyperbolic on G, if it is elliptic, parabolic, hyperbolic at all $x \in G$.

We give a typical example for each type:

(i) The Poisson equation $\Delta u = f(x)$ is elliptic.
(ii) The heat equation $\partial_t u - \Delta u = f(t, x)$ is parabolic (set $x_0 = t$).
(iii) The wave equation $\partial_{tt} u - \Delta u = f(t, x)$ is hyperbolic (set $x_0 = t$).
(iv) The Black–Scholes equation for the value of a European option price $v(t, s)$

$$\partial_t v - \frac{1}{2}\sigma^2 s^2 \partial_{ss} v - rs\partial_s v + rv = 0, \tag{2.3}$$

with volatility $\sigma > 0$ and interest rate $r \ge 0$ is parabolic at $(t, s) \in (0, T) \times (0, \infty)$ and degenerates to an ordinary differential equation as $s \to 0$.

Partial differential equations arising in finance, like the Black–Scholes equation (2.3), are mostly of parabolic type, i.e. they are of the form

$$\partial_t u - \sum_{i,j=1}^{d} a_{ij}(x)\partial_{x_i x_j} u + \sum_{i=1}^{d} b_i(x)\partial_{x_i} u + c(x)u = f(x).$$

Therefore, we introduce the basic concepts for solving parabolic equations in the next section. For illustration purpose, we consider the heat equation. Indeed, setting $s = e^x$, $t = 2\sigma^{-2}\tau$ and

$$v(t, s) = e^{\alpha x + \beta\tau} u(\tau, x), \quad \alpha = 1/2 - r\sigma^{-2}, \quad \beta = -(1/2 + r\sigma^{-2})^2,$$

the Black–Scholes equation (2.3) for $v(t, s)$ can be transformed to the heat equation $\partial_\tau u - \partial_{xx} u = 0$ for $u(\tau, x)$.

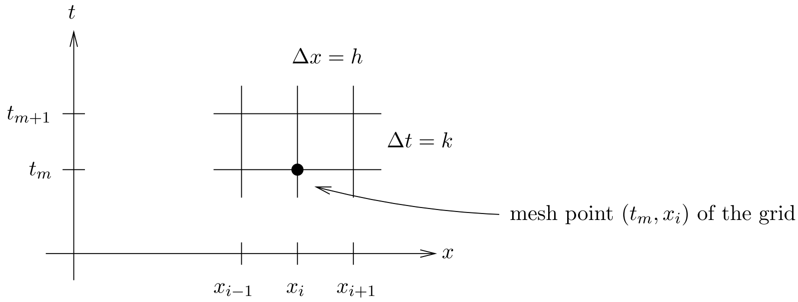

Fig. 2.1 Time-space grid

2.3 Numerical Methods for the Heat Equation

Let the space domain $G = (a, b) \subset \mathbb{R}$ be an open interval and let the time domain $J := (0, T)$ for $T > 0$. Consider the initial–boundary value problem:

$$\begin{aligned}
&\text{Find } u : J \times G \to \mathbb{R} \text{ such that} \\
&\partial_t u - \partial_{xx} u = f(t, x), \qquad && \text{in } J \times G, \\
&u(t, x) = 0, && \text{on } J \times \partial G, \\
&u(0, x) = u_0, && \text{in } G,
\end{aligned} \tag{2.4}$$

where $u(0, x) = u_0$ is the *initial condition* and $u(t, x) = 0$ on the boundary is called *the homogeneous Dirichlet boundary condition*. We explain two numerical methods to find approximations to the solution $u(t, x)$ of the problem (2.4). We start with the finite difference method.

2.3.1 Finite Difference Method

In the finite difference discretization, the domain $J \times G$ is replaced by discrete grid points (t_m, x_i) and the partial derivatives in (2.4) are approximated by difference quotients at the grid points. Let the *space grid points* be given by

$$x_i = a + ih, \; i = 0, 1, \ldots, N + 1, \; h := (b - a)/(N + 1) = \Delta x, \tag{2.5}$$

which are equidistant with mesh width h, and the *time levels* by

$$t_m = mk, \; m = 0, 1, \ldots, M, \; k := T/M = \Delta t. \tag{2.6}$$

The time-space grid is illustrated in Fig. 2.1.

Assume that $f \in C^2(G)$. Then, using Taylor's formula, we have

$$f'(x) = \frac{f(x + h) - f(x)}{h} - \frac{h}{2} f''(\xi), \quad \xi \in (x, x + h).$$

Setting $f_i := f(x_i)$, we obtain as $h \to 0$,

$$f'(x_i) = \frac{f_{i+1} - f_i}{h} + \mathcal{O}(h) =: (\delta_x^+ f)_i + \mathcal{O}(h),$$

where $(\delta_x^+ f)_i$ is called the *one-sided difference quotient* of f with respect to x at x_i. The difference quotient is said to be *accurate of first order* since the remainder term is $\mathcal{O}(h)$ as $h \to 0$. Analogous expressions hold for the time derivative ∂_t.

Higher order finite differences allow obtaining approximations of order $\mathcal{O}(h^p)$ with $p \geq 2$ rather than just $\mathcal{O}(h)$. If the function to be approximated has sufficient *regularity*, we have

$$\begin{aligned}
f'(x_i) &= \frac{f_{i+1} - f_{i-1}}{2h} + \mathcal{O}(h^2) =: (\delta_x f)_i + \mathcal{O}(h^2), && \text{for } f \in C^3(G),\\
f''(x_i) &= \frac{f_{i+1} - 2f_i + f_{i-1}}{h^2} + \mathcal{O}(h^2) =: (\delta_{xx}^2 f)_i + \mathcal{O}(h^2), && \text{for } f \in C^4(G).
\end{aligned}$$

With the difference quotients we turn next to the finite difference discretization of the heat equation (2.4). Let u_i^m denote the approximate value of the solution u at grid point (t_m, x_i), i.e. $u_i^m \approx u(t_m, x_i)$. For a parameter $\theta \in [0, 1]$, we approximate the partial differential operator $\partial_t u - \partial_{xx} u$ at the grid point (t_m, x_i) by the finite difference operator

$$\begin{aligned}
\mathcal{E}_i^m &:= \frac{u_i^{m+1} - u_i^m}{k} - \left[(1-\theta)(\delta_{xx}^2 u)_i^m + \theta(\delta_{xx}^2 u)_i^{m+1}\right]\\
&= \frac{u_i^{m+1} - u_i^m}{k} - \left[(1-\theta)\frac{u_{i+1}^m - 2u_i^m + u_{i-1}^m}{h^2} + \theta\frac{u_{i+1}^{m+1} - 2u_i^{m+1} + u_{i-1}^{m+1}}{h^2}\right],
\end{aligned} \tag{2.7}$$

and replace the partial differential equation (2.4) by the *finite difference equations*

$$\mathcal{E}_i^m = \theta f_i^{m+1} + (1-\theta) f_i^m, \quad i = 1, \dots, N, \quad m = 0, \dots, M-1, \tag{2.8}$$

with initial conditions $u_i^0 = u_0(x_i)$, $i = 1, \dots, N$, and boundary conditions $u_k^m = 0$, $k \in \{0, N+1\}$, $m = 0, \dots, M$. We observe that for $\theta = 0$, u_i^{m+1}, $i = 1, \dots, N$, are given in $\mathcal{E}_i^m = f_i^m$ *explicitly* in terms of u_i^m, i.e. the scheme (2.8) is explicit. For $\theta = 1$, a linear system of equations must be solved for u_i^{m+1} at each time step, i.e. the scheme is *implicit*.

We write (2.8) in matrix form. To this end, we introduce the column vectors

$$\underline{u}^m = (u_1^m, \dots, u_N^m)^\top, \quad \underline{f}^m = (f_1^m, \dots, f_N^m)^\top, \quad \underline{u}_0 = (u_0(x_1), \dots, u_0(x_N))^\top,$$

and the tridiagonal matrices

$$\mathbf{G} = \frac{1}{h^2}\begin{pmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{pmatrix} \in \mathbb{R}^{N\times N}_{\mathrm{sym}}, \quad \mathbf{I} = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \in \mathbb{R}^{N\times N}_{\mathrm{sym}}. \tag{2.9}$$

Then, after multiplication by k, the finite difference scheme (2.8) becomes:

$$\begin{aligned} &\text{Find } \underline{u}^{m+1} \in \mathbb{R}^N \text{ such that for } m = 0, \dots, M-1, \\ &\big(\mathbf{I} + \theta k\mathbf{G}\big)\underline{u}^{m+1} = \big(\mathbf{I} - (1-\theta)k\mathbf{G}\big)\underline{u}^m + k(\theta\underline{f}^{m+1} + (1-\theta)\underline{f}^m), \\ &\underline{u}^0 = \underline{u}_0. \end{aligned} \tag{2.10}$$

We now show that the vectors $\underline{u}^m$ converge towards the exact solution as $k \to 0$ and $h \to 0$.

2.3.2 *Convergence of the Finite Difference Method*

Naturally, by the transition from the PDE (2.4) to the finite difference equations (2.8), which is called the *discretization* of the PDE, an error is introduced, the so-called *discretization error*, which we analyze next. We begin with the definition of a related *consistency error*.

Definition 2.3.1 The consistency error E_i^m at (t_m, x_i) is the difference scheme (2.7) with u_i^m in $\mathcal{E}_i^m$ replaced by $u(t_m, x_i)$.

Using Taylor expansions of the exact solution at the grid point (t_m, x_i), we can readily estimate the consistency errors E_i^m in terms of powers of the mesh width h and the time step size k.

Proposition 2.3.2 *If the exact solution $u(t,x)$ of (2.4) is sufficiently smooth, then, as $h \to 0$, $k \to 0$, the following estimates hold for $m = 1, \dots, M-1$ and $i = 1, \dots, N$:*

$$|E_i^m| \le C(u)(h^2 + k), \quad 0 \le \theta \le 1, \tag{2.11}$$

$$|E_i^m| \le C(u)(h^2 + k^2), \quad \theta = \frac{1}{2}, \tag{2.12}$$

where the constant $C(u) > 0$ depends on the exact solution u and its derivatives.

For the convergence of the FDM, we are interested in estimating the error between the finite difference solution u_i^m and the exact solution $u(t,x)$ at the grid point (t_m, x_i). We collect the discretization errors in the grid points at time t_m in the *error vector* $\underline{\varepsilon}^m$, i.e.

$$\varepsilon_i^m := u(t_m, x_i) - u_i^m, \quad 0 \le i \le N+1, \quad 0 \le m \le M. \tag{2.13}$$

The error vectors $\{\underline{\varepsilon}^m\}_{m=0}^M$ satisfy the difference equation

$$\big(k^{-1}\mathbf{I} + \theta\mathbf{G}\big)\underline{\varepsilon}^{m+1} + \big(-k^{-1}\mathbf{I} + (1-\theta)\mathbf{G}\big)\underline{\varepsilon}^m = \underline{E}^m \tag{2.14}$$

or, in explicit form,

$$\underline{\varepsilon}^{m+1} = \mathbf{A}_\theta \underline{\varepsilon}^m + \underline{\eta}^m, \tag{2.15}$$

where $\underline{\eta}^m := (k^{-1}\mathbf{I} + \theta\mathbf{G})^{-1}\underline{E}^m$, and where $\mathbf{A}_\theta := (k^{-1}\mathbf{I} + \theta\mathbf{G})^{-1}(-k^{-1}\mathbf{I} + (1-\theta)\mathbf{G})$, is called an *amplification matrix*.

The recursion (2.15) shows that the discretization error ε_i^m is related to the consistency error E_i^m. Estimates on ε_i^m can be obtained by taking norms in the recursion (2.15). Using induction on m, we have

Proposition 2.3.3 *For all $M \in \mathbb{N}$, $1 \le m \le M$, one has*

$$\|\underline{\varepsilon}^m\|_{\ell_2} \le \|\mathbf{A}_\theta\|_2^m, \|\underline{\varepsilon}^0\|_{\ell_2} + k \sum_{n=0}^{m-1} \|\mathbf{A}_\theta\|_2^{m-1-n} \|\underline{E}^n\|_{\ell_2}, \tag{2.16}$$

where $\|\underline{\varepsilon}^m\|_{\ell_2}^2 = \sum_{i=0}^{N+1} |\varepsilon_i^m|^2$.

We see from (2.16) that the discretization errors ε_i^m can be controlled in terms of the consistency errors E_i^m *provided* the norm $\|\mathbf{A}_\theta\|_2$ is bounded by 1. The condition that the norm of the amplification matrix $\mathbf{A}_\theta$ is bounded by 1 is a *stability condition* for the FDM. We obtain immediately

Theorem 2.3.4 *If the stability condition*

$$\|\mathbf{A}_\theta\|_2 \le 1 \tag{2.17}$$

holds, then, as $M \to \infty$, $N \to \infty$, the FDM (2.10) *converges and, if $\underline{\varepsilon}^0 = \underline{0}$,*

$$\sup_m \|\underline{\varepsilon}^m\|_{\ell_2} \le T \sup_m \|\underline{E}^m\|_{\ell_2}. \tag{2.18}$$

We want to discuss the validity of the stability condition (2.17) for $\mathbf{G}$ as in (2.9). Therefore, we need the following lemma to obtain the eigenvalues for a tridiagonal matrix. It follows immediately by elementary calculations.

Lemma 2.3.5 *Let* $\mathbf{X} \in \mathbb{R}^{(N-1)\times(N-1)}$ *be a tridiagonal matrix given by*

$$\mathbf{X} = \begin{pmatrix} \alpha & \beta & & \\ \gamma & \alpha & \ddots & \\ & \ddots & \ddots & \beta \\ & & \gamma & \alpha \end{pmatrix}$$

Then, $\mathbf{X}\underline{v}^{(\ell)} = \mu_\ell \underline{v}^{(\ell)}$, $\ell = 1, \dots, N-1$, *with the eigensystem*

$$\mu_\ell = \alpha + 2\beta\sqrt{\beta^{-1}\gamma}\cos\left(N^{-1}\ell\pi\right), \quad \underline{v}^{(\ell)} = \left((\beta^{-1}\gamma)^{j/2}\sin(N^{-1}j\ell\pi)\right)_{j=1}^{N-1}.$$

For $\mathbf{G}$ resulting from the finite differences discretization, we obtain

Corollary 2.3.6 *The eigensystem of the matrix* $\mathbf{G}$ *in* (2.9) *is given by*

$$\mu_\ell = \frac{4}{h^2}\sin^2\left(\frac{\ell\pi}{2(N+1)}\right), \quad \underline{v}^{(\ell)} = \left(\sin\left(\frac{j\ell\pi}{N+1}\right)\right)_{j=1}^{N}, \quad \ell = 1, \dots, N. \quad (2.19)$$

Using Corollary 2.3.6, we find that

$$\lambda_\ell(\mathbf{A}_\theta) = \frac{4(1-\theta)h^{-2}k\sin^2(\ell\pi/(2(N+1))) - 1}{1 + 4\theta h^{-2}k\sin^2(\ell\pi/(2(N+1)))}, \quad \ell = 1, \dots, N.$$

Since $\|\mathbf{A}_\theta\|_2 = \max_\ell |\lambda_\ell|$, we have $\|\mathbf{A}_\theta\|_2 \le 1$, if $2(1-2\theta)h^{-2}k \le 1$. For $0 \le \theta < \frac{1}{2}$, we obtain the so-called *CFL-condition* (after the seminal paper of Courant, Friedrichs and Lewy [43]),

$$\frac{k}{h^2} \le \frac{1}{2(1-2\theta)}. \quad (2.20)$$

Therefore, we obtain directly from Theorem 2.3.4:

Lemma 2.3.7 (Stability of the θ-scheme)

(i) *If* $\frac{1}{2} \le \theta \le 1$, *the scheme* (2.10) *is stable for all* k *and* h.
(ii) *If* $0 \le \theta < \frac{1}{2}$, *the scheme* (2.10) *is stable if and only if the CFL-condition* (2.20) *holds.*

We can now combine the results obtained so far using Lemma 2.3.7, Proposition 2.3.3 and Proposition 2.3.2 to obtain a convergence result for the θ-scheme. We measure the error using the quantity $\sup_m h^{\frac{1}{2}}\|\underline{e}^m\|_{\ell_2}$ which is a discrete version of the $L^\infty(J; L^2(G))$-norm.

Theorem 2.3.8 *If $u \in C^4(\overline{J} \times \overline{G})$, we have*

(i) *For $\frac{1}{2} < \theta \leq 1$ or for $0 \leq \theta < \frac{1}{2}$ and* (2.20),

$$\sup_m h^{\frac{1}{2}} \|\underline{\varepsilon}^m\|_{\ell_2} \leq C(u)(h^2 + k);$$

(ii) *For $\theta = \frac{1}{2}$,*

$$\sup_m h^{\frac{1}{2}} \|\underline{\varepsilon}^m\|_{\ell_2} \leq C(u)(h^2 + k^2),$$

where the constant $C(u) > 0$ depends on the exact solution u and its derivatives.

We next explain the finite element method which is based on variational formulations of the differential equations.

2.3.3 Finite Element Method

For the discretization with finite elements, we use the *method of lines* where we first only discretize in space to obtain a system of coupled ordinary differential equations (ODEs). In a second step, a time discretization scheme is applied to solve the ODEs. We do not require the PDE (2.4) to hold pointwise in space but only in the variational sense. Therefore, we fix $t \in J$ and let $v \in C_0^\infty(G)$ be a smooth test function satisfying $v(a) = v(b) = 0$. We multiply the PDE with v, integrate with respect to the space variable x and use integration by parts to obtain

$$\int_a^b \partial_t u v \,\mathrm{d}x - \int_a^b \partial_{xx} u v \,\mathrm{d}x = \int_a^b f v \,\mathrm{d}x,$$

$$\Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t} \int_a^b u v \,\mathrm{d}x - \underbrace{\Big[\partial_x u(x,t) v(x)\Big]_{x=a}^{x=b}}_{=0} + \int_a^b \partial_x u \partial_x v \,\mathrm{d}x = \int_a^b f v \,\mathrm{d}x.$$

Therefore, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_a^b u v \,\mathrm{d}x + \int_a^b \partial_x u \partial_x v \,\mathrm{d}x = \int_a^b f v \,\mathrm{d}x, \quad \forall v \in C_0^\infty(G). \tag{2.21}$$

Since $C_0^\infty(G)$ is not a closed subspace of $L^2(G)$, we will consider test functions in the *Sobolev space* $H_0^1(G)$ which is the closure of $C_0^\infty(G)$ in the H^1-norm,

$$\|u\|_{H^1(G)}^2 = \|u\|_{L^2(G)}^2 + \|u'\|_{L^2(G)}^2.$$

The Sobolev space $H_0^1(G)$ consists of all continuous functions which are piecewise differentiable and vanish at the boundary. Since (2.21) holds for all $v \in C_0^\infty(G)$, (2.21) also holds for all $v \in H_0^1(G)$ because $C_0^\infty(G)$ is dense in $H_0^1(G)$. The *weak* or *variational formulation* of (2.4) reads:

$$\begin{aligned}
&\text{Find } u \in C(J, H_0^1(G)) \cap C^1(J, L^2(G)) \text{ such that for } t \in J \\
&\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b uv\,\mathrm{d}x + \int_a^b \partial_x u \partial_x v\,\mathrm{d}x = \int_a^b fv\,\mathrm{d}x, \quad \forall v \in H_0^1(G), \\
&u(0,\cdot) = u_0,
\end{aligned} \tag{2.22}$$

where we assume that the initial condition $u_0 \in L^2(G)$. The finite element method is based on the *Galerkin discretization* of (2.22). The idea is to project (2.22) to a finite dimensional subspace $V_N \subset H_0^1(G)$ and to replace (2.22) by:

$$\begin{aligned}
&\text{Find } u_N \in C^1(J, V_N), \text{ such that for } t \in J \\
&\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b u_N v_N\,\mathrm{d}x + \int_a^b \partial_x u_N \partial_x v_N\,\mathrm{d}x = \int_a^b f v_N\,\mathrm{d}x, \quad \forall v_N \in V_N, \\
&u_N(0) = u_{N,0},
\end{aligned} \tag{2.23}$$

where $u_{N,0}$ is an approximation of u_0 in V_N. For example, $u_{N,0} = \mathcal{P}_N u_0$, the L^2-projection of u_0 on V_N, satisfying $\int_G \mathcal{P}_N u_0 v_N\,\mathrm{d}x = \int_G u_0 v_N\,\mathrm{d}x$ for all $v_N \in V_N$.

We show that (2.23) is equivalent to a linear system of ordinary differential equations (in time). Let b_j, $j = 1, \ldots, N$ be a basis of V_N. Since $u_N(t,\cdot) \in V_N$, we have

$$u_N(t,x) = \sum_{j=1}^N u_{N,j}(t) b_j(x),$$

where $u_{N,j}(t)$ denote the time dependent coefficients of u_N with respect to the basis of V_N. Inserting this series representations into (2.23) yields for $v_N(x) = b_i(x)$, $i = 1, \ldots, N$,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b u_N(t,x)v_N(x)\,\mathrm{d}x + \int_a^b \partial_x u_N(t,x)\partial_x v_N(x)\,\mathrm{d}x = \int_a^b f(t,x)v_N(x)\,\mathrm{d}x, \\
\Rightarrow\quad &\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b \sum_{j=1}^N u_{N,j}(t)b_j(x)b_i(x)\,\mathrm{d}x + \int_a^b \sum_{j=1}^N u_{N,j}(t)b_j'(x)b_i'(x)\,\mathrm{d}x \\
&= \int_a^b f(t,x)b_i(x)\,\mathrm{d}x, \\
\Rightarrow\quad &\sum_{j=1}^N \dot{u}_{N,j}\int_a^b b_j(x)b_i(x)\,\mathrm{d}x + \sum_{j=1}^N u_{N,j}\int_a^b b_j'(x)b_i'(x)\,\mathrm{d}x \\
&= \int_a^b f(t,x)b_i(x)\,\mathrm{d}x.
\end{aligned}$$

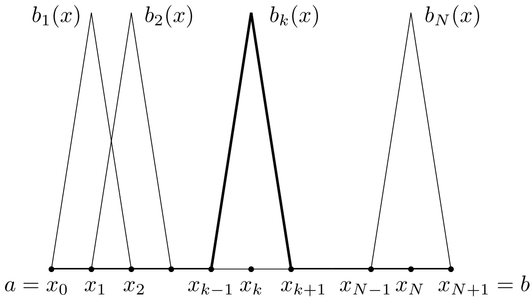

Fig. 2.2 Basis functions $b_i(x)$, $i = 1, \ldots, N$

With matrices $\mathbf{M}, \mathbf{A} \in \mathbb{R}^{N \times N}$ and vector $\underline{f}(t) \in \mathbb{R}^N$ given by

$$\mathbf{M}_{ij} := \int_a^b b_j(x) b_i(x)\,\mathrm{d}x, \quad \mathbf{A}_{ij} := \int_a^b b_j'(x) b_i'(x)\,\mathrm{d}x,$$
$$f_i(t) := \int_a^b f(t,x) b_i(x)\,\mathrm{d}x,$$

we obtain the weak semi-discretization (2.23) in *matrix form*:

$$\begin{aligned} &\text{Find } \underline{u}_N \in C^1(J; \mathbb{R}^N), \text{ such that for } t \in J \\ &\mathbf{M}\underline{\dot{u}}_N(t) + \mathbf{A}\underline{u}_N(t) = \underline{f}(t), \\ &\underline{u}_N(0) = \underline{u}_0, \end{aligned} \tag{2.24}$$

where $\underline{u}_0$ denotes the coefficient vector of $u_{N,0} = \sum_{i=1}^N u_{0,i} b_i$.

For the basis functions b_i of $V_N = \operatorname{span}\{b_i(x) : i = 1, \ldots, N\}$, we take the so-called *hat functions*

$$b_i : [a,b] \to \mathbb{R}_{\geq 0}, \quad b_i(x) = \max\{0, 1 - h^{-1}|x - x_i|\}, \quad i = 1, \ldots, N,$$

as illustrated in Fig. 2.2.

For equidistant mesh points x_i, $i = 1, \ldots, N$ with mesh width h as in (2.5),

$$x_i = a + ih, \ i = 0, 1, \ldots, N+1, \ h := (b-a)/(N+1) = \Delta x,$$

we obtain the matrices, $\mathbf{M}, \mathbf{A} \in \mathbb{R}^{N \times N}_{\text{sym}}$, given by

$$\mathbf{M} = \frac{h}{6}\begin{pmatrix} 4 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & 4 \end{pmatrix}, \quad \mathbf{A} = \frac{1}{h}\begin{pmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{pmatrix}. \tag{2.25}$$

Table 2.1 Difference between finite differences and finite elements

	FDM	FEM
$\underline{u}^m$	vector of $u_i^m \approx u(t_m, x_i)$	coeff. vector of $u_N(t_m, x)$
$\mathbf{B}$	$\mathbf{I} + k\theta\mathbf{G}$	$\mathbf{M} + k\theta\mathbf{A}$
$\mathbf{C}$	$\mathbf{I} - k(1-\theta)\mathbf{G}$ $\mathbf{G} = h^{-2}\mathrm{tridiag}(-1, 2, -1)$	$\mathbf{M} - k(1-\theta)\mathbf{A}$ $\mathbf{A} = h^{-1}\mathrm{tridiag}(-1, 2, -1)$
f_i^m	$f(t_m, x_i)$	$\int_a^b f(t_m, x) b_i(x)\, \mathrm{d}x$

It remains to discretize the ODE (2.24). Proceeding exactly as in the FDM, we choose time levels t_m, $m = 0, \dots, M$ as in (2.6)

$$t_m = mk,\ m = 0, 1, \dots, M,\ k := T/M = \Delta t,$$

and denote $\underline{u}_N^m := u_N(t_m)$ and $\underline{f}^m := \underline{f}(t_m)$. Then, the fully discrete scheme reads:

$$\begin{aligned} &\text{Find } \underline{u}_N^{m+1} \in \mathbb{R}^N \text{ such that for } m = 0, \dots, M-1, \\ &(\mathbf{M} + k\theta\mathbf{A})\underline{u}_N^{m+1} = (\mathbf{M} - k(1-\theta)\mathbf{A})\underline{u}_N^m + k\big(\theta\underline{f}^{m+1} + (1-\theta)\underline{f}^m\big), \\ &\underline{u}_N^0 = \underline{u}_0. \end{aligned} \tag{2.26}$$

Thus, in both the finite difference and the finite element method we have to solve M systems of N linear equations of the form

$$\mathbf{B}\underline{u}^{m+1} = \mathbf{C}\underline{u}^m + k\underline{F}^m, \quad m = 0, \dots, M-1,$$

where $\underline{F}^m = \theta\underline{f}^{m+1} + (1-\theta)\underline{f}^m$. The difference between FDM and FEM is shown in Table 2.1.

From Table 2.1 we see that both discretization schemes for the heat equation lead to similar linear systems of equations to be solved in each timestep. For all the partial differential equations which we will encounter in these note, we will use both finite differences and finite elements for the discretization.

Example 2.3.9 Let $G = (0, 1)$, $T = 1$, and $u(t, x) = e^{-t} x \sin(\pi x)$. We measure the discrete $L^\infty(0, T; L^2(G))$-error defined by $\sup_m h^{\frac{1}{2}} \|\underline{\varepsilon}^m\|_{\ell_2}$ where

$$\|\underline{\varepsilon}^m\|_{\ell_2}^2 := \sum_{i=1}^N |u(t_m, x_i) - u_i^m|^2.$$

For $\theta = 1$ (backward Euler), we let $h = \mathcal{O}(\sqrt{k})$ and obtain first order convergence with respect to the time step k both for FDM and FEM, i.e.

$$\sup_m h^{\frac{1}{2}} \|\underline{\varepsilon}^m\|_{\ell_2} = \mathcal{O}(k). \tag{2.27}$$

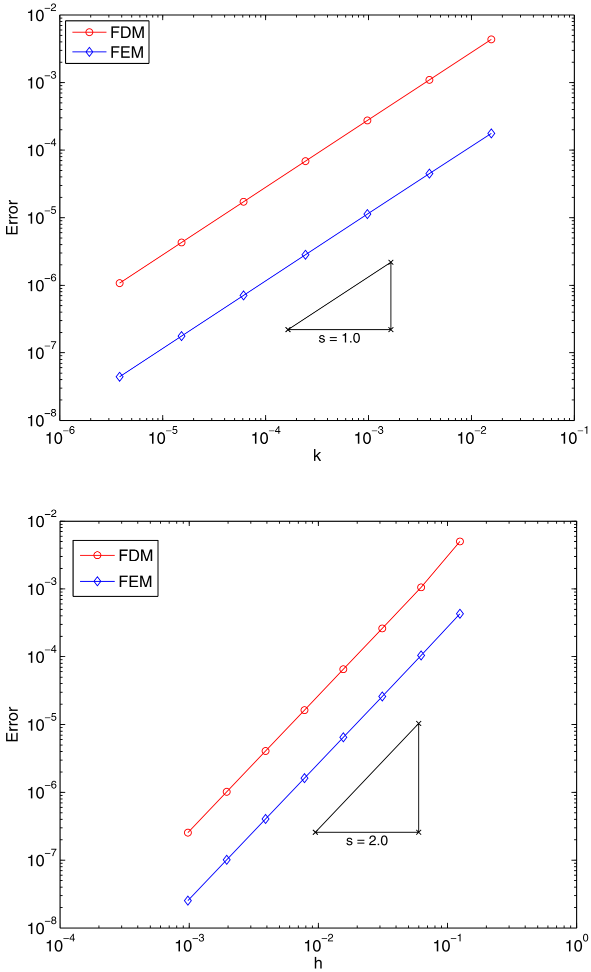

Fig. 2.3 $L^\infty(J; L^2(G))$ convergence rates for $\theta = 1$ (*top*) and $\theta = \frac{1}{2}$ (*bottom*)

For $\theta = \frac{1}{2}$ (Crank–Nicolson), we let $k = \mathcal{O}(h)$ and obtain, in terms of the mesh width h, second order convergence for both FDM and FEM, i.e.

$$\sup_m h^{\frac{1}{2}} \|\underline{\varepsilon}^m\|_{\ell_2} = \mathcal{O}(h^2). \tag{2.28}$$

Both convergence rates are shown in Fig. 2.3.

The convergence rates (2.27)–(2.28) have been shown for the finite difference method in Theorem 2.3.8. In the next chapter, we show that these also hold for the finite element method.

2.4 Further Reading

A nice introduction to the mathematical theory of partial differential equations is given in Evans [65]. The mathematical theory of the finite difference and finite element methods for elliptic problems is introduced in the text of Braess [24], and for parabolic and hyperbolic equations in Larsson and Thomée [112]. Finite difference methods for time dependent problems are studied in more details in Gustafsson et al. [76]. For an elementary introduction to the finite element methods with particular attention to stabilized finite element methods for partial differential equations of the type which arise in finance, we refer to Johnson [99].

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Chapter 3
Finite Element Methods for Parabolic Problems

The finite element methods are an alternative to the finite difference discretization of partial differential equations. The advantage of finite elements is that they give convergent deterministic approximations of option prices under realistic, low smoothness assumptions on the payoff function as, e.g. for binary contracts. The basis for finite element discretization of the pricing PDE is a *variational formulation* of the equation. Therefore, we introduce the Sobolev spaces needed in the variational formulation and give an abstract setting for the parabolic PDEs. All pricing equations satisfied by the price u of plain vanilla contracts which we encounter in these notes have the general form

$$\partial_t u - \mathcal{A}u = f, \quad \text{in } J \times G, \tag{3.1}$$

with the initial condition $u(0,x) = u_0(x)$, $x \in G$, a linear second order partial (integro-) differential operator $\mathcal{A}$, the forcing term f, the space domain $G \subseteq \mathbb{R}^d$ and the time interval $J = (0,T)$. The finite element approximation of the operators $\mathcal{A}$ fits into an abstract parabolic framework which we present next.

3.1 Sobolev Spaces

We now introduce some particular Hilbert spaces which are natural to use in the study of partial differential equations. These spaces consist of functions which are square integrable together with their partial derivatives up to a certain order.

Let $G = (a,b) \subset \mathbb{R}$ be an open, possibly unbounded domain and let first $u \in C^1(\overline{G})$. Integration by parts yields

$$\int_G u'\varphi \,\mathrm{d}x = -\int_G u\varphi' \,\mathrm{d}x, \quad \forall \varphi \in C_0^1(G).$$

N. Hilber et al., *Computational Methods for Quantitative Finance*, Springer Finance,
DOI 10.1007/978-3-642-35401-4_3,

If $u \in L^2(G)$, then u' does not necessarily exist in the classical sense, but we may define u' to be the linear functional

$$u^*(\varphi) = -\int_G u\varphi' \,\mathrm{d}x, \quad \forall \varphi \in C_0^1(G).$$

This functional is said to be a *generalized* or *weak derivative* of u. When u^* is bounded in $L^2(G)$, it follows from Riesz representation theorem (see Theorem 2.1.1) that there exists a unique function $w \in L^2(G)$ such that $u^*(\varphi) = (w, \varphi)$ for all $\varphi \in L^2(G)$, in particular

$$-\int_G u\varphi' \,\mathrm{d}x = \int_G w\varphi \,\mathrm{d}x, \quad \forall \varphi \in C_0^1(G).$$

We then say that the weak derivative belongs to $L^2(G)$ and write $u' = w$. In particular, if $u \in C^1(\overline{G})$, the generalized derivative u' coincides with the classical derivative u'. In a similar way, we can define weak derivatives $D^n u$ of higher order $n \in \mathbb{N}$.

Definition 3.1.1 The linear functional $D^n u$, $n \in \mathbb{N}$ is a *weak derivative* of u if

$$\int_G D^n u\varphi \,\mathrm{d}x = (-1)^n \int_G u D^n \varphi \,\mathrm{d}x, \quad \forall \varphi \in C_0^n(G).$$

We can now define the spaces $H^m(G)$.

Definition 3.1.2 Let $m \in \mathbb{N}$. $H^m(G)$ is the space of all functions whose weak partial derivatives of order $\leq m$ belong to $L^2(G)$, i.e.

$$H^m(G) = \{u \in L^2(G) : D^n u \in L^2(G) \text{ for } n \leq m\}.$$

We equip $H^m(G)$ with the inner product

$$(u, v)_{H^m(G)} = \sum_{n=0}^{m} (D^n u, D^n v)_{L^2(G)},$$

and the corresponding norm

$$\|u\|_{H^m(G)}^2 = (u, u)_{H^m(G)} = \sum_{n=0}^{m} \|D^n u\|_{L^2(G)}^2.$$

We sometimes omit the (G) if the domain is clear from the context. $H^m(G)$ is complete and thus a Hilbert space. The space $H^m(G)$ is an example of a more general class of function spaces, called *Sobolev spaces*.

Definition 3.1.3 Let $p \in \mathbb{N} \cup \{\infty\}$. $W^{m,p}(G)$ is the space of all functions whose weak partial derivatives of order $\leq m$ belong to $L^p(G)$, i.e.

$$W^{m,p}(G) = \{u \in L^p(G) \;:\; D^n u \in L^p(G) \text{ for } n \leq m\}.$$

We equip $W^{m,p}(G)$ with the norm

$$\|u\|^p_{W^{m,p}(G)} = \sum_{n=0}^{m} \|D^n u\|^p_{L^p(G)}.$$

The normed space $W^{m,p}(G)$ is complete and hence a Banach space for $1 \leq p \leq \infty$. Functions $u \in W^{1,p}(G)$ are "essentially" continuous.

Theorem 3.1.4 *Let G be bounded and $u \in W^{1,p}(G)$. Then, there exists a continuous function $\widetilde{u} \in C^0(\overline{G})$ such that $u = \widetilde{u}$ a.e. on G and for all $x_1, x_2 \in \overline{G}$ there holds*

$$\widetilde{u}(x_2) - \widetilde{u}(x_1) = \int_{x_1}^{x_2} u'(\xi)\,\mathrm{d}\xi. \tag{3.2}$$

Proof Fix $y_0 \in G$ and set for any $g \in L^p(G)$,

$$v(x) := \int_{y_0}^{x} g(t)\,\mathrm{d}t, \quad x \in G.$$

Then, $v \in C^0(\overline{G})$ and

$$\begin{aligned}\int_G v\varphi'\,\mathrm{d}x &= \int_G \left(\int_{y_0}^{x} g(t)\,\mathrm{d}t\right)\varphi'(x)\,\mathrm{d}x \\ &= -\int_a^{y_0}\int_x^{y_0} g(t)\varphi'(x)\,\mathrm{d}t\,\mathrm{d}x + \int_{y_0}^{b}\int_{y_0}^{x} g(t)\varphi'(x)\,\mathrm{d}t\,\mathrm{d}x.\end{aligned}$$

Fubini's theorem implies, $\forall\varphi \in C^1_0(G)$,

$$\begin{aligned}\int_G v\varphi'\,\mathrm{d}x &= -\int_a^{y_0} g(t)\int_a^{t}\varphi'(x)\,\mathrm{d}x\,\mathrm{d}t + \int_{y_0}^{b} g(t)\int_t^{b}\varphi'(x)\,\mathrm{d}x\,\mathrm{d}t \\ &= -\int_G g(t)\varphi(t)\,\mathrm{d}t. \end{aligned}\tag{3.3}$$

We set $\overline{u}(x) := \int_{y_0}^{x} u'(\xi)\,\mathrm{d}\xi$. With (3.3) we obtain

$$\int_G \overline{u}\varphi'\,\mathrm{d}x = -\int_G u'\varphi\,\mathrm{d}x, \quad \forall\varphi \in C^1_0(G),$$

and hence with the definition of the weak derivative,

$$\int_G (u-\overline{u})\varphi'\,\mathrm{d}x = 0, \quad \forall\varphi \in C^1_0(G).$$

Therefore, it follows that for a.e. $x \in G$, we have $u(x) - \overline{u}(x) = C$. Putting $\widetilde{u} := \overline{u} + C$, we obtain the result. □

We will also need spaces with boundary conditions where we impose $u = 0$ on ∂G.

Definition 3.1.5 Let $1 \leq p < \infty$. Then, $W_0^{1,p}$ is the closure of C_0^1 in the $W^{1,p}$-norm,

$$W_0^{1,p}(G) = \overline{C_0^1(G)}^{\|\cdot\|_{W^{1,p}(G)}}.$$

The space $W_0^{1,p}(G) \subset W^{1,p}(G)$ is a closed linear subspace. In particular, $H_0^1(G) := W_0^{1,2}(G)$ is again a Hilbert space with the norm $\|\cdot\|_{H^1(G)}$. We have the important *Poincaré inequality*.

Theorem 3.1.6 (Poincaré inequality) *Assume that $G \subset \mathbb{R}$ bounded. Then,*

(i) *There exists a constant $C(|G|, p) > 0$ such that*

$$\|u\|_{L^p(G)} \leq C\,\|u'\|_{L^p(G)}, \quad \forall u \in W_0^{1,p}(G). \tag{3.4}$$

(ii) *Define*

$$W_*^{1,p}(G) := \Big\{u \in W^{1,p}(G) : \int_G u \,\mathrm{d}x = 0\Big\}. \tag{3.5}$$

Then, (3.4) *holds also for all $u \in W_*^{1,p}(G)$, with different C.*

Proof

(i) Let $u \in W_0^{1,p}(G)$, $G = (x_1, x_2)$ be arbitrary, but fixed. By Theorem 3.1.4, there exists $\widetilde{u} \in C^0(\overline{G})$ such that $u = \widetilde{u}$ for a.e. $x \in \overline{G}$ and such that $\widetilde{u}(x_1) = 0$. Therefore, using Hölder's inequality,

$$|\widetilde{u}(x)| = |\widetilde{u}(x) - \widetilde{u}(x_1)| = \left|\int_{x_1}^{x} u'(\xi)\,\mathrm{d}\xi\right| \leq |x - x_1|^{\frac{1}{q}}\ \|u'\|_{L^p(G)},$$

where $\frac{1}{q} + \frac{1}{p} = 1$. Hence, the result follows with $C = (\int_G |x - x_1|^{\frac{p}{q}}\,\mathrm{d}x)^{\frac{1}{p}}$.

(ii) Let $u \in W_*^{1,p}(G)$. Then, there exists $\widetilde{u} \in C^0(\overline{G})$ such that $u = \widetilde{u}$ for a.e. $x \in \overline{G}$ and such that $\int_G \widetilde{u}\,\mathrm{d}x = 0$. Therefore, there is $x^* \in \overline{G}$ such that $\widetilde{u}(x^*) = 0$. We may repeat therefore the proof of (i) with $u(x_1)$ replaced by $u(x^*)$. Taking the supremum over all possible values of x^* gives the result. □

In Chap. 2, we have already introduced the Bochner spaces $L^p(J; \mathcal{H})$ which consist of functions $u : J \to \mathcal{H}$ such that the $L^p(J; \mathcal{H})$-norm is finite. For the theory of parabolic PDEs, it will prove essential to consider maps $u : J \to \mathcal{H}$ which are also differentiable (in time). We call u' the *weak derivative* of u if

$$\int_J u'(t)\varphi(t)\,\mathrm{d}t = -\int_J u(t)\varphi'(t)\,\mathrm{d}t, \quad \forall \varphi \in C_0^1(J).$$

Definition 3.1.7 Let $\mathcal{H}$ be a real Hilbert space with the norm $\|\cdot\|_{\mathcal{H}}$. For $J=(0,T)$ with $T>0$, and $1\le p\le\infty$, the space $W^{1,p}(J;\mathcal{H})$ is defined by

$$W^{1,p}(J;\mathcal{H}) := \{u \in L^p(J;\mathcal{H}) : u' \in L^p(J;\mathcal{H})\},$$

with the norm

$$\|u\|_{W^{1,p}(J;\mathcal{H})} := \begin{cases} (\int_J \|u(t)\|_{\mathcal{H}}^p + \|u'(t)\|_{\mathcal{H}}^p \,\mathrm{d}t)^{1/p} & \text{if } 1\le p<\infty, \\ \operatorname{ess\,sup}_J (\|u(t)\|_{\mathcal{H}} + \|u'(t)\|_{\mathcal{H}}) & \text{if } p=\infty. \end{cases}$$

We again denote by $H^1(J;\mathcal{H}) := W^{1,2}(J;\mathcal{H})$.

3.2 Variational Parabolic Framework

Let $\mathcal{V}\subset\mathcal{H}$ be Hilbert spaces with continuous, dense embedding. We identify $\mathcal{H}$ with its dual $\mathcal{H}^*$ and obtain the triplet

$$\mathcal{V}\subset\mathcal{H}\equiv\mathcal{H}^*\subset\mathcal{V}^*. \tag{3.6}$$

Denote by $(\cdot,\cdot)_{\mathcal{H}}$ the inner product on $\mathcal{H}$, and let $\|\cdot\|_{\mathcal{V}}$, $\|\cdot\|_{\mathcal{H}}$ be the norms on $\mathcal{V}$ and $\mathcal{H}$, respectively. Furthermore, let $J=(0,T)$ with $T>0$, $f\in L^2(J;\mathcal{V}^*)$ and $u_0\in\mathcal{H}$. Consider the variational setting of (3.1):

$$\begin{aligned} &\text{Find } u\in L^2(J;\mathcal{V})\cap H^1(J;\mathcal{V}^*) \text{ such that} \\ &\frac{\mathrm{d}}{\mathrm{d}t}\langle u,v\rangle_{\mathcal{V}^*,\mathcal{V}} + a(u,v) = \langle f,v\rangle_{\mathcal{V}^*,\mathcal{V}}, \ \forall v\in\mathcal{V}, \ \text{ a.e. in } J, \\ &u(0)=u_0, \end{aligned} \tag{3.7}$$

where $\langle\cdot,\cdot\rangle_{\mathcal{V}^*,\mathcal{V}}$ denotes the extension of the $\mathcal{H}$-inner product as *duality pairing* in $\mathcal{V}^*\times\mathcal{V}$. In particular, by Riesz representation theorem, we have $\langle u,v\rangle_{\mathcal{V}^*,\mathcal{V}} = (u,v)_{\mathcal{H}}$, for all $u\in\mathcal{H}$, $v\in\mathcal{V}$. The *bilinear form* $a(\cdot,\cdot):\mathcal{V}\times\mathcal{V}\to\mathbb{R}$ is associated with the operator $\mathcal{A}\in\mathcal{L}(\mathcal{V},\mathcal{V}^*)$ in (3.1) via

$$a(u,v) := -\langle\mathcal{A}u,v\rangle_{\mathcal{V}^*,\mathcal{V}}, \quad \forall u,v\in\mathcal{V},$$

where we denote by $\mathcal{L}(\mathcal{V},\mathcal{W})$ the vector space of linear and continuous operators $\mathcal{A}:\mathcal{V}\to\mathcal{W}$.

Remark 3.2.1

(i) $\frac{\mathrm{d}}{\mathrm{d}t}$ in (3.7) is understood as a weak derivative.
(ii) The choice of the space $\mathcal{V}$ depends on the operator $\mathcal{A}$. Often we can choose $\mathcal{H}=L^2$. The choice of $\mathcal{V}$ is usually the closure of a dense subspace of smooth functions, such as C_0^∞, with respect to the 'energy' norm induced by $\mathcal{A}$.

(iii) We only require $u(0) = u_0$ in $\mathcal{H}$. In particular, for well-posedness of the equation, it is only required that $u_0 \in L^2$, *not* that $u_0 \in \mathcal{V}$. This is important, e.g. for binary contracts, where the payoff u_0 is discontinuous (and, therefore, does not belong to $\mathcal{V}$ in general).

(iv) The bilinear form $a(\cdot,\cdot)$ is, in general, *not* symmetric due to the presence of a drift term in the operator $\mathcal{A}$.

We have the following general result for the existence of weak solutions of the abstract parabolic problem (3.7). A proof is given in Appendix B, Theorem B.2.2. See also [64, 65, 115].

Theorem 3.2.2 *Assume that the bilinear form $a(\cdot,\cdot)$ in (3.7) is continuous, i.e. there is $C_1 > 0$ such that*

$$|a(u,v)| \le C_1 \|u\|_{\mathcal{V}} \|v\|_{\mathcal{V}}, \quad \forall u, v \in \mathcal{V}, \tag{3.8}$$

and satisfies the "Gårding inequality", i.e. there are $C_2 > 0$, $C_3 \ge 0$ such that

$$a(u,u) \ge C_2 \|u\|_{\mathcal{V}}^2 - C_3 \|u\|_{\mathcal{H}}^2, \quad \forall u \in \mathcal{V}. \tag{3.9}$$

Then, problem (3.7) admits a unique solution $u \in L^2(J;\mathcal{V}) \cap H^1(J;\mathcal{V}^*)$. Moreover, $u \in C^0(\overline{J};\mathcal{H})$, and there holds the a priori estimate

$$\|u\|_{C^0(J;\mathcal{H})} + \|u\|_{L^2(J;\mathcal{V})} + \|u\|_{H^1(J;\mathcal{V}^*)} \le C\left(\|u_0\|_{\mathcal{H}} + \|f\|_{L^2(J;\mathcal{V}^*)}\right). \tag{3.10}$$

We give a simple example for the weak formulation (3.7).

Example 3.2.3 Consider the heat equation as in (2.4). Then, the spaces $\mathcal{H}, \mathcal{V}, \mathcal{V}^*$ in (3.6) are $\mathcal{H} = L^2(G)$, $\mathcal{V} = H_0^1(G)$, $\mathcal{V}^* = H^{-1}(G)$, and the bilinear form $a(\cdot,\cdot)$ is given by

$$a(u,v) = \int_G u'(x)v'(x)\,\mathrm{d}x, \quad u, v \in H_0^1(G).$$

The bilinear form is continuous on $\mathcal{V}$, since by the Hölder inequality, for all $u, v \in H_0^1(G)$

$$|a(u,v)| \le \int_G |u'v'|\,\mathrm{d}x \le \|u'\|_{L^2(G)} \|v'\|_{L^2(G)} \le \|u\|_{H^1(G)} \|v\|_{H^1(G)}.$$

Furthermore, we have for all $u \in H_0^1(G)$, by the Poincaré inequality (3.4)

$$\begin{aligned} a(u,u) = \int_G |u'(x)|^2\,\mathrm{d}x &= \frac{1}{2}\|u'\|_{L^2(G)}^2 + \frac{1}{2}\|u'\|_{L^2(G)}^2 \\ &\ge \frac{1}{2C}\|u\|_{L^2(G)}^2 + \frac{1}{2}\|u'\|_{L^2(G)}^2 \end{aligned}$$

$$\geq \frac{1}{2}\min\{C^{-1}, 1\}\big(\|u\|^2_{L^2(G)} + \|u'\|^2_{L^2(G)}\big) = C_2\|u\|^2_{H^1(G)},$$

i.e. (3.9) holds with $C_3 = 0$. Hence, according to Theorem 3.2.2, the variational formulation of the heat equation admits, for $u_0 \in L^2(G)$, $f \in L^2(J; H^{-1}(G))$ a unique weak solution $u \in L^2(J; H^1_0(G)) \cap H^1(J; H^{-1}(G))$.

We can always achieve $C_3 = 0$ in (3.9). If we substitute in (3.1) $v = e^{-\lambda t}u$ with suitably chosen λ and multiply (3.1) by $e^{-\lambda t}$, we find that v satisfies the problem

$$\partial_t v + \mathcal{A}v + \lambda v = e^{-\lambda t} f, \quad \text{in } J \times G,$$

with $v(0, x) = u_0(x)$ in G. Choosing $\lambda > 0$ large enough, the bilinear form $a(u, v) + \lambda(u, v)$ satisfies (3.9) with $C_3 = 0$.

3.3 Discretization

For the discretization we use the method of lines where first (3.7) is only discretized in space to obtain a system of coupled ODEs which are solved in a second step.

Let V_N be a one-parameter family of subspaces $V_N \subset \mathcal{V}$ with finite dimension $N = \dim V_N < \infty$. For each fixed $t \in J$ we approximate the solution $u(t, x)$ of (3.7) by a function $u_N(t) \in V_N$. Furthermore, let $u_{N,0} \in V_N$ be an approximation of u_0. Then, the semidiscrete form of (3.7) is the initial value problem,

$$\begin{aligned} &\text{Find } u_N \in C^1(J; V_N) \text{ such that for } t \in J \\ &(\partial_t u_N, v_N)_{\mathcal{H}} + a(u_N, v_N) = \langle f, v_N\rangle_{\mathcal{V}^*,\mathcal{V}}\,, \ \forall v_N \in V_N, \\ &u_N(0) = u_{N,0}\,, \end{aligned} \tag{3.11}$$

for the approximate solution function $u_N(t) : J \to V_N$. Let V_N be generated by a finite element basis, $V_N = \text{span}\{b_i(x) : 1 \leq i \leq N\}$. We write $u_N \in V_N$ in terms of the basis functions, $u_N(t, x) = \sum_{j=1}^N u_{N,j}(t)\, b_j(x)$, and obtain the matrix form of the semidiscretization (3.11)

$$\begin{aligned} &\text{Find } \underline{u}_N \in C^1(J; \mathbb{R}^N) \text{ such that for } t \in J, \\ &\mathbf{M}\underline{\dot{u}}_N(t) + \mathbf{A}\underline{u}_N(t) = \underline{f}(t)\,, \\ &\underline{u}_N(0) = \underline{u}_0\,, \end{aligned} \tag{3.12}$$

where $\underline{u}_0$ denotes the coefficient vector of $u_{N,0}$. The mass and stiffness matrices and the load vector with respect to the basis of V_N are given by

$$\mathbf{M}_{ij} = (b_j, b_i)_{\mathcal{H}}, \quad \mathbf{A}_{ij} = a(b_j, b_i), \quad f_i(t) = \langle f, b_i\rangle_{\mathcal{V}^*,\mathcal{V}}, \tag{3.13}$$

where $i, j = 1, \dots, N$. Let k_m, $m = 1, \dots, M$, be a sequence of (not necessarily equal sized) time steps and set $t_0 := 0$, $t_m := \sum_{i=1}^m k_i$ such that $t_M = T$. Applying the θ-scheme, we obtain the fully discrete form

$$
\begin{aligned}
&\text{Find } u_N^m \in V_N \text{ such that for } m = 1, \dots, M,\\
&k_m^{-1}(u_N^m - u_N^{m-1}, v_N)_{\mathcal{H}} + a(u_N^{m-1+\theta}, v_N) = \langle f^{m-1+\theta}, v_N \rangle_{\mathcal{V}^*,\mathcal{V}}\,, \ \forall v_N \in V_N,\\
&u_N^0 = u_{N,0},
\end{aligned}
\tag{3.14}
$$

where $u_N^{m+\theta} = \theta u_N(t_{m+1}) + (1-\theta)u_N(t_m)$ and $f^{m+\theta} = \theta f(t_{m+1}) + (1-\theta) f(t_m)$. We can again write (3.14) in matrix notation,

$$
\begin{aligned}
&\text{Find } \underline{u}_N^m \in \mathbb{R}^N \text{ such that for } m = 1, \dots, M,\\
&(\mathbf{M} + k_m\theta\mathbf{A})\underline{u}_N^m = (\mathbf{M} - k_m(1-\theta)\mathbf{A})\underline{u}_N^{m-1} + k_m(\theta \underline{f}^m + (1-\theta)\underline{f}^{m-1}),\\
&\underline{u}_N^0 = \underline{u}_0\,.
\end{aligned}
\tag{3.15}
$$

In the next section, we discuss the implementation of the matrix form (3.15).

3.4 Implementation of the Matrix Form

Let $G = (a, b)$. We describe a scheme to calculate the stiffness matrix $\mathbf{A}$ in case the corresponding bilinear form $a(\cdot,\cdot)$ has the form

$$
a(\varphi, \phi) = \int_G \big(\alpha(x)\varphi'(x)\phi'(x) + \beta(x)\varphi'(x)\phi(x) + \gamma(x)\varphi(x)\phi(x)\big)\,\mathrm{d}x, \tag{3.16}
$$

and the finite element subspace V_N consists of continuous, piecewise linear functions. Let $\mathcal{T} = \{a = x_0 < x_1 < x_2 < \cdots < x_{N+1} = b\}$ be an arbitrary mesh on G. Setting $K_l := (x_{l-1}, x_l)$, $h_l := |K_l| = x_l - x_{l-1}$, $l = 1, \dots, N+1$, we can also write $\mathcal{T} = \{K_l\}_{l=1}^{N+1}$. Define

$$
S_{\mathcal{T}}^1 := \big\{u(x) \in C^0(G) : u|_{K_l} \text{ is linear on } K_l \in \mathcal{T}\big\}. \tag{3.17}
$$

A basis for $S_{\mathcal{T}}^1 = \operatorname{span}\{b_i(x) : i = 0, \dots, N+1\}$ is given by the so-called *hat-functions* where, for $1 \le i \le N$,

$$
b_i(x) := \begin{cases} (x - x_{i-1})/h_i & \text{if } x \in (x_{i-1}, x_i],\\ (x_{i+1} - x)/h_{i+1} & \text{if } x \in (x_i, x_{i+1}),\\ 0 & \text{else,} \end{cases} \tag{3.18}
$$

and

$$
b_0(x) := \begin{cases} (x_1 - x)/h_0 & \text{if } x \in (x_0, x_1),\\ 0 & \text{else,} \end{cases} \tag{3.19}
$$

$$b_{N+1}(x) := \begin{cases} (x - x_N)/h_{N+1} & \text{if } x \in (x_N, x_{N+1}), \\ 0 & \text{else.} \end{cases} \tag{3.20}$$

If the mesh $\mathcal{T}$ is equidistant, i.e. $h_i = h = (b-a)/(N+1)$, we can write

$$b_i(x) = \max\{0, 1 - h^{-1}|x - x_i|\}, \; i = 0, \ldots, N+1.$$

Note that for a given subspace $S^1_{\mathcal{T}}$, there are many different possible choices of basis functions. The choice (3.18)–(3.20) are the basis functions with *smallest support*. The FE subspace to approximate functions with homogeneous Dirichlet boundary conditions is

$$S^1_{\mathcal{T},0} := S^1_{\mathcal{T}} \cap H^1_0(G) = \operatorname{span}\{b_i(x) : i = 1, \ldots, N\}, \tag{3.21}$$

with $\dim S^1_{\mathcal{T},0} = N$.

3.4.1 Elemental Forms and Assembly

We decompose $a(\cdot,\cdot)$ (3.16) into *elemental bilinear forms* $a_l(\cdot,\cdot)$, $l = 1, \ldots, N+1$,

$$\begin{aligned} a(b_j, b_i) &= \int_G \big(\alpha(x) b_j'(x) b_i'(x) + \beta(x) b_j'(x) b_i(x) + \gamma(x) b_j(x) b_i(x)\big)\, \mathrm{d}x \\ &= \sum_{l=1}^{N+1} \int_{K_l} \big(\alpha(x) b_j'(x) b_i'(x) + \beta(x) b_j'(x) b_i(x) + \gamma(x) b_j(x) b_i(x)\big)\, \mathrm{d}x \\ &=: \sum_{l=1}^{N+1} a_l(b_j, b_i). \end{aligned}$$

The restrictions $b_i|_{K_l}$, $i = l-1, l$, are linear and given by the element shape functions $N^1_{K_l} := b_{l-1}(x)|_{K_l}$, $N^2_{K_l} := b_l(x)|_{K_l}$, $l = 1, \ldots, N+1$. The element stiffness matrix $\mathbf{A}_l$ associated to $a_l(\cdot,\cdot)$ can be computed for each element independently. We transform each element $K_l \in \mathcal{T}$ to the so-called *reference element* $\widehat{K} := (-1, 1)$ via an element mapping

$$K_l \ni x = F_{K_l}(\xi) := \frac{1}{2}(x_{l-1} + x_l) + \frac{1}{2} h_l \xi, \quad \xi \in \widehat{K},$$

with derivative $F'_{K_l}(\xi) = h_l/2$. Using the reference element shape functions

$$\widehat{N}_1(\xi) = \frac{1}{2}(1 - \xi), \quad \widehat{N}_2(\xi) = \frac{1}{2}(1 + \xi),$$

which are independent of the element K_l, we have for all $K_l \in \mathcal{T}$,

$$N^1_{K_l}(F_{K_l}(\xi)) = \widehat{N}_1(\xi), \quad N^2_{K_l}(F_{K_l}(\xi)) = \widehat{N}_2(\xi).$$

The reference shape functions are shown in Fig. 3.1.

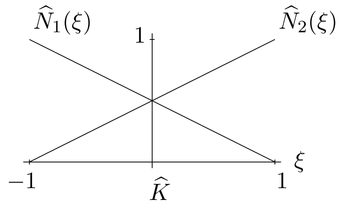

Fig. 3.1 Reference element shape functions on the reference element $\widehat{K}$

Furthermore, for $i, j = 1, 2$ there holds

$$\begin{aligned}(\mathbf{A}_l)_{ij} &= a_l(N_{K_l}^j, N_{K_l}^i)\\ &= \int_{K_l} \Big(\alpha(x)\partial_x N_{K_l}^j \partial_x N_{K_l}^i + \beta(x)\partial_x N_{K_l}^j N_{K_l}^i + \gamma(x) N_{K_l}^j N_{K_l}^i\Big)\,\mathrm{d}x\\ &= \int_{\widehat{K}} \left(\widehat{\alpha}_{K_l}(\xi)\frac{4}{h_l^2}\widehat{N}_j'\widehat{N}_i' + \widehat{\beta}_{K_l}(\xi)\frac{2}{h_l}\widehat{N}_j'\widehat{N}_i + \widehat{\gamma}_{K_l}(\xi)\widehat{N}_j\widehat{N}_i\right)\frac{h_l}{2}\,\mathrm{d}\xi,\end{aligned}$$

where

$$\widehat{\alpha}_{K_l}(\xi) := \alpha\big(F_{K_l}(\xi)\big), \quad \widehat{\beta}_{K_l}(\xi) := \beta\big(F_{K_j}(\xi)\big), \quad \widehat{\gamma}_{K_l}(\xi) := \gamma\big(F_{K_l}(\xi)\big).$$

For later purpose, it is useful to split the integral into three parts

$$(\mathbf{A}_l)_{ij} = (\mathbf{S}_l)_{ij} + (\mathbf{B}_l)_{ij} + (\mathbf{M}_l)_{ij}, \tag{3.22}$$

where

$$\begin{aligned}(\mathbf{S}_l)_{ij} &= \frac{2}{h_l}\int_{\widehat{K}} \widehat{\alpha}_{K_l}\widehat{N}_j'\widehat{N}_i'\,\mathrm{d}\xi, \quad (\mathbf{B}_l)_{ij} = \int_{\widehat{K}} \widehat{\beta}_{K_l}\widehat{N}_j'\widehat{N}_i\,\mathrm{d}\xi,\\ (\mathbf{M}_l)_{ij} &= \frac{h_l}{2}\int_{\widehat{K}} \widehat{\gamma}_{K_l}\widehat{N}_j\widehat{N}_i\,\mathrm{d}\xi.\end{aligned}$$

For general coefficients $\alpha(x)$, $\beta(x)$ and $\gamma(x)$, these integrals cannot be computed exactly. Hence, they have to be approximated by a numerical quadrature

$$\int_{-1}^{1} f(\xi)\,\mathrm{d}\xi \approx \sum_{j=1}^{p} w_j^{(p)} f(\xi_j^{(p)}),$$

using a p-point quadrature rule with quadrature weights $w_j^{(p)}$ and quadrature points $\xi_j^{(p)} \in (-1, 1)$, $j = 1, \ldots, p$.

It remains to construct the stiffness matrix $\mathbf{A}$. The idea is to express the global basis functions $b_i(x)$ in terms of the local shape functions. We have for $i = 1, \ldots, N$,

$$b_i(x) = N_{K_i}^2 + N_{K_{i+1}}^1, \quad b_0(x) = N_{K_1}^1, \quad b_{N+1}(x) = N_{K_{N+1}}^2.$$

Hence, we can assemble the matrix $\mathbf{A} \in \mathbb{R}^{(N+2)\times(N+2)}$ using the following summation

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{K_1} & & \\ & 0 & \\ & & \ddots & \\ & & & 0 \end{pmatrix} + \begin{pmatrix} 0 & & \\ & \mathbf{A}_{K_2} & \\ & & \ddots & \\ & & & 0 \end{pmatrix} + \cdots + \begin{pmatrix} 0 & & \\ & \ddots & \\ & & 0 & \\ & & & \mathbf{A}_{K_{N+1}} \end{pmatrix}. \tag{3.23}$$

As for the stiffness matrix, we also decompose the load vector $\underline{f}$ into elemental loads. For $f(t) \in L^2(G)$, we have

$$(f(t), b_i) = \int_G f(t,x) b_i(x)\,\mathrm{d}x = \sum_{l=1}^{N+1} \int_{K_l} f(t,x) b_i(x)\,\mathrm{d}x =: \sum_{l=1}^{N+1} (f_l(t), b_i).$$

Using again the local shape function, we obtain for $i = 1, 2$,

$$(f_l(t), N^i_{K_l}) = \int_{K_l} f(t,x) N^i_{K_l}\,\mathrm{d}x = \int_{\widehat{K}} \widehat{f}_{K_l}(t,\xi) \widehat{N}_i \frac{h_l}{2}\,\mathrm{d}\xi,$$

where

$$\widehat{f}_{K_l}(t,\xi) := f\big(t, F_{K_l}(\xi)\big).$$

For general functions f, these integrals cannot be computed exactly and have to be approximated by a numerical quadrature rule. To assemble the global load vector $\underline{f}$, we use

$$\underline{f} = \begin{pmatrix} \underline{f}_{K_1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \underline{f}_{K_2} \\ \vdots \\ 0 \end{pmatrix} + \cdots + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \underline{f}_{K_{N+1}} \end{pmatrix}.$$

Remark 3.4.1 For $\mathcal{V} = H^1_0(G)$ we have $V_N = \mathrm{span}\{b_i(x) : i = 1, \dots, N\}$, since $u(x_0) = u(x_{N+1}) = 0$. Hence, we get N degrees of freedom and the reduced matrices $\widetilde{\mathbf{M}}$, $\widetilde{\mathbf{A}} \in \mathbb{R}^{N\times N}$ where the first and last rows and columns of $\mathbf{A}$ in (3.23) are omitted. Similar considerations can be made for the vector $\underline{f}$. If we have non-homogeneous Dirichlet boundary condition, $u(t,a) = \phi_1(t)$, $u(t,b) = \phi_2(t)$ for given functions ϕ_1, ϕ_2, we write $u(t,x) = w(t,x) + \phi(t,x)$, where the function $\phi(t,x)$ satisfies the boundary conditions, $\phi(t,a) = \phi_1(t)$, $\phi(t,b) = \phi_2(t)$. Now, w has again homogeneous Dirichlet boundary conditions and is the solution of

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle w, v\rangle_{\mathcal{V}^*,\mathcal{V}} + a(w,v) = \langle f, v\rangle_{\mathcal{V}^*,\mathcal{V}} - \langle \partial_t \phi, v\rangle_{\mathcal{V}^*,\mathcal{V}} - a(\phi, v).$$

Choosing $\phi(t,x) = \phi_1(t) b_0(x) + \phi_2(t) b_{N+1}(x)$, we obtain the matrix form

$$\widetilde{\mathbf{M}} \underline{\dot{w}}_N(t) + \widetilde{\mathbf{A}} \underline{w}_N(t) = \underline{\widetilde{l}}(t),$$

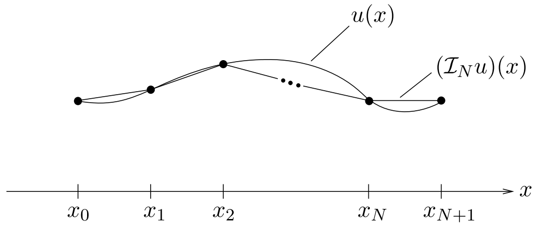

Fig. 3.2 Function $u(x)$ and its piecewise linear approximation $(\mathcal{I}_N u)(x)$

where the load vector $\underline{l}$ is given by

$$\underline{l}(t) = \underline{f}(t) - \mathbf{M}\underline{\dot{\phi}} - \mathbf{A}\underline{\phi},$$

with $\underline{\phi} = (\phi_1(t), 0, \dots, 0, \phi_2(t))^\top \in \mathbb{R}^{N+2}$ denoting the coefficient vector of ϕ. For Neumann boundary conditions, i.e. $\partial_x u(t,a) = \phi_1(t)$, $\partial_x u(t,b) = \phi_2(t)$, the boundary degrees of freedom must be kept.

3.4.2 Initial Data

In the discretization (3.14), we need an approximation of u_0 in V_N. If $u_0 \in H^1(G)$ (e.g. u_0 being the payoff of a put or call option), we can use *nodal interpolation*, i.e. $u_{N,0} = \mathcal{I}_N u_0$. For $u \in H^1(G)$, define the nodal interpolant $\mathcal{I}_N u \in S^1_{\mathcal{T}}$ by

$$(\mathcal{I}_N u)(x) := \sum_{i=0}^{N+1} u(x_i) b_i(x), \tag{3.24}$$

as illustrated in Fig. 3.2. Since $b_i(x_j) = \delta_{ij}$, we get $(\mathcal{I}_N u)(x_i) = u(x_i)$.

For $u \in H^1(G)$, the nodal value $u(x_i)$ is well defined due to Theorem 3.1.4.

Lemma 3.4.2 *The interpolation operator $\mathcal{I}_N : H^1(G) \to S^1_{\mathcal{T}}$ defined in (3.24) is bounded, i.e. there exists a constant C (which is independent of N) such that*

$$\|\mathcal{I}_N u\|_{H^1(G)} \le C \|u\|_{H^1(G)}, \quad \forall u \in H^1(G). \tag{3.25}$$

If we only have $u_0 \in L^2(G)$ (e.g. u_0 being the payoff of a digital option), we cannot use interpolation, but need the L^2-projection of u_0 on V_N, i.e. $u_{N,0} = \mathcal{P}_N u_0$. For $u \in L^2(G)$, the L^2-projection of u on V_N is defined as the solution of

$$\int_G \mathcal{P}_N u v_N \,\mathrm{d}x = \int_G u v_N \,\mathrm{d}x, \quad \forall v_N \in V_N. \tag{3.26}$$

Using the basis b_i, $i = 0, \dots, N+1$, of V_N we can obtain the coefficient vector $\underline{u}_N$ of $\mathcal{P}_N u_0$ as the solution of the linear system

$$\mathbf{M}\underline{u}_N = \underline{f}, \quad \text{with } f_i = \int_G u b_i \,\mathrm{d}x, \quad i = 0, \dots, N+1.$$

We immediately have from (3.26), using Hölder's inequality,

Lemma 3.4.3 *The L^2-projection $\mathcal{P}_N : L^2(G) \to S^1_{\mathcal{T}}$ defined in (3.26) is bounded, i.e.*

$$\|\mathcal{P}_N u\|_{L^2(G)} \leq \|u\|_{L^2(G)}, \quad \forall u \in L^2(G). \tag{3.27}$$

3.5 Stability of the θ-Scheme

For the stability of (3.14), we prove that the finite element solutions satisfy an analog of the estimate (3.10). For this section, we assume the uniform mesh width h in space and constant time steps $k = T/M$. We define

$$\|v\|_a := \big(a(v, v)\big)^{\frac{1}{2}}. \tag{3.28}$$

In the analysis, we will use for $f \in V_N^*$ the following notation:

$$\|f\|_* := \sup_{v_N \in V_N} \frac{(f, v_N)}{\|v_N\|_a}. \tag{3.29}$$

We will also need $\lambda_{\mathcal{A}}$ defined by

$$\lambda_{\mathcal{A}} := \sup_{v_N \in V_N} \frac{\|v_N\|^2}{\|v_N\|_*^2}.$$

In the case $\frac{1}{2} \leq \theta \leq 1$, the θ-scheme is stable for any time step $k > 0$, whereas in the case $0 \leq \theta < \frac{1}{2}$ the time step k must be sufficiently small.

Proposition 3.5.1 *In the case $0 \leq \theta < \frac{1}{2}$, assume*

$$\sigma := k(1 - 2\theta)\lambda_{\mathcal{A}} < 2. \tag{3.30}$$

Then, there are constants C_1 and C_2 independent of h and of k such that the sequence $\{u_N^m\}_{m=0}^M$ of solutions of the θ-scheme (3.14) satisfies the stability estimate

$$\|u_N^M\|_{L^2}^2 + C_1 k \sum_{m=0}^{M-1} \|u_N^{m+\theta}\|_a^2 \leq \|u_N^0\|_{L^2}^2 + C_2 k \sum_{m=0}^{M-1} \|f^{m+\theta}\|_*^2, \tag{3.31}$$

where C_1, C_2 satisfy in the case of $\frac{1}{2} \leq \theta \leq 1$,

$$0 < C_1 < 2, \qquad C_2 \geq \frac{1}{2 - C_1}, \tag{3.32}$$

and in the case of $0 \le \theta < \frac{1}{2}$,

$$0 < C_1 < 2 - \sigma, \qquad C_2 \ge \frac{1 + (4 - C_1)\sigma}{2 - \sigma - C_1}. \tag{3.33}$$

Proof Define

$$X^m := \|u_N^m\|_{L^2}^2 - \|u_N^{m+1}\|_{L^2}^2 + C_2 k \|f^{m+\theta}\|_*^2 - C_1 k \|u_N^{m+\theta}\|_a^2.$$

The assertion follows if we show that $X^m \ge 0$. Then, adding these inequalities for $m = 0, \ldots, M-1$ will obviously give (3.31).

Let $w := u_N^{m+1} - u_N^m$, then $u_N^{m+\theta} = (u_N^m + u_N^{m+1})/2 + (\theta - \frac{1}{2})w$ and

$$\|u_N^{m+1}\|_{L^2}^2 - \|u_N^m\|_{L^2}^2 = (u_N^{m+1} - u_N^m, u_N^{m+1} + u_N^m) = (w, 2u_N^{m+\theta} - (2\theta - 1)w).$$

By the definition of the θ-scheme, we have

$$\begin{aligned}(w, u_N^{m+\theta}) &= k\big(-\mathcal{A}u_N^{m+\theta} + f^{m+\theta}, u_N^{m+\theta}\big) = k\big[-\|u_N^{m+\theta}\|_a^2 + (f^{m+\theta}, u_N^{m+\theta})\big] \\ &\le k\big[-\|u_N^{m+\theta}\|_a^2 + \|f^{m+\theta}\|_* \|u_N^{m+\theta}\|_a\big].\end{aligned}$$

This gives

$$X^m \ge (2\theta - 1)\|w\|_{L^2}^2 + k\big[(2 - C_1)\|u_N^{m+\theta}\|_a^2 - 2\|f^{m+\theta}\|_* \|u_N^{m+\theta}\|_a + C_2 \|f^{m+\theta}\|_*^2\big].$$

In the case of $\frac{1}{2} \le \theta \le 1$, we now obtain $X^m \ge 0$ if condition (3.32) is satisfied.

In the case $0 \le \theta < \frac{1}{2}$, we have by the definition of the θ-scheme that $(w, v_N) = k(-\mathcal{A}u_N^{m+\theta} + f^{m+\theta}, v_N)$, yielding

$$\begin{aligned}\|w\|_{L^2} \le \lambda_{\mathcal{A}}^{1/2} \|w\|_* &\le \lambda_{\mathcal{A}}^{1/2} k \big(\|\mathcal{A}u_N^{m+\theta}\|_* + \|f^{m+\theta}\|_*\big) \\ &= \lambda_{\mathcal{A}}^{1/2} k \big(\|u_N^{m+\theta}\|_a + \|f^{m+\theta}\|_*\big),\end{aligned}$$

since $(\mathcal{A}u_N^{m+\theta}, v_N) \le \|u_N^{m+\theta}\|_a \|v_N\|_a$ gives $\|\mathcal{A}u_N^{m+\theta}\|_* \le \|u_N^{m+\theta}\|_a$ and choosing $v_N := u_N^{m+\theta}$ gives $\|\mathcal{A}u_N^{m+\theta}\|_* \ge \|u_N^{m+\theta}\|_a$. Hence,

$$k^{-1} X^m \ge (2 - C_1 - \sigma)\|u_N^{m+\theta}\|_a^2 - 2(1 + \sigma)\|f^{m+\theta}\|_* \|u_N^{m+\theta}\|_a + (C_2 - \sigma)\|f^{m+\theta}\|_*^2.$$

Therefore, we have $X^m \ge 0$ if conditions (3.30), (3.33) hold. □

Remark 3.5.2 The conditions (3.30), (3.33) are time-step restrictions of CFL[1]-type. Here, time-step restrictions are formulated in terms of the matrix property $\lambda_{\mathcal{A}}$. If

[1]CFL is an acronym for Courant, Friedrich and Lewy who identified an analogous condition as being necessary for the stability of explicit timestepping schemes for first order, hyperbolic equations.

$\mathcal{V} = H_0^1(G)$, and if $V_N = S^1_{\mathcal{T}}$, we obtain $\lambda_{\mathcal{A}} \sim Ch^{-2}$ as $h \downarrow 0$. Hence, we get stability provided the CFL type stability condition

$$k \le Ch^2/(1-2\theta), \quad 0 \le \theta < \frac{1}{2}$$

holds. For $\frac{1}{2} \le \theta \le 1$, the θ-scheme is stable without any time step restriction.

3.6 Error Estimates

Let $u^m(x) = u(t_m, x)$, u_N^m be as in (3.14) and assume V_N consists of linear finite elements, i.e. $V_N = S^1_{\mathcal{T}}$. For $m = 0, \ldots, M-1$, we want to estimate the error $e_N^m(x) := u^m(x) - u_N^m(x)$. Therefore, we split the error

$$e_N^m = (u^m - \mathcal{I}_N u^m) + (\mathcal{I}_N u^m - u_N^m) =: \eta^m + \xi_N^m, \tag{3.34}$$

where $\mathcal{I}_N : V \to V_N$ is the interpolant as defined in (3.24). For a fixed time point t_m, $\eta^m(x) = u(t_m, x) - (\mathcal{I}_N u)(t_m, x) \in V$ is a consistency error for which we now give an error estimate.

3.6.1 Finite Element Interpolation

We prove error estimates of the *interpolation error* $u - \mathcal{I}_N u$.

Proposition 3.6.1 *Let $\mathcal{I}_N : V \to V_N$ be the interpolant as defined in* (3.24). *Then, the following error estimates hold*:

$$\|(u - \mathcal{I}_N u)^{(n)}\|^2_{L^2(G)} \le C \sum_{i=1}^{N+1} h_i^{2(\ell-n)} \|u^{(\ell)}\|^2_{L^2(K_i)}, \; n = 0, 1, \; \ell = 1, 2. \tag{3.35}$$

In particular, if the mesh is uniform, i.e. $h_i = h$,

$$\|(u - \mathcal{I}_N u)^{(n)}\|_{L^2(G)} \le Ch^{\ell-n} \|u^{(\ell)}\|_{L^2(G)}, \; n = 0, 1, \; \ell = 1, 2. \tag{3.36}$$

Proof Consider $\widehat{G} = (0, 1)$ and $\widehat{u} \in H^2(\widehat{G})$. Then, $\widehat{u}' - c \in \widetilde{H}^1(\widehat{G})$ for $c = \int_0^1 \widehat{u}'$. By the Poincaré inequality (3.4), which also holds for the space $\widetilde{H}^1(\widehat{G})$ (see (3.5)),

$$\|\widehat{u}' - c\|_{L^2(\widehat{G})} \le \widehat{C}\, \|\widehat{u}''\|_{L^2(\widehat{G})}. \tag{3.37}$$

With

$$\widehat{\mathcal{I}_N}\widehat{u} := \widehat{u}(0) + \int_0^x c \,\mathrm{d}x = \widehat{u}(0) + cx,$$

we have $(\widehat{\mathcal{I}_N \widehat{u}})(1) = \widehat{u}(0) + c = \widehat{u}(1)$ and $\widehat{u} - \widehat{\mathcal{I}_N \widehat{u}} \in H_0^1(\widehat{G}) \cap H^2(\widehat{G})$. Therefore, by (3.4),

$$\|\widehat{u} - \widehat{\mathcal{I}_N \widehat{u}}\|_{L^2(\widehat{G})} \le \widehat{C} \|\widehat{u}' - (\widehat{\mathcal{I}_N \widehat{u}})'\|_{L^2(\widehat{G})} = \widehat{C} \|\widehat{u}' - c\|_{L^2(\widehat{G})} \le \widehat{C}^2 \|\widehat{u}''\|_{L^2(\widehat{G})}. \tag{3.38}$$

If $G = (0, h)$, $h > 0$, we get from (3.37), (3.38) by scaling to the interval $(0, h)$

$$\|u' - (\mathcal{I}_N u)'\|_{L^2(0,h)} \le \widehat{C} h \|u''\|_{L^2(0,h)}, \tag{3.39}$$

$$\|u - \mathcal{I}_N u\|_{L^2(0,h)} \le \widehat{C}^2 h^2 \|u''\|_{L^2(0,h)}, \tag{3.40}$$

and $(\mathcal{I}_N u)(0) = u(0)$, $(\mathcal{I}_N u)(h) = u(h)$. Applying this to each interval K_i, squaring and summing yields (3.35). □

In Proposition 3.6.1, we estimated the mean square error of $u - \mathcal{I}_N u$. We are often also interested in pointwise error estimates. Some (not optimal) bounds on the pointwise error can be deduced from Proposition 3.6.1 with

Proposition 3.6.2 *Let* $G = (0, 1)$. *For every* $u \in H_0^1(G)$, *the following holds*:

$$\|u\|_{L^\infty(G)}^2 \le 2\|u\|_{L^2(G)} \|u'\|_{L^2(G)}. \tag{3.41}$$

Proof If $u \in H_0^1(G)$, $u = \widetilde{u} \in C^0(\overline{G})$. Let $\xi \in \overline{G}$ be such that $\|u\|_{L^\infty(G)} = \max_x |\widetilde{u}(x)| = |\widetilde{u}(\xi)|$. Then

$$\begin{aligned} \|u\|_{L^\infty(G)}^2 = |\widetilde{u}(\xi)|^2 = |(\widetilde{u}(\xi))^2 - (\widetilde{u}(0))^2| &\le \left| \int_0^\xi (\widetilde{u}(\eta)^2)' \,\mathrm{d}\eta \right| \\ &= 2 \left| \int_0^\xi \widetilde{u}(\eta) \widetilde{u}'(\eta) \,\mathrm{d}\eta \right| \le 2 \|\widetilde{u}\|_{L^2(G)} \|\widetilde{u}'\|_{L^2(G)}. \end{aligned}$$

□

Corollary 3.6.3 *For* $G = (0, 1)$, $u \in H^2(G)$ *and equidistant mesh width* h, *one has*

$$\|u - \mathcal{I}_N u\|_{L^\infty(G)} \le C h^{\frac{3}{2}} \|u''\|_{L^2(G)}, \tag{3.42}$$

as $h \to 0$. *For a general interval* $G = (a, b)$, (3.41), (3.42) *also hold with constants that depend on* $b - a$.

If u has better regularity than just $u \in H^2(G)$, better convergence rates are possible.

Corollary 3.6.4 *For* $G = (0, 1)$, $u \in W^{2,\infty}(G)$ *and equidistant mesh width* h, *one has*

$$\|u - \mathcal{I}_N u\|_{L^\infty(G)} \le C h^2 \|u\|_{W^{2,\infty}(G)}, \tag{3.43}$$

as $h \to 0$.

3.6.2 Convergence of the Finite Element Method

Assume uniform mesh width h in space and constant time steps $k = T/M$ in time. We show now that the computed sequence $\{u_N^m\}$ converges, as $h \to 0$ and $k \to 0$, to the exact solution of (3.7). We have

Theorem 3.6.5 *Assume $u \in C^1(\overline{J}; H^2(G)) \cap C^3(\overline{J}; H^{-1}(G))$. Let $u^m(x) = u(t_m, x)$ and u_N^m be as in (3.14), with $V_N = S_T^1$. Assume for $0 \le \theta < \frac{1}{2}$ also (3.30). Then, the following error bound holds:*

$$\|u^M - u_N^M\|_{L^2(G)}^2 + k \sum_{m=0}^{M-1} \|u^{m+\theta} - u_N^{m+\theta}\|_a^2$$

$$\le Ch^2 \max_{0 \le t \le T} \|u(t)\|_{H^2(G)} + Ch^2 \int_0^T \|\partial_t u(s)\|_{H^1(G)}^2 \, \mathrm{d}s$$

$$+ C \begin{cases} k^2 \int_0^T \|\partial_{tt} u(s)\|_*^2 \, \mathrm{d}s & \textit{if}\, 0 \le \theta \le 1, \\ k^4 \int_0^T \|\partial_{ttt} u(s)\|_*^2 \, \mathrm{d}s & \textit{if}\, \theta = \frac{1}{2}. \end{cases} \tag{3.44}$$

Remark 3.6.6 By the properties (3.8)–(3.9) (with $C_3 = 0$), the norm $\|\cdot\|_a$ in (3.28) is equivalent to the energy-norm $\|\cdot\|_{\mathcal{V}}$. Thus, we see from (3.44) that we have $\|u^M - u_N^M\|_{\mathcal{V}} = \mathcal{O}(h + k)$, i.e. first order convergence in the energy norm, provided the solution $u(t, x)$ is sufficiently smooth. However, one can also prove second order convergence in the L^2-norm, i.e. $\|u^M - u_N^M\|_{L^2(G)} = \mathcal{O}(h^2 + k)$, if $\theta \in [0, 1] \setminus \{1/2\}$, and $\|u^M - u_N^M\|_{L^2(G)} = \mathcal{O}(h^2 + k^2)$ if $\theta = 1/2$. Hence, for continuous, linear finite elements, we obtain the same convergence rates as for the finite difference discretization in Theorem 2.3.8.

The proof of Theorem 3.6.5 will be given in several steps. We define $e_N^m := u^m - u_N^m$ and consider the splitting (3.34), where now $\mathcal{I}_N$ denotes nodal interpolant defined in (3.24). Since we already estimated the consistency error $\eta^m = u^m - \mathcal{I}_N u^m$, we focus on $\xi_N^m \in V_N$.

Lemma 3.6.7 *If $u \in C^1(\overline{J}; H)$, the errors $\{\xi_N^m\}_m$ are solutions of the θ-scheme: Given $\xi_N^0 := \mathcal{I}_N u_0 - u_N^0$, for $m = 0, \dots, M-1$ find $\xi_N^{m+1} \in V_N$ such that $\forall v_N \in V_N$:*

$$k^{-1}(\xi_N^{m+1} - \xi_N^m, v_N) + a\big(\theta \xi_N^{m+1} + (1-\theta)\xi_N^m, v_N\big) = (r^m, v_N) \tag{3.45}$$

where the residuals $r^m = r_1^m + r_2^m + r_3^m$ are given by

$$(r_1^m, v_N) = \big(k^{-1}(u^{m+1} - u^m) - \dot{u}^{m+\theta}, v_N\big),$$

$$(r_2^m, v_N) = \big(k^{-1}(\mathcal{I}_N u^{m+1} - \mathcal{I}_N u^m) - k^{-1}(u^{m+1} - u^m), v_N\big),$$

$$(r_3^m, v_N) = a(\mathcal{I}_N u^{m+\theta} - u^{m+\theta}, v_N)\,.$$

The stability of the θ-scheme, Proposition 3.5.1, gives

Corollary 3.6.8 *Under the assumptions of Proposition* 3.5.1,

$$\|\xi_N^M\|_{L^2(G)}^2 + C_1 k \sum_{m=0}^{M-1} \|\xi_N^{m+\theta}\|_a^2 \le \|\xi_N^0\|_{L^2(G)}^2 + C_2 k \sum_{m=0}^{M-1} \|r^m\|_*^2. \tag{3.46}$$

To prove Theorem 3.6.5, it is therefore sufficient to estimate the residual $\|r^m\|_*$.

Proof of Theorem 3.6.5

(i) Estimate of r_1: for any $v_N \in V_N$, we have

$$|(r_1^m, v_N)| \le \|k^{-1}(u^{m+1} - u^m) - \dot{u}^{m+\theta}\|_* \|v_N\|_a.$$

With

$$k^{-1}(u^{m+1} - u^m) - \dot{u}^{m+\theta} = k^{-1} \int_{t_m}^{t_{m+1}} (s - (1-\theta)t_{m+1} - \theta t_m)\, \ddot{u}\, \mathrm{d}s,$$

we get

$$\begin{aligned}\|k^{-1}(u^{m+1} - u^m) - \dot{u}^{m+\theta}\|_* &\le k^{-1} \int_{t_m}^{t_{m+1}} |s - (1-\theta)t_{m+1} - \theta t_m|\, \|\ddot{u}\|_*\, \mathrm{d}s \\ &\le C_\theta k^{\frac{1}{2}} \left(\int_{t_m}^{t_{m+1}} \|\ddot{u}(s)\|_*^2\, \mathrm{d}s \right)^{\frac{1}{2}}.\end{aligned}$$

(ii) Estimate of r_2: for any $v_N \in V_N$,

$$\begin{aligned}|(r_2^m, v_N)| &\le C \left\| k^{-1}\left[(u^{m+1} - u^m) - \mathcal{I}_N(u^{m+1} - u^m)\right] \right\|_* \|v_N\|_a \\ &= C k^{-1} \left\| (I - \mathcal{I}_N) \int_{t_m}^{t_{m+1}} \dot{u}(s)\, \mathrm{d}s \right\|_* \|v_N\|_a \\ &\le C k^{-1} \int_{t_m}^{t_{m+1}} \|\dot{u} - \mathcal{I}_N \dot{u}\|_*\, \mathrm{d}s\ \|v_N\|_a.\end{aligned}$$

(iii) Estimate of r_3: using the continuity of $a(\cdot,\cdot)$,

$$|(r_3^m, v_N)| \le C\, \|u^{m+\theta} - \mathcal{I}_N u^{m+\theta}\|_a \|v_N\|_a.$$

We have proved that for every $m = 0, 1, \ldots, M-1$,

$$\begin{aligned}\|r^m\|_*^2 \le{} & C k \int_{t_m}^{t_{m+1}} \|\ddot{u}(x)\|_*^2\, \mathrm{d}s \\ & + C k^{-1} \int_{t_m}^{t_{m+1}} \|\dot{u} - \mathcal{I}_N \dot{u}\|_*^2\, \mathrm{d}s + C\|u^{m+\theta} - \mathcal{I}_N u^{m+\theta}\|_a^2.\end{aligned}$$

Inserting into (3.46), we get

$$
\begin{aligned}
&\|e_N^M\|_{L^2(G)}^2 + C_1 k \sum_{m=0}^{M-1} \|e_N^{m+\theta}\|_a^2 \\
&\quad \le 2\left\{ \|\eta^M\|_{L^2(G)}^2 + C_1 k \sum_{m=0}^{M-1} \|\eta^{m+\theta}\|_a^2 + \|\xi_N^M\|_{L^2(G)}^2 + C_1 k \sum_{m=0}^{M-1} \|\xi_N^{m+\theta}\|_a^2 \right\} \\
&\quad \le C\left\{ \|\eta^M\|_{L^2(G)}^2 + C_1 k \sum_{m=0}^{M-1} \|\eta^{m+\theta}\|_a^2 + \|\xi_N^0\|_{L^2(G)}^2 + k\, C_\theta k \int_0^T \|\ddot{u}(s)\|_*^2 \,\mathrm{d}s \right. \\
&\qquad \left. + \int_0^T \|\dot{u} - \mathcal{I}_N \dot{u}\|_*^2 \,\mathrm{d}s + C\, k \sum_{m=0}^{M-1} \|u^{m+\theta} - \mathcal{I}_N u^{m+\theta}\|_a^2 \right\} \\
&\quad \le C\left\{ \|\xi_N^0\|_{L^2(G)}^2 + \|\eta^M\|_{L^2(G)}^2 + k \sum_{m=0}^{M-1} \|\eta^{m+\theta}\|_a^2 \right. \\
&\qquad \left. + \int_0^T \|\dot{\eta}\|_*^2 \,\mathrm{d}s + k^2 \int_0^T \|\ddot{u}(s)\|_*^2 \,\mathrm{d}s \right\}.
\end{aligned}
$$

(iv) The terms involving $\eta = u - \mathcal{I}_N u$ are estimated using the interpolation estimates of Proposition 3.6.1. Furthermore, if u_0 is approximated with the L^2-projection, we have $\|\xi_N^0\|_{L^2(G)} = \|\mathcal{I}_N u_0 - u_{N,0}\|_{L^2(G)} \le \|\mathcal{I}_N u_0 - u_0\|_{L^2(G)} + \|u_0 - u_{N,0}\|_{L^2(G)}$. Since $u \in C^1(\overline{J}; H^2(G))$ by assumption, we have $u_0 \in H^2(G)$. The L^2-projection $u_{N,0}$ is a quasi-optimal approximation of u_0, i.e. $\|u_0 - \mathcal{P}_N u_0\|_{L^2(G)} \le Ch^2 \|u_0\|_{H^2(G)}$. We obtain from Proposition 3.6.1 optimal convergence rates with respect to h, and Theorem 3.6.5 is proved. □

3.7 Further Reading

The basic finite element method is, for example, described in Braess [24] and for parabolic problems in detail by Thomée [154]. Error estimates in a very general framework are also given in Ern and Guermond [64]. In this section, we only considered the θ-scheme for the time discretization. It is also possible to apply finite elements for the time discretization as in Schötzau and Schwab [146, 147] where an hp-discontinuous Galerkin method is used. It yields exponential convergence rates instead of only algebraic ones as in the θ-scheme.

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Chapter 4
European Options in BS Markets

In the last chapters, we explained various methods to solve partial differential equations. These methods are now applied to obtain the price of a European option. We assume that the stock price follows a geometric Brownian motion and show that the option price satisfies a parabolic PDE. The unbounded log-price domain is localized to a bounded domain and the error incurred by the truncation is estimated. It is shown that the variational formulation has a unique solution and the discretization schemes for finite element and finite differences are derived. Furthermore, we describe extensions of the Black–Scholes model, like the constant elasticity of variance (CEV) and the local volatility model.

4.1 Black–Scholes Equation

Let X be the solution of the SDE (1.2), where we assume that the coefficients b, σ are independent of time t and satisfy the assumptions of Theorem 1.2.6. Further assume $r(x)$ to be a bounded and continuous function modeling the riskless interest rate. We want to compute the value of the option with payoff g which is the conditional expectation

$$V(t,x) = \mathbb{E}\left[e^{-\int_t^T r(X_s)\,\mathrm{d}s} g(X_T) \mid X_t = x\right]. \tag{4.1}$$

We show that $V(t,x)$ is a solution of a deterministic partial differential equation. Therefore, we first relate to the process X a differential operator $\mathcal{A}$, the so-called *infinitesimal generator* of the process X.

Proposition 4.1.1 *Let $\mathcal{A}$ denote the differential operator which is, for functions $f \in C^2(\mathbb{R})$ with bounded derivatives, given by*

$$(\mathcal{A}f)(x) = \frac{1}{2}\sigma^2(x)\partial_{xx} f(x) + b(x)\partial_x f(x). \tag{4.2}$$

N. Hilber et al., *Computational Methods for Quantitative Finance*, Springer Finance,
DOI 10.1007/978-3-642-35401-4_4,

Then, the process $M_t := f(X_t) - \int_0^t (\mathcal{A}f)(X_s)\,\mathrm{d}s$ *is a martingale with respect to the filtration of* W.

Proof We apply the Itô formula (1.7) to $f(X_t)$ and obtain, in integral form,

$$f(X_t) = f(X_0) + \int_0^t \partial_x f(X_s)\,\mathrm{d}X_s + \frac{1}{2}\int_0^t \partial_{xx} f(X_s)\sigma^2(X_s)\,\mathrm{d}s.$$

Using $\mathrm{d}X_t = b(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t$, and the definition of $\mathcal{A}$, we have

$$f(X_t) = f(X_0) + \int_0^t (\mathcal{A}f)(X_s)\,\mathrm{d}s + \int_0^t \sigma(X_s) f'(X_s)\,\mathrm{d}W_s.$$

The result follows if we can show that the stochastic integral $\int_0^t \sigma(X_s)\partial_x f(X_s)\,\mathrm{d}W_s$ is a martingale (with respect to the filtration of W). According to Proposition 1.2.7, it is sufficient to show that $\mathbb{E}[\int_0^t |\sigma(X_s)\partial_x f(X_s)|^2\,\mathrm{d}s] < \infty$. Since f has bounded derivatives and σ satisfies (1.4), we obtain

$$\begin{aligned}\mathbb{E}\Big[\int_0^t |\sigma(X_s)|^2 |\partial_x f(X_s)|^2\,\mathrm{d}s\Big] &\le C^2 \sup_{x\in\mathbb{R}} |\partial_x f(x)|^2 \mathbb{E}\Big[\int_0^t (1+|X_s|^2)\,\mathrm{d}s\Big] \\ &\le TC^2 \sup_{x\in\mathbb{R}} |\partial_x f(x)|^2 \big(1+\mathbb{E}[\sup_{0\le s\le T}|X_s|^2]\big) \overset{(1.5)}{<} \infty.\end{aligned}$$

□

Remark 4.1.2 For $t > 0$ denote by X_t^x the solution of the SDE (1.2) starting from x at time 0. Then, since $M_t = f(X_t) - \int_0^t (\mathcal{A}f)(X_s)\,\mathrm{d}s$ is a martingale by Proposition 4.1.1, we know that $\mathbb{E}[M_0] = f(x) = \mathbb{E}[M_t]$. Therefore,

$$\mathbb{E}[f(X_t^x)] = f(x) + \mathbb{E}\left[\int_0^t (\mathcal{A}f)(X_s^x)\,\mathrm{d}s\right].$$

Since by assumption f has bounded derivatives and b, σ satisfy the global Lipschitz and linear growth condition (1.3)–(1.4)

$$\begin{aligned}\mathbb{E}\big[\sup_{0\le s\le T} |(\mathcal{A}f)(X_s^x)|\big] &\le C\mathbb{E}\big[\sup_{0\le s\le T} |\sigma^2(X_s^x)| + |b(X_s^x)|\big] \\ &\le C'\big(1+\mathbb{E}[\sup_{0\le s\le T} |X_s^x|^2]\big) < \infty.\end{aligned}$$

Thus, since $\mathcal{A}f$ and X_t^x are continuous, the dominated convergence theorem gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[f(X_t^x)]|_{t=0} = \lim_{t\to 0}\mathbb{E}\left[\frac{1}{t}\int_0^t (\mathcal{A}f)(X_s^x)\,\mathrm{d}s\right] = (\mathcal{A}f)(x).$$

Therefore, $\mathcal{A}$ is called the *infinitesimal generator* of the process X_t^x.

For the purpose of option pricing, we need a discounted version of Proposition 4.1.1.

Proposition 4.1.3 *Let $f \in C^{1,2}(\mathbb{R} \times \mathbb{R})$ with bounded derivatives in x, let $\mathcal{A}$ be as in (4.2) and assume that $r \in C^0(\mathbb{R})$ is bounded. Then, the process*

$$M_t := e^{-\int_0^t r(X_s)\,\mathrm{d}s} f(t, X_t) - \int_0^t e^{-\int_0^s r(X_\tau)\,\mathrm{d}\tau} (\partial_t f + \mathcal{A}f - rf)(s, X_s)\,\mathrm{d}s,$$

is a martingale with respect to the filtration of W.

Proof Denote by Z the process $Z_t := e^{-\int_0^t r(X_s)\,\mathrm{d}s}$. Then

$$\mathrm{d}(Z_t f(t, X_t)) = \mathrm{d}Z_t f(t, X_t) + Z_t\,\mathrm{d}f(t, X_t),$$

with $\mathrm{d}Z_t = -r(X_t)Z_t\,\mathrm{d}t$, and thus, by the Itô formula (1.7),

$$\begin{aligned}\mathrm{d}(Z_t f(t, X_t)) = &-r(X_t)Z_t f(t, X_t) + Z_t\big(\partial_t f(t, X_t)\,\mathrm{d}t + \partial_x f(t, X_t)\,\mathrm{d}X_t \\ &+ \frac{1}{2}\sigma^2(X_t)\partial_{xx} f(t, X_t)\,\mathrm{d}t\big) \\ =&Z_t\big((-rf(t, X_t) + \partial_t f(t, X_t) + (\mathcal{A}f)(t, X_t))\,\mathrm{d}t \\ &+ \sigma(X_t)\partial_x f(t, X_t)\,\mathrm{d}W_t\big).\end{aligned}$$

Thus, we need to show that $\int_0^t Z_s\sigma(X_s)\partial_x f(s, X_s)\,\mathrm{d}W_s$ is a martingale. But

$$\mathbb{E}\left[\int_0^t |Z_s\sigma(X_s)\partial_x f(s, X_s)|^2\,\mathrm{d}s\right] < \infty,$$

by the boundedness of r and by repeating the estimates in the proof of Proposition 4.1.1. □

We now are able to link the stochastic representation of the option price (4.1) with a parabolic partial differential equation.

Theorem 4.1.4 *Let $V \in C^{1,2}(J \times \mathbb{R}) \cap C^0(\overline{J} \times \mathbb{R})$ with bounded derivatives in x be a solution of*

$$\partial_t V + \mathcal{A}V - rV = 0 \quad \text{in } J \times \mathbb{R}, \qquad V(T, x) = g(x) \quad \text{in } \mathbb{R}, \tag{4.3}$$

with $\mathcal{A}$ as in (4.2). Then, $V(t, x)$ can also be represented as

$$V(t, x) = \mathbb{E}\left[e^{-\int_t^T r(X_s)\,\mathrm{d}s} g(X_T) \mid X_t = x\right].$$

Proof We show the result only for $t=0$. Since $\partial_t V + \mathcal{A}V - rV = 0$, we have, by Proposition 4.1.3, that the process $M_t := e^{-\int_0^t r(X_s)\,\mathrm{d}s} V(t, X_t)$ is a martingale. Thus,

$$\begin{aligned}
V(0,x) &= \mathbb{E}[M_0 \mid X_0 = x] \\
&= \mathbb{E}[M_T \mid X_0 = x] \\
&= \mathbb{E}\Big[e^{-\int_0^T r(X_s)\,\mathrm{d}s} V(T, X_T) \mid X_0 = x\Big] \\
&= \mathbb{E}\Big[e^{-\int_0^T r(X_s)\,\mathrm{d}s} g(X_T) \mid X_0 = x\Big].
\end{aligned}$$

□

Remark 4.1.5 The converse of Theorem 4.1.4 is also true. Any $V(t,x)$ as in (4.1), which is $C^{1,2}(J \times \mathbb{R}) \cap C^0(\overline{J} \times \mathbb{R})$ with bounded derivatives in x, solves the PDE (4.3).

We apply Theorem 4.1.4 to the Black–Scholes model [21]. In the Black–Scholes market, the risky asset's spot-price is modeled by a geometric Brownian motion S_t, i.e. the SDE for this model is as in (1.2), with coefficients $b(t,s) = rs$, $\sigma(t,s) = \sigma s$, where $\sigma > 0$ and $r \geq 0$ denote the (constant) volatility and the (constant) interest rate, respectively. Therefore, the SDE is given by

$$\mathrm{d}S_t = rS_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}W_t.$$

We assume for simplicity that *no dividends are paid*. Based on Theorem 4.1.4, we get that the discounted price of a European contract with payoff $g(s)$, i.e. $V(0,s) = \mathbb{E}[e^{-rT} g(S_T) | S_0 = s]$, is equal to a regular solution $V(0,s)$ of the Black–Scholes equation

$$\partial_t V + \frac{1}{2}\sigma^2 s^2 \partial_{ss} V + rs\partial_s V - rV = 0 \quad \text{in } J \times \mathbb{R}_+. \tag{4.4}$$

The BS equation (4.4) needs to be completed by the *terminal condition*, $V(T,s) = g(s)$, depending on the type of option. Equation (4.4) is a parabolic PDE with the second order "spatial" differential operator

$$(\mathcal{A}f)(s) = \frac{1}{2}\sigma^2 s^2 \partial_{ss} f(s) + rs\partial_s f(s), \tag{4.5}$$

which degenerates at $s = 0$. To obtain a non-degenerate operator with constant coefficients, we switch to the price process $X_t = \log(S_t)$ which solves the SDE

$$\mathrm{d}X_t = \Big(r - \frac{1}{2}\sigma^2\Big)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t.$$

The infinitesimal generator for this process has constant coefficients and is given by

$$(\mathcal{A}^{\mathrm{BS}} f)(x) = \frac{1}{2}\sigma^2 \partial_{xx} f(x) + \Big(r - \frac{1}{2}\sigma^2\Big)\partial_x f(x). \tag{4.6}$$

We furthermore change to *time-to-maturity* $t \to T - t$, to obtain a forward parabolic problem. Thus, by setting $V(t, s) =: v(T - t, \log s)$, the BS equation in real price (4.4) satisfied by $V(t, s)$ becomes the BS equation for $v(t, x)$ in *log-price*

$$\partial_t v - \mathcal{A}^{\mathrm{BS}} v + rv = 0 \quad \text{in } (0, T) \times \mathbb{R}, \tag{4.7}$$

with the initial condition $v(0, x) = g(e^x)$ in $\mathbb{R}$.

Remark 4.1.6 For put and call contracts with strike $K > 0$, it is convenient to introduce the so-called *log-moneyness* variable $x = \log(s/K)$ and setting the option price $V(t, s) := K w(T - t, \log(s/K))$. Then, the function $w(t, x)$ again solves (4.7), with the initial condition $w(0, x) = g(Ke^x)/K$. Thus, the initial condition becomes for a put $w(0, x) = \max\{0, 1 - e^x\}$ and for a call $w(0, x) = \max\{0, e^x - 1\}$ where both payoffs now do not depend on K.

4.2 Variational Formulation

We give the variational formulation of the Black–Scholes equation (4.7) which reads:

$$\begin{aligned}
&\text{Find } u \in L^2(J; H^1(\mathbb{R})) \cap H^1(J; L^2(\mathbb{R})) \text{ such that} \\
&(\partial_t u, v) + a^{\mathrm{BS}}(u, v) = 0, \ \forall v \in H^1(\mathbb{R}), \ \text{a.e. in } J, \\
&u(0) = u_0,
\end{aligned} \tag{4.8}$$

where $u_0(x) := g(e^x)$ and the bilinear form $a^{\mathrm{BS}}(\cdot,\cdot) : H^1(\mathbb{R}) \times H^1(\mathbb{R}) \to \mathbb{R}$ is given by

$$a^{\mathrm{BS}}(\varphi, \phi) := \frac{1}{2}\sigma^2(\varphi', \phi') + (\sigma^2/2 - r)(\varphi', \phi) + r(\varphi, \phi). \tag{4.9}$$

We show that $a^{\mathrm{BS}}(\cdot,\cdot)$ is continuous (3.8) and satisfies a Gårding inequality (3.9) on $\mathcal{V} = H^1(\mathbb{R})$.

Proposition 4.2.1 *There exist constants* $C_i = C_i(\sigma, r) > 0$, $i = 1, 2, 3$, *such that for all* $\varphi, \phi \in H^1(\mathbb{R})$

$$|a^{\mathrm{BS}}(\varphi, \phi)| \le C_1 \|\varphi\|_{H^1(\mathbb{R})} \|\phi\|_{H^1(\mathbb{R})}, \quad a^{\mathrm{BS}}(\varphi, \varphi) \ge C_2 \|\varphi\|^2_{H^1(\mathbb{R})} - C_3 \|\varphi\|^2_{L^2(\mathbb{R})}.$$

Proof We first show continuity. By Hölder's inequality,

$$\begin{aligned}
|a^{\mathrm{BS}}(\varphi, \phi)| &\le \frac{1}{2}\sigma^2 \|\varphi'\|_{L^2(\mathbb{R})} \|\phi'\|_{L^2(\mathbb{R})} + |\sigma^2/2 - r| \|\varphi'\|_{L^2(\mathbb{R})} \|\phi\|_{L^2(\mathbb{R})} \\
&\quad + r \|\varphi\|_{L^2(\mathbb{R})} \|\phi\|_{L^2(\mathbb{R})} \\
&\le C_1(\sigma, r) \|\varphi\|_{H^1(\mathbb{R})} \|\phi\|_{H^1(\mathbb{R})}.
\end{aligned}$$

To show coercivity, note that with $\int_{\mathbb{R}} \varphi' \varphi \, \mathrm{d}x = \frac{1}{2} \int_{\mathbb{R}} (\varphi^2)' \, \mathrm{d}x = 0$ we have

$$\begin{aligned} a^{\mathrm{BS}}(\varphi, \varphi) &= \frac{1}{2}\sigma^2 \|\varphi'\|^2_{L^2(\mathbb{R})} + r\|\varphi\|^2_{L^2(\mathbb{R})} = \frac{1}{2}\sigma^2 \|\varphi\|^2_{H^1(\mathbb{R})} + (r - \sigma^2/2)\|\varphi\|^2_{L^2(\mathbb{R})} \\ &\geq \frac{1}{2}\sigma^2 \|\varphi\|^2_{H^1(\mathbb{R})} - |r - \sigma^2/2| \|\varphi\|^2_{L^2(\mathbb{R})}. \end{aligned}$$ □

Referring to the abstract existence result Theorem 3.2.2 in the spaces $\mathcal{V} = H^1(\mathbb{R})$ and $\mathcal{H} = L^2(\mathbb{R})$, we deduce that the variational problem (4.8) admits a unique weak solution $u \in L^2(J; H^1(\mathbb{R})) \cap H^1(J; L^2(\mathbb{R}))$ for every $u_0 \in L^2(\mathbb{R})$. Since $u_0(x) = g(e^x)$, $u_0 \in L^2(\mathbb{R})$ implies an unrealistic growth condition on the payoff g. In the next section, we reformulate the problem on a bounded domain where this condition can be weakened. In particular, we require the following *polynomial growth condition* on the payoff function: There exist $C > 0$, $q \geq 1$ such that

$$g(s) \leq C(s+1)^q, \quad \text{for all } s \in \mathbb{R}_+. \tag{4.10}$$

This condition is satisfied by the payoff function of all standard contracts like, e.g. plain vanilla European call, put or power options.

4.3 Localization

The unbounded log-price domain $\mathbb{R}$ of the log price $x = \log s$ is truncated to a bounded domain G. In terms of financial modeling, this corresponds to approximating the option price by a knock-out barrier option. Let $G = (-R, R)$, $R > 0$ be an open subset and let $\tau_G := \inf\{t \geq 0 \mid X_t \in G^c\}$ be the first hitting time of the complement set $G^c = \mathbb{R} \backslash G$ by X. Then, the price of a knock-out barrier option in log-price with payoff $g(e^x)$ is given by

$$v_R(t, x) = \mathbb{E}\left[e^{-r(T-t)} g(e^{X_T}) 1_{\{T < \tau_G\}} \mid X_t = x\right]. \tag{4.11}$$

We show that the barrier option price v_R converges to the option price

$$v(t, x) = \mathbb{E}\left[e^{-r(T-t)} g(e^{X_T}) \mid X_t = x\right],$$

exponentially fast in R.

Theorem 4.3.1 *Suppose the payoff function* $g : \mathbb{R} \to \mathbb{R}_{\geq 0}$ *satisfies* (4.10). *Then, there exist* $C(T, \sigma), \gamma_1, \gamma_2 > 0$, *such that*

$$|v(t, x) - v_R(t, x)| \leq C(T, \sigma)\, e^{-\gamma_1 R + \gamma_2 |x|}.$$

Proof Let $M_T = \sup_{\tau \in [t,T]} |X_\tau|$. Then, with (4.10)

$$|v(t, x) - v_R(t, x)| \leq \mathbb{E}\big[g(e^{X_T}) 1_{\{T \geq \tau_G\}} \mid X_t = x\big] \leq C\mathbb{E}\big[e^{qM_T} 1_{\{M_T > R\}} \mid X_t = x\big].$$

Using [143, Theorem 25.18], it suffices to show that there exist a constant $C(T,\sigma) > 0$ such that

$$\mathbb{E}\left[e^{q|X_T|}1_{\{|X_T|>R\}} \mid X_t = x\right] \le C(T,\sigma)\, e^{-\gamma_1 R+\gamma_2|x|}.$$

We have for $\mu = r - \sigma^2/2$, with the transition probability p_{T-t},

$$\begin{aligned}
\mathbb{E}\left[e^{q|X_T|}1_{\{|X_T|>R\}} \mid X_t = x\right] &= \int_{\mathbb{R}} e^{q|z+x|}1_{\{|z+x|>R\}}\, p_{T-t}(z)\,\mathrm{d}z \\
&\le e^{q|x|}\int_{\mathbb{R}} e^{q|z|}1_{\{|z+x|>R\}}\frac{1}{\sqrt{2\pi\sigma^2(T-t)}}\, e^{-(z-\mu(T-t))^2/(2\sigma^2(T-t))}\,\mathrm{d}z \\
&\le C_1(T,\sigma)\, e^{q|x|}\int_{\mathbb{R}} e^{(q+\mu/\sigma^2)|z|}1_{\{|z+x|>R\}}e^{-z^2/(2\sigma^2(T-t))}\,\mathrm{d}z \\
&\le C_1(T,\sigma)\, e^{q|x|}\int_{\mathbb{R}} e^{-(\eta-q-\mu/\sigma^2)(R-|x|)}e^{\eta|z|}e^{-z^2/(2\sigma^2(T-t))}\,\mathrm{d}z \\
&\le C_1(T,\sigma)\, e^{-\gamma_1 R+\gamma_2|x|}\int_{\mathbb{R}} e^{\eta|z|}e^{-z^2/(2\sigma^2(T-t))}\,\mathrm{d}z,
\end{aligned}$$

with $\gamma_1 = \eta - q - \mu/\sigma^2$, and $\gamma_2 = \gamma_1 + q$. Since $\int_{\mathbb{R}} e^{\eta|z|}e^{-z^2/(2\sigma^2(T-t))}\,\mathrm{d}z < \infty$ for any $\eta > 0$, we obtain the required result by choosing $\eta > q + \mu/\sigma^2$. □

Remark 4.3.2 We see from Theorem 4.3.1 that $v_R \to v$ exponentially for a fixed x as $R \to \infty$. The artificial zero Dirichlet barrier type conditions at $x = \pm R$ are *not* describing correctly the asymptotic behavior of the price $v(t,x)$ for large $|x|$. Since the barrier option price v_R is a good approximation to v for $|x| \ll R$, R should be selected substantially larger than the values of x of interest.

The barrier option price v_R can again be computed as the solution of a PDE provided some smoothness assumptions.

Theorem 4.3.3 *Let $v_R(t,x) \in C^{1,2}(J\times\mathbb{R}) \cap C^0(\overline{J}\times\mathbb{R})$ be a solution of*

$$\partial_t v_R + \mathcal{A}^{\mathrm{BS}} v_R - r v_R = 0 \tag{4.12}$$

on $(0,T)\times G$ where the terminal and boundary conditions are given by

$$v_R(T,x) = g(e^x), \quad \forall x \in G, \quad v_R(t,x) = 0 \quad \text{on } (0,T)\times G^c.$$

Then, $v_R(t,x)$ can also be represented as in (4.11).

Now we can restate the problem (4.8) on the bounded domain:

Find $u_R \in L^2(J; H_0^1(G)) \cap H^1(J; L^2(G))$ such that

$$
\begin{aligned}
&(\partial_t u_R, v) + a^{\mathrm{BS}}(u_R, v) = 0,\ \forall v \in H_0^1(G),\ \text{a.e. in } J, \qquad (4.13)\\
&u_R(0) = u_0|_G.
\end{aligned}
$$

By Proposition 4.2.1 and Theorem 3.2.2, the problem (4.13) is well-posed, i.e. there exists a unique solution $u_R \in L^2(0,T; H_0^1(G)) \cap C^0([0,T]; L^2(G))$ which can be approximated by a finite element Galerkin scheme.

4.4 Discretization

We use the finite element and the finite difference method to discretize the Black–Scholes equation. As for the heat equation in Chap. 2, we use the variational formulation of the differential equations for FEM and determine approximate solutions that are piecewise linear. For FDM we replace the derivatives in the differential equation by difference quotients.

4.4.1 Finite Difference Discretization

We discretize the PDE (4.12) directly using finite differences on a bounded domain with homogeneous Dirichlet boundary conditions. Proceeding as in Sect. 2.3.1, we obtain the matrix problem:

$$
\begin{aligned}
&\text{Find } \underline{u}^{m+1} \in \mathbb{R}^N \text{ such that for } m = 0, \ldots, M-1,\\
&\big(\mathbf{I} + \theta k \mathbf{G}^{\mathrm{BS}}\big)\underline{u}^{m+1} = \big(\mathbf{I} - (1-\theta) k \mathbf{G}^{\mathrm{BS}}\big)\underline{u}^{m}, \qquad (4.14)\\
&\underline{u}^0 = \underline{u}_0,
\end{aligned}
$$

where $\mathbf{G}^{\mathrm{BS}} = \sigma^2/2\,\mathbf{R} + (\sigma^2/2 - r)\mathbf{C} + r\mathbf{I}$, is given explicitly with

$$
\mathbf{R} = \frac{1}{h^2}\begin{pmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{pmatrix}, \quad \mathbf{C} = \frac{1}{2h}\begin{pmatrix} 0 & 1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & -1 & 0 \end{pmatrix}.
$$

4.4.2 Finite Element Discretization

We discretize (4.13) using the θ-scheme and the finite element space $V_N = S_{\mathcal{T}}^1 \cap H_0^1(G)$ with $S_{\mathcal{T}}^1$ given as in (3.17). Setting $u_0(x) := g(e^x)$ and proceeding exactly as in Sect. 3.3 with uniform mesh width h and uniform time steps k, we obtain the matrix problem:

$$\begin{aligned}&\text{Find } \underline{u}^{m+1} \in \mathbb{R}^N \text{ such that for } m = 0, \ldots, M-1\\&(\mathbf{M} + k\theta \mathbf{A}^{\mathrm{BS}})\underline{u}^{m+1} = (\mathbf{M} - k(1-\theta)\mathbf{A}^{\mathrm{BS}})\underline{u}^m,\\&\underline{u}^0 = \underline{u}_0,\end{aligned} \tag{4.15}$$

where $\mathbf{M}_{ij} = (b_j, b_i)_{L^2(G)}$ and $\mathbf{A}^{\mathrm{BS}}_{ij} = a^{\mathrm{BS}}(b_j, b_i)$. Let $\mathbf{M}$ be given as in (2.25). Using (3.22), we can compute $\mathbf{A}^{\mathrm{BS}} = \sigma^2/2\mathbf{S} + (\sigma^2/2 - r)\mathbf{B} + r\mathbf{M}$ explicitly with

$$\mathbf{S} = \frac{1}{h}\begin{pmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{pmatrix}, \quad \mathbf{B} = \frac{1}{2}\begin{pmatrix} 0 & 1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & -1 & 0 \end{pmatrix}. \tag{4.16}$$

4.4.3 Non-smooth Initial Data

As already mentioned, the advantage of finite elements is that we have low smoothness assumptions on the initial data u_0, and therefore on the payoff function g. In particular, as shown in Theorem 3.2.2, we have a unique solution for every $u_0 \in L^2(G)$. However, according to Theorem 3.6.5, we need $u_0 \in H^2(G)$ to obtain the optimal convergence rate $\|u - u_N\|_{L^2(J;L^2(G))} = \mathcal{O}(h^2 + k^r)$ where $r = 1$ for $\theta \in [0, 1] \setminus \{1/2\}$ and $r = 2$ for $\theta = 1/2$. This is due to the time discretization since uniform time steps are used. To recover the optimal convergence rate for $u_0 \in H^s(G)$, $0 < s < 2$ we need to use *graded meshes* in time or space. We assume for simplicity $T = 1$.

Let $\lambda : [0, 1] \to [0, 1]$ be a grading function which is strictly increasing and satisfies

$$\lambda \in C^0([0, 1]) \cap C^1((0, 1)), \quad \lambda(0) = 0, \quad \lambda(1) = 1.$$

We define for $M \in \mathbb{N}$ the algebraically graded mesh by the time points,

$$t_m = \lambda\left(\frac{m}{M}\right), \quad m = 0, 1, \ldots, M.$$

It can be shown [146, Remark 3.11] that we obtain again the optimal convergence rate if $\lambda(t) = \mathcal{O}(t^\beta)$ where β depends on r and s, $\beta = \beta(r, s)$ (algebraic grading).

Example 4.4.1 Consider the payoff functions

$$g_c(s) = \begin{cases} s - K & \text{if } s > K, \\ 0 & \text{else,} \end{cases} \qquad g_d(s) = \begin{cases} 1 & \text{if } s > K, \\ 0 & \text{else.} \end{cases}$$

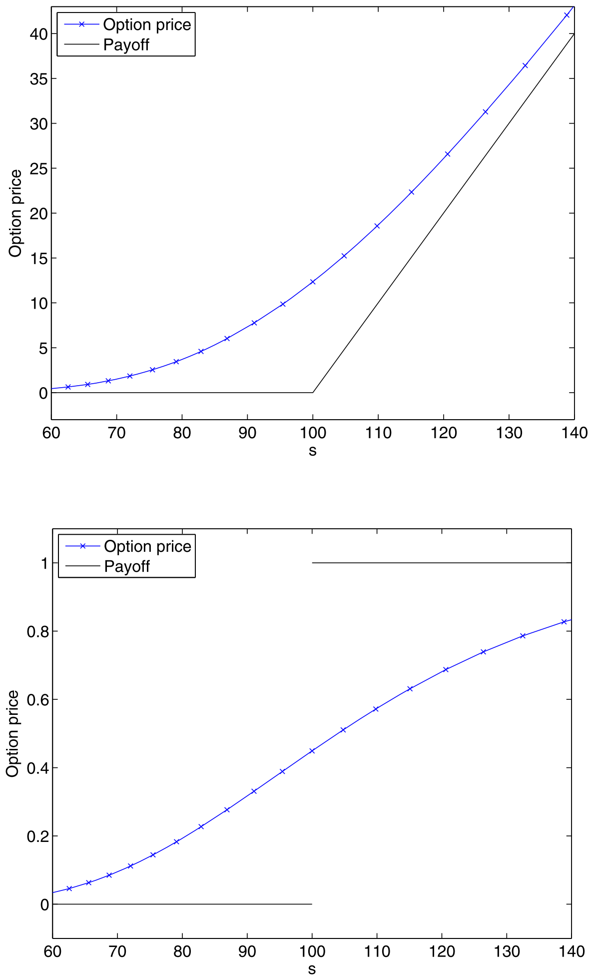


Fig. 4.1 Option price of European call (*top*) and digital (*bottom*) option

We set strike $K = 100$, volatility $\sigma = 0.3$, interest rate $r = 0.01$, maturity $T = 1$. For the discretization we use $N = 300$, $\theta = 1/2$, $R = 3$ and apply the L^2-projection for u_0. As in Remark 4.1.6, we use $K = 1$ in the calculations, for which $R = 3$ is a sufficiently large localization parameter. Using time steps $M = \mathcal{O}(N)$, we obtain the option prices shown in Fig. 4.1.

For $G_0 = (K/2, 3/2K)$ we measure the discrete $L^2(J; L^2(G_0))$-error defined by

$$\sqrt{\sum_{m=1}^{M} k_m h \|\underline{\varepsilon}^m\|_{\ell_2}^2}, \quad \text{with} \quad \|\underline{\varepsilon}^m\|_{\ell_2}^2 := \sum_{i=1}^{N} |u(t_m, x_i) - u_N(t_m, x_i)|^2,$$

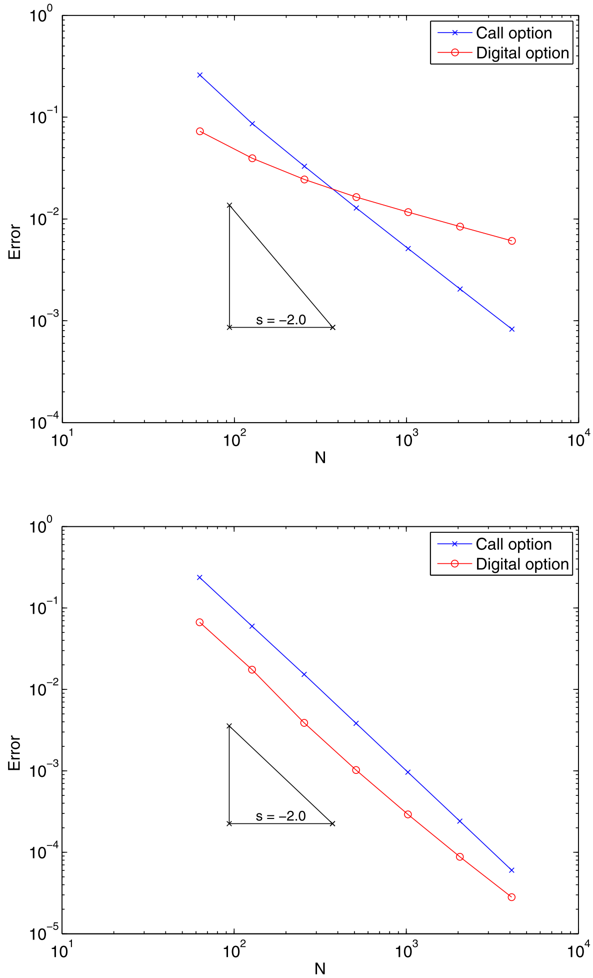


Fig. 4.2 $L^2(J; L^2(G))$ convergence rate for $\theta = 1/2$ using uniform time steps (*top*) and graded time steps (*bottom*)

both with uniform time steps and with graded time steps. The exact values are obtained using the analytic formulas [21]. We use the algebraic grading factor $\beta = 3$ for the call option and the extreme grading factor $\beta = 25$ for the digital option.

It can be seen in Fig. 4.2 that for the call and the digital option we only obtain the optimal convergence rate using a graded time mesh.

Remark 4.4.2 Note that the theory from [146, Chap. 3.3] suggests that choosing $\beta = 10$ is sufficient in the case of a digital option in the Black–Scholes market to obtain full convergence order of the given discretization. But the analysis in [146, Chap. 3.3] is performed in a semi-discrete setting, i.e. there is no discretization error in the space domain. This explains why in our situation, discretizing in space and time, a larger grading factor has to be chosen.

4.5 Extensions of the Black–Scholes Model

We end this chapter by considering two extensions of the Black–Scholes model

$$\mathrm{d}S_t = rS_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}W_t, \quad S_0 = s \geq 0.$$

In the *constant elasticity of variance* (CEV) model, σS_t is replaced by σS_t^ρ for some $0 < \rho < 1$. Another possible extension is to replace the constant volatility σ by a deterministic function $\sigma(s)$ which leads to the so-called *local volatility* models.

4.5.1 CEV Model

Under a unique equivalent martingale measure, the stock price dynamics are given by

$$\mathrm{d}S_t = rS_t\,\mathrm{d}t + \sigma S_t^\rho\,\mathrm{d}W_t, \quad S_0 = s \geq 0, \tag{4.17}$$

where ρ is the *elasticity of variance*. We assume $0 < \rho < 1$. Under this condition, the point 0 is an attainable state. As soon as $S = 0$, we keep S equal to zero, with the resulting process still satisfying (4.17). Note that (4.17) is of the form (1.2) but with $\sigma(t, s) = \sigma s^\rho$, non-Lipschitz. The transformation to log-price, $x = \log s$, will not allow removing the factor s^ρ. A formal application of Theorem 4.1.4 yields that the value $V(t, s)$ of a European vanilla with payoff g is the solution of

$$\partial_t V + \mathcal{A}_\rho^{\mathrm{CEV}} V - rV = 0 \quad \text{in } J \times \mathbb{R}_{\geq 0}, \tag{4.18}$$

with the terminal condition $V(T, s) = g(s)$ and generator

$$(\mathcal{A}_\rho^{\mathrm{CEV}} f)(s) = \frac{1}{2}\sigma^2 s^{2\rho}\partial_{ss} f(s) + rs\partial_s f(s).$$

Note that for $\rho = 1$, the CEV generator is the same as the BS generator. In (4.18), we change to time-to-maturity $t \to T - t$ and localize to a bounded domain $G := (0, R)$, $R > 0$. Thus, we consider

$$\begin{aligned} \partial_t v - \mathcal{A}^{\mathrm{CEV}} v + rv &= 0 && \text{in } J \times G, \\ v &= 0 && \text{on } J \times \{R\}, \\ v(0, s) &= g(s) && \text{in } G. \end{aligned} \tag{4.19}$$

For the variational formulation of (4.19), we multiply the first equation in (4.19) by a test function w and integrate from $s = 0$ to $s = R$. Using $s^{2\rho}\partial_{ss}^2 v = \partial_s(s^{2\rho}\partial_s v) - 2\rho s^{2\rho-1}\partial_s v$, we find, upon integration by parts,

$$\int_0^R s^{2\rho}\partial_{ss} v w\,\mathrm{d}s = -\int_0^R s^{2\rho}\partial_s v\partial_s w\,\mathrm{d}s - 2\rho\int_0^R s^{2\rho-1}\partial_s v w\,\mathrm{d}s + s^{2\rho}\partial_s v w\big|_{s=0}^{s=R}.$$

The boundary terms vanish for $w \in C_0^\infty(G)$.

Define the weighted Sobolev space

$$W_\rho = \overline{C_0^\infty(G)}^{\|\cdot\|_\rho}, \tag{4.20}$$

where the weighted Sobolev norm $\|\cdot\|_\rho$ is defined by

$$\|\varphi\|_\rho^2 := \int_0^R (s^{2\rho}|\partial_s\varphi|^2 + |\varphi|^2)\,\mathrm{d}s, \quad 0 \le \rho \le 1. \tag{4.21}$$

For $\varphi, \phi \in C_0^\infty(G)$, we define the bilinear form

$$\begin{aligned} a_\rho^{\mathrm{CEV}}(\varphi,\phi) &:= \frac{1}{2}\sigma^2 \int_0^R s^{2\rho}\partial_s\varphi\partial_s\phi\,\mathrm{d}s + \rho\sigma^2 \int_0^R s^{2\rho-1}\partial_s\varphi\phi\,\mathrm{d}s \\ &\quad - r\int_0^R s\partial_s\varphi\phi\,\mathrm{d}s + r\int_0^R \varphi\phi\,\mathrm{d}s. \end{aligned} \tag{4.22}$$

The variational formulation of (4.19) is based on the triple of spaces $\mathcal{V} = W_\rho \hookrightarrow \mathcal{H} = L^2(G) = \mathcal{H}^* \hookrightarrow W_\rho^* = \mathcal{V}^*$, and reads:

$$\begin{aligned} &\text{Find } v \in L^2(J; W_\rho) \cap H^1(J; L^2(G)) \text{ such that} \\ &(\partial_t v, w) + a_\rho^{\mathrm{CEV}}(v,w) = 0,\ \forall w \in W_\rho,\ \text{a.e. in } J, \\ &v(0) = g. \end{aligned} \tag{4.23}$$

To establish well-posedness of (4.23), we show continuity and coercivity of the bilinear form $a_\rho^{\mathrm{CEV}}(\cdot,\cdot)$ on W_ρ.

Proposition 4.5.1 *Assume $r > 0$. There exist $C_1, C_2 > 0$ such that for $\varphi, \phi \in W_\rho$*

$$|a_\rho^{\mathrm{CEV}}(\varphi,\phi)| \le C_1\|\varphi\|_\rho\|\phi\|_\rho, \quad \rho \in [0,1]\backslash\{1/2\}, \tag{4.24}$$

$$a_\rho^{\mathrm{CEV}}(\varphi,\varphi) \ge C_2\|\varphi\|_\rho^2, \quad 0 \le \rho \le \frac{1}{2}. \tag{4.25}$$

Proof Let $\varphi \in C_0^\infty(G)$. By Hardy's inequality, for $\varepsilon \neq 1$, $\varepsilon > 0$, and any $R > 0$

$$\Big(\int_0^R s^{\varepsilon-2}|\varphi|^2\,\mathrm{d}s\Big)^{\frac{1}{2}} \le \frac{2}{|\varepsilon-1|}\Big(\int_0^R s^\varepsilon|\partial_s\varphi|^2\,\mathrm{d}s\Big)^{\frac{1}{2}}, \tag{4.26}$$

we find with $\varepsilon = 2\rho \neq 1$, that

$$\begin{aligned} \Big|\int_0^R s^{2\rho-1}\partial_s\varphi\phi\,\mathrm{d}s\Big| &\le \Big(\int_0^R s^{2\rho}(\partial_s\varphi)^2\,\mathrm{d}s\Big)^{\frac{1}{2}}\Big(\int_0^R s^{2\rho-2}\phi^2\,\mathrm{d}s\Big)^{\frac{1}{2}} \\ &\le \|\varphi\|_\rho\frac{2}{|2\rho-1|}\|\phi\|_\rho \end{aligned}$$

and, by the Cauchy–Schwarz inequality,

$$\Big|\int_0^R s\partial_s\varphi\phi\,\mathrm{d}s\Big| \le \|\varphi\|_\rho\Big(\int_0^R s^{2-2\rho}\phi^2\,\mathrm{d}s\Big)^{\frac{1}{2}} \le \|\varphi\|_\rho R^{1-\rho}\,\|\phi\|_{L^2(G)}.$$

Thus, for $\varphi, \phi \in C_0^\infty(G)$, $\rho \neq \frac{1}{2}$, one has

$$|a_\rho^{\mathrm{CEV}}(\varphi,\phi)| \le C(\rho,\sigma,r)\|\varphi\|_\rho\|\phi\|_\rho.$$

Hence, we may extend the bilinear form $a_\rho^{\mathrm{CEV}}(\cdot,\cdot)$ from $C_0^\infty(G)$ to W_ρ by continuity for $\rho \in [0,1]\backslash\{\frac{1}{2}\}$. Furthermore, we have

$$\begin{aligned} a_\rho^{\mathrm{CEV}}(\varphi,\varphi) = \frac{1}{2}\sigma^2\|s^\rho\partial_s\varphi\|_{L^2(G)}^2 + \frac{1}{2}\rho\sigma^2\int_0^R s^{2\rho-1}\partial_s(\varphi^2)\,\mathrm{d}s \\ -\frac{1}{2}r\int_0^R s\partial_s(\varphi^2)\,\mathrm{d}s + r\int_0^R \varphi^2\,\mathrm{d}s. \end{aligned}$$

Integrating by parts, we get, for $0 \le \rho \le \frac{1}{2}$,

$$\int_0^R s^{2\rho-1}\partial_s(\varphi^2)\,\mathrm{d}s = -(2\rho-1)\int_0^R s^{2\rho-2}\varphi^2\,\mathrm{d}s \ge 0.$$

Analogously, $\frac{1}{2}\int_0^R s\partial_s(\varphi^2)\,\mathrm{d}s = -\frac{1}{2}\int_0^R \varphi^2\,\mathrm{d}s$, hence we get for $0 \le \rho \le \frac{1}{2}$

$$a_\rho^{\mathrm{CEV}}(\varphi,\varphi) \ge \frac{1}{2}\sigma^2\|s^\rho\partial_s\varphi\|_{L^2(G)}^2 + \frac{3}{2}r\|\varphi\|_{L^2(G)} \ge \frac{1}{2}\min\{\sigma^2, 3r\}\|\varphi\|_\rho. \qquad \square$$

By Theorem 3.2.2, we deduce

Corollary 4.5.2 *Problem* (4.23) *admits a unique solution* $V \in L^2(J; W_\rho) \cap H^1(J; L^2(G))$ *for* $0 \le \rho < 1/2$.

The previous result addressed only the case $0 \le \rho < \frac{1}{2}$. The case $\frac{1}{2} \le \rho < 1$ (which includes, for $\rho = \frac{1}{2}$, the Heston model and the CIR process) requires a modified variational framework due to the failure of the Hardy inequality (4.26) for $\varepsilon = 1$. Let us develop this framework. We multiply the first equation in (4.19) by an $s^{2\mu}w$, where μ is a parameter to be selected and $w \in C_0^\infty(G)$ is a test function, and integrate from $s = 0$ to $s = R$. We get from (4.19)

$$(\partial_t v, s^{2\mu}w) + a_{\rho,\mu}^{\mathrm{CEV}}(v,w) = 0, \quad \forall w \in C_0^\infty(G), \tag{4.27}$$

where the bilinear form $a_{\rho,\mu}^{\mathrm{CEV}}(\cdot,\cdot)$ is defined by

$$a^{\mathrm{CEV}}_{\rho,\mu}(\varphi,\phi) := \frac{1}{2}\sigma^2\int_0^R s^{2\rho+2\mu}\partial_s\varphi\partial_s\phi\,\mathrm{d}s + \sigma^2(\rho+\mu)\int_0^R s^{2\rho+2\mu-1}\partial_s\varphi\phi\,\mathrm{d}s$$
$$-r\int_0^R s^{1+2\mu}\partial_s\varphi\phi\,\mathrm{d}s + r\int_0^R s^{2\mu}\varphi\phi\,\mathrm{d}s. \tag{4.28}$$

Note that $a^{\mathrm{CEV}}_{\rho,0} = a^{\mathrm{CEV}}_{\rho}$ and $a^{\mathrm{CEV}}_{1} = a^{\mathrm{BS}}$. We introduce the spaces $W_{\rho,\mu}$ as closures of $C_0^\infty(G)$ with respect to the norm

$$\|\varphi\|^2_{\rho,\mu} := \int_0^R \left(s^{2\rho+2\mu}|\partial_s\varphi|^2 + s^{2\mu}|\varphi|^2\right)\mathrm{d}s\,, \tag{4.29}$$

compare with (4.20). Note that $\|\varphi\|_\rho = \|\varphi\|_{\rho,0}$. We now show the analog to Proposition 4.5.1, for $\rho \in [1/2, 1]$.

Proposition 4.5.3 *Assume* $0 \le \rho \le 1$ *and select*

$$\begin{cases} \mu = 0 & \text{if } 0 \le \rho < \frac{1}{2},\ \rho = 1, \\ -\frac{1}{2} < \mu < \frac{1}{2} - \rho & \text{if } \frac{1}{2} \le \rho < 1. \end{cases} \tag{4.30}$$

Assume also $r > 0$. *Then there exist* $C_1, C_2 > 0$ *such that* $\forall\varphi, \phi \in W_{\rho,\mu}$ *the following holds*:

$$|a^{\mathrm{CEV}}_{\rho,\mu}(\varphi,\phi)| \le C_1\|\varphi\|_{\rho,\mu}\|\phi\|_{\rho,\mu}, \tag{4.31}$$
$$a^{\mathrm{CEV}}_{\rho,\mu}(\varphi,\varphi) \ge C_2\|\varphi\|^2_{\rho,\mu}. \tag{4.32}$$

Proof The continuity (4.31) of $a^{\mathrm{CEV}}_{\rho,\mu}$ in $W_{\rho,\mu}(0,R)\times W_{\rho,\mu}(0,R)$ follows from the Cauchy–Schwarz inequality and by Hardy's inequality (4.26) with $\varepsilon = 2(\rho+\mu) \ne 1$

$$|a^{\mathrm{CEV}}_{\rho,\mu}(\varphi,\phi)| \le \frac{1}{2}\sigma^2\|\varphi\|_{\rho,\mu}\|\phi\|_{\rho,\mu} + \sigma^2(\rho+\mu)\|\varphi\|_{\rho,\mu}\left(\int_0^R s^{2\rho+2\mu-2}\phi^2\,\mathrm{d}s\right)^{1/2}$$
$$+ r\|\varphi\|_{\rho,\mu}\left(\int_0^R s^{2+2\mu-2\rho}\phi^2\,\mathrm{d}s\right)^{1/2}$$
$$\le \left(\frac{\sigma^2}{2} + \frac{2\sigma^2(\rho+\mu)}{|2\rho+2\mu-1|} + rR^{1-\rho}\right)\|\varphi\|_{\rho,\mu}\|\phi\|_{\rho,\mu}.$$

Let $\varphi \in C_0^\infty(G)$. We calculate

$$a^{\mathrm{CEV}}_{\rho,\mu}(\varphi,\varphi) = \frac{1}{2}\sigma^2\|s^{\rho+\mu}\partial_s\varphi\|^2_{L^2(G)} + \frac{1}{2}\sigma^2(\rho+\mu)\int_0^R s^{2\rho+2\mu-1}\partial_s(\varphi^2)\,\mathrm{d}s$$
$$-\frac{1}{2}r\int_0^R s^{1+2\mu}\partial_s(\varphi^2)\,\mathrm{d}s + r\|s^\mu\varphi\|^2_{L^2(G)}$$

$$= \frac{1}{2}\sigma^2 \|s^{\rho+\mu}\varphi_s\|^2_{L^2(G)} - \frac{1}{2}\sigma^2(\rho+\mu)(2\rho+2\mu-1)\int_0^R s^{2\rho+2\mu-2}\varphi^2\,\mathrm{d}s$$
$$+ \frac{1}{2}r(1+2\mu)\int_0^R s^{2\mu}\varphi^2\,\mathrm{d}s + r\|s^\mu\varphi\|^2_{L^2(G)}.$$

Given $1/2 \le \rho < 1$, we now choose μ such that $-1/2 \le \mu < 1/2 - \rho$. Then, $2\rho + 2\mu - 1 < 0$, $1 + 2\mu \ge 0$, $\rho + \mu \ge 0$, and we get

$$a^{\mathrm{CEV}}_{\rho,\mu}(\varphi,\varphi) \ge \frac{1}{2}\sigma^2\|s^{\rho+\mu}\partial_s\varphi\|^2_{L^2(G)} + r\|s^\mu\varphi\|^2_{L^2(G)} \ge \frac{1}{2}\min\{\sigma^2, 2r\}\|\varphi\|^2_{\rho,\mu}.$$

By density of $C_0^\infty(0, R)$ in $W_{\rho,\mu}$, we have shown (4.32). □

We are now ready to cast Eq. (4.19) into the abstract parabolic framework. We choose μ as in (4.30) and observe that $\mu \le 0$ then. Hence, if $\mathcal{H}_\mu := L^2(0, R; s^{2\mu}\,\mathrm{d}s)$ denotes the weighted L^2-space corresponding to $W_{\rho,\mu}$, we have the dense inclusions

$$W_{\rho,\mu} \hookrightarrow \mathcal{H}_\mu \cong (\mathcal{H}_\mu)^* \hookrightarrow (W_{\rho,\mu})^*. \tag{4.33}$$

Denote by $(\cdot,\cdot)_\mu$ the inner product in $\mathcal{H}_\mu$. The weak formulation then reads:

$$\begin{aligned}
&\text{Find } v \in L^2(J; W_{\rho,\mu}) \cap H^1(J; \mathcal{H}_\mu) \text{ such that}\\
&(\partial_t v, w)_\mu + a^{\mathrm{CEV}}_{\rho,\mu}(v, w) = 0,\ \forall w \in W_{\rho,\mu},\ \text{ a.e. in } J,\\
&v(0) = g.
\end{aligned} \tag{4.34}$$

Applying Theorem 3.2.2, we have shown

Theorem 4.5.4 *Let $\rho \in [0, 1]$, and assume that μ satisfies (4.30). Then, the problem (4.34) admits a unique solution.*

4.5.2 Local Volatility Models

We replace the constant volatility σ in the Black–Scholes model (1.1) by a deterministic function $\sigma(s)$, i.e. the Black–Scholes model is extended to

$$\mathrm{d}S_t = rS_t\,\mathrm{d}t + \sigma(S_t)S_t\,\mathrm{d}W_t, \tag{4.35}$$

with $r \in \mathbb{R}_{\ge 0}$ and $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$. We assume that the function $s \mapsto s\sigma(s)$ satisfies (1.3)–(1.4), such that the SDE (4.35) admits a unique solution. Thus, the infinitesimal generator $\mathcal{A}$ of the process S is

$$(\mathcal{A}f)(s) = \frac{1}{2}s^2\sigma^2(s)\partial_{ss}f(s) + rs\partial_s f(s).$$

It follows from Theorem 4.1.4 that the option price V in (4.1) solves $\partial_t V + \mathcal{A}V - rV = 0$ in $\overline{J} \times \mathbb{R}_+$, $V(T,s) = g(s)$ in $\mathbb{R}_+$. As in the Black–Scholes model, we switch to time-to-maturity $t \to T - t$, to log-price $x = \ln(s)$ and localize the PDE to a bounded domain $G = (-R, R)$, $R > 0$. Thus, we consider the parabolic problem for $v(t,x) = V(T-t, e^x)$

$$\begin{aligned} \partial_t v - \mathcal{A}^{\mathrm{LV}} v + rv &= 0 && \text{in } J \times G, \\ v &= 0 && \text{on } J \times \partial G, \\ v(0,x) &= g(e^x) && \text{in } G, \end{aligned} \tag{4.36}$$

where, for $\widetilde{\sigma}(x) := \sigma(e^x)$, we denote by $\mathcal{A}^{\mathrm{LV}}$ the operator

$$(\mathcal{A}^{\mathrm{LV}} f)(x) = \frac{1}{2}\widetilde{\sigma}^2(x)\partial_{xx} f(x) + \Big(r - \frac{1}{2}\widetilde{\sigma}^2(x)\Big)\partial_x f(x).$$

The weak formulation to (4.36) reads:

$$\begin{aligned} &\text{Find } u \in L^2(J; H_0^1(G)) \cap H^1(J; L^2(G)) \text{ such that} \\ &(\partial_t u, v) + a^{\mathrm{LV}}(u,v) = 0, \ \forall v \in H_0^1(G), \ \text{a.e. in } J, \\ &u(0) = g, \end{aligned} \tag{4.37}$$

where the bilinear form $a^{\mathrm{LV}}(\cdot,\cdot) : H_0^1(G) \times H_0^1(G) \to \mathbb{R}$ is given by

$$a^{\mathrm{LV}}(\varphi, \phi) := \frac{1}{2}\int_G \widetilde{\sigma}^2 \varphi' \phi' \,\mathrm{d}x + \int_G \big(\widetilde{\sigma}\widetilde{\sigma}' + \widetilde{\sigma}^2/2 - r\big)\varphi' \phi \,\mathrm{d}x + r\int_G \varphi\phi \,\mathrm{d}x.$$

Note that the bilinear form $a^{\mathrm{LV}}(\cdot,\cdot)$ is a particular case of the bilinear form $a(\cdot,\cdot)$ defined in (3.16) by identifying the coefficients $\alpha(x) = \widetilde{\sigma}^2(x)/2$, $\beta(x) = (\widetilde{\sigma}\widetilde{\sigma}')(x) + \widetilde{\sigma}^2(x)/2 - r$ and $\gamma(x) = r$. Hence, the stiffness matrix $\mathbf{A}^{\mathrm{LV}}$ corresponding to the bilinear form $a^{\mathrm{LV}}(\cdot,\cdot)$ can be implemented using the method described in Sect. 3.4.

Proposition 4.5.5 *Assume that $\widetilde{\sigma} : \mathbb{R} \to \mathbb{R}_+$ satisfies $\widetilde{\sigma} \in W^{1,\infty}(G)$ and $\exists \sigma_0 > 0$ such that $\widetilde{\sigma}(x) \geq \sigma_0 > 0$, $\forall x \in G$. Then, there exist constants $C_i > 0$, $i = 1,2,3$, such that for all $\varphi, \phi \in H_0^1(G)$ there holds*

$$\begin{aligned} |a^{\mathrm{LV}}(\varphi,\phi)| &\leq C_1 \|\varphi\|_{H^1(G)} \|\phi\|_{H^1(G)}, \\ |a^{\mathrm{LV}}(\varphi,\varphi)| &\geq C_2 \|\varphi\|^2_{H^1(G)} - C_3 \|\varphi\|^2_{L^2(G)}. \end{aligned}$$

Proof We have

$$\begin{aligned} |a^{\mathrm{LV}}(\varphi,\phi)| \leq\ & \frac{1}{2}\|\widetilde{\sigma}\|^2_{L^\infty(G)} \|\varphi'\|_{L^2(G)} \|\phi'\|_{L^2(G)} + r\|\varphi\|_{L^2(G)}\|\phi\|_{L^2(G)} \\ & + \big(\|\widetilde{\sigma}\|_{L^\infty(G)} \|\widetilde{\sigma}'\|_{L^\infty(G)} + 1/2\|\widetilde{\sigma}\|^2_{L^\infty(G)} + r\big)\|\varphi'\|_{L^2(G)}\|\phi\|_{L^2(G)} \\ \leq\ & C_1 \|\varphi\|_{H^1(G)}\|\phi\|_{H^1(G)}. \end{aligned}$$

Denote by $\overline{\sigma}$ the constant $\overline{\sigma} := \|\widetilde{\sigma}\|_{L^\infty(G)}\|\widetilde{\sigma}'\|_{L^\infty(G)} + 1/2\|\widetilde{\sigma}\|^2_{L^\infty(G)}$. Then, for an arbitrary $\varepsilon > 0$,

$$\begin{aligned} a^{\mathrm{LV}}(\varphi,\varphi) &\geq \frac{1}{2}\sigma_0^2\|\varphi'\|^2_{L^2(G)} + r\|\varphi\|^2_{L^2(G)} - \overline{\sigma}\|\varphi'\|_{L^2(G)}\|\varphi\|_{L^2(G)} \\ &\geq \big(\sigma_0^2/2 - \varepsilon\overline{\sigma}\big)\|\varphi'\|^2_{L^2(G)} + \big(r - \overline{\sigma}/(4\varepsilon)\big)\|\varphi\|^2_{L^2(G)}. \end{aligned}$$

Choosing $\varepsilon = \sigma_0^2/(4\overline{\sigma})$, we have

$$a^{\mathrm{LV}}(\varphi,\varphi) \geq \sigma_0^2/4\|\varphi'\|^2_{L^2(G)} + (r - (\overline{\sigma}/\sigma_0)^2)\|\varphi\|^2_{L^2(G)},$$

which gives, by the Poincaré inequality (3.4),

$$\begin{aligned} a^{\mathrm{LV}}(\varphi,\varphi) &\geq \sigma_0^2/8\|\varphi'\|^2_{L^2(G)} + \sigma_0^2/(8C)\|\varphi\|^2_{L^2(G)} - |r - (\overline{\sigma}/\sigma_0)^2|\|\varphi\|^2_{L^2(G)} \\ &\geq \sigma_0^2/8\min\{1, C^{-1}\}\|\varphi\|^2_{H^1(G)} - C_3\|\varphi\|^2_{L^2(G)}. \end{aligned}$$ □

By Proposition 4.5.5, the bilinear form $a^{\mathrm{LV}}(\cdot,\cdot)$ is continuous and satisfies a Gårding inequality. Hence, for $g \in L^2(G)$, the weak formulation (4.37) admits a unique solution $u \in L^2(J; H_0^1(G)) \cap H^1(J; L^2(G))$.

4.6 Further Reading

To derive the partial differential equations, we followed the line of Lamberton and Lapeyre [109]. Using finite differences to price options was first described in Brennan and Schwartz [26]. A rigorous treatment can be found in Achdou and Pironneau [1]. Finite elements were first applied to finance in Wilmott et al. [161]. Error estimates for non-smooth initial data are given in Thomée [154]. The CEV model was introduced by Cox and Ross [45, 46], and analytic formulas can be found, for example, in Hsu et al. [88]. See also [56]. The probabilistic argument to estimate the localization error is due to Cont and Voltchkova in [41], even in a more general setting of Lévy processes. The local volatility model is used to recover the volatility smile observed in the stock market as shown in Derman and Kani [57] or Dupire [59].

Chapter 5
American Options

Pricing American contracts requires, due to the early exercise feature of such contracts, the solution of optimal stopping problems for the price process. Similar to the pricing of European contracts, the solutions of these problems have a deterministic characterization. Unlike in the European case, the pricing function of an American option does not satisfy a partial differential equation, but a partial differential inequality, or to be more precise, a system of inequalities. We consider the discretization of this inequality both by the finite difference and the finite element method where the latter is approximating the solutions of variational inequalities. The discretization in both cases leads to a sequence of linear complementarity problems (LCPs). These LCPs are then solved iteratively by the PSOR algorithm. Thus, from an algorithmic point of view, the pricing of an American option differs from the pricing of a European option only as in the latter we have to solve linear systems, whereas in the former we need to solve linear complementarity problems. The calculation of the stiffness matrix is the same for both options since the matrix depends on the model and not on the contract.

We assume that the dynamics of the stock price is modeled by a geometric Brownian motion and that no dividends are paid. Under this assumption, the value of an American call contract is equal to the value of the corresponding European call option. Therefore, we focus on put options in the following.

5.1 Optimal Stopping Problem

Recall that a *stopping time* τ for a given filtration $\mathcal{F}_t$ is a random variable taking values in $(0, \infty)$ and satisfying

$$\{\tau \leq t\} \in \mathcal{F}_t, \quad \forall t \geq 0.$$

N. Hilber et al., *Computational Methods for Quantitative Finance*, Springer Finance,
DOI 10.1007/978-3-642-35401-4_5,

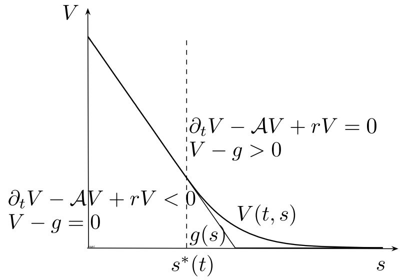


Fig. 5.1 American put option

Denoting by $\mathcal{T}_{t,T}$ the set of all stopping times for S_t with values in the interval (t, T), the value of an American option is given by

$$V(t,s) := \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}\big[e^{-r(\tau - t)} g(S_\tau) \mid S_t = s\big]. \tag{5.1}$$

As for the European vanilla style contracts, there is a close connection between the probabilistic representation (5.1) of the price and a deterministic, PDE based representation of the price. We have

Theorem 5.1.1 *Let $v(t,x)$ be a sufficiently smooth solution of the following system of inequalities*

$$\begin{aligned} \partial_t v - \mathcal{A}^{\mathrm{BS}} v + rv &\geq 0 && \text{in } J \times \mathbb{R}, \\ v(t,x) &\geq g(e^x) && \text{in } J \times \mathbb{R}, \\ (\partial_t v - \mathcal{A}^{\mathrm{BS}} v + rv)(g - v) &= 0 && \text{in } J \times \mathbb{R}, \\ v(0,x) &= g(e^x) && \text{in } \mathbb{R}. \end{aligned} \tag{5.2}$$

Then, $V(T-t, e^x) = v(t,x)$.

A proof can be found in [15], we also refer to [98] for further details. For each $t \in J$ there exists the so-called *optimal exercise price* $s^*(t) \in (0, K)$ such that for all $s \leq s^*(t)$ the value of the American put option is the value of immediate exercise, i.e. $V(t,s) = g(s)$, while for $s > s^*(t)$ the value exceeds the immediate exercise value, see Fig. 5.1. The region $\mathcal{C} := \{(t,s) \mid s > s^*(t)\}$ is called the *continuation region* and the complement $\mathcal{C}^c$ of $\mathcal{C}$ is the *exercise region*. Since the optimal exercise price is not known a priori, it is called a free boundary for the associated pricing PDE and the problem of determining the option price is then a *free boundary problem*. Note that the inequalities (5.2) do not involve the free boundary $s^*(t)$.

Remark 5.1.2 In the Black–Scholes model, the derivative of V at $x = s^*(t)$ is continuous which is known as the *smooth pasting* condition. This does not hold for pure jump models.

5.2 Variational Formulation

The set of admissible solutions for the variational form of (5.2) is the convex set $\mathcal{K}_g \subset H^1(\mathbb{R})$

$$\mathcal{K}_g := \{v \in H^1(\mathbb{R}) : v \geq g \text{ a.e. } x\}. \tag{5.3}$$

The variational form of (5.2) reads:

$$\begin{aligned}
&\text{Find } u \in L^2(J; H^1(\mathbb{R})) \cap H^1(J; L^2(\mathbb{R})) \text{ such that } u(t, \cdot) \in \mathcal{K}_g \text{ and} \\
&(\partial_t u, v - u) + a^{\mathrm{BS}}(u, v - u) \geq 0, \;\; \forall v \in \mathcal{K}_g, \;\; \text{a.e. in } J, \\
&u(0) = g.
\end{aligned} \tag{5.4}$$

Since the bilinear form $a^{\mathrm{BS}}(\cdot, \cdot)$ is continuous and satisfies a Gårding inequality in $H^1(\mathbb{R})$ by Proposition 4.2.1, problem (5.4) admits a unique solution for every payoff $g \in L^\infty(\mathbb{R})$ by Theorem B.2.2 of the Appendix B.

As in the case of plain European vanilla contracts, we localize (5.4) to a bounded domain $G = (-R, R)$, $R > 0$, by approximating the value v in (5.1)

$$v(t, x) := \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}\left[e^{-r(\tau - t)} g(e^{X_\tau}) \mid X_t = x\right],$$

by the value of a barrier option

$$v_R(t, x) = \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}\left[e^{-r(\tau - t)} g(e^{X_\tau}) 1_{\{\tau < \tau_G\}} \mid X_t = x\right]. \tag{5.5}$$

Repeating the arguments in the proof of Theorem 4.3.1, we obtain the estimate for the localization error: there exist constants $C(T, \sigma), \gamma_1, \gamma_2 > 0$ such that

$$|v(t, x) - v_R(t, x)| \leq C(T, \sigma)\, e^{-\gamma_1 R + \gamma_2 |x|},$$

valid for payoff functions $g : \mathbb{R} \to \mathbb{R}$ satisfying the growth condition (4.10). Thus, we consider problem (5.1) restricted on G

$$\begin{aligned}
\partial_t v_R - \mathcal{A}^{\mathrm{BS}} v_R + r v_R \geq 0 \quad & \text{in } J \times G, \\
v_R(t, x) \geq g(e^x)|_G \quad & \text{in } J \times G, \\
(\partial_t v_R - \mathcal{A}^{\mathrm{BS}} v_R + r v_R)(g|_G - v_R) = 0 \quad & \text{in } J \times G, \\
v_R(0, x) = g(e^x)|_G \quad & \text{in } G, \\
v_R(t, \pm R) = g(e^{\pm R}) \quad & \text{in } J.
\end{aligned} \tag{5.6}$$

We cast the truncated problem into variational form. To get a simple convex set for admissible functions and to facilitate the numerical solution, we introduce the *time value* of the option (also called *excess to payoff*) $w_R := v_R - g|_G$ and consider

$$\mathcal{K}_{0,R} := \{v \in H_0^1(G) : v \geq 0 \text{ a.e. } x \in G\}.$$

The variational formulation of the truncated problem reads then

$$\begin{aligned}
&\text{Find } u_R \in L^2(J; H_0^1(G)) \cap H^1(J; L^2(G)) \text{ such that } u_R(t,\cdot) \in \mathcal{K}_{0,R} \text{ and} \\
&(\partial_t u_R, v - u_R) + a^{\mathrm{BS}}(u_R, v - u_R) \geq -a^{\mathrm{BS}}(g, v - u_R), \ \forall v \in \mathcal{K}_{0,R}, \qquad (5.7)\\
&u_R(0) = 0.
\end{aligned}$$

We calculate the right hand side in (5.7) for a put option, i.e. $g(s) = \max\{0, K - s\}$. Let $\varphi \in H_0^1(G)$. Then, by the definition of $a^{\mathrm{BS}}(\cdot,\cdot)$ in (4.9) and integration by parts,

$$\begin{aligned}
-a^{\mathrm{BS}}(g,\varphi) &= -\frac{1}{2}\sigma^2 \int_{-R}^{\ln K} (K - e^x)'\varphi' \,\mathrm{d}x + \left(r - \frac{1}{2}\sigma^2\right) \int_{-R}^{\ln K} (K - e^x)'\varphi \,\mathrm{d}x \\
&\quad - r \int_{-R}^{\ln K} (K - e^x)\varphi \,\mathrm{d}x \\
&= \frac{1}{2}\sigma^2 e^x \varphi \Big|_{-R}^{\ln K} - \frac{1}{2}\sigma^2 \int_{-R}^{\ln K} e^x \varphi \,\mathrm{d}x - \left(r - \frac{1}{2}\sigma^2\right) \int_{-R}^{\ln K} e^x \varphi \,\mathrm{d}x \\
&\quad - r \int_{-R}^{\ln K} (K - e^x)\varphi \,\mathrm{d}x \\
&= \frac{1}{2} K \sigma^2 \varphi(\ln K) - rK \int_{-R}^{\ln K} \varphi \,\mathrm{d}x.
\end{aligned}$$

Since this holds for all φ, $-a^{\mathrm{BS}}(g,\varphi)$ defines the linear functional $f = \frac{1}{2}K\sigma^2\delta_{\ln K} - rK\chi_{\{x \leq \ln K\}} \in H^{-1}(G)$, where $\chi_{\mathcal{B}}$ is the indicator function.

5.3 Discretization

As in the European case, we approximate the option price both by the finite difference and the finite element method. The finite difference method discretizes the partial differential inequalities whereas the finite element method approximates the solution of variational inequalities. In both cases, the discretization leads to a sequence of linear complementarity problems. These LCPs are then solved iteratively by the PSOR algorithm.

5.3.1 Finite Difference Discretization

Discretizing (5.6) with finite differences and the backward Euler scheme, i.e. the θ-scheme with $\theta = 1$, we obtain a sequence of *matrix linear complementarity prob-*

lems

$$\begin{aligned}
&\text{Find } \underline{u}^{m+1} \in \mathbb{R}^N \text{ such that for } m = 0, \dots, M-1,\\
&\big(\mathbf{I} + k\mathbf{G}^{\mathrm{BS}}\big)\underline{u}^{m+1} \geq \underline{u}^m + k\underline{f},\\
&\underline{u}^{m+1} \geq \underline{u}_0,\\
&(\underline{u}^{m+1} - \underline{u}_0)^\top\big((\mathbf{I} + k\mathbf{G}^{\mathrm{BS}})\underline{u}^{m+1} - \underline{u}^m - k\underline{f}\big) = 0,\\
&\underline{u}^0 = \underline{u}_0,
\end{aligned} \tag{5.8}$$

where $\mathbf{G}^{\mathrm{BS}}$ and $\underline{u}_0$ are as in (4.14). The vector $\underline{f}$ accounts for the (non-homogeneous) Dirichlet boundary conditions and is given by $\underline{f} = k(f^-, 0, \dots, 0, f^+)^\top \in \mathbb{R}^N$, with $f^\pm := g(e^{\pm R})(\sigma^2/(2h^2) \mp (\sigma^2/2 - r)/(2h))$. Note that we cannot approximate the time value of the option $w_R = v_R - g|_G$ by the FDM, since the corresponding inequality (5.6) satisfied by w_R involves functionals $f \in H^{-1}(G)$ which cannot be approximated by finite difference quotients.

5.3.2 Finite Element Discretization

We discretize (5.7) using the backward Euler scheme and the finite element space $S^1_{\mathcal{T},0}$ given in (3.21). We obtain

$$\begin{aligned}
&\text{Find } \underline{u}_N^{m+1} \in \mathbb{R}^N_{\geq 0} \text{ such that for } m = 0, \dots, M-1,\\
&(\underline{v} - \underline{u}_N^{m+1})^\top\big(\mathbf{M} + k\mathbf{A}^{\mathrm{BS}}\big)\underline{u}_N^{m+1} \geq (\underline{v} - \underline{u}_N^{m+1})^\top\big(k\underline{f} + \mathbf{M}\underline{u}_N^m\big), \quad \forall \underline{v} \in \mathbb{R}^N_{\geq 0},\\
&\underline{u}_N^0 = \underline{0},
\end{aligned} \tag{5.9}$$

where $\underline{u}_N^m$ is the coefficient vector of $u_N(t_m) \in S^1_{\mathcal{T},0} \cap \mathcal{K}_{0,R}$, and $\mathbf{M}$ and $\mathbf{A}^{\mathrm{BS}}$ are as in the European case, see (4.15). The vector $\underline{f}$ is given by $\underline{f}_i = -a^{\mathrm{BS}}(g, b_i)$. The sequence of inequalities (5.9) can be rewritten as sequence of LCPs.

Lemma 5.3.1 *Denote by* $\mathbf{B} := \mathbf{M} + k\mathbf{A}^{\mathrm{BS}}$, $\underline{F}^m := k\underline{f} + \mathbf{M}\underline{u}_N^m$. *Then, problem* (5.9) *is equivalent to*: *given* $\underline{u}_N^0 = \underline{0}$, *find* $\underline{u}_N^{m+1} \in \mathbb{R}^N$ *such that for* $m = 0, \dots, M-1$,

$$\begin{aligned}
&\mathbf{B}\underline{u}_N^{m+1} \geq \underline{F}^m,\\
&\underline{u}_N^{m+1} \geq \underline{0},\\
&(\underline{u}_N^{m+1})^\top\big(\mathbf{B}\underline{u}_N^{m+1} - \underline{F}^m\big) = 0.
\end{aligned} \tag{5.10}$$

Proof '$\Leftarrow$': Clearly, from $\mathbb{R}^N \ni \underline{u}_N^{m+1} \geq \underline{0}$ follows $\underline{u}_N^{m+1} \in \mathbb{R}^N_{\geq 0}$. From $\mathbf{B}\underline{u}_N^{m+1} \geq \underline{F}^m$, by the definition of $\mathbf{B}$ and $\underline{F}^m$, it follows

$$\underline{w}^{m+1} := \mathbf{M}(\underline{u}_N^{m+1} - \underline{u}_N^m) + k\mathbf{A}^{\mathrm{BS}}\underline{u}_N^{m+1} - k\underline{f} \geq \underline{0}\,.$$

Now,

$$(\underline{v}-\underline{u}_N^{m+1})^\top \underline{w}^{m+1} = \underline{v}^\top \underbrace{\underline{w}^{m+1}}_{\geq \underline{0}} - \underbrace{(\underline{u}_N^{m+1})^\top \underline{w}^{m+1}}_{=0},$$

hence, $(\underline{v}-\underline{u}_N^{m+1})^\top \underline{w}^{m+1} = (\underline{v}-\underline{u}_N^{m+1})^\top (\mathbf{M}(\underline{u}_N^{m+1} - \underline{u}_N^m) + k\mathbf{A}^{\mathrm{BS}}\underline{u}_N^{m+1} - k\underline{f}) \geq 0$ for all $\underline{v} \in \mathbb{R}^N_{\geq 0}$, which is the second line of (5.9).

'$\Rightarrow$': From the second line of (5.9), we have $\forall \underline{v} \in \mathbb{R}^N_{\geq 0}$

$$\underline{v}^\top \underline{w}^{m+1} \geq (\underline{u}_N^{m+1})^\top \underline{w}^{m+1}.$$

Now suppose $(\underline{w}^{m+1})_k < 0$ for some $k \in \{1, \dots, N\}$, and let $(\underline{v})_k \gg 1$. Then the left hand side becomes arbitrarily small, which is a contradiction. Hence $\underline{w}^{m+1} \geq \underline{0}$, which is the inequality $\mathbf{B}\underline{u}_N^{m+1} \geq \underline{F}^m$. Now

$$\underline{u}_N^{m+1} \in \mathbb{R}^N_{\geq 0} \Rightarrow (\underline{u}_N^{m+1})^\top \underline{w}^{m+1} \geq 0.$$

Set in the second line of (5.9) $\underline{v} = \underline{0}$, i.e. $(-\underline{u}_N^{m+1})^\top \underline{w}^{m+1} \geq 0$, hence $(\underline{u}_N^{m+1})^\top \underline{w}^{m+1} \leq 0$. We conclude $(\underline{u}_N^{m+1})^\top \underline{w}^{m+1} = 0$, which is $(\underline{u}_N^{m+1})^\top \big(\mathbf{B}\underline{u}_N^{m+1} - \underline{F}^m\big) = 0$. □

We give a convergence result for the price approximated by FEM. Let $\{u_N^m\}_{m=1}^M$ be the solution of (5.9) and let $u^m := u(t_m, x)$ be the solution of (5.4) at time level t_m. The proof of the next convergence result is shown in [121, Theorem 22].

Theorem 5.3.2 *Assume $u \in C^0((0,T]; H^2(G)) \cap C^{1,1}(\overline{J}; L^2(G))$. Then, there exists a constant $C = C(u) > 0$ such that the following error bound holds*

$$\max_m \|u^m - u_N^m\|_{L^2(G)} + \left(k \sum_{m=1}^M \|u^m - u_N^m\|^2_{H^1(G)}\right)^{1/2} \leq C(k+h).$$

Thus, as in the European case (compare with Theorem 3.6.5), we obtain first order convergence in the energy norm $\|u^M - u_N^M\|_{H^1(G)} = \mathcal{O}(k+h)$, provided that $u(t,x)$ is sufficiently smooth.

5.4 Numerical Solution of Linear Complementarity Problems

In the following, two methods for solving the derived LCPs are described. Both methods are iterative approaches.

Table 5.1 Description of the PSOR algorithm

Choose an initial guess $\underline{x}^0 \geq \underline{c}$.
Choose $\omega \in (0, 1]$ and $\varepsilon > 0$.
For $k = 0, 1, 2, \ldots,$
For $i = 1, \ldots, N,$
$\widetilde{x}_i^{k+1} := \frac{1}{\mathbf{A}_{ii}} \Big(b_i - \sum_{j=1}^{i-1} \mathbf{A}_{ij} x_j^{k+1} - \sum_{j=i+1}^{N} \mathbf{A}_{ij} x_j^k \Big)$
$x_i^{k+1} := \max \left\{ c_i, x_i^k + \omega(\widetilde{x}_i^{k+1} - x_i^k) \right\}$
Next i
If $\lVert \underline{x}^{k+1} - \underline{x}^k \rVert_2 < \varepsilon$ stop else
Next k

5.4.1 Projected Successive Overrelaxation Method

As shown in Eqs. (5.9) and (5.10), the numerical pricing by both the FDM and FEM of an American option reduces to solving (in each time step) an LCP. Both can be written in the abstract form: find $\underline{x} \in \mathbb{R}^N$ such that

$$\begin{aligned} &\mathbf{A}\underline{x} \geq \underline{b}, \\ &\underline{x} \geq \underline{c}, \qquad (5.11) \\ &(\underline{x} - \underline{c})^\top (\mathbf{A}\underline{x} - \underline{b}) = 0. \end{aligned}$$

Problem (5.11) was solved by Cryer [47] using the *projected successive overrelaxation* (PSOR) method for matrices $\mathbf{A}$ being symmetric and positive definite. In applications in finance, however, $\mathbf{A}$ is not symmetric due the presence of a drift term in the infinitesimal generator of the price process. It can be shown that PSOR works also for matrices which are not symmetric, but diagonally dominant.

The algorithm is described in Table 5.1 where we denote by $\mathbf{A}_{ij}$ the entries of $\mathbf{A}$, i.e. $\mathbf{A} = (\mathbf{A}_{ij})_{1 \leq i,j \leq N}$, and we assume that $\mathbf{A}_{ii} \neq 0$, $\forall i$. If the step $x_i^{k+1} := \max\{c_i, x_i^k + \omega(\widetilde{x}_i^{k+1} - x_i^k)\}$ is replaced by

$$x_i^{k+1} = x_i^k + \omega(\widetilde{x}_i^{k+1} - x_i^k),$$

then we get the classical SOR for the solution of $\mathbf{A}\underline{x} = \underline{b}$. This step ensures the inequality condition $\underline{x} \geq \underline{c}$. The parameter ω is a relaxation parameter. Since the PSOR is an iterative algorithm, we have to discuss its convergence. The following result is proven in [1, Chap. 6].

Proposition 5.4.1 *Assume* $\mathbf{A}$ *satisfies*

(i) *There exist constants* $C_1, C_2 > 0$ *such that* $C_1 \underline{v}^\top \underline{v} \leq \underline{v}^\top \mathbf{A} \underline{v} \leq C_2 \underline{v}^\top \underline{v}$, $\forall \underline{v} \in \mathbb{R}^N$;
(ii) *It is diagonally dominant, i.e.* $|\mathbf{A}_{ii}| > \sum_{j \neq i} |\mathbf{A}_{ij}|$, $\forall i$.

Furthermore, assume $\omega \in (0, 1]$. *Then, the sequence* $\{\underline{x}^k\}$ *generated by PSOR converges, as* $k \to \infty$, *to the unique solution* $\underline{x}$ *of* (5.11).

Table 5.2 Description of the primal–dual active set algorithm

Choose an initial guess $\underline{x}^0 \geq \underline{c}$, $\underline{\lambda}^0 \geq 0$.
Choose $\varepsilon > 0$.
For $k = 0, 1, 2, \ldots,$
Set $\mathcal{I}_k = \{i : \lambda_i^k + k_1(c_i^k - x_i^k) \leq 0\}$,
$\mathcal{A}_k = \{i : \lambda_i^k + k_1(c_i^k - x_i^k) > 0\}$
Solve $\mathbf{A}\underline{x}^{k+1} + \underline{\lambda}^{k+1} = \underline{b}$, $\underline{x}^{k+1} = \underline{c}$ on $\mathcal{A}_k$, $\underline{\lambda}^{k+1} = \underline{0}$ on $\mathcal{I}_k$.
If $\|\underline{x}^{k+1} - \underline{x}^k\|_2 < \varepsilon$ stop else
Next k

Note that assumption (i) of Proposition 5.4.1 ensures the existence of a unique solution of (5.11).

5.4.2 Primal–Dual Active Set Algorithm

Problem (5.11) can be formulated as follows: find $\underline{x}, \underline{\lambda} \in \mathbb{R}^N$ such that

$$\begin{aligned} \mathbf{A}\underline{x} + \underline{\lambda} &= \underline{b}, \\ \underline{x} &\geq \underline{c}, \\ \underline{\lambda} &\geq 0, \\ (\underline{x} - \underline{c})^\top \underline{\lambda} &= 0. \end{aligned} \tag{5.12}$$

Problem (5.12) was solved by Ito and Kunisch [84] using the primal–dual active set algorithm that can also be formulated as a semi-smooth Newton method for P-matrices $\mathbf{A}$. A matrix is called a P-matrix if all its principal minors are positive. This can be shown to hold in our setup for the FEM discretization as well as for the FDM discretization for sufficiently small time-step k. The complementarity system in (5.12) can equivalently be expressed as

$$\mathcal{C}(\underline{x}, \underline{\lambda}) = 0, \text{ where } \mathcal{C}(\underline{x}, \underline{\lambda}) := \underline{\lambda} - \max(0, \underline{\lambda} + k_1(\underline{c} - \underline{x})),$$

for each $k_1 > 0$. Therefore, (5.12) is equivalent to

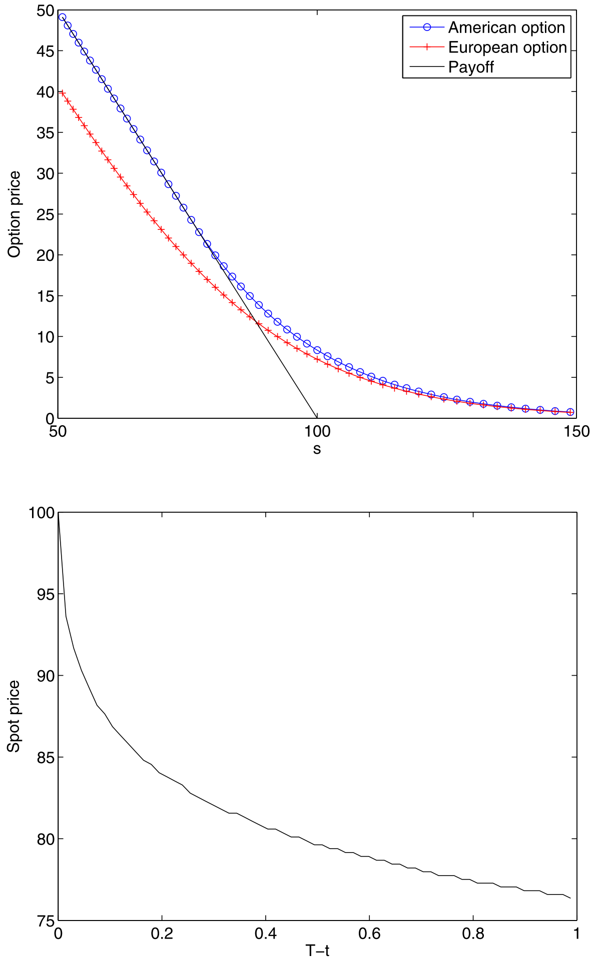
$$\begin{aligned} \mathbf{A}\underline{x} + \underline{\lambda} &= \underline{b}, \\ \mathcal{C}(\underline{x}, \underline{\lambda}) &= 0. \end{aligned} \tag{5.13}$$

The primal–dual active set algorithm is based on using (5.13) for a prediction strategy, i.e. given a pair $(\underline{x}, \underline{\lambda})$, the active and inactive sets are given as

$$\mathcal{I} = \{i : \lambda_i + k_1(c_i - x_i) \leq 0\}, \quad \mathcal{A} = \{i : \lambda_i + k_1(c_i - x_i) > 0\}.$$

This leads to the algorithm given in Table 5.2.

Fig. 5.2 American option price (*top*) and corresponding exercise boundary (*bottom*)



Proposition 5.4.2 *Let* **A** *satisfy* (i) *and* (ii) *from Proposition* 5.4.1 *and let* $(\underline{x}^*, \underline{\lambda}^*)$ *be the unique solution of* (5.13), *then the primal dual active set method converges superlinearly to* $(\underline{x}^*, \underline{\lambda}^*)$, *provided that* $\|\underline{x}^0 - \underline{x}^*\|_2^2 + \|\underline{\lambda}^0 - \underline{\lambda}^*\|_2^2$ *is sufficiently small.*

A proof of this result can be found in [84, Theorem 3.1].

Example 5.4.3 Consider an American put with strike $K = 100$ and maturity $T = 1$. For $\sigma = 0.3$, $r = 0.1$, we compute the price of the option (at $t = T$) as well as the free boundary $s^*(t)$. The results are shown in Fig. 5.2, where we also plot the option price of the corresponding European option, which is, due to the single exercise right, lower than the price of the American contract.

5.5 Further Reading

Approximate, semi-analytic solutions of American option pricing problems can, for example, be found in Barone-Adesi and Whaley [12], Carr [33] or Geske and Johnson [70]. An overview of various methods for pricing an American put in a Black–Scholes setting is given in Barone-Adesi [11]. A rigorous treatment is provided by Jaillet et al. [98] where the Brennan and Schwartz algorithm [25] is used for the discretization. A semi-smooth Newton approach is analyzed in Ito et al. [84, 92] and Hager and Wohlmuth [77]. Holtz and Kunoth [87] provide high-order B-spline approximations for American puts in the Black–Scholes model and the corresponding Greeks.

Computable a posteriori error bounds on the discretization error in the numerical solution of American put contracts in a Black–Scholes setting have been obtained in [126].

Chapter 6
Exotic Options

Options with more sophisticated rules than those for plain vanillas are called *exotic options*. There are different types. *Path dependent* options depend on the whole history of the underlying and not just on the realization at maturity. In particular, we consider *barrier options* which depend on price levels being attained over a period and *Asian options* which depend on the average price of the option's underlying over a period. Furthermore, we look at options which have different exercise styles like *compound options* which are options on options and *swing options* which have multiple exercise rights. We assume that the dynamics of the stock price is modeled by a geometric Brownian motion.

6.1 Barrier Options

Barrier options differ from vanillas in the sense that the option contract is triggered if the price of the underlying hits some barrier $B > 0$. To be more specific, consider a contract which pays a specified amount at maturity T provided during $0 \le t \le T$ the price S does not cross a specified barrier B either from above, the so-called *down-and-out* barrier option, or from below, called *up-and-out* barrier option. If the barrier is crossed before T, the option expires worthless. A *knock-in* barrier option becomes the corresponding European vanilla when the barrier B is crossed at time $0 \le t \le T$, e.g. the up-and-in call becomes the European call when B is crossed from below. We assume here for convenience that B is constant.

A European plain vanilla option pays the same at T as a down-and-out plus a down-and-in with the same barrier B and of the same type (call/put) as the plain vanilla. The standard no arbitrage consideration shows that pricing a knock-in barrier reduces to pricing the corresponding knock-out barrier contract and a plain European vanilla. If we denote the value of a down-and-out barrier option by V_{do}, the value of a up-and-out barrier option by V_{uo} and correspondingly V_{di}, V_{ui} for the knock-in barrier option, we have

$$V_{di}(t,s) = V(t,s) - V_{do}(t,s), \quad V_{ui}(t,s) = V(t,s) - V_{uo}(t,s). \tag{6.1}$$

N. Hilber et al., *Computational Methods for Quantitative Finance*, Springer Finance,
DOI 10.1007/978-3-642-35401-4_6,

Therefore, it is sufficient to consider the prices of knock-out barrier contracts. For notational simplicity, we omit the subscripts.

Let $r \geq 0$ the constant interest rate and let $\tau_B = \inf\{t \geq 0 \mid S_t = B\}$ be the first hitting time of B by the process S. In real price, the value of the down-and-out option V is then given by

$$V_{do}(t,s) = \mathbb{E}\big[e^{r(t-T)} g(S_T) 1_{\{T<\tau_B\}} \mid S_t = s\big],$$

where g is the payoff of the option. The barrier option price V is again the solution of the deterministic BS equation but with different boundary conditions.

Theorem 6.1.1 *Let $V \in C^{1,2}(J \times [B,\infty)) \cap C^0(\overline{J} \times [B,\infty))$ with bounded derivatives in s be a solution of*

$$\partial_t V + \mathcal{A}V - rV = 0 \quad \text{in } J \times (B,\infty), \qquad V(T,s) = g(s) \quad \text{in } (B,\infty),$$

and boundary conditions

$$V(t,s) = 0 \quad \text{in } J \times [0,B],$$

where $\mathcal{A}$ denotes the Black–Scholes generator (4.5). *Then, $V(t,s)$ can be represented as*

$$V(t,s) = \mathbb{E}\big[e^{r(t-T)} g(S_T) 1_{\{T<\tau_B\}} \mid S_t = s\big].$$

Proof We show the result only for $t = 0$. As in the proof of Theorem 4.1.4, we know that the process $M_t := e^{-rt}V(t,S_t)$ is a martingale for $0 \leq t \leq \tau_B$, since $\partial_t V + \mathcal{A}V - rV = 0$ in $J \times [B,\infty)$. Thus,

$$\begin{aligned}
V(0,s) &= \mathbb{E}[M_0 \mid S_0 = s] \\
&= \mathbb{E}[M_{\tau_B \wedge T} \mid S_0 = s] \\
&= \mathbb{E}\big[e^{-r\tau_B \wedge T} V(\tau_B \wedge T, X_{\tau_B \wedge T}) \mid S_0 = s\big] \\
&= \mathbb{E}\big[e^{-rT} V(T,S_T) 1_{\{T<\tau_B\}} \mid S_0 = s\big] \\
&\quad + \mathbb{E}\big[e^{-r\tau_B} V(\tau_B, S_{\tau_B}) 1_{\{T \geq \tau_B\}} \mid S_0 = s\big] \\
&= \mathbb{E}\big[e^{-rT} g(S_T) 1_{\{T<\tau_B\}} \mid S_0 = s\big].
\end{aligned}$$

□

Changing to log-price and time-to-maturity, we obtain the weak formulation:

$$\begin{aligned}
&\text{Find } u \in L^2(J;V) \cap H^1(J;L^2) \text{ such that} \\
&(\partial_t u, v) + a^{\mathrm{BS}}(u,v) = 0, \ \forall v \in V, \ \text{a.e. in } J, \qquad (6.2)\\
&u(0) = u_0,
\end{aligned}$$

where $u_0 = g(e^x)$ and $V = \{u \in H^1((\log(B),\infty)) : u(\log(B)) = 0\}$. As in Sect. 4.3, we can localize the problem to a bounded domain $G = (\log(B), R)$.

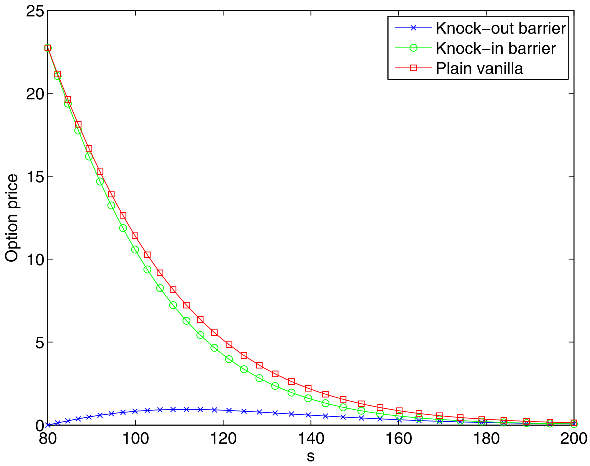


Fig. 6.1 Down-and-out, down-and-in barrier put and plain vanilla put option

Example 6.1.2 Consider a down-and-out and down-and-in put option. Set $K = 100$, $T = 1$, $B = 80$, $\sigma = 0.3$ and $r = 0.01$. We plot the price of both options and the corresponding plain vanilla put price in Fig. 6.1. Here, we computed the down-and-out barrier option using finite elements and applied formula (6.1) to obtain the corresponding down-and-in contract.

6.2 Asian Options

Asian options are path-dependent options where the payoff depends on the price history of the underlying, in particular on the arithmetic average price at maturity,

$$\overline{S}_T := \frac{1}{T - t_0} \int_{t_0}^{T} S(\tau)\,\mathrm{d}\tau. \tag{6.3}$$

The term $T - t_0$ denotes the length of the averaging period. Applying Itô's formula we set $t_0 = 0$. There are different types of options. The *fixed strike* call has payoff $(\overline{S}_T - K)_+$ and the *floating strike* call $(S - \overline{S}_T)_+$. To derive the partial differential equation, we introduce the *new variable*

$$Y(t) = \int_0^t S(\tau)\,\mathrm{d}\tau.$$

Since this history of the asset price is independent of the current price $S(t)$, *we may treat* S, Y *and* t *as independent variables*. The value of an Asian can then be obtained in the form $V(t, S, Y)$. We need a stochastic differential equation for the vector process $(S, Y)^\top$. Applying Itô formula, we obtain that $(S, Y)^\top$ is a diffusion process,

$$\mathrm{d}S_t = rS_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}W_t,$$
$$\mathrm{d}Y_t = S_t\,\mathrm{d}t.$$

We need the two-dimensional version of Proposition 4.1.1.

Proposition 6.2.1 *Let $\mathcal{A}^{\mathrm{AS}}$ denote the differential operator which is, for functions $f \in C^{2,1}(\mathbb{R}\times\mathbb{R})$ with bounded derivatives, given by*

$$(\mathcal{A}^{\mathrm{AS}} f)(x) = \frac{1}{2}\sigma^2 s^2 \partial_{ss} f(s,y) + rs\partial_s f(s,y) + s\partial_y f(s,y). \tag{6.4}$$

Then, the process $M_t := f(S_t, Y_t) - \int_0^t (\mathcal{A}f)(S_\tau, Y_\tau)\,\mathrm{d}\tau$ is a martingale with respect to the filtration of W.

Proof We apply the two-dimensional Itô formula

$$\begin{aligned}\mathrm{d}f(X_t,Y_t) = {}& \frac{\partial f}{\partial x}(X_t,Y_t)\,\mathrm{d}X_t + \frac{\partial f}{\partial y}(X_t,Y_t)\,\mathrm{d}Y_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(X_t,Y_t)\cdot(\mathrm{d}X_t)^2\\ &+ \frac{\partial^2 f}{\partial x \partial y}(X_t,Y_t)\cdot \mathrm{d}X_t\cdot \mathrm{d}Y_t + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(X_t,Y_t)\cdot(\mathrm{d}Y_t)^2,\end{aligned}$$

to the vector process $(S_t, Y_t)^\top \in \mathbb{R}^2_+$ and obtain, using $\mathrm{d}t\cdot\mathrm{d}t = \mathrm{d}t\cdot\mathrm{d}W = 0$,

$$\mathrm{d}f(S_t,Y_t) = (\mathcal{A}^{\mathrm{AS}} f)(S_t,Y_t)\,\mathrm{d}t + \sigma S_t \frac{\partial f}{\partial x} f(S_t,Y_t)\,\mathrm{d}W_t.$$

The result follows since it is shown in Proposition 4.1.1 that the stochastic integral $\int_0^t \sigma S_\tau \partial_x f(S_\tau, Y_\tau)\,\mathrm{d}W_\tau$ is a martingale with respect to the filtration of W. □

Similar arguments as in Theorem 4.1.4 lead to the following PDE for $V(t,s,y)$,

$$\partial_t V + \mathcal{A}^{\mathrm{AS}} V - rV = 0. \tag{6.5}$$

Equation (6.5) is a PDE in two variables, s and y. But it can be reduced to a univariate one,

$$\partial_t H + \frac{1}{2}\sigma^2 (q(t)-z)^2 \partial_{zz} H + r(q(t)-z)\partial_z H = 0 \quad \text{in } J\times\mathbb{R}, \tag{6.6}$$

with the terminal condition $H(T,z) = (z)_+$. The function $q(t)$ depends on the option type. We consider both cases, fixed and floating strike calls:

(i) The *fixed strike call* has the two-dimensional payoff $g(s,y) = (y/T - K)_+$. We set $q(t) = 1 - \frac{t}{T}$, introduce a new variable $z = (y/T-K)/s + q(t)$, and insert the ansatz $V(t,s,y) = sH(t,z)$ into (6.5) to obtain (6.6).

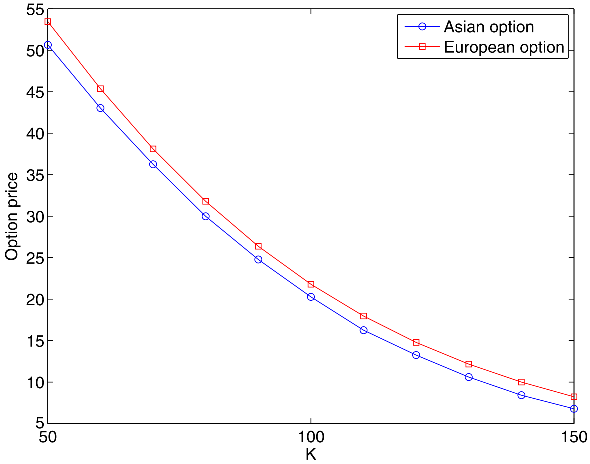


Fig. 6.2 Fixed strike Asian call and European call option with same strike in the same market model

(ii) The *floating strike call* has the two-dimensional payoff $g(s, y) = (s - y/T)_+$. We set $q(t) = t/T$, introduce a new variable $z = q(t) - y/(sT)$, and make the ansatz $V(t, s, y) = sH(t, z)$ in (6.5) to obtain (6.6).

Since Theorem 4.3.1 also holds for path-dependent options and both payoffs, the fixed and the floating strike, satisfy the growth condition (4.10) with respect to the supremum M_T, we can localize the problem (6.5) to a bounded domain $(1/R_s, R_s)^2$ in the variables s and y. This results again in a bounded computational domain $G = (-R_1, R_2)$ for z in (6.6).

Example 6.2.2 Consider a fixed strike Asian option. Set $S = 100$, $T = 1$, $\sigma = 0.3$ and $r = 0.09$. We plot the price of the Asian options and the corresponding European call price for various strike prices K in Fig. 6.2 where we used finite elements for the discretization. It can be seen that Asian and European option prices are both decreasing convex functions in the exercise price K. We also observe the fact that when the time-to-maturity is equal to the length of the averaging period (which is the case here since we set $t_0 = 0$ in (6.3)), European option prices are higher than the corresponding Asian option prices [103].

6.3 Compound Options

Compound options are options on options. Let $V_1(t, s)$ be the option price of a European option with payoff $g_1(s)$ and maturity $T_1 > 0$. Then, the value of a compound option V with payoff $g(s)$ and maturity $0 < T < T_1$ is given by

$$V(t, s) = \mathbb{E}\big[e^{r(t-T)} g(V_1(T, S_T)) \mid S_t = s\big].$$

Applying Theorem 4.1.4, we can obtain the value V_1 of the underlying option by solving the partial differential equation

$$\partial_t V_1 + \mathcal{A}V_1 - rV_1 = 0 \quad \text{in } (0, T_1) \times \mathbb{R}_+, \qquad V_1(T_1, s) = g_1(s) \quad \text{in } \mathbb{R}_+,$$

where $\mathcal{A}$ denotes the Black–Scholes generator (4.5). In a second step, we get the price of the compound option by solving

$$\partial_t V + \mathcal{A}V - rV = 0 \quad \text{in } (0, T) \times \mathbb{R}_+, \qquad V(T, s) = g(V_1(T, s)) \quad \text{in } \mathbb{R}_+.$$

As in the European plain vanilla case, we change to log-price x and localize the problem to a bounded domain $G = (-R, R)$. The corresponding value of the barrier option for the underlying option is denoted with $v_{1,R}$, i.e.

$$v_{1,R}(t, x) = \mathbb{E}\left[e^{-r(T_1-t)} g_1(e^{X_{T_1}}) 1_{\{T_1<\tau_G\}} \mid X_t = x\right], \tag{6.7}$$

and the value of the barrier option for the compound option by v_R, i.e.

$$v_R(t, x) = \mathbb{E}\left[e^{-r(T-t)} g(v_{1,R}(T, X_T)) 1_{\{T<\tau_G\}} \mid X_t = x\right]. \tag{6.8}$$

Again, the compound barrier option price v_R converges to the compound option price v exponentially fast as $R \to \infty$ in a Black–Scholes model.

Theorem 6.3.1 *Suppose the payoff functions $g, g_1 : \mathbb{R} \to \mathbb{R}$ of a compound contract in a Black–Scholes market satisfy* (4.10) *and that g is Lipschitz continuous. Then, there exist $C(T, T_1, \sigma), \gamma_1, \gamma_2 > 0$, such that*

$$|v(t, x) - v_R(t, x)| \leq C(T, T_1, \sigma)\, e^{-\gamma_1 R + \gamma_2 |x|}.$$

Proof Applying Theorem 4.3.1 to the underlying option value v_1, we obtain

$$\left|v_1(t, x) - v_{1,R}(t, x)\right| \leq C_1(T_1, \sigma)\, e^{-\gamma_3 R + \gamma_4 |x|},$$

for some $C_1(T_1, \sigma), \gamma_3, \gamma_4 > 0$. If we denote by $\widetilde{v}_R$ the barrier compound option on v_1,

$$\widetilde{v}_R(t, x) = \mathbb{E}\left[e^{-r(T-t)} g(v_1(T, X_T)) 1_{\{T<\tau_G\}} \mid X_t = x\right],$$

we have with Lipschitz constant C_2

$$\begin{aligned} |v_R(t, x) - \widetilde{v}_R(t, x)| &\leq C_2\, \mathbb{E}\left[\left|v_1(T, X_T) - v_{1,R}(T, X_T)\right| 1_{\{T\geq\tau_G\}} \mid X_t = x\right] \\ &\leq C_1 C_2(T_1, \sigma)\, e^{-\gamma_3 R}\ \mathbb{E}\left[e^{\gamma_4 |X_T|} 1_{\{T<\tau_G\}} \mid X_t = x\right] \\ &\leq C_3(T, T_1, \sigma)\, e^{-\gamma_3 R + \gamma_4 |x|}. \end{aligned}$$

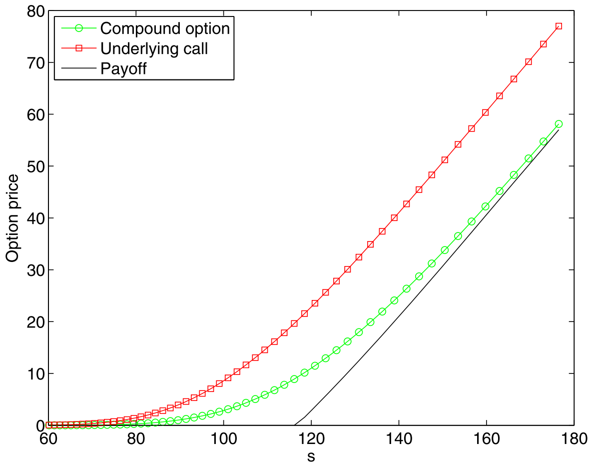


Fig. 6.3 Compound call and underlying call option

We obtain the result by applying Theorem 4.3.1 to $\widetilde{v}_R$,

$$\begin{aligned} |v(t,x) - v_R(t,x)| &\le |v(t,x) - \widetilde{v}_R(t,x)| + |\widetilde{v}_R(t,x) - v_R(t,x)| \\ &\le C_4(T,\sigma)\, e^{-\gamma_5 R + \gamma_6 |x|} + C_3(T, T_1, \sigma)\, e^{-\gamma_3 R + \gamma_4 |x|}, \end{aligned}$$

where the option price v_1 satisfies (4.10) since g_1 satisfies (4.10). □

Therefore, we have the weak formulation on the bounded domain G and in time-to-maturity

$$\begin{aligned} &\text{Find } u \in L^2((0,T); H_0^1(G)) \cap H^1((0,T); L^2(G)) \text{ such that} \\ &(\partial_t u, v) + a^{\mathrm{BS}}(u, v) = 0, \ \forall v \in H_0^1(G), \ \text{a.e. in } (0,T) \\ &u(0) = g(u_1(T_1 - T)), \end{aligned}$$

where the underlying option price u_1 satisfies

$$\begin{aligned} &\text{Find } u_1 \in L^2((0,T_1 - T); H_0^1(G)) \cap H^1((0,T_1 - T); L^2(G)) \text{ such that} \\ &(\partial_t u_1, v) + a^{\mathrm{BS}}(u_1, v) = 0, \ \forall v \in H_0^1(G), \ \text{a.e. in } (0, T_1 - T) \\ &u_1(0) = g_1(e^x)|_G. \end{aligned}$$

Example 6.3.2 Consider a compound call option in a Black–Scholes market. Set $K_1 = 100$, $K = 20$, $T_1 = 1$, $T = 0.5$, $\sigma = 0.3$ and $r = 0.01$. We plot the price of the compound options and the corresponding underlying call price in Fig. 6.3 where we used finite elements for the discretization. It can be seen that the price of the compound option is lower than the price of the underlying option.

6.4 Swing Options

Although swing options appear in various forms, many of them are mathematically of the same type, namely optimal multiple stopping time problems. For example, in energy markets, the delivery of a commodity is limited by capacity constraints usually resulting in a pre-specified refracting time for contracts with several exercise rights. It can be agreed that the refraction period δ which is greater than the minimal delivery time is constant. This separation of two exercise times not only represents an important contract constraint, but also prevents the case of single optimal stopping time problems where all rights are exercised at once.

Let us denote by $\mathcal{T}_{t,T}$ the set of all stopping times for S_t with values in (t, T) and by $\mathcal{T}_{t,\infty}$ the set of all stopping times with values greater or equal than t. For stopping time problems with $p \in \mathbb{N}$ exercise rights, constant refracting period δ and maturity T the following sets are defined.

Definition 6.4.1 The set of admissible stopping time vectors with length $p \in \mathbb{N}$ and refracting time $\delta > 0$ is defined by

$$\begin{aligned}\mathcal{T}_t^{(p)} := \{\tau^{(p)} = (\tau_1, \dots, \tau_p) \mid \tau_i \in \mathcal{T}_{t,\infty} \text{ with } \tau_1 \leq T \text{ a.s. and } \tau_{i+1} - \tau_i \geq \delta \\ \text{for } i = 1, \dots, p-1\}.\end{aligned}$$

Consider a continuous time-dependent payoff function $g : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ where we assume that $g(t, \cdot) = 0$ for $t > T$. The finite horizon multiple stopping time problem with maturity T and $p \in \mathbb{N}$ exercise rights is defined as

$$V^{(p)}(t, s) := \sup_{\tau^{(p)} \in \mathcal{T}_t^{(p)}} \mathbb{E}\left[\sum_{i=1}^{p} e^{-r(\tau_i - t)} g(\tau_i, S_{\tau_i}) \mid S_t = s\right]. \tag{6.9}$$

It is shown in [32] that the multiple stopping time problem can be reduced to a cascade of single stopping time problems, in particular we have

$$V^{(p)}(t, s) = \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}\left[e^{-r(\tau - t)} g^{(p)}(\tau, S_\tau) \mid S_t = s\right],$$

with

$$g^{(p)}(t, s) := \begin{cases} g(t, s) + e^{-r\delta}\mathbb{E}\left[V^{(p-1)}(t + \delta, S_{t+\delta}) \mid S_t = s\right] & \text{if } t \leq T - \delta, \\ g(t, s) & \text{if } t \in (T - \delta, T], \end{cases}$$

$$V^{(0)}(t, s) := 0.$$

Using the results for the American options (5.1.1), the function $u^{(p)}$ in time-to-maturity and log-price is the solution of the variational inequality

$$\begin{aligned}
\partial_t u^{(p)} - \mathcal{A}^{\mathrm{BS}} u^{(p)} + r u^{(p)} &\geq 0 && \text{in } J \times \mathbb{R}, \\
u^{(p)} &\geq \varphi^{(p)} && \text{in } J \times \mathbb{R}, \\
(\partial_t u^{(p)} - \mathcal{A}^{\mathrm{BS}} u^{(p)} + r u^{(p)})(u^{(p)} - \varphi^{(p)}) &= 0 && \text{in } J \times \mathbb{R}, \\
u^{(p)}(0, x) &= \varphi^{(p)}(0, x) && \text{in } \mathbb{R},
\end{aligned}$$

with pth payoff function

$$\varphi^{(p)}(t, x) = \begin{cases} g(T - t, e^x) + w^{(p)}(t, x) & \text{if } t \in [\delta, T), \\ g(T - t, e^x) & \text{if } t \in [0, \delta). \end{cases}$$

This payoff function involves a European option price $w^{(p)}$ which, according to Theorem 4.1.4, satisfies the partial differential equation

$$\begin{aligned}
\partial_\tau w^{(p)} + \mathcal{A}^{\mathrm{BS}} w^{(p)} + r w^{(p)} &= 0 && \text{in } (t - \delta, t) \times \mathbb{R}, \\
w^{(p)}(t - \delta, x) &= u^{(p-1)}(t - \delta, x) && \text{in } \mathbb{R}.
\end{aligned}$$

Similar as for the compound option the pricing problem amounts to pricing iteratively an American option on a European option. Therefore, using the localization arguments from Theorem 6.3.1, we can again truncate the domain to $G = (-R, R)$. As in (5.7), we have the weak formulation

$$\begin{aligned}
&\text{Find } u^{(p)} \in L^2(J; H_0^1(G)) \cap H^1(J; L^2(G)) \text{ such that } u^{(p)}(t, \cdot) \in \mathcal{K}_{0,R} \text{ and} \\
&(\partial_t u^{(p)}, v - u^{(p)}) + a^{\mathrm{BS}}(u^{(p)}, v - u^{(p)}) \geq -a^{\mathrm{BS}}(\varphi^{(p)}, v - u^{(p)}), \ \forall v \in \mathcal{K}_{0,R}, \\
&u^{(p)}(0) = 0,
\end{aligned}$$

where the European option price $w^{(p)}$ is also the weak solution of

$$\begin{aligned}
&\text{Find } w^{(p)} \in L^2((t - \delta, t); H_0^1(G)) \cap H^1((t - \delta, t); L^2(G)) \text{ such that} \\
&(\partial_\tau w^{(p)}, v) + a^{\mathrm{BS}}(w^{(p)}, v) = 0, \ \forall v \in H_0^1(G), \ \text{a.e. in } (t - \delta, t), \\
&w^{(p)}(0) = u^{(p-1)}(t - \delta).
\end{aligned}$$

Example 6.4.2 Consider a swing option. Set $K = 100$, $T = 1$, $\sigma = 0.3$, $r = 0.05$, $\delta = 0.1$ and $p = 5$. We plot the price of the swing options and the corresponding exercise boundary in Fig. 6.4 where we used finite elements for the discretization. It is not surprising that swing and American put option values are similar in appearance. The influence of the refraction period is not visible on the computed swing prices but is clearly seen on the exercise regions. Moreover, it is observed that for $p, p' \in \mathbb{N}$ with $p \geq p'$ the exercise boundaries satisfies the monotonicity

$$s_p^*(t) \geq s_{p'}^*(t), \quad \forall t \in [0, T],$$

and are strictly increasing on the time interval $[(p - 1)\delta, T]$.

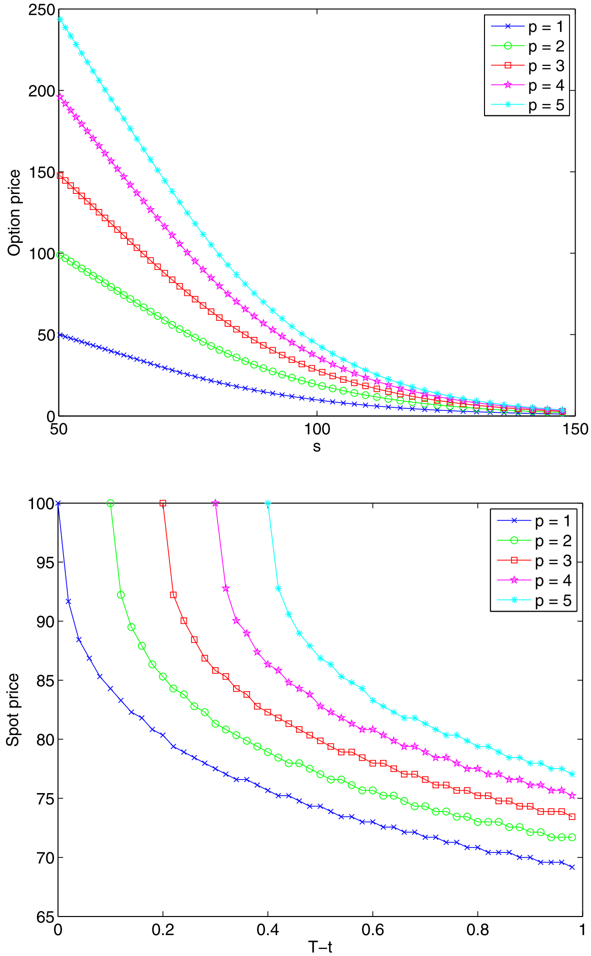


Fig. 6.4 Swing option prices (*top*) and corresponding exercise boundary (*bottom*)

6.5 Further Reading

Pricing of barrier options goes back to Merton [124]. In the recent paper from Zvan [167], an overview over various computational methods is given. Similar partial differential equation formulations for the Asian options are given in Ingersoll [91], Rogers and Shi [142]. A unifying approach is presented by Vecer [157]. Compound options were studied by Geske [69] where also a closed form solution was given. Obtaining swing option prices using a finite difference method was considered in Dahlgren [49] and for finite elements in Wilhelm and Winter [160].

Chapter 7
Interest Rate Models

We consider options on interest rates and present commonly used short rate models to model the time-evolution of the interest rate. Many interest rate derivatives in fixed income markets can then be priced numerically using the computational techniques described in the previous chapter, i.e. they can be interpreted as compound options on bonds.

7.1 Pricing Equation

We model the short rate r_t as a continuous Markovian process satisfying

$$\mathrm{d}r_t = b(t, r_t)\,\mathrm{d}t + \sigma(t, r_t)\,\mathrm{d}W_t, \quad r_0 = r, \tag{7.1}$$

where b and σ satisfy the assumptions of Theorem 1.2.6 and W is a Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$. We are interested in the computation of prices at time t of a zero coupon bond $B(t, r)$. This instrument pays out one unit of currency at maturity T, i.e. $B(T, r) = 1$. Therefore, its price satisfies the following relation

$$B(t, r) = \mathbb{E}\big[e^{-\int_t^T r_s\,\mathrm{d}s} \mid r_t = r\big],$$

where the expectation is taken under the pricing measure $\mathbb{Q}$. We refer to [109, Chap. 6] for details on the choice of measure. Similar as in Chap. 4, $B(t, r)$ can be shown to satisfy a deterministic partial differential equation. Here we consider a general payoff $g(r)$. For a zero coupon bond, $g(r) = 1$.

Theorem 7.1.1 *Let $V \in C^{1,2}(J \times (0, \infty)) \cap C^0(\overline{J} \times [0, \infty)) \cap C^1([0, T) \times [0, \infty))$ with bounded derivatives in r be a solution of*

$$\partial_t V + \mathcal{A}V - rV = 0 \quad \text{in } J \times \mathbb{R}_+, \quad V(T, r) = g(r) \quad \text{in } \mathbb{R}_+, \tag{7.2}$$

with $\mathcal{A}$ given as $\mathcal{A} = b(t, r)\partial_r + \frac{1}{2}\sigma(t, r)^2\partial_{rr}$, where $b(t, r)$, $\sigma(t, r)$ and g are sufficiently smooth and $\sigma(t, r) = 0$ holds if and only if $r = 0$. Then, $V(t, x)$ can also be represented as

$$V(t, r) = \mathbb{E}\big[e^{-\int_t^T r(s)\,\mathrm{d}s} g(r_T) \mid r_t = r\big].$$

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The converse of Theorem 7.1.1 is also true. The proof is given in [63]. Note that under certain assumptions on the coefficients, i.e. if $r = 0$ can be reached with a positive probability, boundary conditions for $V(t, 0)$ have to be imposed in Theorem 7.1.1. We refer to [63] for details on this topic. There are many examples of short rates models. We only state two: the Vasicek model [156] which was one of the first short rate models and the Cox–Ingersoll–Ross (CIR) model.

Example 7.1.2 For the Vasicek model, let α, β, r and σ be positive constants. We consider the following SDE

$$\mathrm{d}r_t = (\alpha - \beta r_t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t, \quad r_0 = r.$$

The corresponding generator is $\mathcal{A} = (\alpha - \beta r)\partial_r + \frac{1}{2}\sigma^2\partial_{rr}$. Note that negative interest rates occur with a positive probability at any time t. This model was applied to modeling in commodity markets by Lucia and Schwartz [116]. For the Cox–Ingersoll–Ross (CIR) model, the sport rate is given by

$$\mathrm{d}r_t = (\alpha - \beta r_t)\,\mathrm{d}t + \sigma\sqrt{r_t}\,\mathrm{d}W_t, \quad r_0 = r, \tag{7.3}$$

which yields the generator $\mathcal{A} = (\alpha - \beta r)\partial_r + \frac{1}{2}\sigma^2 r\partial_{rr}$.

Note that the diffusion coefficient of the CIR model, $\sigma\sqrt{r_t}$ is non-Lipschitz. Existence and uniqueness of a strong solution is nevertheless ensured by Yamada's Theorem [89]. We have

Theorem 7.1.3 *For any standard Brownian motion W on $[0, \infty)$ and $r \geq 0$, there exists a unique, continuous, adapted stochastic process $r_t \subset \mathbb{R}_+$ such that*

$$\mathrm{d}r_t = (\alpha - \beta r_t)\,\mathrm{d}t + \sigma\sqrt{r_t}\,\mathrm{d}W_t \quad \textit{in } [0, \infty), \; r_0 = r. \tag{7.4}$$

For a proof, see [89], p. 221. We give some properties of r_t. Let r_t^r denote the (unique, strong) solution of (7.4), and define

$$\tau_0^r = \inf\{t \geq 0 \mid r_t^r = 0\}, \tag{7.5}$$

with $\inf \emptyset := \infty$. Then it can be shown that

1. If $\alpha \geq \sigma^2/2$ in (7.4), then $\mathbb{P}(\tau_0^r = \infty) = 1$, $\forall r > 0$.
2. If $0 \leq \alpha < \sigma^2/2 \wedge \beta \geq 0$, $\mathbb{P}(\tau_0^r < \infty) = 1$, $\forall r > 0$.
3. If $0 \leq \alpha < \sigma^2/2 \wedge \beta < 0$, $\mathbb{P}(\tau_0^r < \infty) \in (0, 1)$, $\forall r > 0$.

The weak formulation on a bounded domain $G = [0, R]$, $R > 0$, and written in time-to-maturity, for the bond price in the CIR model reads:

$$\begin{aligned}
&\text{Find } v \in L^2(J; W_{1/2,\mu}) \cap H^1(J; \mathcal{H}_\mu) \text{ such that}\\
&(\partial_t v, w)_\mu + a^{\mathrm{CIR}}_{1/2,\mu}(v, w) = 0, \;\; \forall w \in W_{1/2,\mu}, \;\; \text{a.e. in } J,\\
&v(0) = 1,
\end{aligned} \tag{7.6}$$

where we denote by $(\cdot, \cdot)_\mu$ the inner product in $\mathcal{H}_\mu(G)$ and

$$a^{\mathrm{CIR}}(\phi,\varphi) := \frac{1}{2}\sigma^2 \int_0^R r^{1+2\mu}\partial_r\phi\partial_r\varphi\,\mathrm{d}r + \sigma^2\Big(\frac{1}{2}+\mu\Big)\int_0^R r^{2\mu}\partial_r\phi\varphi\,\mathrm{d}r$$
$$-\int_0^R (\alpha-\beta r)r^{2\mu}\partial_r\phi\varphi\,\mathrm{d}r + \int_0^R r^{2\mu+1}\phi\varphi\,\mathrm{d}r.$$

The spaces $W_{1/2,\mu}(G)$ and $\mathcal{H}_\mu(G)$ are defined in (4.29) and (4.33) with $\rho = 1/2$ and $\mu \in (-1/2, 0)$. Applying Theorem 3.2.2, one can show the well-posedness of the pricing problem.

Remark 7.1.4 The localization of the pricing problem (7.2) to a bounded domain cannot be justified as in Theorem 4.3.1 as [143, Theorem 25.18] is not applicable. Instead, we can use a different approach as in Theorem 9.4.1 to rigorously justify the use of homogeneous Dirichlet boundary conditions for large r. Alternatively, we could use the approach described in [136] using local times and leading to homogeneous Neumann conditions for large r.

A quantity that is often considered next to the bond price is the *yield*, which is for a given maturity T given by $Y(t,r) := -\frac{1}{T-t}\log B(t,r)$. It is the constant rate of continuously compounding interest which an investment has to be made starting from $B(t,r)$ units of currency at time t to obtain one unit of currency at maturity. The corresponding graph for different maturities of $T > t$ is called a yield curve and is often considered in the financial context. Typical shapes of yield curves are normal, i.e. monotonously increasing in T, inverse, i.e. monotonously decreasing in T, and humped, i.e. having exactly one local maximum and no local minimum in T.

Example 7.1.5 We consider the CIR model with parameters $\alpha = 0.03$, $\beta = 0.5$ and $\sigma = 0.5$. The bond prices are computed solving PDE (7.6) and the yield obtained postprocessing the bond prices. We plot in Fig. 7.1 the yield curve for different initial values of r_0. It can be seen that the three types of yield curved mentioned above (normal, inverse and humped) can be reproduced. We refer to [102] for further details.

Remark 7.1.6 For other types of interest rate models, such as the Vasicek model, the well-posedness results can be obtained as in Sect. 4.5.2.

7.2 Interest Rate Derivatives

Derivatives in interest rate markets are usually not written on the bonds directly, but on *simply compounded interest rates* which are defined as follows. To emphasize the dependency on the maturity, we now denote the bond price with $B(t,T,r)$.

Definition 7.2.1 The simply compounded interest rate prevailing at time t for the maturity T is denoted by $L(t,T,r)$ and is the constant rate at which an investment

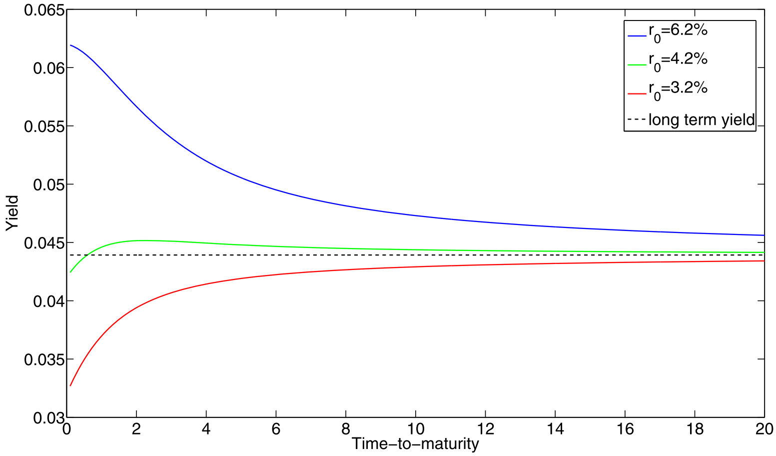

Fig. 7.1 Yield curves in the CIR model with parameters $\alpha = 0.03$, $\beta = 0.5$, $\sigma = 0.5$ and different initial conditions r_0

has to be made to produce an amount of one unit of currency at maturity, starting from $B(t, T, r)$ units of currency at time t, when accruing occurs proportionally to the investment time, i.e.

$$L(t, T, r) := \frac{1 - B(t, T, r)}{(T - t)B(t, T, r)}.$$

We now turn to the definition of *forward rates*. Forward rates are characterized by three time instants, namely the time t at which the rate is considered, its expiry T and its maturity T_1, with $t \le T \le T_1$. Forward rates are interest rates that can be locked in today for an investment in a future time period, and are set consistently with the current structure of zero coupon bonds. We can define a forward rate through a forward rate agreement. This contract gives its holder an interest rate payment for the period between T and T_1. At the maturity T_1 a fixed payment K is exchanged for a variable payment $L(T, T_1, r)$. The value of the contract in T_1 is given as $(T_1 - T)(K - L(T, T_1, r_T))$. Recalling the definition of $L(T, T_1, r_T)$ the value $V(t, r)$ of the contract at t is given as

$$V(t, r) = B(t, T_1, r)(T_1 - T)K - B(t, T, r) + B(t, T_1, r).$$

The value of K that renders the contract fair at time t, i.e. $V(t, r) = 0$, is called *simply compounded forward interest rate*.

Definition 7.2.2 The simply compounded forward interest rate at time t for the expiry $T > t$ with maturity $T_1 > T$ at short rate r is denoted by $F(t, T, T_1)$ and defined by

$$F(t, T, T_1, r) := \frac{1}{T_1 - T}\left(\frac{B(t, T, r)}{B(t, T_1, r)} - 1\right).$$

Table 7.1 Description of the swaption pricing algorithm

Set $v_0 = 1$.
For $j = 2, \ldots, n$,
Solve (7.6) to compute bond price $B(T_1, T_j, r) = v(T_j)$.
Next j
Compute swap value $V^{\text{PFS}}(T_1, r)$ by (7.7)
Set $v_0 = V^{\text{PFS}}(T_1, r)$.
Solve (7.6) to compute swaption price

The counterpart of a future contract on stock markets are swap agreements on interest rates. A *payer interest rate swap* is a contract exchanging fixed and variable payments at certain time instances in the future, called the *tenor structure*. Let $t < T_1 < \cdots < T_n < T$, then the value $V^{\text{PFS}}(t, r)$ is given as

$$V^{\text{PFS}}(t, r) = \mathbb{E}\left[\sum_{i=2}^{n} e^{-\int_t^{T_i} r(s)\,\mathrm{d}s}(T_i - T_{i-1})(L(T_{i-1}, T_i, r_{T_{i-1}}) - K) \mid r_t = r\right],$$

whereas the value of a receiver interest rate swap is given as $V^{\text{RIS}}(t, r) = -V^{\text{PFS}}(t, r)$. Using the definition of a simply compounded forward interest rate (Definition 7.2.2), we can view a payer interest rate swap as a portfolio of forward rate agreements and obtain the following representation for the value $V^{\text{PFS}}(t, r)$

$$\begin{aligned} V^{\text{PFS}}(t, r) &= \mathbb{E}\left[\sum_{i=2}^{n} e^{-\int_t^{T_i} r(s)\,\mathrm{d}s}(T_i - T_{i-1})(F(t, T_{i-1}, T_i, r_t) - K) \mid r_t = r\right] \\ &= B(t, T_1, r) - B(t, T_n, r) - \sum_{i=2}^{n}(T_i - T_{i-1})K B(t, T_i, r). \end{aligned} \tag{7.7}$$

Another example of derivatives on interest rates are swaptions. A *European payer swaption* gives its holder the right but not the obligation to enter a payer interest rate swap at a future time, the swaption maturity. The value $V(t, r)$ of the payer swaption, for which the maturity coincides with the first reset date T_1 is given as

$$\begin{aligned} V(t, r) = \mathbb{E}\Bigg[& e^{-\int_t^{T_1} r(s)\,\mathrm{d}s} \\ & \times \left(\sum_{i=2}^{n} e^{-\int_{T_1}^{T_i} r(s)\mathrm{d}s}(T_i - T_1)(F(T_1, T_{i-1}, T_i, r_{T_1}) - K)\right)_+ \mid r_t = r\Bigg]. \end{aligned}$$

In contrast to a swap, it cannot be decomposed into a portfolio of simpler products. However, we may proceed as in the preceding chapter: first, we compute the value of the swap contract underlying the swaption which can be priced using the PDE (7.2). Second, the swaption can then be priced as a compound option, where the underlying is the zero coupon bond. This is described schematically in Table 7.1.

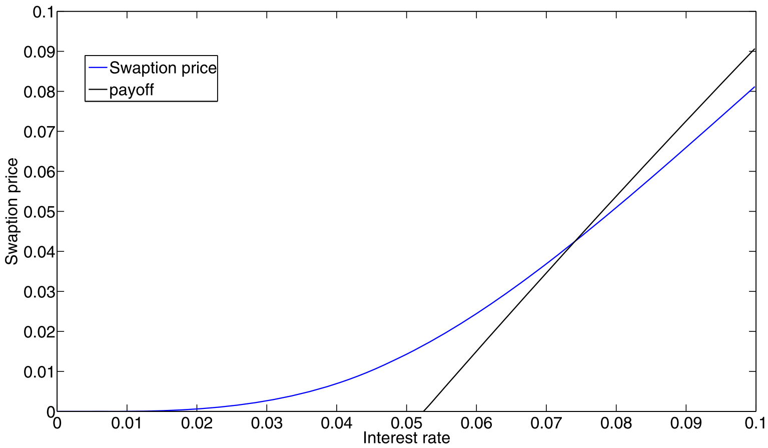

Fig. 7.2 Swaption price in the CIR model

Example 7.2.3 Consider the CIR model with parameters $\alpha = 0.05$, $\beta = 0.2$, $\sigma = 0.3$, $K = 0.05$ and tenor structure $T_1 = 0.25$, $T_2 = 0.5$, $T_3 = 0.75$, $T_4 = 1$, $T_5 = 1.5, \ldots, T_{10} = 4$. The swaption price is shown in Fig. 7.2.

Remark 7.2.4 A widely used extension of the models described above is the consideration of a deterministic shift function, i.e. instead of considering the solution r_t of (7.1), the short rate model $r_t + \varphi_t$ is considered. This extension can be fitted to any observed term structure. The pricing of derivatives in the extended model is analogous to the described method applying a change of variable.

7.3 Further Reading

For an introduction to interest rate models, we refer to Brigo and Mercurio [29]. Libor market models in the Lévy setting were considered by Eberlein and Özkan [60]. For theoretical results, we refer to Keller-Ressel et al. [101]. The equations arising in this context can be solved as described in Chap. 10. Recently, forward rate models have received significant attention, we refer to Carmona and Tehranchi [31] and Hepperger [78] for details. Keller-Ressel and Steiner [102] consider attainable yield curve shapes in an affine setting.

Chapter 8
Multi-asset Options

In Chap. 6, we considered exotic options written on a single underlying. Further examples of exotic options are given by the so-called *multi-asset options*. These are options derived from $d \geq 2$ underlying risky assets, whose price movement can be described by a system of SDEs. The pricing functions of multi-asset options are multivariate functions satisfying a parabolic partial differential equation in d dimensions, together with an appropriate terminal value depending on the type of the option. We distinguish between different types of European multi-asset options. A *basket option* is an option whose payoff is linked to a portfolio or basket of underlier values. The basket can be any weighted sum of underlier values as long as the weights are all positive. A typical example of a basket option is the arithmetic mean put option with payoff $g(s) = \max\{0, \sum_{i=1}^{d} \alpha_i s_i - K\}$. Examples of *rainbow option* are better-of-options, where $g(s) = \max_{1 \leq i \leq d}\{\alpha_i s_i\}$, or maximum call options, with $g(s) = \max_{1 \leq i \leq d}\{\max\{0, s_i - K_i\}\}$. A *quanto option* (also called *cross-currency option*) is a vanilla option on a foreign underlying, but with a payout in domestic currency. The payout of the option is converted to the domestic currency at expiration, at a predefined exchange rate s_2. The payoff function of a call is $g(s_1, s_2) = s_2 \max\{0, s_1 - K\}$.

8.1 Pricing Equation

As in the case of a single underlying, the price of a multi-asset option on d assets is given as the conditional expectation

$$V(t, x) = \mathbb{E}\big[e^{-\int_t^T r(X_s)\,\mathrm{d}s} g(X_T) \mid X_t = x\big],$$

where $X_t = (X_t^1, \ldots, X_t^d)^\top$ is an $\mathbb{R}^d$-valued stochastic process modeling the dynamics of the d assets, $r \in C^0(\mathbb{R}^d; \mathbb{R}_{\geq 0})$ is the deterministic interest rate, $g : \mathbb{R}^d \to \mathbb{R}_{\geq 0}$ denotes the payoff of the option and $\mathbb{R}_{\geq 0}$ the non-negative real numbers.

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We describe the process X in more detail. Let W be an $\mathbb{R}^n$-valued Brownian motion. We assume that the ith component of the process X evolves according to

$$\mathrm{d}X_t^i = b_i(X_t)\,\mathrm{d}t + \sum_{j=1}^{n} \Sigma_{ij}(X_t)\,\mathrm{d}W_t^j, \quad X_0^i = Z^i, \quad i = 1, \dots, d. \tag{8.1}$$

Herewith, we assume the coefficients $b : \mathbb{R}^d \to \mathbb{R}^d$, $\Sigma : \mathbb{R}^d \to \mathbb{R}^{d\times n}$ satisfy the usual Lipschitz continuity and linear growth condition, i.e. there exists a constant $C > 0$ such that for all $x, y \in \mathbb{R}^d$

$$|b(x) - b(y)| + |\Sigma(x) - \Sigma(y)| \leq C|x - y|, \tag{8.2}$$

$$|b(x)| + |\Sigma(x)| \leq C(1 + |x|), \tag{8.3}$$

where by $|\cdot| : \mathbb{R}^{m\times n} \to \mathbb{R}_{\geq 0}$, $A \mapsto |A| := |A|_F = \sqrt{\sum_{1\leq i\leq m,\, 1\leq j\leq n} |A_{ij}|^2}$ we denote the Frobenius norm (Euclidean norm) of an $m \times n$-matrix A. We further assume the existence of a random variable $Z = (Z^1, \dots, Z^d)^\top \in \mathbb{R}^d$ which is independent of the σ-algebra generated by W and satisfies $\mathbb{E}[|Z|^2] < \infty$. Under these assumptions, the existence and uniqueness result of the scalar case (Theorem 1.2.6) carries over to the SDE (8.1). To prove a multidimensional analog to Theorem 4.1.4, we need a generalization of Propositions 4.1.1, 6.2.1 to arbitrary dimensions. To this end, denote by $\mathcal{Q} := \Sigma\Sigma^\top$ the covariance matrix of the process X.

Remark 8.1.1 We note that $\int_0^t \mathcal{Q}_s\,\mathrm{d}s = \langle X, X\rangle_t$, where $\langle X, X\rangle_t$ denotes the quadratic variation process.

Proposition 8.1.2 *Denote by $\mathcal{A}$ the infinitesimal generator of X which is, for functions $f \in C^2(\mathbb{R}^d)$ with bounded derivatives, given by*

$$(\mathcal{A}f)(x) = \frac{1}{2}\,\mathrm{tr}[\mathcal{Q}(x)D^2 f(x)] + b(x)^\top \nabla f(x), \tag{8.4}$$

using the notation as in (2.2). *Then, the process $M_t := f(X_t) - \int_0^t (\mathcal{A}f)(X_s)\,\mathrm{d}s$ is a martingale with respect to the filtration of W.*

Proof By the multidimensional Itô formula, we have with $\mathrm{d}\langle X^i, X^j\rangle_t = \mathcal{Q}_{ij}(X_t)\,\mathrm{d}t$

$$\begin{aligned}
\mathrm{d}f(X_t) &= \sum_{i=1}^{d} \partial_{x_i} f(X_t)\,\mathrm{d}X_t^i + \frac{1}{2}\sum_{i,j=1}^{d} \partial_{x_i x_j} f(X_t)\,\mathrm{d}\langle X^i, X^j\rangle_t \\
&= b(X_t)^\top \nabla f(X_t)\,\mathrm{d}t + \sum_{i=1}^{d} \partial_{x_i} f(X_t) \sum_{j=1}^{n} \Sigma_{ij}(X_t)\,\mathrm{d}W_t^j \\
&\quad + \frac{1}{2}\sum_{i,j=1}^{d} (D^2 f)_{ij}(X_t)\mathcal{Q}_{ij}(X_t)\,\mathrm{d}t \\
&= \Big(b(X_t)^\top \nabla f(X_t) + \frac{1}{2}\,\mathrm{tr}[\mathcal{Q}(X_t)D^2 f(X_t)]\Big)\,\mathrm{d}t \\
&\quad + \sum_{i=1}^{d} \partial_{x_i} f(X_t) \sum_{j=1}^{n} \Sigma_{ij}(X_t)\,\mathrm{d}W_t^j.
\end{aligned}$$

Using the boundedness of the derivatives of f, the linear growth of the coefficient Σ (8.3) and proceeding as in the proof of Proposition 4.1.1, we find that the process $\int_0^t \sum_{i=1}^d \sum_{j=1}^n \partial_{x_i} f(X_\tau)\Sigma_{i,j}(X_\tau)\,\mathrm{d}W_\tau^j$, is a martingale with respect to the filtration of W. □

Repeating the arguments which lead to Theorem 4.1.4 yields

Theorem 8.1.3 *Let $V \in C^{1,2}(J \times \mathbb{R}^d) \cap C^0(\overline{J} \times \mathbb{R}^d)$ with bounded derivatives in x be a solution of*

$$\partial_t V + \mathcal{A}V - rV = 0 \quad \text{in } J \times \mathbb{R}^d, \quad V(T,x) = g(x) \quad \text{in } \mathbb{R}^d, \tag{8.5}$$

with $\mathcal{A}$ as in (8.4). Then, $V(t,x)$ can also be represented as

$$V(t,x) = \mathbb{E}\big[e^{-\int_t^T r(X_s)\,\mathrm{d}s} g(X_T) \mid X_t = x\big].$$

We now consider the pricing of a multi-asset option in a Black–Scholes market model, i.e. we assume in (8.1) that $b_i(s) = rs_i$, $\Sigma_{ij}(s) = \boldsymbol{\Sigma}_{ij}s_i$, $1 \le i, j \le d$, with $r \ge 0$ and $\boldsymbol{\Sigma}_{ij} \ge 0$ constants. Furthermore, we denote by $\mu \in \mathbb{R}^d$ the column vector given by $\mu := (\mathcal{Q}_{11}/2 - r, \dots, \mathcal{Q}_{dd}/2 - r)^\top$. Switching to log-price $x_i = \log(s_i)$, $i = 1 \dots, d$, and to time-to-maturity $t \to T - t$, we obtain that $v(t, x_1, \dots, x_d) := V(T-t, e^{x_1}, \dots, e^{x_d})$ solves

$$\partial_t v - \mathcal{A}^{\mathrm{BS}} v + rv = 0 \quad \text{in } J \times \mathbb{R}^d, \quad v(0,x) = g(e^x) \quad \text{in } \mathbb{R}^d. \tag{8.6}$$

Herewith, by a slight abuse of notation, we let $g(e^x) := g(e^{x_1}, \dots, e^{x_d})$, and $\mathcal{A}^{\mathrm{BS}}$ denotes the differential operator with constant coefficients

$$(\mathcal{A}^{\mathrm{BS}} f)(x) := \frac{1}{2}\operatorname{tr}[\mathcal{Q}D^2 f(x)] - \mu^\top \nabla f(x).$$

8.2 Variational Formulation

The variational formulation of (8.6) will require Sobolev spaces for functions of several variables. For $m \in \mathbb{N}$ and a bounded Lipschitz and simply connected domain $G \subseteq \mathbb{R}^d$, it is sufficient to introduce $H^m(G)$ as

$$H^m(G) := \big\{u \in L^2(G) : D^{\mathbf{n}} u \in L^2(G) \text{ for } |\mathbf{n}| \le m\big\}. \tag{8.7}$$

Here, $D^{\mathbf{n}}u$ has to be understood in the weak sense, i.e. $D^{\mathbf{n}}u$ is the weak derivative of u satisfying $\int_G D^{\mathbf{n}}u\varphi\,\mathrm{d}x = (-1)^{|\mathbf{n}|}\int_G uD^{\mathbf{n}}\varphi\,\mathrm{d}x$, $\forall \varphi \in C_0^n(G)$. The space $H^m(G)$ is equipped with the norm

$$\|u\|^2_{H^m(G)} := \sum_{|\mathbf{n}| \le m} \|D^{\mathbf{n}}u\|^2_{L^2(G)},$$

in particular, $\|u\|^2_{H^1(G)} = \int_G |u|^2\,\mathrm{d}x + \int_G |\nabla u|^2\,\mathrm{d}x$. For $G \subset \mathbb{R}^d$ as above, we introduce spaces $H_0^m(G)$ consisting of trace-zero functions,

$$H_0^m(G) := \overline{C_0^\infty(G)}^{\|\cdot\|_{H^m(G)}}.$$

We are now able to give the weak formulation of Eq. (8.6). It reads

$$\begin{aligned}
&\text{Find } u \in L^2(J; H^1(\mathbb{R}^d)) \cap H^1(J; L^2(\mathbb{R}^d)) \text{ such that} \\
&(\partial_t u, v) + a^{\mathrm{BS}}(u, v) = 0, \ \forall v \in H^1(\mathbb{R}^d), \ \text{a.e. in } J, \\
&u(0) = u_0,
\end{aligned} \tag{8.8}$$

where $u_0(x) := g(e^x)$ and the bilinear form $a^{\mathrm{BS}}(\cdot,\cdot) : H^1(\mathbb{R}^d) \times H^1(\mathbb{R}^d) \to \mathbb{R}$ is given by

$$a^{\mathrm{BS}}(\varphi, \phi) = \frac{1}{2}\int_{\mathbb{R}^d} (\nabla\varphi)^\top \mathcal{Q}\nabla\phi \,\mathrm{d}x + \int_{\mathbb{R}^d} \mu^\top \nabla\varphi\phi \,\mathrm{d}x + r\int_{\mathbb{R}^d} \varphi\phi \,\mathrm{d}x. \tag{8.9}$$

Proposition 8.2.1 *Assume the symmetric matrix $\mathcal{Q}$ is positive definite, i.e. there exists a constant $\gamma > 0$ such that $x^\top \mathcal{Q} x \geq \gamma x^\top x$, $\forall x \in \mathbb{R}^d$. Then there exist constants $C_i > 0$, $i = 1, 2, 3$, such that for all $\varphi, \phi \in H^1(\mathbb{R}^d)$ the following holds*:

$$\begin{aligned}
|a^{\mathrm{BS}}(\varphi, \phi)| &\leq C_1 \|\varphi\|_{H^1(\mathbb{R}^d)} \|\phi\|_{H^1(\mathbb{R}^d)}, \\
a^{\mathrm{BS}}(\varphi, \varphi) &\geq C_2 \|\varphi\|^2_{H^1(\mathbb{R}^d)} - C_3 \|\varphi\|^2_{L^2(\mathbb{R}^d)}.
\end{aligned}$$

Proof By Hölder's inequality,

$$\begin{aligned}
|a^{\mathrm{BS}}(\varphi, \phi)| \leq{}& \frac{1}{2}\sum_{i,j=1}^d |\mathcal{Q}_{ij}| \int_{\mathbb{R}^d} |\nabla\varphi||\nabla\phi| \,\mathrm{d}x + \sum_{i=1}^d |\mu_i| \int_{\mathbb{R}^d} |\nabla\varphi||\phi| \,\mathrm{d}x \\
&+ r\int_{\mathbb{R}^d} |\varphi||\phi| \,\mathrm{d}x \leq C_1 \|\varphi\|_{H^1(\mathbb{R}^d)} \|\phi\|_{H^1(\mathbb{R}^d)}.
\end{aligned}$$

Furthermore, by the positive definiteness of $\mathcal{Q}$ and $\int_{\mathbb{R}^d} \partial_{x_i}\varphi\varphi \,\mathrm{d}x = \frac{1}{2}\int_{\mathbb{R}^d} \partial_{x_i}(\varphi^2) \,\mathrm{d}x = 0$,

$$\begin{aligned}
a^{\mathrm{BS}}(\varphi, \varphi) &\geq \frac{1}{2}\gamma \int_{\mathbb{R}^d} |\nabla\varphi|^2 \,\mathrm{d}x + r\int_{\mathbb{R}^d} |\varphi|^2 \,\mathrm{d}x \\
&\geq \frac{1}{2}\gamma \|\varphi\|^2_{H^1(\mathbb{R}^d)} - |r - \gamma/2| \|\varphi\|^2_{L^2(\mathbb{R}^d)}.
\end{aligned}$$

□

We apply the abstract existence result Theorem 3.2.2 in the spaces $\mathcal{V} = H^1(\mathbb{R}^d)$, $\mathcal{H} = L^2(\mathbb{R}^d)$ and obtain, for every $u_0 \in L^2(\mathbb{R}^d)$, a unique solution to the problem (8.8). As in the one-dimensional case described in Sect. 4.3, we need to reformulate the problem on a bounded domain where the condition $u_0(x) = g(e^x) \in L^2(\mathbb{R}^d)$ can be weakened. We require similar to (4.10) the following polynomial growth condition on the multidimensional payoff function: There exist $C > 0$, $q \geq 1$ such that

$$g(s_1, \ldots, s_d) \leq C\Big(\sum_{i=1}^d s_i + 1\Big)^q \quad \text{for all } s \in \mathbb{R}^d_{\geq 0}. \tag{8.10}$$

This condition is satisfied by all standard multi-asset options like, e.g. basket, rainbow, spread or power options.

8.3 Localization

The unbounded domain $\mathbb{R}^d$ of the log-price is truncated to a bounded domain $G = (-R, R)^d \subset \mathbb{R}^d$, $R > 0$. This again corresponds to approximating the option price by a knock-out barrier option

$$v_R(t, x) = \mathbb{E}\Big[e^{-r(T-t)} g(e^{X_T}) 1_{\{T<\tau_G\}} \mid X_t = x\Big], \tag{8.11}$$

where $x \in \mathbb{R}^d$ and $X = (X^1, \ldots, X^d)^\top$ is now a d-dimensional Brownian motion. We show that the barrier option price v_R again converges to the option price exponentially fast in R.

Theorem 8.3.1 *Suppose the payoff function* $g : \mathbb{R}^d \to \mathbb{R}_{\geq 0}$ *satisfies* (8.10)*. Then, there exist* $C(d, T, \mathcal{Q})$, $\gamma_1, \gamma_2 > 0$, *such that*

$$|v(t, x) - v_R(t, x)| \leq C(d, T, \mathcal{Q}) e^{-\gamma_1 R + \gamma_2 \|x\|_\infty}.$$

Proof Let $M_T = \sup_{\tau \in [t,T]} \|X_\tau\|_\infty$. Then, with (8.10)

$$\begin{aligned} |v(t, x) - v_R(t, x)| &\leq \mathbb{E}\big[g(e^{X_T}) 1_{\{T \geq \tau_G\}} \mid X_t = x\big] \\ &\leq C(d)\, \mathbb{E}\big[e^{q M_T} 1_{\{M_T > R\}} \mid X_t = x\big]. \end{aligned}$$

It suffices to show that there exist a constant $C(d, T, \mathcal{Q}) > 0$ such that

$$\mathbb{E}\big[e^{q\|X_T\|_\infty} 1_{\{\|X_T\|_\infty > R\}} \mid X_t = x\big] \leq C(d, T, \mathcal{Q})\, e^{-\gamma_1 R + \gamma_2 \|x\|_\infty}.$$

We have, following the proof of Theorem 4.3.1,

$$\begin{aligned} \mathbb{E}\big[e^{q\|X_T\|_\infty} 1_{\{\|X_T\|_\infty > R\}} \mid X_t = x\big] &= \int_{\mathbb{R}^d} e^{q\|z+x\|_\infty} 1_{\{\|z+x\|_\infty > R\}} p_{T-t}(z)\, \mathrm{d}z \\ &\leq C(d, T, \mathcal{Q}) e^{q\|x\|_\infty} \sum_{i=1}^d \int_{\mathbb{R}^d} e^{(q + \|\mathcal{Q}^{-1}\mu\|_\infty)|z_i|} 1_{\{\|z+x\|_\infty > R\}} e^{-z^\top \mathcal{Q}^{-1} z/(2(T-t))}\, \mathrm{d}z \\ &\leq C(d, T, \mathcal{Q}) e^{q\|x\|_\infty} \\ &\quad \times \sum_{i=1}^d \int_{\mathbb{R}^d} e^{-(\eta - q - \|\mathcal{Q}^{-1}\mu\|_\infty)(R - \|x\|_\infty)} e^{\eta|z_i|} e^{-z^\top \mathcal{Q}^{-1} z/(2(T-t))}\, \mathrm{d}z \\ &\leq C(d, T, \mathcal{Q}) e^{-\gamma_1 R + \gamma_2 \|x\|_\infty} \sum_{i=1}^d \int_{\mathbb{R}} e^{\eta|z_i|} e^{-z_i^2/(2\sigma_i^2(T-t))}\, \mathrm{d}z_i, \end{aligned}$$

with $\gamma_1 = \eta - q - \|\mathcal{Q}^{-1}\mu\|_\infty$, and $\gamma_2 = \gamma_1 + q$. Since $\int_{\mathbb{R}} e^{\eta|z_i|} e^{-z_i^2/(2\sigma_i^2(T-t))}\, \mathrm{d}z_i < \infty$ for all $i = 1, \ldots, d$, and any $\eta > 0$, we obtain the required result by choosing $\eta > q + \|\mathcal{Q}^{-1}\mu\|_\infty$. □

The price v_R of the barrier option is, under the smoothness assumption $v_R \in C^{1,2}(J \times \mathbb{R}^d) \cup C^0(\overline{J} \times \mathbb{R}^d)$, and after switching to time-to-maturity $t \to T - t$, a solution of the PDE

$$\begin{aligned}
&\partial_t v_R - \mathcal{A}^{\mathrm{BS}} v_R + r v_R = 0 \quad \text{in } J \times G, \\
&v_R(t,x) = 0 \quad \text{in } J \times G^c, \\
&v_R(0,x) = g(e^x) \quad \text{on } G.
\end{aligned} \tag{8.12}$$

The weak formulation for the price v_R of the barrier option on the bounded domain $G = (-R, R)^d$ reads:

$$\begin{aligned}
&\text{Find } u_R \in L^2(J; H_0^1(G)) \cap H^1(J; L^2(G)) \text{ such that} \\
&(\partial_t u_R, v) + a^{\mathrm{BS}}(u_R, v) = 0, \ \forall v \in H_0^1(G), \ \text{a.e. in } J, \\
&u_R(0) = u_0|_G.
\end{aligned} \tag{8.13}$$

8.4 Discretization

We derive the matrix problems analogous to the univariate case (4.14) and (4.15). Due to the product structure of the domain $G = (-R, R)^d$ and since the coefficients of the operator $\mathcal{A}^{\mathrm{BS}}$ are constant, the matrices $\mathbf{G}^{\mathrm{BS}}$ and $\mathbf{A}^{\mathrm{BS}}$ are Kronecker products of matrices corresponding to univariate problems. To describe the ideas and to simplify the notation, we focus in the derivation on the two-dimensional case.

Definition 8.4.1 The *Kronecker product* of the matrices $\mathbf{X} \in \mathbb{R}^{m\times n}$ and $\mathbf{Y} \in \mathbb{R}^{p\times q}$ is given by

$$\mathbf{Z} := \mathbf{X} \otimes \mathbf{Y} := \begin{pmatrix} \mathbf{X}_{11}\mathbf{Y} & \mathbf{X}_{12}\mathbf{Y} & \cdots & \mathbf{X}_{1n}\mathbf{Y} \\ \mathbf{X}_{21}\mathbf{Y} & \mathbf{X}_{22}\mathbf{Y} & \cdots & \mathbf{X}_{2n}\mathbf{Y} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{X}_{m1}\mathbf{Y} & \mathbf{X}_{m2}\mathbf{Y} & \cdots & \mathbf{X}_{mn}\mathbf{Y} \end{pmatrix} \in \mathbb{R}^{mp\times nq}.$$

For later purpose, it is sufficient to consider matrices $\mathbf{X} \in \mathbb{R}^{N_1\times N_1}$, $\mathbf{Y} \in \mathbb{R}^{N_2\times N_2}$ which are quadratic. Then, also $\mathbf{Z} = \mathbf{X} \otimes \mathbf{Y}$ is quadratic, $\mathbf{Z} \in \mathbb{R}^{N\times N}$ with $N := N_1 N_2$. It follows by Definition 8.4.1 that an arbitrary entry $\mathbf{Z}_{j,j'}$, $1 \le j, j' \le N$ is then given by

$$\mathbf{X}_{i_1,i_1'} \mathbf{Y}_{i_2,i_2'} = \mathbf{Z}_{N_2(i_1-1)+i_2, N_2(i_1'-1)+i_2'} =: \mathbf{Z}_{j,j'} \tag{8.14}$$

where $\mathbf{X}_{i_1,i_1'}$, $1 \le i_1, i_1' \le N_1$, and $\mathbf{Y}_{i_2,i_2'}$, $1 \le i_2, i_2' \le N_2$, are the entries of $\mathbf{X}$ and $\mathbf{Y}$, respectively.

8.4.1 Finite Difference Discretization

We consider the truncated PDE (8.12) in two space variables, where the spacial operator $\mathcal{A}^{\mathrm{BS}}$ simplifies to

$$\mathcal{A}^{\mathrm{BS}} = \frac{1}{2}\mathcal{Q}_{11}\partial_{x_1x_1} + \mathcal{Q}_{12}\partial_{x_1x_2} + \frac{1}{2}\mathcal{Q}_{22}\partial_{x_2x_2} - \mu_1\partial_{x_1} - \mu_2\partial_{x_2}.$$

To discretize $\mathcal{A}^{\mathrm{BS}}$ by finite difference quotients, we define, for $N_1, N_2 \in \mathbb{N}$, a grid $\mathcal{G}$ on $G = (-R, R)^2$ by

$$\mathcal{G} := \{(x_{1,i_1}, x_{2,i_2}) \mid 1 \le i_1 \le N_1,\ 1 \le i_2 \le N_2\},$$

where the grid points (x_{1,i_1}, x_{2,i_2}) are given by

$$\begin{aligned}(x_{1,i_1}, x_{2,i_2}) &= (-R + i_1 h_1, -R + i_2 h_2),\\ h_k &:= 2R/(N_k + 1), \quad 1 \le i_k \le N_k, \quad k = 1, 2.\end{aligned}$$

For $f \in C^4(G)$, we set $f_{i_1,i_2} := f(x_{1,i_1}, x_{2,i_2})$ for the function value of f at the grid point $(x_{1,i_1}, x_{2,i_2}) \in \mathcal{G}$ and consider the difference quotients

$$\begin{aligned}\partial_{x_1x_1} f(x_{1,i_1}, x_{2,i_2}) &= h_1^{-2}(f_{i_1-1,i_2} - 2f_{i_1,i_2} + f_{i_1+1,i_2}) + \mathcal{O}(h_1^2)\\ &=: (\delta^2_{x_1x_1} f)_{i_1,i_2} + \mathcal{O}(h_1^2),\\ \partial_{x_2x_2} f(x_{1,i_1}, x_{2,i_2}) &= h_2^{-2}(f_{i_1,i_2-1} - 2f_{i_1,i_2} + f_{i_1,i_2+1}) + \mathcal{O}(h_2^2)\\ &=: (\delta^2_{x_2x_2} f)_{i_1,i_2} + \mathcal{O}(h_2^2).\end{aligned}$$

Furthermore,

$$\begin{aligned}\partial_{x_1x_2} f(x_{1,i_1}, x_{2,i_2}) &= \partial_{x_1}[(2h_2)^{-1}(f_{i_1,i_2+1} - f_{i_1,i_2-1}) + \mathcal{O}(h_2^2)]\\ &= (4h_1h_2)^{-1}\big(f_{i_1+1,i_2+1} - f_{i_1-1,i_2+1} - f_{i_1+1,i_2-1} + f_{i_1-1,i_2-1}\big)\\ &\quad + \mathcal{O}(h_1^2) + \mathcal{O}(h_2^2)\\ &=: (\delta^2_{x_1x_2} f)_{i_1,i_2} + \mathcal{O}(h_1^2) + \mathcal{O}(h_2^2),\end{aligned}$$

as well as

$$\begin{aligned}\partial_{x_1} f(x_{1,i_1}, x_{2,i_2}) &= (2h_1)^{-1}(f_{i_1+1,i_2} - f_{i_1-1,i_2}) + \mathcal{O}(h_1^2) =: (\delta_{x_1} f)_{i_1,i_2} + \mathcal{O}(h_1^2),\\ \partial_{x_2} f(x_{1,i_1}, x_{2,i_2}) &= (2h_2)^{-1}(f_{i_1,i_2+1} - f_{i_1,i_2-1}) + \mathcal{O}(h_2^2) =: (\delta_{x_2} f)_{i_1,i_2} + \mathcal{O}(h_2^2).\end{aligned}$$

Denote by $v^m_{i_1,i_2} \approx v(t_m, x_{1,i_1}, x_{2,i_2})$ an approximation of the option value v on time level t_m at the grid point $(x_{1,i_1}, x_{2,i_2}) \in \mathcal{G}$. We now replace the PDE, $\partial_t v - \mathcal{A}^{\mathrm{BS}} v + rv = 0$, by the finite difference equations

$$\mathcal{E}^m_{i_1,i_2} = 0, \quad 1 \le i_k \le N_k, \quad k = 1, 2, \quad m = 0, \ldots, M - 1, \tag{8.15}$$

with the initial condition $v^0_{i_1,i_2} = g(e^{x_1^{i_1}}, e^{x_2^{i_2}})$ and homogeneous boundary conditions $v^m_{0,i_2} = v^m_{i_1,0} = 0$, $i_k = 0, \ldots, N_k + 1$, $m = 0, \ldots, M$. In (8.15), $\mathcal{E}^m_{i_1,i_2}$ is the finite difference operator given by

$$\begin{aligned}\mathcal{E}^m_{i_1,i_2} := k^{-1}\big(v^{m+1}_{i_1,i_2} - v^m_{i_1,i_2}\big) - \big[\theta(\mathcal{F}v)^{m+1}_{i_1,i_2} + (1-\theta)(\mathcal{F}v)^m_{i_1,i_2}\big] + r\theta v^{m+1}_{i_1,i_2}\\ + r(1-\theta)v^m_{i_1,i_2},\end{aligned}$$

where

$$(\mathcal{F}v)^m_{i_1,i_2} := \frac{1}{2}\sum_{k=1}^{2} \mathcal{Q}_{kk}(\delta^2_{x_k x_k} v)^m_{i_1,i_2} + \mathcal{Q}_{12}(\delta^2_{x_1 x_2} v)^m_{i_1,i_2} - \sum_{k=1}^{2} \mu_k(\delta_{x_k} v)^m_{i_1,i_2}.$$

We write (8.15) in matrix form. Setting, for $N := N_1 N_2$,

$$\underline{u}^m = (u^m_1, \dots, u^m_N)^\top, \quad u^m_j = u^m_{N_2(i_1-1)+i_2} := v^m_{i_1,i_2},$$

we find that (8.15) is equivalent to

$$\begin{aligned}
&\text{Find } \underline{u}^{m+1} \in \mathbb{R}^N \text{ such that for } m = 0, \dots, M-1,\\
&\big(\mathbf{I} + \theta \Delta t \mathbf{G}^{\mathrm{BS}}\big)\underline{u}^{m+1} = \big(\mathbf{I} - (1-\theta)\Delta t\, \mathbf{G}^{\mathrm{BS}}\big)\underline{u}^m, \\
&\underline{u}^0 = \underline{u}_0,
\end{aligned} \tag{8.16}$$

where the matrices $\mathbf{I}$ and $\mathbf{G}^{\mathrm{BS}}$ are given by

$$\begin{aligned}
\mathbf{I} &:= \mathbf{I}^1 \otimes \mathbf{I}^2,\\
\mathbf{G}^{\mathrm{BS}} &:= \frac{1}{2}\mathcal{Q}_{11}\mathbf{R}^1 \otimes \mathbf{I}^2 - \mathcal{Q}_{12}\mathbf{C}^1 \otimes \mathbf{C}^2 + \frac{1}{2}\mathcal{Q}_{11}\mathbf{I}^1 \otimes \mathbf{R}^2\\
&\quad + \mu_1 \mathbf{C}^1 \otimes \mathbf{I}^2 + \mu_2 \mathbf{I}^1 \otimes \mathbf{C}^2 + r\mathbf{I}^1 \otimes \mathbf{I}^2.
\end{aligned}$$

Herewith, we denote by $\mathbf{R}^k, \mathbf{C}^k, \mathbf{I}^k \in \mathbb{R}^{N_k \times N_k}$, the matrices given in (4.14) with respect to the coordinate direction x_k, $k = 1, 2$.

8.4.2 *Finite Element Discretization*

In the two-dimensional case, the bilinear form $a^{\mathrm{BS}}(\cdot,\cdot)$ in (8.9) simplifies to

$$a^{\mathrm{BS}}(\varphi,\phi) = \frac{1}{2}\sum_{\ell,k=1}^{2} \mathcal{Q}_{\ell k}\int_G \partial_{x_\ell}\varphi \partial_{x_k}\phi \,\mathrm{d}x + \sum_{k=1}^{2}\mu_k \int_G \partial_{x_k}\varphi\phi\,\mathrm{d}x + r\int_G \varphi\phi\,\mathrm{d}x. \tag{8.17}$$

The discretization of the weak formulation (8.13) relies on finite element spaces $V_N \subset H^1_0(G)$ which are spanned by products of hat-functions. To be more precise, for $N_1, N_2 \in \mathbb{N}$ we consider

$$\begin{aligned}
V_N &:= \mathrm{span}\{\varphi_j(x_1,x_2) \mid 1 \le j \le N_1 N_2\}\\
&= \mathrm{span}\{\varphi_{N_2(i_1-1)+i_2}(x_1,x_2) \mid 1 \le i_k \le N_k,\ k = 1,2\}\\
&= \mathrm{span}\{b_{i_1}(x_1) b_{i_2}(x_2) \mid 1 \le i_k \le N_k,\ k = 1,2\},
\end{aligned} \tag{8.18}$$

where $b_{i_k}(x_k)$ are the univariate hat-functions as in (3.18). Note that $\dim V_N = N := N_1 N_2$. If the mesh sizes $h_k = 2R/(N_k+1)$ in both coordinate directions are constant, we have $b_{i_k}(x_k) = \max\{0, 1 - h_k^{-1}|x_k - x_{k,i_k}|\}$, $i_k = 1, \dots, N_k$, with mesh points $x_{k,i_k} := -R + h_k i_k$, $i_k = 0, \dots, N_k + 1$.

We now calculate the stiffness matrix $\mathbf{A}^{\mathrm{BS}} \in \mathbb{R}^{N\times N}$ using the finite element space V_N. We have, by (8.17),

$$\begin{aligned}
\mathbf{A}^{\mathrm{BS}}_{j,j'} &= a^{\mathrm{BS}}(\varphi_{j'},\varphi_j) = a^{\mathrm{BS}}(\varphi_{N_2(i_1'-1)+i_2'},\varphi_{N_2(i_1-1)+i_2}) = a^{\mathrm{BS}}(b_{i_1'}b_{i_2'}, b_{i_1}b_{i_2})\\
&= \frac{\mathcal{Q}_{11}}{2}\int_G \partial_{x_1}(b_{i_1'}b_{i_2'})\partial_{x_1}(b_{i_1}b_{i_2})\,\mathrm{d}x + \frac{\mathcal{Q}_{12}}{2}\int_G \partial_{x_1}(b_{i_1'}b_{i_2'})\partial_{x_2}(b_{i_1}b_{i_2})\,\mathrm{d}x\\
&\quad + \frac{\mathcal{Q}_{21}}{2}\int_G \partial_{x_2}(b_{i_1'}b_{i_2'})\partial_{x_1}(b_{i_1}b_{i_2})\,\mathrm{d}x + \frac{\mathcal{Q}_{22}}{2}\int_G \partial_{x_2}(b_{i_1'}b_{i_2'})\partial_{x_2}(b_{i_1}b_{i_2})\,\mathrm{d}x\\
&\quad + \mu_1\int_G \partial_{x_1}(b_{i_1'}b_{i_2'})b_{i_1}b_{i_2}\,\mathrm{d}x + \mu_2\int_G \partial_{x_2}(b_{i_1'}b_{i_2'})b_{i_1}b_{i_2}\,\mathrm{d}x\\
&\quad + r\int_G b_{i_1'}b_{i_2'}b_{i_1}b_{i_2}\,\mathrm{d}x\\
&= \frac{\mathcal{Q}_{11}}{2}\int_{-R}^{R} b'_{i_1'}b'_{i_1}\,\mathrm{d}x_1\int_{-R}^{R} b_{i_2'}b_{i_2}\,\mathrm{d}x_2 + \frac{\mathcal{Q}_{12}}{2}\int_{-R}^{R} b'_{i_1'}b_{i_1}\,\mathrm{d}x_1\int_{-R}^{R} b_{i_2'}b'_{i_2}\,\mathrm{d}x_2\\
&\quad + \frac{\mathcal{Q}_{21}}{2}\int_{-R}^{R} b_{i_1'}b'_{i_1}\,\mathrm{d}x_1\int_{-R}^{R} b'_{i_2'}b_{i_2}\,\mathrm{d}x_2 + \frac{\mathcal{Q}_{22}}{2}\int_{-R}^{R} b_{i_1'}b_{i_1}\,\mathrm{d}x_1\int_{-R}^{R} b'_{i_2'}b'_{i_2}\,\mathrm{d}x_2\\
&\quad + \mu_1\int_{-R}^{R} b'_{i_1'}b_{i_1}\,\mathrm{d}x_1\int_{-R}^{R} b_{i_2'}b_{i_2}\,\mathrm{d}x_2 + \mu_2\int_{-R}^{R} b_{i_1'}b_{i_1}\,\mathrm{d}x_1\int_{-R}^{R} b'_{i_2'}b_{i_2}\,\mathrm{d}x_2\\
&\quad + r\int_{-R}^{R} b_{i_1'}b_{i_1}\,\mathrm{d}x_1\int_{-R}^{R} b_{i_2'}b_{i_2}\,\mathrm{d}x_2.
\end{aligned}$$

Using that the matrices $\mathbf{S}$, $\mathbf{B}$ and $\mathbf{M}$ are given by $\mathbf{S}_{i,i'} = \int b'_{i'}b'_i$, $\mathbf{B}_{i,i'} = \int b'_{i'}b_i$ and $\mathbf{M}_{i,i'} = \int b_{i'}b_i$, see, e.g. (3.22), and noting that $\int b_{i'}b'_i = -\int b'_{i'}b_i = -\mathbf{B}_{i,i'}$, we obtain

$$\begin{aligned}
\mathbf{A}^{\mathrm{BS}}_{j,j'} &= \frac{\mathcal{Q}_{11}}{2}\mathbf{S}_{i_1,i_1'}\mathbf{M}_{i_2,i_2'} + \frac{\mathcal{Q}_{12}}{2}\mathbf{B}_{i_1,i_1'}(-\mathbf{B})_{i_2,i_2'} + \frac{\mathcal{Q}_{21}}{2}(-\mathbf{B})_{i_1,i_1'}\mathbf{B}_{i_2,i_2'}\\
&\quad + \frac{\mathcal{Q}_{22}}{2}\mathbf{M}_{i_1,i_1'}\mathbf{S}_{i_2,i_2'} + \mu_1\mathbf{B}_{i_1,i_1'}\mathbf{M}_{i_2,i_2'} + \mu_2\mathbf{M}_{i_1,i_1'}\mathbf{B}_{i_2,i_2'} + r\mathbf{M}_{i_1,i_1'}\mathbf{M}_{i_2,i_2'}.
\end{aligned}$$

Denote by $\mathbf{S}^k$ the matrix with entries $\mathbf{S}_{i_k,i_k'}$, $k = 1,2$, and analogously for $\mathbf{B}^k$, $\mathbf{M}^k$. Using that the covariance matrix $\mathcal{Q}$ is symmetric and Eq. (8.14), we have shown

Proposition 8.4.2 *Assume $d = 2$ in (8.9) and assume that the finite element space V_N is as in (8.18). Then, the stiffness matrix $\mathbf{A}^{\mathrm{BS}}$ to the bilinear form $a^{\mathrm{BS}}(\cdot,\cdot)$ is given by*

$$\begin{aligned}
\mathbf{A}^{\mathrm{BS}} &= \frac{\mathcal{Q}_{11}}{2}\mathbf{S}^1\otimes\mathbf{M}^2 - \mathcal{Q}_{12}\mathbf{B}^1\otimes\mathbf{B}^2 + \frac{\mathcal{Q}_{22}}{2}\mathbf{M}^1\otimes\mathbf{S}^2\\
&\quad + \mu_1\mathbf{B}^1\otimes\mathbf{M}^2 + \mu_2\mathbf{M}^1\otimes\mathbf{B}^2 + r\mathbf{M}^1\otimes\mathbf{M}^2.
\end{aligned}$$

Since the Kronecker product is distributive, we can reduce the number of products by

$$\mathbf{A}^{\mathrm{BS}} = \left(\frac{\mathcal{Q}_{11}}{2}\mathbf{S}^1 + \mu_1 \mathbf{B}^1 + r\mathbf{M}^1 \right) \otimes \mathbf{M}^2 + (-\mathcal{Q}_{12}\mathbf{B}^1 + \mu_2 \mathbf{M}^1) \otimes \mathbf{B}^2 + \frac{\mathcal{Q}_{22}}{2}\mathbf{M}^1 \otimes \mathbf{S}^2.$$

Proposition 8.4.2 can be extended to arbitrary dimensions $d > 2$. The corresponding finite element space V_N is spanned by the d-fold tensor products of univariate hat-functions, i.e.

$$\begin{aligned} V_N &:= \operatorname{span}\{\varphi_j(x_1, \dots, x_d) \mid 1 \le j \le N\} \\ &= \operatorname{span}\{b_{i_1}(x_1) \cdots b_{i_d}(x_d) \mid 1 \le i_k \le N_k,\ k = 1, \dots, d\}, \end{aligned} \tag{8.19}$$

with $\dim V_N = N := \prod_{k=1}^{d} N_k$. The index j in (8.19) is defined via the indices $i_1, \dots, i_d$ by $j := \sum_{k=1}^{d-1} \prod_{\ell=1}^{d-k} N_{\ell+k}(i_k - 1) + i_d$; compare with (8.18). The stiffness matrix $\mathbf{A}^{\mathrm{BS}} \in \mathbb{R}^{N \times N}$ becomes

$$\begin{aligned} \mathbf{A}^{\mathrm{BS}} = &\frac{1}{2}\sum_{i=1}^{d} \mathcal{Q}_{ii} \bigotimes_{k=1}^{i-1} \mathbf{M}^k \otimes \mathbf{S}^i \otimes \bigotimes_{k=i+1}^{d} \mathbf{M}^k \\ &- \sum_{i=1}^{d-1} \sum_{j=i+1}^{d} \mathcal{Q}_{ij} \bigotimes_{k=1}^{i-1} \mathbf{M}^k \otimes \mathbf{B}^i \otimes \bigotimes_{k=i+1}^{j-1} \mathbf{M}^k \otimes \mathbf{B}^j \otimes \bigotimes_{k=j+1}^{d} \mathbf{M}^k \\ &+ \sum_{k=1}^{d} \mu_k \bigotimes_{k=1}^{i-1} \mathbf{M}^k \otimes \mathbf{B}^i \otimes \bigotimes_{k=i+1}^{d} \mathbf{M}^k + r \bigotimes_{k=1}^{d} \mathbf{M}^k. \end{aligned} \tag{8.20}$$

If we furthermore discretize (8.13) by the θ-scheme in time, we obtain the matrix problem

$$\begin{aligned} &\text{Find } \underline{u}^{m+1} \in \mathbb{R}^N \text{ such that for } m = 0, \dots, M-1 \\ &(\mathbf{M} + \theta \Delta t \mathbf{A}^{\mathrm{BS}})\underline{u}^{m+1} = (\mathbf{M} - (1-\theta)\Delta t \mathbf{A}^{\mathrm{BS}})\underline{u}^m, \\ &\underline{u}_N^0 = \underline{u}_0. \end{aligned} \tag{8.21}$$

Note that if $N_1 = \cdots = N_d$, then the mesh width is the same in each coordinate direction, and the matrices in the representation of $\mathbf{A}^{\mathrm{BS}}$ satisfy $\mathbf{M} := \mathbf{M}^1 = \cdots = \mathbf{M}^d$, $\mathbf{B} := \mathbf{B}^1 = \cdots = \mathbf{B}^d$ and $\mathbf{S} := \mathbf{S}^1 = \cdots = \mathbf{S}^d$. Hence, in this case, we have to calculate only the matrices $\mathbf{M}$, $\mathbf{B}$ and $\mathbf{S}$, independent of the dimension d.

We give a convergence result that generalizes Theorem 3.6.5 to arbitrary dimensions. For simplicity, we assume that the mesh width is the same in each coordinate direction, $h := h_1 = \cdots = h_d$. Let $u := u_R$ be the unique solution of (8.13), and let $u_N(t_m, \cdot) \in V_N$ be its finite element approximation at time level t_m with corresponding coefficient vector $\underline{u}^m$ obtained from (8.21). As in the one-dimensional case, we split the error $e_N^m(x) := u(t_m, x) - u_N(t_m, x) =: u^m - u_N^m$ as

$$e_N^m = \left(u^m - \mathcal{P}_N u^m\right) + \left(\mathcal{P}_N u^m - u_N^m\right) =: \eta^m + \xi_N^m, \tag{8.22}$$

where $\mathcal{P}_N : \mathcal{V} \to V_N$ is a projector or a quasi-interpolant. The reason why we cannot rely on a multivariate version of the nodal interpolant $\mathcal{I}_N$ as in (3.24) is that functions $u \in \mathcal{V} = H^1(G)$ for $G \subset \mathbb{R}^d$ are not necessarily continuous. In fact, the compact Sobolev embedding $H^m(G) \subset C^0(\overline{G})$ only holds if $m > d/2$. We assume that the operator $\mathcal{P}_N$ satisfies the following approximation property (compare with (3.36))

$$\|u - \mathcal{P}_N u\|_{H^t(G)} \le C h^{s-t} \|u\|_{H^s(G)}, \quad t = 0, 1, \quad 0 \le t \le s \le 2. \tag{8.23}$$

The construction of $\mathcal{P}_N$ having the property (8.23) for a large class of finite element spaces including the tensor product space V_N in (8.19) can be found, e.g. in [27, Sect. 4.8] or [64, Sect. 1.6]. Thus, replacing in the proof of Theorem 3.6.5 the nodal interpolant $\mathcal{I}_N$ by $\mathcal{P}_N$ and using the approximation property (8.23), we find

Theorem 8.4.3 *Assume* $u_R \in C^1(\overline{J}; H^2(G)) \cap C^3(\overline{J}; H^{-1}(G))$. *Let* $u^m(x) := u_R(t_m, x)$ *and* $u_N^m := u_N(t_m, x) \in V_N$, *defined via* (8.21), *with* V_N *as in* (8.19), *where* $N_1 = \cdots = N_d$. *Assume for* $0 \le \theta < \frac{1}{2}$ *also* (3.30). *Then, the following error bound holds*:

$$\begin{aligned} &\|u^M - u_N^M\|^2_{L^2(G)} + k \sum_{m=0}^{M-1} \|u^{m+\theta} - u_N^{m+\theta}\|^2_{H^1(G)} \\ &\le C h^2 \max_{0 \le t \le T} \|u(t)\|_{H^2(G)} + C h^2 \int_0^T \|\partial_t u(s)\|^2_{H^1(G)} \, \mathrm{d}s \\ &\quad + C \begin{cases} k^2 \int_0^T \|\partial_{tt} u(s)\|^2_* \, \mathrm{d}s & \text{if } 0 \le \theta \le 1, \\ k^4 \int_0^T \|\partial_{ttt} u(s)\|^2_* \mathrm{d}s & \text{if } \theta = \frac{1}{2}. \end{cases} \end{aligned}$$

As in the univariate case, using $k = \mathcal{O}(h)$ and $\theta = 1/2$, one can show that $\|u^M - u_N^M\|^2_{L^2(G)} = \mathcal{O}(h^2)$. Since $n := N_1 = \cdots = N_d = \mathcal{O}(h^{-1})$, the dimension of the finite element space V_N is $N := \dim V_N = n^d$, and the rate of convergence with respect to the mesh width h can be expressed in terms of the number of degrees of freedom N

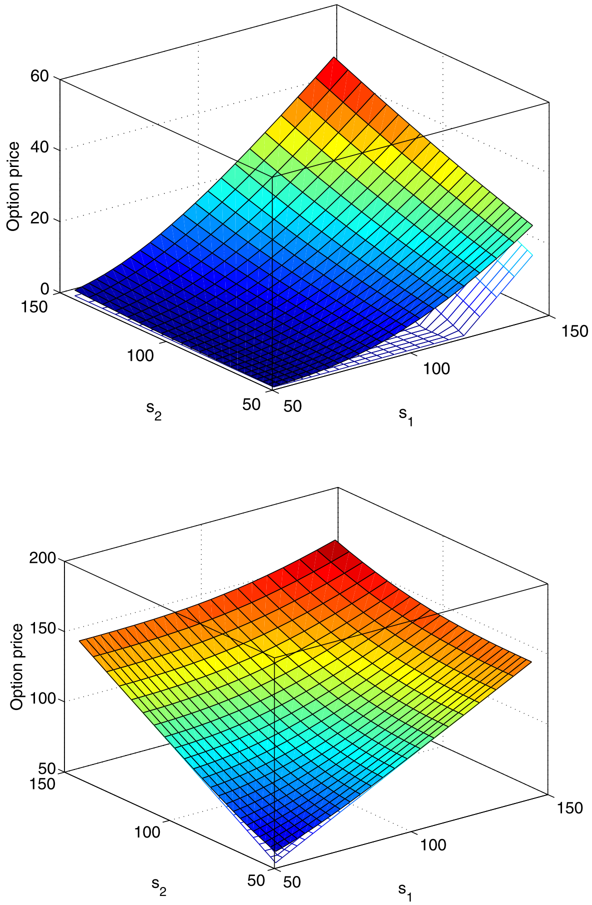
$$\|u^M - u_N^M\|^2_{L^2(G)} = \mathcal{O}(N^{-2/d}). \tag{8.24}$$

The exponential decay of the rate of convergence with respect the dimension d of the problem, or equivalently, the exponential growth of N with respect to d is called the *curse of dimension*. Therefore, full tensor product spaces V_N lead to inefficient schemes for large d.

Example 8.4.4 For $d = 2$ we consider two different options: A basket option with payoff $g(s) = (\alpha_1 s_1 + \alpha_2 s_2 - K)_+$, and a better-of-option with payoff $g(s) = \max\{s_1, s_2\}$. We set strike $K = 100$, and maturity $T = 1$, covariance matrix $\mathcal{Q} = (\sigma_i \sigma_j \rho_{ij})_{1 \le i,j \le 2}$ with volatilities $\sigma = (0.4, 0.1)$ and correlation $\rho_{12} = 0.2$. Furthermore, we let the interest rate $r = 0.01$ and $\alpha_1 = 0.7$, $\alpha_2 = 0.3$ in the payoff of the basket option. The option values are shown in Fig. 8.1 where we used finite elements for the discretization.

Using analytic formulas (see, e.g. [165]), we can compute the L^∞-convergence rate on a subset $G_0 = (K/2, 3/2K)^2 \subset G$ at maturity $t = T$. The convergence rates are shown in Fig. 8.2. It can be seen that we obtain the optimal convergence rate

Fig. 8.1 Basket (*top*) and better-of-option price (*bottom*)



$\mathcal{O}(h^2)$ with respect to the mesh width h for both options. But if we plot the convergence rate in terms of degrees of freedom N, we see the "curse of dimension" and only obtain $\mathcal{O}(N^{-1})$ instead of the optimal convergence rate $\mathcal{O}(N^{-2})$.

8.5 Further Reading

Analytic formulas for basket options on two underlying were derived by Zhang [165, Chap. 27]. To avoid the 'curse of dimension', several authors (see Leentvaar and Oosterlee [114] or Reisinger and Wittum [139] and the references therein) use finite differences on the so-called sparse grids. For the finite element discretization, it is possible to use sparse tensor product spaces for the discretization as in von Petersdorff and Schwab [159]. In both cases, we are able to reduce the complexity in the number of degrees of freedom from $\mathcal{O}(h^{-d})$ to $\mathcal{O}(h^{-1}|\log h|^{d-1})$. Therefore, even

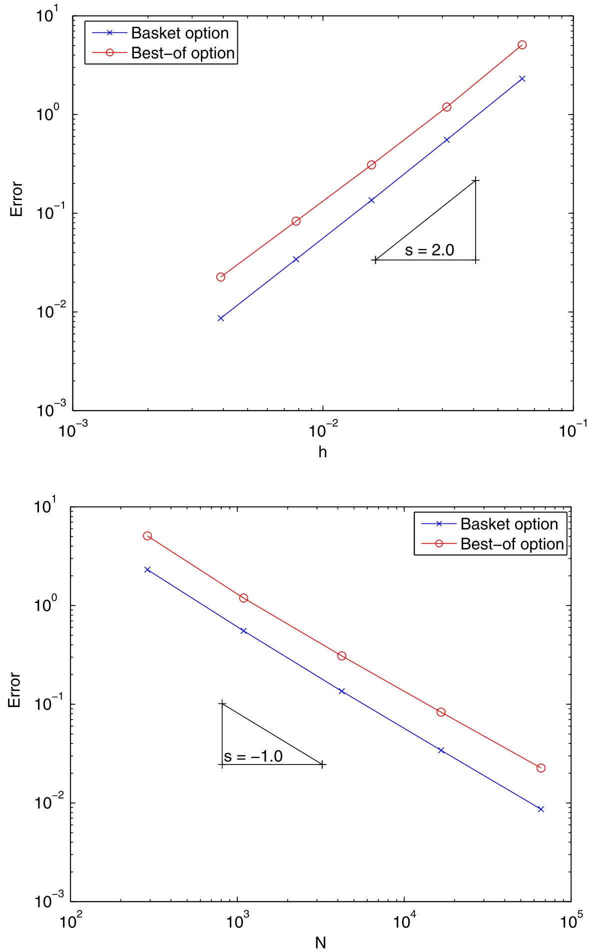


Fig. 8.2 Convergence rates for a two-dimensional Black–Scholes model with respect to the mesh width (*top*) and degrees of freedom (*bottom*)

for large dimensions d, multivariate options can be priced. We explain this in more detail in Chap. 13.

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Chapter 9
Stochastic Volatility Models

In Sect. 4.5, we considered local volatility models as an extension of the Black–Scholes model. These models replace the constant volatility by a deterministic volatility function, i.e. the volatility is a deterministic function of s and t. In *stochastic volatility* (SV) models, the volatility is modeled as a function of at least one additional stochastic process $Y^1, \dots, Y^{n_v}$, $n_v \geq 1$. Such models can explain some of the empirical properties of asset returns, such as volatility clustering and the leverage effect. These models can also account for long term smiles and skews.

We focus on SV models which are pure diffusion models, i.e. the processes $Y^1, \dots, Y^{n_v}$ used to model the volatility are diffusions. We will also assume that there is only one risky underlying S. The flexibility offered by SV models comes at the cost of an increase in dimension. In SV models, the price S is no longer Markovian, since the price process is determined not only by its value but also by the level of volatility. To regain a Markov process, one must consider the $(n_v + 1)$-dimensional process $(S, Y^1, \dots, Y^{n_v})^\top$. This results in pricing PDEs in $n_v + 1$ space dimensions: each additional source of randomness Y^i gives an additional dimension in the pricing equation. Hence, we can use the discretization techniques developed for the pricing of multi-asset options also for the pricing of options in SV models.

9.1 Market Models

A large class of diffusion SV models can described as follows. Consider the asset price process S following the SDE

$$\mathrm{d}S_t = \mu S_t \,\mathrm{d}t + \sigma_t S_t \,\mathrm{d}W_t^1, \tag{9.1}$$

where $\sigma = \{\sigma_t : t \geq 0\}$ is called the volatility process. A widely used model for σ is

$$\sigma_t = \xi(Y_t^1, \dots, Y_t^{n_v}), \tag{9.2}$$

where $\xi : \mathbb{R}^{n_v} \to \mathbb{R}_{\geq 0}$ is some function and $Y := (Y^1, \dots, Y^{n_v})^\top$, $n_v \geq 1$, is an $\mathbb{R}^{n_v}$-valued diffusion process. The dynamics of the vector process $Z :=$

N. Hilber et al., *Computational Methods for Quantitative Finance*, Springer Finance,
DOI 10.1007/978-3-642-35401-4_9,

$(S, Y^1, \dots, Y^{n_v})^\top \in \mathbb{R}^{n_v+1}$ can be described by the system of SDEs (compare with (8.1))

$$\mathrm{d}Z_t = b(Z_t)\,\mathrm{d}t + \Sigma(Z_t)\,\mathrm{d}W_t, \quad Z_0 = z, \tag{9.3}$$

where W is $\mathbb{R}^{n_v+1}$-valued Brownian motion, and the coefficients $b : \mathbb{R}^{n_v+1} \to \mathbb{R}^{n_v+1}$, $\Sigma : \mathbb{R}^{n_v+1} \to \mathbb{R}^{(n_v+1)\times(n_v+1)}$ satisfy the Lipschitz-continuity (8.2) and the linear growth condition (8.3). We again denote by $\mathcal{Q} = \Sigma\Sigma^\top$.

9.1.1 Heston Model

The Heston model is given by $\xi(y) = \sqrt{y}$ with $n_v = 1$ in (9.2) and Y following a CIR process, i.e. $\mathrm{d}Y_t = \alpha(m - Y_t)\,\mathrm{d}t + \beta\sqrt{Y_t}\,\mathrm{d}\widehat{W}_t$. The Brownian motion $\widehat{W}$ might be correlated with the Brownian motion W^1 in (9.1) that drives the asset price S. We thus introduce a Brownian motion W^2 independent of W^1 and write $\widehat{W}_t = \rho W_t^1 + \sqrt{1-\rho^2}\,W_t^2$, with $\rho \in [-1, 1]$ being the instantaneous correlation coefficient. Under a non-unique EMM, the coefficients b, Σ in (9.3) for the Heston model are therefore given by

$$b(z) = \begin{pmatrix} rs \\ \alpha(m-y) - \lambda(s, y) \end{pmatrix}, \tag{9.4}$$

$$\Sigma(z) = \begin{pmatrix} s\sqrt{y} & 0 \\ \beta\rho\sqrt{y} & \beta\sqrt{1-\rho^2}\sqrt{y} \end{pmatrix}, \tag{9.5}$$

where $z = (s, y)$ and the function λ appearing in (9.4) represents the price of volatility risk. Different choices of λ are given in the literature, for example, $\lambda(s, y) = c\sqrt{y}$ (together with $\rho = 0$) in [6] and $\lambda(s, y) = cy$ in [79]. For simplicity, we will choose $\lambda \equiv 0$ in the following.

9.1.2 Multi-scale Model

We assume that each component of $Y = (Y^1, \dots, Y^{n_v})$ evolves according to

$$\mathrm{d}Y_t^k = c_k(Y_t^k)\,\mathrm{d}t + g_k(Y_t^k)\,\mathrm{d}\widehat{W}_t^k, \quad k = 1, \dots, n_v, \tag{9.6}$$

with coefficients $c_k, g_k : \mathbb{R} \to \mathbb{R}$ smooth and at most linearly growing.

As in the Heston model, the Brownian motions W^1 and $\widehat{W}_t^k$, $k = 1, \dots, n_v$, are not independent to each other, but are allowed to have a correlation structure $L \in \mathbb{R}^{(n_v+1)\times(n_v+1)}$ of the form

$$(W_t^1, \widehat{W}_t^1, \dots, \widehat{W}_t^{n_v})^\top = LW_t$$

with $W = (W^1, \dots, W^{n_v+1})^\top$ a standard (n_v+1)-dimensional Brownian motion and

$$L = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \rho_1 & & & \\ \vdots & & \widetilde{L} & \\ \rho_{n_v} & & & \end{pmatrix},$$

where $\widetilde{L} \in \mathbb{R}^{n_v \times n_v}$ is defined as

$$\widetilde{L}_{ij} := \begin{cases} 0 & \text{if } i < j, \\ \left(1 - \rho_i^2 - \sum_{k=1}^{j-1} \widetilde{\rho}_{ki}^2\right)^{1/2} & \text{if } i = j, \\ \widetilde{\rho}_{ji} & \text{if } i > j, \end{cases} \quad i, j = 1, \ldots, n_v.$$

The constants ρ_i, $i = 1, \ldots, n_v$, and $\widetilde{\rho}_{ij}$, $i, j = 1, \ldots, n_v$, satisfy $|\rho_i| \leq 1$, $\rho_i^2 + \sum_{k=1}^{j-1} \widetilde{\rho}_{ki}^2 \leq 1$.

Example 9.1.1 The case $n_v = 1$ is introduced in [68]. The process $Y_t^1 = Y_t$ follows a mean-reverting Ornstein–Uhlenbeck (OU) process, i.e. the coefficients in (9.6) are given by

$$c_1(y) = \alpha(m - y), \quad g_1(y) = \beta,$$

with $\alpha > 0$ the rate of mean-reversion, $m > 0$ the long-run mean level of volatility and $\beta \in \mathbb{R}$. Here, $\widetilde{L} = \widetilde{L}_{11} = \sqrt{1 - \rho^2}$, with $|\rho| \leq 1$. Note that this model includes in particular the model of Stein–Stein [153] ($\xi(y) = |y|$, $\rho = 0$) and the model of Scott [150] ($\xi(y) = e^y$, $\rho = 0$).

Example 9.1.2 The case $n_v = 2$ is studied in [67]. The first component of Y_t is assumed to follow a mean-reverting OU process modeling a fast scale volatility factor, while the second component is a diffusion process, modeling a slow scale volatility factor. In particular, the coefficients in (9.6) are

$$\begin{aligned} c_1(y_1) &= \alpha_1(m - y_1), & g_1(y_1) &= \alpha_2, \\ c_2(y_2) &= \alpha_3 c(y_2), & g_2(y_2) &= \alpha_4 g(y_2), \end{aligned}$$

with $\alpha_1 > 0$ "large", $\alpha_2 = \mathcal{O}(\sqrt{\alpha_1})$ as well as with $\alpha_3 > 0$ "small", and $\alpha_4 = \mathcal{O}(\sqrt{\alpha_3})$. The functions $c, g : \mathbb{R} \to \mathbb{R}$ are assumed to be smooth and at most linearly growing.

Introducing random volatility renders the market model incomplete so that there is—unlike in the constant volatility case—no unique measure equivalent to the physical measure under which the discounted stock price is a martingale. We consider the $(n_v + 1)$-dimensional standard Brownian motion under the risk-neutral measure

$$\begin{aligned} \widehat{W}_t &= (\widehat{W}_t^1, \widehat{W}_t^2, \ldots, \widehat{W}_t^{n_v+1})^\top \\ &:= (W_t^1, W_t^2, \ldots, W_t^{n_v+1})^\top \\ &\quad + \left[\int_0^t \frac{\mu - r}{\xi(Y_s)} \, \mathrm{d}s, \int_0^t \gamma_1(Y_s) \, \mathrm{d}s, \ldots, \int_0^t \gamma_{n_v}(Y_s) \, \mathrm{d}s \right]^\top, \end{aligned}$$

where it is assumed that the market prices of volatility risk $\gamma_i(y)$, $i = 1, \ldots, n_v$, are smooth bounded functions of $y = (y_1, \ldots, y_{n_v})$ only (compare with [68, Chap. 2], [67]). We introduce the combined market prices of volatility risk Λ_i defined by

$$\Lambda_i(y) = \frac{\rho_i(\mu - r)}{\xi(y)} + \sum_{k=1}^{i} \gamma_k(y) L_{i+1,k+1}, \quad i = 1, \ldots, n_v. \tag{9.7}$$

Under a risk neutral probability measure, the dynamics of $Z := (S, Y^1, \ldots, Y^{n_v})^\top$ is as in (9.3), with coefficients b, Σ given by

$$b(z) = \begin{pmatrix} rs \\ c_1(y_1) - g_1(y_1)\Lambda_1(y) \\ \vdots \\ c_p(y_p) - g_p(y_p)\Lambda_p(y) \end{pmatrix}, \tag{9.8}$$

$$\Sigma(z) = \mathrm{diag}\big(s\xi(y), g_1(y_1), \ldots, g_2(y_2)\big)L, \tag{9.9}$$

where $z := (s, y_1, \ldots, y_{n_v}) \in \mathbb{R}^{n_v+1}$. In the next section, we derive the pricing equation for European options under these SV models.

9.2 Pricing Equation

As in the BS model, we switch in (9.1) to the log-price process $X = \ln(S)$. We therefore consider instead of the process $(S, Y^1, \ldots, Y^{n_v})^\top$ the process $Z := (X, Y^1, \ldots, Y^{n_v})^\top$. Assuming constant interest rate $r \geq 0$ and setting $z := (x, y_1, \ldots, y_{n_v}) \in \mathbb{R}^{n_v+1}$, we are interested in calculating the option value

$$V(t, z) := \mathbb{E}\big[e^{-r(T-t)} g(e^{X_T}) \mid Z_t = z\big]. \tag{9.10}$$

Note that the option price is a function of $z = (x, y_1, \ldots, y_{n_v})$ and not just x, since we have to condition on $Z_t = z$ and not just on $X_t = x$. The reason for this is that the process Z is Markovian (the unique strong solution to the SDE (9.3) is a Markov process, see, e.g. [3, Theorem 6.4.5]), but the process X is not. Proceeding as in Sect. 8.1, we find that the function $v(t, z) := V(T - t, z)$ is a solution of the PDE

$$\begin{aligned} \partial_t v - \mathcal{A}v + rv &= 0 \quad \text{in } J \times \mathbb{R} \times G^Y, \\ v(0, z) = v_0(z) &:= g(e^x) \quad \text{in } \mathbb{R} \times G^Y, \end{aligned} \tag{9.11}$$

where $(\mathcal{A}f)(z) := \frac{1}{2}\mathrm{tr}[\mathcal{Q}(z)D^2 f(z)] + b(z)^\top \nabla f(z)$ is the infinitesimal generator of the process Z and $G^Y \subseteq \mathbb{R}^{n_v}$ denotes the state space of the process Y.

Consider now the Heston model with coefficients (9.4)–(9.5) (recall that we set $\lambda = 0$). Since the coefficient Σ in (9.5) is not Lipschitz continuous, we can only formally conclude that the infinitesimal generator $\mathcal{A} =: \mathcal{A}^{\mathrm{H}}$ appearing in the pricing equation (9.11) is given in log-price by

$$(\mathcal{A}^{\mathrm{H}} f)(z) := \frac{1}{2} y \partial_{xx} f(x, y) + \beta \rho y \partial_{xy} f(x, y) + \frac{1}{2} \beta^2 y \partial_{yy} f(x, y) + \left(r - \frac{1}{2} y\right) \partial_x f(x, y) + \alpha(m - y) \partial_y f(x, y). \tag{9.12}$$

Furthermore, we have $G^Y = \mathbb{R}_{\geq 0}$. To cast the pricing equation (9.11) corresponding to the Heston model in a variational formulation and to establish its well-posedness, we change variables

$$\widetilde{v}(t, x, \widetilde{y}) := v(t, x, 1/4\widetilde{y}^2). \tag{9.13}$$

The pricing equation for $\widetilde{v}$ becomes,

$$\begin{aligned} &\partial_t \widetilde{v} - \widetilde{\mathcal{A}}^{\mathrm{H}} \widetilde{v} + r \widetilde{v} = 0 \quad \text{in } J \times \mathbb{R} \times \mathbb{R}_{\geq 0}, \\ &\widetilde{v}_0 = g(e^x) \quad \text{in } \mathbb{R} \times \mathbb{R}_{\geq 0}, \end{aligned} \tag{9.14}$$

with

$$\begin{aligned} (\widetilde{\mathcal{A}}^{\mathrm{H}} f)(x, \widetilde{y}) := &\frac{1}{8} \widetilde{y}^2 \partial_{xx} f(x, \widetilde{y}) + \frac{1}{2} \rho \beta \widetilde{y} \partial_{x\widetilde{y}} f(x, \widetilde{y}) + \frac{1}{2} \beta^2 \partial_{\widetilde{y}\widetilde{y}} f(x, \widetilde{y}) \\ &+ \left(r - \frac{1}{8} \widetilde{y}^2\right) \partial_x f(x, \widetilde{y}) \\ &+ \frac{1}{2} \left(-\alpha \widetilde{y} + \frac{4\alpha m - \beta^2}{\widetilde{y}}\right) \partial_{\widetilde{y}} f(x, \widetilde{y}). \end{aligned} \tag{9.15}$$

In the multi-scale SV model, assume that the market price of volatility risk $\gamma_i = 0$, $i = 1, \ldots, n_v$. Furthermore, assume that the functions ξ, c_k, g_k, $k = 1, \ldots, n_v$, are such that the coefficients b, Σ in (9.8)–(9.9) satisfy (8.2)–(8.3). Then, the generator $\mathcal{A} =: \mathcal{A}^{\mathrm{MS}}$ of the multi-scale process Z is given by

$$\begin{aligned} (\mathcal{A}^{\mathrm{MS}} f)(z) = &\frac{1}{2} \xi^2(y) \partial_{xx} f(z) + \frac{1}{2} \sum_{i=1}^{n_v} g_i^2(y_i) \partial_{y_i y_i} f(z) \\ &+ \xi(y) \sum_{i=1}^{n_v} \rho_i g_i(y_i) \partial_{x y_i} f(z) + \sum_{\substack{i=1 \\ j>i}}^{n_v} q_{i,j}(y) \partial_{y_i y_j} f(z) \\ &+ \left(r - \frac{1}{2} \xi^2(y)\right) \partial_x f(z) + \sum_{i=1}^{n_v} \left(c_i(y_i) - g_i(y_i) \rho_i \frac{\mu - r}{\xi(y)}\right) \partial_{y_i} f(z), \end{aligned}$$

where the coefficients $q_{i,j}(y)$ are given by $q_{i,j}(y) := g_i(y_i) g_j(y_j)(\rho_i \rho_j + \sum_{k=1}^{i} \widetilde{L}_{j,k} \widetilde{L}_{i,k})$. Hence, for the mean reverting OU model considered in Example 9.1.1, we find

$$\begin{aligned} (\mathcal{A}^{\mathrm{MS}} f)(z) = &\frac{1}{2} \xi^2(y) \partial_{xx} f(x, y) + \frac{1}{2} \beta^2 \partial_{yy} f(x, y) + \xi(y) \rho \beta \partial_{xy} f(x, y) \\ &+ \left(r - \frac{1}{2} \xi^2(y)\right) \partial_x f(x, y) \\ &+ \left(\alpha(m - y) - \beta \rho \frac{\mu - r}{\xi(y)}\right) \partial_y f(x, y), \end{aligned} \tag{9.16}$$

where $z = (x, y_1) = (x, y)$.

9.3 Variational Formulation

We consider the weak formulation of the pricing equation (9.11). For the ensuing numerical analysis, it is convenient to multiply the value of the option v in (9.11) with an exponentially decaying factor, i.e. we consider

$$w := ve^{-\eta}, \tag{9.17}$$

where $\eta \in C^2(\mathbb{R}^{n_v+1})$ is assumed to be at most polynomially growing at infinity. The function w solves

$$\begin{aligned} &\partial_t w - (\mathcal{A} + \mathcal{A}_\eta)w + rw = 0 \quad \text{in } J \times \mathbb{R} \times G^Y, \\ &w(0, z) = w_0 := g(e^x)e^{-\eta} \quad \text{in } \mathbb{R} \times G^Y, \end{aligned} \tag{9.18}$$

where the operator $\mathcal{A}_\eta$ is given by

$$\mathcal{A}_\eta := \Sigma_\eta^\top \nabla + \frac{1}{2}\operatorname{tr}\big[\mathcal{Q}D^2\eta\big] + \frac{1}{2}(\Sigma_\eta + 2b)^\top \nabla\eta. \tag{9.19}$$

The coefficient $\Sigma_\eta \in \mathbb{R}^{n_v+1}$ appearing in (9.19) is defined by

$$\Sigma_\eta = (\Sigma_\eta^1, \ldots, \Sigma_\eta^{n_v+1})^\top, \quad \Sigma_\eta^k := \sum_{i=1}^{n_v+1} \mathcal{Q}_{ik}\partial_{x_i}\eta.$$

Consider now the pricing equation of the (transformed) Heston model (9.14). For notational simplicity, we drop "$\sim$" in $\widetilde{v}$ and $\widetilde{\mathcal{A}}$. Consider the change of variables (9.17) with $\eta = \eta(x, y) := \frac{1}{2}\kappa y^2$, $\kappa > 0$. By the definition of $\mathcal{A}_\eta$ (9.19), it follows that the pricing equation for $w := (v - v_0)e^{-\eta}$ in the Heston model becomes

$$\begin{aligned} &\partial_t w - \mathcal{A}_\kappa^{\mathrm{H}} w + rw = f_\kappa^{\mathrm{H}} \quad \text{in } J \times \mathbb{R} \times \mathbb{R}_{\geq 0}, \\ &w(0, x, y) = 0 \quad \text{in } \mathbb{R} \times \mathbb{R}_{\geq 0}, \end{aligned} \tag{9.20}$$

where $f_\kappa^{\mathrm{H}} := e^{-\kappa/2y^2}(\mathcal{A}^{\mathrm{H}} v_0 - rv_0)$ and

$$\begin{aligned} (\mathcal{A}_\kappa^{\mathrm{H}} f)(z) := &\frac{1}{8}y^2\partial_{xx}f(x, y) + \frac{1}{2}\rho\beta y\partial_{xy}f(x, y) + \frac{1}{2}\beta^2\partial_{yy}f(x, y) \\ &+ \left(r - \frac{1}{8}y^2 + \frac{1}{2}\beta\kappa\rho y^2\right)\partial_x f(x, y) \\ &+ \frac{1}{2}\left((2\beta^2\kappa - \alpha)y + \frac{4\alpha m - \beta^2}{y}\right)\partial_y f(x, y) \\ &+ \left(\frac{1}{2}y^2\kappa(\beta^2\kappa - \alpha) + 2\alpha\kappa m\right)f(x, y). \end{aligned} \tag{9.21}$$

Let $G := \mathbb{R} \times \mathbb{R}_{\geq 0}$ and denote by $(\cdot,\cdot)$ the $L^2(G)$-inner product, i.e. $(\varphi, \phi) = \int_G \varphi\phi \,\mathrm{d}x\,\mathrm{d}y$. We associate to $-\mathcal{A}_\kappa^{\mathrm{H}} + r$ the bilinear form $a_\kappa^H(\cdot,\cdot)$ via

$$a_\kappa^H(\varphi, \phi) := \big((-\mathcal{A}_\kappa^{\mathrm{H}} + r)\varphi, \phi\big), \quad \varphi, \phi \in C_0^\infty(G).$$

Integration by parts yields

$$
\begin{aligned}
a_\kappa^{\mathrm{H}}(\varphi,\phi) &= \frac{1}{8}(y\partial_x\varphi, y\partial_x\phi) + \frac{1}{2}\beta^2(\partial_y\varphi,\partial_y\phi) + \frac{1}{2}\rho\beta(y\partial_x\varphi,\partial_y\phi)\\
&\quad + \frac{1}{2}[\rho\beta - 2r](\partial_x\varphi,\phi) + \frac{1}{8}[1-4\beta\kappa\rho](y\partial_x\varphi, y\phi)\\
&\quad - \frac{1}{2}[2\beta^2\kappa-\alpha](y\partial_y\varphi,\phi) - \frac{1}{2}[4\alpha m-\beta^2](y^{-1}\partial_y\varphi,\phi)\\
&\quad - \frac{1}{2}\kappa[\beta^2\kappa-\alpha](y\varphi,y\phi) - [2\kappa\alpha m - r](\varphi,\phi)\\
&=: \sum_{k=1}^{9} b_k(\varphi,\phi).
\end{aligned}
\tag{9.22}
$$

Define the weighted Sobolev space

$$
V := \overline{C_0^\infty(G)}^{\|\cdot\|_V}, \tag{9.23}
$$

where the closure is taken with respect to the norm

$$
\|v\|_V^2 := \|y\partial_x v\|_{L^2(G)}^2 + \|\partial_y v\|_{L^2(G)}^2 + \|\sqrt{1+y^2}\,v\|_{L^2(G)}^2. \tag{9.24}
$$

Theorem 9.3.1 *Assume that* $0<\kappa<\alpha/\beta^2$ *and that*

$$
1 - 2|4\alpha m/\beta^2 - 1| > \rho^2.
$$

Then, there exist constants $C_i>0$, $i=1,2,3$, *such that for all* $\varphi,\phi\in V$ *one has*

$$
\begin{aligned}
|a_\kappa^{\mathrm{H}}(\varphi,\phi)| &\le C_1\|\varphi\|_V\|\phi\|_V,\\
a_\kappa^{\mathrm{H}}(\varphi,\varphi) &\ge C_2\|\varphi\|_V^2 - C_3\|\varphi\|_{L^2(G)}^2.
\end{aligned}
$$

Proof Let $\varphi,\phi\in C_0^\infty(G)$. We first show continuity of the bilinear form $a_\kappa^{\mathrm{H}}(\cdot,\cdot)$. The only terms in (9.22) which cannot be estimated directly by an application of Cauchy–Schwarz are $(\partial_x\varphi,\phi)$ and $(y^{-1}\partial_y\varphi,\phi)$. For both terms, the continuity follows from the Hardy inequality (4.26)

$$
\|y^{-1}\phi\|_{L^2(G)} \le 2\|\partial_y\phi\|_{L^2(G)}. \tag{9.25}
$$

Indeed,

$$
\begin{aligned}
&\frac{1}{2}\big|[4\alpha m-\beta^2]\big(y^{-1}\partial_y\varphi,\phi\big)\big|\\
&\quad\le C\|\partial_y\varphi\|_{L^2(G)}\|y^{-1}\phi\|_{L^2(G)} \overset{(9.25)}{\le} \|\partial_y\varphi\|_{L^2(G)}\|\partial_y\phi\|_{L^2(G)},
\end{aligned}
$$

and similarly

$$
\frac{1}{2}\big|[\rho\beta-2r]\big(\partial_x\varphi,\phi\big)\big| \le C\|y\partial_x\varphi\|_{L^2(G)}\|y^{-1}\phi\|_{L^2(G)} \le C\|y\partial_x\varphi\|_{L^2(G)}\|\partial_y\phi\|_{L^2(G)}.
$$

To this end, it follows from the triangle inequality $|a_\kappa^{\mathrm{H}}(\varphi,\phi)|\le C_1\|\varphi\|_V\|\phi\|_V$. To prove the Gårding inequality, we consider each term in (9.22) separately. We have, since $(y\partial_y\varphi,\varphi)=\frac{1}{2}(\partial_y(\varphi^2),y) = -\frac{1}{2}\|\varphi\|_{L^2(G)}^2$,

$$b_1(\varphi,\varphi) = \frac{1}{8}\|y\partial_x\varphi\|^2_{L^2(G)}, \quad b_2(\varphi,\varphi) = \frac{1}{2}\beta^2\|\partial_y\varphi\|^2_{L^2(G)},$$
$$b_6(\varphi,\varphi) = \frac{1}{4}[2\beta^2\kappa - \alpha]\|\varphi\|^2_{L^2(G)},$$

$$b_8(\varphi,\varphi) = \frac{1}{2}\kappa[\alpha - \kappa\beta^2]\|y\varphi\|^2_{L^2(G)}, \quad b_9(\varphi,\varphi) = [r - 2\kappa\alpha m]\|\varphi\|^2_{L^2(G)},$$

and, using $(\partial_x\varphi,\varphi) = 1/2(\partial_x(\varphi)^2, 1) = 0$, $(y\partial_x\varphi, y\varphi) = 1/2(\partial_x(\varphi)^2, y^2) = 0$, we find that $b_4(\varphi,\varphi) = b_5(\varphi,\varphi) = 0$. Furthermore, for $\varepsilon > 0$,

$$\begin{aligned}
b_3(\varphi,\varphi) &\geq -\frac{1}{2}|\rho\beta|\|y\partial_x\varphi\|_{L^2(G)}\|\partial_y\varphi\|_{L^2(G)} \\
&\geq -\frac{1}{2}\varepsilon\|y\partial_x\varphi\|^2_{L^2(G)} - \frac{1}{2}\frac{\rho^2\beta^2}{4\varepsilon}\|\partial_y\varphi\|^2_{L^2(G)}, \\
b_7(\varphi,\varphi) &\geq -\frac{1}{2}|4\alpha m - \beta^2|\|\partial_y\varphi\|_{L^2(G)}\|y^{-1}\varphi\|_{L^2(G)} \overset{(9.25)}{\geq} -|4\alpha m - \beta^2|\|\partial_y\varphi\|^2_{L^2(G)}.
\end{aligned}$$

Collecting these estimates yields

$$a^{\mathrm{H}}_\kappa(\varphi,\varphi) \geq c_1\|y\partial_x\varphi\|^2_{L^2(G)} + c_2\|\partial_y\varphi\|_{L^2(G)} + c_3\|y\varphi\|^2_{L^2(G)} + c_4\|\varphi\|^2_{L^2(G)},$$

with $c_1 = \frac{1}{8} - \frac{1}{2}\varepsilon$, $c_2 = \frac{1}{2}\beta^2 - \frac{1}{2}\frac{\beta^2\rho^2}{4\varepsilon} - |4\alpha m - \beta^2|$, $c_3 = \frac{1}{2}\kappa[\alpha - \kappa\beta^2]$ and $c_4 = r - 2\kappa\alpha m + \frac{1}{4}[2\beta^2\kappa - \alpha]$. We choose ε sufficiently close to $\frac{1}{4}$ such that $c_1 > 0$. Thus, in order for c_2 to be positive, we must have

$$\frac{1}{2}\beta^2 > \frac{1}{2}\beta^2\rho^2 + |4\alpha m - \beta^2| \;\Leftrightarrow\; 1 > \rho^2 + 2|4\alpha m/\beta^2 - 1|.$$

Furthermore, if $0 < \kappa < \alpha/\beta^2$, then we have $c_3 > 0$. Hence,

$$\begin{aligned}
a^{\mathrm{H}}_\kappa(\varphi,\varphi) &\geq c_1\|y\partial_x\varphi\|^2_{L^2(G)} + c_2\|\partial_y\varphi\|_{L^2(G)} + c_3\|\sqrt{1+y^2}\,\varphi\|^2_{L^2(G)} \\
&\quad + (c_4 - c_3)\|\varphi\|^2_{L^2(G)} \\
&\geq \min\{c_1, c_2, c_3\}\|\varphi\|^2_V - |c_4 - c_3|\|\varphi\|^2_{L^2(G)}.
\end{aligned}$$

Since by definition $C^\infty_0(G)$ is dense in V, we may extend the bilinear form $a^{\mathrm{H}}_\kappa(\cdot,\cdot)$ continuously to V. □

By the abstract well-posedness result Theorem 3.2.2 in the triple of spaces $\mathcal{V} = V \subset L^2(G) = \mathcal{H} \equiv \mathcal{H}^* \subset V^*$, we conclude that the weak formulation to the (transformed) Heston model (9.20),

$$\begin{aligned}
&\text{Find } w \in L^2(J;V) \cap H^1(J;L^2(G)) \text{ such that} \\
&(\partial_t w, v) + a^{\mathrm{H}}_\kappa(w,v) = \langle f^{\mathrm{H}}_\kappa, v\rangle_{V^*,V}, \;\; \forall v \in V, \text{ a.e. in } J, \\
&w(0) = 0,
\end{aligned} \tag{9.26}$$

admits a unique solution for every $f^{\mathrm{H}}_\kappa \in V^*$.

The weak formulation (9.26) addresses the Heston model. Since the generator $\mathcal{A}^{\mathrm{H}}$ (9.15) has the same structure as the generator $\mathcal{A}^{\mathrm{MS}}$ (9.16) of the multi-scale model of Example 9.1.1, i.e. $n_v = 1$, $\xi(y) = |y|$, we also obtain the well-posedness of the weak formulation for this model.

We derive the bilinear form $a^{\mathrm{SV}}(\cdot,\cdot)$ for the general diffusion SV model in (9.11), (9.18) and give its weak formulation. The operator $\mathcal{A} + \mathcal{A}_\eta - r$ in (9.18) can be written as (compare with (9.19))

$$\begin{aligned}(\mathcal{A}^{\mathrm{SV}} f)(z) &:= (\mathcal{A} f + \mathcal{A}_\eta f - r f)(z)\\ &:= \frac{1}{2}\operatorname{tr}[\mathcal{Q}(z) D^2 f(z)] + \mu(z)^\top \nabla f(z) + c(z) f(z),\end{aligned} \tag{9.27}$$

where $\mu := \Sigma_\eta + b$, and $c := \frac{1}{2}\operatorname{tr}[\mathcal{Q} D^2 \eta] + \frac{1}{2}(\Sigma_\eta + 2b)^\top \nabla\eta - r$. Denote by $G := \mathbb{R} \times G^Y$ the state space of the process $Z = (\ln(S), Y^1, \ldots, Y^{n_v})^\top$. Then, the bilinear form associated to the operator $\mathcal{A}^{\mathrm{SV}}$

$$a^{\mathrm{SV}}(\varphi,\phi) := \left(-\mathcal{A}^{\mathrm{SV}}\varphi, \phi\right), \quad \varphi,\phi \in C_0^\infty(G),$$

becomes, after integration by parts,

$$a^{\mathrm{SV}}(\varphi,\phi) := \frac{1}{2}\int_G (\nabla\varphi)^\top \mathcal{Q}\nabla\phi \,\mathrm{d}x + \frac{1}{2}\int_G (\underline{\nabla}\mathcal{Q} - 2\mu)^\top \nabla\varphi\phi \,\mathrm{d}x - \int_G c\varphi\phi \,\mathrm{d}x. \tag{9.28}$$

Herewith, we denote by $\underline{\nabla} : [W^{1,\infty}(G)]^{(n_v+1)\times(n_v+1)} \to [L^\infty(G)]^{n_v+1}$, $A \mapsto \underline{\nabla}A$, the operator which takes the divergence of the columns $\underline{A}_k := (A_{1,k}, \ldots, A_{n_v+1,k})^\top$, $k = 1, \ldots, n_v + 1$, of the matrix A, i.e.

$$\underline{\nabla}A := \left(\operatorname{div}(\underline{A}_1), \ldots, \operatorname{div}(\underline{A}_{n_v+1})\right)^\top.$$

Note that for constant coefficients $\mathcal{Q}$, μ and c, the bilinear form $a^{\mathrm{SV}}(\cdot,\cdot)$ reduces to the bilinear form $a^{\mathrm{BS}}(\cdot,\cdot)$ in (8.9) of the multivariate Black–Scholes model.

The variational formulation to (9.18) reads:

$$\begin{aligned}&\text{Find } w \in L^2(J;V) \cap H^1(J;L^2(G)) \text{ such that}\\ &(\partial_t w, v) + a^{\mathrm{SV}}(w,v) = 0, \ \forall v \in V, \ \text{a.e. in } J,\\ &w(0) = w_0.\end{aligned} \tag{9.29}$$

Here, we assume that the Hilbert space V is given by $V := \overline{C_0^\infty(G)}^{\|\cdot\|_V}$, where the norm $\|\cdot\|_V$ depends on the model. We also assume that $a^{\mathrm{SV}}(\cdot,\cdot) : V \times V \to \mathbb{R}$ is continuous and satisfies a Gårding inequality, and that $w_0 \in L^2(G)$.

9.4 Localization

We consider the pricing equation (9.18) truncated to a bounded domain $G_R := \prod_{k=1}^{n_v+1}(a_k, b_k)$, $b_k > a_k \in \mathbb{R}$,

$$\begin{aligned}\partial_t w_R - \mathcal{A}^{\mathrm{SV}} w_R + r w_R &= 0 \text{ in } J \times G_R,\\ w_R(0,z) &= w_0 \text{ in } G_R.\end{aligned} \tag{9.30}$$

Equation (9.30) needs to be complemented with appropriate boundary conditions which depend on the model under consideration. Localizing the pricing equation to a bounded domain induces an error which we now estimate using the example of the Heston model. To this end, consider the weak formulation of the Heston model, truncated to a bounded domain $G_R := (-R_1, R_1) \times (-R_2, R_2)$,

$$
\begin{aligned}
&\text{Find } w_R \in L^2(J; \widetilde{V}) \cap H^1(J; L^2(G_R)) \text{ such that}\\
&(\partial_t w_R, v) + a_\kappa^{\mathrm{H}}(w_R, v) = \langle f_\kappa^{\mathrm{H}}, v\rangle_{\widetilde{V}^*, \widetilde{V}}, \;\forall v \in \widetilde{V}, \text{ a.e. in } J, \qquad (9.31)\\
&w_R(0) = 0.
\end{aligned}
$$

Herewith, we denote by $\widetilde{V}$ the space $\widetilde{V} := \overline{C_0^\infty(G_R)}^{\|\cdot\|_V}$, with norm $\|\cdot\|_V$ as in (9.24). Note that the truncated problem (9.31) admits a unique solution, since, by Theorem 9.3.1, the bilinear form $a_\kappa^{\mathrm{H}}(\cdot,\cdot) : \widetilde{V} \times \widetilde{V} \to \mathbb{R}$ is continuous and satisfies a Gårding inequality also on $\widetilde{V} \times \widetilde{V}$.

Now, let w denote the solution of (9.26) on $G = \mathbb{R}^2$, and let w_R denote the solution of (9.31). Denote by $\widehat{w}_R$ the zero extension of w_R to $G := \mathbb{R} \times \mathbb{R}_+$ and let $e_R := \widehat{w}_R - w$ be the localization error. Denoting by $G_{R/2} := (-R_1/2, R_1/2) \times (-R_2/2, R_2/2)$ and repeating the arguments in the proof of [81, Theorem 3.6], we obtain

Theorem 9.4.1 *Let $\phi_R = \phi_R(x, y) \in C_0^\infty(G_R)$ denote a cut-off function with the following properties*

$$\phi_R \geq 0, \quad \phi_R \equiv 1 \text{ on } G_{R/2} \text{ and } \|\nabla \phi_R\|_{L^\infty(G_R)} \leq C$$

for some constant $C > 0$ independent of $R_1, R_2 \geq 1$. Then, there exist constants $c = c(T)$, $\varepsilon > 0$ which are independent of R_1, R_2 such that

$$\|\phi_R e_R(t, \cdot)\|^2_{L^2(G_R)} + \int_0^t \|\phi_R e_R(s, \cdot)\|_V^2 \, \mathrm{d}s \leq c e^{-\varepsilon(R_1 + R_2)}.$$

9.5 Discretization

We discuss the implementation of the stiffness matrix $\mathbf{A}^{\mathrm{SV}}$ of the general diffusion SV model $\mathcal{A}^{\mathrm{SV}}$ in (9.27). Under the assumption that the coefficients $\mathcal{Q}$, μ and c of $\mathcal{A}^{\mathrm{SV}}$ can be written as (a sum of) products of univariate functions, we show that $\mathbf{A}^{\mathrm{SV}}$ can be represented as sums of Kronecker products of matrices corresponding to univariate problems, as in Sect. 8.4.

We will write $x = (x_1, \ldots, x_d)$ instead of $z = (\ln(s), y_1, \ldots, y_{n_v+1})$ to unify the notation and assume the following product structure of the coefficients $\mathcal{Q}$, μ and c

Assumption 9.5.1 The coefficients $\mathcal{Q}$, μ and c are given by

$$\mathcal{Q}_{ij}(x) = \prod_{k=1}^d q_{ij}^k(x_k), \quad \mu_i(x) = \prod_{k=1}^d \mu_i^k(x_k), \quad c(x) = \prod_{k=1}^d c_k(x_k),$$

for univariate functions $q_{ij}^k, \mu_i^k, c_k : \mathbb{R} \to \mathbb{R}$, $1 \leq i, j, k \leq d$.

9.5.1 Finite Difference Discretization

We define weighted versions of the matrices **R**, **C** and **I** given in (4.14). For $w : \mathbb{R} \to \mathbb{R}$ define matrices in $\mathbb{R}^{N_k \times N_k}$ with respect to kth coordinate direction

$$\mathbf{R}^{w(x_k)} := \frac{1}{h_k^2} \begin{pmatrix} 2w(x_{k,1}) & -w(x_{k,1}) & & \\ -w(x_{k,2}) & \ddots & \ddots & \\ & \ddots & \ddots & -w(x_{k,N_k-1}) \\ & & -w(x_{k,N_k}) & 2w(x_{k,N_k}) \end{pmatrix},$$

$$\mathbf{C}^{w(x_k)} := \frac{1}{2h_k} \begin{pmatrix} 0 & w(x_{k,1}) & & \\ -w(x_{k,2}) & \ddots & \ddots & \\ & \ddots & \ddots & w(x_{k,N_k-1}) \\ & & -w(x_{k,N_k}) & 0 \end{pmatrix},$$

as well as

$$\mathbf{I}^{w(x_k)} := \operatorname{diag}\big(w(x_{k,1}), \ldots, w(x_{k,N_k})\big).$$

Here, we denote by x_{k,i_k} a grid point of a grid in the kth coordinate direction, i.e. $x_{k,i_k} = a_k + h_k i_k$, $h_k = \frac{b_k - a_k}{N_k+1}$. The following definition will help to simplify the notation.

Definition 9.5.2 For an arbitrary permutation $\sigma : \{1, \ldots, d\} \to \{1, \ldots, d\}$, $\{1, \ldots, d\} \mapsto \{\sigma(1), \ldots, \sigma(d)\}$ and matrices $\mathbf{X}^{w(x_k)}$, $1 \le k \le d$, we denote by ${}^s(\mathbf{X}^{w(x_{\sigma(1)})} \otimes \cdots \otimes \mathbf{X}^{w(x_{\sigma(d)})})$ the sorted Kronecker product with factors sorted by increasing indices, i.e.

$${}^s\big(\mathbf{X}^{w(x_{\sigma(1)})} \otimes \cdots \otimes \mathbf{X}^{w(x_{\sigma(d)})}\big) := \mathbf{X}^{w(x_1)} \otimes \cdots \otimes \mathbf{X}^{w(x_d)}.$$

Using the finite difference quotients $\delta^2_{x_i x_j}$, δ_{x_i} on the grid

$$\mathcal{G} := \big\{(x_{1,i_1}, \ldots, x_{d,i_d}) \mid 1 \le i_k \le N_k,\ 1 \le k \le d\big\} \subset G_R,$$

and proceeding exactly as in Sect. 8.4.1, we find that the finite difference matrix $\mathbf{G}^{\mathrm{SV}}$ corresponding to (9.27) is given by

$$\begin{aligned} \mathbf{G}^{\mathrm{SV}} := \frac{1}{2} \sum_{i=1}^{d} {}^s\Big(\mathbf{R}^{q^i_{ii}(x_i)} \otimes \bigotimes_{k \neq i} \mathbf{I}^{q^k_{ii}(x_k)}\Big) \\ - \frac{1}{2} \sum_{\substack{i,j=1 \\ j \neq i}}^{d} {}^s\Big(\mathbf{C}^{q^i_{ij}(x_i)} \otimes \mathbf{C}^{q^j_{ij}(x_j)} \otimes \bigotimes_{k \notin \{i,j\}} \mathbf{I}^{q^k_{ij}(x_k)}\Big) \\ - \sum_{i=1}^{d} {}^s\Big(\mathbf{C}^{\mu^i_i(x_i)} \otimes \bigotimes_{k \neq i} \mathbf{I}^{\mu^k_i(x_k)}\Big) - \bigotimes_{k} \mathbf{I}^{c_k(x_k)}. \end{aligned}$$

Assuming homogeneous Dirichlet boundary conditions on ∂G_R, discretizing in time with θ-scheme and denoting again by $N := \prod_{k=1}^{d} N_k = |\mathcal{G}|$ the number of points in the grid $\mathcal{G}$, we obtain the fully discrete finite difference scheme for the truncated pricing equation (9.30)

$$\begin{aligned}
&\text{Find } \underline{w}^{m+1} \in \mathbb{R}^N \text{ such that for } m = 0, \ldots, M-1\\
&(\mathbf{I} + \theta\Delta t\mathbf{G}^{\mathrm{SV}})\underline{w}^{m+1} = (\mathbf{I} - (1-\theta)\Delta t\mathbf{G}^{\mathrm{SV}})\underline{w}^{m},\\
&\underline{w}^0 = \underline{w}_0.
\end{aligned}$$

Example 9.5.3 Consider the multi-scale SV model with $n_v = 1$ factor as described in Example 9.1.1 with $\rho = 0$. By (9.16), the coefficients are $q_{11}^1(x_1) = 1$, $q_{11}^2(x_2) = \xi^2(x_2)$, $q_{22}^1(x_1) = 1$, $q_{22}^2(x_2) = \beta^2$, $\mu_1^1(x_1) = 1$, $\mu_1^2(x_2) = r - \frac{1}{2}\xi^2(x_2)$, $\mu_2^1(x_1) = 1$, $\mu_2^2(x_2) = \alpha(m - x_2)$ and $c_1(x_1) = 1$, $c_2(x_2) = -r$. The remaining coefficients are zero. Hence, the finite difference matrix becomes

$$\begin{aligned}
\mathbf{G}^{\mathrm{MS}} = {}& \frac{1}{2}\mathbf{R}^1 \otimes \mathbf{I}^{\xi^2(x_2)} + \frac{1}{2}\mathbf{I}^1 \otimes \mathbf{R}^{\beta^2}\\
& - \mathbf{C}^1 \otimes \mathbf{I}^{r-\frac{1}{2}\xi^2(x_2)} - \mathbf{I}^1 \otimes \mathbf{C}^{\alpha(m-x_2)} - \mathbf{I}^1 \otimes \mathbf{I}^{-r}.
\end{aligned}$$

Since for any constants γ_i and any weights w_i, $i = 1, 2$, one has $\mathbf{X}^{\gamma_1 w_1 + \gamma_2 w_2} = \gamma_1\mathbf{X}^{w_1} + \gamma_2\mathbf{X}^{w_2}$, and since the Kronecker product is distributive, we simplify the above expression to

$$\mathbf{G}^{\mathrm{MS}} = \frac{1}{2}\mathbf{R}^1 \otimes \mathbf{I}^{\xi^2(x_2)} - \mathbf{C}^1 \otimes \mathbf{Y}_2 + \mathbf{I}^1 \otimes \mathbf{Y}_1,$$

where

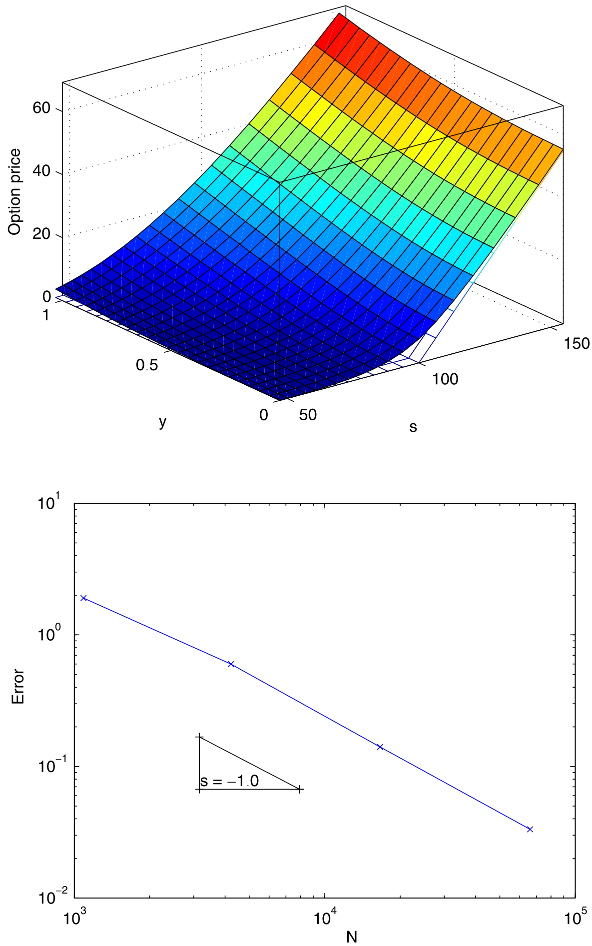
$$\mathbf{Y}_1 := \frac{1}{2}\beta^2\mathbf{R}^1 - \alpha m\mathbf{C}^1 + \alpha\mathbf{C}^{x_2} + \mathbf{I}^1, \qquad \mathbf{Y}_2 := r\mathbf{I}^1 - \frac{1}{2}\mathbf{I}^{\xi^2(x_2)}.$$

We now approximate the value of a call option in the model of Stein–Stein, i.e. $\xi(x_2) = |x_2|$, whose exact price can be found in [145]. To this end, we set $K = 100$, $T = 1/2$ for the contract parameters and $\alpha = 1$, $\beta = 1/\sqrt{2}$, $\rho = 0$, $m = 0.2$, $r = 0$ for the model parameters. The computed option price is plotted in Fig. 9.1. We also give the rate of convergence $\mathcal{O}(N^{-1})$ of the L^∞-error with respect to the number of grid points $N = |\mathcal{G}|$. The error is measured on the domain $(K/2, 3/2K) \times (0, 1)$.

9.5.2 *Finite Element Discretization*

As for the finite difference discretization, we have to introduce weighted matrices. For $w : \mathbb{R} \to \mathbb{R}$, define the matrices $\mathbf{S}^{w(x_k)}, \mathbf{B}^{w(x_k)}, \mathbf{M}^{w(x_k)} \in \mathbb{R}^{N_k \times N_k}$ with entries given by

Fig. 9.1 Option price (*top*) and convergence rate (*bottom*) for the mean reverting OU model



$$\mathbf{S}^{w(x_k)}_{i_k,i'_k} := \int_{a_k}^{b_k} b'_{i'_k}(x_k) b'_{i_k}(x_k) w(x_k)\, \mathrm{d}x_k,$$

$$\mathbf{B}^{w(x_k)}_{i_k,i'_k} := \int_{a_k}^{b_k} b'_{i'_k}(x_k) b_{i_k}(x_k) w(x_k)\, \mathrm{d}x_k,$$

$$\mathbf{M}^{w(x_k)}_{i_k,i'_k} := \int_{a_k}^{b_k} b_{i'_k}(x_k) b_{i_k}(x_k) w(x_k)\, \mathrm{d}x_k,$$

where by $b_{i_k} : (a_k, b_k) \to \mathbb{R}_{\geq 0}$ we denote the hat-function $b_{i_k}(x_k) := \max\{0, 1 - h_k^{-1}|x_k - x_{k,i_k}|\}$ with respect to the kth coordinate direction on a uniform mesh $x_{k,i_k} := a_k + i_k h_k$, $i_k = 0, \ldots, N_k + 1$, with mesh width $h_k := (b_k - a_k)/(N_k + 1)$.

We now prove a representation of the stiffness matrix $\mathbf{A}^{\mathrm{SV}}$ corresponding to the bilinear form $a^{\mathrm{SV}}(\cdot,\cdot)$ in (9.28).

Proposition 9.5.4 *Let Assumption 9.5.1 hold, and assume the finite element space V_N is given by* (8.19). *Then, the matrix $\mathbf{A}^{\mathrm{SV}} \in \mathbb{R}^{N\times N}$ is given by*

$$
\begin{aligned}
\mathbf{A}^{\mathrm{SV}} := {} & \frac{1}{2}\sum_{i=1}^{d}{}^{s}\Big(\mathbf{S}^{q_{ii}^{i}(x_i)} \otimes \bigotimes_{k\neq i}\mathbf{M}^{q_{jj}^{k}(x_k)}\Big) \\
& - \frac{1}{2}\sum_{\substack{i,j=1\\ j\neq i}}^{d}{}^{s}\Big(\mathbf{B}^{q_{ij}^{i}(x_i)} \otimes \big(\mathbf{B}^{q_{ij}^{j}(x_j)} + \mathbf{M}^{\frac{\mathrm{d}}{\mathrm{d}x_j} q_{ij}^{j}(x_j)}\big) \otimes \bigotimes_{k\notin\{i,j\}}\mathbf{M}^{q_{ij}^{k}(x_k)}\Big) \\
& + \frac{1}{2}\sum_{i=1}^{d}{}^{s}\Big(\mathbf{B}^{\frac{\mathrm{d}}{\mathrm{d}x_i} q_{ii}^{i}(x_i)} \otimes \bigotimes_{k\neq i}\mathbf{M}^{q_{ii}^{k}(x_k)}\Big) + \frac{1}{2}\sum_{\substack{i,j=1\\ j\neq i}}^{d}{}^{s}\Big(\mathbf{B}^{q_{ii}^{j}(x_j)} \otimes \bigotimes_{k\neq j}\mathbf{M}^{\widetilde{q}_{ij}^{k}(x_k)}\Big) \\
& - \sum_{i=1}^{d}{}^{s}\Big(\mathbf{B}^{\mu_i^{i}(x_i)} \otimes \bigotimes_{k\neq i}\mathbf{M}^{\mu_i^{k}(x_k)}\Big) - \bigotimes_{k}\mathbf{M}^{c_k(x_k)},
\end{aligned}
$$

with weights

$$
\widetilde{q}_{ij}^{k}(x_k) := \begin{cases} q_{ij}^{k}(x_k) & \text{if } k \neq i, \\ \frac{\mathrm{d}}{\mathrm{d}x_k} q_{ij}^{k}(x_k) & \text{if } k = i. \end{cases}
$$

Proof The proof follows the lines which led to Proposition 8.4.2. Since the coefficients $q_{ij}(x) = \prod_k q_{ij}^{k}(x_k)$ are not constant, we obtain additional terms due to integration by parts, i.e. $\int_{a_k}^{b_k} q_{ij}^{k}(x_k) b_{i_k'} b_{i_k}' \,\mathrm{d}x_k = -\int_{a_k}^{b_k} (q_{ik}^{k}(x_k) b_{i_k'})' b_{i_k}\,\mathrm{d}x_k = -\int_{a_k}^{b_k} q_{ik}^{k}(x_k) b_{i_k'}' b_{i_k}\,\mathrm{d}x_k - \int_{a_k}^{b_k} (q_{ik}^{k})'(x_k) b_{i_k'} b_{i_k}\,\mathrm{d}x_k = -\mathbf{B}_{i_k,i_k'}^{q_{ik}^{k}(x_k)} - \mathbf{M}_{i_k,i_k'}^{\frac{\mathrm{d}}{\mathrm{d}x_k} q_{ik}^{k}(x_k)}$. □

Note that the matrices appearing in the representation of $\mathbf{A}^{\mathrm{SV}}$ have to be implemented as discussed in Sect. 3.4, since the coefficients are not constant.

Example 9.5.5 Consider the transformed Heston model with operator $-\mathcal{A}_{\kappa}^{\mathrm{H}} + r$ in (9.21) and coefficients $q_{11}^{2}(x_2) = \frac{1}{4}x_2^2$, $q_{12}^{2}(x_2) = q_{21}^{2}(x_2) = \frac{1}{2}\beta\rho x_2$, $q_{22}^{2}(x_2) = \beta^2$, $\mu_1^{2}(x_2) = \frac{1}{8}(-1+4\beta\kappa\rho)x_2^2 + r$, $\mu_2^{2}(x_2) = \frac{1}{2}(2\beta^2\kappa - \alpha)x_2 + (4\alpha m - \beta^2)(2x_2^{-1})$ and $c_2(x_2) = \frac{1}{2}\kappa(\beta^2\kappa - \alpha)x_2^2 + 2\alpha\kappa m - r$ (the coefficients depending on x_1 are equal to 1). By Proposition 9.5.4, we find

$$
\begin{aligned}
\mathbf{A}_{\kappa}^{\mathrm{H}} = {} & \frac{1}{8}\mathbf{S}^{1} \otimes \mathbf{M}^{x_2^2} + \frac{1}{2}\mathbf{M}^{1} \otimes \mathbf{S}^{\beta^2} - \frac{1}{2}\mathbf{B}^{1} \otimes \big(\mathbf{B}^{\frac{1}{2}\beta\rho x_2} + \mathbf{M}^{\frac{1}{2}\beta\rho}\big) - \frac{1}{2}\mathbf{B}^{1} \otimes \mathbf{B}^{\frac{1}{2}\beta\rho x_2} \\
& + \frac{1}{2}\mathbf{B}^{1} \otimes \mathbf{M}^{\frac{1}{2}\beta\rho} - \mathbf{B}^{1} \otimes \mathbf{M}^{\frac{1}{8}(-1+4\beta\kappa\rho)x_2^2 + r} - \mathbf{M}^{1} \otimes \mathbf{B}^{\frac{1}{2}(2\beta^2\kappa-\alpha)x_2 + \frac{4\alpha m - \beta^2}{2x_2}} \\
& - \mathbf{M}^{1} \otimes \mathbf{M}^{\frac{1}{2}\kappa(\beta^2\kappa-\alpha)x_2^2 + 2\alpha\kappa m - r}.
\end{aligned}
$$

The above expression can be simplified to

$$
\mathbf{A}_{\kappa}^{\mathrm{H}} = \frac{1}{8}\mathbf{S}^{1} \otimes \mathbf{M}^{x_2^2} + \mathbf{B}^{1} \otimes \mathbf{Y}_1 + \mathbf{M}^{1} \otimes \mathbf{Y}_2, \tag{9.32}
$$

where the matrices $\mathbf{Y}_1$, $\mathbf{Y}_2$ are given by

$$\begin{aligned}\mathbf{Y}_1 &:= -\frac{1}{2}\beta\rho\mathbf{B}^{x_2} + \frac{1}{8}(1-4\beta\kappa\rho)\mathbf{M}^{x_2^2} - r\mathbf{M}^1,\\ \mathbf{Y}_2 &:= \frac{1}{2}\beta^2\mathbf{S}^1 + \frac{1}{2}(\alpha-2\beta^2\kappa)\mathbf{B}^{x_2} - \frac{1}{2}(4\alpha m-\beta^2)\mathbf{B}^{x_2^{-1}}\\ &\quad + \frac{1}{2}\kappa(\alpha-\beta^2\kappa)\mathbf{M}^{x_2^2} + (r-2\alpha\kappa m)\mathbf{M}^1.\end{aligned}$$

Applying the θ-scheme to discretize in time, we finally obtain the fully discrete scheme to approximate the solution w of the (truncated) weak formulation (9.29)

$$\begin{aligned}&\text{Find } \underline{w}^{m+1} \in \mathbb{R}^N \text{ such that for } m = 0,\dots,M-1\\ &\big(\mathbf{M}+\theta\Delta t\mathbf{A}^{\mathrm{SV}}\big)\underline{w}^{m+1} = \big(\mathbf{M}-(1-\theta)\Delta t\mathbf{A}^{\mathrm{SV}}\big)\underline{w}^m,\\ &\underline{w}_N^0 = \underline{w}_0.\end{aligned} \tag{9.33}$$

Note that (9.33) gives approximations $w_N(t_m, x, y_1, \dots, y_p) = w_N(t_m, z)$ to the function $w(t_m, z)$ of (9.18). To obtain approximations $v_N(t_m, z)$ to $v(t_m, z)$ of (9.11), we have, by (9.17), to set $v_N(t_m, z) = w_N(t_m, z)e^{\eta(z)}$.

Example 9.5.6 We consider a European call with strike $K = 100$ and maturity $T = 0.5$ within the Heston model, for which we set the parameters $\alpha = 2.5$, $\beta = 0.5$, $\rho = -0.5$, $m = 0.025$, $r = 0$. As shown in Fig. 9.2, the option price at maturity converges in the L^∞-norm at the rate $\mathcal{O}(N^{-1})$.

9.6 American Options

American options in stochastic volatility models can be obtained similar to the Black–Scholes case by replacing the Black–Scholes operator $\mathcal{A}^{\mathrm{BS}}$ by the corresponding stochastic volatility operator. The value of an American option is given as

$$V(t,z) := \sup_{\tau\in\mathcal{T}_{t,T}} \mathbb{E}\big[e^{-r(T-t)}g(e^{X_T}) \mid Z_t = z\big],$$

where $\mathcal{T}_{t,T}$ denotes the set of all stopping times for Z. A similar result to Theorem 5.1.1 is not available for general stochastic volatility models due to the possible degeneracy of the coefficients in the SDE. We therefore make the following assumption.

Assumption 9.6.1 Let $v(t,z)$ be a sufficiently smooth solution of the following system of inequalities

$$\begin{aligned}\partial_t v - \mathcal{A}v + rv \geq 0 &\quad \text{in } J\times\mathbb{R}\times G^Y,\\ v(t,x) \geq g(e^x) &\quad \text{in } J\times\mathbb{R}\times G^Y,\\ (\partial_t v - \mathcal{A}v + rv)(g-v) = 0 &\quad \text{in } J\times\mathbb{R}\times G^Y,\\ v(0,z) = g(e^x) &\quad \text{in } \mathbb{R}\times G^Y,\end{aligned} \tag{9.34}$$

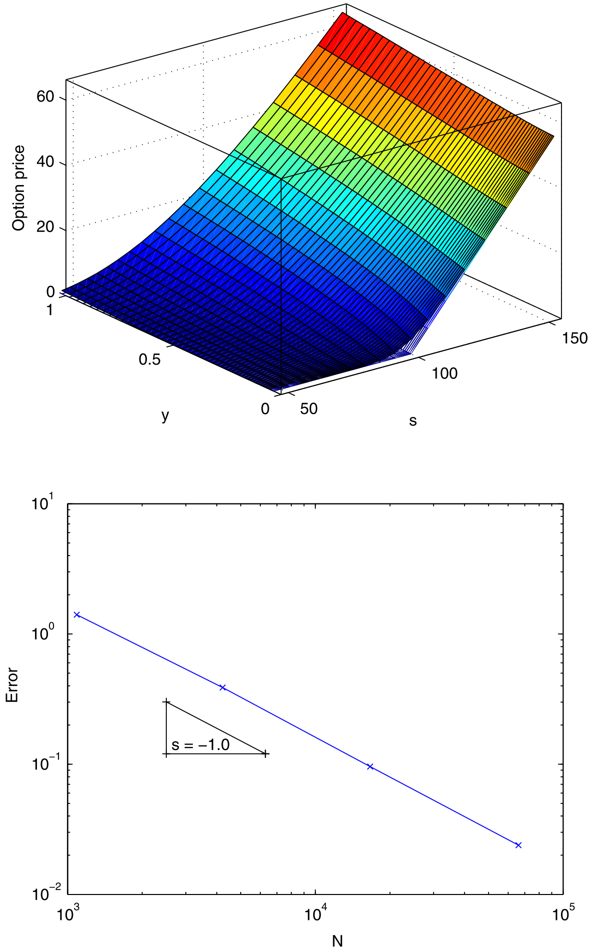


Fig. 9.2 Option price (*top*) and convergence rate (*bottom*) for the Heston model

where $\mathcal{A}$ is the infinitesimal generator of the process Z and G^Y denotes the state space of Y. Then, $V(T-t,z)=v(t,z)$.

We perform the same transformations as described in Sect. 9.3 and obtain the following system of inequalities for $w := (v - v_0)e^{-\eta}$ in the Heston model

$$\begin{aligned} \partial_t w - \mathcal{A}_\kappa^{\mathrm{H}} w + rw \geq f_\kappa^{\mathrm{H}} \quad & \text{in } J \times \mathbb{R} \times \mathbb{R}_+, \\ w(t,x) \geq 0 \quad & \text{in } J \times \mathbb{R} \times \mathbb{R}_+, \\ (\partial_t w - \mathcal{A}_\kappa^{\mathrm{H}} w + rw)w = 0 \quad & \text{in } J \times \mathbb{R} \times \mathbb{R}_+, \\ w(0,x,y) = 0 \quad & \text{in } \mathbb{R} \times \mathbb{R}_+. \end{aligned} \tag{9.35}$$

The set of admissible solutions in the Heston model for the variational form of (9.35) is the convex set $\mathcal{K}_0$ given as

$$\mathcal{K}_0 := \left\{ v \in V \,|\, v \geq 0 \text{ a.e. } z \in \mathbb{R} \times \mathbb{R}_+ \right\},$$

where V is given in (9.23). The variational formulation of (9.35) reads:

$$\begin{aligned}
&\text{Find } w \in L^2(J;V)\cap H^1(J;L^2(G)) \text{ such that } w(t,\cdot)\in\mathcal{K}_0 \text{ and}\\
&(\partial_t w, v-w) + a_\kappa^{\mathrm{H}}(w, v-w) \geq \langle f_\kappa^{\mathrm{H}}, v-w\rangle_{V^*,V},\ \forall v\in\mathcal{K}_0,\ \text{a.e. in } J,\\
&w(0)=0.
\end{aligned} \tag{9.36}$$

Since the bilinear form $a_\kappa^{\mathrm{H}}(\cdot,\cdot)$ is continuous and satisfies a Gårding inequality in V by Theorem 9.3.1, problem (9.36) admits a unique solution for every payoff $g \in L^\infty(\mathbb{R})$ by Theorem B.2.2 of the Appendix B. We localize the problem to a bounded domain as in Sect. 9.4 and obtain the following problem for

$$\mathcal{K}_{0,R} := \left\{v\in\widetilde{V} | v\geq 0 \text{ a.e. } z\in G_R\right\},$$

namely

$$\begin{aligned}
&\text{Find } w \in L^2(J;\widetilde{V})\cap H^1(J;L^2(G_R)) \text{ such that } w(t,\cdot)\in\mathcal{K}_{0,R} \text{ and}\\
&(\partial_t w, v-w) + a^{\mathrm{H}}(w, v-w) \geq \langle f_\kappa^{\mathrm{H}}, v-w\rangle_{\widetilde{V}^*,\widetilde{V}},\ \forall v\in\mathcal{K}_{0,R},\ \text{a.e. in } J,\\
&w(0)=0.
\end{aligned} \tag{9.37}$$

For a general diffusion SV model, the weak localized formulation reads:

$$\begin{aligned}
&\text{Find } w \in L^2(J;\widetilde{V})\cap H^1(J;L^2(G_R)) \text{ such that } w(t,\cdot)\in\mathcal{K}_{0,R} \text{ and}\\
&(\partial_t w, v-w) + a^{\mathrm{SV}}(w, v-w) \geq -a^{\mathrm{SV}}(w_0, v-w),\ \forall v\in\mathcal{K}_{0,R},\ \text{a.e. in } J,\\
&w(0)=0,
\end{aligned} \tag{9.38}$$

where $\mathcal{K}_{0,R}$ and $\widetilde{V}$ depend on the model. Discretization using finite differences or finite elements in space and the backward Euler scheme in time as explained in Sect. 9.5 leads to the following sequence of linear complementary problems for (9.37). Given $\underline{w}_N^0 = \underline{0}$, find $\underline{w}_N^{m+1}\in\mathbb{R}^N$ such that for $m=0,\dots,M-1$,

$$\begin{aligned}
&\mathbf{B}\underline{w}_N^{m+1} \geq \underline{F}^m,\\
&\underline{w}_N^{m+1} \geq \underline{0},\\
&(\underline{w}_N^{m+1})^\top\big(\mathbf{B}\underline{w}_N^{m+1} - \underline{F}^m\big) = 0,
\end{aligned} \tag{9.39}$$

where $\mathbf{B} := \mathbf{M} + k\mathbf{A}_\kappa^{\mathrm{H}}$, $\underline{F}^m := k\underline{f} + \mathbf{M}\underline{u}_N^m$ and $\underline{f}_i = \langle f_\kappa^{\mathrm{H}}, b_i\rangle_{\widetilde{V}^*,\widetilde{V}}$. For a general SV model, we obtain the following system discretizing (9.38). Given $\underline{w}_N^0=\underline{0}$, find $\underline{w}_N^{m+1}\in\mathbb{R}^N$ such that for $m=0,\dots,M-1$,

$$\begin{aligned}
&\mathbf{B}\underline{w}_N^{m+1} \geq \underline{F}^m,\\
&\underline{w}_N^{m+1} \geq \underline{0},\\
&(\underline{w}_N^{m+1})^\top\big(\mathbf{B}\underline{w}_N^{m+1} - \underline{F}^m\big) = 0,
\end{aligned} \tag{9.40}$$

where $\mathbf{B} := \mathbf{M} + k\mathbf{A}^{\mathrm{SV}}$, $\underline{F}^m := k\underline{f} + \mathbf{M}\underline{u}_N^m$ and $\underline{f}_i = -a^{\mathrm{SV}}(w_0, b_i)$.

Example 9.6.2 We consider an American put with strike $K = 100$ and maturity $T = 0.5$ within the Heston model, for which we set the parameters $\alpha = 2.5$, $\beta = 0.5$, $\rho = -0.5$, $m = 0.06$, $r = 0$. Figure 9.3 depicts the option prices as well as the behavior of the free boundary at $t = T$.

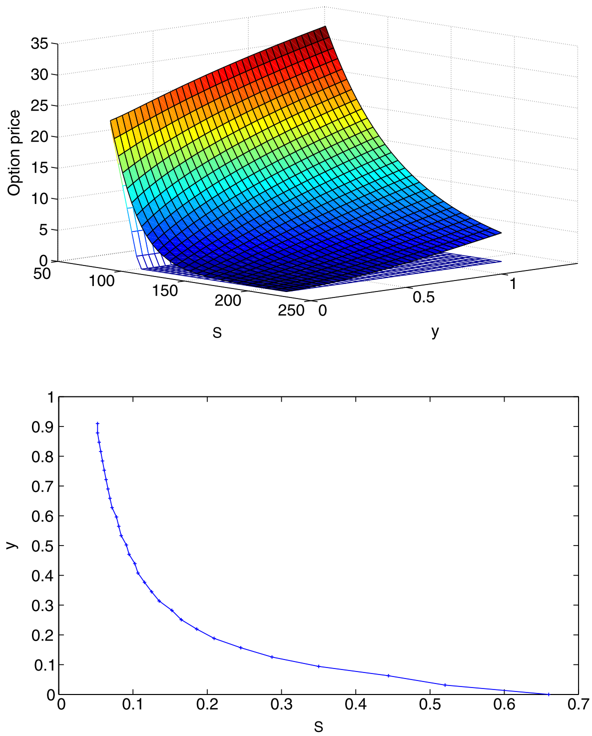


Fig. 9.3 Option price (*top*) and free boundary (*bottom*) for the Heston model

9.7 Further Reading

Background information to stochastic volatility, in general, can be found in Shepard [151] and the references therein. Stochastic volatility (diffusion models) connected to derivative pricing is considered in Fouque et al. [68]. The diffusion SV models we have considered in this chapter can be extended by adding jumps. Bates [13, 14], for example, adds log-normal jumps to the Heston model. The model of Barndorff-Nielsen and Shepard (BNS) [10] describes the volatility as non-Gaussian mean reverting OU process. Introducing SV into an exponential Lévy model via time change was suggested by Carr et al. [37]. The pricing of European and American options via finite difference or finite element methods for diffusion SV models can be found in Achdou and Tchou [2], Ikonen and Toivanen [90] as well as Zvan et al. [166]. Benth and Groth [16] consider option pricing in the BNS model under the minimal entropy EMM. Stochastic volatility models with jumps are described in Chap. 15.

Chapter 10
Lévy Models

One problem with the Black–Scholes model is that empirically observed log returns of risky assets are not normally distributed, but exhibit significant skewness and kurtosis. If large movements in the asset price occur more frequently than in the BS-model of the same variance, the tails of the distribution of X_t, $t > 0$, should be "fatter" than in the Black–Scholes case. Another problem is that observed log-returns occasionally appear to change discontinuously. Empirically, certain price processes with no continuous component have been found to allow for a considerably better fit of observed log returns than the classical BS model. Pricing derivative contracts on such underlyings becomes more involved mathematically and also numerically since partial integro-differential equations must be solved. We consider a class of price processes which can be purely discontinuous and which contains the Wiener process as special case, the class of *Lévy processes*. Lévy processes contain most processes proposed as realistic models for log-returns.

10.1 Lévy Processes

We start by recalling essential definitions and properties of Lévy processes and refer to [18] and [143] for a thorough introduction to Lévy processes.

Definition 10.1.1 An adapted, càdlàg stochastic process $X = \{X_t : t \geq 0\}$ on $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ with values in $\mathbb{R}$ such that $X_0 = 0$ is called a *Lévy process* if it has the following properties:

(i) Independent increments: $X_t - X_s$ is independent of $\mathcal{F}_s$, $0 \leq s < t < \infty$;
(ii) Stationary increments: $X_t - X_s$ has the same distribution as X_{t-s}, $0 \leq s < t < \infty$;
(iii) Stochastically continuous: $\lim_{t\to s} X_t = X_s$, where the limit is taken in probability.

N. Hilber et al., *Computational Methods for Quantitative Finance*, Springer Finance,
DOI 10.1007/978-3-642-35401-4_10,

Note that for an adapted, càdlàg stochastic process $X = \{X_t : t \geq 0\}$ starting in $X_0 = 0$ on $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ (i) and (ii) imply (iii). We can associate to $X = \{X_t : t \in [0, T]\}$ a random measure J_X on $[0, T] \times \mathbb{R}$,

$$J_X(\omega, \cdot) = \sum_{t \in [0,T]}^{\Delta X_t \neq 0} 1_{(t, \Delta X_t)},$$

which is called the *jump measure*. For any measurable subset $B \subset \mathbb{R}$, $J_X([0, t] \times B)$ counts then the number of jumps of X occurring between 0 and t whose amplitude belongs to B. The intensity of J_X is given by the Lévy measure.

Definition 10.1.2 Let X be a Lévy process. The measure ν on $\mathbb{R}$ defined by

$$\nu(B) = \mathbb{E}[\#\{t \in [0, 1] : \Delta X_t \neq 0,\ \Delta X_t \in B\}], \quad B \in \mathcal{B}(\mathbb{R}),$$

is called the *Lévy measure* of X. $\nu(B)$ is the expected number, per unit time, of jumps whose size belongs to B.

The Lévy measure satisfies $\int_{\mathbb{R}} 1 \wedge z^2 \nu(\mathrm{d}z) < \infty$. Using the Lévy–Itô decomposition, we see that every Lévy process is uniquely defined by the drift γ, the variance σ^2 and the Lévy measure ν. The triplet (σ^2, ν, γ) is the *characteristic triplet* of the process X.

Theorem 10.1.3 (Lévy–Itô decomposition) *Let X be a Lévy process and ν its Lévy measure. Then, there exist a γ, σ and a standard Brownian motion W such that*

$$\begin{aligned} X_t &= \gamma t + \sigma W_t + \int_0^t \int_{|x| \geq 1} x J_X(\mathrm{d}s, \mathrm{d}x) \\ &\quad + \lim_{\varepsilon \downarrow 0} \int_0^t \int_{\varepsilon \leq |x| \leq 1} x (J_X(\mathrm{d}s, \mathrm{d}x) - \nu(\mathrm{d}x)\,\mathrm{d}s) \\ &= \gamma t + \sigma W_t + \sum_{0 \leq s \leq t} \Delta X_s 1_{\{|\Delta X_s| \geq 1\}} \\ &\quad + \lim_{\varepsilon \downarrow 0} \int_0^t \int_{\varepsilon \leq |x| \leq 1} x \widetilde{J}_X(\mathrm{d}s, \mathrm{d}x), \end{aligned} \tag{10.1}$$

where J_X is the jump measure of X.

Proof See [143, Chap. 4]. □

The characteristic triplet could also be derived from the Lévy–Khinchine representation.

Theorem 10.1.4 (Lévy–Khinchine representation) *Let X be a Lévy process with characteristic triplet (σ^2, ν, γ). Then, for $t \geq 0$,*

$$\begin{aligned} &\mathbb{E}[e^{i\xi X_t}] = e^{-t\psi(\xi)}, \quad \xi \in \mathbb{R}, \\ &\textit{with } \psi(\xi) = -i\gamma\xi + \frac{1}{2}\sigma^2\xi^2 + \int_{\mathbb{R}} \big(1 - e^{i\xi z} + i\xi z 1_{\{|z| \leq 1\}}\big)\nu(\mathrm{d}z). \end{aligned} \tag{10.2}$$

Proof See [3, Chap. 1.2.4]. □

Note that in (10.2) the integral with respect to the Lévy measure exists since the integrand is bounded outside of any neighborhood of 0 and

$$1 - e^{i\xi z} + i\xi z\, 1_{\{|z|\le 1\}} = \mathcal{O}(z^2) \quad \text{as } |z| \to 0.$$

But there are many other ways to obtain an integrable integrand. We could, for example, replace $1_{\{|z|\le 1\}}$ by any bounded measurable function $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(z) = 1 + \mathcal{O}(|z|)$ as $|z| \to 0$ and $f(z) = \mathcal{O}(1/|z|)$ as $|z| \to \infty$. Different choices of f do not affect σ^2 and ν. But γ depends on the choice of the truncation function. If the Lévy measure satisfies $\int_{|z|\le 1} |z|\nu(\mathrm{d}z) < \infty$, we can use the zero function as f and get

$$\psi(\xi) = -i\gamma_0\xi + \frac{1}{2}\sigma^2\xi^2 + \int_{\mathbb{R}} \left(1 - e^{i\xi z}\right)\nu(\mathrm{d}z), \tag{10.3}$$

with $\gamma_0 \in \mathbb{R}$. We denote this representation by the triplet $(\sigma^2, \nu, \gamma_0)_0$. Furthermore, if $\int_{\mathbb{R}} \nu(\mathrm{d}z) < \infty$, i.e. X is a compound Poisson process, we can rewrite (10.3) as

$$\psi(\xi) = -i\gamma_0\xi + \frac{1}{2}\sigma^2\xi^2 + \lambda \int_{\mathbb{R}} (1 - e^{i\xi z})\nu_0(\mathrm{d}z), \tag{10.4}$$

with $\lambda = \int_{\mathbb{R}} \nu(\mathrm{d}z)$ and $\nu_0 = \nu/\lambda$. We say that X is of *finite activity* with jump intensity λ and jump size distribution ν_0. If the Lévy measure satisfies $\int_{|z|>1} |z|\nu(\mathrm{d}z) < \infty$, then, letting f be a constant function 1, we obtain

$$\psi(\xi) = -i\gamma_c\xi + \frac{1}{2}\sigma^2\xi^2 + \int_{\mathbb{R}} \left(1 - e^{i\xi z} + i\xi z\right)\nu(\mathrm{d}z), \tag{10.5}$$

with triplet $(\sigma^2, \nu, \gamma_c)_c$ where γ_c is called the center of X since $\mathbb{E}[X_t] = \gamma_c t$. We use the representation (10.5) instead of (10.2) throughout this work but omit the subscript c for simplicity. No arbitrage considerations require Lévy processes employed in mathematical finance to be martingales. The following result gives sufficient conditions of the characteristic triplet to ensure this.

Lemma 10.1.5 *Let X be a Lévy process with characteristic triplet (σ^2, ν, γ). Assume, $\int_{|z|>1} |z|\nu(\mathrm{d}z) < \infty$ and $\int_{|z|>1} e^z\nu(\mathrm{d}z) < \infty$. Then, e^X is a martingale with respect to the filtration $\mathbb{F}$ of X if and only if*

$$\frac{\sigma^2}{2} + \gamma + \int_{\mathbb{R}} (e^z - 1 - z)\nu(\mathrm{d}z) = 0.$$

Proof We obtain for $0 \le t < s$ using the independent and stationary increments property

$$\begin{aligned}\mathbb{E}[e^{X_s} \mid \mathcal{F}_t] &= \mathbb{E}[e^{X_t + X_s - X_t} \mid \mathcal{F}_t] = e^{X_t}\mathbb{E}[e^{X_s - X_t}] \\ &= e^{X_t}\mathbb{E}[e^{X_{s-t}}] = e^{X_t}e^{(t-s)\psi(-i)}.\end{aligned}$$

Therefore, setting $\psi(-i) = 0$ and using the Lévy–Khinchine formula (10.5) yields

$$\frac{\sigma^2}{2} + \gamma + \int_{\mathbb{R}} (e^z - 1 - z)\nu(\mathrm{d}z) = 0.$$

Note that $\psi(-i)$ is well-defined due to [143, Theorem 25.17]. □

10.2 Lévy Models

Financial models with jumps fall into two categories: *Jump–diffusion* models have a nonzero Gaussian component and a jump part which is a compound Poisson process with finitely many jumps in every time interval. On the other hand, *infinite activity* models have an infinite number of jumps in every interval of positive measure. A Brownian motion component is not necessary for infinite activity models since the dynamics of the jumps is already rich enough to generate nontrivial small-time behavior. If there is no Brownian motion component, the models are called *pure jump* models.

10.2.1 Jump–Diffusion Models

A Lévy process of jump–diffusion has the following form

$$X_t = \gamma_0 t + \sigma W_t + \sum_{i=1}^{N_t} Y_i \tag{10.6}$$

where N is a Poisson process with intensity λ counting the jumps of X, and Y_i are the jump sizes which are modeled by i.i.d. random variables with distribution ν_0. The Lévy measure is given by $\nu = \lambda\nu_0$. To define the parametric model completely, we must specify the distribution ν_0 of the jump sizes.

In the *Merton model* [125], jumps in the log-price X are assumed to have a Gaussian distribution, i.e. $Y_i \sim N(\mu, \delta^2)$, and therefore,

$$\nu_0(\mathrm{d}z) = \frac{1}{\sqrt{2\pi\delta^2}}\, e^{-(z-\mu)^2/(2\delta^2)}\,\mathrm{d}z. \tag{10.7}$$

In the *Kou model* [107], the distribution of jump sizes is an asymmetric exponential with a density of the form

$$\nu_0(\mathrm{d}z) = \big(p\beta_+ e^{-\beta_+ z} 1_{\{z>0\}} + (1-p)\beta_- e^{-\beta_-|z|} 1_{\{z<0\}}\big)\,\mathrm{d}z, \tag{10.8}$$

with $\beta_+, \beta_- > 0$ governing the decay of the tails for the distribution of the positive and negative jump sizes and $p \in [0, 1]$ representing the probability of an upward jump. The probability distribution of returns of this model has semi-heavy tails.

10.2.2 Pure Jump Models

A popular class of processes is obtained by *subordination* of a Brownian motion with drift.

Definition 10.2.1 A Lévy process X is called a *subordinator* if the sample paths of X are a.s. nondecreasing, i.e. $t_1 \geq t_2 \Rightarrow X_{t_1} \geq X_{t_2}$ a.s.

Since $X_0 = 0$, it immediately follows by the definition that $X_t \geq 0$ a.s. $\forall t \geq 0$. The characteristic triplet of a subordinator has the following properties, see, e.g. [40, Proposition 3.10].

Lemma 10.2.2 *Let G be a subordinator. Then the characteristic triplet (σ^2, ν, γ) of G satisfies $\sigma^2 = 0$, $\gamma \geq 0$ and $\nu((-\infty, 0]) = 0$, $\int_{\mathbb{R}_+} 1 \wedge z\, \nu(\mathrm{d}z) < \infty$.*

By Lemma 10.2.2 and (10.3), the characteristic exponent ψ of a subordinator is given by $\psi(\xi) = -i\gamma\xi + \int_{\mathbb{R}_+} (1 - e^{i\xi z})\nu(\mathrm{d}z)$.

Now let G be a subordinator and W a standard Brownian motion. Then, we can construct a Lévy process X by time changing W

$$X_t = \sigma W_{G_t} + \theta G_t, \quad \sigma > 0,\ \theta \in \mathbb{R},\ t \in [0, T].$$

As an example we use as the subordinator a gamma process to obtain a *variance gamma process* [118]. We consider a gamma process G with Lévy density $k_G(s) = e^{-\frac{s}{\vartheta}}(\vartheta s)^{-1} 1_{\{s>0\}}$. Then, using [143, Theorem 30.1], the Lévy measure of X is given for $B \in \mathcal{B}(\mathbb{R})$ by

$$\begin{aligned} \nu(B) &= \int_B \int_0^\infty \frac{1}{\sqrt{2\pi s\sigma^2}} e^{-\frac{(z-\theta s)^2}{2s\sigma^2}} \frac{1}{\vartheta s} e^{-\frac{s}{\vartheta}}\, \mathrm{d}s\, \mathrm{d}z \\ &= \frac{1}{\vartheta\sqrt{2\pi\sigma^2}} \int_B e^{\theta s/\sigma^2} \int_0^\infty s^{-\frac{1}{2}-1} e^{-\frac{z^2}{2\sigma^2}\frac{1}{s} - (\frac{\theta^2}{2\sigma^2} + \frac{1}{\vartheta})s}\, \mathrm{d}s\, \mathrm{d}z. \end{aligned}$$

Using [74, Formula 3.471 (9)] to integrate the second integral, we obtain the Lévy measure

$$\nu(\mathrm{d}z) = \left(c\frac{e^{-\beta_+|z|}}{|z|} 1_{\{z>0\}} + c\frac{e^{-\beta_-|z|}}{|z|} 1_{\{z<0\}} \right) \mathrm{d}z, \tag{10.9}$$

with $c = 1/\vartheta$, $\beta_+ = c\,2/(\sqrt{2\sigma^2/\vartheta + \theta^2} + \theta)$, $\beta_- = c\,2/(\sqrt{2\sigma^2/\vartheta + \theta^2} - \theta)$.

The variance gamma process is a special case of the *tempered stable process* (for $c = c_+ = c_-$ also called *CGMY* process in [36] or *KoBoL* in [23]) which has a Lévy density of the form

$$\nu(\mathrm{d}z) = \left(c_+\frac{e^{-\beta_+|z|}}{|z|^{1+\alpha}} 1_{\{z>0\}} + c_-\frac{e^{-\beta_-|z|}}{|z|^{1+\alpha}} 1_{\{z<0\}} \right) \mathrm{d}z, \tag{10.10}$$

for $c_+, c_-, \beta_+, \beta_- > 0$ and $0 \leq \alpha < 2$.

The *normal inverse Gaussian process* (NIG) was proposed in [8] and has the Lévy density

$$\nu(\mathrm{d}z) = \frac{\delta\alpha}{\pi}\frac{e^{\beta z}K_1(\alpha|z|)}{|z|}\,\mathrm{d}z,$$

where K_1 denotes the modified Bessel function of the third kind with index 1 and $\alpha > 0$, $-\alpha < \beta < \alpha$, $\delta > 0$. The NIG model is a special case of the *generalized hyperbolic model* [62].

10.2.3 Admissible Market Models

We make the following assumptions on our market models.

Assumption 10.2.3 Let X be a Lévy process with characteristic triplet (σ^2, ν, γ) and Lévy density $k(z)$ where $\nu(\mathrm{d}z) = k(z)\,\mathrm{d}z$.

(i) There are constants $\beta_- > 0$, $\beta_+ > 1$ and $C > 0$ such that

$$k(z) \le C\begin{cases} e^{-\beta_-|z|}, & z < -1, \\ e^{-\beta_+ z}, & z > 1. \end{cases} \tag{10.11}$$

(ii) Furthermore, there exist constants $0 < \alpha < 2$ and $C_+ > 0$ such that

$$k(z) \le C_+\frac{1}{|z|^{1+\alpha}}, \quad 0 < |z| < 1. \tag{10.12}$$

(iii) If $\sigma = 0$, we assume additionally that there is a $C_- > 0$ such that

$$\frac{1}{2}(k(z) + k(-z)) \ge C_-\frac{1}{|z|^{1+\alpha}}, \quad 0 < |z| < 1. \tag{10.13}$$

Note that due to the semi-heavy tails (10.11) the Lévy measure satisfies $\int_{|z|>1}|z|\nu(\mathrm{d}z) < \infty$ and $\int_{|z|>1}e^z\nu(\mathrm{d}z) < \infty$. All Lévy processes described before satisfy these assumptions except for the variance gamma model. Here, $\alpha = 0$ in (10.13) which is not allowed. Nevertheless, we will show in the numerical example that the finite element discretization still converges to the option value with optimal rate.

10.3 Pricing Equation

As in the Black–Scholes case, we assume the risk-neutral dynamics of the underlying asset price is given by

$$S_t = S_0 e^{rt+X_t},$$

where X is a Lévy process with characteristic triplet (σ^2, ν, γ) under a non-unique EMM. As shown in Lemma 10.1.5, the martingale condition implies

$$\gamma = -\frac{\sigma^2}{2} - \int_{\mathbb{R}} (e^z - 1 - z)\nu(\mathrm{d}z).$$

We again want to compute the value of the option in log-price with payoff g which is the conditional expectation

$$v(t,x) = \mathbb{E}[e^{-r(T-t)} g(e^{rT+X_T}) \mid X_t = x].$$

We show that $v(t,x)$ is a solution of a deterministic *partial integro-differential equation* (PIDE). To prove the Feynman–Kac theorem for Lévy processes, we need a generalization of Proposition 4.1.1.

Proposition 10.3.1 *Let X be a Lévy process with characteristic triplet (σ^2, ν, γ) where the Lévy measure satisfies* (10.11). *Denote by $\mathcal{A}$ the integro-differential operator*

$$(\mathcal{A}f)(x) = \frac{1}{2}\sigma^2 \partial_{xx} f(x) + \gamma \partial_x f(x) + \int_{\mathbb{R}} (f(x+z) - f(x) - z\partial_x f(x))\nu(\mathrm{d}z), \tag{10.14}$$

for functions $f \in C^2(\mathbb{R})$ with bounded derivatives. Then, the process $M_t := f(X_t) - \int_0^t (\mathcal{A}f)(X_s)\,\mathrm{d}s$ is a martingale with respect to the filtration of X.

Proof We need the Itô formula for Lévy processes which is given in differential form by

$$\mathrm{d}f(X_t) = \partial_x f(X_{t-})\,\mathrm{d}X_t + \frac{\sigma^2}{2}\partial_{xx} f(X_t)\,\mathrm{d}t + f(X_t) - f(X_{t-}) - \Delta X_t \partial_x f(X_{t-}).$$

Using the Lévy–Itô decomposition

$$\mathrm{d}X_t = \gamma\,\mathrm{d}t + \sigma\,\mathrm{d}W_t + \int_{\mathbb{R}\setminus\{0\}} z\widetilde{J}_X(\mathrm{d}t, \mathrm{d}z),$$

we obtain

$$\begin{aligned}
\mathrm{d}f(X_t) &= \partial_x f(X_{t-})\gamma\,\mathrm{d}t + \sigma\partial_x f(X_{t-})\,\mathrm{d}W_t \\
&\qquad + \partial_x f(X_{t-})\int_{\mathbb{R}\setminus\{0\}} z\widetilde{J}_X(\mathrm{d}t, \mathrm{d}z) + \frac{\sigma^2}{2}\partial_{xx} f(X_t)\,\mathrm{d}t \\
&\qquad + \int_{\mathbb{R}\setminus\{0\}} f(X_{t-}+z) - f(X_{t-}) - z\partial_x f(X_{t-}) J_X(\mathrm{d}t, \mathrm{d}z) \\
&= (\mathcal{A}f)(X_{t-})\,\mathrm{d}t + \sigma(X_t)\partial_x f(X_{t-})\,\mathrm{d}W_t \\
&\qquad + \int_{\mathbb{R}\setminus\{0\}} (f(X_{t-}+z) - f(X_{t-}))\widetilde{J}_X(\mathrm{d}t, \mathrm{d}z).
\end{aligned}$$

As already showed in Proposition 4.1.1, the process $\int_0^t \sigma(X_s)\partial_x f(X_s)\,\mathrm{d}W_s$ is a martingale. Therefore, it remains to show that $\int_0^t \int_{\mathbb{R}\setminus\{0\}} (f(X_{s-}+z) - f(X_{s-}))\widetilde{J}_X(\mathrm{d}s, \mathrm{d}z)$

is a martingale. Using a generalization of Proposition 1.2.7 for Lévy processes (see, e.g. [40, Proposition 8.8]) it is sufficient to show that $\mathbb{E}[\int_0^t \int_{\mathbb{R}} |f(X_{t-}+z) - f(X_{t-})|^2 \nu(\mathrm{d}z)] < \infty$. We have

$$\mathbb{E}\Big[\int_0^t \int_{\mathbb{R}} |f(X_{t-}+z) - f(X_{t-})|^2 \nu(\mathrm{d}z)\Big] \leq T \sup_{x\in\mathbb{R}} |\partial_x f(x)|^2 \int_{\mathbb{R}} |z|^2 \nu(\mathrm{d}z) < \infty,$$

since $f \in C^2(\mathbb{R})$ has bounded derivatives and ν satisfies (10.11). □

Remark 10.3.2 Using the Lévy–Khinchine representation (see Theorem 10.1.4), we derive from (10.14) the following formula for the action of the operator $\mathcal{A}$ on oscillating exponents

$$\begin{aligned}(-\mathcal{A})e^{i\xi x} &= \frac{1}{2}\sigma^2\xi^2 e^{i\xi x} - \gamma i\xi e^{i\xi x} - \int_{\mathbb{R}} \big(e^{i\xi(x+z)} - e^{i\xi x} - iz\xi e^{i\xi x}\big)\nu(\mathrm{d}z)\\ &= \psi(\xi)e^{i\xi x},\end{aligned} \tag{10.15}$$

with $\xi \in \mathbb{R}$. For $f \in \mathcal{S}(\mathbb{R})$, the space of functions in $C^\infty(\mathbb{R})$ vanishing at infinity faster than any negative power of $\sqrt{1+|x|^2}$, we can write

$$f(x) = \int_{\mathbb{R}} e^{i\xi x}\widehat{f}(\xi)\,\mathrm{d}\xi, \tag{10.16}$$

where $\widehat{f}(\xi) := (2\pi)^{-1}\int_{\mathbb{R}} e^{-i\xi x} f(x)\mathrm{d}x$ is the Fourier transform of f. By applying (10.15) to the representation (10.16), using Fubini's theorem and the convergence theorem of Lebesgue, we obtain

$$(-\mathcal{A}f)(x) = \int_{\mathbb{R}} (-\mathcal{A}e^{i\xi x})\widehat{f}(\xi)\,\mathrm{d}\xi = \int_{\mathbb{R}} \psi(\xi)e^{i\xi x}\widehat{f}(\xi)\,\mathrm{d}\xi. \tag{10.17}$$

Thus, $-\mathcal{A}$ is a *pseudo-differential operator* (PDO) with symbol ψ.

Repeating the arguments which lead to Theorem 4.1.4 yields

Theorem 10.3.3 *Let $v \in C^{1,2}(J\times\mathbb{R}) \cap C^0(\overline{J}\times\mathbb{R})$ with bounded derivatives in x be a solution of*

$$\partial_t v + \mathcal{A}v - rv = 0 \quad \text{in } J\times\mathbb{R}, \qquad v(T,x) = g(e^x) \quad \text{in } \mathbb{R}, \tag{10.18}$$

where $\mathcal{A}$ as in (10.14) *with drift $r+\gamma$. Then, $v(t,x)$ can also be represented as*

$$v(t,x) = \mathbb{E}[e^{-r(T-t)}g(e^{rT+X_T}) \mid X_t = x].$$

We again change to *time-to-maturity* $t \to T-t$, to obtain a forward parabolic problem. Thus, by setting $u(t,s) =: e^{rt}v(T-t, x-(\gamma+r)t)$, we remove the drift γ and the interest rate r. Then, u satisfies

$$\partial_t u - \mathcal{A}^{\mathrm{J}} u = 0 \quad \text{in } (0,T)\times\mathbb{R}, \tag{10.19}$$

with the initial condition $u(0,x) = g(e^x)$ in $\mathbb{R}$ and

$$(\mathcal{A}^{\mathrm{J}} f)(x) = \frac{1}{2}\sigma^2\partial_{xx} f(x) + \int_{\mathbb{R}} (f(x+z) - f(x) - z\partial_x f(x))\nu(\mathrm{d}z). \tag{10.20}$$

The *removal of drift* is important for stability of the numerical algorithm. For notational simplicity, we also removed the interest rate r.

Remark 10.3.4 If the Lévy measure satisfies $\int_{|z|\le 1} |z|\nu(\mathrm{d}z) < \infty$, we remove the drift γ_0 which is given, due to the martingale condition, by

$$\gamma_0 = -\frac{\sigma^2}{2} - \int_{\mathbb{R}} (e^z - 1)\nu(\mathrm{d}z),$$

and obtain the generator

$$(\mathcal{A}^{\mathrm{J}} f)(x) = \frac{1}{2}\sigma^2 \partial_{xx} f(x) + \int_{\mathbb{R}} (f(x+z) - f(x))\nu(\mathrm{d}z).$$

10.4 Variational Formulation

For the variational formulation, we need Sobolev spaces of fractional order, i.e. $H^s(\mathbb{R})$, $s > 0$. For any integer m, we defined the H^m-norm using the derivative ∂^m (see Sect. 3.1). Equivalently, we can also define the norm using the Fourier transformation. In particular, for $\partial^m u \in L^2(\mathbb{R})$ we have, by using Plancherel's theorem,

$$\int_{\mathbb{R}} |\partial^m u(x)|^2 \,\mathrm{d}x = 2\pi \int_{\mathbb{R}} \left|\widehat{\partial^m u}(\xi)\right|^2 \mathrm{d}\xi = 2\pi \int_{\mathbb{R}} |\xi|^{2m} |\widehat{u}(\xi)|^2 \,\mathrm{d}\xi,$$

where $\widehat{u}(\xi) = (2\pi)^{-1} \int e^{-i\xi z} u(z)\,\mathrm{d}z$ denotes the Fourier transform of u. Therefore, we can define an equivalent H^s-norm for $s > 0$ via

$$\|u\|^2_{H^s(\mathbb{R})} := \int_{\mathbb{R}} (1 + |\xi|)^{2s} |\widehat{u}(\xi)|^2 \,\mathrm{d}\xi,$$

and set $H^s(\mathbb{R}) := \{u \in L^2(\mathbb{R}) : \|u\|_{H^s(\mathbb{R})} < \infty\}$. We also set

$$\rho = \begin{cases} 1 & \text{if } \sigma > 0, \\ \alpha/2 & \text{if } \sigma = 0, \end{cases}$$

with α given in Assumption 10.2.3. The variational formulation of the PIDE (10.19) reads

$$\begin{aligned} &\text{Find } u \in L^2(J; H^\rho(\mathbb{R})) \cap H^1(J; L^2(\mathbb{R})) \text{ such that} \\ &(\partial_t u, v) + a^{\mathrm{J}}(u, v) = 0, \ \forall v \in H^\rho(\mathbb{R}), \quad \text{a.e. in } J, \\ &u(0) = u_0, \end{aligned} \tag{10.21}$$

where $u_0(x) := g(e^x)$ and the bilinear form $a^{\mathrm{J}}(\cdot,\cdot) : H^\rho(\mathbb{R}) \times H^\rho(\mathbb{R}) \to \mathbb{R}$ is given by

$$a^{\mathrm{J}}(\varphi, \phi) := \frac{1}{2}\sigma^2(\varphi', \phi') - \int_{\mathbb{R}} \int_{\mathbb{R}} (\varphi(x+z) - \varphi(x) - z\varphi'(x))\phi(x)\nu(\mathrm{d}z)\,\mathrm{d}x. \tag{10.22}$$

Remark 10.4.1 Let $\varphi, \phi \in C_0^\infty(\mathbb{R})$. Then, by the definition of the bilinear form $a^{\mathrm{J}}(\cdot,\cdot)$ and (10.17), we find

$$\begin{aligned} a^{\mathrm{J}}(\varphi, \phi) &= \int_{\mathbb{R}} (-\mathcal{A}^{\mathbf{J}}\varphi)(x)\phi(x)\,\mathrm{d}x = \int_{\mathbb{R}}\int_{\mathbb{R}} \psi(\xi)e^{ix\xi}\widehat{\varphi}(\xi)\,\mathrm{d}\xi\,\phi(x)\,\mathrm{d}x \\ &= \int_{\mathbb{R}} \psi(\xi)\widehat{\varphi}(\xi)\int_{\mathbb{R}} e^{ix\xi}\phi(x)\,\mathrm{d}x\,\mathrm{d}\xi = \int_{\mathbb{R}} \psi(\xi)\widehat{\varphi}(\xi)\overline{\widehat{\phi}(\xi)}\,\mathrm{d}\xi. \end{aligned}$$

We want to show that $a^{\mathrm{J}}(\cdot,\cdot)$ is continuous (3.8) and satisfies a Gårding inequality (3.9) on $\mathcal{V} = H^\rho(\mathbb{R})$. By Remark 10.4.1, it is sufficient to study the characteristic exponent ψ. We do this in the following lemma.

Lemma 10.4.2 *Let X be a Lévy process with characteristic triplet $(\sigma^2, \nu, 0)$ where the Lévy measure satisfies (10.12), (10.13). Then, there exist constants $C_i > 0$, $i = 1, 2, 3,$ for all $\xi \in \mathbb{R}$ holds*

$$\Re\psi(\xi) \geq C_1|\xi|^{2\rho}, \quad |\psi(\xi)| \leq C_2|\xi|^{2\rho} + C_3.$$

Proof Consider first $\sigma = 0$ and denote $k^{\mathrm{sym}} = \frac{1}{2}(k(z) + k(-z))$. Using (10.13) and $1 - \cos(z) = 2\sin(z/2)^2 \geq C_1 z^2$ for $|z| \leq 1$, we obtain

$$\begin{aligned} \Re\psi(\xi) &= \int_{\mathbb{R}} (1 - \cos(\xi z))k^{\mathrm{sym}}(z)\,\mathrm{d}z \geq C_1\int_0^{1/|\xi|} \xi^2 z^2 k^{\mathrm{sym}}(z)\,\mathrm{d}z \\ &\geq C_1 C_- \int_0^{1/|\xi|} \xi^2 z^{1-\alpha}\,\mathrm{d}z = \frac{C_1 C_-}{2-\alpha}|\xi|^\alpha. \end{aligned}$$

For an upper bound, we first consider $\alpha < 1$. Then, with $\int_{|z|\leq 1} |z|\nu(\mathrm{d}z) < \infty$ and (10.12),

$$\begin{aligned} |\psi(\xi)| &= \left|\int_{\mathbb{R}} \left(1 - e^{i\xi z}\right)\nu(\mathrm{d}z)\right| \leq \int_{-1/|\xi|}^{1/|\xi|} |e^{i\xi z} - 1|\nu(\mathrm{d}z) + C_2 \\ &\leq 2\int_{-1/|\xi|}^{1/|\xi|} |\xi z|\nu(\mathrm{d}z) + C_2 \\ &\leq 4C_+\int_0^{1/|\xi|} |\xi||z|^{-\alpha}\,\mathrm{d}z + C_2 \leq \frac{4C_+}{1-\alpha}|\xi|^\alpha + C_2. \end{aligned}$$

Similarly, we obtain for $1 \leq \alpha < 2$ that

$$\begin{aligned} |\psi(\xi)| &= \left|\int_{\mathbb{R}} \left(1 - e^{i\xi z} + i\xi z\right)\nu(\mathrm{d}z)\right| \leq \int_{-1/|\xi|}^{1/|\xi|} |e^{i\xi z} - 1 - i\xi z|\nu(\mathrm{d}z) + C_3 + C_4|\xi| \\ &\leq \int_{-1/|\xi|}^{1/|\xi|} |\xi z|^2\nu(\mathrm{d}z) + C_3 + C_4|\xi| \leq 2C_+\int_0^{1/|\xi|} |\xi|^2|z|^{1-\alpha}\,\mathrm{d}z + C_3 + C_4|\xi| \\ &\leq C_5|\xi|^\alpha + C_6. \end{aligned}$$

For $\sigma > 0$, we immediately have

$$\Re\psi(\xi) = \frac{1}{2}\sigma^2\xi^2 + \int_{\mathbb{R}} (1 - \cos(\xi z))\nu(\mathrm{d}z) \geq \frac{1}{2}\sigma^2\xi^2,$$
$$|\psi(\xi)| \leq \frac{1}{2}\sigma^2\xi^2 + \int_{\mathbb{R}} |e^{i\xi z} - 1 - i\xi z|\nu(\mathrm{d}z) \leq C_7\xi^2 + C_8. \qquad \square$$

Lemma 10.4.2 shows that ψ satisfies the so-called sector condition $|\Im\psi(\xi)| \leq C\Re\psi(\xi)$, for all $\xi \in \mathbb{R}$. Using this, we can show

Theorem 10.4.3 *Let* X *be a Lévy process with characteristic triplet* $(\sigma^2, \nu, 0)$ *where the Lévy measure satisfies* (10.12), (10.13). *Then, there exist constants* $C_i > 0$, $i = 1, 2, 3$, *such that for all* $\varphi, \phi \in H^\rho(\mathbb{R})$ *one has*

$$|a^{\mathrm{J}}(\varphi, \phi)| \leq C_1\|\varphi\|_{H^\rho(\mathbb{R})}\|\phi\|_{H^\rho(\mathbb{R})}, \quad a^{\mathrm{J}}(\varphi, \varphi) \geq C_2\|\varphi\|^2_{H^\rho(\mathbb{R})} - C_3\|\varphi\|^2_{L^2(\mathbb{R})}.$$

Proof Using Lemma 10.4.2, there exist positive constants $C_1, C_2 > 0$ such that

$$\Re\psi(\xi) \geq C_1|\xi|^{2\rho}, \quad |\psi(\xi)| \leq C_2\big(|\xi|^{2\rho} + 1\big).$$

Then, we have, using Hölder's inequality and Remark 10.4.1,

$$\begin{aligned}
|a^{\mathrm{J}}(\varphi, \phi)| &= \left|\int_{\mathbb{R}} \psi(\xi)\widehat{\varphi}(\xi)\overline{\widehat{\phi}(\xi)}\,\mathrm{d}\xi\right| \\
&\leq C_2\int_{\mathbb{R}} (1 + |\xi|^{2\rho})|\widehat{\varphi}(\xi)\overline{\widehat{\phi}(\xi)}|\,\mathrm{d}\xi \\
&\leq \widetilde{C}_2\int_{\mathbb{R}} (1 + |\xi|)^{2\rho}|\widehat{\varphi}(\xi)\overline{\widehat{\phi}(\xi)}|\,\mathrm{d}\xi \\
&\leq \widetilde{C}_2\Big(\int_{\mathbb{R}} (1 + |\xi|)^{2\rho}|\widehat{\varphi}(\xi)|^2\,\mathrm{d}\xi\Big)^{1/2}\Big(\int_{\mathbb{R}} (1 + |\xi|)^{2\rho}|\widehat{\phi}(\xi)|^2\,\mathrm{d}\xi\Big)^{1/2} \\
&= \widetilde{C}_2\|\varphi\|_{H^\rho(\mathbb{R})}\|\phi\|_{H^\rho(\mathbb{R})},
\end{aligned}$$

where we used the fact that there exists a $c > 0$ such that

$$0 < c \leq \frac{(1 + |\xi|)^{2\rho}}{1 + |\xi|^{2\rho}} \leq \frac{1}{c} < \infty, \quad \forall\xi \in \mathbb{R}.$$

Furthermore, to prove the Gårding inequality, one finds

$$a^{\mathrm{J}}(\varphi, \varphi) = \int_{\mathbb{R}} \Re\psi(\xi)|\widehat{\varphi}(\xi)|^2\,\mathrm{d}\xi = \int_{\mathbb{R}} (C_1 + \Re\psi(\xi))|\widehat{\varphi}(\xi)|^2\,\mathrm{d}\xi - C_1\int_{\mathbb{R}} |\widehat{\varphi}(\xi)|^2\,\mathrm{d}\xi,$$

and

$$\begin{aligned}
\int_{\mathbb{R}} (C_1 + \Re\psi(\xi))|\widehat{\varphi}(\xi)|^2\,\mathrm{d}\xi &\geq C_1\int_{\mathbb{R}} (1 + |\xi|^{2\rho})|\widehat{\varphi}(\xi)|^2\,\mathrm{d}\xi \\
&\geq \widetilde{C}_1\int_{\mathbb{R}} (1 + |\xi|)^{2\rho}|\widehat{\varphi}(\xi)|^2\mathrm{d}\xi. \qquad \square
\end{aligned}$$

10.5 Localization

As in the Black–Scholes case, the unbounded range $\mathbb{R}$ of the log price $x = \log s$ is truncated to a bounded computational domain $G = (-R, R)$, $R > 0$. Let $\tau_G = \inf\{t \geq 0 \mid X_t \in G^c\}$ be the first hitting time of the complement set $G^c = \mathbb{R}\backslash G$ by X. Note that for diffusion models, the paths of X are continuous and therefore

$$\inf\{t \geq 0 \mid X_t \in G^c\} = \inf\{t \geq 0 \mid X_t = \pm R\}.$$

This does not hold for jump models where the process can jump out of the domain G instantaneously. As before we denote the value of the knock-out barrier option in log-price by v_R.

Theorem 10.5.1 *Suppose the payoff function $g : \mathbb{R} \to \mathbb{R}_{\geq 0}$ satisfies* (4.10). *Let X be a Lévy process with characteristic triplet $(\sigma^2, \nu, 0)$ where the Lévy measure satisfies* (10.11) *with $\beta_+, \beta_- > q$ and q as in* (4.10). *Then, there exist $C(T, \sigma, \nu), \gamma_1, \gamma_2 > 0$, such that*

$$|v(t,x) - v_R(t,x)| \leq C(T,\sigma,\nu)\, e^{-\gamma_1 R + \gamma_2 |x|}.$$

Proof Let $M_T = \sup_{\tau \in [t,T]} |X_\tau|$. Then, with (4.10)

$$|v(t,x) - v_R(t,x)| \leq \mathbb{E}\big[g(e^{X_T}) 1_{\{T \geq \tau_G\}} \mid X_t = x\big] \leq C \mathbb{E}\big[e^{q M_T} 1_{\{M_T > R\}} \mid X_t = x\big].$$

Using [143, Theorem 25.18], similar to the proof of Theorem 4.3.1, it suffices to show that

$$\begin{aligned}
\mathbb{E}\big[e^{q|X_T|} 1_{\{|X_T| > R\}} \mid X_t = x\big] &= \int_{\mathbb{R}} e^{q|z+x|} 1_{\{|z+x|>R\}} p_{T-t}(z)\, \mathrm{d}z \\
&\leq e^{q|x|} \int_{\mathbb{R}} e^{-(\eta - q)(R - |x|)} e^{\eta |z|} p_{T-t}(z)\, \mathrm{d}z \\
&\leq e^{-\gamma_1 R + \gamma_2 |x|} \int_{\mathbb{R}} e^{\eta|z|} p_{T-t}(z)\, \mathrm{d}z,
\end{aligned}$$

with $\gamma_1 = \eta - q$ and $\gamma_2 = \gamma_1 + q$. Using [143, Theorem 25.3], we obtain the equivalence

$$\int_{\mathbb{R}} e^{\eta|z|} p_{T-t}(z)\, \mathrm{d}z < \infty \quad \Leftrightarrow \quad \int_{|z|>1} e^{\eta|z|} \nu(\mathrm{d}z) < \infty,$$

which follows from (10.11). □

We define the space

$$\widetilde{H}^s(G) := \{u|_G : u \in H^s(\mathbb{R}),\ u|_{\mathbb{R}\backslash G} = 0\}.$$

For $s + 1/2 \notin \mathbb{N}$, the space $\widetilde{H}^s(G)$ coincides with $H_0^s(G)$, the closure of $C_0^\infty(G)$ with respect to the norm of $H^s(G)$. For any function u with support in G, we denote by $\widetilde{u}$ its extension by zero to all of $\mathbb{R}$. Then, the bilinear form on the bounded domain G is given by $a_R^{\mathrm{J}}(u,v) = a^{\mathrm{J}}(\widetilde{u}, \widetilde{v})$, and we have continuity and a Gårding inequality on $\widetilde{H}^\rho(G)$. Now we can restate the problem (10.21) on the bounded domain:

$$
\begin{aligned}
&\text{Find } u_R \in L^2(J;\widetilde{H}^\rho(G)) \cap H^1(J;L^2(G)) \text{ such that}\\
&(\partial_t u_R, v) + a_R^{\mathrm{J}}(u_R, v) = 0,\ \forall v \in \widetilde{H}^\rho(G),\ \text{a.e. in } J, \qquad (10.23)\\
&u_R(0) = u_0|_G.
\end{aligned}
$$

By Theorem 10.4.3 and the inclusion $\widetilde{H}^\rho(G) \subset H^\rho(\mathbb{R})$, we conclude that $a_R^J(\cdot,\cdot)$ satisfies a Gårding inequality and is continuous. Therefore, the problem (10.23) is well-posed by Theorem 3.2.2, i.e. there exists a unique solution u_R of (10.23).

10.6 Discretization

Since the diffusion part has been already discussed in Sect. 4.4, we set $\sigma = 0$ and only consider pure jump models. The main problem is the singularity of the Lévy measure at $z = 0$. Therefore, we integrate the jump generator by parts twice, to obtain for $f \in \widetilde{H}^2(G)$ with $G = (-r, r)$.

$$
\begin{aligned}
&\int_0^\infty (f(x+z) - f(x) - zf'(x))k(z)\,\mathrm{d}z\\
&\quad = \underbrace{(f(x+z) - f(x) - zf'(x))k^{(-1)}(z)}_{=0}|_{z=0}^{z=\infty}\\
&\qquad - \int_0^\infty (f'(x+z) - f'(x))k^{(-1)}(z)\,\mathrm{d}z\\
&\quad = -\underbrace{(f'(x+z) - f'(x))k^{(-2)}(z)}_{=0}|_{z=0}^{z=\infty} + \int_0^\infty f''(x+z)k^{(-2)}(z)\,\mathrm{d}z,
\end{aligned}
$$

where $k^{(-i)}(z)$ is the ith antiderivative of k vanishing at $\pm\infty$, i.e.

$$
k^{(-i)}(z) = \begin{cases} \int_{-\infty}^z k^{(-i+1)}(x)\,\mathrm{d}x & \text{if } z < 0,\\ -\int_z^\infty k^{(-i+1)}(x)\,\mathrm{d}x & \text{if } z > 0. \end{cases}
$$

Therefore, we obtain

$$
(\mathcal{A}^{\mathrm{J}} f)(x) = \int_{\mathbb{R}} f''(x+z)k^{(-2)}(z)\,\mathrm{d}z. \tag{10.24}
$$

Similarly, we can write the bilinear form for $\varphi, \phi \in H_0^1(G)$ as

$$
a^{\mathrm{J}}(\varphi, \phi) := \int_G \int_G \varphi'(y)\phi'(x)k^{(-2)}(y-x)\,\mathrm{d}y\,\mathrm{d}x. \tag{10.25}
$$

The second antiderivative $k^{(-2)}(z)$ may still be singular at $z = 0$, but the singularity is integrable, i.e. $\int_{-\infty}^\infty |k^{(-2)}(z)|\,\mathrm{d}z < \infty$. We again use the finite element and the finite difference method to discretize the partial integro-differential equation.

10.6.1 Finite Difference Discretization

As in the Black–Scholes case, we replace the domain $J \times G$ by discrete grid points (t_m, x_i) and approximate the partial derivatives in (10.24) by difference quotients at the grid points. Let the grid points in the log-price coordinate x be given by

$$x_i = -R + ih, \quad i = 0, 1, \dots, N+1,\ h := 2R/(N+1) = \Delta x, \tag{10.26}$$

which are equidistant with mesh width h, and the time levels by

$$t_m = m\Delta t,\ m = 0, 1, \dots, M,\ \Delta t := T/M.$$

We define the weights for $j = 0, 1, 2, \dots,$

$$\nu_j^+ = \int_{jh}^{(j+1)h} k^{(-2)}(z)\,\mathrm{d}z = k^{(-3)}((j+1)h) - k^{(-3)}(jh),$$
$$\nu_j^- = \int_{-(j+1)h}^{-jh} k^{(-2)}(z)\,\mathrm{d}z = k^{(-3)}(-jh) - k^{(-3)}(-(j+1)h),$$

which satisfy $\sum_{j=0}^{\infty}(\nu_j^+ + \nu_j^-) = \int_{\mathbb{R}} k^{(-2)}(z)\,\mathrm{d}z < \infty$. Then, we can discretize the generator (10.24) for $f \in C_0^4(G)$ at the mesh point x_i, $i = 1, \dots, N$, by

$$\begin{aligned}\int_0^{R-x_i} \partial_{xx} f(x_i + z) k^{(-2)}(z)\mathrm{d}z &= \sum_{j=0}^{N-i} \partial_{xx} f(x_i + jh)\nu_j^+ + \mathcal{O}(h)\\ &= \sum_{j=0}^{N-i} (\delta_{xx}^2 f)_{i+j}\nu_j^+ + \mathcal{O}(h) + \mathcal{O}(h^2),\\ \int_{-R+x_i}^{0} \partial_{xx} f(x_i + z) k^{(-2)}(z)\,\mathrm{d}z &= \sum_{j=0}^{i-1} (\delta_{xx}^2 f)_{i-j}\nu_j^- + \mathcal{O}(h).\end{aligned}$$

Using the θ-scheme for the time discretization, we obtain

$$\begin{aligned}&\text{Find } \underline{u}^{m+1} \in \mathbb{R}^N \text{ such that for } m = 0, \dots, M-1,\\ &\big(\mathbf{I} + \theta\Delta t\mathbf{G}^{\mathrm{J}}\big)\underline{u}^{m+1} = \big(\mathbf{I} - (1-\theta)\Delta t\mathbf{G}^{\mathrm{J}}\big)\underline{u}^m,\\ &\underline{u}^0 = \underline{u}_0,\end{aligned} \tag{10.27}$$

with $\mathbf{G}^{\mathrm{J}} \in \mathbb{R}^{N\times N}$ defined for $i = 1, \dots, N$,

$$\begin{aligned}\mathbf{G}^{\mathrm{J}}_{i,i} &= \frac{1}{h^2}\big(2(\nu_0^+ + \nu_0^-) - \nu_1^+ - \nu_1^-\big),\\ \mathbf{G}^{\mathrm{J}}_{i,i+1} &= \frac{1}{h^2}\big(2\nu_1^+ - \nu_2^+ - \nu_0^+ - \nu_0^-\big),\\ \mathbf{G}^{\mathrm{J}}_{i,i-1} &= \frac{1}{h^2}\big(2\nu_1^- - \nu_2^- - \nu_0^- - \nu_0^+\big),\\ \mathbf{G}^{\mathrm{J}}_{i,i+j} &= \frac{1}{h^2}\big(2\nu_j^+ - \nu_{j+1}^+ - \nu_{j-1}^+\big), \quad j = 2, \dots, N-i,\\ \mathbf{G}^{\mathrm{J}}_{i,i-j} &= \frac{1}{h^2}\big(2\nu_j^- - \nu_{j+1}^- - \nu_{j-1}^-\big), \quad j = 2, \dots, i-1.\end{aligned}$$

Remark 10.6.1 If the Lévy measure satisfies $\int_{|z|\le 1} |z|\nu(\mathrm{d}z) < \infty$, we have the generator

$$(\mathcal{A}^{\mathrm{J}} f)(x) = \int_{\mathbb{R}} (f(x+z) - f(x))\nu(\mathrm{d}z) = \int_{\mathbb{R}} f'(x+z)k^{(-1)}(z)\,\mathrm{d}z,$$

and furthermore, if $\int_{|z|\le 1} \nu(\mathrm{d}z) < \infty$, we have (with $\lambda = \int_{\mathbb{R}} \nu(\mathrm{d}z)$)

$$(\mathcal{A}^{\mathrm{J}} f)(x) = \int_{\mathbb{R}} f(x+z)k(z)\,\mathrm{d}z - \lambda f(x).$$

In both cases, similar discretization schemes can be derived.

10.6.2 Finite Element Discretization

We consider again the mesh (10.26) with uniform mesh width h and the finite element space $V_N = S^1_{\mathcal{T}} \cap H^1_0(G)$ with $S^1_{\mathcal{T}} = \operatorname{span}\{b_i(x) : i = 1, \ldots, N\}$. We need to compute the stiffness matrix $\mathbf{A}^{\mathrm{J}}_{ij} = a^{\mathrm{J}}(b_j, b_i)$, where the bilinear form $a^{\mathrm{J}}(\cdot,\cdot)$ is given by (10.25). We define auxiliary variables for $j = 0, 1, 2, \ldots,$

$$\begin{aligned} k^+_j &= \int_0^h \int_{jh}^{(j+1)h} k^{(-2)}(y-x)\,\mathrm{d}y\,\mathrm{d}x, \\ k^-_j &= \int_{jh}^{(j+1)h} \int_0^h k^{(-2)}(y-x)\,\mathrm{d}y\,\mathrm{d}x, \end{aligned}$$

and write for $j \ge i$

$$\begin{aligned} a^{\mathrm{J}}(b_j, b_i) &= \int_{x_{i-1}}^{x_{i+1}} \int_{x_{j-1}}^{x_{j+1}} b'_j(y)b'_i(x)k^{(-2)}(y-x)\,\mathrm{d}y\,\mathrm{d}x \\ &= \frac{1}{h^2}\Big(\int_{x_{i-1}}^{x_i} \int_{x_{j-1}}^{x_j} k^{(-2)}(y-x)\,\mathrm{d}y\,\mathrm{d}x - \int_{x_{i-1}}^{x_i} \int_{x_j}^{x_{j+1}} k^{(-2)}(y-x)\,\mathrm{d}y\,\mathrm{d}x \\ &\quad - \int_{x_i}^{x_{i+1}} \int_{x_{j-1}}^{x_j} k^{(-2)}(y-x)\,\mathrm{d}y\,\mathrm{d}x + \int_{x_i}^{x_{i+1}} \int_{x_j}^{x_{j+1}} k^{(-2)}(y-x)\,\mathrm{d}y\,\mathrm{d}x\Big) \\ &= \frac{1}{h^2}\Big(\int_0^h \int_{(j-i)h}^{(j-i+1)h} k^{(-2)}(y-x)\,\mathrm{d}y\,\mathrm{d}x \\ &\quad - \int_0^h \int_{(j-i+1)h}^{(j-i+2)h} k^{(-2)}(y-x)\,\mathrm{d}y\,\mathrm{d}x \\ &\quad - \int_0^h \int_{(j-i-1)h}^{(j-i)h} k^{(-2)}(y-x)\,\mathrm{d}y\,\mathrm{d}x \\ &\quad + \int_0^h \int_{(j-i)h}^{(j-i+1)h} k^{(-2)}(y-x)\,\mathrm{d}y\,\mathrm{d}x\Big). \end{aligned}$$

Then, using the θ-scheme for the time discretization, we obtain the matrix problem

$$\begin{aligned}&\text{Find } \underline{u}^{m+1} \in \mathbb{R}^N \text{ such that for } m = 0, \dots, M-1\\&(\mathbf{M} + \theta\Delta t\mathbf{A}^{\mathrm{J}})\underline{u}^{m+1} = (\mathbf{M} - (1-\theta)\Delta t\mathbf{A}^{\mathrm{J}})\underline{u}^m,\\&\underline{u}_N^0 = \underline{u}_0,\end{aligned} \tag{10.28}$$

with $\mathbf{A}^{\mathrm{J}} \in \mathbb{R}^{N\times N}$ defined for $i = 1, \dots, N$,

$$\begin{aligned}\mathbf{A}^{\mathrm{J}}_{i,i} &= \frac{1}{h^2}\big(2k_0^+ - k_1^+ - k_1^-\big),\\\mathbf{A}^{\mathrm{J}}_{i,i+j} &= \frac{1}{h^2}\big(2k_j^+ - k_{j+1}^+ - k_{j-1}^+\big), \quad j = 1, \dots, N-i,\\\mathbf{A}^{\mathrm{J}}_{i,i-j} &= \frac{1}{h^2}\big(2k_j^- - k_{j+1}^- - k_{j-1}^-\big), \quad j = 1, \dots, i-1.\end{aligned}$$

The variables k_j^+, k_j^- can be expressed in closed form using $k^{(-3)}$ and $k^{(-4)}$:

$$\begin{aligned}k_0^+ &= \int_0^h \int_0^h k^{(-2)}(y-x)\,\mathrm{d}y\,\mathrm{d}x\\&= \int_0^h \Big(\int_0^x k^{(-2)}(y-x)\,\mathrm{d}y + \int_x^h k^{(-2)}(y-x)\,\mathrm{d}y\Big)\,\mathrm{d}x\\&= \int_0^h \big(k^{(-3)}(0^-) - k^{(-3)}(-x) + k^{(-3)}(h-x) - k^{(-3)}(0^+)\big)\,\mathrm{d}x\\&= h\big(k^{(-3)}(0^-) - k^{(-3)}(0^+)\big) + k^{(-4)}(-h) - k^{(-4)}(0^-) - k^{(-4)}(0^+)\\&\quad + k^{(-4)}(h),\\k_j^+ &= -2k^{(-4)}(jh) + k^{(-4)}((j-1)h) + k^{(-4)}((j+1)h), \quad j = 1, \dots, N-1.\end{aligned}$$

Note that, using finite elements, we computed the matrix entries exactly (assuming that the antiderivatives $k^{(-3)}$ and $k^{(-4)}$ of the density function $k(z)$ are explicitly available) on the subspace $V_N \subset V$, and therefore still obtain the optimal convergence rate $\mathcal{O}(h^2)$ whereas for finite differences we approximated the integral and only obtain convergence rate $\mathcal{O}(h)$.

Remark 10.6.2 As already noted in Remark 10.6.1, if the Lévy measure ν satisfies the assumptions $\int_{|z|\le 1} |z|\nu(\mathrm{d}z) < \infty$ or $\int_{|z|\le 1} \nu(\mathrm{d}z) < \infty$, we have the bilinear forms

$$\begin{aligned}a^{\mathrm{J}}(\varphi, \phi) &= \int_G \int_G \varphi'(y)\phi(x)k^{-1}(y-x)\,\mathrm{d}y\,\mathrm{d}x,\\a^{\mathrm{J}}(\varphi, \phi) &= \int_G \int_G \varphi(y)\phi(x)k(y-x)\,\mathrm{d}y\,\mathrm{d}x - \lambda\int_G \varphi(x)\phi(x)\,\mathrm{d}x,\end{aligned}$$

and similar discretization schemes can be derived.

Example 10.6.3 To have an exact solution available, we consider the variance gamma model [118] with parameter $\sigma = 0.3$, $\vartheta = 0.25$ and $\theta = -0.3$. The density of the variance gamma distribution for the log-returns has not only “fatter” tails

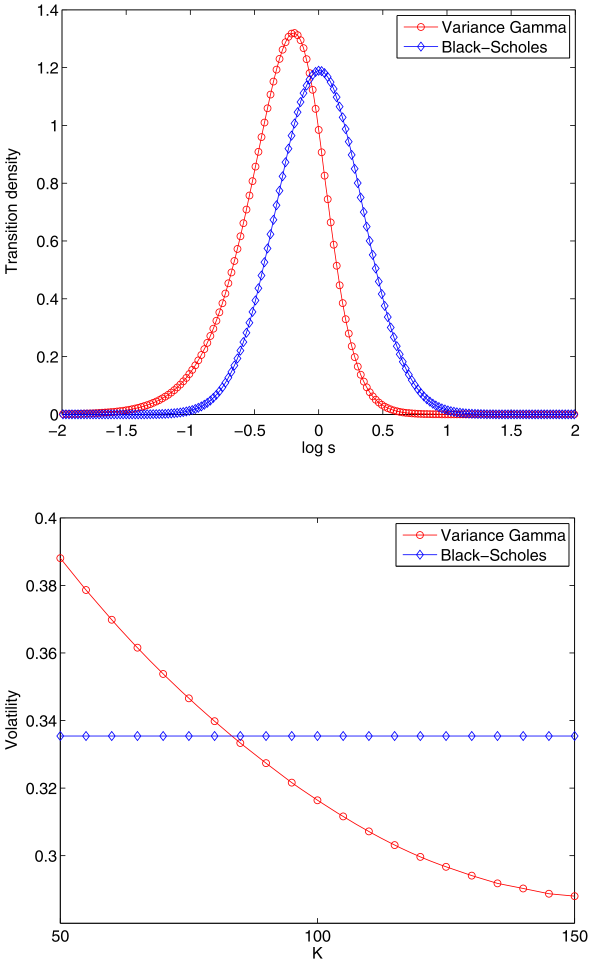


Fig. 10.1 Probability density for the variance gamma model (*top*) and the implied volatility (*bottom*)

than in the Black–Scholes case but is skewed as shown in Fig. 10.1. For comparison, we also plot the Gaussian density with the same variance. Additionally, we plot the implied volatility of a European put option at $S = 100$ for various strikes K. Here, one nicely sees the so-called volatility smile. Note that in the Black–Scholes case the implied volatility is constant.

For $T = 1$ and $K = 100$, we compute the L^∞-error at maturity $t = T$ on the subset $G_0 = (K/2, 3/2K)$. We use constant time steps with $M = \mathcal{O}(N)$ and the Crank–Nicolson scheme. It can be seen in Fig. 10.2 that for the finite element method we again obtain the optimal convergence rate $\mathcal{O}(h^2)$ rather than $\mathcal{O}(h)$ for the finite difference method.

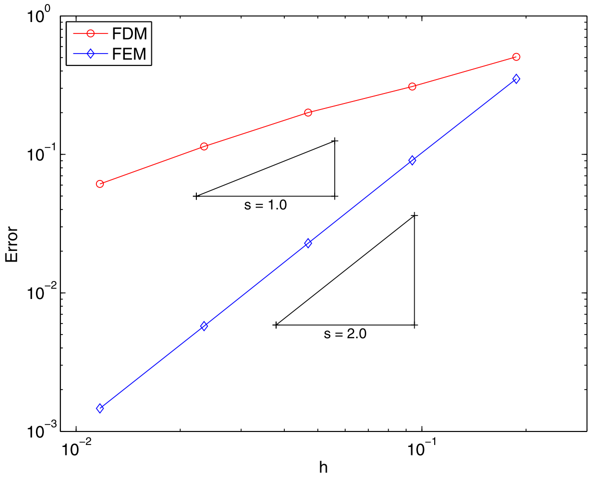


Fig. 10.2 Convergence rate for the variance gamma model

10.7 American Options Under Exponential Lévy Models

American options can be obtained similarly to the Black–Scholes case just by replacing the Black–Scholes operator $\mathcal{A}^{\mathrm{BS}}$ by the jump operator $\mathcal{A}^{\mathrm{J}}$. The value of an American option in log-price

$$v(t,x) := \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}\big[e^{-r(\tau-t)} g(e^{r\tau + X_{\tau-}}) \mid X_t = x\big],$$

is, provided some smoothness assumption on v, the solution of a parabolic integro-differential inequality

$$\begin{aligned}
\partial_t u - \mathcal{A}^{\mathrm{J}} u &\geq 0 && \text{in } J \times \mathbb{R},\\
u(t,x) &\geq \widetilde{g}(t,x) && \text{in } J \times \mathbb{R},\\
(\partial_t u - \mathcal{A}^{\mathrm{J}} u)(\widetilde{g} - u) &= 0 && \text{in } J \times \mathbb{R},\\
u(0,x) &= g(e^x) && \text{in } \mathbb{R},
\end{aligned}$$

where we set $u(t,x) =: e^{rt} v(T-t, x-(\gamma+r)t)$ and $\widetilde{g}(t,x) = e^{rt} g(e^{x-(\gamma+r)t})$. As shown in Theorem 10.5.1, we can localize the problem to a bounded domain $G = (-R, R)$. For the variational formulation, we again consider $w_R := u_R - \widetilde{g}|_G$, the time value of the option, and obtain

Find $u_R \in L^2(J; H_0^1(G)) \cap H^1(J; L^2(G))$ such that $u_R(t,\cdot) \in \mathcal{K}_{0,R}$ and

$$(\partial_t u_R, v - u_R) + a^{\mathrm{J}}(u_R, v - u_R) - (\partial_t \widetilde{g}, v - u_R) - a^{\mathrm{J}}(\widetilde{g}, v - u_R), \ \forall v \in \mathcal{K}_{0,R},$$

$$u_R(0) = 0.$$

Discretization using finite differences or finite elements as explained in Sect. 10.6 leads to the following sequence of linear complementary problems with $\underline{u}^0 = u_0$:

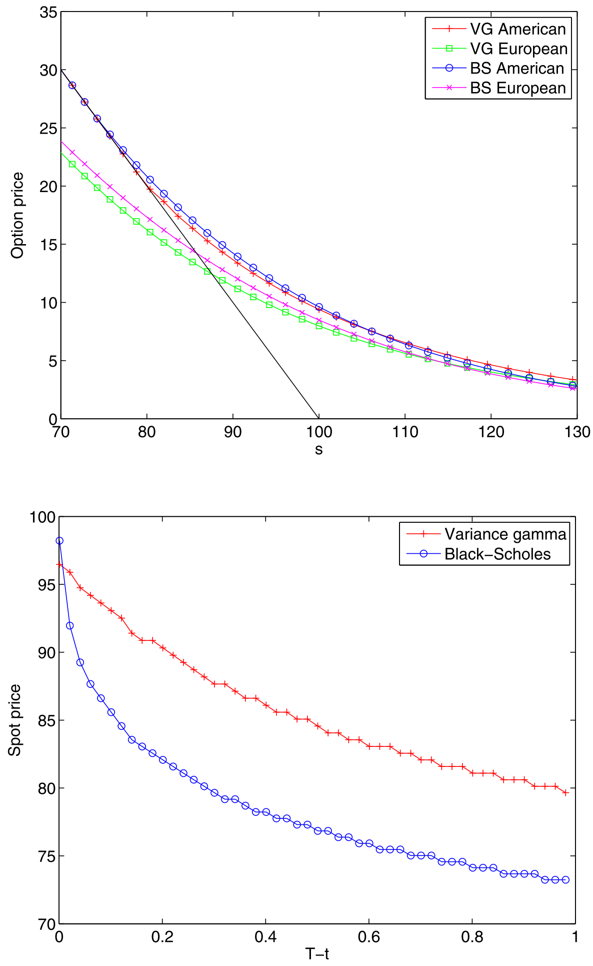


Fig. 10.3 Option prices for the VG model (*top*) and the exercise boundary (*bottom*)

Find $\underline{u}^{m+1} \in \mathbb{R}^N$ such that for $m = 0, \ldots, M-1$,

$$\begin{aligned}
&\mathbf{B}\underline{u}^{m+1} \geq \underline{F}^m,\\
&\underline{u}^{m+1} \geq \underline{g}^m,\\
&(\underline{u}^{m+1})^\top \big(\mathbf{B}\underline{u}^{m+1} - \underline{F}^m\big) = 0.
\end{aligned}$$

Example 10.7.1 As in Example 10.6.3 we use the variance gamma model with parameters $\sigma = 0.3$, $\vartheta = 0.25$ and $\theta = -0.3$. For $T = 1$ and $K = 100$, we compute the price of a European and American put option and compare these with the corresponding Black–Scholes prices. It can be seen in Fig. 10.3 that the smooth pasting condition (see Remark 5.1.2) does not hold for Lévy models. We also observe that in contrast to the Black–Scholes model the exercise boundary values in a Lévy model never reach the option's strike price, i.e. $s^*(t) < K$, as proven in [110, Theorem 4.4].

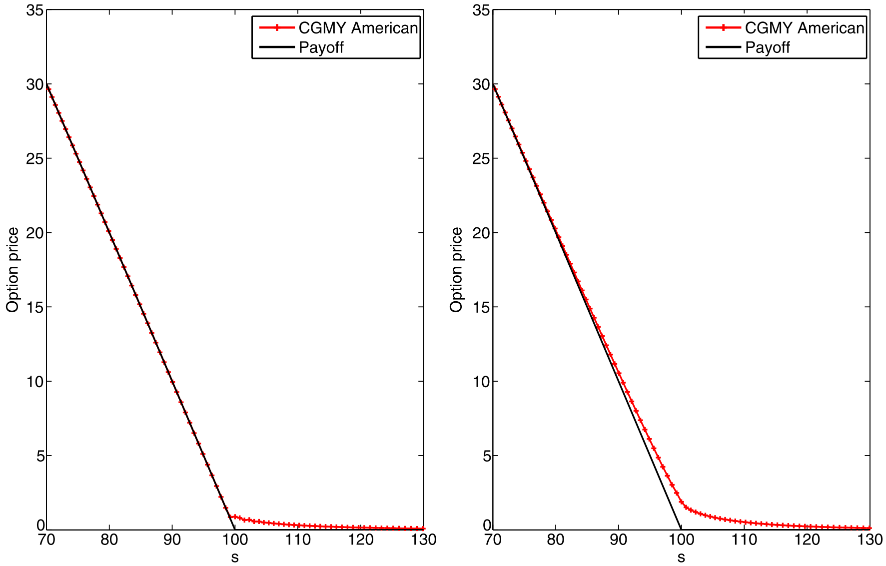


Fig. 10.4 Loss of the smooth pasting condition, CGMY model with $\mu \approx -0.0018$, $C = 0.5$, $G = 5$, $M = 5.4$, $Y = 0.1$, $r = 0.13$ (*left*) and CGMY model with $\widetilde{\mu} \approx 0.0478$, $C = 1$, $G = 6$, $M = 5.4$, $Y = 0.1$, $r = 0.01$ (*right*) and $h = 0.01$ in both cases

For American style contracts on underlyings whose log-returns are described by pure jump Lévy models, the analytic behavior of the American put price and the free exercise boundary can be considerably more involved than in the Black–Scholes model. For prices of American style contracts on underlyings modeled by pure jump Lévy processes, it is shown in [111, Theorem 4.2] that the smooth pasting principle fails for admissible market models if the model is of finite variation and $\mu = \int_{\mathbb{R}}(e^z - 1)_+ \nu(\mathrm{d}z) - r \le 0$. The smooth pasting principle holds for admissible market models (in particular, for pure jump models) of infinite variation and for pure jump Lévy models of finite variation which satisfy $\widetilde{\mu} = \int_{\mathbb{R}}(e^z - 1)\nu(\mathrm{d}z) - r > 0$, see [111, Theorem 4.1] and Fig. 10.4. For an admissible market model, the free boundary $t \mapsto s^*(t)$ satisfies $s^*(t) > 0$ for $t \in [0, T)$ and is a continuous mapping [110, Proposition 4.1 and Theorem 4.2]. We get the following result:

$$
\begin{aligned}
\lim_{t \to T} s^*(t) &= K \quad \text{for } \mu \le 0, \\
\lim_{t \to T} s^*(t) &= K^* \quad \text{for } \mu > 0,
\end{aligned}
$$

where $K^* < K$. Note that $\mu > 0$ holds for all infinite variation market models, and hence there is an instant jump of the exercise boundary from the strike price K to K^*. We refer to [110, Theorem 4.4] for a proof. The early exercise of an American put if the interest rate is zero is not optimal, i.e. for the free boundary we have $s^*(T) \to 0$ for $r \to 0$ [110, Remark 4.7].

10.8 Further Reading

For a comprehensive introduction to Lévy processes, we refer to Sato [143] and Bertoin [18]. An overview for various Lévy models is given in Cont and Tankov [40] and Schoutens [148]. A similar localization argument to Theorem 10.5.1 is given by Cont and Voltchkova in [41] under stronger assumptions on the initial condition. A finite difference method for the discretization of the PIDE similar to Sect. 10.6.1 was proposed by Biswas [20]. In Cont and Voltchkova [41, 42], small jumps were approximated by an artificial diffusion and a finite difference discretization of the PIDE with small jumps truncated was proposed in this case. Similar techniques for $d = 1$ and $d = 2$ are also shown in Briani et al. [28].

Due to the non-locality of the integro-differential operator in the infinitesimal generator of a Lévy process, the corresponding stiffness matrix becomes fully populated, no matter whether we use a finite difference or a finite element discretization. In the latter case, however, one can "compress" the matrix to a sparsely populated matrix, without loss of asymptotic convergence rate of discretization errors, provided the basis functions which span the finite element space are properly chosen. One instance of such basis functions are wavelets, which we will introduce and explain in Chap. 12.

Lévy copulas [100] are used for parametric constructions of Lévy processes in d-dimensions. These models are explained in detail in Chap. 14.

Chapter 11
Sensitivities and Greeks

A key task in financial engineering is the fast and accurate calculation of sensitivities of market models with respect to model parameters. This becomes necessary, for example, in model calibration, risk analysis and in the pricing and hedging of certain derivative contracts. Classical examples are variations of option prices with respect to the spot price or with respect to time-to-maturity, the so-called "Greeks" of the model. For classical, diffusion type models and plain vanilla type contracts, the Greeks can be obtained analytically. With the trends to more general market models of jump–diffusion type and to more complicated contracts, closed form solutions are generally not available for pricing and calibration. Thus, prices and model sensitivities have to be approximated numerically.

Here, we consider the general class of Markov processes X, including stochastic volatility and Lévy models as described before. We distinguish between two classes of sensitivities. The sensitivity of the solution V to variation of a model parameter, like the Greek Vega ($\partial_\sigma V$) and the sensitivity of the solution V to a variation of state spaces such as the Greek Delta ($\partial_x V$). We show that an approximation for the first class can be obtained as a solution of the pricing PIDE with a right hand side depending on V. For the second class, a finite difference like differentiation procedure is presented which allows obtaining the sensitivities from the forward price without additional calls to the forward solver.

11.1 Option Pricing

We consider the process X to model the dynamics of a single underlying, a basket or an underlying and its "background" volatility drivers in case of stochastic volatility models. For notational simplicity, only we assume that the interest rate is zero, i.e. $r = 0$. As shown in the previous sections provided some smoothness assumptions, the fair price of a European style contingent claim with payoff g and underlying X, i.e.

$$v(t,x) = \mathbb{E}\big[g(X_T) \mid X_t = x\big],$$

N. Hilber et al., *Computational Methods for Quantitative Finance*, Springer Finance, DOI 10.1007/978-3-642-35401-4_11,

is the solution of

$$\partial_t v + \mathcal{A}v = 0 \quad \text{in } J \times \mathbb{R}^d, \quad v(T,x) = g(x) \text{ in } \mathbb{R}^d, \tag{11.1}$$

where $\mathcal{A}$ denotes the infinitesimal generator of X. We consider processes X where $\mathcal{A}$ is given by

$$\begin{aligned}(\mathcal{A}f)(x) =& \frac{1}{2}\operatorname{tr}[\mathcal{Q}(x)D^2 f(x)] + b(x)^\top \nabla f(x) + c(x)f(x) \\ &+ \int_{\mathbb{R}^d} \big(f(x+z) - f(x) - z^\top \nabla f(x)\big)\nu(\mathrm{d}z),\end{aligned} \tag{11.2}$$

where $\mathcal{Q} : \mathbb{R}^d \to \mathbb{R}^{d\times d}$, $b : \mathbb{R}^d \to \mathbb{R}^d$, $c : \mathbb{R}^d \to \mathbb{R}$ and ν a Lévy measure in $\mathbb{R}^d$ satisfying $\int_{\mathbb{R}^d} \min\{1, |z|^2\}\nu(\mathrm{d}z) < \infty$ and $\int_{|z|>1} |z_i|\nu(\mathrm{d}z) < \infty$, $i = 1,\ldots,d$.

Definition 11.1.1 We call a process X a parametric Markovian market model with admissible parameter set $\mathcal{S}_\eta$, if

(i) For all $\eta \in \mathcal{S}_\eta$ X is a strong Markov process with respect to $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$,
(ii) The infinitesimal generator $\mathcal{A}$ of the semigroup generated by X has the form (11.2), and the mapping $\mathcal{S}_\eta \ni \eta \to \{\mathcal{Q}, b, c, \nu\}$ is infinitely differentiable.

Examples of Markov processes X and their infinitesimal generators are given by the one-dimensional diffusion (4.2), the multidimensional diffusion (8.4), the general stochastic volatility model (9.27) and the one-dimensional Lévy process (10.14).

We calculate the sensitivities of the solution v of (11.1) with respect to parameters in the infinitesimal generator $\mathcal{A}$ and with respect to solution arguments x and t. We write $\mathcal{A}(\eta_0)$ for a fixed parameter $\eta_0 \in \mathcal{S}_\eta$ to emphasize the dependence of $\mathcal{A}$ on η_0 and change the time to time-to-maturity $t \to T - t$. For sensitivity computation, it will be crucial below to admit a non-trivial right hand side. Accordingly, we consider the parabolic problem

$$\partial_t u - \mathcal{A}(\eta_0)u = f \quad \text{in } J \times \mathbb{R}^d, \quad u(0,x) = u_0 \text{ in } \mathbb{R}^d, \tag{11.3}$$

with $u_0 = g$. For the numerical implementation, we truncate (11.3) to a bounded domain $G \subset \mathbb{R}^d$ and impose boundary conditions on ∂G. Typically, G is d-dimensional hypercube, i.e. $G = \prod_{k=1}^d (a_k, b_k)$ for some $a_k, b_k \in \mathbb{R}$, $b_k > a_k$, $k = 1,\ldots,d$, as shown in the localization Theorems 4.3.1, 8.3.1, 9.4.1 and 10.5.1.

With a parametric Markovian market model X in the sense of Definition 11.1.1 with parameter set $\mathcal{S}_\eta$ and infinitesimal generator $\mathcal{A}(\eta_0)$ as in (11.2), $\eta_0 \in \mathcal{S}_\eta$, we associate to $\mathcal{A}(\eta_0)$ the Dirichlet form $a(\eta_0;\cdot,\cdot) : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ via

$$a(\eta_0; u, v) := -\langle \mathcal{A}(\eta_0)u, v\rangle_{\mathcal{V}^*,\mathcal{V}}, \quad u, v \in \mathcal{V},$$

where we consider the abstract setting as given in Sect. 3.2 with Hilbert spaces $\mathcal{V} \subset \mathcal{H} \equiv \mathcal{H}^* \subset \mathcal{V}^*$.

We assume that $a(\eta_0;\cdot,\cdot):\mathcal{V}\times\mathcal{V}\to\mathbb{R}$ satisfies continuity (3.8) and the Gårding inequality (3.9) for all $\eta_0\in\mathcal{S}_\eta$. In general, the space $\mathcal{V}$ may depend on the parameter η_0, and we should write $\mathcal{V}_{\eta_0}$. For notational simplicity, we drop the subscript η_0. For $f\in L^2(J;\mathcal{V}^*)$ and $u_0\in\mathcal{H}$, the weak formulation of the problem (11.3) is given by:

$$
\begin{aligned}
&\text{Find } u\in L^2(J;\mathcal{V})\cap H^1(J;\mathcal{H}) \text{ such that}\\
&(\partial_t u, v)_{\mathcal{H}} + a\big(\eta_0; u, v\big) = \langle f, v\rangle_{\mathcal{V}^*,\mathcal{V}}, \quad \forall v\in\mathcal{V}, \\
&u(0,\cdot) = u_0 .
\end{aligned}
\tag{11.4}
$$

Under the assumption that (3.8)–(3.9) hold for every model parameter $\eta_0\in\mathcal{S}_\eta$, the problem (11.4) admits a unique solution. We assume that $\mathcal{V}$ is a Sobolev-type space with smoothness index r, i.e.

$$
\mathcal{V}=\widetilde{H}^r(G),\quad \mathcal{H}=\widetilde{H}^0(G)=L^2(G). \tag{11.5}
$$

Note that r depends on the order of the operator $\mathcal{A}(\eta_0)$. We also assume that the solution $u(\eta_0)$ to (11.4) has higher regularity in space, $u(\eta_0)(t)\in\widetilde{H}^s(G)\subset\widetilde{H}^r(G)$ for $t\in J$, where $\widetilde{H}^s(G)$ is again a Sobolev-type space with smoothness index s.

11.2 Sensitivity Analysis

For a parametric Markovian market model X in the sense of Definition 11.1.1, we distinguish two classes of sensitivities:

(i) The sensitivity of the solution u to a variation $\mathcal{S}_\eta\ni\eta_s:=\eta_0+s\delta\eta$, $s>0$, of an input parameter $\eta_0\in\mathcal{S}_\eta$. Typical examples are the Greeks Vega ($\partial_\sigma u$), Rho ($\partial_r u$) and Vomma ($\partial_{\sigma\sigma} u$). Other sensitivities which are not so commonly used in the financial community are the sensitivity of the price with respect to the jump intensity or the order of the process that models the underlying.
(ii) The sensitivity of the solution u to a variation of arguments t, x. Typical examples are the Greeks Theta ($\partial_t u$), Delta ($\partial_x u$) and Gamma ($\partial_{xx} u$).

11.2.1 Sensitivity with Respect to Model Parameters

Let $\mathcal{C}$ be a Banach space over a domain $G\subset\mathbb{R}^d$. $\mathcal{C}$ is the space of parameters or coefficients in the operator $\mathcal{A}$ and $\mathcal{S}_\eta\subseteq\mathcal{C}$ is the set of admissible coefficients. We denote by $u(\eta_0)$ the unique solution to (11.4) and introduce the derivative of $u(\eta_0)$ with respect to $\eta_0\in\mathcal{S}_\eta$ as the mapping $D_{\eta_0}u(\eta_0):\mathcal{C}\to\mathcal{V}$

$$
\widetilde{u}(\delta\eta):=D_{\eta_0}u(\eta_0)(\delta\eta):=\lim_{s\to0^+}\frac{1}{s}\big(u(\eta_0+s\delta\eta)-u(\eta_0)\big),\quad \delta\eta\in\mathcal{C}.
$$

We also introduce the derivative of $\mathcal{A}(\eta_0)$ with respect to $\eta_0 \in \mathcal{S}_\eta$

$$\begin{aligned}\widetilde{\mathcal{A}}(\delta\eta)\varphi &:= D_{\eta_0}\mathcal{A}(\eta_0)(\delta\eta)\varphi \\ &:= \lim_{s\to 0^+} \frac{1}{s}\big(\mathcal{A}(\eta_0 + s\delta\eta)\varphi - \mathcal{A}(\eta_0)\varphi\big), \quad \varphi \in \mathcal{V}, \quad \delta\eta \in \mathcal{C}.\end{aligned}$$

We assume that $\widetilde{\mathcal{A}}(\delta\eta) \in \mathcal{L}(\widetilde{\mathcal{V}}, \widetilde{\mathcal{V}}^*)$ with $\widetilde{\mathcal{V}}$ a real and separable Hilbert space satisfying

$$\widetilde{\mathcal{V}} \subseteq \mathcal{V} \subset \mathcal{H} \cong \mathcal{H}^* \subset \mathcal{V}^* \subseteq \widetilde{\mathcal{V}}^*.$$

We further assume that there exists a real and separable Hilbert space $\overline{\mathcal{V}} \subseteq \widetilde{\mathcal{V}}$ such that $\widetilde{\mathcal{A}}v \in \mathcal{V}^*$, $\forall v \in \overline{\mathcal{V}}$. We have the following relation between $D_{\eta_0}u(\eta_0)(\delta\eta)$ and u.

Lemma 11.2.1 *Let* $\widetilde{\mathcal{A}}(\delta\eta) \in \mathcal{L}(\widetilde{\mathcal{V}}, \widetilde{\mathcal{V}}^*)$, $\forall \delta\eta \in \mathcal{C}$ *and* $u(\eta_0) : J \to \overline{\mathcal{V}}$, $\eta_0 \in \mathcal{S}_\eta$ *be the unique solution to*

$$\partial_t u(\eta_0) - \mathcal{A}(\eta_0)u(\eta_0) = 0 \quad \text{in } J \times \mathbb{R}^d, \quad u(\eta_0)(0,\cdot) = g(x) \quad \text{in } \mathbb{R}^d. \tag{11.6}$$

Then, $\widetilde{u}(\delta\eta)$ *solves*

$$\partial_t \widetilde{u}(\delta\eta) - \mathcal{A}(\eta_0)\widetilde{u}(\delta\eta) = \widetilde{\mathcal{A}}(\delta\eta)u(\eta_0) \quad \text{in } J \times \mathbb{R}^d, \quad \widetilde{u}(\delta\eta)(0,\cdot) = 0 \quad \text{in } \mathbb{R}^d. \tag{11.7}$$

Proof Since $u(\eta_0)(0) = g$ does not depend on η_0, its derivative with respect to η is 0. Now let $\eta_s := \eta_0 + s\delta\eta$, $s > 0$, $\delta\eta \in \mathcal{C}$. Subtract from the equation $\partial_t u(\eta_s)(t) - \mathcal{A}(\eta_s)u(\eta_s)(t) = 0$ Eq. (11.6) and divide by s to obtain

$$\begin{aligned}\partial_t \frac{1}{s}\big(u(\eta_s)(t) - u(\eta_0)(t)\big) &- \frac{1}{s}\big(\mathcal{A}(\eta_s) - \mathcal{A}(\eta_0)\big)u(\eta_s)(t) \\ &- \frac{1}{s}\mathcal{A}(\eta_0)\big(u(\eta_s)(t) - u(\eta_0)(t)\big) = 0.\end{aligned}$$

Taking $\lim_{s\to 0^+}$ gives Eq. (11.7). □

We associate to the operator $\widetilde{\mathcal{A}}(\delta\eta)$ the Dirichlet form $\widetilde{a}(\delta\eta;\cdot,\cdot) : \widetilde{\mathcal{V}} \times \widetilde{\mathcal{V}} \to \mathbb{R}$ which is given by

$$\widetilde{a}(\delta\eta; u, v) = -\langle \widetilde{\mathcal{A}}(\delta\eta)u, v\rangle_{\widetilde{\mathcal{V}}^*, \widetilde{\mathcal{V}}}.$$

The variational formulation to (11.7) reads:

$$\begin{aligned}&\text{Find } \widetilde{u}(\delta\eta) \in L^2(J;\mathcal{V}) \cap H^1(J;\mathcal{H}) \text{ such that} \\ &(\partial_t \widetilde{u}(\delta\eta), v)_{\mathcal{H}} + a\big(\eta_0; \widetilde{u}(\delta\eta), v\big) = -\widetilde{a}\big(\delta\eta; u(\eta_0), v\big), \quad \forall v \in \mathcal{V}, \\ &\widetilde{u}(\delta\eta)(0) = 0\,.\end{aligned} \tag{11.8}$$

Note that (11.8) has a unique solution $\widetilde{u}(\delta\eta) \in \mathcal{V}$ due to the assumptions on $a(\eta_0;\cdot,\cdot)$, $\widetilde{\mathcal{A}}$ and $u(\eta_0) \in \overline{\mathcal{V}}$.

Example 11.2.2 (Black–Scholes model) We consider a one-dimensional diffusion process X with the infinitesimal generator

$$(\mathcal{A}^{\mathrm{BS}} f)(x) = \frac{1}{2}\sigma^2 \partial_{xx} f(x) - \frac{1}{2}\sigma^2 \partial_x f(x).$$

For the sensitivity of the price with respect to the volatility σ, the set of admissible parameters $\mathcal{S}_\eta$ is $\mathcal{S}_\eta = \mathbb{R}_+$ with $\eta = \sigma$. We have

$$(\widetilde{\mathcal{A}}(\delta\sigma) f)(x) = \delta\sigma\sigma_0 \partial_{xx} f(x) - \delta\sigma\sigma_0 \partial_x f(x) \quad \in \mathcal{L}(\mathcal{V}, \mathcal{V}^*),$$

with $\delta\sigma \in \mathbb{R} = \mathcal{C}$. The Dirichlet form $\widetilde{a}(\delta\sigma; \cdot, \cdot)$ appearing in the weak formulation (11.8) of $\widetilde{u}(\delta\sigma)$ is given by

$$\widetilde{a}(\delta\sigma; \varphi, \phi) = \delta\sigma\sigma_0(\partial_x \varphi, \partial_x \phi) + \delta\sigma\sigma_0(\partial_x \varphi, \phi).$$

In this setting, $\widetilde{\mathcal{V}} = \mathcal{V} = H_0^1(G)$.

Example 11.2.3 (Tempered stable model) We consider a one-dimensional pure jump process X with the tempered stable density k as in (10.10) and infinitesimal generator

$$(\mathcal{A}^{\mathrm{J}} f)(x) = \int_{\mathbb{R}} (f(x+z) - f(x) - z f'(x)) k(z)\, \mathrm{d}z.$$

For the sensitivity of the price with respect to the jump intensity parameter α of the Lévy process X, we have $\mathcal{S}_\eta = (0, 2)$ with $\eta = \alpha$ and

$$(\widetilde{\mathcal{A}}(\delta\alpha) f)(x) = \delta\alpha \int_{\mathbb{R}} (f(x+z) - f(x) - z f'(x)) \widetilde{k}(z)\, \mathrm{d}z \in \mathcal{L}(\widetilde{\mathcal{V}}, \widetilde{\mathcal{V}}^*),$$

where the kernel $\widetilde{k}$ is given by

$$\widetilde{k}(z) := -\ln|z| k(z).$$

It is easy to check that $\int_{|z|\le 1} z^2 \widetilde{k}(z)\, \mathrm{d}z < \infty$, $\int_{|z|>1} \widetilde{k}(z)\, \mathrm{d}z < \infty$. In this setting, $\widetilde{\mathcal{V}} = \widetilde{H}^{\alpha/2+\varepsilon}(G) \subset \widetilde{H}^{\alpha/2}(G) = \mathcal{V}$, $\varepsilon > 0$.

For the discretization of (11.7), (11.8), we can use either finite differences or finite elements. Here, we consider the finite element method and obtain the matrix form of (11.8)

Find $\underline{\widetilde{u}}^{m+1} \in \mathbb{R}^N$ such that for $m = 0, \ldots, M-1$,

$$(\mathbf{M} + \theta\Delta t\mathbf{A})\underline{\widetilde{u}}^{m+1} = (\mathbf{M} - (1-\theta)\Delta t\mathbf{A})\underline{\widetilde{u}}^{m} - \Delta t\widetilde{\mathbf{A}}(\theta\underline{u}^{m+1} + (1-\theta)\underline{u}^{m}),$$

$$\underline{\widetilde{u}}^0 = \underline{0},$$

where $\widetilde{\mathbf{A}}$ is matrix of the Dirichlet form $\widetilde{a}(\delta\eta; \cdot, \cdot)$ in the basis of V_N, $\widetilde{\mathbf{A}}_{ij} = \widetilde{a}(\delta\eta; b_j, b_i)$ for $1 \le i, j \le N$, and $\underline{u}^{m+1}$, $m = 0, \ldots, M-1$, is the coefficient vector

Table 11.1 Algorithm to compute sensitivities with respect to model parameters

Choose $\eta_0 \in \mathcal{S}_\eta$, $\delta\eta \in \mathcal{C}$.
Calculate the matrices $\mathbf{M}$, $\mathbf{A}$ and $\widetilde{\mathbf{A}}$.
Let $\underline{u}^0$ be the coefficient vector of u_N^0 in the basis of V_N.
Set $\underline{\widetilde{u}}^0 = \underline{0}$.
For $j = 0, 1, \dots, M-1$,
$\quad \underline{u}^1 \leftarrow \text{solve}\big(\mathbf{M} + \theta\Delta t\mathbf{A}, (\mathbf{M} - (1-\theta)\Delta t\mathbf{A})\underline{u}^0\big)$
$\quad$ Set $\underline{f} := \widetilde{\mathbf{A}}(\theta\underline{u}^1 + (1-\theta)\underline{u}^0)$.
$\quad \underline{\widetilde{u}}^1 \leftarrow \text{solve}\big(\mathbf{M} + \theta\Delta t\mathbf{A}, \mathbf{M} - (1-\theta)\Delta t\mathbf{A})\underline{\widetilde{u}}^0 - \Delta t\underline{f}\big)$
$\quad$ Set $\underline{u}^0 := \underline{u}^1$, $\underline{\widetilde{u}}^0 := \underline{\widetilde{u}}^1$.
Next j

of the finite element solution $u_N(t_{m+1}, x) \in V_N$ to (11.4). The resulting algorithm is illustrated as pseudo-code in Table 11.1. Here, we denote by $\underline{y} \leftarrow \text{solve}(\mathbf{B}, \underline{x})$ the output of a generic solver for a linear system $\mathbf{B}\underline{x} = \underline{y}$.

We assume the following approximation property of the space V_N: For all $u \in \widetilde{H}^s(G)$ with $r \le s \le p+1$, there exists a $u_N \in V_N$ such that for $0 \le \tau \le r$ (with r as in (11.5))

$$\|u - u_N\|_{\widetilde{H}^\tau(G)} \le Ch^{s-\tau}\|u\|_{\widetilde{H}^s(G)}. \tag{11.9}$$

We further assume the existence of a projector $\mathcal{P}_N : \mathcal{V} \to V_N$ which satisfies (11.9) for $u_N = \mathcal{P}_N u$. Similar to Theorem 3.6.5, we obtain the following convergence result.

Theorem 11.2.4 *Assume* $u, \widetilde{u} \in C^1(\overline{J}; \mathcal{V}) \cap C^3(\overline{J}; \mathcal{V}^*)$. *Assume for* $0 \le \theta < \frac{1}{2}$ *also* (3.30). *Then, the following error bound holds*:

$$\begin{aligned}
&\|\widetilde{u}^M - \widetilde{u}_N^M\|^2_{L^2(G)} + \Delta t\sum_{m=0}^{M-1}\|\widetilde{u}^{m+\theta} - \widetilde{u}_N^{m+\theta}\|^2_{\mathcal{V}} \\
&\le C\sum_{v\in\{u,\widetilde{u}\}}\begin{cases}(\Delta t)^2\int_0^T\|\ddot{v}(\tau)\|_*^2\,\mathrm{d}\tau, & \theta\in[0,1]\\ (\Delta t)^4\int_0^T\|\dddot{v}(\tau)\|_*^2\,\mathrm{d}\tau, & \theta = \frac{1}{2}\end{cases} \\
&\quad + Ch^{2(s-r)}\sum_{v\in\{u,\widetilde{u}\}}\int_0^T\|\dot{v}(\tau)\|^2_{\widetilde{H}^{s-r}(G)}\,\mathrm{d}\tau \\
&\quad + Ch^{2(s-r)}\max_{0\le t\le T}\|u(t)\|^2_{\widetilde{H}^s(G)}.
\end{aligned}$$

Theorem 11.2.4 shows that if the error between the exact and the approximate price satisfies $\|u^m - u_N^m\|_{L^2(G)} = \mathcal{O}(h^{s-r}) + \mathcal{O}((\Delta t)^\kappa)$, the error between the exact and approximate sensitivity preserves the same convergence rates both in space and time, i.e. $\|\widetilde{u}^m - \widetilde{u}_N^m\|_{L^2(G)} = \mathcal{O}(h^{s-r}) + \mathcal{O}((\Delta t)^\kappa)$.

11.2.2 Sensitivity with Respect to Solution Arguments

We discuss the computation of $\mathcal{D}^{\mathbf{n}}u = \partial_{x_1}^{n_1} \cdots \partial_{x_d}^{n_d} u$ for an arbitrary multi-index $\mathbf{n} \in \mathbb{N}_0^d$, where $\mathbf{n} = (n_1, \ldots, n_d)$. For $\mu \in \mathbb{Z}^d$ and $h > 0$, we define the translation operator $T_h^{\mu}\varphi(x) = \varphi(x + \mu h)$ and the forward difference quotient $\partial_{h,j}\varphi(x) = h^{-1}(T_h^{e_j}\varphi(x) - \varphi(x))$, where e_j, $j = 1, \ldots, d$, denotes the jth standard basis vector in $\mathbb{R}^d$. For $\mathbf{n} \in \mathbb{N}_0^d$, we denote by $\partial_h^{\mathbf{n}}\varphi = \partial_{h,1}^{n_1} \cdots \partial_{h,d}^{n_d}\varphi$ and by $\mathcal{D}_h^{\mathbf{n}}$ the difference operator of order $n \geq 0$

$$\mathcal{D}_h^{\mathbf{n}}\varphi := \sum_{\gamma, |\mathbf{n}|=n} C_{\gamma,\mathbf{n}} T_h^{\gamma} \partial_h^{\mathbf{n}}\varphi.$$

Definition 11.2.5 The difference operator $\mathcal{D}_h^{\mathbf{n}}$ of order $|\mathbf{n}| = n$ and mesh width h is called an approximation to the derivative $\mathcal{D}^{\mathbf{n}}$ of order $s \in \mathbb{N}_0$ if for any $G_0 \subset G$ there holds

$$\|\mathcal{D}^{\mathbf{n}}\varphi - \mathcal{D}_h^{\mathbf{n}}\varphi\|_{\widetilde{H}^r(G_0)} \leq Ch^s \|\varphi\|_{\widetilde{H}^{s+r+n}(G)}, \forall \varphi \in \widetilde{H}^{s+r+n}(G). \tag{11.10}$$

Using finite elements for the discretization with basis $b_1, \ldots, b_N$ of V_N, the action of $\mathcal{D}_h^{\mathbf{n}}$ to $v_N \in V_N$ can be realized as matrix–vector multiplication $\underline{v}_N \mapsto \mathbf{D}_h^{\mathbf{n}}\underline{v}_N$, where

$$\mathbf{D}_h^{\mathbf{n}} = \left(\mathcal{D}_h^{\mathbf{n}}b_1, \ldots, \mathcal{D}_h^{\mathbf{n}}b_N\right) \in \mathbb{R}^{N \times N},$$

and $\underline{v}_N$ is the coefficient vector of v_N with respect to the basis of V_N.

Example 11.2.6 Let V_N be as in (3.17), the space of piecewise linear continuous functions on [0, 1] vanishing at the end points 0, 1. For $\alpha, \beta, \gamma \in \mathbb{R}$ and $\mu \in \mathbb{N}_0$, we denote by $\operatorname{diag}_\mu(\alpha, \beta, \gamma)$ the matrices

$$\operatorname{diag}_\mu(\alpha, \beta, \gamma) = \begin{pmatrix} \cdots & 0 & \alpha & \beta & \gamma & 0 & \cdots & \\ & \cdots & 0 & \alpha & \beta & \gamma & 0 & \cdots \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

where the entries β are on the μth lower diagonal. Then, the matrices $\mathbf{Q}_h$ of the forward difference quotient ∂_h and $\mathbf{T}_\mu$ of the translation operator T_h^{μ}, respectively, are given by

$$\mathbf{Q}_h = h^{-1}\operatorname{diag}_0(0, -1, 1), \quad \mathbf{T}_\mu = \operatorname{diag}_\mu(0, 1, 0).$$

Hence, for example, we have for the centered finite difference quotient

$$\mathcal{D}_h^2\varphi(x) = h^{-2}(\varphi(x + h) - 2\varphi(x) + \varphi(x - h))$$

Table 11.2 Algorithm to compute sensitivities with respect to arguments of solution

Choose $\eta_0 \in \mathcal{S}_\eta$.
Calculate the matrices $\mathbf{M}$, $\mathbf{A}$ and $\mathbf{D}_h^\alpha$.
Let $\underline{u}^0$ be the coefficient vector of u_N^0 in the basis of V_N.
For $j = 0, 1, \dots, M-1$
$\quad \underline{u}^1 \leftarrow \text{solve}\big(\mathbf{M} + \theta\Delta t\mathbf{A}, (\mathbf{M} - (1-\theta)\Delta t\mathbf{A})\underline{u}^0\big)$
$\quad$ Set $\underline{u}^0 := \underline{u}^1$.
Next j
Set $\underline{v} := \mathbf{D}_h^\alpha \underline{u}^1$.

of order 2 in one dimension $\mathbf{D}_h^2 = \mathbf{T}_{-1}\mathbf{Q}_h^2 = h^{-2}\,\text{diag}_0(1, -2, 1)$. In the multidimensional case where V_N is given by (8.19), the matrix $\mathbf{D}_h^{\mathbf{n}}$ is given by

$$\mathbf{D}_h^{\mathbf{n}} = \sum_{\gamma, |\mathbf{n}|=n} C_{\gamma,\mathbf{n}} \mathbf{T}_{\gamma_1} \otimes \cdots \otimes \mathbf{T}_{\gamma_d} \mathbf{Q}_h^{n_1} \otimes \cdots \otimes \mathbf{Q}_h^{n_d}.$$

In Table 11.2, the algorithm how to obtain an approximation to the derivative $\mathcal{D}^{\mathbf{n}} u(T, x)$ at maturity T is illustrated. The vector $\underline{v} \in \mathbb{R}^N$ is the coefficient vector of $\mathcal{D}_h^{\mathbf{n}} u_N^M$ in the basis of V_N.

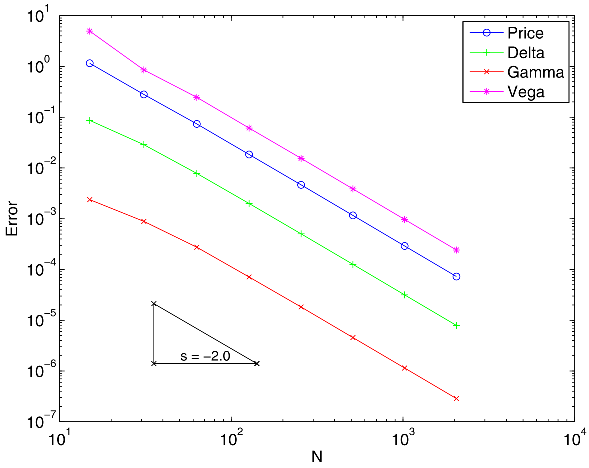
We have the following convergence result for the approximation of sensitivities with respect to solution arguments.

Theorem 11.2.7 *Assume $u \in C^1(\overline{J}; \mathcal{V}) \cap C^3(\overline{J}; \mathcal{V}^*)$ and for $0 \le \theta < \frac{1}{2}$ also* (3.30). *Assume that the approximation $\partial_h^\beta u_N^0$ is quasi-optimal in $L^2(G)$ for all $\beta \le \alpha$. Assume further that $\mathcal{D}_h^{\mathbf{n}}$ approximates $\mathcal{D}^{\mathbf{n}}$ in the sense of Definition* 11.2.5. *Then,*

$$\begin{aligned}
&\|\mathcal{D}^{\mathbf{n}} u^M - \mathcal{D}_h^{\mathbf{n}} u_N^M\|_{L^2(G_0)}^2 + \Delta t \sum_{m=0}^{M-1} \|\mathcal{D}^{\mathbf{n}} u^{m+\theta} - \mathcal{D}_h^{\mathbf{n}} u_N^{m+\theta}\|_{\mathcal{V}}^2 \\
&\quad \le C \begin{cases} (\Delta t)^2 \int_0^T \|\ddot{u}(\tau)\|_*^2 \,\mathrm{d}\tau, & \theta \in [0,1] \\ (\Delta t)^4 \int_0^T \|\dddot{u}(\tau)\|_*^2 \,\mathrm{d}\tau, & \theta = \frac{1}{2} \end{cases} + Ch^{2(s-r)} \int_0^T \|\dot{u}(\tau)\|_{\widetilde{H}^{s-r}(G)}^2 \,\mathrm{d}\tau \\
&\qquad + Ch^{2(s-r)} \max_{0 \le t \le T} \|u(t)\|_{\widetilde{H}^s(G)}^2.
\end{aligned}$$

11.3 Numerical Examples

In this section, we compute various sensitivities for different models. We choose models where the price is known in closed form so that we are able to compute the errors between the exact price/sensitivities and their approximations. We measure the L^∞-norm of the error on a subset $G_0 \subset G$ at maturity $t = T$. In all computations, we discretize by linear finite elements and the Crank–Nicolson scheme where the time steps are chosen sufficiently small.



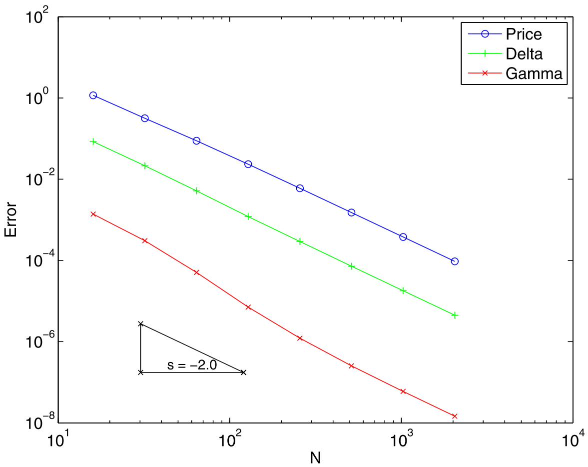


Fig. 11.1 Convergence rates of Greeks for a European put in the Black–Scholes (*top*) and variance gamma model (*bottom*)

11.3.1 One-Dimensional Models

We consider two models, the Black–Scholes model and the variance gamma model. For both models, we consider a European put with strike $K = 100$, maturity $T = 1.0$ and interest rate $r = 0.01$, and we calculate the Greeks Delta, $\Delta = \partial_s V$, and Gamma, $\Gamma = \partial_{ss} V$. For the Black–Scholes model, we additionally compute the Vega, $\mathcal{V} = \partial_\sigma V$. We choose for both models the parameter $\sigma = 0.3$ and additionally for the variance gamma model $\nu = 0.04$, $\theta = -0.2$. The convergence rates on $G_0 = (K/2, 3/2K)$ are shown in Fig. 11.1. As predicted in Theorems 11.2.4 and 11.2.7, all Greeks converge with the optimal rate as the price V itself.

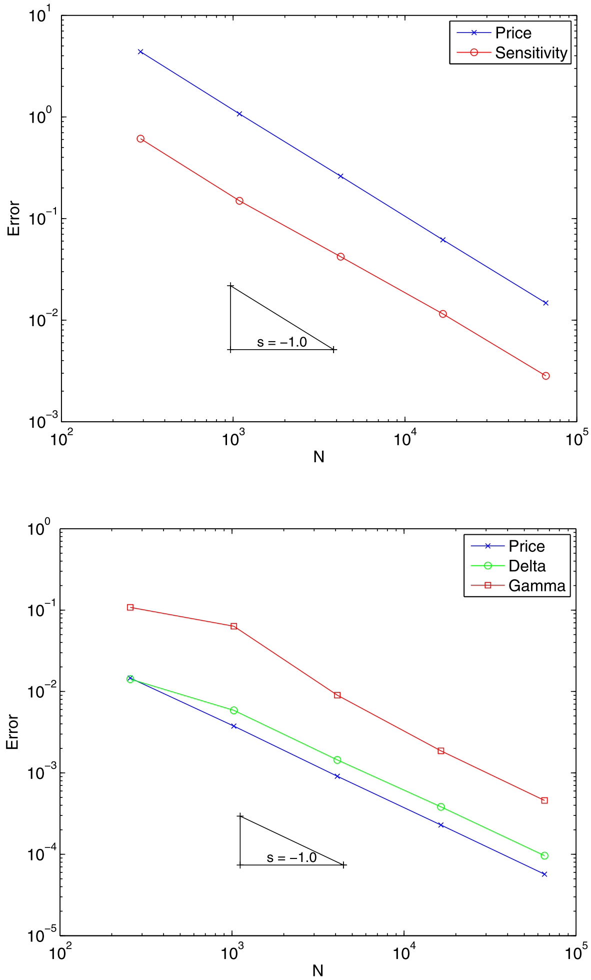


Fig. 11.2 Convergence rates of sensitivities in the Heston stochastic volatility (*top*) and a two-dimensional Black–Scholes model (*bottom*)

11.3.2 Multivariate Models

We first consider the Heston stochastic volatility model, see (9.21) for the (transformed) infinitesimal generator. We calculate the sensitivity $\widetilde{u}(\delta\rho)$ with respect to correlation ρ of the Brownian motions that drive the underlying and the volatility. According to (9.21), the operator $\widetilde{\mathscr{A}}^{\mathrm{H}}_{\kappa}(\delta\rho)$ is given by $\widetilde{\mathscr{A}}^{\mathrm{H}}_{\kappa}(\delta\rho) = \frac{1}{2}\delta\rho\beta(y\partial_{xy} + \kappa y^2\partial_x)$, hence the corresponding stiffness matrix $\widetilde{\mathbf{A}}^{\mathrm{H}}_{\kappa}$ becomes $\widetilde{\mathbf{A}}^{\mathrm{H}}_{\kappa} = -\frac{1}{2}\beta\mathbf{B}^1 \otimes (\mathbf{B}^{x_2} + \kappa\mathbf{M}^{x_2^2})$. We consider a European call with strike $K = 100$ and maturity $T = 0.5$. The model parameters for the sensitivity run are $\alpha = 2.5$, $\beta = 0.5$ and $m = 0.025$. Additionally, for the sensitivity with respect to ρ, we let $\rho_0 = -0.4$. The convergence rates on $G_0 = (K/2, 3/2K) \times (0.04, 1)$ are shown in Fig. 11.2.

We also consider a multi-asset option with payoff $g(s_1, s_2) = (s_1^{1/2} s_2^{1/2} - K)$ for the Black–Scholes model in dimension $d = 2$, strike $K = 1$ and maturity $T = 1.0$. We calculate the Greeks Delta, $\Delta_1 = \partial_{s_1}$, and Gamma, $\Gamma_{11} = \partial_{s_1 s_1}$. The parameters are $\sigma = (0.4, 0.1)$, $\rho_{12} = 0.2$. The convergence rates on $G_0 = (K/2, 3/2K)^2$ are shown in Fig. 11.2. Again we find that computed prices and sensitivities converge with the same rate.

11.4 Further Reading

In this section, we closely followed Hilber et al. [83]. Analytic formulas for the Greeks in diffusion type models and plain vanilla type contracts can be found in Reiss and Wyst [140]. Automatic differentiation of a finite element code is used to approximate Greeks in Achdou and Pironneau [1].

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Part II
Advanced Techniques and Models

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Chapter 12
Wavelet Methods

In the previous sections, we developed various algorithms for the efficient pricing of derivative contracts when the price of the underlying is a one-dimensional diffusion, a multidimensional diffusion, a general stochastic volatility or a one-dimensional Lévy process. In this part, we introduce variational numerical methods for pricing under yet more general processes with the aim of achieving *linear complexity*.

We say an asset pricing algorithm has *linear complexity* if it yields, for one payoff function, a vector of N option prices at maturity $T > 0$ in essentially, i.e. up to powers of $\log N$, $\mathcal{O}(N)$ operations and in essentially $\mathcal{O}(N)$ memory. It is of *order* $s > 0$, if the error in the computed option prices decreases asymptotically, as $N \to \infty$, and is $\mathcal{O}(N^{-s})$.

Consider, for example, the valuation of a European plain vanilla contract under either a local volatility or under constant elasticity of variance (see Sect. 4.5) assumptions. We use piecewise linear finite elements on a mesh of width $h > 0$ in log-price space and the θ-scheme in time with time step $k = \mathcal{O}(h)$. The number of degrees of freedom equals the number of grid points, i.e. $N = \mathcal{O}(h^{-1})$. There are $\mathcal{O}(k^{-1}) = \mathcal{O}(N)$ number of time steps to reach $t = T$ and in each time step, a linear system of equations with a tridiagonal matrix must be solved. Direct solution of a tridiagonal, linear system with a diagonally dominant coefficient matrix requires $\mathcal{O}(N)$ operations. Hence continuous, piecewise linear finite element methods in space and implicit time-stepping are of complexity $\mathcal{O}(N^2)$ and of accuracy at best $\mathcal{O}(h^2) = \mathcal{O}(N^{-2})$.[1] For a European plain vanilla contract in a Lévy market (see Sect. 10.6), we obtain an overall complexity of $\mathcal{O}(N^3)$ operations since the matrix to be inverted in each time step is fully populated and of size N.

These examples show that we only have superlinear complexity. Our goal in this chapter is to develop deterministic pricing algorithms of linear complexity. This is achieved by the following ingredients:

(i) High order, *discontinuous Galerkin time stepping* from $t = 0$ to T, to exploit analyticity of the transition semigroup of the price process,

[1] High order timestepping [146, 147] and the so-called hp-Finite Element Methods can improve on this substantially, at the expense of more involved implementations.

N. Hilber et al., *Computational Methods for Quantitative Finance*, Springer Finance, DOI 10.1007/978-3-642-35401-4_12, © Springer-Verlag Berlin Heidelberg 2013

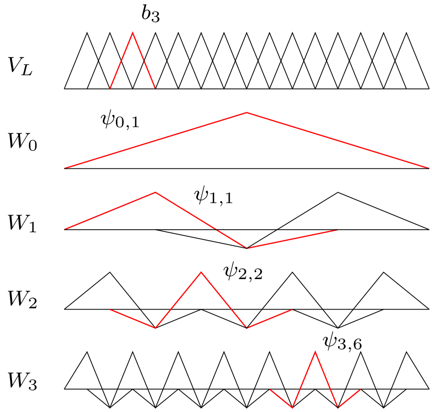


Fig. 12.1 Single-scale space V_L and its decomposition into multiscale wavelet spaces W_ℓ for $L = 3$

(ii) Iterative solution and judicious preconditioning of the linear systems in each time step,

(iii) *Wavelet finite element bases* for the *preconditioning* and the *matrix compression* of the nonlocal operators arising with jump type price processes.

We start with explaining spline wavelets on an interval.

12.1 Spline Wavelets

As in Sect. 3.3, we discretize the domain $G = (a, b)$ by equidistant mesh points

$$a = x_0 < x_1 < x_2 < \cdots < x_{N_L+1} = b,$$

where we assume the number N_L satisfies $N_L = 2^{L+1} - 1$ with $L \in \mathbb{N}_0$ and use the notation $V_L = V_{N_L}$. Then, we have the nested spaces with $2, 4, \ldots, 2^{L+1}$ subintervals

$$V_0 \subset V_1 \subset \cdots \subset V_L,$$

and $\dim V_\ell = 2^{\ell+1} - 1 =: N_\ell$. In Sect. 3.3, we generated V_ℓ by a *single-scale* basis $V_\ell = \operatorname{span}\{b_{\ell,j}(x) : 1 \le j \le N_\ell\}$. Here, we change notation and we write $b_{\ell,j}$ instead of b_j to indicate the refinement level. Wavelets constitute a so-called *hierarchical* or *multi-scale* basis. We start with $\{\psi_{0,1}\}$ for the space V_0. Then, we add basis functions $\{\psi_{1,1}, \psi_{1,2}\}$ such that $\operatorname{span}\{\psi_{0,1}, \psi_{1,1}, \psi_{1,2}\} = V_1$. Similarly, we add again basis functions $\{\psi_{2,1}, \psi_{2,2}, \psi_{2,3}, \psi_{2,4}\}$ such that $\operatorname{span}\{\psi_{0,1}, \psi_{1,1}, \psi_{1,2}, \psi_{2,1}, \psi_{2,1}, \psi_{2,3}, \psi_{2,4}\} = V_2$, and so on. Therefore, we introduce for $\ell \in \mathbb{N}_0$ the complement spaces $W_\ell = \operatorname{span}\{\psi_{\ell,k} \ : \ k \in \nabla_\ell\}$ where $\nabla_\ell := \{1, \ldots, 2^\ell\}$ such that $V_\ell = V_{\ell-1} \oplus W_\ell$, $\ell \ge 1$ and $V_0 = W_0$. This decomposition is illustrated in Fig. 12.1.

We assume that the wavelets $\psi_{\ell,k}$ have compact support $|\operatorname{supp} \psi_{\ell,k}| \le C2^{-\ell}$, are normalized in $L^2(G)$, i.e. $\|\psi_{\ell,k}\|_{L^2} = 1$, and $\Phi_\ell := \{b_{\ell,j}(x) : 1 \le j \le N_\ell\}$ has approximation order p. In addition, we associated with Φ_ℓ a dual basis, $\widetilde{\Phi}_\ell = \{\widetilde{b}_{\ell,j} : 1 \le j \le N_\ell\}$, i.e. one has $\langle b_{\ell,j}, \widetilde{b}_{\ell,j'}\rangle = \delta_{j,j'}$, $1 \le j, j' \le N_\ell$. The approximation order of $\widetilde{\Phi}_\ell$ is denoted by $\widetilde{p}$ and we assume $p \le \widetilde{p}$.

Example 12.1.1 (Piecewise linear wavelets) We define the wavelet functions $\psi_{\ell,k}$ as the following piecewise linear functions. Let $h_\ell = 2^{-\ell-1}(b-a)$ and $c_\ell := \sqrt{3}/2 \cdot (2h_\ell)^{-1/2}$. For $\ell = 0$ we have $N_0 = 1$ and $\psi_{0,1}$ is the function with value $2c_0$ at $x = a + h_0$. For $\ell \ge 1$ the wavelet $\psi_{\ell,1}$ has the values $\psi_{\ell,1}(a + h_\ell) = 2c_\ell$, $\psi_{\ell,1}(a + 2h_\ell) = -c_\ell$ and zero at all other nodes. The wavelet $\psi_{\ell,2^\ell}$ has the values $\psi_{\ell,2^\ell}(b - h_\ell) = 2c_\ell$, $\psi_{\ell,2^\ell}(b - 2h_\ell) = -c_\ell$ and zero at all other nodes. The wavelet $\psi_{\ell,k}$ with $1 < k < 2^\ell$ has the values $\psi_{\ell,k}(a + (2k-2)h_\ell) = -c_\ell$, $\psi_{\ell,k}(a + (2k-1)h_\ell) = 2c_\ell$, $\psi_{\ell,k}(a + 2kh_\ell) = -c_\ell$ and zero at all other nodes. For $\ell = 0, \dots, 3$, these wavelets are plotted in Fig. 12.1. The constants c_ℓ are chosen such that the wavelets $\psi_{\ell,k}$ are normalized in $L^2(G)$. Note that these biorthogonal wavelets W_ℓ are not orthogonal on $V_{\ell-1}$. But the inner wavelets $\psi_{\ell,k}$ with $1 < k < 2^\ell$ have two vanishing moments, i.e. $\int \psi_{\ell,k}(x)x^n \,\mathrm{d}x = 0$ for $n = 0, 1$. The approximation order of V_ℓ is $p = 2$.

Since $V_L = \operatorname{span}\{\psi_{\ell,k} : 0 \le \ell \le L, k \in \nabla_\ell\}$, we have a unique decomposition

$$u = \sum_{\ell=0}^{L} u_\ell = \sum_{\ell=0}^{L} \sum_{k \in \nabla_\ell} u_{\ell,k}\psi_{\ell,k},$$

for any $u \in V_L$ with $u_\ell \in W_\ell$. Furthermore, any $u \in \widetilde{H}^s(G)$, $0 \le s \le p$ admits a representation as an infinite wavelet series,

$$u = \sum_{\ell=0}^{\infty} u_\ell = \sum_{\ell=0}^{\infty} \sum_{k \in \nabla_\ell} u_{\ell,k}\psi_{\ell,k}, \tag{12.1}$$

which converges in $\widetilde{H}^s(G)$. The coefficients $u_{\ell,k}$ are the so-called wavelet coefficients of the function u.

12.1.1 Wavelet Transformation

For $u \in V_L$ we want to show how to obtain the multi-scale wavelet coefficients $d_{\ell,k} := u_{\ell,k}$ in $u = \sum_{\ell=0}^{L} \sum_{k \in \nabla_\ell} d_{\ell,k}\psi_{\ell,k}$, from the single-scale coefficients $c_{\ell,j}$ in $u = \sum_{j=1}^{N_L} c_{\ell,j} b_{\ell,j}$.

For the wavelets $\psi_{\ell,k}$ as in Example 12.1.1 and hat functions $b_{\ell,j}$, we have

$$\psi_{\ell,k} = -0.5 b_{\ell,2k-2} + b_{\ell,2k-1} - 0.5 b_{\ell,2k}, \quad 1 < k < 2^\ell,$$

$$\psi_{\ell,1} = b_{\ell,1} - 0.5 b_{\ell,2},$$

$$\psi_{\ell,2^\ell} = -0.5 b_{\ell,2^{\ell+1}-2} + b_{\ell,2^{\ell+1}-1},$$

$$b_{\ell-1,j} = 0.5b_{\ell,2j-1} + b_{\ell,2j} + 0.5b_{\ell,2j+1},$$

where we set the normalization factors $c_\ell = 0.5$ for simplicity. For $u = \sum_{j=1}^{N_L} c_{L,j} b_{L,j}$, we have

$$\sum_{j=1}^{N_{\ell+1}} c_{\ell+1,j} b_{\ell+1,j} = \sum_{j=1}^{N_\ell} c_{\ell,j} b_{\ell,j} + \sum_{k \in \nabla_{\ell+1}} d_{\ell+1,k} \psi_{\ell+1,k},$$

and therefore,

$$\begin{aligned}
c_{\ell+1,2j+1} &= 0.5c_{\ell,j} + 0.5c_{\ell,j+1} + d_{\ell+1,j+1}, \quad j = 2, \dots, N_\ell - 1, \\
c_{\ell+1,2j} &= c_{\ell,j} - 0.5d_{\ell+1,j} - 0.5d_{\ell+1,j+1}, \quad j = 2, \dots, N_\ell, \\
c_{\ell+1,1} &= 0.5c_{\ell,1} + d_{\ell+1,1}, \\
c_{\ell+1,N_{\ell+1}} &= 0.5c_{\ell,N_\ell} + d_{\ell+1,2^{\ell+1}}.
\end{aligned} \tag{12.2}$$

This can be written in matrix form

$$\begin{pmatrix} c_{\ell+1,1} \\ c_{\ell+1,2} \\ \vdots \\ \vdots \\ c_{\ell+1,N_{\ell+1}-1} \\ c_{\ell+1,N_{\ell+1}} \end{pmatrix} = \begin{pmatrix} 1 & 0.5 & & & & \\ -0.5 & \ddots & -0.5 & & & \\ & 0.5 & \ddots & 0.5 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -0.5 & \ddots & -0.5 \\ & & & & 0.5 & 1 \end{pmatrix} \begin{pmatrix} d_{\ell+1,1} \\ c_{\ell,1} \\ \vdots \\ \vdots \\ c_{\ell,N_\ell} \\ d_{\ell+1,2^{\ell+1}} \end{pmatrix}.$$

Now, starting with $u = \sum_{j=1}^{N_L} c_{L,j} b_{L,j}$, we can compute using the decomposition algorithm (12.2) the coefficients $c_{L-1,j}$ and $d_{L,k}$. Iteratively, we decompose $c_{\ell+1,j}$ into $c_{\ell,j}$ and $d_{\ell+1,j}$ until we have the series representation $u = \sum_{\ell=0}^{L} \sum_{k \in \nabla_\ell} d_{\ell,k} \psi_{\ell,k}$. Similarly, we can obtain the single-scale coefficients $c_{L,j}$ from the multi-scale wavelet coefficients $d_{\ell,k}$.

12.1.2 Norm Equivalences

For preconditioning of the large systems which are solved at each time step, we require *wavelet norm equivalences*. These are analogous to the classical Parseval relation in Fourier analysis which allow expressing Sobolev norms of a periodic function u in terms of (weighted) sums of its Fourier coefficients. Wavelets allow for analogous statements: the Parseval equation is replaced by appropriate inequalities and the function u need not be periodic. Since in the pure jump case $\sigma = 0$ we obtain fractional order Sobolev spaces $\widetilde{H}^s(G)$, it is essential that the wavelet norm equivalences hold in the whole range of Sobolev spaces, i.e. from $L^2(G)$ to $H_0^1(G)$. For $u \in L^2(G)$, there holds

$$\|u\|_{L^2(G)}^2 \sim \sum_{\ell=0}^{\infty} \|u_\ell\|_{L^2(G)}^2 \sim \sum_{\ell=0}^{\infty} \sum_{k \in \nabla_\ell} |u_{\ell,k}|^2.$$

The mapping $u \mapsto u_0 + \cdots + u_\ell$ defines a continuous projector $\mathcal{P}_\ell : L^2(G) \to V_\ell$. For general Sobolev spaces $\widetilde{H}^s(G)$, we have the direct (or Jackson type) estimate,

$$\|u - \mathcal{P}_\ell u\|_{L^2(G)} \leq C 2^{-ls} \|u\|_{\widetilde{H}^s(G)}, \quad 0 \leq s \leq p.$$

For $u \in V_\ell$ we also have the inverse (or Bernstein-type) estimates,

$$\|u\|_{\widetilde{H}^s(G)} \leq C 2^{ls} \|u\|_{L^2(G)}, \quad s < p - 1/2.$$

Using the inverse estimate and the series representation (12.1), we have

$$\|u\|_{\widetilde{H}^s(G)}^2 \sim \sum_{\ell=0}^{\infty} \|u_\ell\|_{\widetilde{H}^s(G)}^2 \leq C \sum_{\ell=0}^{\infty} \sum_{k \in \nabla_\ell} 2^{2ls} |u_{\ell,k}|^2, \quad 0 \leq s < p - 1/2.$$

In the other direction, we have for $u \in \widetilde{H}^p(G)$

$$\begin{aligned} \|u_\ell\|_{L^2(G)} = \|\mathcal{P}_\ell u - \mathcal{P}_{\ell-1} u\|_{L^2(G)} &\leq \|\mathcal{P}_\ell u - u\|_{L^2(G)} + \|\mathcal{P}_{\ell-1} u - u\|_{L^2(G)} \\ &\leq C(2^{-\ell p} + 2^p 2^{-\ell p}) \|u\|_{\widetilde{H}^p(G)}, \end{aligned}$$

and therefore,

$$\sum_{\ell=0}^{L} \sum_{k \in \nabla_\ell} 2^{2lp} |u_{\ell,k}|^2 \leq C' L \|u\|_{\widetilde{H}^p(G)}^2.$$

Unfortunately, we do not quite obtain the required bound. This estimate is sharp. For $0 \leq s < p$, one can modify this argument by using the modulus of continuity and obtain

$$\|u\|_{\widetilde{H}^s(G)}^2 \sim \sum_{\ell=0}^{\infty} \|u_\ell\|_{\widetilde{H}^s(G)}^2 \sim \sum_{\ell=0}^{\infty} \sum_{k \in \nabla_\ell} 2^{2ls} |u_{\ell,k}|^2, \quad 0 \leq s < p - 1/2. \tag{12.3}$$

12.2 Wavelet Discretization

Let X be a Lévy process with characteristic triplet $(0, \nu, 0)$ satisfying Assumption 10.2.3. As shown in Chap. 10, the option price can be obtained by the solution of the PIDE

$$\partial_t u - \mathcal{A} u = 0 \quad \text{in } J \times G, \quad u(0, x) = u_0 \quad \text{in } G = (-R, R), \tag{12.4}$$

with $\mathcal{A}: \widetilde{H}^{\alpha/2}(G) \to \widetilde{H}^{-\alpha/2}(G)$, $(\mathcal{A}f)(x) = \int_{\mathbb{R}}(f(x+z) - f(x) - z\partial_x f(x))\nu(\mathrm{d}z)$. The infinitesimal generator $\mathcal{A}$ is a pseudodifferential operator of order α. The corresponding variational formulation reads

$$\begin{aligned}
&\text{Find } u \in L^2(J; \widetilde{H}^{\alpha/2}(G)) \cap H^1(J; L^2(G)) \text{ such that} \\
&(\partial_t u, v) + a(u, v) = 0, \ \forall v \in \widetilde{H}^{\alpha/2}(G), \ \text{a.e. in } J, \\
&u(0) = u_0,
\end{aligned} \tag{12.5}$$

with bilinear form

$$\begin{aligned}
a(u, v) &= -\langle \mathcal{A}u, v\rangle_{\widetilde{H}^{-\alpha/2}, \widetilde{H}^{\alpha/2}} \\
&= \int_G \int_G u'(y)v'(x)k^{-2}(y-x)\,\mathrm{d}y\,\mathrm{d}x, \quad u, v \in \widetilde{H}^{\alpha/2}(G).
\end{aligned}$$

As before, we discretize this equation successively in space and time. Contrary to what has been done previously, however, we now use wavelet basis functions in (log-price) space.

12.2.1 Space Discretization

Let $\mathcal{T}_0 = \{x_0 = -R < x_1 = 0 < x_2 = R\}$ be a coarse partition of G. Furthermore, define the mesh $\mathcal{T}_\ell$, for $\ell \in \mathbb{N}$, recursively by bisection of each interval in $\mathcal{T}^{\ell-1}$. We denote our computational mesh obtained in this way as $\mathcal{T}_L$, for some $L \in \mathbb{N}_0$, with mesh size $h = R2^{-L}$. The finite element space $V_\ell \subset \widetilde{H}^{\alpha/2}(G)$ used for the spatial discretization is the space of all continuous piecewise polynomials of approximation order p on the triangulation $\mathcal{T}_\ell$ which vanish on the boundary ∂G.

The semi-discrete problem corresponding to (12.5) reads:

$$\begin{aligned}
&\text{Find } u_L \in H^1(J; V_L) \text{ such that} \\
&(\partial_t u_L, v_L) + a(u_L, v_L) = 0, \ \forall v_L \in V_L, \ \text{a.e. in } J, \\
&u_L(0) = u_{L,0},
\end{aligned} \tag{12.6}$$

where $u_{L,0} = \mathcal{P}_{N_L}$ is the L^2 projection of u_0 onto V_L. We have the following a priori result on the spatial semi-discretization which is shown in [123].

Theorem 12.2.1 *Assume the Lévy density k satisfies Assumption* 10.2.3. *Then, for $t > 0$, the following error estimate holds*:

$$\|u(t) - u_L(t)\|_{L^2(G)} \le C \min\{1, h^p t^{-\frac{p}{\alpha}}\}.$$

Here, $C > 0$ is a constant independent of h and t, and u, u_L are the solutions of (12.5) *and* (12.6), *respectively.*

Using the basis $\{\psi_{\ell,k} : 0 \le \ell \le L, k \in \nabla_\ell\}$ of V_L, we need to compute the stiffness matrix entries $\mathbf{A}_{(\ell',k'),(\ell,k)} = a(\psi_{\ell,k}, \psi_{\ell',k'})$. Since $\mathbf{A}$ is, in general, densely populated, we use wavelet compression to reduce the number of non-zero entries to $\mathcal{O}(N_L)$. We need the following smoothness assumption on the Lévy density k:

$$|\partial^n k(z)| \le C_0\, C^n n! |z|^{-\alpha-1-n}, \quad z \ne 0, \quad \forall n \in \mathbb{N}_0, \tag{12.7}$$

for $C_0, C > 0$. These kind of estimates are called Caldéron–Zygmund estimates.

12.2.2 Matrix Compression

The compression scheme is based on the fact that the matrix entries $\mathbf{A}_{(\ell',k'),(\ell,k)}$ can be estimated a priori and therefore neglected if these are smaller than some cut-off parameter. There are two reasons for an entry to be omitted. Either the distance of the supports $\operatorname{supp}\psi_{\ell,k}$ and $\operatorname{supp}\psi_{\ell',k'}$ or the distance of the singular supports (the singular support of a wavelet is that subset of G where the wavelet is not smooth) is large enough. The distance of support is denoted by

$$\delta := \operatorname{dist}\{\operatorname{supp}\psi_{\ell,k}, \operatorname{supp}\psi_{\ell',k'}\},$$

and the distance of singular support

$$\delta^{\text{sing}} := \begin{cases} \operatorname{dist}\{\operatorname{singsupp}\psi_{\ell,k}, \operatorname{supp}\psi_{\ell',k'}\} & \text{if } \ell \le \ell', \\ \operatorname{dist}\{\operatorname{supp}\psi_{\ell,k}, \operatorname{singsupp}\psi_{\ell',k'}\} & \text{else.} \end{cases}$$

Theorem 12.2.2 *Let X be a Lévy process with Lévy density k satisfying* (12.7). *Define the compression scheme by*

$$\widetilde{\mathbf{A}}_{(\ell',k'),(\ell,k)} = \begin{cases} 0 & \text{if } \delta > \mathcal{B}_{\ell,\ell'}, \\ 0 & \text{if } \delta < c\, 2^{-\min\{\ell,\ell'\}} \text{ and } \delta^{\text{sing}} > \widetilde{\mathcal{B}}_{\ell,\ell'}, \\ \mathbf{A}_{(\ell',k'),(\ell,k)}, & \text{else,} \end{cases}$$

with cut-of parameter

$$\mathcal{B}_{\ell,\ell'} = a \max\left\{2^{-\min\{\ell,\ell'\}}, 2^{\frac{2L(p-\alpha/2)-(\ell+\ell')(p+\widetilde{p})}{2\widetilde{p}+\alpha}}\right\}, \quad a > 1,$$

$$\widetilde{\mathcal{B}}_{\ell,\ell'} = \widetilde{a} \max\left\{2^{-\max\{\ell,\ell'\}}, 2^{\frac{2L(p-\alpha/2)-(\ell+\ell')p-\max\{\ell,\ell'\}\widetilde{p}}{\widetilde{p}+\alpha}}\right\}, \quad \widetilde{a} > 1.$$

The number of non-zero entries for the compressed matrix $\widetilde{\mathbf{A}}$ is $\mathcal{O}(N_L)$.

A proof can be found in [51, Theorem 11.1]. We call a compression scheme *first compression* if $\delta > \mathcal{B}_{\ell,\ell'}$, i.e. if the distance of the wavelets is large enough, and *second compression* if $\delta^{\text{sing}} > \widetilde{\mathcal{B}}_{\ell,\ell'}$ (and $\delta < c\, 2^{-\min\{\ell,\ell'\}}$), i.e. if the distance of the singular support is large enough (compare with Fig. 12.2).

We give an example for the matrix compression.

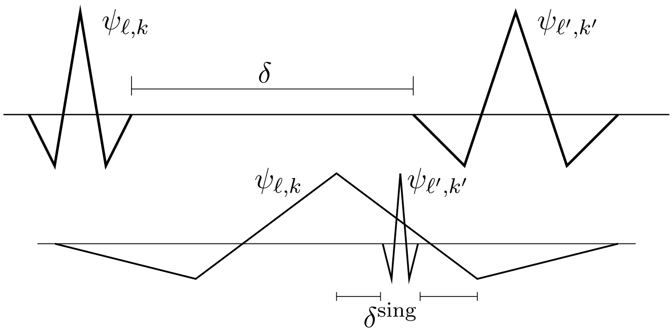


Fig. 12.2 First compression: a matrix entry $\mathbf{A}_{(\ell',k'),(\ell,k)}$ is set to 0 if δ is large enough (*top*). Second compression: a matrix entry $\mathbf{A}_{(\ell',k'),(\ell,k)}$ is set to 0 if δ^{sing} is large enough (*bottom*)

Example 12.2.3 Let $a = \widetilde{a} = 1$, $p = 2$, $\widetilde{p} = 4$, $\alpha = 0.5$ and $L = 7$. The corresponding compression scheme is plotted in Fig. 12.3. Zero entries due to the first compression are left white, zero entries due to the second compression are colored red and non-zero entries blue regardless of their size. Additionally, we plot the number of non-zero entries which grow like $\mathcal{O}(N)$. For $L = 7$ there are only 14 % non-zero entries.

The matrix compression induces instead of (12.6) a perturbed semi-discretization

$$\begin{aligned}
&\text{Find } \widetilde{u}_L \in H^1(J; V_L) \text{ such that}\\
&(\partial_t \widetilde{u}_L, v_L) + \widetilde{a}(\widetilde{u}_L, v_L) = 0,\ \forall v_L \in V_L,\ \text{a.e. in } J,\\
&\widetilde{u}_L(0) = u_{L,0}.
\end{aligned} \tag{12.8}$$

We have the following analog to Theorem 12.2.1 which states that the convergence rate of the numerical solutions obtained from space semi-discretization with matrix compression do not deteriorate.

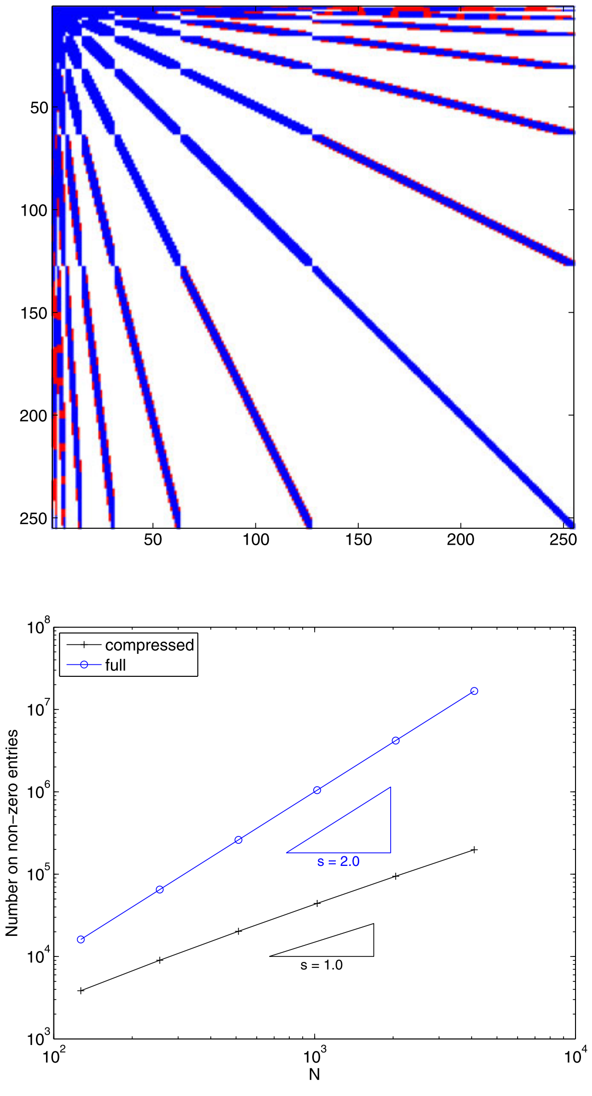
Theorem 12.2.4 *Assume the Lévy density k satisfies Assumption* 10.2.3 *and* (12.7). *Moreover, let* $\varepsilon = a^{-2(\widetilde{p}+\alpha/2)} + \widetilde{a}^{-(\widetilde{p}+\alpha)}$ *be sufficiently small. Then, for* $t > 0$, *we have the following a priori error estimate for the perturbed semi-discrete problem* (12.8):

$$\|u(t) - \widetilde{u}_h(t)\|_{L^2(G)} \le C \min\{1, h^p t^{-\frac{p}{\alpha}}\},$$

where $C > 0$ *is a constant independent of h and t, and u is the solution of* (12.5).

This can be proven combining the arguments given in [51, Theorem 10.1] and [123, Theorem 2].

Fig. 12.3 Wavelet compression for level $L = 7$ (*top*) and number of non-zero entries (*bottom*)



12.2.3 Multilevel Preconditioning

One of the advantages of multi-scale discretizations is their ability to precondition large linear systems due to the norm equivalences. With (12.3) for $s = 0$ we have for every $u \in V_L$ with coefficient vector $\underline{u} \in \mathbb{R}^{N_L}$ that

$$\langle \underline{u}, \mathbf{M}\underline{u} \rangle = \|u\|^2_{L^2(G)} \sim |\underline{u}|^2.$$

Therefore, the condition number of $\mathbf{M}$ is bounded, independent of the level L, i.e. $\kappa(\mathbf{M}) < c$, $\forall L \in \mathbb{N}$. Denote by $\mathbf{D}$ the diagonal matrix with entries $2^{\alpha\ell}$ for an index corresponding to level ℓ. Then, (12.3) for $s = \alpha/2$ implies that

$$\langle \underline{u}, \mathbf{A}\underline{u} \rangle \sim \|u\|^2_{\widetilde{H}^{\alpha/2}(G)} \sim \langle \underline{u}, \mathbf{D}\underline{u} \rangle.$$

Written in terms of $\widehat{\underline{u}} = \mathbf{D}^{1/2}\underline{u}$, we have

$$\langle \widehat{\underline{u}}, \mathbf{D}^{-1/2}\mathbf{A}\mathbf{D}^{-1/2}\widehat{\underline{u}} \rangle \sim |\widehat{\underline{u}}|^2. \tag{12.9}$$

And so we obtain that the condition number $\kappa(\mathbf{D}^{-1/2}\mathbf{A}\mathbf{D}^{-1/2})$ is bounded, independent of the level L.

Example 12.2.5 We compute the condition number for several operators with order ρ on various number of degrees of freedom N in Fig. 12.4. It can be seen that in the hat basis the condition numbers grow like $\mathcal{O}(N^\rho)$. Using the multi-scale wavelet basis with preconditioning, the condition numbers are bounded, independent of N.

We next address the time-discretization to obtain a fully discrete algorithm. As before, we could use a θ-scheme to perform the timestepping. However, since the parabolic problem generates an analytic semigroup, we present now a high-order, discontinuous Galerkin (hp-dG) timestepping scheme.

12.3 Discontinuous Galerkin Time Discretization

For $0 < T < \infty$ and $M \in \mathbb{N}$, let $\mathcal{M} = \{J_m\}_{m=1}^M$ be a partition of $J = (0, T)$ into M subintervals $J_m = (t_{m-1}, t_m)$, $m = 1, \dots, M$, with

$$0 = t_0 < t_1 < t_2 < \cdots < t_M = T.$$

Moreover, denote by $k_m = t_m - t_{m-1}$ the length of J_m. For $u \in H^1(\mathcal{M}, V_L) = \{v \in L^2(J, V_L) : v|_{J_m} \in H^1(J_m, V_L), m = 1, \dots, M\}$, define the one-sided limits

$$\begin{aligned} u_+^m &= \lim_{s\to 0^+} u(t_m + s), \quad & m = 0, \dots, M-1, \\ u_-^m &= \lim_{s\to 0^+} u(t_m - s), \quad & m = 1, \dots, M, \end{aligned}$$

and the jumps

$$[\![u]\!]_m = u_+^m - u_-^m, \qquad m = 1, \dots, M-1.$$

To each time step J_m, we associate an approximation order $r_m \geq 0$. The orders are collected in the degree vector $\mathbf{r} = (r_1, \dots, r_M)$. We introduce the following space of functions which are discontinuous in time:

$$S^{\mathbf{r}}(\mathcal{M}, V_L) = \{u \in L^2(J, V_L) : u|_{J_m} \in S^{r_m}(J_m, V_L),\ m = 1, \dots, M\},$$

where $S^{r_m}(J_m)$ denotes the space of polynomials of degree at most r_m on J_m.

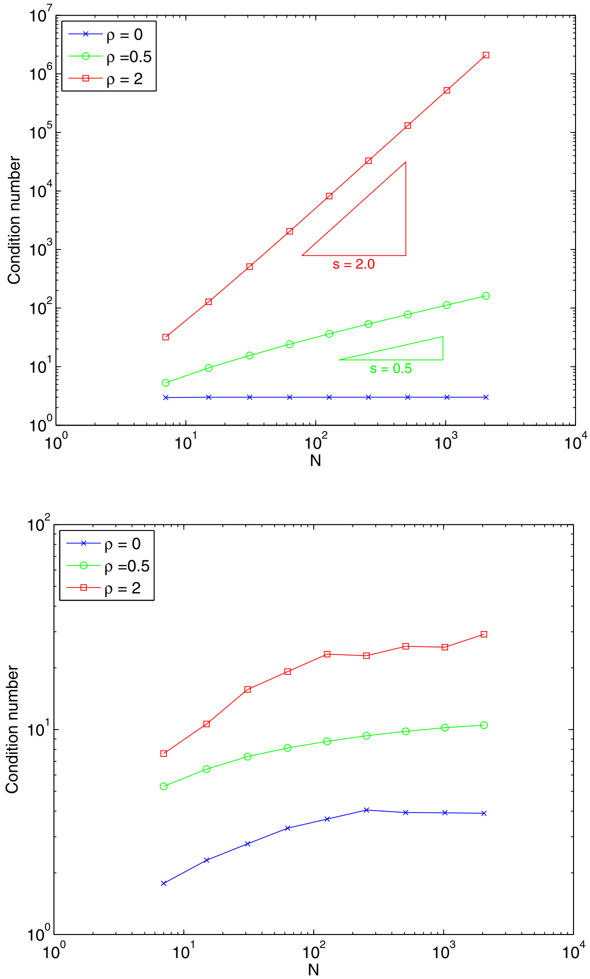


Fig. 12.4 Condition number for hat functions (*top*) and wavelets with preconditioning (*bottom*)

Consider the problem (12.8) where we drop the $\tilde{}$ for notational simplicity. For a test function $w \in C^1(J, V_L)$ with $w(T) = 0$, we integrate the variational formulation with respect to the time variable t and use integration by parts to obtain

$$\int_J \Big((\partial_t u_L, w) + a(u_L, w) \Big)\, \mathrm{d}t = 0$$
$$\Rightarrow \quad \int_J \Big(-(u_L, w') + a(u_L, w) \Big)\, \mathrm{d}t = (u_{L,0}, w(0)).$$

Replacing u_L by a function $U \in S^{\mathbf{r}}(\mathcal{M}, V_L)$ and integrating by parts in each J_m, we obtain with $w^m = w(t_m)$

$$-\int_J (u_L, w')\,\mathrm{d}t = -\sum_{m=1}^{M} \Big((U, w)|_{t_{m-1}}^{t_m} - \int_{J_m} (U', w)\,\mathrm{d}t \Big)$$
$$= \int_J (U', w)\,\mathrm{d}t + \sum_{m=1}^{M-1} ([\![U]\!]_m, w^m) + (U_+^0, w^0).$$

Therefore, we obtain the fully discrete scheme: Find $U \in S^{\mathbf{r}}(\mathcal{M}, V_L)$ such that for all $W \in S^{\mathbf{r}}(\mathcal{M}, V_L)$

$$\int_J \Big((U', W) + a(U, W) \Big)\,\mathrm{d}t + \sum_{m=1}^{M-1} ([\![U]\!]_m, W_+^m) + (U_+^0, W_+^0) = (u_{L,0}, W_+^0). \tag{12.10}$$

The solution operator of the parabolic problem generates a holomorphic semigroup. Therefore, the solution $u(t)$ is analytic with respect to t for all $t > 0$. However, due to the non-smoothness of the initial data, the solution may be singular at $t = 0$. By the use of the so-called geometric time discretization, the low regularity of the solution at $t = 0$ can be resolved.

Definition 12.3.1 We call a partition $\mathcal{M}_{M,\gamma} = \{J_m\}_{m=1}^M$ of the time interval $J = (0, T)$, $0 < T < \infty$, *geometric* with M time steps $J_m = (t_{m-1}, t_m)$, $m = 1, \ldots, M$, and grading factor $\gamma \in (0, 1)$ if

$$t_0 = 0, \qquad t_m = T\gamma^{M-m}, \quad 1 \le m \le M.$$

A polynomial degree vector $\mathbf{r} = (r_1, \ldots, r_M)$ is called *linear* with slope $\mu > 0$ on $\mathcal{M}_{M,\gamma}$ if $r_1 = 0$ and $r_m = \lfloor \mu m \rfloor$, $m = 2, \ldots, M$, where $\lfloor \mu m \rfloor = \max\{q \in \mathbb{N}_0 \,:\, q \le \mu m\}$.

We have the following a priori error estimate on the hp-dG scheme [123, Theorem 3].

Theorem 12.3.2 *Let $u_0 \in \widetilde{H}^s(G)$, $0 < s \le 1$ and the assumptions of Theorem 12.2.4 be fulfilled. Then, there exist $\mu_0, m_0 > 0$ such that for all geometric partitions $\mathcal{M}_{M,\gamma}$ with $M \ge m_0 |\log h|$, and all polynomial degree vectors $\mathbf{r}$ on $\mathcal{M}_{M,\gamma}$ with slope $\mu > \mu_0$, the fully discrete solution U obtained by (12.10) satisfies*

$$\|u(T) - U(T)\|_{L^2(G)} \le Ch^p, \tag{12.11}$$

where $C > 0$ is a constant independent of mesh width h, and u is the solution of the parabolic problem (12.5).

We now study the linear systems resulting from the hp-dG method (12.10). We show that they may be solved iteratively, without causing a loss in the rates of convergence in the error estimate (12.11), by the use of an incomplete GMRES method. Furthermore, we prove that the overall complexity is linear (up to logarithmic terms).

12.3.1 Derivation of the Linear Systems

The hp-dG time stepping scheme (12.10) corresponds to a linear system of size $(r_m+1)N_L$ to be solved in each time step $m = 1, \ldots, M$. Let $\{\widehat{\varphi}_j : j = 0, \ldots, r_m\}$ be a basis of the polynomial space $S^{r_m}(-1, 1)$. We also refer to $\widehat{\varphi}_j$ as the reference time shape functions. On the time interval $J_m = (t_{m-1}, t_m)$, the time shape functions $\varphi_{m,j}$ are then defined as $\varphi_{m,j} = \widehat{\varphi}_j \circ F_m^{-1}$, where F_m is the mapping from the reference interval $(-1, 1)$ to J_m given by

$$F_m(\widehat{t}) = \frac{1}{2}(t_{m-1} + t_m) + \frac{1}{2}k_m\widehat{t}.$$

Since the semi-discrete approximation $U|_{J_m}$ and the test function $W|_{J_m}$ in (12.10) are both in $S^{r_m}(J_m, V_L)$, they can be written in terms of the basis $\{\varphi_{m,j} : j = 0, \ldots, r_m\}$,

$$U|_{J_m}(x,t) = \sum_{j=0}^{r_m} U_{m,j}(x)\varphi_{m,j}(t), \quad W|_{J_m}(x,t) = \sum_{j=0}^{r_m} W_{m,j}(x)\varphi_{m,j}(t).$$

We choose normalized Legendre polynomials as reference time shape functions, i.e.

$$\widehat{\varphi}_j(\widehat{t}) = \sqrt{j+1/2} \cdot L_j(\widehat{t}), \quad j \in \mathbb{N}_0, \tag{12.12}$$

where L_j are the usual Legendre polynomials of degree j on $(-1, 1)$.

Example 12.3.3 The first four reference time shape functions of the form (12.12) are

$$\begin{aligned}
\widehat{\varphi}_0(\widehat{t}) &= \sqrt{1/2}, \\
\widehat{\varphi}_1(\widehat{t}) &= \sqrt{3/2} \cdot \widehat{t}, \\
\widehat{\varphi}_2(\widehat{t}) &= \sqrt{5/2} \cdot (3\widehat{t}^2 - 1)/2, \\
\widehat{\varphi}_3(\widehat{t}) &= \sqrt{7/2} \cdot (5\widehat{t}^3 - 3\widehat{t})/2.
\end{aligned}$$

The variational formulation (12.10) then reads: For $m = 1, \dots, M$, find $(U_{m,j})_{j=0}^{r_m} \in V_L^{r_m+1}$ such that for all $(W_{m,i})_{i=0}^{r_m} \in V_L^{r_m+1}$ the following holds:

$$\sum_{i,j=0}^{r_m} \mathbf{C}_{ij}^m (U_{m,j}, W_{m,i}) + \frac{k_m}{2} \sum_{i,j=0}^{r_m} \mathbf{I}_{ij}^m \, a(U_{m,j}, W_{m,i}) = \sum_{i=0}^{r_m} f_{m,i},$$

where $f_{m,i} = \widehat{\varphi}_i^+(-1)(U_{m-1}(t_{m-1}), W_{m,i})$, with $U_0(t_0) = u_{L,0} \in V_L$, and for $i, j = 1, \dots, r_m$,

$$\mathbf{C}_{ij}^m = \int_{-1}^{1} \widehat{\varphi}_j' \widehat{\varphi}_j \, \mathrm{d}\widehat{t} + \widehat{\varphi}_j^+(-1)\widehat{\varphi}_i^+(-1), \qquad \mathbf{I}_{ij}^m = \int_{-1}^{1} \widehat{\varphi}_j \widehat{\varphi}_i \, \mathrm{d}\widehat{t} = \delta_{ij}.$$

The matrices $\mathbf{C}^m$ and $\mathbf{I}^m$, $m = 1, \dots, M$, are independent of the time step and can be calculated in a preprocessing step. Their size, however, depends on the corresponding approximation order r_m.

Denoting by $\mathbf{M}$ and $\mathbf{A}$ the mass and (wavelet compressed) stiffness matrix with respect to $(\cdot,\cdot)$ and $a(\cdot,\cdot)$, (12.10) takes the matrix form

$$\begin{aligned} &\text{Find } \underline{u}^m \in \mathbb{R}^{(r_m+1)N_L} \text{ such that for } m = 1, \dots, M \\ &\Big(\mathbf{C}^m \otimes \mathbf{M} + \frac{k}{2}\mathbf{I}^m \otimes \mathbf{A}\Big)\underline{u}^m = (\underline{\varphi}^m \otimes \mathbf{M})\underline{u}^{m-1}, \\ &\underline{u}^0 = \underline{u}_0, \end{aligned} \tag{12.13}$$

where $\underline{u}^m$ denotes the coefficient vector of $U|_{J_m}$ and $\underline{\varphi}^m := (\widehat{\varphi}_1^+(-1), \dots, \widehat{\varphi}_{r_m+1}^+(-1))^\top \in \mathbb{R}^{r_m+1}$. Furthermore, $\underline{u}_0 \in \mathbb{R}^{N_L}$ is the coefficient vector of $u_{L,0}$ with respect to the wavelet basis of V_L.

For notational simplicity, we consider for the rest of this section a generic time step, omit the index m and write $\mathbf{C}$ and $\mathbf{I}$ for the matrices $\mathbf{C}^m$ and $\mathbf{I}^m$, respectively, $\underline{u}$, $\underline{\varphi}$ for the vectors appearing in (12.13), and r for the approximation order. Furthermore, we denote the right hand side by $\underline{f} = (\underline{\varphi}^m \otimes \mathbf{M})\underline{u}^{m-1}$.

12.3.2 Solution Algorithm

The system (12.13) of size $(r+1)N_L$ can be reduced to solving $r+1$ linear systems of size N_L. To this end, let $\mathbf{C} = \mathbf{Q}\mathbf{T}\mathbf{Q}^\mathsf{H}$ be the Schur decomposition of the $(r+1) \times (r+1)$ matrix $\mathbf{C}$ with a unitary matrix $\mathbf{Q}$ and an upper triangular matrix $\mathbf{T}$. Note that the diagonal of $\mathbf{T}$ contains the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_{r+1}$ of $\mathbf{C}$. Then, multiplying (12.13) by $\mathbf{Q}^\mathsf{H} \otimes \mathbf{I}$ from the left results in the linear system

$$\Big(\mathbf{T} \otimes \mathbf{M} + \frac{k}{2}\mathbf{I} \otimes \mathbf{A}\Big)\underline{W} = \underline{g} \quad \text{with} \quad \underline{w} = (\mathbf{Q}^\mathsf{H} \otimes \mathbf{I})\underline{u}, \quad \underline{g} = (\mathbf{Q}^\mathsf{H} \otimes \mathbf{I})\underline{f}.$$

This system is block-upper-triangular. With $\underline{w} = (\underline{w}_0, \underline{w}_1, \ldots, \underline{w}_r)$, $\underline{w}_j \in \mathbb{C}^{N_L}$, we obtain its solution by solving

$$\Big(\lambda_{j+1}\mathbf{M} + \frac{k}{2}\mathbf{A}\Big)\underline{w}_j = \underline{s}_j \tag{12.14}$$

for $j = r, \ldots, 0$, where $\underline{s}_j = \underline{g}_j - \sum_{l=j+1}^{r} \mathbf{T}_{j+1,l+1}\mathbf{M}\underline{w}_l$. By (12.14), an hp-dG time step of order r amounts to solving $r+1$ linear systems with a matrix of the form

$$\mathbf{B} = \lambda\mathbf{M} + \frac{k}{2}\mathbf{A},$$

where λ is an eigenvalue of $\mathbf{C}$. For the preconditioning of the linear system, we define the matrix $\mathbf{S}$ and the scaled matrix $\widehat{\mathbf{B}} \in \mathbb{R}^{N_L} \times \mathbb{R}^{N_L}$ by

$$\mathbf{S} = \Big(\mathrm{Re}(\lambda)\mathbf{I} + \frac{k}{2}\mathbf{D}\Big)^{\frac{1}{2}}, \qquad \widehat{\mathbf{B}} = \mathbf{S}^{-1}\mathbf{B}\mathbf{S}^{-1}, \tag{12.15}$$

where $\mathbf{D}$ is the diagonal preconditioner with entries $2^{\alpha l}$ as in (12.9). The preconditioned linear equations corresponding to (12.14) are solved approximately with incomplete GMRES(m_0) iteration (restarted every $m_0 \geq 1$ iterations). We then obtain [123, Theorem 4]

Theorem 12.3.4 *Let the assumptions of Theorem 12.3.2 hold. Then, choosing the number and order of time steps such that $M = r = \mathcal{O}(|\log h|)$ and in each time step $n_G = \mathcal{O}(|\log h|)^5$ GMRES iterations implies that*

$$\|u(T) - U^{dG}(T)\|_{L^2(G)} \leq Ch^p, \tag{12.16}$$

where U^{dG} denotes the (perturbed) hp-dG approximation of the exact solution u to (12.4) obtained by the incomplete GMRES(m_0) method.

Applying the matrix compression techniques, the judicious combination of geometric mesh refinement and linear increase of polynomial degrees in the hp-dG time-stepping scheme, an appropriate number of GMRES iterations, results in linear (up to some logarithmic terms) overall complexity of the fully discrete scheme (12.10) for the solution of the parabolic problem (12.4).

Example 12.3.5 As in Example 10.6.3, we consider the variance gamma model [118] with parameter $\sigma = 0.3$, $\vartheta = 0.25$ and $\theta = -0.3$. For the compression scheme, we use $a = 1$, $a' = 1$, $p = 2$, $\widetilde{p} = 2$. For $L = 8$, the absolute value of the entries in the stiffness matrix $\mathbf{A}$ and the compressed matrix $\widetilde{\mathbf{A}}$ are shown in Fig. 12.5. Here, large entries are colored red. For the stiffness matrix, blue entries are small but non-zero whereas for the compressed matrix blue entries are zero either due to the first or second compression. One clearly sees that the compression scheme neglects small entries.

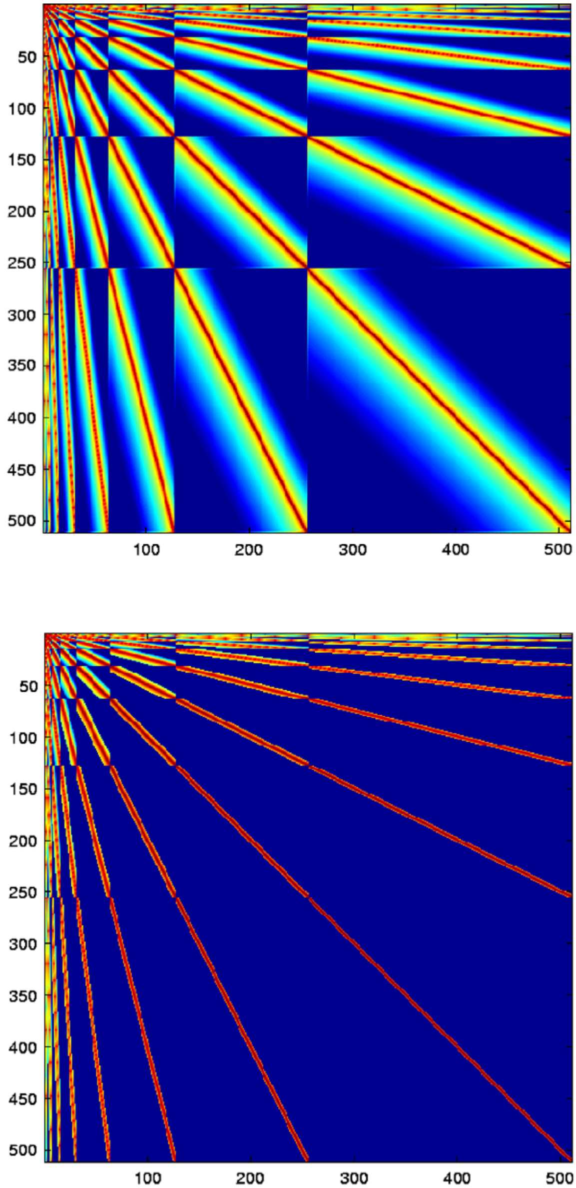


Fig. 12.5 Stiffness matrix $\mathbf{A}$ (*top*) and compressed matrix $\widetilde{\mathbf{A}}$ (*bottom*) for level $L = 8$

For a European call option with maturity $T = 1$ and strike $K = 100$, we compute the L^∞-error at maturity $t = T$ on the subset $G_0 = (K/2, 3/2K)$. In the discretization, we use for the hat functions the Crank–Nicolson scheme with the constant time steps $M = \mathcal{O}(N)$ and for the wavelet basis the hp-dG time stepping with $M = \mathcal{O}(\log N)$ graded time steps. It can be seen in Fig. 12.6 that for the (perturbed)

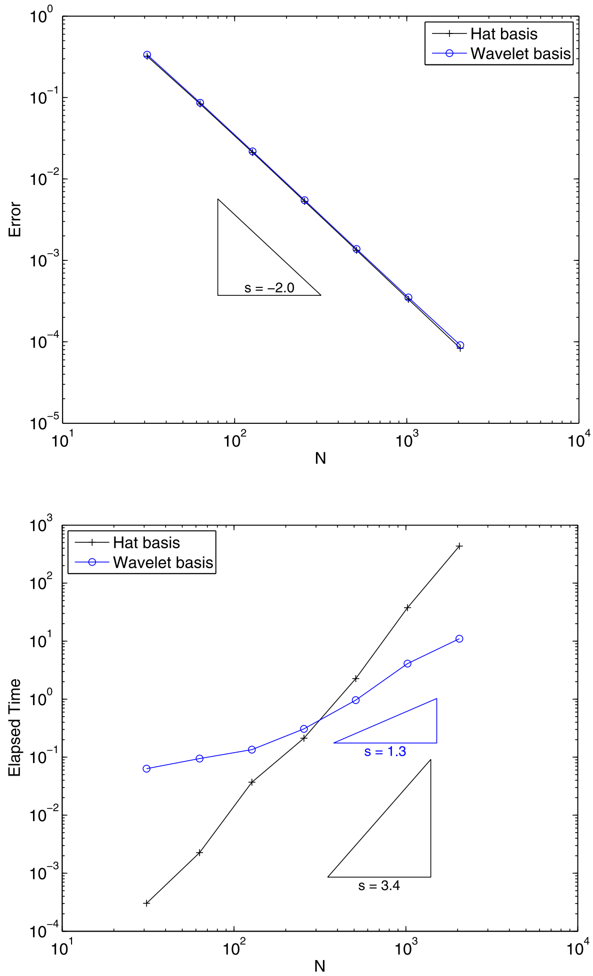


Fig. 12.6 Convergence rate for the variance gamma model (*top*) and elapsed time (in seconds) to solve the linear systems (*bottom*)

hp-dG approximation we still obtain the optimal convergence rate $\mathcal{O}(N^{-2})$ but only need (up to log terms) $\mathcal{O}(N)$ seconds to solve the linear systems instead of $\mathcal{O}(N^3)$.

12.4 Further Reading

For a general introduction to wavelets, we refer to Daubechies [52], whereas a focus on the numerical solution of operator equations by wavelet methods can be found, e.g. in Cohen [38], Dahmen [50], Urban [155], and the references therein. The wavelet compression of the stiffness matrix corresponding to singular integral operators (similar to the jump operator of Lévy processes) is discussed in, e.g. Dahmen [51] and in von Petersdorff and Schwab [158].

In one dimension, Matache et al. [122, 123] have introduced a wavelet-based finite element scheme to solve the variational PIDE formulation of European-style options. This was subsequently also applied to American-type contracts as in [121, 160]. In Farkas et al. [66, 134, 163], the wavelet-based approach was extended to multidimensional Lévy models. A short overview of the method can also be found in Hilber et al. [82].

Computable *a posteriori* estimates of the discretization error for parabolic integro-differential variational inequalities which arise for American put type contracts in exponential Lévy models were obtained in [130].

Chapter 13
Multidimensional Diffusion Models

In the previous chapter, we introduced a spline wavelet basis $\{\psi_{\ell,k}\}$ for the discretization of infinitesimal generators $\mathcal{A}$ of Lévy processes X. The wavelet basis served two purposes:

(i) *Preconditioning of linear systems*: We showed that the wavelet basis implies in the context of hp-dG timestepping that the (generalized) condition number of the linear systems to be solved in each time step is bounded independently of jump intensity α, volatility σ and time step k. This robust preconditioning was crucial for efficient iterative solution of the linear systems for the widely varying time steps which occurred in connection with the hp-dG timestepping and was crucial for the overall log-linear complexity of the proposed scheme, and

(ii) *Compression of linear systems*: The compression of the stiffness matrix based on a priori analysis reduces the number of non-zero matrix entries from $\mathcal{O}(N^2)$ to $\mathcal{O}(N)$ and does not deteriorate the optimal convergence rate of the scheme.

In the present chapter, we develop efficient pricing algorithms for *multivariate problems*, such as the pricing of multi-asset options and the pricing of options in stochastic volatility models, which exploit a third feature of the wavelet basis, namely that wavelets constitute a *hierarchic basis* of the univariate finite element space V_L. This allows constructing the so-called *sparse tensor product* subspaces for the numerical solution of d-dimensional pricing problems with complexity essentially equal to that of one-dimensional problems. The tensor product structure of the infinitesimal generators of the multidimensional BS operator and of stochastic volatility models implies that the stiffness matrices corresponding to these operators in tensor product wavelet bases can be built from tensor products of banded matrices corresponding to univariate operators. Here, we will show that the same convergence rates can, in fact, be achieved with *sparse tensor products of univariate matrices*. These matrices are not sparse anymore and should therefore never be formed to maintain low complexity of the pricing algorithm. To obtain linear complexity pricing algorithms, sparse tensor product matrices are stored in factored form and are applied to a vector in log-linear overall cost. Using wavelet preconditioning, the condition number of

N. Hilber et al., *Computational Methods for Quantitative Finance*, Springer Finance,
DOI 10.1007/978-3-642-35401-4_13,

the sparse tensor product matrices is once again bounded independently of the mesh width. We see, therefore, that the increased computational complexity of the pricing of multi-asset options or options in SV market models can be practically removed by the sparse tensor product construction.

Dimensionality reduction by principal component analysis is investigated in order to price options on indices by considering the whole vector process of all of their constituents.

13.1 Sparse Tensor Product Finite Element Spaces

Consider the space $V_L = \mathrm{span}\{\psi_{\ell,k} : 0 \le \ell \le L,\ k \in \nabla_\ell\}$ as in Chap. 12, where the wavelets $\psi_{\ell,k} : (0,1) \to \mathbb{R}$ are assumed to be generated from a single-scale basis Φ_L of approximation order p. Now, let $G := (0,1)^d$, $d > 1$ and define the full tensor product space $\mathcal{V}_L$ as the d-fold tensor product of V_L, i.e.

$$\mathcal{V}_L := \bigotimes_{1\le i\le d} V_L. \tag{13.1}$$

As an example, consider the continuous, piecewise linear wavelets of Example 12.1.1. Then, $\mathcal{V}_L$ is the same space as in (8.19). Writing $\psi_{\boldsymbol{\ell},\mathbf{k}}(x) := \psi_{\ell_1,k_1}(x_1)\cdots\psi_{\ell_d,k_d}(x_d)$ for an arbitrary tensor product wavelet, $\mathcal{V}_L$ can be written as

$$\mathcal{V}_L = \mathrm{span}\{\psi_{\boldsymbol{\ell},\mathbf{k}} \ :\ 0 \le \ell_i \le L,\ k_i \in \nabla_{\ell_i}, i = 1,\dots,d\}.$$

Using the decomposition of $V_L = V_{L-1} \oplus W_L$, $V_0 = W_0$ into its increment spaces, we also can write $\mathcal{V}_L$ in terms of increment spaces

$$\mathcal{V}_L = \bigoplus_{0\le \ell_i\le L} W^{\ell_1} \otimes \cdots \otimes W^{\ell_d}.$$

Since $\dim W^{\ell_i} = \mathcal{O}(2^{\ell_i})$, the space $\mathcal{V}_L$ has $\mathcal{O}(2^{Ld})$ degrees of freedom which grow exponentially with increasing dimension d. To avoid this “curse of dimension”, we introduce the sparse tensor product space

$$\begin{aligned}\widehat{\mathcal{V}}_L &:= \mathrm{span}\{\psi_{\boldsymbol{\ell},\mathbf{k}} \ :\ 0 \le \ell_1 + \cdots + \ell_d \le L,\ k_i \in \nabla_{\ell_i}, i = 1,\dots,d\}\\ &= \bigoplus_{0\le \ell_1+\cdots+\ell_d\le L} W_{\ell_1} \otimes \cdots \otimes W_{\ell_d}.\end{aligned} \tag{13.2}$$

The difference between the tensor product space $\mathcal{V}_L$ and the sparse tensor product space $\widehat{\mathcal{V}}_L$ is shown in Fig. 13.1 for level $L = 3$ and $d = 2$ using wavelets as described in Example 12.1.1.

Lemma 13.1.1 *The dimension* $\widehat{N}_L := \dim \widehat{\mathcal{V}}_L$ *of* $\widehat{\mathcal{V}}_L$ *is* $\widehat{N}_L = \mathcal{O}(2^L L^{d-1})$.

Proof Let $\mathcal{I}_L := \{\boldsymbol{\ell} \in \mathbb{N}_0^d \mid |\boldsymbol{\ell}|_1 \in (L-1, L]\}$ and

$$K_L := \dim \sum_{\boldsymbol{\ell}\in\mathcal{I}_L} W^{\ell_1} \otimes \cdots \otimes W^{\ell_d} \le C \sum_{\boldsymbol{\ell}\in\mathcal{I}_L} 2^{|\boldsymbol{\ell}|_1} = C2^L \sum_{\boldsymbol{\ell}\in\mathcal{I}_L} 1$$

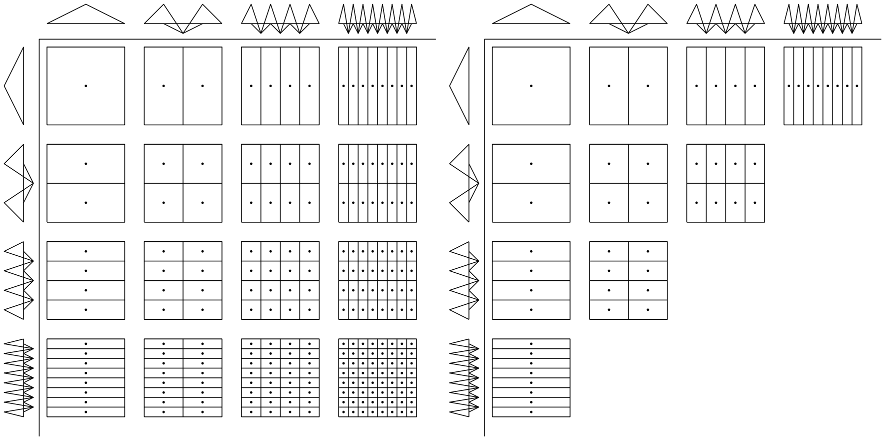


Fig. 13.1 Tensor product (*left*) and sparse tensor product (*right*) for $d = 2$

Table 13.1 Dimension of full and sparse tensor product spaces

d	L	5	6	7	8	9	10
2	$\widehat{N}_L$	$3.21 \cdot 10^2$	$7.69 \cdot 10^2$	$1.79 \cdot 10^3$	$4.10 \cdot 10^3$	$9.22 \cdot 10^3$	$2.05 \cdot 10^4$
	N_L	$3.97 \cdot 10^2$	$1.61 \cdot 10^4$	$6.50 \cdot 10^4$	$2.61 \cdot 10^5$	$1.05 \cdot 10^6$	$4.19 \cdot 10^6$
6	$\widehat{N}_L$	$1.06 \cdot 10^4$	$4.02 \cdot 10^4$	$1.42 \cdot 10^5$	$4.71 \cdot 10^5$	$1.50 \cdot 10^6$	$4.57 \cdot 10^6$
	N_L	$6.25 \cdot 10^{10}$	$4.20 \cdot 10^{12}$	$2.75 \cdot 10^{14}$	$1.78 \cdot 10^{16}$	$1.15 \cdot 10^{18}$	$7.36 \cdot 10^{19}$
10	$\widehat{N}_L$	$7.75 \cdot 10^4$	$3.98 \cdot 10^5$	$1.86 \cdot 10^6$	$8.09 \cdot 10^6$	$3.30 \cdot 10^7$	$1.28 \cdot 10^8$
	N_L	$9.85 \cdot 10^{17}$	$1.09 \cdot 10^{21}$	$1.16 \cdot 10^{24}$	$1.21 \cdot 10^{27}$	$1.26 \cdot 10^{30}$	$1.29 \cdot 10^{33}$

so that $\dim \widehat{\mathcal{V}}_L = K_0 + \cdots + K_L$. With $K_L \le (\#\mathcal{I}_L) C 2^L = C' L^{d-1} 2^L$ the claim follows. □

It is illustrative to compare the dimension $N_L := \dim \mathcal{V}_L$ of the full tensor product space with the dimension $\widehat{N}_L$ of the sparse tensor product space, see Table 13.1. According to Lemma 13.1.1, the spaces $\widehat{\mathcal{V}}_L$ have considerably smaller dimension than $\mathcal{V}_L$. On the other hand, they do have similar approximation properties as $\mathcal{V}_L$, provided the function to be approximated is sufficiently smooth. To characterize the smoothness, we introduce the Sobolev space $\mathcal{H}^{\mathbf{n}}(G)$, $\mathbf{n} \in \mathbb{N}_0$, of all measurable functions $u : G \to \mathbb{R}$ such that the norm,

$$\|u\|_{\mathcal{H}^{\mathbf{n}}(G)} := \Bigg(\sum_{\substack{0 \le m_i \le n_i \\ i=1,\dots,d}} \|D^{\mathbf{m}} u\|^2_{L^2(G)} \Bigg)^{1/2},$$

is finite. That is,

$$\mathcal{H}^{\mathbf{n}}(G) = \bigotimes_{i=1}^{d} H^{n_i}(I), \tag{13.3}$$

where $I := [0, 1]$. For arbitrary $\mathbf{s} \in \mathbb{R}^d_{\geq 0}$, we define $\mathcal{H}^{\mathbf{s}}(G)$ by interpolation. For later purpose, we additionally introduce anisotropic Sobolev spaces, i.e. spaces which consist of functions with different smoothness in different coordinate directions. Recall the definition of $H^s(\mathbb{R})$ for $s \geq 0$ via the Fourier transform $\|u\|^2_{H^s(\mathbb{R})} := \int_{\mathbb{R}} (1 + |\xi|)^{2s} |\widehat{u}(\xi)|^2 \,\mathrm{d}\xi$. Naturally, this extends to *isotropic* Sobolev spaces of multivariate functions via

$$\|u\|^2_{H^s(\mathbb{R}^d)} := \int_{\mathbb{R}^d} \left(1 + |\xi|\right)^{2s} |\widehat{u}(\xi)|^2 \,\mathrm{d}\xi,$$

where $|\xi| = (\sum_{j=1}^d x_j^2)^{1/2}$. Similarly, for a multi-index $\mathbf{s} \in \mathbb{R}^d_{\geq 0}$, we can define *anisotropic* Sobolev spaces $H^{\mathbf{s}}(\mathbb{R}^d)$ with norm

$$\|u\|^2_{H^{\mathbf{s}}(\mathbb{R}^d)} := \int_{\mathbb{R}^d} \sum_{j=1}^{d} (1 + \xi_j^2)^{s_j} |\widehat{u}(\xi)|^2 \,\mathrm{d}\xi.$$

It is useful to notice that by [129, Sect. 9.2] the spaces $H^{\mathbf{s}}(\mathbb{R}^d)$ admit an intersection structure, and we have

$$H^{\mathbf{s}}(\mathbb{R}^d) = \bigcap_{j=1}^{d} H_j^{s_j}(\mathbb{R}^d) \quad \text{and} \quad \|u\|^2_{H^{\mathbf{s}}(\mathbb{R}^d)} \sim \sum_{j=1}^{d} \|(1 + \xi_j^2)^{s_j/2} \widehat{u}\|^2_{L^2(\mathbb{R}^d)}. \tag{13.4}$$

Similarly to the one dimensional case, we finally define the space

$$\widetilde{H}^{\mathbf{s}}(G) := \left\{ u|_G \;:\; u \in H^{\mathbf{s}}(\mathbb{R}^d),\; u|_{\mathbb{R}^d \setminus G} = 0 \right\}. \tag{13.5}$$

Note that, for example, the space $\mathcal{H}^{\mathbf{1}}(G)$ is different from the space $H^1(G)$: $u \in \mathcal{H}^{\mathbf{1}}(G)$ implies $\partial_{x_1} \cdots \partial_{x_d} u \in L^2(G)$, but $u \in H^1(G)$ is equivalent to $u, \partial_{x_1} u, \ldots, \partial_{x_d} u \in L^2(G)$. The following holds for $s \geq 0$:

$$H^s(G) = H^s(I) \otimes L^2(I) \otimes \cdots \otimes L^2(I) \cap \cdots \cap L^2(I) \otimes L^2(I) \otimes \cdots \otimes H^s(I).$$

For a function $u \in L^2(G)$, we have as a consequence of (12.1) and (13.1) the series representation

$$u = \sum_{\ell_i=0}^{\infty} \sum_{k_i \in \nabla_{\ell_i}} u_{\boldsymbol{\ell},\mathbf{k}} \psi_{\boldsymbol{\ell},\mathbf{k}}. \tag{13.6}$$

Using the norm equivalences (12.3) and the underlying tensor product structure (13.3), we obtain

$$\|u\|^2_{\mathcal{H}^{\mathbf{s}}(G)} \lesssim \sum_{\ell_i=0}^{\infty} \sum_{k_i \in \nabla_{\ell_i}} (2^{2s_1\ell_1 + \cdots + 2s_d\ell_d}) |u_{\boldsymbol{\ell},\mathbf{k}}|^2 \lesssim \|u\|^2_{\mathcal{H}^{\mathbf{s}}(G)}, \tag{13.7}$$

for $0 \le s_i \le p - 1/2$, $i = 1, \dots, d$. Similarly, due to the intersection structure (13.4), we obtain

$$\|u\|^2_{\widetilde{H}^{\mathbf{s}}(G)} \lesssim \sum_{\ell_i=0}^{\infty} \sum_{k_i \in \nabla_{\ell_i}} (2^{2s_1\ell_1} + \cdots + 2^{2s_d\ell_d})|u_{\boldsymbol{\ell},\mathbf{k}}|^2 \lesssim \|u\|^2_{\widetilde{H}^{\mathbf{s}}(G)}, \tag{13.8}$$

for $0 \le s_i \le p - 1/2$, $i = 1, \dots, d$.

We study the approximation property of the sparse tensor product space $\widehat{\mathcal{V}}_L$ for functions $u \in \mathcal{H}^{\mathbf{s}}(G)$. To this end, we define the sparse projection operator $\widehat{P}_L : L^2(G) \to \widehat{\mathcal{V}}_L$ by truncating the wavelet expansion (13.6)

$$\widehat{P}_L u := \sum_{|\boldsymbol{\ell}|_1 \le L} \sum_{k_i \in \nabla_{\ell_i}} u_{\boldsymbol{\ell},\mathbf{k}} \psi_{\boldsymbol{\ell},\mathbf{k}}.$$

For a multi-index $\boldsymbol{\alpha} \in \mathbb{R}^d_{>0}$, denote by $\boldsymbol{\alpha}_* := \min\{\alpha_i\}$. The next result is taken from [133].

Theorem 13.1.2 *Assume $0 \le s_i \le p - 1/2$ and $s_i < t_i \le p$, $i = 1, \dots, d$. Then, for $u \in \widetilde{H}^{\mathbf{s}}(G)$ there holds*

$$\|u - \widehat{P}_L u\|_{\widetilde{H}^{\mathbf{s}}(G)} \le \begin{cases} C 2^{-(\mathbf{t}-\mathbf{s})_* L} \|u\|_{\mathcal{H}^{\mathbf{t}}(G)} & \text{if } \mathbf{s} \ne 0 \text{ or } t_i \ne p, \forall i, \\ C 2^{-(\mathbf{t}-\mathbf{s})_* L} L^{(d-1)/2} \|u\|_{\mathcal{H}^{\mathbf{t}}(G)} & \text{otherwise.} \end{cases}$$

We can also state the approximation rate in terms of the dimension of the sparse tensor product space $\widehat{N}_L = \mathcal{O}(2^L L^{d-1})$. For example, if $u \in \mathcal{H}^p(G)$, we obtain

$$\|u - \widehat{P}_L u\|_{L^2(G)} \le C \widehat{N}_L^{-p} (\log_2 \widehat{N}_L)^{(p+1/2)(d-1)+\varepsilon} \|u\|_{\mathcal{H}^p(G)}, \quad \forall \varepsilon > 0.$$

Hence, for the wavelets of polynomial degree 1 with approximation order $p = 2$ as in Example 12.1.1, the approximation rate is, up to logarithmic terms, the same as in one dimension, compare with the finite element interpolation estimate (3.36), where with $h = \mathcal{O}(N^{-1})$ we have $\|u - \mathcal{I}_N u\|_{L^2(G)} \le C N^{-2} \|u\|_{H^2(G)}$. Thus, the curse of dimension $\|u - P_L u\|_{L^2(G)} \le C N_L^{-p/d} \|u\|_{H^p(G)}$ (see also (8.24)) of the full tensor product space $\mathcal{V}_L$ can be avoided by the sparse tensor product space.

In the next section, we use the sparse tensor product space $\widehat{\mathcal{V}}_L$ to discretize the weak formulation of the pricing equation for multi-asset options in a BS market.

13.2 Sparse Wavelet Discretization

Recall the weak formulation of the localized pricing problem of multi-asset options (8.13). Its space semi-discretization using the sparse tensor product space $\widehat{\mathcal{V}}_L \subset H^1_0(G)$ reads

$$\begin{aligned} &\text{Find } \widehat{u}_L \in L^2(J; \widehat{\mathcal{V}}_L) \cap H^1(J; \widehat{\mathcal{V}}_L^*) \text{ such that} \\ &(\partial_t \widehat{u}_L, v) + a^{\mathrm{BS}}(\widehat{u}_L, v) = 0, \ \forall v \in \widehat{\mathcal{V}}_L, \quad \text{a.e. in } J, \\ &\widehat{u}_L(0) = \widehat{u}_{L,0}. \end{aligned} \tag{13.9}$$

Herewith, the bilinear form $a^{\mathrm{BS}}(\cdot,\cdot)$ is given by $a^{\mathrm{BS}}(\varphi,\phi)=\frac{1}{2}\int_G(\nabla\varphi)^\top\mathcal{Q}\nabla\phi\,\mathrm{d}x+\int_G\mu^\top\nabla\varphi\phi\,\mathrm{d}x+r\int_G\varphi\phi\,\mathrm{d}x$ with covariance matrix $\mathcal{Q}=\boldsymbol{\Sigma}\boldsymbol{\Sigma}^\top$ and drift vector $\mu=(1/2\mathcal{Q}_{11}-r,\dots,1/2\mathcal{Q}_{dd}-r)^\top$. Furthermore, $\widehat{u}_{L,0}$ is the $L^2(G)$ projection of the payoff g onto $\widehat{\mathcal{V}}_L$, i.e.

$$(\widehat{u}_{L,0},v)=(g,v),\quad \forall v\in\widehat{\mathcal{V}}_L. \tag{13.10}$$

We have to calculate the matrix

$$\begin{aligned}\mathbf{A}=\left(\mathbf{A}_{(\boldsymbol{\ell}',\mathbf{k}')(\boldsymbol{\ell},\mathbf{k})}\right)_{\substack{|\boldsymbol{\ell}|_1,|\boldsymbol{\ell}'|_1\le L\\ \mathbf{k}'\in\nabla_{\boldsymbol{\ell}'},\mathbf{k}\in\nabla_{\boldsymbol{\ell}}}}&:=\left(a^{\mathrm{BS}}(\psi_{\boldsymbol{\ell},\mathbf{k}},\psi_{\boldsymbol{\ell}',\mathbf{k}'})\right)_{\substack{|\boldsymbol{\ell}|_1,|\boldsymbol{\ell}'|_1\le L\\ \mathbf{k}'\in\nabla_{\boldsymbol{\ell}'},\mathbf{k}\in\nabla_{\boldsymbol{\ell}}}},\\ \psi_{\boldsymbol{\ell},\mathbf{k}},\psi_{\boldsymbol{\ell}',\mathbf{k}'}\in\widehat{\mathcal{V}}_L,&\end{aligned}\tag{13.11}$$

where the set $\nabla_{\boldsymbol{\ell}}$ is given by

$$\nabla_{\boldsymbol{\ell}}:=\{k_i\ :\ 1\le k_i\le 2^{\ell_i},\ |\boldsymbol{\ell}|_1\le L,\ i=1,\dots,d\}.$$

In (8.20), we have the representation of the stiffness matrix $\mathbf{A}=\mathbf{A}^{\mathrm{BS}}$ in the full tensor product space spanned by (single-scale) hat functions: the matrix $\mathbf{A}$ is given by a sum of Kronecker product of matrices corresponding to univariate problems. It turns out that a similar representation of the stiffness matrix also holds for the sparse tensor product space $\widehat{\mathcal{V}}_L$ defined in (13.2).

To this end, we calculate the matrices $\mathbf{S}^i$, $\mathbf{B}^i$ and $\mathbf{M}^i$ with respect to the wavelet basis $\{\psi_{\ell,k}\}$, i.e. for $1\le i\le d$ we let

$$\mathbf{M}^i:=\left(\int_{-R}^{R}\psi_{\ell,k}(x)\psi_{\ell',k'}(x)\,\mathrm{d}x\right)_{\substack{0\le\ell',\ell\le L\\ k'\in\nabla_{\ell'},k\in\nabla_\ell}}, \tag{13.12}$$

$$\mathbf{S}^i:=\left(\int_{-R}^{R}\psi'_{\ell,k}(x)\psi'_{\ell',k'}(x)\,\mathrm{d}x\right)_{\substack{0\le\ell',\ell\le L\\ k'\in\nabla_{\ell'},k\in\nabla_\ell}}, \tag{13.13}$$

$$\mathbf{B}^i:=\left(\int_{-R}^{R}\psi'_{\ell,k}(x)\psi_{\ell',k'}(x)\,\mathrm{d}x\right)_{\substack{0\le\ell',\ell\le L\\ k'\in\nabla_{\ell'},k\in\nabla_\ell}}. \tag{13.14}$$

Note that for the wavelets $\psi_{\ell,k}$ as in Example 12.1.1, the matrices $\mathbf{S}^i$, $\mathbf{B}^i$ and $\mathbf{M}^i$ can be obtained by first calculating them with respect to the basis spanned by hat-functions $b_{\ell,j}$ (see (4.16)), and then applying the wavelet transformation (12.2).

Let $\mathbf{X}^i$ be any matrix given by (13.12)–(13.14). We view the matrix $\mathbf{X}^i$ as a collection of block matrices, i.e.

$$\mathbf{X}^i=(\mathbf{X}^i_{\ell',\ell})_{0\le\ell',\ell\le L},\quad \text{where}\quad \mathbf{X}^i_{\ell',\ell}:=(\mathbf{X}^i_{(\ell',k'),(\ell,k)})_{k'\in\nabla_{\ell'},k\in\nabla_\ell},$$

and define a sparse tensor product $\mathbf{X}^1\,\widehat{\otimes}\,\mathbf{X}^2\,\widehat{\otimes}\cdots\widehat{\otimes}\,\mathbf{X}^d$ by tensor products of block matrices

$$\mathbf{X}^1\,\widehat{\otimes}\cdots\widehat{\otimes}\,\mathbf{X}^d:=\left(\mathbf{X}^1_{\ell'_1,\ell_1}\otimes\cdots\otimes\mathbf{X}^d_{\ell'_d,\ell_d}\right)_{0\le|\boldsymbol{\ell}'|_1,|\boldsymbol{\ell}|_1\le L}$$

Table 13.2 Algorithm to compute matrix–vector multiplication

Set $v = u$
For $j = 0, 1, \dots, d$
For $\lvert \boldsymbol{\ell}' \rvert_1 = 0, 1, \dots, L$
Compute $v_{\boldsymbol{\ell}',\mathbf{k}'} = \sum_{\ell_j, k_j} X^j_{(\ell'_j,k'_j),(\ell_j,k_j)} v_{\boldsymbol{\ell},\mathbf{k}}, \quad \forall \mathbf{k}',$
with $\ell_i = \ell'_i,\ k_i = k'_i, \quad \forall i \neq j.$
Next $\boldsymbol{\ell}$
Next j

for multi-indices $\boldsymbol{\ell} = (\ell_1, \dots, \ell_d)$, $\boldsymbol{\ell}' = (\ell'_1, \dots, \ell'_d)$. Thus, the stiffness matrix $\mathbf{A}$ (13.11) of the bilinear form $a^{\mathrm{BS}}(\cdot,\cdot)$ can be written as

$$\mathbf{A} = \frac{1}{2}\sum_{i=1}^{d} \mathcal{Q}_{ii} \widehat{\bigotimes_{1\le k\le i-1}} \mathbf{M}^k \,\widehat{\otimes}\, \mathbf{S}^i \,\widehat{\otimes} \widehat{\bigotimes_{i+1\le k\le d}} \mathbf{M}^k$$
$$-\sum_{i=1}^{d-1}\sum_{j=i+1}^{d} \mathcal{Q}_{ij} \widehat{\bigotimes_{1\le k\le i-1}} \mathbf{M}^k \,\widehat{\otimes}\, \mathbf{B}^i \,\widehat{\otimes} \widehat{\bigotimes_{i+1\le k\le j-1}} \mathbf{M}^k \,\widehat{\otimes}\, \mathbf{B}^j \,\widehat{\otimes} \widehat{\bigotimes_{j+1\le k\le d}} \mathbf{M}^k$$
$$+\sum_{k=1}^{d} \mu_k \widehat{\bigotimes_{1\le k\le i-1}} \mathbf{M}^k \,\widehat{\otimes}\, \mathbf{B}^i \,\widehat{\otimes} \widehat{\bigotimes_{i+1\le k\le d}} \mathbf{M}^k + r \widehat{\bigotimes_{1\le k\le d}} \mathbf{M}^k.$$

Computing the matrix $\mathbf{A}$ for $d \gg 1$ requires too much memory. However, solving linear systems involving $\mathbf{A}$ by iterative methods like GMRES requires only a matrix–vector multiplication $\underline{u} \mapsto \mathbf{A}\underline{u}$. Using the tensor product structure, this can be done without computing the matrix $\mathbf{A}$.

Let $\mathbf{A} = \mathbf{X}^1 \,\widehat{\otimes} \cdots \widehat{\otimes}\, \mathbf{X}^d$ and $u_L \in \widehat{\mathcal{V}}_L$. We again view the coefficient vector $\underline{u}_L$ of u_L as a collection of block coefficient vectors,

$$\underline{u}_L = \left(\underline{u}_{\boldsymbol{\ell}}\right)_{0\le|\boldsymbol{\ell}|_1\le L} \quad \text{where} \quad \underline{u}_{\boldsymbol{\ell}} = \left(u_{\boldsymbol{\ell},\mathbf{k}}\right)_{1\le k_i\le M_{\ell_i}}.$$

The matrix–vector multiplication

$$\underline{v}_L = \mathbf{A}\underline{u}_L = \left(\mathbf{X}^1_{\ell'_1,\ell_1} \otimes \cdots \otimes \mathbf{X}^d_{\ell'_d,\ell_d}\right)_{0\le|\ell'|_1,|\ell|_1\le L} \left(\underline{u}_{\boldsymbol{\ell}}\right)_{0\le|\boldsymbol{\ell}|_1\le L}$$

is defined by

$$v_{\boldsymbol{\ell}',\mathbf{k}'} = \sum_{|\boldsymbol{\ell}|_1 < L}\ \sum_{1\le k_i\le M_{\ell_i}} X^1_{(\ell'_1,k'_1),(\ell_1,k_1)} \cdots X^d_{(\ell'_d,k'_d),(\ell_d,k_d)} u_{\boldsymbol{\ell},\mathbf{k}}.$$

This can be computed iteratively as shown in Table 13.2.

13.3 Fully Discrete Scheme

We apply the discontinuous Galerkin time discretization (see Sect. 12.3) to the semi-discrete problem (13.9) and arrive at the fully discrete scheme: Find $U \in S^{\mathbf{r}}(\mathcal{M}, \widehat{\mathcal{V}}_L)$ such that for all $W \in S^{\mathbf{r}}(\mathcal{M}, \widehat{\mathcal{V}}_L)$

$$\int_J \Big((U', W) + a^{\mathrm{BS}}(U, W)\Big)\,\mathrm{d}t + \sum_{m=1}^{M-1} ([\![U]\!]_m, W_+^m) + (U_+^0, W_+^0) = (\widehat{u}_{L,0}, W_+^0). \tag{13.15}$$

Proceeding exactly as in Sect. 12.3, (13.15) is equivalent to solving M times $r+1$ linear systems of size $\widehat{N}_L$

$$\Big(\lambda_{j+1}\mathbf{M} + \frac{k}{2}\mathbf{A}\Big)\underline{w}_j = \underline{s}_j,$$

where the stiffness matrix $\mathbf{A}$ is given in (13.11), $\mathbf{M} = \widehat{\bigotimes}_{1\le k\le d} \mathbf{M}^k$ is the mass matrix and $\underline{w}_j, \underline{s}_j \in \mathbb{C}^{\widehat{N}_L}$.

Due to the norm equivalence (13.8) valid for the energy space $H_0^1(G)$, we define, analogously to the one dimensional case, the diagonal matrix $\mathbf{D}$ with entries $2^{2\ell_1} + \cdots + 2^{2\ell_d}$. Then, we obtain a diagonal preconditioner $\mathbf{S} := (\Re\lambda_{j+1}\mathbf{I} + \frac{k}{2}\mathbf{D})^{1/2}$ for the matrix $\lambda_{j+1}\mathbf{M} + \frac{k}{2}\mathbf{A}$ as in (12.15). The following result is proven in [159] and generalizes Theorem 12.3.4 to dimensions $d \ge 2$.

Theorem 13.3.1 *Assume that the payoff $g|_G \in \widetilde{H}^{\vartheta}(G)$ for some $0 < \vartheta \le 1$. Choose the number and order of time steps such that $M = r = \mathcal{O}(L)$ and use in each time step $\mathcal{O}(L)^5$ GMRES iterations. Then, the fully discrete Galerkin scheme with incomplete GMRES gives an approximate solution $U^{dG}(T) \in \widehat{\mathcal{V}}_L$ to the exact solution $u(T)$ of* (8.13) *at maturity with accuracy*

$$\|u(T) - U^{dG}(T)\|_{L^2(G)} \le C\widehat{N}_L^{-s}(\log_2 \widehat{N}_L)^{(d-1)s+\varepsilon}, \quad s := p - 1 + \frac{p-1}{dp-1},$$

and can be computed with at most $\mathcal{O}(\widehat{N}_L(\log_2 \widehat{N}_L)^{7+\varepsilon})$ operations, for all $\varepsilon > 0$.

Recall that $\widehat{u}_{L,0}$ on the left hand side of (13.15) is the L^2-projection of the payoff g onto $\widehat{\mathcal{V}}_L$ (13.10). Thus, its corresponding coefficient vector $\underline{u}_0$ appearing in (12.13) with respect to the basis of $\widehat{\mathcal{V}}_L$ is the unique solution of the system $\mathbf{M}\underline{u}_0 = \underline{g}$ with the right hand side $\underline{g} \in \mathbb{R}^{\widehat{N}_L}$ given by

$$g_{(\boldsymbol{\ell},\mathbf{k})} = \int_G g(x_1, \ldots, x_d)\psi_{\ell_1,k_1}(x_1)\cdots\psi_{\ell_d,k_d}(x_d)\,\mathrm{d}x_1\cdots\mathrm{d}x_d, \quad |\boldsymbol{\ell}|_1 \le L, \ \mathbf{k} \in \nabla_{\boldsymbol{\ell}}.$$

If g factorizes, i.e. $g(x_1, \ldots, x_d) = \prod_{j=1}^d g_j(x_j)$ for some univariate $g_j : \mathbb{R} \to \mathbb{R}$, $j = 1, \ldots, d$, then

$$g_{(\boldsymbol{\ell},\mathbf{k})} = \prod_{j=1}^d \int_{a_j}^{b_j} g_j(x_j)\psi_{\ell_j,k_j}(x_j)\,\mathrm{d}x_j.$$

However, for most payoffs the factorizing property does not hold. To avoid numerical quadrature, we use integration by parts to find, in the sense of distributions,

$$g_{(\boldsymbol{\ell},\mathbf{k})} = \int_G g^{(-2)}(x_1,\dots,x_d)\psi''_{\ell_1,k_1}(x_1)\cdots\psi''_{\ell_d,k_d}(x_d)\,\mathrm{d}x_1\cdots\mathrm{d}x_d, \tag{13.16}$$

where

$$g^{(-k)}(x) := \int_{[x_0,x]} g^{(-k+1)}(y)\,\mathrm{d}y, \quad x\in\mathbb{R}^d, \quad k\geq 1,$$

for a suitable $x_0\in\mathbb{R}\cup\{-\infty\}$. Let $\psi_{\ell_i,k_i}\in V_L$ be a continuous, piecewise linear spline wavelet and denote its singular support by

$$\operatorname{singsupp}\psi_{\ell_i,k_i} =: \{x^1_{\ell_i,k_i},\dots,x^{n_i}_{\ell_i,k_i}\}.$$

Then, the integral in (13.16) becomes

$$g_{(\boldsymbol{\ell},\mathbf{k})} = \sum_{\substack{1\leq j_i\leq n_i\\ 1\leq i\leq d}} g^{(-2)}\big(x^{j_1}_{\ell_1,k_1},\dots,x^{j_d}_{\ell_d,k_d}\big)\omega^{j_1}_{\ell_1,k_1}\cdots\omega^{j_d}_{\ell_d,k_d},$$

where the weights $\omega^1_{\ell_i,k_i},\dots,\omega^{n_i}_{\ell_i,k_i}\in\mathbb{R}$ depend only on the wavelet ψ_{ℓ_i,k_i}. As an example, consider the L^2-normalized wavelets $\psi_{\ell_i,k_i}:(a_i,b_i)\to\mathbb{R}$ defined on an interval (a_i,b_i) as described in Example 12.1.1. Then

$$(\omega^1_{\ell_i,k_i},\omega^2_{\ell_i,k_i},\omega^3_{\ell_i,k_i},\omega^4_{\ell_i,k_i},\omega^5_{\ell_i,k_i}) = \sqrt{3}(b_i-a_i)^{-\frac{3}{2}}2^{\frac{3}{2}\ell_i}(-1,4,-6,4,-1),$$

if $\ell_i\geq 1$ and ψ_{ℓ_i,k_i} is an interior wavelet, and

$$(\omega^1_{\ell_i,1},\omega^2_{\ell_i,1},\omega^3_{\ell_i,1},\omega^4_{\ell_i,1}) = \sqrt{3}(b_i-a_i)^{-\frac{3}{2}}2^{\frac{3}{2}\ell_i}(2,-5,4,-1),$$
$$(\omega^1_{\ell_i,N_{\ell_i}},\omega^2_{\ell_i,N_{\ell_i}},\omega^3_{\ell_i,N_{\ell_i}},\omega^4_{\ell_i,N_{\ell_i}}) = \sqrt{3}(b_i-a_i)^{-\frac{3}{2}}2^{\frac{3}{2}\ell_i}(-1,4,-5,2),$$

if $\ell_i\geq 1$ and $\psi_{\ell_i,1},\psi_{\ell_i,N_{\ell_i}}$ is a left or a right boundary wavelet, respectively.

13.4 Diffusion Models

We consider a process Z to model the dynamics of the underlying stock prices and of the *background* volatility drivers in case of stochastic volatility models. If Z is Markovian, the fair price of a European style contingent claim with underlying Z is given by

$$u(t,z) = \mathbb{E}^{\mathbb{Q}}\big[e^{-r(T-t)}g(Z_T)\mid Z_t=z\big]. \tag{13.17}$$

We model the market Z by the stochastic differential equation (SDE)

$$\mathrm{d}Z_t = \mu(Z_t)\,\mathrm{d}t + \Sigma(Z_t)\,\mathrm{d}W_t, \quad Z_0 = z. \tag{13.18}$$

Herewith, for $G\subseteq\mathbb{R}^d$, the coefficients $\mu: G\to\mathbb{R}^d$, $\Sigma: G\to\mathbb{R}^{d\times d}$ are assumed to satisfy (1.3). We consider next two kinds of market dynamics, namely the Black–Scholes model and stochastic volatility models.

13.4.1 Aggregated Black–Scholes Models

In the following, a full-rank Black–Scholes model and low-rank Black–Scholes model are introduced. Option pricing under the low-rank Black–Scholes model leads to lower dimensional PDEs compared to the full rank Black–Scholes model, introducing an additional error due to the rank reduction.

Consider d assets $S = (S^1, \dots, S^d)$ with spot price dynamics $Z^i = S^i$ given by

$$\mathrm{d}S_t^i = \mu_i S_t^i \mathrm{d}t + \sum_{j=1}^{d} \boldsymbol{\Sigma}_{ij} S_t^i \mathrm{d}W_t^j, \quad i = 1, \dots, d, \tag{13.19}$$

where $W = \{W_t : t \geq 0\}$ is a standard Brownian motion in $\mathbb{R}^d$ and

$$\mu := \left(\mu_i\right)_{1 \leq i \leq d} \in \mathbb{R}^d, \tag{13.20}$$

$$\Sigma := \left(\boldsymbol{\Sigma}_{ij}\right)_{1 \leq i, j \leq d} \in \mathbb{R}^{d \times d} \tag{13.21}$$

are the constant drift vector and volatility matrix, respectively, with the assumption that rank $\boldsymbol{\Sigma} = d$. The state space domain is given by $G = \mathbb{R}^d$. Under the unique EMM, the log-price dynamics $X^i := \log S^i$ are given by

$$\mathrm{d}X_t^i = \eta_i \,\mathrm{d}t + \sum_{j=1}^{d} \boldsymbol{\Sigma}_{ij} \,\mathrm{d}W_t^j, \quad i = 1, \dots, d, \tag{13.22}$$

where $\eta_i := (r - 1/2\mathcal{Q}_{ii})$, $i = 1, \dots, d$ and $\mathcal{Q} := \boldsymbol{\Sigma}\boldsymbol{\Sigma}^\top \in \mathbb{R}^{d \times d}_{\mathrm{sym.}}$ denotes the volatility covariance matrix. Since $\mathcal{Q}$ is symmetric positive definite, there exists an orthogonal matrix $\mathbf{U} \in \mathbb{R}^{d \times d}$ such that $\mathbf{U}\mathcal{Q}\mathbf{U}^\top = \mathbf{D}$ with diagonal matrix $\mathbf{D} := \mathrm{diag}(s_1^2, \dots, s_d^2)$, $s_1 \geq \dots \geq s_d > 0$. Without loss of generality, we rescale time in (13.19) such that $t \to t^* = s_1^2 t$, yielding $\mathbf{D}^* := \mathrm{diag}(s_1^{*2}, \dots, s_d^{*2})$ with normalized $s_1^* = 1$, $s_i^* = s_i/s_1$, $i = 2, \dots, d$. In the remainder, we drop the * and define a process $Y := \{\mathbf{U}X_t : t \geq 0\}$ with dynamics

$$\mathrm{d}Y_t^i = \lambda_i \,\mathrm{d}t + s_i \,\mathrm{d}W_t^i, \quad i = 1, \dots, d, \tag{13.23}$$

where $\lambda := \mathbf{U}\eta$. The components $Y^1, \dots, Y^d$ now satisfy the system of d decoupled SDEs (13.23).

Let $1 \leq \widetilde{d} < d$ be a parameter and define $\widehat{\mathbf{D}} := \mathrm{diag}(\hat{s}_1^2, \dots, \hat{s}_d^2) \in \mathbb{R}^{d \times d}$ with

$$\hat{s}_i = \begin{cases} s_i, & 1 \leq i \leq \widetilde{d}, \\ 0, & \widetilde{d} + 1 \leq i \leq d, \end{cases} \tag{13.24}$$

and $\widehat{\boldsymbol{\Sigma}} := \mathbf{U}^\top \widehat{\mathbf{D}}^{\frac{1}{2}}$ with $\mathbf{U}$ and s_i, $i = 1, \dots, d$, as before. Consider the log-price process $\widehat{X} := \{\widehat{X}_t : t \geq 0\}$ with dynamics

$$\mathrm{d}\widehat{X}_t^i = \widehat{\eta}_i \,\mathrm{d}t + \sum_{j=1}^{\widetilde{d}} \widehat{\boldsymbol{\Sigma}}_{ij} \,\mathrm{d}W_t^j, \quad i = 1, \dots, d, \tag{13.25}$$

where $\widehat{\eta}_i := (r - 1/2\widehat{Q}_{ii})$, $i = 1, \dots, d$ and $\widehat{Q} = \widehat{\boldsymbol{\Sigma}}\widehat{\boldsymbol{\Sigma}}^\top \in \mathbb{R}^{d\times d}_{\text{sym.}}$ and W_t^j, $j = 1, \dots, \widetilde{d}$, as in (13.19). We designate the process $\widehat{X}$ as *rank-reduced* from d to $\widetilde{d}$ since, under the change of basis induced by $\mathbf{U}$, the process $\widehat{Y} := \{\mathbf{U}\widehat{X}_t : t \geq 0\}$ has dynamics

$$\begin{aligned} \mathrm{d}\widehat{Y}_t^i &= \widehat{\lambda}_i \mathrm{d}t + \hat{s}_i\, \mathrm{d}W_t^i \\ &= \widehat{\lambda}_i\, \mathrm{d}t + \mathbb{1}_{\{1\leq i\leq \widetilde{d}\}} s_i\, \mathrm{d}W_t^i, \quad i = 1, \dots, d, \end{aligned} \tag{13.26}$$

where $\widehat{\lambda} := \mathbf{U}\widehat{\eta}$. The components $\widehat{Y}^{\widetilde{d}+1}, \dots, \widehat{Y}^d$ are therefore deterministic. Under such market dynamics, the price of a European contingent claim (13.17) becomes

$$\begin{aligned} u(t, \hat{x}) &= v(t, \hat{y}) \\ &= \widehat{v}(t, \hat{y}_1, \dots, \hat{y}_{\widetilde{d}}; \hat{y}_{\widetilde{d}+1}, \dots, \hat{y}_d) \\ &= \mathbb{E}\big[e^{-r(T-t)} \widehat{f}(e^{\widehat{Y}_T^1}, \dots, e^{\widehat{Y}_T^{\widetilde{d}}}; e^{\hat{y}_{\widetilde{d}+1}}, \dots, e^{\hat{y}_d}) \,|\, (\widehat{Y}_t^1, \dots, \widehat{Y}_t^{\widetilde{d}}) = (\hat{y}_1, \dots, \hat{y}_{\widetilde{d}})\big], \end{aligned}$$

with $\hat{y} := \mathbf{U}\hat{x}$ and

$$\begin{aligned} \widehat{f}(e^{\hat{y}}) &= \widehat{f}(e^{\hat{y}_1}, \dots, e^{\hat{y}_{\widetilde{d}}}; e^{\hat{y}_{\widetilde{d}+1}}, \dots, e^{\hat{y}_d}) \\ &= f(e^{\hat{y}_1}, \dots, e^{\hat{y}_{\widetilde{d}}}, e^{\hat{y}_{\widetilde{d}+1} + \widehat{\lambda}_{\widetilde{d}+1}(T-t)}, \dots, e^{\hat{y}_d + \widehat{\lambda}_d(T-t)}), \end{aligned} \tag{13.27}$$

herewith $f(e^y) := g(e^x)$. The price $u(t, \hat{x}) = \widehat{v}(t, \hat{y}_1, \dots, \hat{y}_{\widetilde{d}}; \hat{y}_{\widetilde{d}+1}, \dots, \hat{y}_d)$ becomes a solution of a $\widetilde{d}$-dimensional PDE with the initial condition depending on the parameters $\hat{y}_{\widetilde{d}+1}, \dots, \hat{y}_d$.

Suppose a time rescaled d-dimensional Black–Scholes market model with log-price process X as in (13.19) has a volatility covariance matrix Q of full rank. Given $0 \leq \varepsilon \ll 1$, assume that a principal component analysis of Q, i.e. $\mathbf{U}Q\mathbf{U}^\top = \mathbf{D} = \operatorname{diag}(s_1^2, \dots, s_d^2)$ with $s_1 = 1 \geq \cdots \geq s_d > 0$ yields

$$s_i^2 \leq \varepsilon, \quad i = \widetilde{d} + 1, \dots, d,$$

for some $\widetilde{d} = \widetilde{d}(\varepsilon)$. This suggests the d-dimensional dynamics to be mainly driven by $\widetilde{d} < d$ *ε-aggregated price processes* (see, e.g. [139]). We denote $\widehat{X}^\varepsilon$ the ε-aggregated rank-$\widetilde{d}$ process of X with $\hat{s}_{\widetilde{d}+1} = \cdots = \hat{s}_d = 0$ and define the ε-residual market process, i.e. the aggregation remainder, $R^\varepsilon := X - \widehat{X}^\varepsilon$. From (13.22) and (13.25), we have that

$$\mathrm{d}R_t^{\varepsilon,i} = \mathrm{d}(X - \widehat{X}^\varepsilon)_t^i = (\eta_i - \widehat{\eta}_i)\, \mathrm{d}t + \sum_{j=1}^d (\boldsymbol{\Sigma}_{ij} - \widehat{\boldsymbol{\Sigma}}_{ij})\, \mathrm{d}W_t^j. \tag{13.28}$$

Under the change of basis induced by $\mathbf{U}$, the process $T^\varepsilon := \{\mathbf{U}R_t^\varepsilon : t \geq 0\}$, i.e. the fluctuation of X about the ε-aggregate $\widehat{X}^\varepsilon$, has therefore dynamics

$$\begin{aligned} \mathrm{d}T_t^{\varepsilon,i} &= (\mathbf{U}(\eta - \widehat{\eta}))_i\, \mathrm{d}t + \sum_{j=1}^d (\mathbf{D} - \widehat{\mathbf{D}})_{ij}\, \mathrm{d}W_t^j \\ &= \begin{cases} (\lambda_i - \widehat{\lambda}_i)\, \mathrm{d}t, & 1 \leq i \leq \widetilde{d}(\varepsilon), \\ (\lambda_i - \widehat{\lambda}_i)\, \mathrm{d}t + s_i\, \mathrm{d}W_i^t, & \widetilde{d}(\varepsilon) + 1 \leq i \leq d. \end{cases} \end{aligned} \tag{13.29}$$

Lemma 13.4.1 *Given $\widetilde{d} = \widetilde{d}(\varepsilon)$, there holds*

$$|\lambda_i - \widehat{\lambda}_i| \leq \frac{d}{2} \sum_{j=\widetilde{d}+1}^{d} s_j^2, \quad i = 1, \dots, d.$$

Proof From the definitions of η and λ, we have that

$$\begin{aligned}|\lambda_i - \widehat{\lambda}_i| &= \left|\sum_{k=1}^{d} U_{ik}(\eta_k - \widehat{\eta}_k)\right| \leq \sum_{k=1}^{d} |\eta_k - \widehat{\eta}_k| = \frac{1}{2}\sum_{k=1}^{d} |Q_{kk} - \widehat{Q}_{kk}| \\ &= \frac{1}{2}\sum_{k=1}^{d}\left|\sum_{j=1}^{d} U_{jk}^2(D_{jj} - \widehat{D}_{jj})\right| \leq \frac{1}{2}\sum_{k=1}^{d}\sum_{j=\widetilde{d}+1}^{d} s_j^2 \\ &= \frac{d}{2}\sum_{j=\widetilde{d}+1}^{d} s_j^2, \quad i = 1, \dots, d.\end{aligned}$$

□

Remark 13.4.2 From (13.29) and Lemma 13.4.1, we conclude that the fluctuation components $T^{\varepsilon,i}$, $i = 1, \dots, \widetilde{d}(\varepsilon)$, are pure drifts of order ε. Furthermore, note that, upon setting

$$d \to \widehat{d} = d - \widetilde{d}(\varepsilon), \quad t \to \widehat{t} = s^2_{\widetilde{d}(\varepsilon)+1} t,$$

the fluctuation components $T^{\varepsilon,i}$, $i = \widetilde{d}(\varepsilon) + 1, \dots, d$, again define a $\widehat{d}$-dimensional full-rank market of type (13.19)–(13.21) with timescale $\widehat{t}$, allowing, in principle, for recursive ε-rank aggregation.

We again consider the d-dimensional Black–Scholes market model with log-price process X and its ε-aggregate rank $\widetilde{d}$ process $\widehat{X}^\varepsilon$ of the previous section, and we estimate the error of approximating $u(t, x)$ by $\widehat{u}(t, \hat{x}^\varepsilon)$ in (13.17).

Theorem 13.4.3 *Assume that the payoff g is Lipschitz. Then, there exists a constant $C(x)$ independent of ε such that*

$$|u(t, x) - \widehat{u}(t, \hat{x}^\varepsilon)| \leq C(x) \sum_{i=\widetilde{d}(\varepsilon)+1}^{d} s_i^2.$$

Proof We introduce the artificial process $\widetilde{Y}^\varepsilon$ with dynamics

$$\mathrm{d}\widetilde{Y}_t^{\varepsilon,i} = \lambda_i \,\mathrm{d}t + \mathbb{1}_{\{1 \leq i \leq \widetilde{d}(\varepsilon)\}} s_i \,\mathrm{d}W_t^i, \quad i = 1, \dots, d.$$

Under the change of basis induced by $\mathbf{U}$, we have

$$\begin{aligned}|u(t, x) - \widehat{u}(t, \hat{x}^\varepsilon)| &= |v(t, y) - \widehat{v}(t, \hat{y}^\varepsilon)| \\ &\leq |v(t, y) - \widehat{v}(t, \tilde{y})| + |\widehat{v}(t, \tilde{y}) - \widehat{v}(t, \hat{y}^\varepsilon)|. \quad (13.30)\end{aligned}$$

The two terms in (13.30) are estimated separately. Since g is globally Lipschitz, we have for the first term, where $f(e^y) = g(e^x)$ and constants may change between lines,

$$
\begin{aligned}
|v(t,y) - \widehat{v}(t,\tilde{y})| &= \left|\mathbb{E}\left[f(e^{y+Y_{T-t}}) - f(e^{y+\widetilde{Y}^{\varepsilon}_{T-t}})\right]\right| \\
&\le C \sum_{i=1}^{d} \mathbb{E}\left[\left|e^{y_i+Y^i_{T-t}} - e^{y_i+\widetilde{Y}^{\varepsilon,i}_{T-t}}\right|\right] \\
&= C \sum_{i=\widetilde{d}(\varepsilon)+1}^{d} e^{y_i+\lambda_i(T-t)}\,\mathbb{E}\left[\left|e^{s_i W^i_{T-t}} - 1\right|\right] \\
&= C \sum_{i=\widetilde{d}(\varepsilon)+1}^{d} e^{y_i+\lambda_i(T-t)} \int_{\mathbb{R}} |e^z - 1| e^{-z^2/(2s_i^2)}\,\mathrm{d}z \\
&\le C \sum_{i=\widetilde{d}(\varepsilon)+1}^{d} e^{y_i} s_i^2 \le C(y) \sum_{i=\widetilde{d}(\varepsilon)+1}^{d} s_i^2.
\end{aligned}
$$

Similarly, using Lemma (13.4.1), we have for the second term

$$
\begin{aligned}
\left|\widehat{v}(t,\tilde{y}) - \widehat{v}(t,\hat{y}^{\varepsilon})\right| &= \left|\mathbb{E}\left[f(e^{y+\widetilde{Y}^{\varepsilon}_{T-t}}) - f(e^{y+\widehat{Y}^{\varepsilon}_{T-t}})\right]\right| \\
&\le C \sum_{i=1}^{d} \mathbb{E}\left[\left|e^{y_i+\widetilde{Y}^{\varepsilon,i}_{T-t}} - e^{y_i+\widehat{Y}^{\varepsilon,i}_{T-t}}\right|\right] \\
&= C \sum_{i=1}^{d} e^{y_i}\,\mathbb{E}\left[e^{\mathbb{1}_{\{1\le i\le \widetilde{d}(\varepsilon)\}} s_i W^i_{T-t}}\right] \left|e^{\lambda_i(T-t)} - e^{\widehat{\lambda}_i(T-t)}\right| \\
&\le C(y) \sum_{i=1}^{d} e^{y_i+\frac{1}{2}s_i^2(T-t)} |\lambda_i - \widehat{\lambda}_i| \le C(y) \sum_{i=1}^{d} e^{y_i}(1+s_i^2) \sum_{j=\widetilde{d}(\varepsilon)+1}^{d} s_j^2 \\
&\le C(y) \sum_{j=\widetilde{d}(\varepsilon)+1}^{d} s_j^2.
\end{aligned}
$$

Since $y = \mathbf{U}x$, $C(y) = C'(x)$, which completes the proof. □

13.4.2 Stochastic Volatility Models

Similarly to the one-dimensional case, multivariate stochastic volatility models replace the constant volatilities $\boldsymbol{\Sigma}_{ij}$ in the Black–Scholes model (13.19) by stochastic processes $\Sigma_{ij} = f_{ij}(Y)$, where f_{ij} are non-negative functions and Y is an additional source of randomness, which is modeled by an Itô diffusion in $\mathbb{R}^d$.

We consider the stochastic volatility extension of the Black–Scholes model as described in [68, Chap. 10.6]. We set $Z := (X, Y)$, where X describes again the log-price dynamics of $n > 1$ assets and Y is an $\mathbb{R}^n$-valued Itô diffusion describing the stochastic volatility $\Sigma_{ij} = f_{ij}(Y)$. In particular, we assume that each Y^i evolves according to the SDE

$$\mathrm{d}Y_t^i = c_i(Y_t^i)\,\mathrm{d}t + b_i(Y_t^i)\,\mathrm{d}\widetilde{W}_t^i, \quad Y_0^i = y^i, \quad i = 1, \ldots, n.$$

We pose the following assumptions: the state space domain of Y is $G^Y \subseteq \mathbb{R}^n$, and the coefficients $c_k, b_k : G^Y \to \mathbb{R}$ are globally Lipschitz-continuous and at most linearly growing. Furthermore, the $\mathbb{R}^n$-valued standard Brownian motion $(\widetilde{W}_t)_{t\geq 0}$ is correlated to the $\mathbb{R}^n$-valued standard Brownian motion $(W_t)_{t\geq 0}$ that drives the process X by $\widetilde{W}^k = \sum_{j=1}^n \rho_{jk} W^j + \rho^* \widehat{W}^k$, where $(W, \widehat{W})$ is a standard Brownian motion in $\mathbb{R}^d$ with $d = 2n$, and $\rho_k^* := (1 - \sum_{j=1}^n \rho_{jk}^2)^{1/2}$.

Denoting by $z := (x, y) = (x_1, \ldots, x_n, y_1, \ldots, y_n)$, the coefficients μ, Σ in (13.18) under a non-unique EMM are given by

$$\mu(z) := \big(r - 1/2 f_{11}^2(y), \ldots, r - 1/2 f_{nn}^2(y), c_1(y_1), \ldots, c_n(y_n)\big)^\top \in \mathbb{R}^d, \tag{13.31}$$

$$\Sigma(z) := \begin{pmatrix} \Sigma^X(z) & 0 \\ \Sigma^Y(z) & D(z) \end{pmatrix} \in \mathbb{R}^{d\times d}, \tag{13.32}$$

where the matrices Σ^X, Σ^Y, $D \in \mathbb{R}^{n\times n}$ are

$$\Sigma^X(z) := \big(f_{ij}(y)\big)_{1\leq i,j\leq n}, \quad \Sigma^Y(z) := \big(\rho_{ji} b_i(y_i)\big)_{1\leq i,j\leq n},$$
$$D(z) := \operatorname{diag}\big(\rho_1^* b_1(y_1), \ldots, \rho_n^* b_n(y_n)\big).$$

The smooth functions $f_{ij} : G^Y \to \mathbb{R}_+$ are assumed to be bounded from below and above. The state space domain of the pair process $Z = (X, Y)$ is $G = \mathbb{R}^n \times G^Y$.

The infinitesimal generator $\mathcal{A}$ of the semigroup generated by the process Z is given by

$$\mathcal{A} := -\frac{1}{2}\operatorname{tr}\big[\mathcal{Q}(z)D^2\big] - \mu(z)^\top \nabla, \tag{13.33}$$

with $\mathcal{Q} = \boldsymbol{\Sigma}\boldsymbol{\Sigma}^\top$.

Example 13.4.4 (Volatility processes) In [68, Chap. 10.6], it is assumed that each volatility component Y^k follows a mean-reverting Ornstein–Uhlenbeck process, i.e. $c_k(y_k) = \alpha_k(m_k - y_k)$, $b_k(y_k) = \beta_k$, $1 \leq k \leq n$. Here, $\alpha_k > 0$ is called the rate of mean reversion and $m_k \geq 0$ is the long-run mean level of Y^k. Under an EMM, the drift term c_k becomes $c_k(y) = \alpha_k(m_k - y_k) - \beta_k \Lambda_k(y)$, for some volatility risk premium $\Lambda(y) = (\Lambda_1(y), \ldots, \Lambda_n(y))^\top$. See [68, Chap. 2.5] for a representation of Λ in the one dimensional case $n = 1$.

13.5 Numerical Examples

We give numerical examples using the wavelet finite element discretization as described in Sect. 13.3.

13.5.1 Full-Rank d-Dimensional Black–Scholes Model

We consider the geometric call option with payoff

$$g(x) = \max(0, e^{\sum_{i=1}^d \alpha_i x_i} - K), \quad x \in \mathbb{R}^d,$$

for strike $K = 1$. The antiderivatives of g for $\alpha_i > 0$, $i = 1, \ldots, d$ are given by

$$g^{(-2)}(x) = \prod_{i=1}^d \alpha_i^{-2} \left(e^{\sum_{i=1}^d \alpha_i x_i} - \sum_{k=0}^{2d} \frac{1}{k!} \left(\sum_{i=1}^d \alpha_i x_i - \log K \right)^k \right) 1_{\{\sum_{i=1}^d \alpha_i x_i \geq \log K\}}.$$

We first set $d = 2$ and solve problem (2.24) for various mesh widths $h = 2^{-L}$. Using interest rate $r = 0.01$, covariance $\mathcal{Q} = (\sigma_i \sigma_j \rho_{ij})_{1 \leq i,j \leq d}$, $\sigma_1 = 0.4$, $\sigma_2 = 0.1$, $\rho_{12} = 0.2$ and weights $\alpha_i = 1/d$, $i = 1, \ldots, d$, we plot the convergence rate of the L^2-error

$$\|e_L\| := \|u(T, \cdot) - u_L(T, \cdot)\|_{L^2(G_0)}, \quad G_0 = (K/2, 3/2K)^2,$$

at maturity $T = 1$ in Fig. 13.2. To compare the rates we also solved the problem on full grid. In the top picture, it can be seen that sparse grid has (up to log terms) the same rate as full grid. To better show the advantage of a sparse grid, we additionally plot the convergence rate in terms of degrees of freedom. For the full grid, we have $N_L = \mathcal{O}(2^{2L})$ and for the sparse grid $\widehat{N}_L = \mathcal{O}(L\,2^L)$. The convergence rate in the full grid shows the "curse of dimension", whereas for the sparse grid we still obtain the optimal rate (up to log terms).

For $d > 2$ we set $\sigma_i = 0.3$, $i = 1, \ldots, d$ and $\rho_{i,j} = 0$, $i = 1, \ldots, d-2$, $j = i + 2, \ldots, d$, $\rho_{12} = \rho$, $\rho_{i,i+1} = \rho\sqrt{1-\rho^2}$, $i = 1, \ldots, d-1$, with $\rho = 0.1$ and weights $\alpha_1 = 1$, $\alpha_i = 0$, $i = 2, \ldots, d$. We plot the convergence rate of the absolute error at the point $S_0 = (K, \ldots, K)$ in Fig. 13.3.

For low dimensions $d \in \{2, 3, 4\}$, the second order convergence rate on the sparse grid can be well observed over all levels. For higher dimensions $d \in \{6, 8\}$, the log terms prevail at low levels, hence the flattened behavior of the convergence curves which then exhibit the expected second order rates at finer discretizations. Despite the smoothness of the solution, we report a steeply increasing constant in the rates as d is raised, which forces us to already set $L = 11$ for $d = 8$ in order to have a relative L^2-error on the order of 10^{-2} which currently prevents us from reasonably increasing d beyond 8. The size of the constant can be traced back to the initial condition u_h^0, the H-projection of the payoff g onto V_h (Sect. 13.3), showing similar relative L^2-errors.

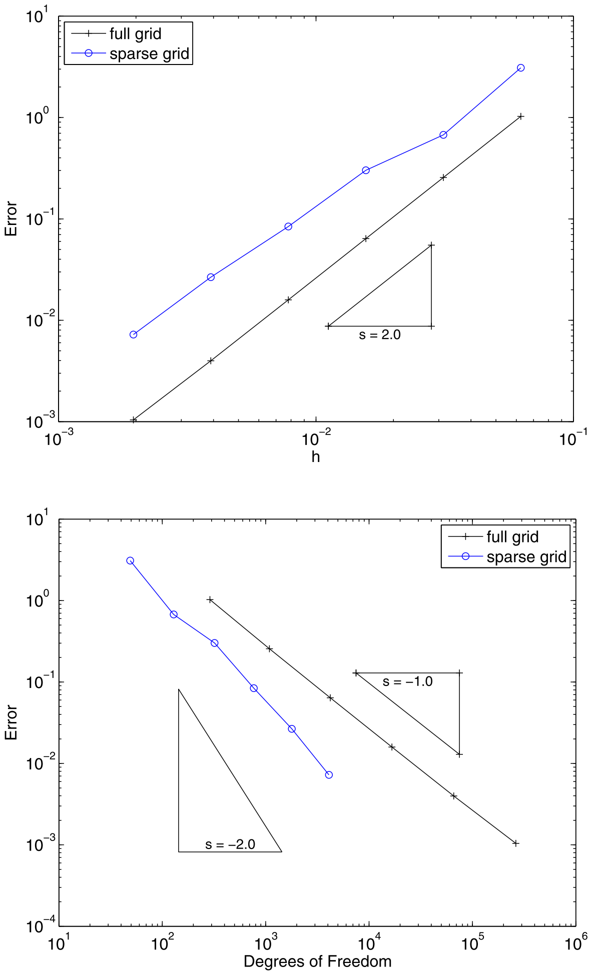


Fig. 13.2 Convergence rate of the wavelet discretization in terms of the mesh width h (*top*) and in terms of degrees of freedom (*bottom*)

13.5.2 Low-Rank $\widetilde{d}$-Dimensional Black–Scholes

We consider the geometric call option as in Sect. 13.5.1 written on d underlyings under the Black–Scholes model and we are now interested in the case of larger dimensions, i.e. $d > 8$. This occurs when pricing contingent claims on stock indices considering all d price processes in comparison to handling the index as one single process. Straightforward computations in such high dimensions would currently require too high discretization levels as previously noted. Instead, we rely on the dimensionality reduction by ε-aggregation to identify a rank $\widetilde{d}$ ε-aggregated process driving a d-dimensional market. In particular, we focus on the Dow Jones industrial index where $d = 30$. We compute the principal components of the volatility covariance matrix $\mathcal{Q} := \mathbf{U}^\top \mathbf{D} \mathbf{U} \in \mathbb{R}^{d \times d}$ of their realized daily log-returns over 252 periods resulting in the spectrum $(s_1^2, \ldots, s_d^2)$, normalized by s_1^2 as in Sect. 13.4.1

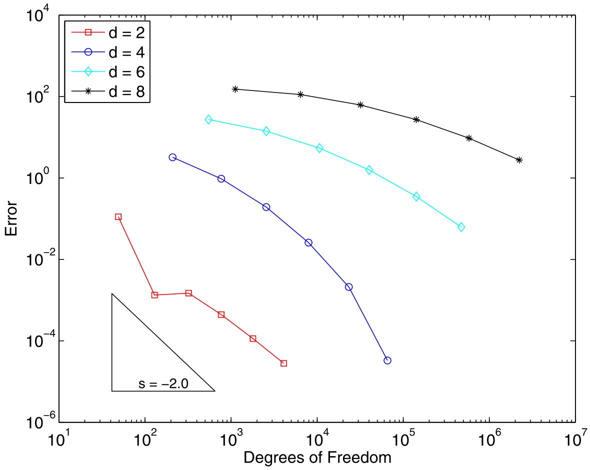


Fig. 13.3 Convergence rate of the BS model in different dimensions d

and shown in Fig. 13.4 (top). Similar eigenvalue decompositions for the DAX index are presented in [139]. We define the *recovery ratio*

$$\eta_{\widetilde{d}} := \left(\sum_{i=1}^{\widetilde{d}} s_i^2\right)\left(\sum_{i=1}^{d} s_i^2\right)^{-1}, \qquad \widetilde{d} = 1, \ldots, d,$$

shown in Fig. 13.4 (bottom), whose values for $\widetilde{d} = 1, \ldots, 5$ are reported in Table 13.3. The eigenvalues are observed to decay exponentially and by virtue of Theorem 13.4.3, we may therefore expect sufficiently accurate results for $\widetilde{d} \leq 5$.

Let $\widehat{\mathbf{D}} := \operatorname{diag}(\hat{s}_1^2, \ldots, \hat{s}_d^2) \in \mathbb{R}^{d\times d}$ with $\hat{s}$ as in (13.24) for some $2 \leq \widetilde{d} \leq 5$ which defines the rank-reduced processes $\widehat{X}$ and $\widehat{Y}$ defined by (13.25) and (13.26), respectively. We approximate the exact solution $u(t, x) = v(t, y)$ of the d-dimensional problem (8.12) by a parametrized $\widetilde{d}$-dimensional option price $\widehat{v}(t, \hat{y}) = \widehat{v}(t, \hat{y}_1, \ldots, \hat{y}_{\widetilde{d}}; \hat{y}_{\widetilde{d}+1}, \ldots, \hat{y}_d)$ for $\hat{y} = (\hat{y}_1, \ldots, \hat{y}_d) \in G := (-R, R)^d$. We therefore consider the option prices $\widehat{v}_R$ satisfying

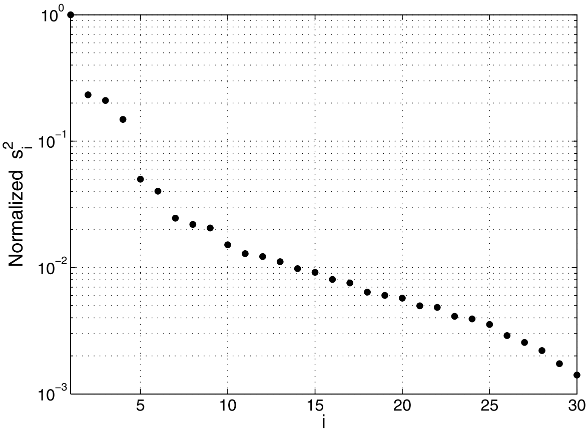
$$\begin{aligned} \partial_t \widehat{v}_R + \widehat{\mathcal{A}}\widehat{v}_R + r\widehat{v}_R &= 0 && \text{in } J \times \widehat{G}_R^{\widetilde{d}}, \\ \widehat{v}_R(t, \cdot) &= 0 && \text{on } J \times \partial\widehat{G}_R^{\widetilde{d}}, \\ \widehat{v}_R(0, \hat{y}) &= \widehat{f}(e^{\hat{y}})|_{\widehat{G}_R^{\widetilde{d}}} && \text{in } \widehat{G}_R^{\widetilde{d}}, \end{aligned} \tag{13.34}$$

where $\widehat{G}_R^{\widetilde{d}} := (-R, R)^{\widetilde{d}}$. The rank-$\widetilde{d}$ operator $\widehat{\mathcal{A}}$ is now given by

$$\widehat{\mathcal{A}} := -\frac{1}{2}\operatorname{tr}\big[\widehat{\mathbf{D}}\widehat{D}^2\big] - \widehat{\lambda}^\top \widehat{\nabla},$$

with $\widehat{\lambda}$ as in (13.26), rank-$\widetilde{d}$ differential operators $\widehat{\nabla}$, and $\widehat{D}^2$

$$\widehat{\nabla} := (\partial_{\hat{y}_1}, \ldots, \partial_{\hat{y}_{\widetilde{d}}}, 0, \ldots, 0)^\top, \quad \widehat{D}^2 := \left(\begin{array}{c|c} (\partial^2_{\hat{y}_i \hat{y}_j})_{1\leq i,j\leq \widetilde{d}} & 0 \\ \hline 0 & 0 \end{array}\right),$$



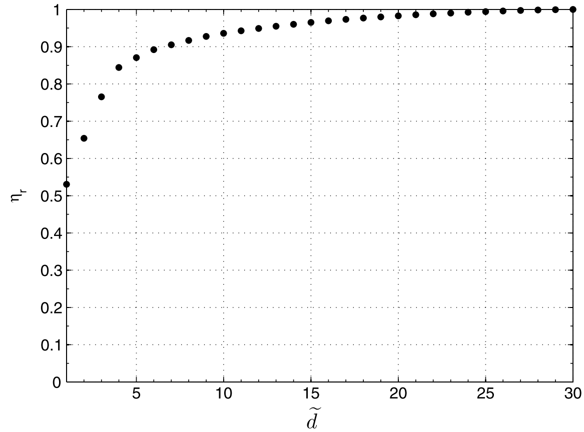


Fig. 13.4 Eigenvalue spectrum (normalized by s_1^2) of the realized volatility covariance matrix of the constituents of the Dow Jones index over a period of 252 days (*top*) and recovery ratio $\eta_{\widetilde{d}}$, $\widetilde{d} = 1, \dots, d$ (*bottom*)

Table 13.3 First eigenvalues $s_{\widetilde{d}}^2$, $\widetilde{d} = 1, \dots, 5$, of the Dow Jones realized volatility covariance matrix Q and recovery ratio $\eta_{\widetilde{d}}$

$\widetilde{d}$	$s_{\widetilde{d}}$	$s_{\widetilde{d}}^2$	$\eta_{\widetilde{d}}$
1	0.2052	0.0421	0.5307
2	0.0990	0.0098	0.6542
3	0.0940	0.0088	0.7655
4	0.0791	0.0062	0.8443
5	0.0459	0.0021	0.8708

respectively, and

$$\begin{aligned}\widehat{f}(e^{\hat{y}}) &= f(e^{\hat{y}_1}, \dots, e^{\hat{y}_{\widetilde{d}}}, e^{\hat{y}_{\widetilde{d}+1} + \widehat{\lambda}_{\widetilde{d}+1} T}, \dots, e^{\hat{y}_d + \widehat{\lambda}_d T}) \\ &= \max(0, e^{\sum_{i=1}^{\widetilde{d}} \alpha_i \hat{y}_i + \sum_{i=\widetilde{d}+1}^{d} \hat{y}_i + \widehat{\lambda}_i T} - K).\end{aligned}$$

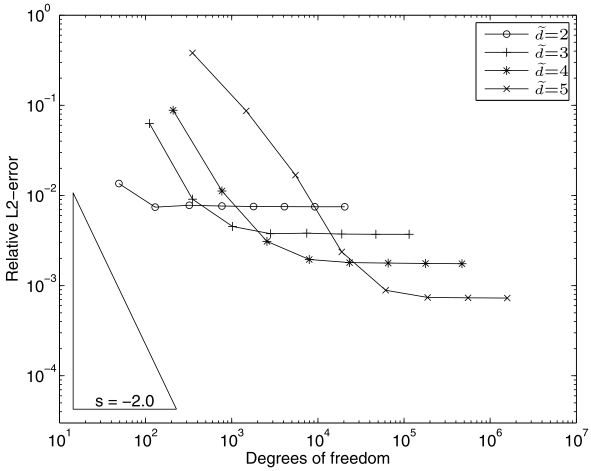


Fig. 13.5 Convergence rates of the approximation of a 30 dimensional option price on the Dow Jones by $\widetilde{d} = 2, \ldots, 5$ dimensional options in the Black–Scholes model

We numerically solve (13.34) for $\widetilde{d} = 2, \ldots, 5$ and various mesh widths $h = 2^{-L}$ with the Dow Jones realized volatility covariance matrix $\mathcal{Q}$ whose principal components are plotted in Fig. 13.4 (top), weights $\alpha_i = 0.3$, $i = 1, \ldots, d$, maturity $T = 1$, strike $K = 1$ and interest rate $r = 0.045$, and plot the convergence rate of the relative L^2-error

$$\|e_L\| := \frac{\|v(T,\cdot) - \widehat{v}_L(T,\cdot)\|_{L^2(G_1)}}{\|v(T,\cdot)\|_{L^2(G_1)}}, \quad G_1 = (3/4K, 5/4K) \times \{K\} \times \cdots \times \{K\},$$

at maturity $T = 1$ in Fig. 13.5. The flattening behavior of the convergence rates is explained by further expanding the error into (a) an ε-aggregation error made by artificially setting volatilities to zero (Theorem 13.4.3) and (b) a discretization error [159] as

$$\begin{aligned}\|v(t,y) - \widehat{v}_L(t,\hat{y})\| &= \|v(t,y) - \widehat{v}(t,\hat{y}) + \widehat{v}(t,\hat{y}) - \widehat{v}_L(t,\hat{y})\| \\ &\le \underbrace{\|v(t,y) - \widehat{v}(t,\hat{y})\|}_{(a)} + \underbrace{\|\widehat{v}(t,\hat{y}) - \widehat{v}_L(t,\hat{y})\|}_{(b)}.\end{aligned}$$

It follows that the lower bounds observed in Fig. 13.5 therefore stem from the ε-aggregation error which diminishes as $\widetilde{d}$ is increased.

13.6 Further Reading

A rigorous error analysis for the numerical solution of high dimensional parabolic equations is given in von Petersdorff and Schwab [159], where a wavelet based sparse grid discretization is used in conjunction with a discontinuous Galerkin time-discretization. Reisinger and Wittum [139] use a finite difference based discretization on sparse grids employing the combination technique. Leentvaar and Oosterlee [113] use high order finite difference discretizations on sparse grids for the pricing of basket options in up to 5 dimensions. The construction of the sparse tensor product space $\widehat{\mathcal{V}}_L$ can be further developed to general sparse approximation spaces

$$\widehat{\mathcal{V}}_{L,\beta} := \bigoplus_{\beta|\boldsymbol{\ell}|_\infty - |\boldsymbol{\ell}|_1 \geq \beta L - (L+d-1)} W_{\ell_1} \otimes \cdots \otimes W_{\ell_d},$$

where $\beta \in \mathbb{R}$ and $\boldsymbol{\ell} \in \mathbb{N}^d$. Griebel and Knapek show in [75] that using such approximation spaces one can remove the logarithmic term L^{d-1}. Indeed, for $0 < \beta < 1$, one has $\dim \widehat{\mathcal{V}}_{L,\beta} = \mathcal{O}(2^L)$. Furthermore, they have similar approximation properties as $\widehat{\mathcal{V}}_L$.

Chapter 14
Multidimensional Lévy Models

In this chapter, we extend the one-dimensional Lévy models described in Chap. 10 to multidimensional Lévy models. Since the law of a Lévy process X is time-homogeneous, it is completely characterized by its characteristic triplet $(\mathcal{Q}, \nu, \gamma)$. The drift γ has no effect on the dependence structure between the components of X. The dependence structure of the Brownian motion part of X is given by its covariance matrix $\mathcal{Q}$. For purposes of financial modeling, it remains to specify a parametric dependence structure of the purely discontinuous part of X which can be done by using Lévy copulas.

14.1 Lévy Processes

We associate to the multidimensional Lévy process $X = \{X_t : t \in [0, T]\}$ the Lévy measure

$$\nu(B) = \mathbb{E}\left[\#\{t \in [0,1] \ : \ \Delta X_t \neq 0, \ \Delta X_t \in B\}\right], \quad B \in \mathcal{B}(\mathbb{R}^d).$$

The Lévy measure satisfies $\int_{\mathbb{R}^d} 1 \wedge |z|^2 \, \nu(\mathrm{d}z) < \infty$, where we denote by $|z|^2 = \sum_{i=1}^d z_i^2$. Using the Lévy–Khinchine representation, we see that every Lévy process is uniquely defined by a drift vector γ, a positive definite matrix $\mathcal{Q} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ and the Lévy measure ν.

Theorem 14.1.1 (Lévy–Khinchine representation) *Let X be a Lévy process with state space $\mathbb{R}^d$ and characteristic triplet $(\mathcal{Q}, \nu, \gamma)$. Assume $\int_{|z|>1} |z| \, \nu(\mathrm{d}z) < \infty$. Then, for $t \geq 0$,*

$$\begin{aligned} &\mathbb{E}\left[e^{i\langle \xi, X_t\rangle}\right] = e^{-t\psi(\xi)}, \quad \xi \in \mathbb{R}^d, \\ &\textit{with } \psi(\xi) = -i\langle \gamma, \xi\rangle + \frac{1}{2}\langle \xi, \mathcal{Q}\xi\rangle + \int_{\mathbb{R}^d} \left(1 - e^{i\langle \xi, z\rangle} + i\langle \xi, z\rangle\right)\nu(\mathrm{d}z). \end{aligned} \tag{14.1}$$

N. Hilber et al., *Computational Methods for Quantitative Finance*, Springer Finance, DOI 10.1007/978-3-642-35401-4_14,

In what follows, we denote by X^i, $i = 1, \ldots, d$, the coordinate projection of the multidimensional process $X = (X^1, \ldots, X^d)^\top \in \mathbb{R}^d$. Coordinate projections of Lévy processes are again Lévy processes.

Proposition 14.1.2 *Let $X = (X^1, \ldots, X^d)^\top$ be a Lévy process with state space $\mathbb{R}^d$ and characteristic triplet $(\mathcal{Q}, \nu, \gamma)$. Assume $\int_{|z|>1} |z|\, \nu(\mathrm{d}z) < \infty$. Then, the marginal processes X^j, $j = 1, \ldots, d$, are again Lévy processes with the characteristic triplet $(\mathcal{Q}_{jj}, \nu_j, \gamma_j)$ where the marginal Lévy measures are defined as*

$$\nu_j(B) := \nu(\{x \in \mathbb{R}^d : x_j \in B \backslash \{0\}\}), \quad \forall B \in \mathcal{B}(\mathbb{R}), \quad j = 1, \ldots, d.$$

If all the marginal Lévy processes X^i satisfy the martingale condition as in Lemma 10.1.5, i.e. e^{X^i} are martingales with respect to the filtration of X^i, $i = 1, \ldots, d$, they are also martingales with respect to the filtration $\mathbb{F}$ of $X = (X^1, \ldots, X^d)^\top$.

Lemma 14.1.3 *Let X be a Lévy process with state space $\mathbb{R}^d$ and characteristic triplet $(\mathcal{Q}, \nu, \gamma)$. Assume, $\int_{|z|>1} |z|\, \nu(\mathrm{d}z) < \infty$ and $\int_{|z|>1} e^{z_j} \nu_j(\mathrm{d}z) < \infty$, $j = 1, \ldots, d$. Then, e^{X^j}, $j = 1, \ldots, d$, are martingales with respect to the filtration $\mathbb{F}$ of X if and only if*

$$\frac{\mathcal{Q}_{jj}}{2} + \gamma_j + \int_{\mathbb{R}} \left(e^{z_j} - 1 - z_j\right) \nu_j(\mathrm{d}z) = 0, \quad j = 1, \ldots, d.$$

Proof As in the proof of Lemma 10.1.5, we have

$$\mathbb{E}\left[e^{X_s^j} \mid \mathcal{F}_t\right] = e^{X_t^j} e^{(t-s)\psi(-ie_j)}.$$

Therefore, setting $\psi(-ie_j) = 0$, $j = 1, \ldots, d$, the Lévy–Khinchine formula (14.1) yields

$$\frac{\mathcal{Q}_{jj}}{2} + \gamma_j + \int_{\mathbb{R}^d} \left(e^{z_j} - 1 - z_j\right) \nu(\mathrm{d}z) = 0, \quad j = 1, \ldots, d.$$

The result follows with the definition of the marginal Lévy density ν_j given in Proposition 14.1.2. □

14.2 Lévy Copulas

We first give some definitions. The *F-volume* of $(a, b]$, $a, b \in \overline{\mathbb{R}}^d$ for a function $F : S \to \overline{\mathbb{R}}$, $S \subset \overline{\mathbb{R}}^d$ is defined by

$$V_F((a, b]) := \sum_{u \in \{a_1, b_1\} \times \cdots \times \{a_d, b_d\}} (-1)^{N(u)} F(u),$$

where $N(u) = |\{k : u_k = a_k\}|$.

Definition 14.2.1 A function $F : S \to \overline{\mathbb{R}}$, $S \subset \overline{\mathbb{R}}^d$ is called *d-increasing* if $V_F\left((a, b]\right) \geq 0$ for all $a, b \in S$ with $a \leq b$ and $\overline{(a, b]} \subset S$.

Examples of d-increasing functions are distribution functions of random vectors $X \in \mathbb{R}^d$, $F(x_1, \ldots x_d) = \mathbb{P}[X_1 \leq x_1, \ldots, X_d \leq x_d]$, or more generally,

$$F(x_1, \ldots x_d) = \mu\left((-\infty, x_1], \ldots, (-\infty, x_d]\right),$$

where μ is a finite measure on $\mathcal{B}(\mathbb{R}^d)$. F is clearly d-increasing since the F-volume is just $V_F\left((a, b]\right) = \mu\left((a, b]\right)$ for every $a \leq b$. For parametric modeling dependence structures of multidimensional jump processes, margins play an important role.

Definition 14.2.2 Let $F : \overline{\mathbb{R}}^d \to \overline{\mathbb{R}}$ be a d-increasing function which satisfies $F(u) = 0$ if $u_i = 0$ for at least one $i \in \{1, \ldots, d\}$. For any index set $\mathcal{I} \subset \{1, \ldots, d\}$ the $\mathcal{I}$-margin of F is the function $F^{\mathcal{I}} : \overline{\mathbb{R}}^{|\mathcal{I}|} \to \overline{\mathbb{R}}$

$$F^{\mathcal{I}}(u^{\mathcal{I}}) := \lim_{a\to\infty} \sum_{(u_j)_{j\in\mathcal{I}^c} \in \{-a,\infty\}^{|\mathcal{I}^c|}} \Big(\prod_{j\in\mathcal{I}^c} \operatorname{sgn} u_j\Big) F(u_1, \ldots, u_d).$$

Since the Lévy measure is a measure on $\mathcal{B}(\mathbb{R}^d)$, it is possible to define a suitable notion of a copula. However, one has to take into account that the Lévy measure is possibly infinite at the origin.

Definition 14.2.3 A function $F : \overline{\mathbb{R}}^d \to \overline{\mathbb{R}}$ is called a *Lévy copula* if

(i) $F(u_1, \ldots, u_d) \neq \infty$ for $(u_1, \ldots, u_d) \neq (\infty, \ldots, \infty)$,
(ii) $F(u_1, \ldots, u_d) = 0$ if $u_i = 0$ for at least one $i \in \{1, \ldots, d\}$,
(iii) F is d-increasing,
(iv) $F^{\{i\}}(u) = u$ for any $i \in \{1, \ldots, d\}$, $u \in \mathbb{R}$.

In the following, we prove some useful properties of Lévy copulas.

Lemma 14.2.4 *Let F be a Lévy copula. Then,*

$$0 \leq \prod_{j=1}^{d} \operatorname{sgn} u_j F(u_1, \ldots, u_d) \leq \min\{|u_1|, \ldots, |u_d|\} \quad \forall u \in \mathbb{R}^d,$$

and $\prod_{j=1}^{d} \operatorname{sgn} u_j F(u)$ is nondecreasing in the absolute value of each argument $|u_j|$.
Furthermore, Lévy copulas are Lipschitz continuous, i.e.

$$|F(v_1, \ldots, v_d) - F(u_1, \ldots, u_d)| \leq \sum_{i=1}^{d} |v_i - u_i| \quad \forall u, v \in \mathbb{R}^d.$$

Proof Let $u \in \mathbb{R}^d$, $u_1 \geq 0$ and $0 \leq a_1 \leq u_1$. Set $b_1 = u_1$ and for $2 \leq j \leq d$ set $a_j = 0$, $b_j = u_j$ if $u_j \geq 0$ otherwise $a_j = u_j$, $b_j = 0$. Since F is d-increasing and grounded

$$
\begin{aligned}
V_F\left((a,b]\right) &= \sum_{v\in\{a_1,b_1\}\times\cdots\times\{a_d,b_d\}} (-1)^{N(v)} F(v) \\
&= \prod_{j=2}^{d} \operatorname{sgn} u_j F(u_1,\ldots,u_d) - \prod_{j=2}^{d} \operatorname{sgn} u_j F(a_1,u_2\ldots,u_d) \geq 0\,.
\end{aligned}
$$

Similarly for $u_1 < 0$. This gives the lower bound with $a_1 = 0$.

Let $\mathcal{I} = \{i\} \subset \{1,\ldots,d\}$. Then,

$$
\begin{aligned}
&\prod_{j=1}^{d} \operatorname{sgn} u_j F(u_1,\ldots,u_i,\ldots u_d) \\
&\quad\leq \operatorname{sgn} u_i \lim_{n\to\infty} \left(\prod_{j\in\mathcal{I}^c} \operatorname{sgn} u_j\right) F(n\operatorname{sgn} u_1,\ldots,u_i,\ldots,n\operatorname{sgn} u_d) \\
&\quad\leq \operatorname{sgn} u_i \lim_{n\to\infty} \sum_{(v_j)_{j\in\mathcal{I}^c}\in\{-n,\infty\}^{|\mathcal{I}^c|}} \left(\prod_{j\in\mathcal{I}^c} \operatorname{sgn} v_j\right) F(v_1,\ldots,u_i,\ldots,v_d) \\
&\quad= \operatorname{sgn} u_i\ F^i(u_i) = |u_i|\ .
\end{aligned}
$$

Since $i \in \{1,\ldots,d\}$ is arbitrary, we obtain the upper bound.

To simplify notation, we suppose $0 \leq u_i \leq v_i$, $i = 1,\ldots,d$. The general case can be treated similarly. We have

$$
\begin{aligned}
&|F(v_1,\ldots,v_d) - F(u_1,\ldots,u_d)| \\
&\quad= |V_F((0,v_1]\times\cdots\times(0,v_d]) - V_F((0,u_1]\times\cdots\times(0,u_d])| \\
&\quad\leq \sum_{i=1}^{d} \lim_{a\to\infty} V_F\big((-a,\infty]^{i-1}\times(u_i,v_i]\times(-a,\infty]^{d-i}\big) \\
&\quad= \sum_{i=1}^{d} \big(F^i(v_i) - F^i(u_i)\big) \\
&\quad= \sum_{i=1}^{d} |v_i - u_i|\,.
\end{aligned}
$$

□

We also need tail integrals of Lévy processes.

Definition 14.2.5 Let X be a Lévy process with state space $\mathbb{R}^d$ and Lévy measure ν. The *tail integral* of X is the function $U : \mathbb{R}^d\backslash\{0\} \to \mathbb{R}$ given by

$$
U(x_1,\ldots,x_d) = \prod_{i=1}^{d} \operatorname{sgn}(x_j)\nu\Big(\prod_{j=1}^{d} I(x_j)\Big)\,,
$$

where

$$I(x) = \begin{cases} (x, \infty) & \text{for } x \geq 0, \\ (-\infty, x] & \text{for } x < 0. \end{cases}$$

Furthermore, for $\mathcal{I} \subset \{1, \dots, d\}$ nonempty the $\mathcal{I}$-marginal tail integral $U^{\mathcal{I}}$ of X is the tail integral of the process $X^{\mathcal{I}} := (X^i)_{i \in \mathcal{I}}$.

The next result, [100, Theorem 3.6], shows that essentially any Lévy process $X = (X^1, \dots, X^d)^\top$ can be built from univariate marginal processes X^j, $j = 1, \dots, d$ and Lévy copulas. It can be viewed as a version of Sklar's theorem for Lévy copulas.

Theorem 14.2.6 (Sklar's theorem for Lévy copulas) *For any Lévy process X with state space $\mathbb{R}^d$, there exists a Lévy copula F such that the tail integrals of X satisfy*

$$U^{\mathcal{I}}(x^{\mathcal{I}}) = F^{\mathcal{I}}((U_i(x_i))_{i \in \mathcal{I}}), \tag{14.2}$$

for any nonempty $\mathcal{I} \subset \{1, \dots, d\}$ and any $x^{\mathcal{I}} \in \mathbb{R}^{|\mathcal{I}|} \backslash \{0\}$. The Lévy copula F is unique on $\prod_{i=1}^d \overline{\operatorname{Ran} U_i}$.

Conversely, let F be a d-dimensional Lévy copula and U_i, $i = 1, \dots, d$, tail integrals of univariate Lévy processes. Then, there exits a d-dimensional Lévy process X such that its components have tail integrals U_i and its marginal tail integrals satisfy (14.2). *The Lévy measure ν of X is uniquely determined by F and U_i, $i = 1, \dots, d$.*

Using partial integration, we can write the multidimensional Lévy measure in terms of the Lévy copula.

Lemma 14.2.7 *Let $f(z) \in C^\infty(\mathbb{R}^d)$ be bounded and vanishing on a neighborhood of the origin. Furthermore, let X be a d-dimensional Lévy process with Lévy measure ν, Lévy copula F and marginal Lévy measures ν_j, $j = 1, \dots, d$. Then,*

$$\begin{aligned} \int_{\mathbb{R}^d} f(z)\nu(\mathrm{d}z) = & \sum_{j=1}^d \int_{\mathbb{R}} f(0 + z_j)\nu_j(\mathrm{d}z_j) \\ & + \sum_{j=2}^d \sum_{\substack{|\mathcal{I}|=j \\ \mathcal{I}_1 < \dots < \mathcal{I}_j}} \int_{\mathbb{R}^j} \partial^{\mathcal{I}} f(0 + z^{\mathcal{I}}) F^{\mathcal{I}}((U_k(z_k))_{k \in \mathcal{I}})\, \mathrm{d}z^{\mathcal{I}}. \end{aligned} \tag{14.3}$$

Proof We proceed by induction with respect to the dimension d:

For $d = 1$, integration by parts yields

$$\begin{aligned} \int_0^\infty f(z)\nu(\mathrm{d}z) = & -\lim_{b \to \infty} f(b)\nu(I(b)) + \lim_{a \to 0+} f(a)\nu(I(a)) \\ & + \int_0^\infty \partial_1 f(z)\nu(I(z))\, \mathrm{d}z, \end{aligned}$$

$$\int_{-\infty}^{0} f(z)\nu(\mathrm{d}z) = \lim_{a\to 0-} f(a)\nu\left(I(a)\right) - \lim_{b\to -\infty} f(b)\nu\left(I(b)\right) - \int_{-\infty}^{0} \partial_1 f(z)\nu\left(I(z)\right)\mathrm{d}z\,,$$

and since f is bounded,

$$\int_{\mathbb{R}} f(z)\nu(\mathrm{d}z) = f(0)\lim_{a\to 0+}\left(\nu\left(I(a)\right)+\nu\left(I(-a)\right)\right) + \int_{\mathbb{R}} \partial_1 f(z)\mathrm{sgn}(z)\nu\left(I(z)\right)\mathrm{d}z\,.$$

Abusing notation, we write

$$\nu(\mathbb{R}) := \lim_{a\to 0+}\left(\nu\left(I(a)\right)+\nu\left(I(-a)\right)\right).$$

With f vanishing on a neighborhood of 0, we therefore find $f(0)\nu(\mathbb{R}) = 0$.

For the multidimensional case, i.e. for $d > 1$, we use the Lévy measure of $X^{\mathcal{I}}$ which is given by

$$\nu^{\mathcal{I}}(B) := \nu\Big(\{x\in\mathbb{R}^d : (x_i)_{i\in\mathcal{I}} \in B\setminus\{0\}\}\Big), \quad \forall B \in \mathcal{B}\big(\mathbb{R}^{|\mathcal{I}|}\big).$$

We show by induction with respect to the dimension d that

$$\begin{aligned}\int_{\mathbb{R}^d} f(z)\nu(\mathrm{d}z) &= f(0,\dots,0)\nu(\mathbb{R},\dots,\mathbb{R})\\ &\quad + \sum_{i=1}^{d}\int_{\mathbb{R}} \partial_i f(0,\dots,z_i,\dots,0)\mathrm{sgn}(z_i)\nu_i\left(I(z_i)\right)\mathrm{d}z_i\\ &\quad + \sum_{i=2}^{d}\sum_{\substack{|\mathcal{I}|=i\\ \mathcal{I}_1<\dots<\mathcal{I}_i}}\int_{\mathbb{R}^i} \partial^{\mathcal{I}} f(0+z^{\mathcal{I}})\prod_{j\in\mathcal{I}}\mathrm{sgn}(z_j)\nu^{\mathcal{I}}\Big(\prod_{j\in\mathcal{I}} I(z_j)\Big)\mathrm{d}z^{\mathcal{I}}\,.\end{aligned}$$

With $f(0,\dots,0)\nu(\mathbb{R},\dots,\mathbb{R}) = 0$, the definition of the tail integrals and Theorem 14.2.6 we then have the required result.

For the induction step $d-1 \to d$, using integration by parts and the induction hypothesis, we obtain

$$\begin{aligned}\int_{\mathbb{R}^d} f(z)\nu(\mathrm{d}z) &= \int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}} f(z',z_d)\nu(\mathrm{d}z',\mathrm{d}z_d)\\ &= \int_{\mathbb{R}^{d-1}} f(z',0)\nu(\mathrm{d}z',\mathbb{R})\\ &\quad + \int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}} \partial_d f(z',z_d)\,\mathrm{sgn}(z_d)\nu\left(\mathrm{d}z', I(z_d)\right)\mathrm{d}z_d\\ &= f(0,\dots,0)\nu(\mathbb{R},\dots,\mathbb{R})\\ &\quad + \sum_{i=1}^{d-1}\int_{\mathbb{R}} \partial_i f(0,\dots,z_i,\dots,0)\mathrm{sgn}(z_i)\nu_i\left(I(z_i)\right)\mathrm{d}z_i\end{aligned}$$

$$+\sum_{i=2}^{d-1}\sum_{\substack{|\mathcal{I}|=i\\ \mathcal{I}_1<\cdots<\mathcal{I}_i}}\int_{\mathbb{R}^i}\partial^{\mathcal{I}}f(0+z^{\mathcal{I}})\prod_{j\in\mathcal{I}}\operatorname{sgn}(z_j)\nu^{\mathcal{I}}\Big(\prod_{j\in\mathcal{I}}I(z_j)\Big)\mathrm{d}z^{\mathcal{I}}$$
$$+\int_{\mathbb{R}}\partial_d f(0,\ldots,0,z_d)\operatorname{sgn}(z_d)\nu\,(\mathbb{R},\ldots,\mathbb{R},I(z_d))$$
$$+\sum_{i=1}^{d-1}\int_{\mathbb{R}}\int_{\mathbb{R}}\partial_i\partial_d f(0,\ldots,z_i,\ldots,0,z_d)$$
$$\times\operatorname{sgn}(z_i)\operatorname{sgn}(z_d)\nu_{i,d}\,(I(z_i),I(z_d))\,\mathrm{d}z_i\,\mathrm{d}z_d$$
$$+\sum_{i=2}^{d-1}\sum_{\substack{|\mathcal{I}|=i\\ \mathcal{I}_1<\cdots<\mathcal{I}_i}}\int_{\mathbb{R}^i}\int_{\mathbb{R}}\partial^{\mathcal{I}}\partial_d f(z^{\{\mathcal{I},d\}})$$
$$\times\prod_{j\in\{\mathcal{I},d\}}\operatorname{sgn}(z_j)\nu^{\{\mathcal{I},d\}}\Big(\prod_{j\in\{\mathcal{I},d\}}I(z_j)\Big)\mathrm{d}z^{\mathcal{I}}\,\mathrm{d}z_d\,,$$

which is the claimed result. □

Using Lemma 14.2.7, we immediately obtain

Corollary 14.2.8 *Let $X=(X^1,\ldots,X^d)^\top$ be a d-dimensional square integrable Lévy process with characteristic triplet $(0,\nu,\gamma)$. Then,*

$$\operatorname{Cov}(X_t^i,X_t^j)=t\int_{\mathbb{R}^d}z_iz_j\,\nu(\mathrm{d}z)=t\int_{\mathbb{R}^2}F^{\{i,j\}}(U_i(z_i),U_j(z_j))\,\mathrm{d}z_i\,\mathrm{d}z_j,\quad\forall i\neq j\,,$$

where F is the Lévy copula from Theorem 14.2.6.

We conclude with examples of Lévy copulas.

Example 14.2.9 Examples of Lévy copulas are:

(i) Independence Lévy copula

$$F(u_1,\ldots,u_d)=\sum_{i=1}^{d}u_i\prod_{j\neq i}1_{\{\infty\}}(u_j)\,. \tag{14.4}$$

(ii) Complete dependence Lévy copula

$$F(u_1,\ldots,u_d)=\min\{|u_1|,\ldots,|u_d|\}1_K(u_1,\ldots,u_d)\prod_{j=1}^{d}\operatorname{sgn}u_j\,, \tag{14.5}$$

where $K:=\{x\in\mathbb{R}^d:\operatorname{sgn}(x_1)=\cdots=\operatorname{sgn}(x_d)\}$.

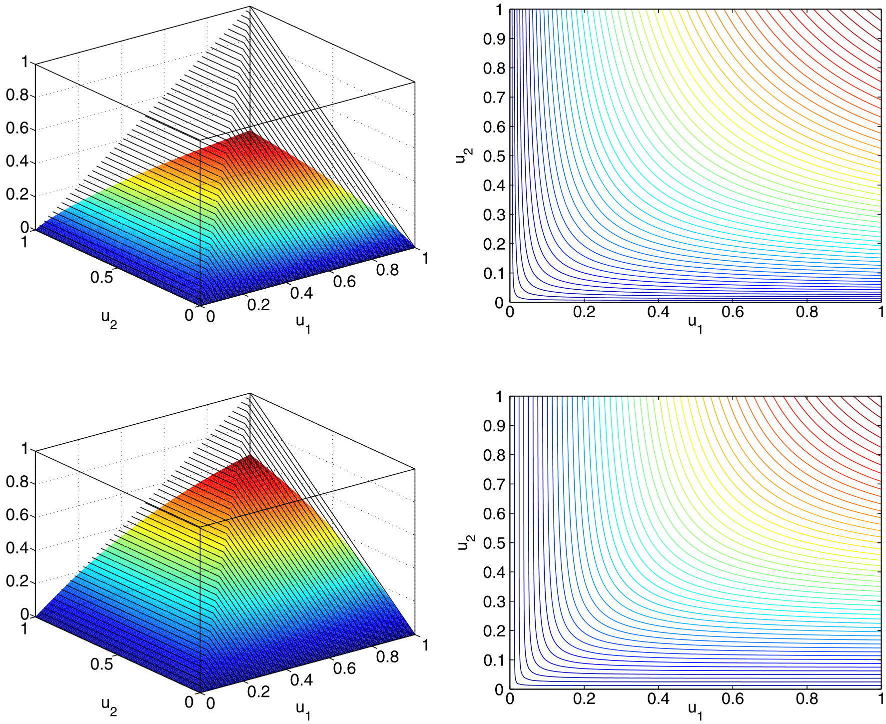


Fig. 14.1 Clayton copula (14.6) in $d = 2$ for $\vartheta = 0.5$ (*top*) and $\vartheta = 1.5$ (*bottom*)

(iii) Clayton Lévy copulas

$$F(u_1, \ldots, u_d) = 2^{2-d} \left(\sum_{i=1}^{d} |u_i|^{-\vartheta} \right)^{-\frac{1}{\vartheta}} \left(\eta 1_{\{u_1 \cdots u_d \geq 0\}} - (1-\eta) 1_{\{u_1 \cdots u_d \leq 0\}} \right), \tag{14.6}$$

where $\vartheta > 0$ and $\eta \in [0, 1]$. For $\eta = 1$ and $\vartheta \to 0$, F converges to the independence Lévy copula, for $\eta = 1$ and $\vartheta \to \infty$ to the complete dependence Lévy copula. In Fig. 14.1, the Clayton copula in $d = 2$ for $\vartheta = 0.5, 1.5$ and $\eta = 1$ is plotted. We include the upper bound $\min\{|u_1|, |u_2|\}$ and additionally give the corresponding contour plot.

An important class of Lévy copulas are the so-called 1-homogeneous copulas.

Definition 14.2.10 A Lévy copula is called 1-homogeneous if for any $r > 0$ the following holds:

$$F(ru_1, \ldots, ru_d) = rF(u_1, \ldots, u_d),$$

for all $(u_1, \ldots, u_d)^\top \in \mathbb{R}^d$.

14.3 Lévy Models

If $\{X_t : t \geq 0\}$ is a Brownian motion on $\mathbb{R}^d$ then, for any $r > 0$, the process $\{X_{rt} : t \geq 0\}$ is identical in law with the process $\{r^{1/2} X_t : t \geq 0\}$. This property is called *self-similarity* of a stochastic process with index 2. There are many self-similar Lévy processes other than the Brownian motion, the so-called stable processes.

Definition 14.3.1 Let $0 < \alpha < 2$. A Lévy process $X = \{X_t : t \geq 0\}$ with state space $\mathbb{R}^d$ is called *α-stable* if the distribution μ of X at $t = 1$ is α-stable, i.e. for any $r > 0$ there exists $c \in \mathbb{R}^d$ such that

$$\widehat{\mu}(z)^r = \widehat{\mu}(r^{\frac{1}{\alpha}} z) e^{i\langle c,z\rangle}.$$

It is shown in [143, Theorem 14.3] that any Lévy process with the characteristic triplet $(\mathcal{Q}, \nu, \gamma)$ has an α-stable probability measure if and only if $\mathcal{Q} = 0$ and if there is a finite measure λ on the unit sphere $S = \{x \in \mathbb{R}^d \;:\; |x| = 1\}$ such that

$$\nu(B) = \int_S \lambda(\mathrm{d}\xi) \int_0^\infty 1_B(r\xi) \frac{1}{r^{1+\alpha}} \mathrm{d}r, \quad B \in \mathcal{B}(\mathbb{R}^d).$$

A simple example of an α-stable Lévy process on $\mathbb{R}^d$ is given by the Lévy measure

$$\nu(\mathrm{d}z) = \sum_{j=1}^{2^d} c_j \, |z|^{-d-\alpha} \, 1_{Q_j} \mathrm{d}z, \tag{14.7}$$

where $c_j \geq 0$, $\sum_{j=1}^{2^d} c_j > 0$ and Q_j denoting the jth quadrant. Note that for $d = 1$ this is the only possible α-stable process. The corresponding marginal processes X^i, $i = 1, \ldots, d$ of X are again α-stable processes in $\mathbb{R}$ with Lévy measure $\nu_i(\mathrm{d}z) = \widetilde{c}_i \, |z|^{-1-\alpha} \, \mathrm{d}z$ where $\widetilde{c}_i$ depend on α, d and c_j, $j = 1, \ldots, 2^d$.

For $d > 1$ the notation of stable processes can be extended by using non-singular matrices for scaling.

Definition 14.3.2 Let $\mathbf{Q} \in \mathbb{R}^{d\times d}$ be a matrix with positive eigenvalues. A Lévy process $X = \{X_t : t \geq 0\}$ with state space $\mathbb{R}^d$ is called **Q**-*stable* if for any $r > 0$ there exist a $c \in \mathbb{R}^d$ such that the distribution μ of X at $t = 1$ satisfies

$$\widehat{\mu}(z)^r = \widehat{\mu}(r^{\mathbf{Q}^\top} z) e^{i\langle c,z\rangle},$$

where $r^{\mathbf{Q}} = \sum_{n=0}^{\infty} (n!)^{-1} (\log r)^n \mathbf{Q}^n$.

For $\mathbf{Q} = \operatorname{diag}((1/\alpha, \ldots, 1/\alpha))$, $0 < \alpha < 2$, we again obtain α-stable processes. An extension of (isotropic) α-stable processes are anisotropic $\boldsymbol{\alpha}$-stable processes for an $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d)$ with $0 < \alpha_i < 2$, $i = 1, \ldots, d$.

Definition 14.3.3 Let $0 < \alpha_i < 2$, $i = 1, \ldots, d$ and $\mathbf{Q} = \operatorname{diag}\{\alpha_i^{-1} : i = 1, \ldots, d\}$. A Lévy process $X = \{X_t : t \geq 0\}$ with state space $\mathbb{R}^d$ is called $\boldsymbol{\alpha}$-stable if the distribution μ of X at $t = 1$ is $\mathbf{Q}$-stable, i.e. for any $r > 0$ there exists a $c \in \mathbb{R}^d$ such that

$$\widehat{\mu}(z)^r = \widehat{\mu}(r^{\frac{1}{\alpha_1}} z_1, \dots, r^{\frac{1}{\alpha_d}} z_d) e^{i\langle c, z\rangle}. \tag{14.8}$$

Since we have for the characteristic function $\widehat{\mu}(z) = e^{-\psi(z)}$, it follows from (14.8) that the characteristic exponent of an $\boldsymbol{\alpha}$-stable process satisfies for any $r > 0$

$$\Re\psi(r^{\frac{1}{\alpha_1}}\xi_1, \dots, r^{\frac{1}{\alpha_d}}\xi_d) = r\Re\psi(\xi), \quad \forall \xi \in \mathbb{R}^d. \tag{14.9}$$

We assume that the Lévy measure ν has a Lévy density k, i.e. $\nu(\mathrm{d}z) = k(z)\mathrm{d}z$ and obtain for $\mathcal{Q} = 0$ that

$$\begin{aligned}
&\Re\psi(r^{\frac{1}{\alpha_1}}\xi_1, \dots, r^{\frac{1}{\alpha_d}}\xi_d) \\
&= \int_{\mathbb{R}^d} \left(1 - \cos\left(\sum_{i=1}^d r^{\frac{1}{\alpha_i}}\xi_i z_i\right)\right) k^{\mathrm{sym}}(z)\,\mathrm{d}z \\
&= \int_{\mathbb{R}^d} (1 - \cos\langle \xi, z\rangle)\, k^{\mathrm{sym}}(r^{-\frac{1}{\alpha_1}} z_1, \dots, r^{-\frac{1}{\alpha_d}} z_d) r^{-\frac{1}{\alpha_1} - \dots - \frac{1}{\alpha_d}}\,\mathrm{d}z.
\end{aligned}$$

Now using (14.9), the Lévy density has to satisfy

$$k^{\mathrm{sym}}(r^{-\frac{1}{\alpha_1}} z_1, \dots, r^{-\frac{1}{\alpha_d}} z_d) = r^{1+\frac{1}{\alpha_1}+\dots+\frac{1}{\alpha_d}} k^{\mathrm{sym}}(z_1, \dots, z_d)\,. \tag{14.10}$$

A simple example of an $\boldsymbol{\alpha}$-stable Lévy process on $\mathbb{R}^d$ is given by the Lévy measure

$$\nu(\mathrm{d}z) = \sum_{j=1}^{2^d} c_j \left(\sum_{i=1}^d |z_i|^{\alpha_i}\right)^{-1-\frac{1}{\alpha_1}-\dots-\frac{1}{\alpha_d}} 1_{Q_j}\,\mathrm{d}z, \tag{14.11}$$

where $c_j \geq 0$, $\sum_{j=1}^{2^d} c_j > 0$. The corresponding marginal processes X^i, $i = 1, \dots, d$ of X are again α-stable processes in $\mathbb{R}$ with Lévy measure $\nu_i(\mathrm{d}z) = \widetilde{c}_i\, |z|^{-1-\alpha_i}\,\mathrm{d}z$ where $\widetilde{c}_i$ depend on d, $\boldsymbol{\alpha}$ and c_j, $j = 1, \dots, 2^d$. We plot the density (14.11) for $d = 2$, $\boldsymbol{\alpha} = (0.5, 1.2)$ and $c_j = 1$, $j = 1, \dots, 4$ in Fig. 14.2.

14.3.1 Subordinated Brownian Motion

As in the one-dimensional case, we can obtain Lévy processes by *subordination*. For $d > 1$ there are two possibilities. Using a one-dimensional increasing process or subordinator $G = \{G_t : t \geq 0\}$, the resulting process is given by

$$X_t^i = W_{G_t}^i + \theta_i G_t, \quad \theta_i \in \mathbb{R}, \quad t \in [0, T],$$

for $i = 1, \dots, d$ where $W = (W^1, \dots, W^d)^\top$ is a vector of d Brownian motions with covariance matrix $\mathcal{Q} = (\sigma_i\sigma_j\rho_{ij})_{1\leq i,j\leq d}$. Here, σ_i^2, $i = 1, \dots, d$, is the variance of

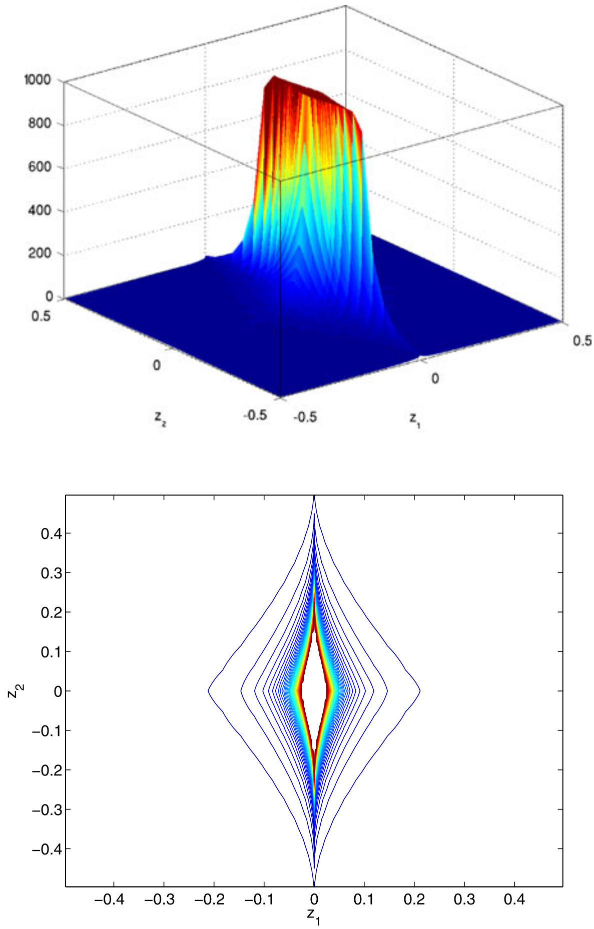


Fig. 14.2 Anisotropic $\boldsymbol{\alpha}$-stable density in $d = 2$ for $\boldsymbol{\alpha} = (0.5, 1.2)$ and corresponding contour plot

the one-dimensional Brownian motions W^i, and ρ_{ij} the correlation of the Brownian motions W^i and W^j. But we can also use a d-dimensional Lévy process $G = (G^1, \dots, G^d)^\top$ which is componentwise increasing in each coordinate to obtain

$$X_t^i = W_{G_t^i}^i + \theta_i G_t^i, \quad \theta_i \in \mathbb{R}, \quad t \in [0, T],$$

for $i = 1, \dots, d$. This is called *multivariate subordination*.

As an example we use as the one-dimensional subordinator a gamma process to obtain a multidimensional variance gamma process [117]. As in the one-dimensional case (see Sect. 10.2.2), we consider a gamma process G with Lévy density $k_G(s) = e^{-\frac{s}{\vartheta}}(\vartheta s)^{-1} 1_{\{s>0\}}$. Then, the Lévy measure of X is given for $B \in \mathcal{B}(\mathbb{R}^d)$ by

$$
\begin{aligned}
\nu(B) &= \int_B \int_0^\infty (2\pi)^{\frac{-d}{2}} \det \mathcal{Q}^{-\frac{1}{2}} s^{-\frac{d}{2}} e^{-\langle z-\theta s, \mathcal{Q}^{-1}(z-\theta s)\rangle/(2s)} e^{-\frac{s}{\vartheta}} (\vartheta s)^{-1} \,\mathrm{d}s\,\mathrm{d}z \\
&= \int_B (2\pi)^{\frac{-d}{2}} \vartheta^{-1} \det \mathcal{Q}^{-\frac{1}{2}} e^{\langle \theta, \frac{\mathcal{Q}^{-1}+\mathcal{Q}^{-\top}}{2} z\rangle} \\
&\quad \times \int_0^\infty s^{-\frac{d}{2}-1} e^{-\frac{\langle z, \mathcal{Q}^{-1} z\rangle}{2s} - (\frac{\langle \theta, \mathcal{Q}^{-1}\theta\rangle}{2} + \frac{1}{\vartheta}) s} \mathrm{d}s\,\mathrm{d}z \\
&= \int_B (2\pi)^{\frac{-d}{2}} \vartheta^{-1} \det \mathcal{Q}^{-\frac{1}{2}} e^{\langle \theta, \frac{\mathcal{Q}^{-1}+\mathcal{Q}^{-\top}}{2} z\rangle} \int_0^\infty s^{-\frac{d}{2}-1} e^{-\beta \frac{1}{s} - \gamma s} \mathrm{d}s\,\mathrm{d}z\,,
\end{aligned}
$$

where $\beta = \langle z, \mathcal{Q}^{-1} z\rangle/2$ and $\gamma = \langle \theta, \mathcal{Q}^{-1}\theta\rangle/2 + 1/\vartheta$. Integrating the second integral, we obtain the Lévy measure

$$
\nu(\mathrm{d}z) = 2\,(2\pi)^{\frac{-d}{2}} \vartheta^{-1} \det \mathcal{Q}^{-\frac{1}{2}} e^{\langle \theta, \frac{\mathcal{Q}^{-1}+\mathcal{Q}^{-\top}}{2} z\rangle} \left(\frac{\beta}{\gamma}\right)^{-d/4} K_{-d/2}\left(2\sqrt{\beta\gamma}\right) \mathrm{d}z, \tag{14.12}
$$

where $K_{-d/2}(\xi)$ is the modified Bessel function of the second kind. For small ξ, we have $K_{-d/2}(\xi) \sim \xi^{-d/2}$, and therefore $\nu(\mathrm{d}z) \sim \langle z, \mathcal{Q}^{-1} z\rangle^{-d/2}\,\mathrm{d}z \sim |z|^{-d}\,\mathrm{d}z$ since $\mathcal{Q} > 0$. The marginal processes X^i, $i = 1, \ldots, d$ of X are variance gamma processes on $\mathbb{R}$ with Lévy measure $\nu_i(\mathrm{d}z) = \vartheta^{-1} e^{\theta_i/\sigma_i^2 z} e^{-\sqrt{2/\vartheta + \theta_i^2/\sigma_i^2}/\sigma_i |z|} |z|^{-1}\,\mathrm{d}z$. We plot the density (10.9) for $d = 2$, $\theta = (-0.1, -0.2)$, $\sigma = (0.3, 0.4)$, $\rho_{12} = 0.5$ and $\vartheta = 1$ in Fig. 14.3.

14.3.2 Lévy Copula Models

Lévy copulas F allow parametric constructions of multivariate jump densities from univariate ones. Let $U_1, \ldots, U_d$ be one-dimensional tail integrals with Lévy density $k_1, \ldots, k_d$, and let F be a Lévy copula such that $\partial_1 \cdots \partial_d F$ exists in the sense of distributions. Then,

$$
k(x_1, \ldots, x_d) = \partial_1 \cdots \partial_d F|_{\xi_1 = U_1(x_1), \ldots, \xi_d = U_d(x_d)} k_1(x_1) \cdots k_d(x_d) \tag{14.13}
$$

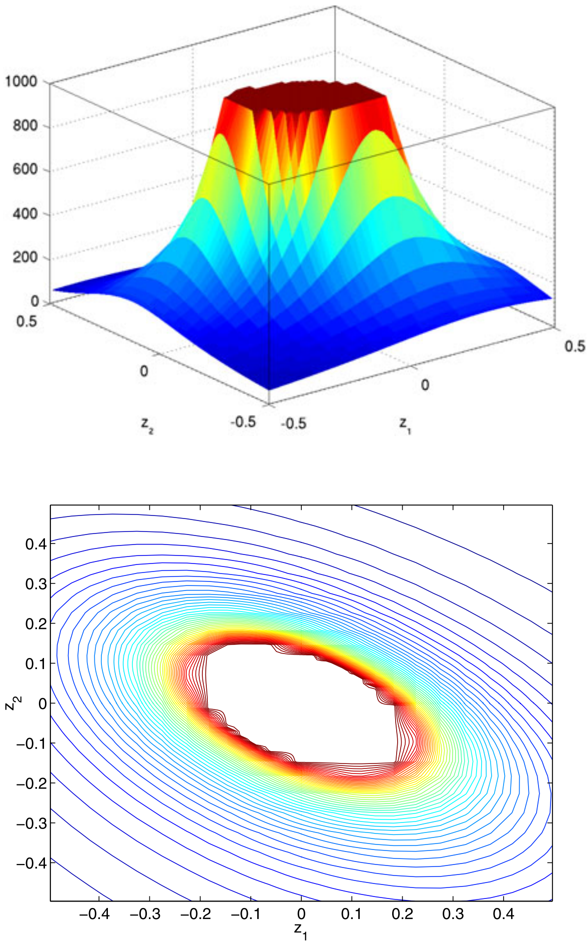
is the jump density of a d-variate Lévy measure with marginal Lévy densities $k_1, \ldots, k_d$. For example, we can use the Clayton Lévy copula (see Definition 14.6)

$$
F(u_1, \ldots, u_d) = 2^{2-d} \left(\sum_{i=1}^d |u_i|^{-\vartheta}\right)^{-\frac{1}{\vartheta}} \left(\eta 1_{\{u_1 \cdots u_d \geq 0\}} - (1-\eta) 1_{\{u_1 \cdots u_d \leq 0\}}\right),
$$

where $\vartheta > 0$, $\eta \in [0, 1]$ and consider α-stable marginal Lévy densities, $k_i(z) = |z|^{-1-\alpha_i}$, $0 < \alpha_i < 2$, $i = 1, \ldots, d$. This leads to the d-dimensional Lévy density

$$
\begin{aligned}
k(z) = 2^{2-d} \prod_{i=1}^d &\left(1 + (i-1)\vartheta\right) \alpha_i^{\vartheta+1} |z_i|^{\alpha_i \vartheta - 1} \left(\sum_{i=1}^d \alpha_i^\vartheta |z_i|^{\alpha_i \vartheta}\right)^{-\frac{1}{\vartheta} - d} \\
&\cdot \left(\eta 1_{\{z_1 \cdots z_d \geq 0\}} + (1-\eta) 1_{\{z_1 \cdots z_d \leq 0\}}\right).
\end{aligned} \tag{14.14}
$$

Fig. 14.3 Variance gamma density in $d = 2$ and corresponding contour plot



Note that this is again an anisotropic $\boldsymbol{\alpha}$-stable process. We plot the density (14.14) for $d = 2$, $\vartheta = 0.5$, $\eta = 0.5$ and $\boldsymbol{\alpha} = (0.5, 1.2)$ in Fig. 14.4.

14.3.3 Admissible Models

We make the following assumptions on our models.

Assumption 14.3.4 Let X be a d-dimensional Lévy process with characteristic triplet $(\mathcal{Q}, \nu, \gamma)$, Lévy density k and marginal Lévy densities k_i, $i = 1, \ldots, d$.

(i) There are constants $\beta_i^- > 0$, $\beta_i^+ > 1$, $i = 1, \ldots, d$ and $C > 0$ such that

$$k_i(z) \le C \begin{cases} e^{-\beta_i^-|z|}, & z < -1, \\ e^{-\beta_i^+ z}, & z > 1. \end{cases} \tag{14.15}$$

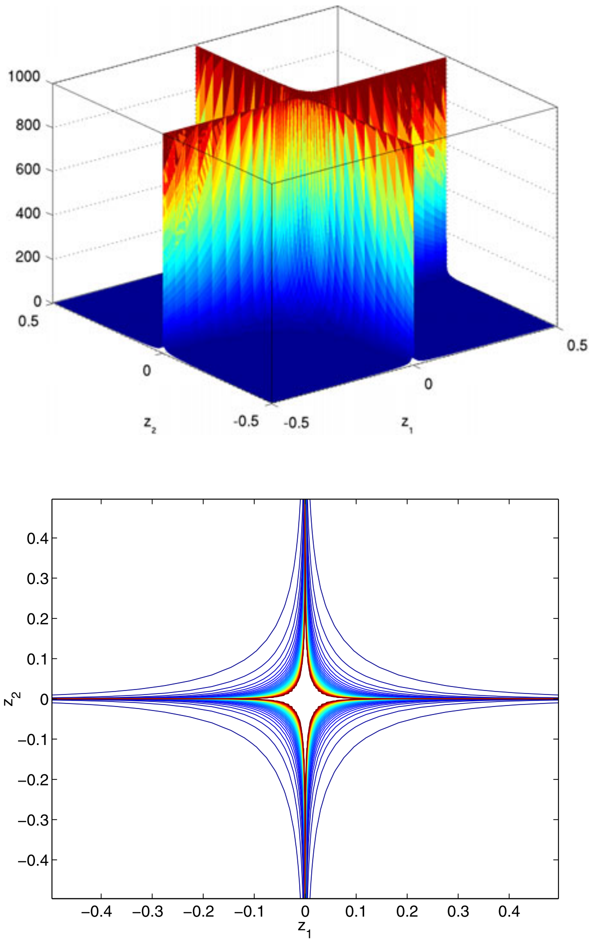


Fig. 14.4 Anisotropic $\boldsymbol{\alpha}$-stable Lévy copula density (14.14) in $d = 2$ for $\boldsymbol{\alpha} = (0.5, 1.2)$ and corresponding contour plot

(ii) Furthermore, we assume there exist an $\boldsymbol{\alpha}$-stable process X^0 with Lévy density k^0 and $C_+ > 0$ such that

$$k(z) \leq C_+ k^0(z), \quad 0 < |z| < 1. \tag{14.16}$$

(iii) If $\mathcal{Q}$ is not positive definite, we assume additionally that there is a $C_- > 0$ such that

$$k^{\text{sym}}(z) \geq C_- k^{0,\text{sym}}(z), \quad 0 < |z| < 1. \tag{14.17}$$

For $d = 1$, the assumptions (14.15)–(14.17) coincide with the assumptions (10.11)–(10.13). In the case that the marginal processes X^i, $i = 1, \ldots, d$ are independent, it is equivalent that the corresponding marginal one-dimensional densities k_i, $i = 1, \ldots, d$ satisfy (10.11)–(10.13).

14.4 Pricing Equation

As before we assume the risk-neutral dynamics of the underlying asset price is given by

$$S_t^i = S_0^i e^{rt+X_t^i}, \quad i = 1, \dots, d,$$

where X is a d-dimensional Lévy process with characteristic triplet $(\mathcal{Q}, \nu, \gamma)$ under a non-unique EMM. As shown in Lemma 14.1.3, the martingale condition implies

$$\gamma_j = -\frac{\mathcal{Q}_{jj}}{2} - \int_{\mathbb{R}} \left(e^{z_j} - 1 - z_j\right) \nu_j(\mathrm{d}z), \quad j = 1, \dots, d.$$

We again show that the value of the option in log-price $v(t, x)$ is a solution of a multidimensional PIDE.

Proposition 14.4.1 *Let X be a Lévy process with state space $\mathbb{R}^d$ and characteristic triplet $(\mathcal{Q}, \nu, \gamma)$ where the Lévy measure satisfies (14.15). Denote by $\mathcal{A}$ the integro-differential operator*

$$\begin{aligned}(\mathcal{A}f)(x) &= \frac{1}{2}\mathrm{tr}[\mathcal{Q}D^2 f(x)] + \gamma^\top \nabla f(x) \\ &\quad + \int_{\mathbb{R}^d} \left(f(x+z) - f(x) - z^\top \nabla_x f(x)\right) \nu(\mathrm{d}z), \end{aligned} \tag{14.18}$$

for functions $f \in C^2(\mathbb{R}^d)$ with bounded derivatives. Then, the process $M_t := f(X_t) - \int_0^t (\mathcal{A}f)(X_s)\,\mathrm{d}s$ is a martingale with respect to the filtration of X.

Proof Let $\boldsymbol{\Sigma} = (\boldsymbol{\Sigma}_{ij})_{1 \le i,j \le d}$ be given such that $\boldsymbol{\Sigma}\boldsymbol{\Sigma}^\top = \mathcal{Q}$. Proceeding as in the proof of Proposition 10.3.1, we obtain using the Itô formula for multidimensional Lévy processes and the Lévy–Itô decomposition

$$\begin{aligned}\mathrm{d}f(X_t) &= \sum_{i=1}^d \partial_{x_i} f(X_{t-})\,\mathrm{d}X_t^i + \frac{1}{2}\sum_{i,j=1}^d \mathcal{Q}_{ij}\partial_{x_i x_j} f(X_t)\,\mathrm{d}t \\ &\quad + f(X_t) - f(X_{t-}) - \sum_{i=1}^d \Delta X_t^i \partial_{x_i} f(X_{t-}) \\ &= \gamma^\top \nabla f(X_{t-})\,\mathrm{d}t + \sum_{i=1}^d \partial_{x_i} f(X_{t-}) \sum_{j=1}^d \boldsymbol{\Sigma}_{ij}\,\mathrm{d}W_t^j \\ &\quad + \sum_{i=1}^d \partial_{x_i} f(X_{t-}) \int_{\mathbb{R}^d \setminus \{0\}} z_i \widetilde{J}_X(\mathrm{d}t, \mathrm{d}z) \\ &\quad + \frac{1}{2}\mathrm{tr}[\mathcal{Q}D^2 f(X_t)]\,\mathrm{d}t + \int_{\mathbb{R}^d \setminus \{0\}} \Bigg(f(X_{t-} + z) - f(X_{t-}) \\ &\quad - \sum_{i=1}^d z_i \partial_{x_i} f(X_{t-})\Bigg) J_X(\mathrm{d}t, \mathrm{d}z)\end{aligned}$$

$$= (\mathcal{A}f)(X_t)\,\mathrm{d}t + \sum_{i=1}^{d} \partial_{x_i} f(X_{t-}) \sum_{j=1}^{d} \boldsymbol{\Sigma}_{ij}\,\mathrm{d}W_t^j$$
$$+ \int_{\mathbb{R}^d} (f(X_{t-} + z) - f(X_{t-}))\widetilde{J}_X(\mathrm{d}t, \mathrm{d}z).$$

As already shown in Proposition 8.1.2, $\sum_{i=1}^{d} \partial_{x_i} f(X_{t-}) \sum_{j=1}^{d} \boldsymbol{\Sigma}_{ij}\mathrm{d}W_t^j$ is a martingale. Since $f \in C^2(\mathbb{R}^d)$ and the Lévy measure ν satisfies (14.15), we have similar to Proposition 10.3.1 that also $\int_{\mathbb{R}^d}(f(X_{t-} + z) - f(X_{t-}))\widetilde{J}_X(\mathrm{d}t, \mathrm{d}z)$ is a martingale. □

Repeating the arguments which lead to Theorem 4.1.4 yields

Theorem 14.4.2 *Let $v \in C^{1,2}(J \times \mathbb{R}^d) \cap C^0(\overline{J} \times \mathbb{R}^d)$ with bounded derivatives in x be a solution of*

$$\partial_t v + \mathcal{A}v - rv = 0 \quad \text{in } J \times \mathbb{R}^d, \quad v(T, x) = g(e^{x_1}, \dots, e^{x_d}) \quad \text{in } \mathbb{R}^d, \tag{14.19}$$

with $\mathcal{A}$ as in (14.18) with drift $r + \gamma$. Then, $v(t, x)$ can also be represented as

$$v(t, x) = \mathbb{E}\left[e^{-r(T-t)} g(e^{rT + X_T}) \mid X_t = x\right].$$

We again change to *time-to-maturity* $t \to T - t$ and remove the drift γ and the interest rate r by setting $u(t, s) =: e^{rt} v(T - t, x_1 - (\gamma_1 + r)t, \dots, x_d - (\gamma_d + r)t)$.

14.5 Variational Formulation

For the variational formulation, we need anisotropic Sobolev spaces of fractional order as defined in (13.4). Let $\mathcal{Q} = (\sigma_i \sigma_j \rho_{ij})_{1 \le i,j \le d}$, where ρ_{ij} is the correlation of the Brownian motion W^i and W^j. We set $\boldsymbol{\rho} = (\rho_1, \dots, \rho_d)$ with

$$\rho_i = \begin{cases} 1 & \text{if } \sigma_i > 0, \\ \alpha_i/2 & \text{if } \sigma_i = 0, \end{cases}$$

with $\boldsymbol{\alpha}$ given in Assumption 14.3.4. The variational formulation of the PIDE reads

$$\begin{aligned} &\text{Find } u \in L^2(J; H^{\boldsymbol{\rho}}(\mathbb{R}^d)) \cap H^1(J; L^2(\mathbb{R}^d)) \text{ such that} \\ &(\partial_t u, v) + a^{\mathrm{J}}(u, v) = 0, \ \forall v \in H^{\boldsymbol{\rho}}(\mathbb{R}^d), \quad \text{a.e. in } J, \\ &u(0) = u_0, \end{aligned} \tag{14.20}$$

where $u_0(x) := g(e^{x_1}, \dots, e^{x_d})$ and the bilinear form $a^{\mathrm{J}}(\cdot,\cdot) : H^{\boldsymbol{\rho}}(\mathbb{R}^d) \times H^{\boldsymbol{\rho}}(\mathbb{R}^d) \to \mathbb{R}$ is given by

$$a^{\mathrm{J}}(\varphi, \phi) := \frac{1}{2}\int_{\mathbb{R}^d} (\nabla\varphi)^\top \mathcal{Q} \nabla\phi\,\mathrm{d}x$$
$$- \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left(\varphi(x + z) - \varphi(x) - \sum_{i=1}^{d} z_i \varphi_{x_i}(x)\right)\phi(x)\nu(\mathrm{d}z)\,\mathrm{d}x.$$

For a multidimensional Lévy process satisfying the Assumptions 14.3.4, we again have a sector condition as shown in [134].

Lemma 14.5.1 *Let X be a Lévy process with characteristic triplet $(\mathcal{Q}, \nu, 0)$ where the Lévy measure satisfies (14.16), (14.17). Then, there exist constants $C_i > 0$, $i = 1, 2, 3$,*

$$\Re\psi(\xi) \geq C_1 \sum_{i=1}^{d} |\xi|^{2\rho_i}, \quad |\psi(\xi)| \leq C_2 \sum_{i=1}^{d} |\xi|^{2\rho_i} + C_3.$$

Using this, we can show as in Theorem 10.4.3

Theorem 14.5.2 *Let X be a Lévy process with characteristic triplet $(\mathcal{Q}, \nu, 0)$ where the Lévy measure satisfies (14.16), (14.17). Then, there exist constants $C_i > 0$, $i = 1, 2, 3$, such that for all $\varphi, \phi \in H^{\boldsymbol{\rho}}(\mathbb{R}^d)$ the following holds:*

$$|a^{\mathrm{J}}(\varphi, \phi)| \leq C_1 \|\varphi\|_{H^{\boldsymbol{\rho}}(\mathbb{R}^d)} \|\phi\|_{H^{\boldsymbol{\rho}}(\mathbb{R}^d)}, \quad a^{\mathrm{J}}(\varphi, \varphi) \geq C_2 \|\varphi\|^2_{H^{\boldsymbol{\rho}}(\mathbb{R}^d)} - C_3 \|\varphi\|^2_{L^2(\mathbb{R}^d)}.$$

In particular, for every $u_0 \in L^2(\mathbb{R}^d)$ there exists a unique solution to the problem (14.20).

For multidimensional payoffs g which satisfy the growth condition (8.10), we can localized to a bounded domain $G = (-R, R)^d \subset \mathbb{R}^d$, $R > 0$ which again corresponds to approximating the option price by a knock-out barrier option. Similar to Theorem 8.3.1 and Theorem 10.5.1, we obtain that the barrier option price converges to the option price exponentially fast in R if the Lévy measure satisfies (14.15). Therefore, we have the localized problem

$$\begin{aligned} &\text{Find } u_R \in L^2(J; \widetilde{H}^{\boldsymbol{\rho}}(G)) \cap H^1(J; L^2(G)) \text{ such that} \\ &(\partial_t u_R, v) + a^{\mathrm{J}}(u_R, v) = 0, \ \forall v \in \widetilde{H}^{\boldsymbol{\rho}}(G), \text{ a.e. in } J, \\ &u_R(0) = u_0|_G, \end{aligned} \tag{14.21}$$

which as a unique solution for every payoff g satisfying the growth condition (8.10).

14.6 Wavelet Discretization

Since the diffusion part has been already discussed in Sect. 8.4, we set $\mathcal{Q} = 0$ and only consider pure jump models. The main problem is as in the one-dimensional case the singularity of the Lévy measure at the origin $z = 0$ and additionally on the axis $z_i = 0$, $i = 1, \ldots, d$. Therefore, we integrate by parts twice the integro-differential expression for the jump generator as in Lemma 14.2.7 to obtain for $\phi, \varphi \in \widetilde{H}^1(G)$

$$a^{\mathrm{J}}(\varphi,\phi) = \sum_{i=1}^{d} \int_{\mathbb{R}} \int_{G} \partial_i \varphi(x+z_i) \partial_i \phi(x) k_i^{-2}(z_i)\, \mathrm{d}x\, \mathrm{d}z_i - \sum_{i=2}^{d} \sum_{\substack{|\mathcal{I}|=i \\ \mathcal{I}_1<\cdots<\mathcal{I}_i}} \int_{\mathbb{R}^i} \int_{G} \partial^{\mathcal{I}} \varphi(x+z^{\mathcal{I}}) \phi(x) U^{\mathcal{I}}(z^{\mathcal{I}})\, \mathrm{d}x\, \mathrm{d}z^{\mathcal{I}}. \tag{14.22}$$

Using the spline-wavelet basis $\psi_{\boldsymbol{\ell},\mathbf{k}} = \psi_{\ell_1,k_1} \cdots \psi_{\ell_d,k_d}$, $0 \le \ell_1 + \cdots + \ell_d \le L$, $k_i \in \nabla_{\ell_i}$ of $\widehat{V}_L$ (see Sect. 13.1) of the Finite Element space, we need to compute the stiffness matrix

$$\mathbf{A}^{\mathrm{J}}_{(\boldsymbol{\ell}',\mathbf{k}'),(\boldsymbol{\ell},\mathbf{k})} = \sum_{i=1}^{d} \int_{\mathbb{R}} \int_{G} \partial_i \psi_{\boldsymbol{\ell},\mathbf{k}}(x+z_i) \partial_i \psi_{\boldsymbol{\ell}',\mathbf{k}'}(x) k_i^{-2}(z_i)\, \mathrm{d}x\, \mathrm{d}z_i - \sum_{i=2}^{d} \sum_{\substack{|\mathcal{I}|=i \\ \mathcal{I}_1<\cdots<\mathcal{I}_i}} \int_{\mathbb{R}^i} \int_{G} \partial^{\mathcal{I}} \psi_{\boldsymbol{\ell},\mathbf{k}}(x+z^{\mathcal{I}}) \psi_{\boldsymbol{\ell}',\mathbf{k}'}(x) U^{\mathcal{I}}(z^{\mathcal{I}})\, \mathrm{d}x\, \mathrm{d}z^{\mathcal{I}}.$$

We define the one-dimensional mass matrix $\mathbf{M}^i$ as in (13.12) and additionally

$$\mathbf{A}^{i}_{(\ell',k'),(\ell,k)} := \int_{-R}^{R} \int_{-R}^{R} \psi'_{\ell,k}(y) \psi'_{\ell',k'}(x) k_i^{-2}(y-x)\, \mathrm{d}y\, \mathrm{d}x\,,$$

$$\mathbf{A}^{\mathcal{I}}_{(\boldsymbol{\ell}'_{\mathcal{I}},\mathbf{k}'_{\mathcal{I}}),(\boldsymbol{\ell}_{\mathcal{I}},\mathbf{k}_{\mathcal{I}})} := -\int_{[-R,R]^{|\mathcal{I}|}} \int_{[-R,R]^{|\mathcal{I}|}} \partial^{\mathcal{I}} \psi_{\boldsymbol{\ell}_{\mathcal{I}},\mathbf{k}_{\mathcal{I}}}(y) \psi_{\boldsymbol{\ell}'_{\mathcal{I}},\mathbf{k}'_{\mathcal{I}}}(x) U^{\mathcal{I}}(y-x)\, \mathrm{d}y\, \mathrm{d}x\,,$$

where $\boldsymbol{\ell}_{\mathcal{I}} = (\ell_i)_{i\in\mathcal{I}}$, $0 \le \ell_i \le L$, $\mathbf{k}_{\mathcal{I}} = (k_i)_{i\in\mathcal{I}}$, $k_i \in \nabla_{\ell_i}$, $\mathcal{I} \subset \{1,\ldots,d\}$, $|\mathcal{I}| > 1$. Then, we can write the jump stiffness matrix as

$$\mathbf{A}^{\mathrm{J}}_{(\boldsymbol{\ell}',\mathbf{k}'),(\boldsymbol{\ell},\mathbf{k})} = \sum_{i=1}^{d} \sum_{\substack{|\mathcal{I}|=i \\ \mathcal{I}_1<\cdots<\mathcal{I}_i}} \mathbf{A}^{\mathcal{I}}_{(\boldsymbol{\ell}'_{\mathcal{I}},\mathbf{k}'_{\mathcal{I}}),(\boldsymbol{\ell}_{\mathcal{I}},\mathbf{k}_{\mathcal{I}})} \prod_{j\in\mathcal{I}^c} \mathbf{M}^{j}_{(\ell'_j,k'_j),(\ell_j,k_j)}.$$

As in the diffusion case, we can then compute the jump stiffness matrix $\mathbf{A}^{\mathrm{J}}$ as a sparse tensor product using the matrices $\mathbf{A}^{\mathcal{I}}$ and $\mathbf{M}^j$.

Since the Black–Scholes operator is a local operator, there are only $\mathcal{O}(2^L L^{d-1})$ non-zero entries in $\mathbf{A}^{\mathrm{BS}}$. But as already discussed in the one-dimensional case, the stiffness matrix for the jump part is densely populated. Again using wavelet compression, we can reduce the number of non-zero entries in $\mathbf{A}^{\mathrm{J}}$ to $\mathcal{O}(2^L L^{2(d-1)})$. We also need a smoothness assumption on the Lévy density k

$$|\partial^{\mathbf{n}} k(z)| \le C_0\, C^{|\mathbf{n}|}\, |\mathbf{n}|! \|z\|_\infty^{-\alpha} \prod_{i=1}^{d} |z_i|^{-n_i-1}, \quad \forall z_i \ne 0, \tag{14.23}$$

for $C_0, C > 0$, $\alpha = \|\boldsymbol{\alpha}\|_\infty$ and a multi-index $\mathbf{n} = (n_1,\ldots,n_d) \in \mathbb{N}_0^d$.

14.6.1 Wavelet Compression

To define the compression scheme, we need to introduce some notation. Consider tensor product wavelets $\psi_{\boldsymbol{\ell},\mathbf{k}} = \psi_{\ell_1,k_1} \otimes \cdots \otimes \psi_{\ell_d,k_d}$, $\psi_{\boldsymbol{\ell}',\mathbf{k}'} = \psi_{\ell_1',k_1'} \otimes \cdots \otimes \psi_{\ell_d',k_d'}$. The distance of support in each coordinate direction is denoted by

$$\delta_{x_i} := \operatorname{dist}\{\operatorname{supp}\psi_{\ell_i,k_i}, \operatorname{supp}\psi_{\ell_i',k_i'}\},$$

for $i = 1, \ldots, d$, and the distance of singular support

$$\delta_{x_i}^{\text{sing}} := \begin{cases} \operatorname{dist}\{\operatorname{singsupp}\psi_{\ell_i,k_i}, \operatorname{supp}\psi_{\ell_i',k_i'}\} & \text{if } \ell_i \le \ell_i', \\ \operatorname{dist}\{\operatorname{supp}\psi_{\ell_i,k_i}, \operatorname{singsupp}\psi_{\ell_i',k_i'}\} & \text{else.} \end{cases}$$

Define

$$\begin{aligned} \widetilde{L}_{\boldsymbol{\ell},\boldsymbol{\ell}'} := &\begin{cases} L(p-\alpha/2) - p\,|\boldsymbol{\ell}| & \text{if } p(L-|\boldsymbol{\ell}|) \ge \alpha/2(L-|\boldsymbol{\ell}|_\infty), \\ -\alpha/2\,|\boldsymbol{\ell}|_\infty & \text{else} \end{cases} \\ &+ \begin{cases} L(p-\alpha/2) - p\left|\boldsymbol{\ell}'\right| & \text{if } p(L-\left|\boldsymbol{\ell}'\right|) \ge \alpha/2(L-\left|\boldsymbol{\ell}'\right|_\infty), \\ -\alpha/2\left|\boldsymbol{\ell}'\right|_\infty & \text{else} \end{cases} \end{aligned} \tag{14.24}$$

and $m_i := \ell_i + \ell_i' - 2\min\{\ell_i, \ell_i'\}$. Furthermore, we denote the index sets $\mathcal{I}^c_{\boldsymbol{\ell},\boldsymbol{\ell}'}, \mathcal{I}_{\boldsymbol{\ell},\boldsymbol{\ell}'} \subset \{1, \ldots, d\}$ by

$$\mathcal{I}^c_{\boldsymbol{\ell},\boldsymbol{\ell}'} = \left\{ i \in \{1, \ldots, d\} \;:\; \delta_{x_i} > 2^{-\min\{\ell_i,\ell_i'\}} \right\}, \quad \mathcal{I}_{\boldsymbol{\ell},\boldsymbol{\ell}'} = \{1, \ldots, d\} \backslash \mathcal{I}^c_{\boldsymbol{\ell},\boldsymbol{\ell}'}, \tag{14.25}$$

and set

$$\begin{aligned} \beta^i_{\boldsymbol{\ell},\boldsymbol{\ell}} &= \widetilde{L}_{\boldsymbol{\ell},\boldsymbol{\ell}'} - \widetilde{p}(\ell_i + \ell_i') + \alpha \sum_{j \ne i} \min\{\ell_j, \ell_j'\} + \frac{1}{2} \sum_{j \in \mathcal{I}_{\boldsymbol{\ell},\boldsymbol{\ell}'}} m_j - \widetilde{p} \sum_{j \in \mathcal{I}^c_{\boldsymbol{\ell},\boldsymbol{\ell}'} \backslash \{i\}} m_j, \\ \widetilde{\beta}^i_{\boldsymbol{\ell},\boldsymbol{\ell}} &= \widetilde{L}_{\boldsymbol{\ell},\boldsymbol{\ell}'} - \widetilde{p}\max\{\ell_i, \ell_i'\} + \alpha \sum_{j \ne i} \min\{\ell_j, \ell_j'\} + \frac{1}{2} \sum_{j \in \mathcal{I}_{\boldsymbol{\ell},\boldsymbol{\ell}'} \backslash \{i\}} m_j - \widetilde{p} \sum_{j \in \mathcal{I}^c_{\boldsymbol{\ell},\boldsymbol{\ell}'}} m_j. \end{aligned} \tag{14.26}$$

The cut-off parameters are now defined by

$$\begin{aligned} \mathcal{B}^i_{\boldsymbol{\ell},\boldsymbol{\ell}'} &= a_i \max\left\{ 2^{-\min\{\ell_i,\ell_i'\}}, 2^{\beta^i_{\boldsymbol{\ell},\boldsymbol{\ell}}/(2\widetilde{p}+\alpha)} \right\}, \quad a_i > 1, \\ \widetilde{\mathcal{B}}^i_{\boldsymbol{\ell},\boldsymbol{\ell}'} &= \widetilde{a}_i \max\left\{ 2^{-\max\{\ell_i,\ell_i'\}}, 2^{\widetilde{\beta}^i_{\boldsymbol{\ell},\boldsymbol{\ell}}/(\widetilde{p}+\alpha)} \right\}, \quad \widetilde{a}_i > 1. \end{aligned}$$

We define a multidimensional version of Theorem 12.2.2. A proof can be found in [134, Theorem 4.6.3].

Theorem 14.6.1 *Let X be a Lévy process with Lévy density k satisfying (14.23). Define the compression scheme by*

$$\widetilde{\mathbf{A}}^{\mathrm{J}}_{(\boldsymbol{\ell}',\mathbf{k}'),(\boldsymbol{\ell},\mathbf{k})} = \begin{cases} 0 & \textit{if } \exists i \in \mathcal{I}^c_{\boldsymbol{\ell},\boldsymbol{\ell}'} \;:\; \delta_{x_i} > \mathcal{B}^i_{\boldsymbol{\ell},\boldsymbol{\ell}'}, \\ 0 & \textit{if } \exists i \in \mathcal{I}_{\boldsymbol{\ell},\boldsymbol{\ell}'} \;:\; \delta^{\text{sing}}_{x_i} > \widetilde{\mathcal{B}}^i_{\boldsymbol{\ell},\boldsymbol{\ell}'}, \\ \mathbf{A}^{\mathrm{J}}_{(\boldsymbol{\ell}',\mathbf{k}'),(\boldsymbol{\ell},\mathbf{k})} & \textit{else.} \end{cases}$$

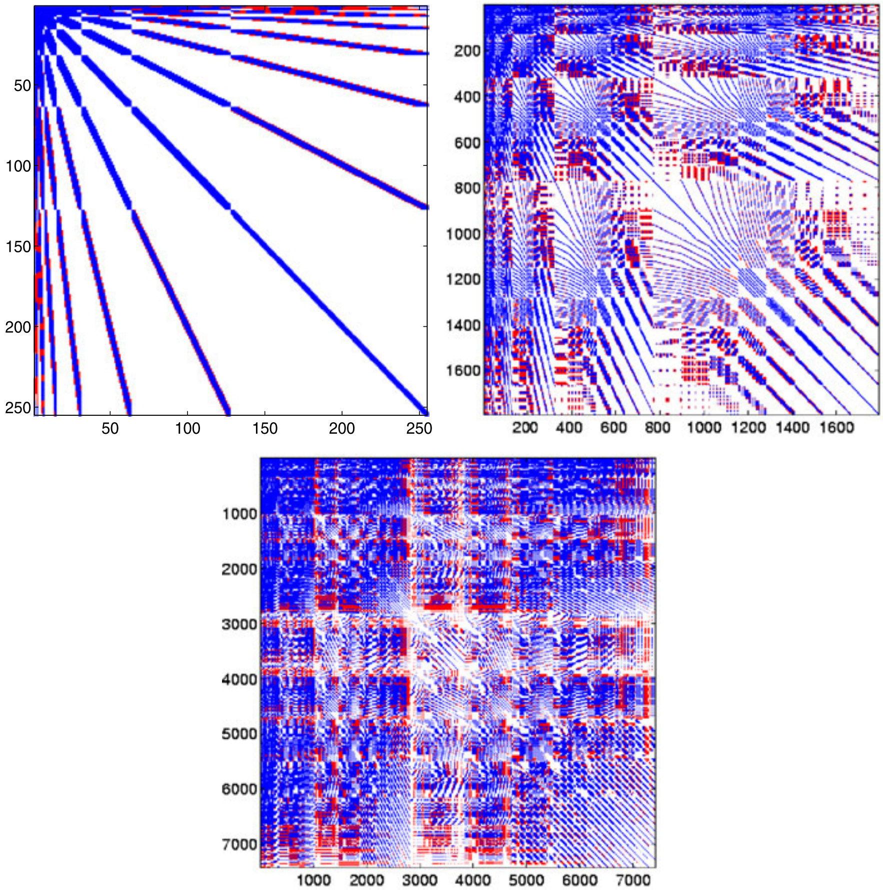


Fig. 14.5 Wavelet compression of the Lévy measure for level $L = 7$ in $d = 1, 2, 3$

If $\widetilde{p} > 2dp - (d + 1)\alpha$ and $\alpha \leq 2/d$, the number of non-zero entries for the compressed matrix $\widetilde{\mathbf{A}}^{\mathrm{J}}$ is $\mathcal{O}(2^L L^{2(d-1)})$.

We give an example for the matrix compression for various dimensions.

Example 14.6.2 Let $a = 1$, $a' = 1$, $p = 2$, $\widetilde{p} = 4$, $\alpha = 0.5$ and $L = 7$. The corresponding compression scheme is plotted in Fig. 14.5 for $d = 1, 2, 3$. Zero entries due to the first compression are left white, zero entries due to the second compression are colored red and non-zero entries are blue regardless of their size. For $d = 1$, there are 14 % of non-zero entries, for $d = 2$ one has 28 % and for $d = 3$, we have 43 %.

14.6.2 Fully Discrete Scheme

We can again use the norm equivalences (13.8) to precondition our linear systems. Denote by $\mathbf{D}$ the diagonal matrix with entries $2^{\alpha_1 \ell_1} + \cdots + 2^{\alpha_d \ell_d}$ for an index corresponding to level $\boldsymbol{\ell} = (\ell_1, \ldots, \ell_d)$. Then, (13.8) for $s_i = \alpha_i/2$, $i = 1, \ldots, d$ implies that

$$\langle \underline{u}, \widetilde{\mathbf{A}}^{\mathrm{J}} \underline{u} \rangle \sim \|u\|^2_{\widetilde{H}^{\boldsymbol{\alpha}/2}(G)} \sim \langle \underline{u}, \mathbf{D}\underline{u} \rangle,$$

and we obtain that the condition number $\kappa(\mathbf{D}^{-1/2}\widetilde{\mathbf{A}}^{\mathrm{J}}\mathbf{D}^{-1/2})$ is bounded, independent of the level L.

Using the hp-dG timestepping method (see Sect. 12.3) for the time discretization, we have the (perturbed) fully discrete scheme

$$\begin{aligned} &\text{Find } \underline{u}^m \in \mathbb{R}^{(r_m+1)\widehat{N}_L} \text{ such that for } m = 1, \ldots, M, \\ &\Big(\mathbf{C}^m \otimes \mathbf{M} + \frac{k}{2}\mathbf{I}^m \otimes \widetilde{\mathbf{A}}^{\mathrm{J}}\Big)\underline{u}^m = (\underline{\varphi}^m \otimes \mathbf{M})\underline{u}^{m-1}, \qquad (14.27) \\ &\underline{u}^0 = \underline{u}_0 \,. \end{aligned}$$

The linear systems can again be decoupled and preconditioned using $\mathbf{D}$. We have the following extension to the one-dimensional result Theorem 12.3.4.

Theorem 14.6.3 *Let X be a Lévy process with characteristic triplet $(\mathcal{Q}, \nu, \gamma)$ where $\mathcal{Q} > 0$ and Lévy density k satisfies Assumption 14.3.4 and (14.23). Moreover, let $\max_{i=1}^{d}\{a_i^{-2(\widetilde{p}+\alpha/2)} + \widetilde{a}_i^{-(\widetilde{p}+\alpha)}\}$ be sufficiently small and assume that the payoff $g|_G \in \widetilde{H}^s(G)$ for some $0 < s \le 1$. Choose $M = r = \mathcal{O}(L)$ and use in each time step $\mathcal{O}(L^5)$ GMRES iterations. Then, the fully discrete Galerkin scheme with incomplete GMRES gives*

$$\|u(T) - U^{dG}(T)\|_{L^2(G)} \le C\widehat{N}_L^{-s}(\log_2 \widehat{N}_L)^{(d-1)s+\varepsilon}, \qquad s := p - 1 + \frac{p-1}{dp-1},$$

where $C > 0$ is a constant independent of h and U^{dG} denotes the (perturbed) hp-dG approximation.

We give a numerical example.

Example 14.6.4 Let $d = 2$ and consider two independent variance gamma processes [118] with parameter $\sigma = 0.3$, $\vartheta = 0.25$ and $\theta = -0.3$. We set the compression parameter $a_i = \widetilde{a}_i = 1$, $i = 1, \ldots, 2$, $p = 2$, $\widetilde{p} = 2$. For $L = 8$, the absolute value of the entries in the stiffness matrix $\mathbf{A}^{\mathrm{J}}$ and the compressed matrix $\widetilde{\mathbf{A}}^{\mathrm{J}}$ are shown in Fig. 14.6. As in Example 12.3.5, large entries are colored red. One again clearly sees that the compression scheme neglects small entries.

For a geometric basket option with maturity $T = 1$ and strike $K = 1$, we compute in Fig. 14.7 the L^∞-error at maturity $t = T$ on the subset $G_0 = (K/2, 3/2K)^2$. In the discretization, we use the sparse tensor wavelet basis and the (perturbed) hp-dG time stepping with $M = \mathcal{O}(L)$ graded time steps. As in Sect. 13.5.1, we also solved the problem on the full grid to illustrate the "curse of dimension".

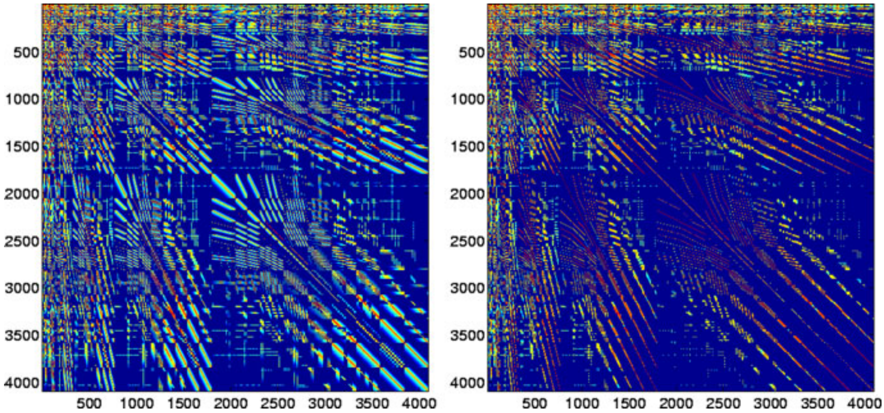


Fig. 14.6 Stiffness matrix $\mathbf{A}^{\mathrm{J}}$ (*left*) and compressed matrix $\widetilde{\mathbf{A}}^{\mathrm{J}}$ (*right*) for level $L = 8$

14.7 Application: Impact of Approximations of Small Jumps

In this section, we consider a regularization of the (multivariate) Lévy measure where small jumps are either neglected or approximated by an artificial Brownian motion. This Gaussian approximation is often proposed to simulate Lévy processes or to price options using finite differences. Applying the methods developed in the previous chapters gives accurate numerical schemes for either model. We use our scheme to study and compare the error of diffusion approximations of small jumps in multivariate Lévy models via accurate numerical solutions of the corresponding PIDEs for various types of contracts.

14.7.1 Gaussian Approximation

Let X be a d-dimensional Lévy process with the characteristic exponent

$$\psi(\xi) = -i\langle\gamma,\xi\rangle + \int_{\mathbb{R}^d}\left(1 - e^{i\langle\xi,z\rangle} + i\langle\xi,z\rangle\right)\nu(\mathrm{d}z),$$

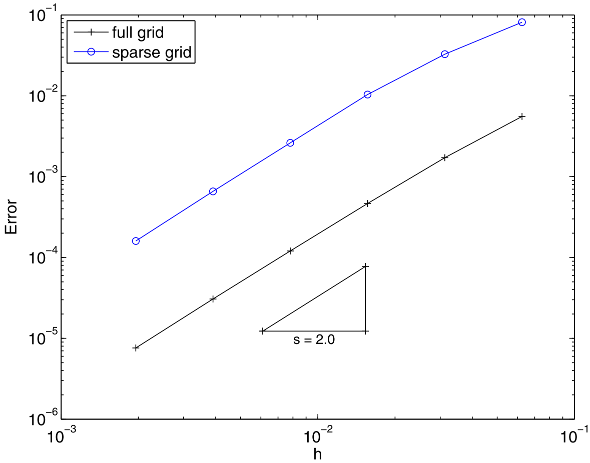
where we assume $\int_{|z|>1}|z|\,\nu(\mathrm{d}z) < \infty$. For $\varepsilon > 0$ let ν_ε be a measure such that $\nu^\varepsilon = \nu - \nu_\varepsilon$ is a finite measure and $\int_{\mathbb{R}^d}|z|^2\nu_\varepsilon(\mathrm{d}z) < \infty$. Then, the characteristic exponent can be decomposed into two parts

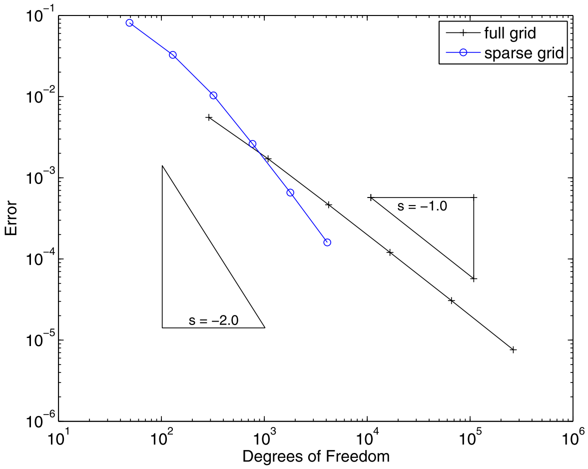
$$\psi(\xi) = \underbrace{-i\langle\gamma^\varepsilon,\xi\rangle + \int_{\mathbb{R}^d}\left(1 - e^{i\langle\xi,z\rangle}\right)\nu^\varepsilon(\mathrm{d}z)}_{\psi^\varepsilon(\xi)} + \underbrace{\int_{\mathbb{R}^d}\left(1 - e^{i\langle\xi,z\rangle} + i\langle\xi,z\rangle\right)\nu_\varepsilon(\mathrm{d}z)}_{\psi_\varepsilon(\xi)}, \tag{14.28}$$

where $\gamma_i^\varepsilon = \gamma_i - \int_{\mathbb{R}} z_i\,\nu_i^\varepsilon(\mathrm{d}z_i)$, $i = 1,\dots,d$. Correspondingly, we can decompose X into its small and large jump parts

$$X_t = \gamma^\varepsilon t + N_t^\varepsilon + X_{\varepsilon,t} = X_t^\varepsilon + X_{\varepsilon,t}, \tag{14.29}$$

Fig. 14.7 Convergence rate of the wavelet discretization in terms of the mesh width h (*top*) and in terms of degrees of freedom (*bottom*)





where N^ε is a compound Poisson process with jump measure ν^ε. The small jump part X_ε is independent of N^ε and has the covariance matrix $\mathcal{Q}_\varepsilon = \int_{\mathbb{R}^d} zz^\top \nu_\varepsilon(\mathrm{d}z)$. We assume $\mathcal{Q}_\varepsilon$ is non-singular. Let $\boldsymbol{\Sigma}_\varepsilon$ be a non-singular matrix such that $\boldsymbol{\Sigma}_\varepsilon \boldsymbol{\Sigma}_\varepsilon^\top = \mathcal{Q}_\varepsilon$. X_ε can be approximated by a d-dimensional standard Brownian motion W independent of N^ε. The next theorem [39, Theorem 3.1] shows that the process $\boldsymbol{\Sigma}_\varepsilon^{-1} X_\varepsilon$ converges *in distribution* to W as $\varepsilon \to 0$.

Theorem 14.7.1 *Let X be a d-dimensional Lévy process with characteristic triplet $(0, \nu, \gamma)$. Assume that $\mathcal{Q}_\varepsilon$ is non-singular for every $\varepsilon \in (0, 1]$ and that for every $\delta > 0$ there holds*

$$\int_{\langle \mathcal{Q}_\varepsilon^{-1} z, z\rangle > \delta} \langle \mathcal{Q}_\varepsilon^{-1} z, z\rangle \nu_\varepsilon(\mathrm{d}z) \to 0, \quad \textit{as } \varepsilon \to 0.$$

Assume further that for some family of non-singular matrices $\{\boldsymbol{\Sigma}_\varepsilon\}_{\varepsilon\in(0,1]}$ *there holds*

$$\boldsymbol{\Sigma}_\varepsilon^{-1} \mathcal{Q}_\varepsilon \boldsymbol{\Sigma}_\varepsilon^{-\top} \to \mathbf{I}_d, \quad \text{as } \varepsilon \to 0,$$

where $\mathbf{I}_d$ *denotes the identity matrix in* $\mathbb{R}^d$. *Then, for all* $\varepsilon \in (0,1]$ *there exists a càdlàg process* R^ε *such that*

$$X_t \overset{(d)}{=} \gamma^\varepsilon t + \boldsymbol{\Sigma}_\varepsilon W_t + N_t^\varepsilon + R_t^\varepsilon, \tag{14.30}$$

in the sense of equality of finite dimensional distributions. Furthermore, we have for all $T > 0$, $\sup_{t\in[0,T]} |\boldsymbol{\Sigma}_\varepsilon^{-1} R_t^\varepsilon| \xrightarrow{(\mathbb{P})} 0$, *as* $\varepsilon \to 0$ *where* γ^ε, N^ε *are given in* (14.29) *and* W *is a* d*-dimensional standard Brownian motion independent of* N^ε.

We give an example of the decomposition (14.28) into small and large jumps.

Example 14.7.2 Let $X = (X^1, \dots, X^d)^\top$ be a d-dimensional Lévy process with Lévy measure ν and marginal Lévy measures ν_i, $i = 1, \dots, d$. To obtain ν^ε in $d = 1$, we simply cut off the small jumps, i.e.

$$\nu^\varepsilon = \nu 1_{\{|z|>\varepsilon\}}. \tag{14.31}$$

For $d > 1$ the Lévy measure ν^ε could be obtained by $\nu^\varepsilon = \nu 1_{\{\|z\|_\infty > \varepsilon\}}$ where jumps are neglected if the jump size in all directions is small. But the corresponding one-dimensional Lévy measures ν_i^ε, $i = 1, \dots, d$ are then not of the form (14.31). If we choose

$$\nu^\varepsilon = \nu 1_{\{\min\{z_1,\dots,z_d\}>\varepsilon\}}, \tag{14.32}$$

the corresponding one-dimensional Lévy measures ν_i^ε, $i = 1, \dots, d$ again satisfy (14.31). We consider the Clayton Lévy copula model as explained in Sect. 14.3.2 with the density k given by (14.14) for $d = 2$, $\vartheta = 0.5$, $\eta = 0.5$ and $\boldsymbol{\alpha} = (0.5, 1.2)$. The corresponding regularized density k^ε, $\nu^\varepsilon(\mathrm{d}z) = k^\varepsilon(z)\mathrm{d}z$ as in (14.32) for $\varepsilon = 0.01$ is plotted in Fig. 14.8.

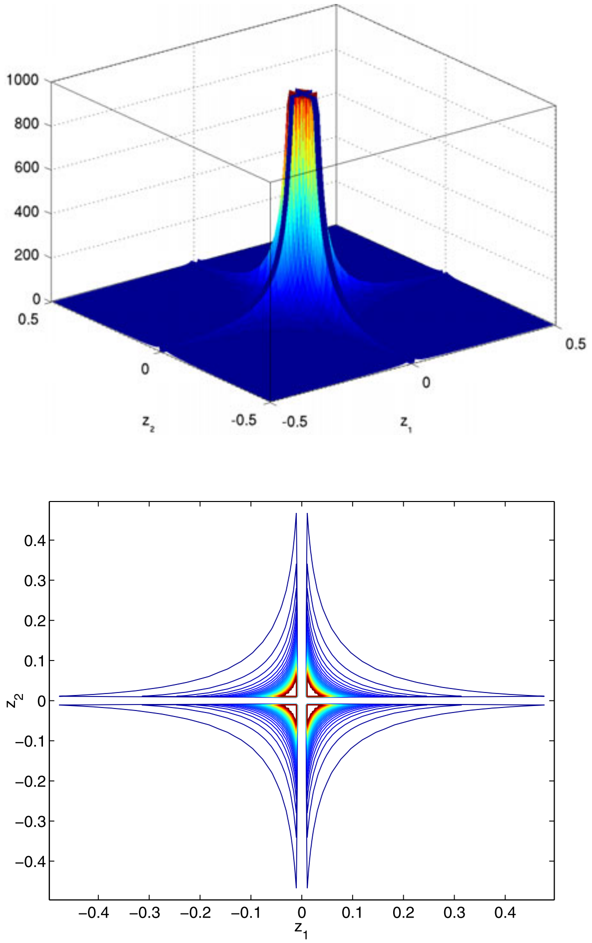
We now consider a d-dimensional pure jump process X with characteristic triplet $(0, \nu, \gamma)$ where the Lévy measure ν satisfies (10.11). Let γ be chosen according to Lemma 10.1.5 such that e^{X^j}, $j = 1, \dots, d$ are martingales. The covariance matrix is given by $\mathcal{Q} = \int_{\mathbb{R}^d} zz^\top \nu(\mathrm{d}z)$. For any $\varepsilon > 0$ the process X can be approximated by a compound Poisson process Y_1^ε as in (14.29) where the small jumps are neglected as in (14.32),

$$Y_{1,t}^\varepsilon = \gamma_1^\varepsilon t + N_t^\varepsilon. \tag{14.33}$$

The characteristic triplet of Y_1^ε is $(0, \nu^\varepsilon, \gamma_1^\varepsilon)$ and γ_1^ε is again such that $e^{Y_1^{\varepsilon,j}}$, $j = 1, \dots, d$ are martingales. A better approximation can be obtained by replacing the small jumps with a Brownian motion which yields a jump–diffusion process Y_2^ε,

$$Y_{2,t}^\varepsilon = \boldsymbol{\Sigma}_\varepsilon W_t + \gamma_2^\varepsilon t + N_t^\varepsilon, \tag{14.34}$$

Fig. 14.8 Regularized anisotropic $\boldsymbol{\alpha}$-stable Lévy copula density for $\boldsymbol{\alpha} = (0.5, 1.2)$, $\varepsilon = 0.01$ and corresponding contour plot



with characteristic triplet $(\mathcal{Q}_\varepsilon, \nu^\varepsilon, \gamma_2^\varepsilon)$. The processes W and N are independent. Y_2^ε has the same covariance matrix as X and drift $\gamma_{2,j}^\varepsilon = \gamma_{1,j}^\varepsilon - \mathcal{Q}_{\varepsilon,jj}/2$, $j = 1, \ldots, d$. For $\varepsilon \to \infty$ we obtain a diffusion process $Y_t^\infty = \boldsymbol{\Sigma} W_t + \gamma_\infty t$ with the covariance matrix $\mathcal{Q} = \boldsymbol{\Sigma}\boldsymbol{\Sigma}^\top$ and drift $\gamma_{\infty,j} = -\mathcal{Q}_{jj}/2$, $j = 1, \ldots, d$.

There are two sources of error. We have a discretization error using a mesh width $h > 0$ and a modeling error using $\varepsilon > 0$. To assess the impact of $\varepsilon > 0$, we use the wavelet discretization for $\varepsilon = 0$ and $\varepsilon > 0$. Here, h is chosen so small that the discretization error is negligible in comparison to the truncation error due to cut-off of jumps of size smaller than ε.

14.7.2 Basket Options

Consider a basket option $u(t,x)$ with payoff $g(x)$ where the log price processes of the underlyings are given by the pure jump process $X = (X^1, \dots, X^d)^\top$ and correspondingly $u_1^\varepsilon(t,x)$, $u_2^\varepsilon(t,x)$ for the processes Y_1^ε, Y_2^ε. We want to study the error $|u(T,x) - u_i^\varepsilon(T,x)|$ for $\varepsilon \to 0$, $i = 1,2$. Since we adjusted the drift to preserve the martingale property, we additionally introduce the processes

$$Z_{i,t}^\varepsilon = X_t + (\gamma_i^\varepsilon - \gamma)t, \quad i = 1,2,$$

which have the same drift as Y_i^ε and the same Lévy measure as X.

Proposition 14.7.3 *Assume g is Lipschitz continuous. Then, there are $C_1, C_2 > 0$ such that*

$$\left|\mathbb{E}(g(x+X_T)) - \mathbb{E}(g(x+Z_{1,T}^\varepsilon))\right| \le C_1 \sum_{j=1}^d \int_{-\varepsilon}^{\varepsilon} \left|z_j\right|^2 \nu_j(\mathrm{d}z_j), \quad \forall x \in \mathbb{R}^d, \tag{14.35}$$

$$\left|\mathbb{E}(g(x+X_T)) - \mathbb{E}(g(x+Z_{2,T}^\varepsilon))\right| \le C_2 \sum_{j=1}^d \int_{-\varepsilon}^{\varepsilon} \left|z_j\right|^3 \nu_j(\mathrm{d}z_j), \quad \forall x \in \mathbb{R}^d. \tag{14.36}$$

Proof We have for $i = 1,2$,

$$\begin{aligned}\left|\mathbb{E}(g(x+X_T)) - \mathbb{E}(g(x+Z_{i,T}^\varepsilon))\right| &\le \mathbb{E}\left|g(x+X_T) - g(x+X_T+(\gamma_i^\varepsilon-\gamma)T)\right| \\ &\le T \sum_{j=1}^d \left|\gamma_{i,j}^\varepsilon - \gamma_j\right|.\end{aligned}$$

Furthermore,

$$\begin{aligned}\left|\gamma_{1,j}^\varepsilon - \gamma_j\right| &= \left|\int_{\mathbb{R}^d} \left(e^{z_j} - 1 - z_j\right) \nu_\varepsilon(\mathrm{d}z)\right| \le \int_{-\varepsilon}^{\varepsilon} \int_0^{|z_j|} e^s \left|z_j - s\right| \mathrm{d}s\, \nu_j(\mathrm{d}z) \\ &\le \frac{e^\varepsilon}{2} \int_{-\varepsilon}^{\varepsilon} \left|z_j\right|^2 \nu_j(\mathrm{d}z), \quad j = 1, \dots, d, \\ \left|\gamma_{2,j}^\varepsilon - \gamma_j\right| &= \left|\frac{\mathcal{Q}_{\varepsilon,jj}}{2} - \int_{\mathbb{R}^d} \left(e^{z_j} - 1 - z_j\right) \nu_\varepsilon(\mathrm{d}z)\right| \\ &\le \frac{1}{2} \left|\int_{-\varepsilon}^{\varepsilon} \int_0^{|z_j|} e^s (z_j - s)^2 \mathrm{d}s\, \nu_j(\mathrm{d}z)\right| \\ &\le \frac{e^\varepsilon}{6} \int_{-\varepsilon}^{\varepsilon} \left|z_j\right|^3 \nu_j(\mathrm{d}z), \quad j = 1, \dots, d.\end{aligned}$$

□

The same error estimates are also obtained for the compound Poisson and Gaussian approximation.

Proposition 14.7.4 *Assume $g \in C^2(\mathbb{R}^d)$. Then, there is $C_1 > 0$*

$$\left|\mathbb{E}(g(x + Z^{\varepsilon}_{1,T})) - \mathbb{E}(g(x + Y^{\varepsilon}_{1,T}))\right| \le C_1 \sum_{j=1}^{d} \int_{-\varepsilon}^{\varepsilon} \left|z_j\right|^2 \nu_j(\mathrm{d}z_j), \quad \forall x \in \mathbb{R}^d. \tag{14.37}$$

Furthermore, assume $g \in C^4(\mathbb{R}^d)$ and $\int_{\mathbb{R}^d} |z|\, \nu(\mathrm{d}z) < \infty$. Then, for some $C_2 > 0$

$$\left|\mathbb{E}(g(x + Z^{\varepsilon}_{2,T})) - \mathbb{E}(g(x + Y^{\varepsilon}_{2,T}))\right| \le C_2 \sum_{j=1}^{d} \int_{-\varepsilon}^{\varepsilon} \left|z_j\right|^3 \nu_j(\mathrm{d}z_j), \quad \forall x \in \mathbb{R}^d. \tag{14.38}$$

Proof Consider the Taylor series expansion of $g(x)$ at x_0

$$g(x) = g(x_0) + \nabla g(x_0) \cdot (x - x_0) + \frac{1}{2}(x - x_0) \cdot D^2 g(x_0)(x - x_0) + \mathcal{O}(|x - x_0|^3),$$

where $D^2 g$ is the Hessian matrix of g. Define $R^{\varepsilon} = Z^{\varepsilon}_1 - Y^{\varepsilon}_1$. The Lévy process R^{ε} has Lévy measure ν_{ε} and is independent of Y^{ε}_1. Since $Z^{\varepsilon,j}_{1,T}$ and $Y^{\varepsilon,j}_{1,T}$, $j = 1, \ldots, d$, have the same expected value, we have $\mathbb{E}(R^{\varepsilon,j}_T) = 0$, $j = 1, \ldots, d$. Thus, we obtain

$$\begin{aligned}
&\left|\mathbb{E}(g(x + Z^{\varepsilon}_{1,T})) - \mathbb{E}(g(x + Y^{\varepsilon}_{1,T}))\right| \\
&= \left|\sum_{j=1}^{d} \mathbb{E}\Big(\partial_j g\big(x + Y^{\varepsilon}_{1,T}\big)\Big)\, \mathbb{E}\big(R^{\varepsilon,j}_T\big) + \sum_{j=1}^{d}\sum_{k=1}^{d} \mathcal{O}\Big(\mathbb{E}\big(R^{\varepsilon,j}_T R^{\varepsilon,k}_T\big)\Big)\right| \\
&\le C_1 \sum_{j=1}^{d} \mathbb{E}\Big(\big(R^{\varepsilon,j}_T\big)^2\Big).
\end{aligned}$$

Equation (14.37) follows from

$$\mathbb{E}\Big(\big(R^{\varepsilon,j}_T\big)^2\Big) = \int_{-\varepsilon}^{\varepsilon} \left|z_j\right|^2 \nu_j(\mathrm{d}z_j), \quad j = 1, \ldots, d.$$

Furthermore, $Y^{\varepsilon}_{2,t} = Y^{\varepsilon}_{1,t} + \boldsymbol{\Sigma}_{\varepsilon} W_t + (\gamma^{\varepsilon}_2 - \gamma^{\varepsilon}_1)t$, where the standard Brownian motion W is independent of Y^{ε}_1. We set $\widetilde{x} = x + (\gamma^{\varepsilon}_2 - \gamma^{\varepsilon}_1)T$ and obtain

$$\begin{aligned}
&\left|\mathbb{E}(g(x + Z^{\varepsilon}_{2,T})) - \mathbb{E}(g(x + Y^{\varepsilon}_{2,T}))\right| \\
&= \left|\mathbb{E}(g(\widetilde{x} + Z^{\varepsilon}_{1,T})) - \mathbb{E}(g(\widetilde{x} + Y^{\varepsilon}_{1,T})) + \mathbb{E}(g(\widetilde{x} + Y^{\varepsilon}_{1,T})) - \mathbb{E}(g(x + Y^{\varepsilon}_{2,T}))\right| \\
&= \Bigg|\sum_{j=1}^{d}\sum_{k=1}^{d} \frac{1}{2}\mathbb{E}\Big(\partial_j \partial_k g\big(\widetilde{x} + Y^{\varepsilon,j}_{1,T}\big)\Big)\mathbb{E}\Big(\big(R^{\varepsilon,j}_T R^{\varepsilon,k}_T\big)\Big) + \sum_{j=1}^{d} \mathcal{O}\Big(\mathbb{E}\big(|R^{\varepsilon,j}_T|^3\big)\Big) \\
&\quad - \sum_{j=1}^{d}\sum_{k=1}^{d} \frac{1}{2}\mathbb{E}\Big(\partial_j \partial_k g\big(\widetilde{x} + Y^{\varepsilon,j}_{1,T}\big)\Big)\mathcal{Q}_{\varepsilon,jk} + \mathcal{O}\Big(\mathbb{E}\big(|\boldsymbol{\Sigma}_{\varepsilon} W_T|^4\big)\Big)\Bigg| \\
&\le C_2 \sum_{j=1}^{d} \left(\int_{-\varepsilon}^{\varepsilon} \left|z_j\right|^3 \nu_j(\mathrm{d}z_j) + \left(\int_{-\varepsilon}^{\varepsilon} \left|z_j\right|^2 \nu_j(\mathrm{d}z_j)\right)^2\right).
\end{aligned}$$

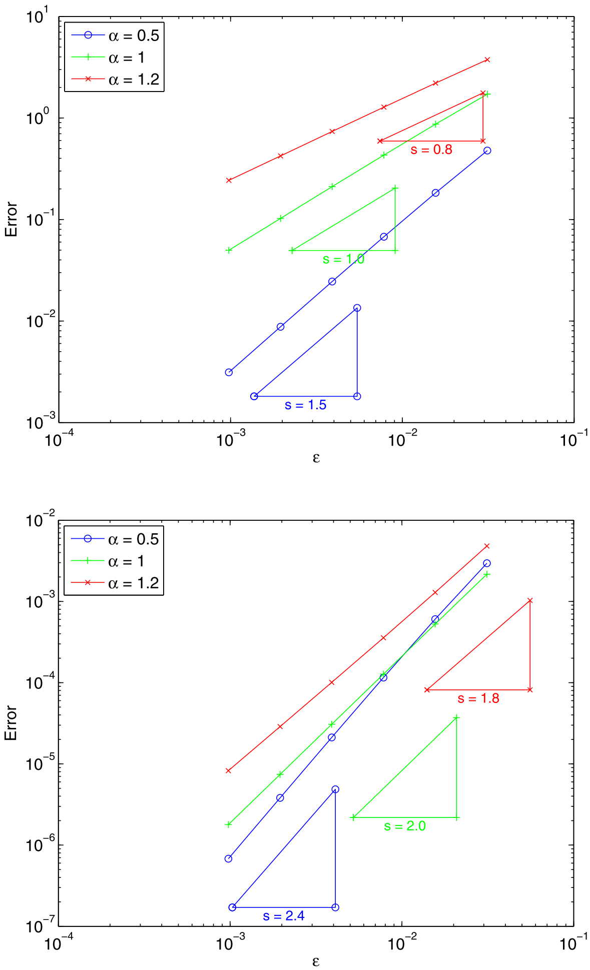


Fig. 14.9 Convergence rates with respect to ε for Y_1^ε (*top*) and Y_2^ε (*bottom*) in $d = 1$

Now with $c_j = \int_{-\varepsilon}^{\varepsilon} |z_j| \, \nu_j(\mathrm{d}z_j) < \infty, \; j = 1, \ldots, d$ and Jensen's inequality, we have

$$\left(\int_{-\varepsilon}^{\varepsilon} |z_j|^2 \, \nu_j(\mathrm{d}z_j)\right)^2 = c_j^2 \left(\int_{-\varepsilon}^{\varepsilon} |z_j| \, \frac{|z_j| \nu_j(\mathrm{d}z_j)}{c_j}\right)^2 \le c_j^2 \int_{-\varepsilon}^{\varepsilon} |z_j|^2 \, \frac{|z_j| \nu_j(\mathrm{d}z_j)}{c_j}$$

$$= c_j \int_{-\varepsilon}^{\varepsilon} |z_j|^3 \, \nu_j(\mathrm{d}z_j), \quad j = 1, \ldots, d. \qquad \square$$

Using Propositions 14.7.3 and 14.7.4, we immediately obtain

Corollary 14.7.5 *Assume the Lévy measure* ν *satisfies* (10.12) *with* $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d)$. *Then, for* $g \in C^4(\mathbb{R}^d)$

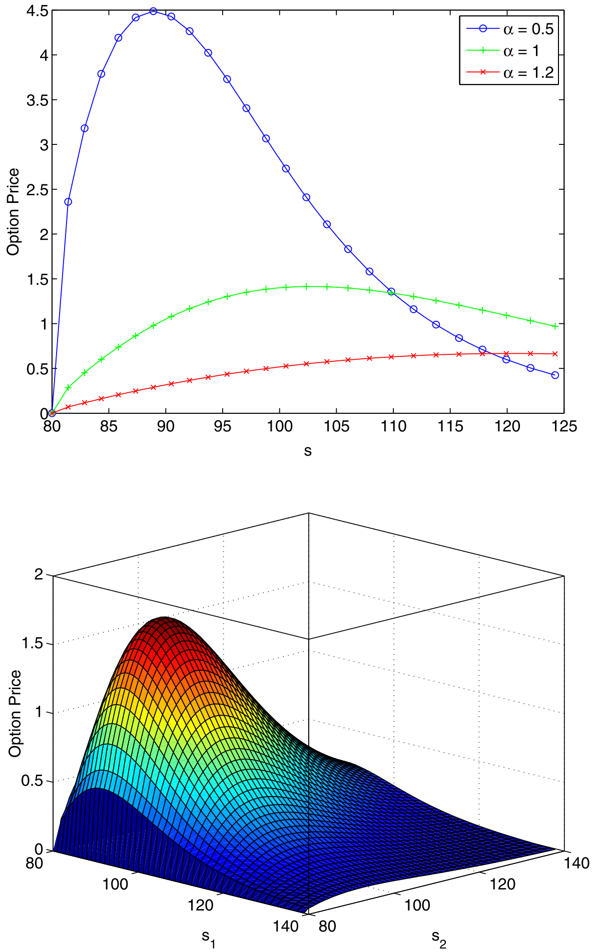


Fig. 14.10 Barrier option price in $d = 1$ (*top*) and $d = 2$ with barrier $B = 80$ and strike $K = 100$

$$\left|\mathbb{E}(g(x + X_T^{\varepsilon})) - \mathbb{E}(g(x + Y_{1,T}^{\varepsilon}))\right| \le \widetilde{C}_1\, \varepsilon^{2-\max\{\alpha_1,\dots,\alpha_d\}}, \quad \forall x \in \mathbb{R}^d,\ 0 < \alpha_j < 2,$$
$$\left|\mathbb{E}(g(x + X_T^{\varepsilon})) - \mathbb{E}(g(x + Y_{2,T}^{\varepsilon}))\right| \le \widetilde{C}_2\, \varepsilon^{3-\max\{\alpha_1,\dots,\alpha_d\}}, \quad \forall x \in \mathbb{R}^d,\ 0 < \alpha_j < 1.$$

These convergence rates can also be shown numerically even for $g \notin C^4(\mathbb{R}^d)$ and $\alpha > 1$.

Example 14.7.6 Let $d = 1$ and consider the tempered stable density (10.10),

$$k(z) = c\,\frac{e^{-\beta_+|z|}}{|z|^{1+\alpha}}1_{\{z>0\}} + c\,\frac{e^{-\beta_-|z|}}{|z|^{1+\alpha}}1_{\{z<0\}}.$$

We compute the price of a put option with maturity $T = 0.5$, strike $K = 100$ and interest rate $r = 0.01$. We set $c = 1$, $\beta^- = 10$, $\beta^+ = 15$ and compute for Y_1^{ε}, Y_2^{ε}

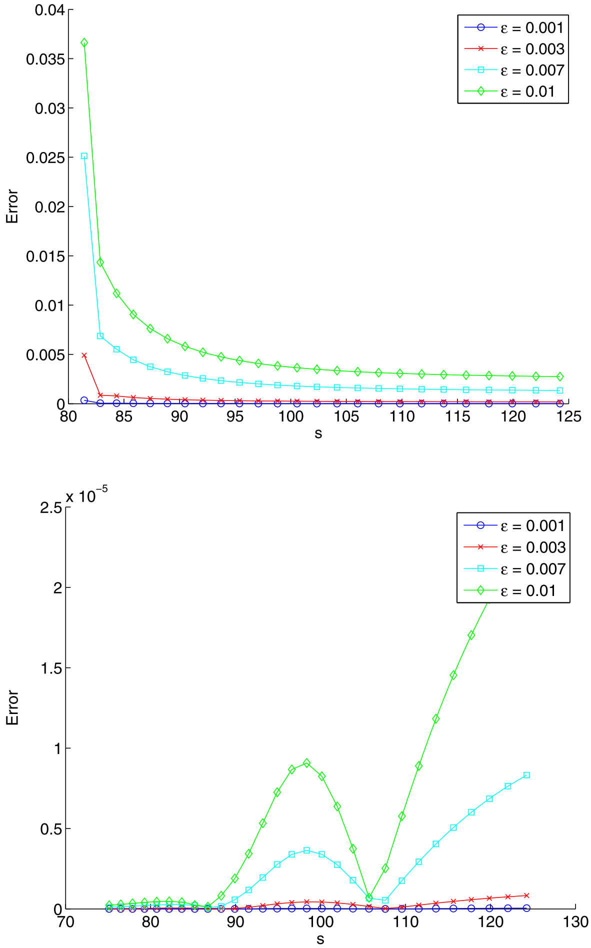


Fig. 14.11 Relative error for various values of ε using Y_2^ε (14.34) in place of X in (14.29) for a barrier option (*top*) and a non-barrier basket option (*bottom*) in $d = 1$ with barrier $B = 80$ and strike $K = 100$

the convergence rate with respect to ε at $s = 100$ using various α's. As shown in Fig. 14.9, the rates $2 - \alpha$ and $3 - \alpha$ are always obtained.

14.7.3 Barrier Options

Propositions 14.7.3 and 14.7.4 do not hold for barrier options since the option price is not smooth at the boundary ∂G. In particular, it is shown in $d = 1$ for tempered stable densities with $1 < \alpha < 2$ and $c^+ = c^-$ that the derivative of the option price behaves in log-price like $|x - \log B|^{\alpha/2-1}$ as $x \to \log B$ (see, e.g. [108]). Therefore, one obtains a large error at the boundary by approximating X with Y_2^ε.

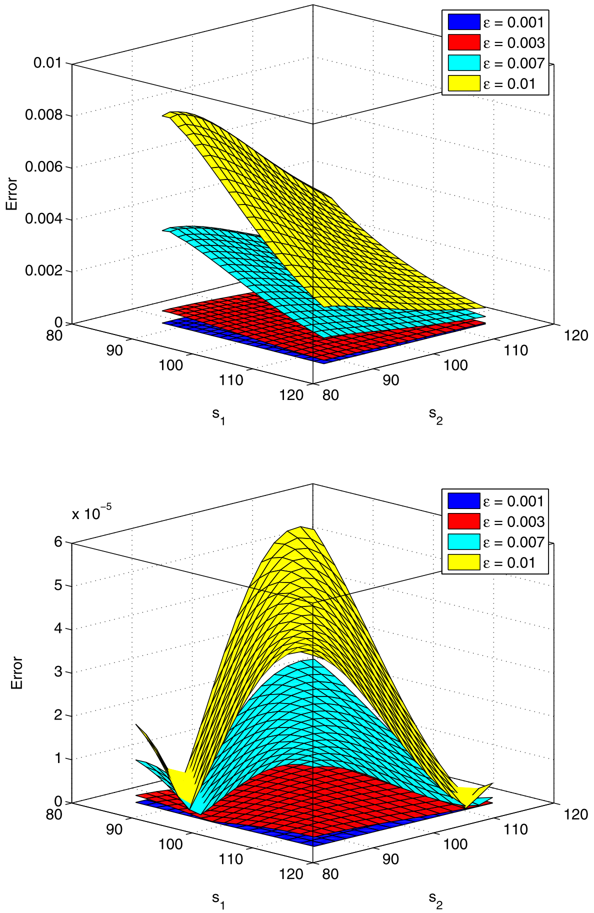


Fig. 14.12 Relative error for various values of ε using Y_2^ε (14.34) in place of X in (14.29) for a barrier option (*top*) and a non-barrier basket option (*bottom*) in $d = 2$ with barrier $B = 80$ and strike $K = 100$

Example 14.7.7 Let $d = 1$ and $d = 2$. We consider again a pure jump process ($\mathcal{Q} \equiv 0$) with one or two (independent) marginal densities of tempered stable type, i.e.

$$k_i(z) = c_i \frac{e^{-\beta_-^+|z|}}{|z|^{1+\alpha_i}} 1_{\{z>0\}} + c_i \frac{e^{-\beta_-^-|z|}}{|z|^{1+\alpha_i}} 1_{\{z<0\}}, \quad i = 1, \ldots, d\,.$$

We compute the price of a down-and-out basket option, $g(s) = (K - \frac{1}{d}\sum_{i=1}^d s_i)_+$, on the domain $D = [B, \infty)^d$ with barrier $B = 80$, maturity $T = 0.5$, strike $K = 100$ and interest rate $r = 0.01$. We set $c_1 = c_2 = 1$, $\beta_1^- = 10$, $\beta_1^+ = 15$, $\beta_2^- = 9$, $\beta_2^+ = 16$, $\alpha_1 = 0.5$ and $\alpha_2 = 0.7$. The option price is shown in Fig. 14.10 where for $d = 1$ we additionally plot the price for $\alpha = 1, 1.2$ to show the behavior of the option price close to the barrier.

The relative error for approximating X by Y_2^ε is plotted in Figs. 14.11 and 14.12.

Additionally, we also show the corresponding error for the non-barrier basket option. As expected, the relative error is significantly higher for a barrier option close to the barrier.

14.8 Further Reading

Lévy copulas are introduced by Kallsen and Tankov [100]. These are used in Farkas et al. [66] to study pure jump processes built from 1-homogeneous Lévy copulas and univariate marginal Lévy processes with symmetric tempered stable margins. The domain of the infinitesimal generator $\mathcal{A}$ is characterized and it is shown that the corresponding variational problem is well-posed. The multivariate, nonsymmetric case was studied in Reich et al. [134], i.e. when the univariate marginal Lévy processes are tempered stable, but with possibly nonsymmetric margins. In this chapter, we closely followed Winter [163]. For a more detailed description of the numerical scheme, in particular the numerical integration of the matrix coefficients, see [163, 164].

Chapter 15
Stochastic Volatility Models with Jumps

In Chap. 9, we considered pure diffusion stochastic volatility models. In particular, we assumed that the vector process $Z = (X, Y^1, \dots, Y^{n_v})^\top$ of the log-price process $X = \ln(S)$ and the $n_v \geq 1$ additional processes which describe the volatility $\sigma_t = \xi(Y_t^1, \dots, Y_t^{n_v})$ satisfies the SDE $\mathrm{d}Z_t = b(Z_t)\,\mathrm{d}t + \Sigma(Z_t)\,\mathrm{d}W_t$. We extend these models by adding jumps to it.

15.1 Market Models

Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ be a filtered complete probability space satisfying the usual hypotheses (see Sect. 1.2). Let $(W_t)_{t\geq 0}$ be an n-dimensional standard Brownian motion and J an independent Poisson random measure $\mathbb{R}_+ \times \mathbb{R} \setminus \{0\}$ with associated compensator $\widetilde{J}$ and intensity measure ν, where we assume that ν is a Lévy measure. We assume that the filtration is generated by the two mutually independent processes W and J and that W and J are independent of $\mathcal{F}_0$. Let $d := n_v + 1$ be the dimension of the process $Z \in \mathbb{R}^d$, whose dynamics evolve according to the SDE

$$\begin{aligned}\mathrm{d}Z_t = b(Z_{t-})\,\mathrm{d}t + \Sigma(Z_{t-})\,\mathrm{d}W_t + \int_{|\zeta|<c} \varsigma_s(Z_{t-}, \zeta)\widetilde{J}(\mathrm{d}t, \mathrm{d}\zeta)\\ + \int_{|\zeta|\geq c} \varsigma_l(Z_{t-}, \zeta) J(\mathrm{d}t, \mathrm{d}\zeta).\end{aligned} \tag{15.1}$$

We assume that the coefficients $b : \mathbb{R}^d \to \mathbb{R}^d$, $\Sigma : \mathbb{R}^d \to \mathbb{R}^{d\times n}$, $\varsigma : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$, $\varsigma \in \{\varsigma_s, \varsigma_l\}$ are Lipschitz continuous and satisfy a linear growth condition: There exists a constant $C > 0$ such that for all $z, z' \in \mathbb{R}^d$, $\zeta \in \mathbb{R}$

$$\begin{aligned} |b(z) - b(z')| + |\Sigma(z) - \Sigma(z')| &\leq C|z - z'|,\\ |b(z)| + |\Sigma(z)| &\leq C(1 + |z|),\\ |\varsigma(z, \zeta) - \varsigma(z', \zeta)| &\leq C(1 \wedge |\zeta|)|z - z'|,\\ |\varsigma(z, \zeta)| &\leq C(1 \wedge |\zeta|)(1 + |z|).\end{aligned} \tag{15.2}$$

N. Hilber et al., *Computational Methods for Quantitative Finance*, Springer Finance,
DOI 10.1007/978-3-642-35401-4_15,

Under these assumptions, there exists a unique solution to (15.1), and if we additionally assume that $\mathbb{E}[|Z_0|^2] < \infty$, then $\mathbb{E}[|Z_t|^2] < \infty$, $\forall t \geq 0$, and there exists a constant $C(t) > 0$ such that

$$\mathbb{E}[|Z_t|^2] \leq C(t)\big(1 + \mathbb{E}[|Z_0|^2]\big); \tag{15.3}$$

see, e.g. [3].

15.1.1 Bates Models

The models of Bates [13, 14] are combinations of the jump–diffusion model of Merton (10.6)–(10.7) and the stochastic volatility model of Heston (9.4)–(9.5). Let $n_v \in \{1, 2\}$. Under a risk-neutral probability measure, the log-price dynamics $X_t = \ln(S_t)$ of the risky underlying are

$$\mathrm{d}X_t = \left(r - \varkappa\lambda_t - \frac{1}{2}\sum_{i=1}^{n_v} Y_t^i\right)\mathrm{d}t + \sum_{i=1}^{n_v}\sqrt{Y_t^i}\,\mathrm{d}W_t^i + \mathrm{d}J_t,$$

where J is a compound Poisson process with state dependent intensity $\lambda_t = \lambda_0 + \sum_{i=1}^{n_v}\lambda_i Y_t^i$, $\lambda_i \geq 0$, $i = 0, \ldots, n_v$, and jump size distribution ν_0 as in (10.7), with $\varkappa := \int_{\mathbb{R}}(e^\zeta - 1)\nu_0(\mathrm{d}\zeta) = e^{\mu+\delta^2/2} - 1$. Each of the processes Y_t^i follows a CIR process, i.e.

$$\mathrm{d}Y_t^i = \alpha_i(m_i - Y_t^i)\,\mathrm{d}t + \beta_i\sqrt{Y_t^i}\,\mathrm{d}\widehat{W}_t^i, \quad i = 1, \ldots, n_v.$$

The Brownian motions W^1, W^2 are independent, and the Brownian motions $\widehat{W}^i$ satisfy $\widehat{W}_t^i = \rho_i W_t^i + \sqrt{1-\rho_i^2}\,W_t^{i+n_v}$, with independent Brownian motions W^{i+n_v}, $i = 1, \ldots, n_v$. Thus, the coefficients b, Σ and ς_l in (15.1) are, for the case $n_v = 2$, given by: $c = 0$,

$$b(z) = \begin{pmatrix} r - \lambda_0\varkappa - \sum_{i=1}^{n_v}\left(\frac{1}{2} + \lambda_i\kappa\right)y_i \\ \alpha_1(m_1 - y_1) \\ \alpha_2(m_2 - y_2) \end{pmatrix}, \tag{15.4}$$

$$\Sigma(z) = \begin{pmatrix} \sqrt{y_1} & \sqrt{y_2} & 0 & 0 \\ \beta_1\rho_1\sqrt{y_1} & 0 & \beta_1\sqrt{1-\rho_1^2}\sqrt{y_2} & 0 \\ 0 & \beta_2\rho_2\sqrt{y_2} & 0 & \beta_2\sqrt{1-\rho_2^2}\sqrt{y_2} \end{pmatrix}, \tag{15.5}$$

$$\varsigma_l(z,\zeta) = \big(\zeta, 0, 0\big)^\top. \tag{15.6}$$

The Lévy density $k(\zeta)$ of $\nu_0(\mathrm{d}\zeta)$ depends on y_1, y_2 via

$$k(\zeta) = \left(\lambda_0 + \sum_{k=1}^{n_v}\lambda_i y_i\right)\frac{1}{\sqrt{2\pi\delta^2}}\,e^{-(\zeta-\mu)^2/(2\delta^2)}. \tag{15.7}$$

Note that for $n_v = 1$ with $\lambda_1 = 0$ we obtain the first model of Bates [13], which further reduces to the Heston model for $\lambda_0 = 0$.

15.1.2 BNS Model

The stochastic volatility model suggested by Barndorff-Nielsen and Shepard (BNS) [10] specifies the volatility σ as an Ornstein–Uhlenbeck process, driven by a pure jump subordinator $\{L_t : t \geq 0\}$ (for the definition of a subordinator, see Definition 10.2.1). The construction of structure preserving equivalent martingale measure $\mathbb{Q}$ is discussed in [128]. In particular, it is shown in [128, Theorem 3.2] that the dynamics of the pair process $(X_t, \sigma_t^2)_{t\geq 0}$ under $\mathbb{Q}$ is given by

$$\begin{aligned} \mathrm{d}X_t &= (r - \lambda\varkappa(\rho) - 1/2\sigma_t^2)\,\mathrm{d}t + \sigma_t\,\mathrm{d}W_t + \rho\,\mathrm{d}L_{\lambda t}, \\ \mathrm{d}\sigma_t^2 &= -\lambda\sigma_t^2\,\mathrm{d}t + \mathrm{d}L_{\lambda t}. \end{aligned} \tag{15.8}$$

Here, $\rho \leq 0$ is a correlation parameter and $\lambda > 0$. The constant $\varkappa(\rho)$ is defined as

$$\varkappa(\rho) = \int_{\mathbb{R}_+} \big(e^{\rho\zeta} - 1\big) w(\zeta) k(\zeta)\,\mathrm{d}\zeta, \qquad \rho < 0, \tag{15.9}$$

where $w : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies

$$\int_{\mathbb{R}_+} \big(\sqrt{w(\zeta)} - 1\big)^2 k(\zeta)\,\mathrm{d}\zeta < \infty,$$

and k is the Lévy density of L under $\mathbb{P}$. Let $\nu^{\mathbb{Q}}(\mathrm{d}\zeta) = \lambda w(\zeta)k(\zeta)\,\mathrm{d}\zeta$ be the Lévy measure of L under $\mathbb{Q}$ and let $Z = (X, Y) := (X, \sigma^2)$. From (15.8), we readily deduce that this model fits into (15.1), with $c = 0$ and coefficients b, Σ, ς_l given by

$$b(z) = \begin{pmatrix} r - \lambda\varkappa(\rho) - \frac{1}{2}y \\ -\lambda y \end{pmatrix}, \quad \Sigma(z) = \begin{pmatrix} \sqrt{y} \\ 0 \end{pmatrix}, \quad \varsigma_l(z,\zeta) = \begin{pmatrix} \rho\zeta \\ \zeta \end{pmatrix}. \tag{15.10}$$

It is shown in that the process $(X_t, \sigma_t)_{t\geq 0}$ is Markovian, so that $(X_t, \sigma_t^2)_{t\geq 0}$ is also Markovian (the Markov property is invariant under bijective mappings).

15.2 Pricing Equations

Let $Z = (X, Y^1, \ldots, Y^{n_v})$ be the unique solution of (15.1) and let $z := (x, y_1, \ldots, y_{n_v})$. As in the previous chapters, we show that the fair value of a European derivative

$$v(t,z) := \mathbb{E}\big[e^{-r(T-t)} g(e^{X_T}) \mid Z_t = z\big]$$

solves a parabolic partial integro-differential equation in $\mathbb{R}^{n_v+1}$. To this end, we consider a generalization of Proposition 8.1.2.

Proposition 15.2.1 *Let the Lévy measure satisfy* (10.11). *Denote by* $\mathcal{Q}(z) := (\Sigma\Sigma^\top)(z)$ *and by* $\mathcal{A}$ *the infinitesimal generator of* Z *which is, for functions* $f \in C^2(\mathbb{R}^d)$ *with bounded derivatives, given by*

$$
\begin{aligned}
(\mathcal{A}f)(z) = {} & \frac{1}{2}\mathrm{tr}[\mathcal{Q}(z)D^2 f(z)] + b(z)^\top \nabla f(z) \\
& + \int_{|\zeta|<c} \big(f(z+\varsigma_s(z,\zeta)) - f(z) - \varsigma_s(z,\zeta)^\top \nabla f(z)\big)\nu(\mathrm{d}\zeta) \\
& + \int_{|\zeta|\geq c} \big(f(z+\varsigma_l(z,\zeta)) - f(z)\big)\nu(\mathrm{d}\zeta).
\end{aligned} \tag{15.11}
$$

Then, the process $M_t := f(Z_t) - \int_0^t (\mathcal{A}f)(Z_s)\mathrm{d}s$ *is a martingale with respect to the filtration of* Z.

Proof By the Itô formula (see, e.g. [132]), we have

$$
\begin{aligned}
\mathrm{d}f(Z_t) = {} & \sum_{i=1}^{d} \partial_{z_i} f(Z_{t-})\,\mathrm{d}Z_t^i + \frac{1}{2}\sum_{i,j=1}^{d} \partial_{z_i z_j} f(Z_{t-})\,\mathrm{d}\langle Z^i, Z^j\rangle_t^c \\
& + f(Z_t) - f(Z_{t-}) - \sum_{i=1}^{d} \partial_{z_i} f(Z_{t-})\Delta Z_t^i \\
= {} & b(Z_{t-})^\top \nabla f(Z_{t-})\,\mathrm{d}t + \sum_{i=1}^{d} \partial_{x_i} f(Z_{t-}) \sum_{j=1}^{n} \Sigma_{ij}(Z_{t-})\,\mathrm{d}W_t^j \\
& + \frac{1}{2}\sum_{i,j=1}^{d} (D^2 f)_{ij}(Z_{t-})\mathcal{Q}_{ij}(Z_{t-})\,\mathrm{d}t \\
& + \int_{|\zeta|<c} \varsigma_s(Z_{t-},\zeta)^\top \nabla f(Z_{t-})\widetilde{J}(\mathrm{d}t,\mathrm{d}\zeta) \\
& + \int_{|\zeta|\geq c} \varsigma_l(Z_{t-},\zeta)^\top \nabla f(Z_{t-}) J(\mathrm{d}t,\mathrm{d}\zeta) \\
& + \int_{|\zeta|<c} \big(f(Z_{t-}+\varsigma_s(Z_{t-},\zeta)) - f(Z_{t-}) - \varsigma_s(Z_{t-},\zeta)^\top \nabla f(Z_{t-})\big) \\
& \times J(\mathrm{d}t,\mathrm{d}\zeta) \\
& + \int_{|\zeta|\geq c} \big(f(Z_{t-}+\varsigma_l(Z_{t-},\zeta)) - f(Z_{t-}) - \varsigma_l(Z_{t-},\zeta)^\top \nabla f(Z_{t-})\big) \\
& \times J(\mathrm{d}t,\mathrm{d}\zeta) \\
= {} & (\mathcal{A}f)(Z_{t-}) + \sum_{i=1}^{d} \partial_{z_i} f(Z_{t-}) \sum_{j=1}^{n} \Sigma_{ij}(Z_{t-})\,\mathrm{d}W_t^j \\
& + \int_{|\zeta|<c} \big(f(Z_{t-}+\varsigma_s(Z_{t-},\zeta)) - f(Z_{t-})\big)\widetilde{J}(\mathrm{d}t,\mathrm{d}\zeta) \\
& + \int_{|\zeta|\geq c} \big(f(Z_{t-}+\varsigma_l(Z_{t-},\zeta)) - f(Z_{t-})\big)\widetilde{J}(\mathrm{d}t,\mathrm{d}\zeta) \\
= {} & (\mathcal{A}f)(Z_{t-}) + \mathrm{d}M_t^1 + \mathrm{d}M_t^2 + \mathrm{d}M_t^3.
\end{aligned}
$$

In the proof of Proposition 8.1.2, we already showed that M_t^1 is a martingale. We show that $M_t^2 = \int_0^t \int_{|\zeta|<c} (f(Z_{\tau-} + \varsigma_s(Z_{\tau-}, \zeta)) - f(Z_{\tau-})) \widetilde{J}(\mathrm{d}\tau, \mathrm{d}\zeta)$ is a martingale. Indeed (compare also with the proof of Proposition 10.3.1),

$$
\begin{aligned}
&\mathbb{E}\bigg[\int_0^t \int_{|\zeta|<c} \big|f(Z_{\tau-} + \varsigma_s(Z_{\tau-}, \zeta)) - f(Z_{\tau-})\big|^2 \nu(\mathrm{d}\zeta)\,\mathrm{d}\tau\bigg] \\
&\quad\le C \max_{1\le i\le d} \sup_{z\in\mathbb{R}^d} |\partial_{z_i} f(z)|^2\, \mathbb{E}\bigg[\int_0^t \int_{|\zeta|<c} |\varsigma_s(Z_{\tau-}, \zeta)|^2 \nu(\mathrm{d}\zeta)\,\mathrm{d}\tau\bigg] \\
&\quad\le \widetilde{C} \int_{|\zeta|<c} \min\{1, |\zeta|^2\} \nu(\mathrm{d}\zeta) \mathbb{E}\bigg[\int_0^t (1 + |Z_{\tau-}|^2)\,\mathrm{d}\tau\bigg] < \infty,
\end{aligned}
$$

where we used (15.2) and (15.3) and the fact that $\int_{\mathbb{R}} 1 \wedge \zeta^2 \nu(\mathrm{d}\zeta) < \infty$. The same considerations can be made to show that M_t^3 is a martingale, where we employ (10.11). □

We repeat the arguments which lead to Theorem 4.1.4 and obtain, using Proposition 15.2.1, a generalization of Theorem 8.1.3

Theorem 15.2.2 *Let $d := n_v + 1$, and let $G \subseteq \mathbb{R}^d$ be the state space of the process Z. Let $v \in C^{1,2}(J \times \mathbb{R}^d) \cap C^0(\overline{J} \times \mathbb{R}^d)$ with bounded derivatives in $z = (x, y_1, \ldots, y_{n_v})$ be a solution of*

$$
\partial_t v - \mathcal{A}v + rv = 0 \quad \text{in } J \times G, \qquad v(0, z) = g(e^x) \quad \text{in } G, \tag{15.12}
$$

with $\mathcal{A}$ as in (15.2.1). Then, $v(t, z) = V(T - t, z)$ can also be represented as

$$
V(t, z) = \mathbb{E}\big[e^{-r(T-t)} g(e^{X_T}) \mid Z_t = z\big].
$$

Consider the Bates models. It follows by (15.4)–(15.7) that the generator $\mathcal{A} =: \mathcal{A}^{\mathrm{B}}$ is given by

$$
\begin{aligned}
(\mathcal{A}^{\mathrm{B}} f)(z) := {}& \frac{1}{2} \sum_{i=1}^{n_v} y_i \partial_{xx} f(z) + \sum_{i=1}^{n_v} \beta_i \rho_i y_i \partial_{x y_i} f(z) + \frac{1}{2} \sum_{i=1}^{n_v} \beta_i^2 y_i \partial_{y_i y_i} f(z) \\
& + \bigg[r - \lambda_0 \varkappa - \sum_{i=1}^{n_v} \Big(\frac{1}{2} + \lambda_i \kappa\Big) y_i\bigg] \partial_x f(z) + \sum_{i=1}^{n_v} \alpha_i (m_i - y_i) \partial_{y_i} f(z) \\
& + \Big(\lambda_0 + \sum_{i=1}^{n_v} \lambda_i y_i\Big) \int_{\mathbb{R}} \big(f(x + \zeta, y_1, \ldots, y_{n_v}) - f(z)\big) \nu_0(\mathrm{d}\zeta),
\end{aligned}
\tag{15.13}
$$

with ν_0 as in (10.7). If $n_v = 1$ and $\lambda_0 = \lambda_1 = 0$, then $\mathcal{A}^{\mathrm{B}}$ reduces to $\mathcal{A}^{\mathrm{H}}$ in (9.12).

As a second example, we give the generator $\mathcal{A} := \mathcal{A}^{\mathrm{S}}$ of the BNS model. From (15.10) we deduce

$$
\begin{aligned}
(\mathcal{A}^{\mathrm{S}} f)(z) := {} & \frac{1}{2} y \partial_{xx} f(z) + (r - \lambda \varkappa(\rho) - y/2) \partial_x f(z) - \lambda y \partial_y f(z) \\
& + \int_{\mathbb{R}_+} \big(f(x + \rho\zeta, y + \zeta) - f(z) \big) \nu^{\mathbb{Q}}(\mathrm{d}\zeta),
\end{aligned}
$$

with $\nu^{\mathbb{Q}}(\mathrm{d}\zeta) = \lambda w(\zeta) k(\zeta)\,\mathrm{d}\zeta$. Note that there is no diffusion component in the second coordinate direction $\partial_{yy} f(z)$.

Remark 15.2.3 The operator $\mathcal{A}^{\mathrm{S}}$ is the generator of the process $Z = (X, Y)$ in (15.8) under a structure preserving EMM. However, one can consider the pricing of options under the minimal entropy martingale measure (MEMM). For $\rho = 0$, this is done by Benth and Meyer-Brandis [17], see also [16]. In particular, they derive that $\mathcal{A}^{\mathrm{S}}$ under the MEMM is given by

$$
\begin{aligned}
(\mathcal{A}^{\mathrm{S}}(t) f)(z) := {} & \frac{1}{2} y \partial_{xx} f(z) + (r - y/2) \partial_x f(z) - \lambda y \partial_y f(z) \\
& + \lambda \int_{\mathbb{R}_+} \big(f(x, y + \zeta) - f(z) \big) \frac{H(t, y + \zeta)}{H(t, y)} \nu(\mathrm{d}\zeta), \quad (15.14)
\end{aligned}
$$

where $H : [0, T] \times \mathbb{R}_+ \to \mathbb{R}$ is solution of the PIDE

$$
\begin{aligned}
& \partial_t H - \lambda y \partial_y H - r(y) H \\
& \quad + \lambda \int_{\mathbb{R}_+} \big(H(t, y + \zeta) - H(t, y) \big) \nu(\mathrm{d}\zeta) = 0 \quad \text{in } [0, T) \times \mathbb{R}_+, \\
& \qquad\qquad\qquad\qquad\qquad\qquad\quad H(T, y) = 1 \quad \text{in } \mathbb{R}_+.
\end{aligned}
$$

Herewith, $r(y) = \frac{1}{2}(\mu y^{-1/2} + \beta y^{1/2})^2$, where $\mu, \beta \in \mathbb{R}$.

15.3 Variational Formulation

We derive the variational formulation of the pricing PIDE (15.12) exemplarily for the Bates model with $n_v = 1$, $\lambda_1 = 0$ and the BNS model.

Consider the operator $\mathcal{A}^{\mathrm{B}}$ in (15.13) and let $n_v = 1$, $\lambda_1 = 0$. For this choice of parameters, we can write

$$
(\mathcal{A}^{\mathrm{B}} f)(z) = (\mathcal{A}^{\mathrm{H}} f)(z) + (\widehat{\mathcal{A}} f)(z),
$$

where $\mathcal{A}^{\mathrm{H}}$ is the generator of the Heston model (9.12) and the operator $\widehat{\mathcal{A}}$ is given by

$$
(\widehat{\mathcal{A}} f)(z) := -\lambda_0 \kappa \partial_x f(z) + \lambda_0 \int_{\mathbb{R}_+} \big(f(x + \zeta, y) - f(z) \big) \nu_0(\mathrm{d}\zeta).
$$

To prepare the variational formulation, we perform in the pricing equation (15.12) the variable transformation (9.13), i.e. we set $\widetilde{v}(t, x, \widetilde{y}) := v(t, x, 1/4\widetilde{y}^2)$. The operator $\mathcal{A}^{\mathrm{B}}$ changes accordingly to $\widetilde{\mathcal{A}}^{\mathrm{B}} = \widetilde{\mathcal{A}}^{\mathrm{H}} + \widehat{\mathcal{A}}$, where the transformed Heston operator $\widetilde{\mathcal{A}}^{\mathrm{H}}$ is given in (9.15). Additionally, we consider the change of variables (9.17)

$w := (\widetilde{v} - \widetilde{v}_0)e^{-\eta}$, where $\widetilde{v}_0 = \widetilde{v}(0, x, \widetilde{y}) = g(e^x)$, and where $\eta(x, \widetilde{y}) = \kappa/2\widetilde{y}^2$. Thus, the transformed time value of the option w solves

$$\begin{aligned} &\partial_t w - \mathcal{A}_\kappa^{\mathrm{B}} w + rw = f_\kappa^{\mathrm{B}} \text{ in } J \times \mathbb{R} \times \mathbb{R}_{\geq 0}, \\ &w(0, x, \widetilde{y}) = 0 \text{ in } \mathbb{R} \times \mathbb{R}_{\geq 0}, \end{aligned} \tag{15.15}$$

where $f_\kappa^{\mathrm{B}} := e^{-\kappa/2\widetilde{y}^2}(\widetilde{\mathcal{A}}^{\mathrm{B}}\widetilde{v}_0 - r\widetilde{v}_0)$ and the operator $\mathcal{A}_\kappa^{\mathrm{B}} := \mathcal{A}_\kappa^{\mathrm{H}} + \widehat{\mathcal{A}}$, where $\mathcal{A}_\kappa^{\mathrm{H}}$ is as in (9.21).

As for the Heston model, we drop "$\sim$", let $G := \mathbb{R} \times \mathbb{R}_{\geq 0}$ and denote by $(\cdot, \cdot)$ the $L^2(G)$-inner product, i.e. $(\varphi, \phi) = \int_G \varphi\phi \mathrm{d}x\mathrm{d}y$. We associate to $-\mathcal{A}_\kappa^{\mathrm{B}} + r$ the bilinear form $a_\kappa^B(\cdot, \cdot)$ via

$$a_\kappa^{\mathrm{B}}(\varphi, \phi) := \big((-\mathcal{A}_\kappa^{\mathrm{B}} + r)\varphi, \phi\big), \quad \varphi, \phi \in C_0^\infty(G).$$

We find, since $\int_{\mathbb{R}} \nu_0(\mathrm{d}\zeta) = 1$,

$$a_\kappa^{\mathrm{B}}(\varphi, \phi) = a_\kappa^{\mathrm{H}}(\varphi, \phi) + \lambda_0 \varkappa(\partial_x\varphi, \phi) + \lambda_0(\varphi, \phi) - \lambda_0\left(\int_{\mathbb{R}} \varphi(x + \zeta, y)\nu_0(\mathrm{d}\zeta), \phi\right), \tag{15.16}$$

where the bilinear form $a_\kappa^{\mathrm{H}}(\cdot, \cdot)$ is given in (9.22). Let $V := \overline{C_0^\infty(G)}^{\|\cdot\|_V}$, where the closure is taken with respect to norm $\|\cdot\|_V$ defined in (9.24).

Theorem 15.3.1 *Assume that* $0 < \kappa < \alpha/\beta^2$ *and that*

$$1 - 2|4\alpha m/\beta^2 - 1| > \rho^2.$$

Then, there exist constants $C_i > 0$, $i = 1, 2, 3$, *such that for all* $\varphi, \phi \in V$ *there holds*

$$\begin{aligned} |a_\kappa^{\mathrm{B}}(\varphi, \phi)| &\leq C_1\|\varphi\|_V\|\phi\|_V, \\ a_\kappa^{\mathrm{B}}(\varphi, \varphi) &\geq C_2\|\varphi\|_V^2 - C_3\|\varphi\|_{L^2(G)}^2. \end{aligned}$$

Proof The continuity of $a_\kappa^{\mathrm{B}}(\cdot, \cdot)$ follows from the continuity of $a_\kappa^{\mathrm{H}}(\cdot, \cdot)$, by the Hardy inequality $|\lambda_0\varkappa(\partial_x\varphi, \phi)| \leq |\lambda_0||\varkappa|\|y\partial_x\varphi\|_{L^2(G)}\|y^{-1}\varphi\|_{L^2(G)} \leq C\|y\partial_x\varphi\|_{L^2(G)} \times \|\partial_y\varphi\|_{L^2(G)}$, and Young's inequality $\|\int_{\mathbb{R}} \varphi(x + \zeta, y)\nu_0(\mathrm{d}\zeta)\|_{L^2(G)} \leq \|\varphi\|_{L^2(G)}$. Furthermore,

$$a_\kappa^{\mathrm{B}}(\varphi, \varphi) \geq c_1\|y\partial_x\varphi\|_{L^2(G)}^2 + c_2\|\partial_y\varphi\|_{L^2(G)} + c_3\|y\varphi\|_{L^2(G)}^2 + c_4\|\varphi\|_{L^2(G)}^2,$$

with c_1, c_2, c_3 as in the proof of Theorem 9.3.1, and $c_4 = r - 2\lambda_0 - 2\kappa\alpha m$. Now deduce as in the proof of Theorem 9.3.1. □

Consequently, the weak formulation to the (transformed) Bates model (15.15)

$$\begin{aligned} &\text{Find } w \in L^2(J; V) \cap H^1(J; L^2(G)) \text{ such that} \\ &(\partial_t w, v) + a_\kappa^{\mathrm{B}}(w, v) = \langle f_\kappa^{\mathrm{B}}, v\rangle_{V^*,V}, \ \forall v \in V, \ \text{ a.e. in } J, \\ &w(0) = 0 \end{aligned} \tag{15.17}$$

admits a unique solution for every $f_\kappa^{\mathrm{B}} \in V^*$.

Consider now the pricing equation (15.12) in the BNS model with $\mathcal{A} = \mathcal{A}^{\mathrm{S}}$ given in (15.14). The functional setting for this pricing equation does not fit into the abstract parabolic framework described in Sect. 3.2 due to absence of diffusion with respect to the volatility coordinate y (compare with the operator $\mathcal{A}^{\mathrm{S}}$). The order of the jump operator appearing in $\mathcal{A}^{\mathrm{S}}$ is $\alpha < 1$, since the driving process (of volatility) is a subordinator. Thus, the first order term ∂_y is dominant. It is therefore desirable to remove this term, see also Sect. 10.3.

Consider in (15.12) the change of variables $w(t,x,y) := v(t, x + \lambda\varkappa t, e^{\lambda t} y)$, and assume for simplicity $r = 0$. Let $G_R := (-R_1, R_1) \times (0, R_2)$ and let $\Gamma_0 := (-R_1, R_1) \times \{0\}$ and $\Gamma_1 := \partial G_R \setminus \Gamma_0$. Instead of the pricing equation (15.12) for v on the unbounded domain $G = \mathbb{R} \times \mathbb{R}_+$, we consider the pricing equation for w on the bounded domain G_R, i.e.

$$\begin{aligned} \partial_t w + \big(\widetilde{\mathcal{A}}(t) + \widetilde{\mathcal{A}}^{\mathrm{J}}(t)\big) w &= 0 && \text{in} \quad J \times G_R, \\ w &= 0 && \text{on} \quad J \times \Gamma_1, \\ w(0) &= g(e^x) && \text{in} \quad G_R, \end{aligned} \tag{15.18}$$

where the change of variables induces the operator $\widetilde{\mathcal{A}}(t) + \widetilde{\mathcal{A}}^{\mathrm{J}}(t)$ given by

$$\begin{aligned} \widetilde{\mathcal{A}}(t) &:= -\frac{1}{2} e^{\lambda t} y (\partial_{xx} - \partial_x), \\ (\widetilde{\mathcal{A}}^{\mathrm{J}}(t)\varphi)(x,y,t) &:= -\lambda \int_{\mathbb{R}_+} \big[\varphi(x + \rho z, y + e^{-\lambda t} z, t) - \varphi(x,y,t)\big] k^w(z)\,\mathrm{d}z. \end{aligned}$$

We will now derive a variational formulation for problem (15.18). To this end, denote by (φ, ϕ) the $L^2(G_R)$-inner product. For $\varphi, \phi \in C_0^\infty(G_R)$, we associate with $\widetilde{\mathcal{A}}(t) + \widetilde{\mathcal{A}}^{\mathrm{J}}(t)$ the bilinear form

$$a(t; \varphi, \phi) := \big(\widetilde{\mathcal{A}}(t)\varphi, \phi\big) + \big(\widetilde{\mathcal{A}}^{\mathrm{J}}(t)\varphi, \phi\big) = a^{\mathrm{C}}(t; \varphi, \phi) + a^{\mathrm{J}}(t; \varphi, \phi).$$

Define the weighted Sobolev space

$$W := \overline{C_0^\infty(G_R)}^{\|\cdot\|_W},$$

where the norm $\|\cdot\|_W$ is given by

$$\|v\|_W^2 := \|\sqrt{y}\,\partial_x v\|_{L^2(G_R)}^2 + \|v\|_{L^2(G_R)}^2.$$

Lemma 15.3.2 *The bilinear form $a^{\mathrm{C}}(t; \cdot, \cdot) : W \times W \to \mathbb{R}$ is continuous and satisfies a Gårding inequality, i.e. there exist constants $C_i > 0$, $i = 1, 2, 3$, such that $\forall \varphi, \phi \in W$ and $\forall t \in J$ there holds*

$$|a^{\mathrm{C}}(t; \varphi, \phi)| \le C_1 \|\varphi\|_W \|\phi\|_W, \qquad a^{\mathrm{C}}(t; \varphi, \varphi) \ge C_2 \|\varphi\|_W^2 - C_3 \|\varphi\|_{L^2(G_R)}^2.$$

Proof By integration by parts, we have

$$a^{\mathrm{C}}(t; \varphi, \phi) = \frac{1}{2} e^{\lambda t} (y \partial_x \varphi, \partial_x \phi) + \frac{1}{2} e^{\lambda t} (y \partial_x \varphi, \phi),$$

and hence, since $(y\partial_x\varphi, \varphi) = 1/2(y, \partial_x(\varphi^2)) = 0$,

$$a^{\mathrm{C}}(t; \varphi, \varphi) = \frac{1}{2}e^{\lambda t}\|\sqrt{y}\partial_x\varphi\|^2_{L^2(G_R)} \geq \frac{1}{2}\|\varphi\|^2_W - \frac{1}{2}\|\varphi\|^2_{L^2(G_R)}.$$

The continuity of the term $e^{\lambda t}/2(y\partial_x\varphi, \partial_x\phi)$ is clear. Furthermore,

$$\begin{aligned}\frac{1}{2}e^{\lambda t}|(y\partial_x\varphi, \phi)| &\leq \frac{1}{2}e^{\lambda T}\|\sqrt{y}\partial_x\varphi\|_{L^2(G_R)}\|\sqrt{y}\phi\|_{L^2(G_R)} \\ &\leq \frac{1}{2}e^{\lambda T}\|y\|^{1/2}_{L^\infty(G_R)}\|\sqrt{y}\partial_x\varphi\|_{L^2(G_R)}\|\phi\|_{L^2(G_R)}.\end{aligned}$$ □

According to Lemma 15.3.2, the function space W is an appropriate space for the bilinear form $a^{\mathrm{C}}(t; \cdot, \cdot)$, and it remains to characterize the space for the bilinear form $a^{\mathrm{J}}(t; \cdot, \cdot)$. As in the one-dimensional case (see Theorem 10.4.3), this space is a Sobolev space of fractional order. The next lemma is proved in [80].

Lemma 15.3.3 *Assume the Lévy measure ν of the subordinator L in* (15.8) *admits a density $k : \mathbb{R}_+ \to \mathbb{R}$ of the form $k(\zeta) = ce^{-\beta\zeta}\zeta^{-1-\alpha}$, with $c > 0$, $\beta > 1$ and $0 < \alpha < 1$. Then, there exist constants $C_4, C_5, C_6 > 0$ such that $\forall t \in J$ there holds*

$$\begin{aligned}|a^{\mathrm{J}}(t; \varphi, \phi)| &\leq C_4\|\varphi\|_{\widetilde{H}^{\alpha/2}(G_R)}\|\phi\|_{\widetilde{H}^{\alpha/2}(G_R)}, \\ a^{\mathrm{J}}(t; \varphi, \varphi) &\geq C_5\|\varphi\|^2_{\widetilde{H}^{\alpha/2}(G_R)} - C_6\|\varphi\|^2_{L^2(G_R)}.\end{aligned}$$

Consider now the space

$$V := W \cap \widetilde{H}^{\alpha/2}(G_R)$$

equipped with the norm $\|v\|^2_V := \|v\|^2_W + \|v\|^2_{\widetilde{H}^{\alpha/2}(G_R)}$.

By combining Lemmas 15.3.2 and 15.3.3, we obtain

Theorem 15.3.4 *Under the assumptions of Lemma* 15.3.3, *there exist constants C_7, C_8, $C_9 > 0$ such that the bilinear form $a(t; \cdot, \cdot) : V \times V \to \mathbb{R}$ satisfies $\forall\varphi, \phi \in V$ and $\forall t \in J$*

$$|a(t; \varphi, \phi)| \leq C_7\|\varphi\|_V\|\phi\|_V, \qquad a(t; \varphi, \varphi) \geq C_8\|\varphi\|^2_V - C_9\|\varphi\|^2_{L^2(G_R)}.$$

Proof By definition of $a(t; \cdot, \cdot)$, we have

$$\begin{aligned}a(t; \varphi, \varphi) &= a^{\mathrm{C}}(t; \varphi, \varphi) + a^{\mathrm{J}}(t; \varphi, \varphi) \\ &\geq \frac{1}{2}\|\sqrt{y}\partial_x\varphi\|^2_{L^2(G_R)} - (1/2 + C_9)\|\varphi\|^2_{L^2(G_R)} + C_5\|\varphi\|^2_{\widetilde{H}^{\alpha/2}(G_R)} \\ &\geq \min\{1/2, C_5\}\|\varphi\|^2_V - (1 + C_9)\|\varphi\|^2_{L^2(G_R)}.\end{aligned}$$

Furthermore,

$$\begin{aligned}|a(t;\varphi,\phi)| &\le |a^{\mathrm{C}}(t;\varphi,\phi)| + |a^{\mathrm{J}}(t;\varphi,\phi)| \\ &\le C_1\|\varphi\|_W\|\phi\|_W + C_4\|\varphi\|_{\widetilde{H}^{\alpha/2}(G_R)}\|\phi\|_{\widetilde{H}^{\alpha/2}(G_R)} \\ &\le 2\max\{C_4, C_5\}\|\varphi\|_V\|\phi\|_V. \end{aligned} \qquad \square$$

We deduce that the weak formulation to (15.18)

$$\begin{aligned}&\text{Find } w \in L^2(J;V)\cap H^1(J;L^2(G_R)) \text{ such that}\\ &(\partial_t w, v) + a(t;w,v) = 0, \ \forall v \in V, \quad \text{a.e. in } J, \\ &w(0) = g(e^x)\end{aligned} \tag{15.19}$$

admits a unique solution for every $g \in L^2(G_R)$.

15.4 Wavelet Discretization

As in the previous chapters, the discretization of the weak formulation (9.29) of the general pure diffusion SV model, (15.17) of the jump–diffusion SV model of Bates or (15.19) of the BNS model is based on the sparse tensor product space $\widehat{\mathcal{V}}_L$ and the hp-dG time stepping scheme.

In order to describe the stiffness matrix $\mathbf{A}$ for these SV models, we introduce as in Chap. 9, weighted matrices $\mathbf{M}^{w(x_i)}$, $\mathbf{B}^{w(x_i)}$ and $\mathbf{S}^{w(x_i)}$. In contrast to Chap. 9, however, we need to define them with respect to the wavelet basis $\{\psi_{\ell,k}\}$, compare with (13.12)–(13.14),

$$\mathbf{M}^{w(x_i)} := \left(\int_{a_i}^{b_i} \psi_{\ell_i,k_i}(x_i)\psi_{\ell_i',k_i'}(x_i)w(x_i)\,\mathrm{d}x_i\right)_{\substack{0\le \ell_i',\ell_i\le L\\ k_i'\in\nabla_{\ell_i'},k_i\in\nabla_{\ell_i}}}, \tag{15.20}$$

$$\mathbf{S}^{w(x_i)} := \left(\int_{a_i}^{b_i} \psi'_{\ell_i,k_i}(x_i)\psi'_{\ell_i',k_i'}(x_i)w(x_i)\,\mathrm{d}x_i\right)_{\substack{0\le \ell_i',\ell_i\le L\\ k_i'\in\nabla_{\ell_i'},k_i\in\nabla_{\ell_i}}}, \tag{15.21}$$

$$\mathbf{B}^{w(x_i)} := \left(\int_{a_i}^{b_i} \psi'_{\ell_i,k_i}(x_i)\psi_{\ell_i',k_i'}(x_i)w(x_i)\,\mathrm{d}x_i\right)_{\substack{0\le \ell_i',\ell_i\le L\\ k_i'\in\nabla_{\ell_i'},k_i\in\nabla_{\ell_i}}}. \tag{15.22}$$

As an example, consider the (transformed) Bates model with operator $\mathcal{A}_\kappa^{\mathrm{B}}$ as in (15.15). Then, the corresponding stiffness matrix $\mathbf{A}_\kappa^{\mathrm{B}}$ is given by

$$\mathbf{A}_\kappa^{\mathrm{B}} = \mathbf{A}_\kappa^{\mathrm{H}} + \lambda_0\varkappa\mathbf{B}^1\,\widehat{\otimes}\,\mathbf{M}^1 - \lambda_0\mathbf{A}^{\mathrm{J}}\,\widehat{\otimes}\,\mathbf{M}^1,$$

where $\mathbf{A}_\kappa^{\mathrm{H}}$ is as in (9.32) (where $\otimes$ has to be replaced by $\widehat{\otimes}$) and $\mathbf{A}^{\mathrm{J}}$ is the stiffness matrix to the jump operator $\int_{\mathbb{R}}(f(x+\zeta)-f(x))\nu_0(\mathrm{d}\zeta)$. Note that $\mathbf{A}^{\mathrm{J}}$ can be implemented and (wavelet-) compressed as described in Chap. 12.

Applying the hp-dG time stepping to the semi-discrete problem leads then after decoupling, as explained in the previous chapters, to linear systems of the form

$$\underbrace{\left(\lambda\mathbf{M} + k/2\mathbf{A}\right)}_{=:\mathbf{B}}\underline{w} = \underline{s}, \tag{15.23}$$

with $\lambda \in \mathbb{C}$. Again, we want to construct a diagonal preconditioner for the matrix $\mathbf{B}$. Since the underlying energy space $\mathcal{V}$ (i.e. the space on which the bilinear form $a(\cdot,\cdot) : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ is continuous and satisfies a Gårding inequality) for most of the SV models is a weighted Sobolev space, we can no longer rely on the norm equivalences (12.3), (13.8) to construct the preconditioners, but have to use norm equivalences for weighted Sobolev spaces [19]. To this end, we recall some facts on these.

The wavelets $\psi_{\ell,k}$ and the weighting function w defined on the interval $(0, 1)$ are assumed to satisfy the following assumptions.

Assumption 15.4.1

(i) The wavelets have one vanishing moment: $\int_0^1 \psi_{\ell,k}(x)\,\mathrm{d}x = 0$.

(ii) The wavelets $\psi_{\ell,k}$ and their duals $\widetilde{\psi}_{\ell,k}$ belong to $W^{1,\infty}(0, 1)$. Furthermore, they satisfy: There exist constants $C > 0$, $\beta, \widetilde{\beta} \in \mathbb{N}_0$, $\gamma + \beta > -1/2$, $-\gamma + \widetilde{\beta} > -1/2$, such that for $j \in \{0, 1\}$ and for $x \in [0, 2^{-\ell}]$ there holds

$$|(\psi_{\ell,k})^{(j)}(x)| \le C 2^{(j+1/2)\ell}(2^\ell x)^{\beta-j}, \quad k \in \mathcal{I},$$
$$|(\widetilde{\psi}_{\ell,k})^{(j)}(x)| \le C 2^{(j+1/2)\ell}(2^\ell x)^{\widetilde{\beta}-j}, \quad j \in \widetilde{\mathcal{I}},$$

where the index sets $\mathcal{I}$ and $\widetilde{\mathcal{I}}$ are given by $\mathcal{I} := \{i \in \mathbb{N} : \beta - 1 \le i \le 2^{\ell-1},\ 0 \in \operatorname{supp} \psi_{\ell,i}\}$ and $\widetilde{\mathcal{I}} := \{i \in \mathbb{N} : \beta - 1 \le i \le 2^{\ell-1},\ 0 \in \operatorname{supp} \widetilde{\psi}_{\ell,i}\}$, respectively.

(iii) The nonnegative weighting function $w(\cdot)$ belongs to $W^{1,\infty}(\varepsilon, 1)$ for every $\varepsilon > 0$ and satisfies: there exists a constant $C_w > 0$ such that for $j \in \{0, 1\}$

$$C_w^{-1} \le \frac{w^{(j)}(x)}{x^{\gamma-j}} \le C_w, \tag{15.24}$$

with $\gamma \in \mathbb{R}$ as in (ii).

The following weighted norm equivalences are proved in [19, Theorem 3.3, Theorem 5.1].

Proposition 15.4.2 *Let Assumption 15.4.1 hold and assume that* (12.3) *holds for* $s = 0$. *Then, for any* $u = \sum_{\ell=\ell_0}^{\infty} \sum_{k\in\nabla_\ell} u_{\ell,k}\psi_{\ell,k}$ *and* $j \in \{0, 1\}$ *the following norm equivalence holds*:

$$\|u^{(j)}\|^2_{L^2_w(0,1)} := \int_0^1 |u^{(j)}(x)|^2 w^2(x)\,\mathrm{d}x \simeq \sum_{\ell=\ell_0}^{\infty} \sum_{k\in\nabla_\ell} 2^{2\ell j} w^2(2^{-\ell}k)|u_{\ell,k}|^2. \tag{15.25}$$

We need a tensorized version of Proposition 15.4.2.

Corollary 15.4.3 *Let* $d \ge 1$ *and let* $w(x) := \prod_{j=1}^d w_j(x_j)$. *Assume that* $w_j : \mathbb{R} \to \mathbb{R}_+$, $j = 1, \dots, d$, *satisfies Assumption 15.4.1*(iii). *Assume further that* $\psi_{\boldsymbol{\ell},\mathbf{k}}(x) := \psi_{\ell_1,k_1} \otimes \cdots \otimes \psi_{\ell_d,k_d}$ *with* ψ_{ℓ_i,k_i} *satisfying Assumption 15.4.1*(i)–(ii). *Then, for any multi-index* $\boldsymbol{\alpha} \in \mathbb{N}_0^d$ *with* $|\boldsymbol{\alpha}|_\infty \le 1$ *and any*

$$u = \sum_{\ell_i=\ell_0}^{\infty} \sum_{k_i\in\nabla_{\ell_i}} u_{\boldsymbol{\ell},\mathbf{k}}\psi_{\boldsymbol{\ell},\mathbf{k}}$$

there holds

$$\|D^{\boldsymbol{\alpha}} u\|^2_{L^2_w([0,1]^d)} := \int_{[0,1]^d} |D^{\boldsymbol{\alpha}} u(x)|^2 w^2(x)\,\mathrm{d}x$$
$$\simeq \sum_{\ell_i \geq \ell_0} \sum_{k_i \in \nabla_{\ell_i}} 2^{2\langle \boldsymbol{\ell}, \boldsymbol{\alpha}\rangle} \prod_{i=1}^d w_i^2(2^{-\ell_i} k_i) |u_{\boldsymbol{\ell},\mathbf{k}}|^2.$$

Proof Let $\boldsymbol{\alpha} = (0, \dots, 0)$. As in [19], for $1 \leq i \leq d$, let

$$\big(M_i\big)_{(\ell_i,k_i),(\ell_i',k_i')} = \left(\frac{\int_0^1 w_i^2(x_i)\psi_{\ell_i,k_i}(x_i)\psi_{\ell_i',k_i'}(x_i)\,\mathrm{d}x_i}{w_i(2^{-\ell_i}k_i)w_i(2^{-\ell_i'}k_i')}\right)_{(\ell_i,k_i),(\ell_i',k_i')}$$
$$:= \Big(\big(\psi_{\ell_i,k_i}, \psi_{\ell_i',k_i'}\big)_{w_i}\Big)_{(\ell_i,k_i),(\ell_i',k_i')}.$$

We estimate by Cauchy–Schwarz

$$\|u\|^2_{L^2_w([0,1]^d)} = \sum_{\substack{\ell_1,\dots,\ell_d \\ \ell_1',\dots,\ell_d'}} \sum_{\substack{k_i \in \nabla_{\ell_i} \\ k_i' \in \nabla_{\ell_i'}}} u_{\boldsymbol{\ell},\mathbf{k}} u_{\boldsymbol{\ell}',\mathbf{k}'}$$
$$\times \prod_{i=1}^d w_i(2^{-\ell_i}k_i) w_i(2^{-\ell_i'}k_i')\big(\psi_{\ell_i,k_i}, \psi_{\ell_i',k_i'}\big)_{w_i}$$
$$\leq \|M_1 \otimes \cdots \otimes M_d\|_2 \sum_{\ell_1,\dots,\ell_d} \sum_{k_i \leq \nabla_{\ell_i}} \prod_{i=1}^d w_i^2(2^{-\ell_i}k_i)|u_{\boldsymbol{\ell},\mathbf{k}}|^2.$$

Since $\|M_1 \otimes \cdots \otimes M_d\|_2 \leq \prod_{i=1}^d \|M_i\|_2$ and $\|M_i\|_2 \leq c_i$ by [19, Theorem 3.2], this shows the upper estimate. The lower estimate follows by a duality argument (see the proof of [19, Theorem 3.3] for $d = 1$). Now let $\boldsymbol{\alpha}$ be such that $|\boldsymbol{\alpha}|_\infty = 1$. Then the claim follows by the same arguments as for the case $\boldsymbol{\alpha} = (0, \dots, 0)$ and by (15.25) with $j = 1$. □

For most stochastic volatility models under consideration, the function spaces $\mathcal{V}$ for which the corresponding bilinear form is continuous and satisfies a Gårding inequality are weighted Sobolev spaces. In particular, these spaces are equipped with a norm of the form

$$\|v\|^2_{\mathcal{V}} := \sum_{|\boldsymbol{\alpha}|_1 \leq 1} \|D^{\boldsymbol{\alpha}} v\|^2_{L^2_{w^{\boldsymbol{\alpha}}}(G)}, \tag{15.26}$$

where the weight $w^{\boldsymbol{\alpha}} : \mathbb{R}^d \to \mathbb{R}_+$ depending on $\boldsymbol{\alpha}$ is given by $w^{\boldsymbol{\alpha}}(x) := \prod_{i=1}^d w_i^{\boldsymbol{\alpha}}(x_i)$ for univariate weights $w_i^{\boldsymbol{\alpha}} : \mathbb{R} \to \mathbb{R}_+$. We assume that all the $d(d+1)$ weights $w_i^{\boldsymbol{\alpha}}$ satisfy Assumption 15.4.1(iii).

Example 15.4.4 Consider the norm for the Bates model (9.24). Then, $w_1^{(1,0)}(x_1) = 1$, $w_2^{(1,0)}(x_2) = x_2$, $w_1^{(0,1)}(x_1) = w_2^{(0,1)}(x_2) = 1$, as well as $w_1^{(0,0)}(x_1) = 1$ and $w_2^{(0,0)}(x_2) = \sqrt{1+x_2^2}$.

According to Proposition 15.4.2 and Corollary 15.4.3, we define diagonal preconditioners as follows, see also [19, Sect. 5]. Denote by $\mathbf{D}_{w^{\boldsymbol{\alpha}}}^{(i)}$ the diagonal matrix (corresponding to the ith coordinate direction)

$$\big(\mathbf{D}_{w^{\boldsymbol{\alpha}}}^{(i)}\big)_{(\ell_i,k_i),(\ell_i',k_i')} := 2^{2\alpha_i\ell_i}(w_i^{\boldsymbol{\alpha}})^2(2^{-\ell_i}k_i)\delta_{\ell_i,\ell_i'}\delta_{k_i,k_i'} \in \mathbb{R}^{\dim V_L \times \dim V_L},$$

and set

$$\mathbf{D}_{w^{\boldsymbol{\alpha}}} := \sum_{|\boldsymbol{\alpha}|_1 \le 1} \mathbf{D}_{w^{\boldsymbol{\alpha}}}^{(1)} \,\widehat{\otimes} \cdots \widehat{\otimes}\, \mathbf{D}_{w^{\boldsymbol{\alpha}}}^{(d)} \in \mathbb{R}^{\widehat{N}_L \times \widehat{N}_L}. \tag{15.27}$$

For the matrix $\mathbf{B} = \lambda\mathbf{M} + k/2\mathbf{A}$ in (15.23), now define the preconditioner

$$\mathbf{D} := \big(\Re(\lambda)\mathbf{I} + k/2\mathbf{D}_{w^{\boldsymbol{\alpha}}}\big)^{1/2}. \tag{15.28}$$

The next lemma is proven in [80].

Lemma 15.4.5 *Let the assumptions of Corollary* 15.4.3 *hold. Assume the bilinear form* $a(\cdot,\cdot) : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ *satisfies* (3.8)–(3.9) *and assume, without loss of generality, that* $C_3 = 0$ *in* (3.9). *Further assume that the space* $\mathcal{V}$ *is equipped with the norm given in* (15.26). *Let the matrices* $\mathbf{B}$ *and* $\mathbf{D}$ *be given by* (15.23) *and* (15.28), *respectively. Then, the preconditioned matrix*

$$\widehat{\mathbf{B}} := \mathbf{D}^{-1}\mathbf{B}\mathbf{D}^{-1},$$

satisfies: There exists a constant c independent of L, λ and k such that

$$\lambda_{\min}\big((\widehat{\mathbf{B}} + \widehat{\mathbf{B}}^{\mathrm{H}})/2\big)\|\widehat{\mathbf{B}}\|^{-1} \ge c. \tag{15.29}$$

Note that for $\mathbf{A} \in \mathbb{R}^{\widehat{N}_L \times \widehat{N}_L}$ symmetric, the quantity $\lambda_{\min}\big((\widehat{\mathbf{B}} + \widehat{\mathbf{B}}^{\mathrm{H}})/2\big)\|\widehat{\mathbf{B}}\|^{-1}$ is equal to $1/\kappa(\widehat{\mathbf{B}})$. Thus, estimate (15.29) is equivalent to the boundedness (from above) of the condition number of $\widehat{\mathbf{B}}$.

Example 15.4.6 (CEV model) Consider the CEV model (4.17) with $0 < \rho < 0.5$. According to Proposition 4.5.1, the preconditioner for stiffness matrix $\mathbf{A}$ of this model is given by $\mathbf{D} := (\mathbf{D}_{x^\rho} + \mathbf{D}_1)^{1/2}$. Figure 15.1 shows the condition number of $\mathbf{D}^{-1}\mathbf{A}\mathbf{D}^{-1}$ for different values of ρ.

By combining Theorem 13.3.1 with Lemma 15.4.5, we obtain the following convergence result for the approximated option price in the Bates model.

Theorem 15.4.7 *Let the assumptions of Theorem* 13.3.1 *and Lemma* 15.4.5 *hold. Then, choosing the number and order of time steps such that* $M = r = \mathcal{O}(L)$ *and*

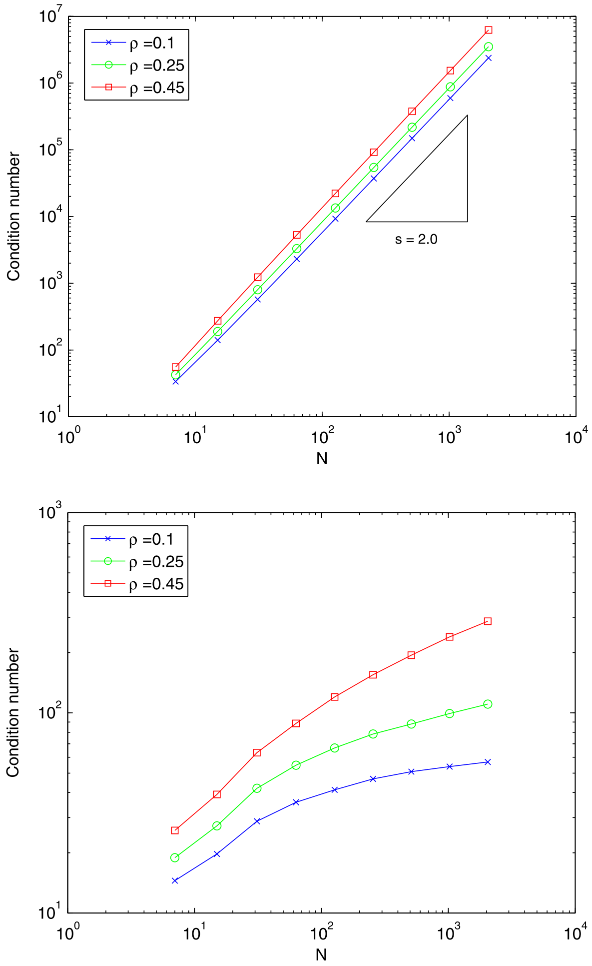


Fig. 15.1 Condition number for hat functions (*top*) and wavelets with preconditioning (*bottom*)

using in each time step $\mathcal{O}(L^5)$ *GMRES iterations, the fully discrete Galerkin scheme with incomplete GMRES gives*

$$\|u(T) - U^{dG}(T)\|_{L^2(G)} \le C\widehat{N}_L^{-s}(\log_2 \widehat{N}_L)^{(d-1)s+\varepsilon}, \quad s := p - 1 + \frac{p-1}{dp-1}.$$

Remark 15.4.8 The numerical experiments in the examples below show that the convergence rate of Theorem 15.4.7 is likely not optimal. Indeed, the experiments suggest that the error measured in L^2 satisfies the same estimate as the sparse grid projector $\widehat{P}_L$ (at least for the wavelets of Example 12.1.1), i.e.

$$\|u(T) - U^{dG}(T)\|_{L^2(G)} \le C\widehat{N}_L^{-p}(\log_2 \widehat{N}_L)^{(d-1)(p+1/2)+\varepsilon};$$

compare with Theorem 13.1.2.

Example 15.4.9 We approximate the price of a European call with strike $K = 1$ and maturity $T = 1/2$ within the Bates model (see (15.13) for its infinitesimal generator). For the one-factor model ($n_v = 1$), we choose the model parameters

$$(\alpha, \beta, \lambda_0, \lambda_1, \rho, m, \mu, \delta, r) = (2.5, 0.5, 0.5, 0, -0.5, 0.025, 0, 0.2, 0),$$

and for the two-factor model ($n_v = 2$) we let

$$\begin{aligned}&(\alpha_1, \alpha_2, \beta_1, \beta_2, \lambda_0, \lambda_1, \lambda_2, \rho_1, \rho_2, m_1, m_2, \mu, \delta, r)\\ &\quad = (1.6, 0.9, 0.5, 0.2, 0.1, 5, 0.3, -0.1, -0.3, 0.039, 0.011, -0.08, 0.1, 0).\end{aligned}$$

To obtain rates of convergence on the sparse tensor product space $\widehat{\mathcal{V}}_L$, we take $L = 4, \dots, 9$. Furthermore, in the hp-dG time stepping, we take the partition $\mathcal{M}_{M,\gamma}$ of $(0, T)$ with $M = L$, $\gamma = 0.3$ and let the slope $\mu = 0.4$ in the polynomial degree vector (see Definition 12.3.1). We measure both the error $e := u(T) - U^{dG}(T)$ in the L^2- and L^∞-norm on the domain $G_0 = (-0.25, 0.75) \times (0.01, 1.21)$ for the one-factor model and on $G_0 = (-0.25, 0.75) \times (0.25, 0.64) \times (0.25, 0.64)$ for the two-factor model. To illustrate once more the superiority of $\widehat{\mathcal{V}}_L$, we also approximate the option value for the two-factor model using the full tensor product space $\mathcal{V}_L$ with $L = 3, \dots, 6$. We find for the one-factor model

$$\|e\|_{L^2(G_0)} = \mathcal{O}\big(\widehat{N}_L^{-2}(\log_2 \widehat{N}_L)^{2.75}\big), \quad \|e\|_{L^\infty(G_0)} = \mathcal{O}\big(\widehat{N}_L^{-2}(\log_2 \widehat{N}_L)^{3.9}\big),$$

whereas for the two-factor model we have

$$\|e\|_{L^2(G_0)} = \mathcal{O}\big(\widehat{N}_L^{-2}(\log_2 \widehat{N}_L)^{5.1}\big), \quad \|e\|_{L^\infty(G_0)} = \mathcal{O}\big(\widehat{N}_L^{-2}(\log_2 \widehat{N}_L)^{5.7}\big).$$

The experimental L^2-rates are in very good agreement with the approximation property of the projector $\widehat{P}_L$, compare with Theorem 13.1.2 and Remark 15.4.8, while the rate in the L^∞-norm is slightly smaller. Note that the curse of dimension is clearly visible in the rate $\|e\|_{L^2(G_0)} = \|e\|_{L^\infty(G_0)} = \mathcal{O}(N_L^{-2/3})$ of the full tensor product space, compare with Fig. 15.2.

Example 15.4.10 We consider the BNS model and its corresponding (transformed) pricing equation (15.18). We compute the price and its sensitivity with respect to ρ of a European call with strike $K = 1$ and maturity $T = 0.5$. We choose for the background driving subordinator $L_{\lambda t}$ an IG(a, b)–OU process, for which (with $w = 1$) there holds $\varkappa(\rho) = a\rho(b^2 - 2\rho)^{-1/2}$, $k(z) = \frac{a}{2\sqrt{2\pi}} z^{-3/2}(1 + b^2 z)e^{-\frac{1}{2}b^2 z}$, see, e.g. [128] and (15.9) for the definition of $\varkappa(\rho)$. We take the parameters $(\rho, \lambda, a, b) = (0, 2.5, 0.09, 12)$. Rates of convergence (for the error measured in the L^2-norm) are calculated on the domain $G_0 = (-2, 0.5) \times (0.22, 1.1)$ using sparse tensor product spaces $\widehat{\mathcal{V}}_L$, $L = 5, \dots, 9$, and time step $\Delta t = 5 \cdot 10^{-4}$ in the backward Euler time stepping. A closed form solution using the characteristic function of the log-price process can be found in [9]. The resulting rates of convergence are shown in Fig. 15.3. We observe that for both subordinators the price and sensitivity converge with same rate, which is $\|e\|_{L^2(G_0)} = \mathcal{O}(\widehat{N}_L^{-2}(\log_2 \widehat{N}_L)^{7.5})$.

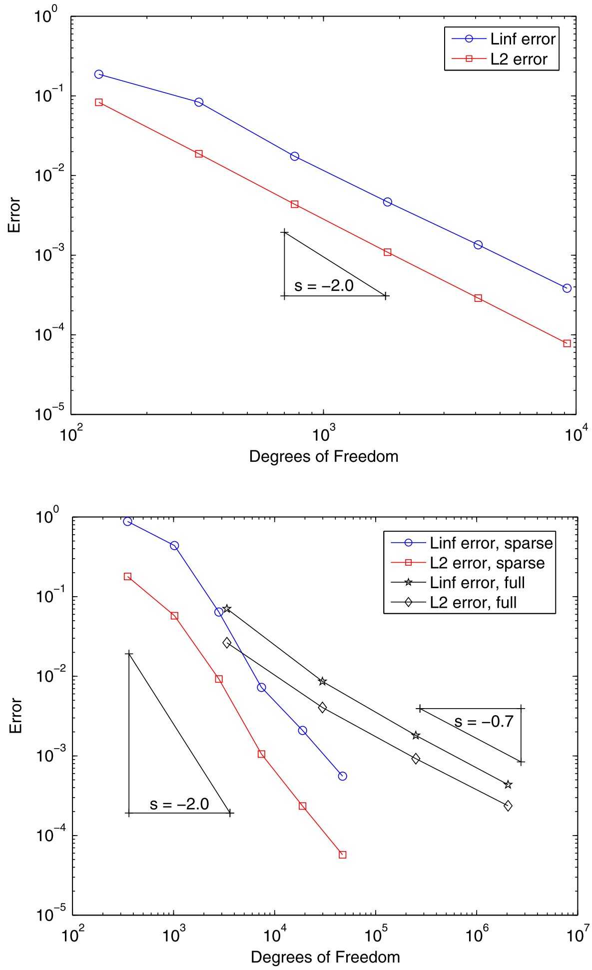


Fig. 15.2 Convergence rate of the one-factor (*top*) and two-factor (*bottom*) Bates model on sparse tensor product space

15.5 Further Reading

A stochastic volatility model with jumps comparable to the BNS model is the so-called COGARCH$(1, 1)$ process introduced by Klüppelberg et al. [105]. This model is a continuous time version of the popular GARCH model in discrete time and states that the asset log-price X is given by $\mathrm{d}X_t = \sigma_t \,\mathrm{d}L_t$, where L is a Lévy process and the jumps ΔL of this Lévy process are also used to define the volatility process σ. The model is generalized to the COGARCH(p, q) process by Brockwell et al. [30].

A large class of stochastic volatility models is described by Carr et al. [37] where the stock price process S is given as the ordinary or the stochastic exponential of a stochastic volatility process $Z_t = L_{Y_t}$, which is obtained by subordinating a Lévy process L to Y (which, for example, is given by the CIR model).

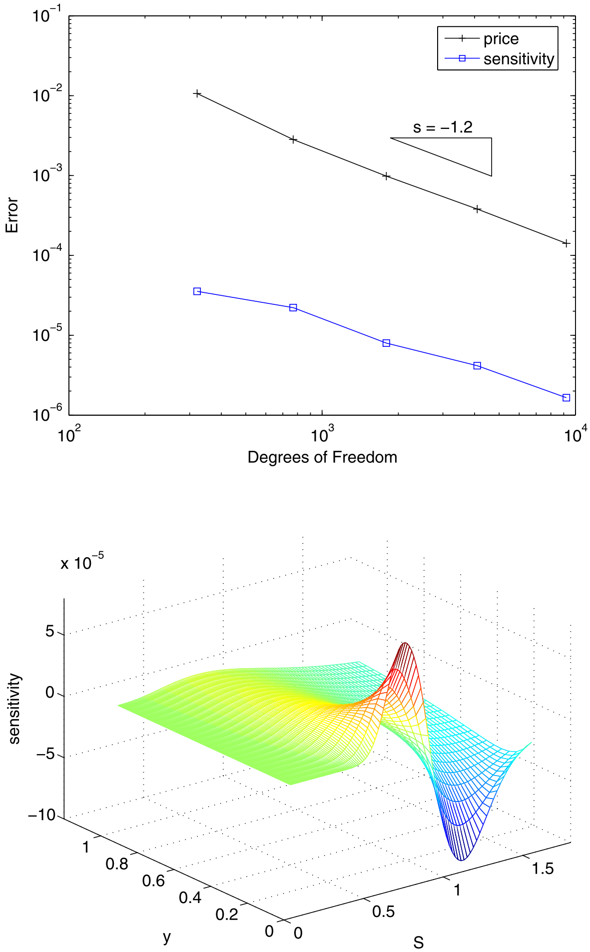


Fig. 15.3 *Top*: Convergence rate of price $u(\rho_0)$ and sensitivity $\widetilde{u}(\delta\rho)$ for IG(a, b)–OU BNS model. *Bottom*: Computed sensitivity $\widetilde{u}(\delta\rho)$ for IG(a, b)–OU

More recent works consider multivariate stochastic volatility models, where the SV models for one underlying are extended to $d > 1$ assets. In such models, the matrix Σ in (8.1) is defined, for example, as the solution of a matrix-valued SDE both without and with jumps. We refer to, e.g. Cuchiero et al. [48], Dimitroff et al. [58], Muhle-Karbe et al. [127] and the references therein.

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Chapter 16
Multidimensional Feller Processes

In this chapter, we extend the setting of Chap. 14 to a more general class of processes. We consider a large class of Markov processes in the following. Under certain assumptions we can apply the theory of pseudodifferential operators in order to analyse the arising pricing equations. The dependence structure of the purely discontinuous part of the market model X is described using Lévy copulas. Wavelets are used for the discretization and preconditioning of the arising PIDEs, which are of variable order with the order depending on the jump state.

16.1 Pseudodifferential Operators

We introduce a class of stochastic processes which are characterized via the symbol of their generator. Semimartingales are a well-investigated class of stochastic processes that is sufficiently rich to include most of the stochastic processes commonly employed in financial modelling while still being closed under various operations such as conditional expectations and stopping. Semimartingales can be well understood via their (generally stochastic) semimartingale characteristic, we refer to the standard reference [97] for details. Here, we restrict ourselves to a class of processes with deterministic, but generally state-space dependent characteristic triplets including Lévy processes, affine processes and many local volatility models. We consider an $\mathbb{R}^d$-valued Markov process X and the corresponding family of linear operators $(T_{s,t})$ for $0 \le s \le t < \infty$ given by

$$(T_{s,t}(f))(x) = \mathbb{E}[f(X(t))|X(s) = x],$$

for each $f \in B_b(\mathbb{R}^d)$, $x \in \mathbb{R}^d$. Here, $B_b(\mathbb{R}^d)$ denotes the space of bounded Borel measurable functions on $\mathbb{R}^d$. In the following, we call a Markov process X normal if its associated semigroup T satisfies

$$T_{s,t}(B_b(\mathbb{R}^d)) \subset B_b(\mathbb{R}^d) . \tag{16.1}$$

We recall the following properties:

N. Hilber et al., *Computational Methods for Quantitative Finance*, Springer Finance,
DOI 10.1007/978-3-642-35401-4_16, © Springer-Verlag Berlin Heidelberg 2013

(i) $T_{s,t}$ is a linear operator on $B_b(\mathbb{R}^d)$ for each $0 \le s \le t < \infty$.
(ii) $T_{s,s} = I$ for each $s \ge 0$.
(iii) $T_{r,s}T_{s,t} = T_{r,t}$, whenever $0 \le r \le s \le t < \infty$.
(iv) $f \ge 0$ implies $T_{s,t} f \ge 0$ for all $0 \le s \le t < \infty$ and $f \in B_b(\mathbb{R}^d)$.
(v) $\|T_{s,t}\| \le 1$ for each $0 \le s \le t < \infty$, i.e. $T_{s,t}$ is a contraction.
(vi) $T_{s,t}(1) = 1$ for all $t \ge 0$.

Here, I denotes the identity operator on $B_b(\mathbb{R}^d)$. If we restrict ourselves to time-homogeneous Markov processes satisfying (16.1), we obtain directly from the above properties that the family of operators $T_t := T_{0,t}$ forms a positivity preserving contraction semigroup. The *infinitesimal generator* $\mathcal{A}$ with domain $\mathcal{D}(\mathcal{A})$ of such a process X with semigroup $(T_t)_{t\ge 0}$ is defined by the strong pointwise limit

$$\mathcal{A}u := \lim_{t\to 0^+} \frac{1}{t}\,(T_t u - u) \tag{16.2}$$

for all functions $u \in \mathcal{D}(\mathcal{A}) \subset B_b(\mathbb{R}^d)$ for which the limit (16.2) exists with respect to the sup-norm. We call $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ the *generator* of X. Generators of normal Markov processes admit the *positive maximum principle*, i.e.

$$\text{if} \quad u \in \mathcal{D}(\mathcal{A}) \quad \text{and} \quad \sup_{x\in\mathbb{R}^d} u(x) = u(x_0) > 0, \quad \text{then} \quad (\mathcal{A}u)(x_0) \le 0. \tag{16.3}$$

Furthermore, they admit a pseudodifferential representation (e.g. [22, 44, 94, 95]):

Theorem 16.1.1 *Let $\mathcal{A}$ be an operator with domain $\mathcal{D}(\mathcal{A})$, where $C_0^\infty(\mathbb{R}^d) \subset \mathcal{D}(\mathcal{A})$ and $\mathcal{A}(C_0^\infty(\mathbb{R}^d)) \subset C(\mathbb{R})$, where $C_0^\infty(\mathbb{R}^d)$ denotes the space of smooth functions with support compactly contained in $\mathbb{R}^d$. Then, $\mathcal{A}|_{C_0^\infty(\mathbb{R}^d)}$ is a pseudodifferential operator,*

$$(\mathcal{A}u)(x) := -(2\pi)^{-d/2} \int_{\mathbb{R}^d} \psi(x,\xi)\hat{u}(\xi)e^{i\langle x,\xi\rangle}\,\mathrm{d}\xi, \tag{16.4}$$

where $u \in C_0^\infty(\mathbb{R}^d)$ and $\hat{u}$ is the Fourier transform of u. The symbol $\psi(x,\xi) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ is locally bounded in (x,ξ). The function $\psi(\cdot,\xi)$ is measurable for every ξ and $\psi(x,\cdot)$ is a negative definite function, see [94, *Definition* 3.6.5], *for every x, which admits the* Lévy–Khintchine representation

$$\begin{aligned}\psi(x,\xi) &= c(x) - i\langle b(x),\xi\rangle + \frac{1}{2}\langle \xi, Q(x)\xi\rangle \\ &\quad + \int_{0\neq z\in\mathbb{R}^d} \left(1 - e^{i\langle z,\xi\rangle} + \frac{i\langle z,\xi\rangle}{1+|z|^2}\right)\nu(x,\mathrm{d}z).\end{aligned} \tag{16.5}$$

Here, $c : \mathbb{R}^d \to \mathbb{R}$, $b : \mathbb{R}^d \to \mathbb{R}$ and $Q : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ are functions and $\nu(x,\cdot)$ is a measure on $\mathbb{R}^d$ for fixed $x \in \mathbb{R}$ with

$$\mathbb{R}^d \ni x \mapsto \int_{z\neq 0} (1 \wedge |z|^2)\nu(x,\mathrm{d}z) \tag{16.6}$$

continuous and bounded.

The tuple $(c(x), b(x), Q(x), \nu(x, \mathrm{d}z))$ in (16.5) is called the *characteristics* of the Markov process X. We sometimes denote $\mathcal{A}$ by $-\psi(x, D)$. In the following, we set $c(x) = 0$ for notational convenience and restrict ourselves to a certain kind of normal Markov processes, the so-called Feller processes ([3, Theorem 3.1.8] states (16.1) for a Feller process, see also [141, p. 83]). These can be defined via the semigroup $(T_t)_{t\geq 0}$ generated by the corresponding process X. A semigroup $(T_t)_{t\geq 0}$ is called Feller if it satisfies

(i) T_t maps $C_\infty(\mathbb{R}^d)$, the continuous functions on $\mathbb{R}^d$ vanishing at infinity, into itself:
$$T_t : C_\infty(\mathbb{R}^d) \to C_\infty(\mathbb{R}^d) \quad \text{boundedly.}$$
(ii) T_t is strongly continuous, i.e. $\lim_{t\to 0^+} \|u - T_t u\|_{L^\infty(\mathbb{R}^d)} = 0$ for all $u \in C_\infty(\mathbb{R}^d)$.

Spatially homogeneous Feller processes are Lévy processes (cf., e.g. [18, 143]). Their characteristics, the *Lévy characteristics*, do not depend on x, see Chaps. 10 and 14.

Example 16.1.2 A standard Brownian motion has the characteristics $(0, 1, 0)$. An $\mathbb{R}$-valued Lévy process has characteristics $(b, Q, \nu(\mathrm{d}z))$, for real numbers b, $Q \geq 0$ and a jump measure ν with $\int_{0\neq z\in\mathbb{R}} \min(1, z^2)\nu(\mathrm{d}z) < \infty$.

It is interesting to ask which symbols correspond to PDOs that are generators of Feller processes. This martingale problem is discussed in the following theorem due to [144].

Theorem 16.1.3 *Let $\psi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ be a negative definite symbol, i.e. a measurable and locally bounded function in both variables (x, ξ) that admits for each $x \in \mathbb{R}^d$ a Lévy–Khinchine representation (16.5). If*

(a) $\sup_{x\in\mathbb{R}^d} |\psi(x, \xi)| \leq \kappa(1 + |\xi|^2)$ *for all* $\xi \in \mathbb{R}^d$,
(b) $\xi \mapsto \psi(x, \xi)$ *is uniformly continuous at* $\xi = 0$,
(c) $x \mapsto \psi(x, \xi)$ *is continuous for all* $\xi \in \mathbb{R}^d$,

then $(-\psi(x, D), C_0^\infty(\mathbb{R}^d))$ extends to a Feller generator.

Remark 16.1.4 We show well-posedness of the pricing equations in Sect. 16.5. For the existence of a process with a given symbol, it is sufficient to require (a) from Theorem 16.1.3 and $\psi(x, 0) = 0$ for all $x \in \mathbb{R}^d$, cf. [85, Theorem 3.15], (b) and (c) are required to obtain a Feller process.

In the Lévy case, existence of a Lévy process can be proven for any Lévy symbol. This does not hold for Feller processes. For (financial) applications, it is more convenient to consider the characteristic triplet instead of the symbol. We therefore make the following assumption on the characteristic triplet in the remainder.

Assumption 16.1.5 The characteristic triplet $(b(x), Q(x), \nu(x, \mathrm{d}z))$ of a Feller process in $\mathbb{R}^d$ satisfies the following conditions:

(I) $(b(x), Q(x), \nu(x, \mathrm{d}z))$ is a Lévy triplet for all fixed $x \in \mathbb{R}^d$.
(II) The mapping $x \mapsto \int_{B\cap\mathbb{R}^d\setminus\{0\}}(1\wedge|z|^2)\nu(x, \mathrm{d}z)$ is continuous for all $B \in \mathcal{B}(\mathbb{R}^d)$.
(III) There exists a Lévy measure $\overline{\nu}(z)$ s.t.

$$0 \le \int_{B\cap\mathbb{R}^d\setminus\{0\}} (1 \wedge |z|^2)\nu(x, \mathrm{d}z) \le \int_{B\cap\mathbb{R}^d\setminus\{0\}} (1 \wedge |z|^2)\overline{\nu}(\mathrm{d}z) < \infty,$$

for all $x \in \mathbb{R}^d$, $B \in \mathcal{B}(\mathbb{R}^d)$.
(IV) The functions $x \mapsto b(x)$ and $x \mapsto Q(x)$ are continuous and bounded.

Our aim is to conclude that there exists a Feller process whose generator is a PDO for a symbol that satisfies Assumption 16.1.5. Therefore, it suffices to verify the assumptions of Theorem 16.1.3.

Lemma 16.1.6 *Let* $(b(x), Q(x), \nu(x, \mathrm{d}z))$ *be the characteristic triplet of a process* X *taking values in* $\mathbb{R}^d$ *that satisfies Assumption* 16.1.5. *Then,* $(-\psi(x, D), C_0^\infty(\mathbb{R}^d))$ *extends to a Feller generator, where* $\psi(x, \xi)$ *is given by*

$$\psi(x, \xi) = -i\langle b(x), \xi\rangle + \frac{1}{2}\langle \xi, Q(x)\xi\rangle + \int_{\mathbb{R}^d\setminus\{0\}} \left(1 - e^{i\langle z,\xi\rangle} + \frac{i\langle z, \xi\rangle}{1+|z|^2}\right)\nu(x, \mathrm{d}z). \tag{16.7}$$

Proof Condition (I) of Assumption 16.1.5 implies that the corresponding Feller symbol is negative definite. Conditions (III) and (IV) imply (a) of 16.1.3, Conditions (II) and (III) imply (b), and (c) follows from (II) and (IV). □

Remark 16.1.7 Note that real price market models, as well as Ornstein–Uhlenbeck models do not fit into our modelling framework due to Assumption (a) in Theorem 16.1.3, as they do not admit a uniform estimate in the state space variable. The numerical methods presented in the following can in many cases be straightforwardly extended to this kind of models.

In order to apply available tools from pseudodifferential calculus, we need to impose stronger assumptions on the characteristic triplets of the considered processes. We state the assumptions needed at the end of Sect. 16.4. In particular, smoothness of the characteristic triplet in the state variable x. Numerical experiments indicate strongly that these assumptions can be weakened.

16.2 Variable Order Sobolev Spaces

For later use, we shall introduce anisotropic and variable order Sobolov spaces. We start with the definition of fractional order isotropic spaces and define for a positive non-integer $s \geq 0$ and $u \in \mathcal{S}^*(\mathbb{R}^d)$, where $\mathcal{S}^*(\mathbb{R}^d)$ denotes the space of tempered distributions,

$$\|u\|^2_{H^s(\mathbb{R}^d)} := \int_{\mathbb{R}^d} (1+|\xi|^2)^s \left|\hat{u}(\xi)\right|^2 \mathrm{d}\xi. \tag{16.8}$$

Similarly, we can define anisotropic Sobolev spaces $H^{\mathbf{s}}(\mathbb{R}^d)$ with norm $\|\cdot\|_{H^{\mathbf{s}}}$ given by

$$\|u\|^2_{H^{\mathbf{s}}(\mathbb{R}^d)} := \int_{\mathbb{R}^d} \sum_{j=1}^{d} (1+\xi_j^2)^{s_j} \left|\hat{u}(\xi)\right|^2 \mathrm{d}\xi, \tag{16.9}$$

for any multi-index $\mathbf{s} \geq 0$. The consideration of certain symbol classes will be useful for the definition of the variable order Sobolev spaces. We set $\langle\xi\rangle := (1+|\xi|^2)^{1/2}$ for notational convenience.

Definition 16.2.1 Let $0 \leq \delta < \rho \leq 1$ and let $m(x) \in C^\infty(\mathbb{R}^d)$ be a real-valued function with bounded derivatives on $\mathbb{R}^d$ of arbitrary order. Then, the symbol $\psi(x,\xi)$ belongs to the class $S^{m(x)}_{\rho,\delta}$ of symbols of variable order $m(x)$ if $\psi(x,\xi) \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ and $m(x) = s + \widetilde{m}(x)$ with $\widetilde{m} \in \mathcal{S}(\mathbb{R}^d)$ a tempered function, and if, for every $\alpha, \beta \in \mathbb{N}_0^d$ there is a constant $c_{\alpha,\beta}$ such that

$$\forall x,\xi \in \mathbb{R}^d: \quad |D_x^\beta D_\xi^\alpha \psi(x,\xi)| \leq c_{\alpha,\beta} \langle\xi\rangle^{m(x)-\rho|\alpha|+\delta|\beta|}. \tag{16.10}$$

The variable order pseudodifferential operators $\mathcal{A}(x,D) \in \Psi^{m(x)}_{\rho,\delta}$ correspond to symbols $\psi(x,\xi) \in S^{m(x)}_{\rho,\delta}$ by

$$\mathcal{A}(x,D)u(x) := \frac{1}{2\pi} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i\langle x-y,\xi\rangle} \psi(x,\xi) u(y)\, \mathrm{d}y\, \mathrm{d}\xi, \qquad u \in C_0^\infty(\mathbb{R}^d). \tag{16.11}$$

We are now able to define an isotropic Sobolev space of variable order $H^{m(x)}(\mathbb{R}^d)$, $m(x) \geq 0$, using the variable order Riesz potential $\Lambda^{m(x)}$ with symbol $\psi(x,\xi) = \langle\xi\rangle^{m(x)}$. Clearly, $\psi(x,\xi)$ is an element of $S^{m(x)}_{1,\delta}$ for $\delta \in (0,1)$. The norm on $H^{m(x)}(\mathbb{R}^d)$ is given as

$$\|u\|^2_{H^{m(x)}(\mathbb{R}^d)} := \|\Lambda^{2m(x)} u\|^2_{L^2(\mathbb{R}^d)} + \|u\|^2_{L^2(\mathbb{R}^d)}.$$

Note that for $\psi(x,\xi) = 1$, we obtain the usual $L^2(\mathbb{R}^d)$-norm. For $\psi(x,\xi) = (1 + |\xi|^s)$, we obtain the norm given in (16.8), which follows by applying Plancherel's theorem. Now we turn to the definition of anisotropic variable order Sobolev spaces. In analogy to Definition 16.2.1, we start with the definition of an appropriate symbol class.

Definition 16.2.2 Let $\mathbf{m}(x) = s + \widetilde{\mathbf{m}}(x)$, $\widetilde{\mathbf{m}}(x) : \mathbb{R}^d \to \mathbb{R}^d$ with each component of $\widetilde{\mathbf{m}}(x)$ being a tempered function and $s \in \mathbb{R}^d_+$, $0 \leq \delta < \rho \leq 1$. We define the symbol class $S^{\mathbf{m}(x)}_{\rho,\delta}$ as the set of all $\psi(x,\xi) \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ such that for all multi-indices $\alpha, \beta \in \mathbb{N}_0^d$ there exists a constant $C_{\alpha,\beta} > 0$ with

$$\forall x, \xi \in \mathbb{R}^d : \quad \left| D_x^\beta D_\xi^\alpha \psi(x,\xi) \right| \le C_{\alpha,\beta} \sum_{i=1}^d (1+\xi_i^2)^{(m_i(x)-\rho\alpha_i+\delta|\beta|)/2}. \tag{16.12}$$

An anisotropic Sobolev space of variable order can now be defined using the variable order Riesz potential $\Lambda^{\mathbf{m}(x)}$ with symbol $\psi(x,\xi) = \langle \xi \rangle^{\mathbf{m}(x)} := \sum_{i=1}^n (1 + \xi_i^2)^{\frac{1}{2} m_i(x)}$, $m_i(x_i) \ge 0$, $i = 1, \dots, d$. Clearly, $\psi(x,\xi)$ is an element of $S^{\mathbf{m}(x)}_{1,\delta}$ for $\delta \in (0,1)$. The norm on $H^{\mathbf{m}(x)}$ is given by

$$\|u\|^2_{H^{\mathbf{m}(x)}} := \|\Lambda^{2\mathbf{m}(x)} u\|^2_{L^2(\mathbb{R}^d)} + \|u\|^2_{L^2(\mathbb{R}^d)}.$$

There is an alternative representation of the above space, when $\mathbf{m}(x)$ admits the following representation $\mathbf{m}(x) = (m_1(x_1), \dots, m_d(x_d))$. This is very useful for the proof of norm equivalences, which play a crucial role in wavelet discretization theory, we refer to [137] for details. We consider the anisotropic Sobolev spaces $H_i^{m_i(x_i)}$ of variable order $m_i(x_i)$ in direction x_i, equipped with the following norms:

$$\|u\|^2_{H_i^{m(x)}} := \|\Lambda_i^{2m_i(x_i)} u\|^2_{L^2(\mathbb{R}^d)} + \|u\|^2_{L^2(\mathbb{R}^d)},$$

where $\Lambda_i^{m_i(x_i)}$ is a pseudodifferential operator with symbol $(1+|\xi_i|)^{m_i(x_i)}$. It then follows by the elementary inequality

$$C_1 \left| \sum_{i=1}^d a_i \right|^2 \le \sum_{i=1}^d a_i^2 \le C_2 \left| \sum_{i=1}^d a_i \right|^2,$$

with $a_i > 0$ and C_1, C_2 only dependent on d, that

$$\|u\|^2_{H^{m(x)}(\mathbb{R}^d)} \sim \sum_{j=1}^d \|u\|^2_{H_j^{m_j(x_j)}(\mathbb{R}^d)}, \tag{16.13}$$

and therefore,

$$H^{\mathbf{m}(x)}(\mathbb{R}^d) = \bigcap_{j=1}^d H_j^{m_j(x_j)}(\mathbb{R}^d). \tag{16.14}$$

On the bounded set $G = (\mathbf{a}, \mathbf{b}) = \prod_{i=1}^d (a_i, b_i) \subset \mathbb{R}^d$, we define for a variable order $\mathbf{m}(x)$, $\mathbf{a} \le x \le \mathbf{b}$ the space

$$\widetilde{H}^{\mathbf{m}(x)}(G) := \left\{ u|_G \;\middle|\; u \in H^{\mathbf{m}(x)}(\mathbb{R}^d),\ u|_{\mathbb{R}^d \setminus \overline{G}} = 0 \right\}.$$

This space coincides with the closure of $C_0^\infty(G)$ (the space of smooth functions with support compactly contained in G) with respect to the norm

$$\|u\|_{\widetilde{H}^{\mathbf{m}(x)}(G)} := \|\widetilde{u}\|_{H^{\mathbf{m}(x)}(\mathbb{R}^d)}, \tag{16.15}$$

where $\widetilde{u}$ is the zero extension of u to all of $\mathbb{R}^d$. The spaces of order $\mathbf{m}(x) \leq 0$, $\forall x \in \mathbb{R}^d$, are defined by duality. We have

$$H^{\mathbf{m}(x)}(G) := (\widetilde{H}^{-\mathbf{m}(x)}(G))^*,$$

where duality is understood with respect to the "pivot" space $L^2(G)$, i.e. $L^2(G)^* \cong L^2(G)$.

Remark 16.2.3 In the Black–Scholes case, $H^1(\mathbb{R}^d)$ is obtained as the domain of the corresponding bilinear form while $H_0^1(G)$ is the domain in the localized case, see Chap. 4. In the Lévy case, we obtain anisotropic Sobolev spaces as in (16.9) and the spaces $\widetilde{H}^{\mathbf{s}}(G)$ in the localized case for $Q = 0$, see Chap. 14. For $Q > 0$ the domains are equal to those in the Black–Scholes case, cf. [134, Theorem 4.8].

Several examples of Feller processes are provided in this section. The methods presented in this work are not applicable to all of them. But admissible market models are derived ensuring the well-posedness of the corresponding pricing equations and the applicability of finite element methods. Subordination can be used to construct Feller processes. However, the structure of the symbol is more involved than in the Lévy setting. We also present a construction of Feller processes using Lévy copulas.

16.3 Subordination

Many Lévy models in the context of option pricing are constructed via subordination of a Brownian motion by a corresponding stochastic clock, e.g. an NIG process [8] or a VG process [119]. We describe a similar construction for Feller processes and point out similarities and differences to the Lévy case. Bernstein functions play a crucial role in the representation of subordinators.

Definition 16.3.1 A function $f(x) \in C^\infty(0, \infty)$ is called a Bernstein function if

$$f \geq 0, \quad (-1)^k \frac{\partial^k f(x)}{\partial x^k} \leq 0, \quad \forall k \in \mathbb{N}.$$

Example 16.3.2 The functions $f_1(x) = c$, $f_2(x) = cx$ and $f_3(x) = 1 - e^{-cx}$, $c \geq 0$ are Bernstein functions.

Bernstein functions admit the following representation.

Theorem 16.3.3 *For any Bernstein function $f(x)$ there exists a measure μ on $(0, \infty)$, with*

$$\int_{0+}^{\infty} \frac{s}{1+s} \mu(\mathrm{d}s) < \infty$$

such that for $x > 0$ and positive constants a, b

$$f(x) = a + bx + \int_{0+}^{\infty} (1 - e^{-xs})\mu(\mathrm{d}s).$$

Proof See [94, Theorem 3.9.4]. □

If we choose as Lévy process a subordinator, Bernstein functions allow a complete characterization of the process. It is based on the observation that subordinators can be described via their convolution semigroup.

Definition 16.3.4 A family $(\eta_t)_{t\geq 0}$ of bounded Borel measures on $\mathbb{R}$ is called convolution semigroup on $\mathbb{R}$ if the following conditions are fulfilled:

(i) $\eta_t(\mathbb{R}) \leq 1$, for all $t \geq 0$,
(ii) $\eta_s * \eta_t = \eta_{s+t}$, $s, t \geq 0$ and $\mu_0 = \delta_0$,
(iii) $\eta_t \to \delta_0$ vaguely as $t \to 0$,

where δ_0 denotes the Dirac measure at 0. By vague convergence of a sequence $(\eta_t)_{t>0}$ of measures to η_0 we mean that for all continuous functions with compact support $u \in C_0(\mathbb{R})$, we have

$$\lim_{t\to 0} \int_{\mathbb{R}} u(x)\eta_t(\mathrm{d}x) = \int_{\mathbb{R}} u(x)\eta_0(\mathrm{d}x).$$

The relation between convolution semigroups and Bernstein functions is given in the following theorem.

Theorem 16.3.5 *Let $f : (0, \infty) \to \mathbb{R}$ be a Bernstein function. Then, there exists a unique convolution semigroup $(\eta_t)_{t\geq 0}$ supported on $[0, \infty)$ such that*

$$\mathcal{L}(\eta_t)(x) = e^{-tf(x)}, \quad x > 0 \text{ and } t > 0, \tag{16.16}$$

holds, where $\mathcal{L}$ denotes the Laplace transform, i.e. $\mathcal{L}(\eta_t)(x) := \int_0^{\infty} e^{-zx}\eta_t(\mathrm{d}z)$, for appropriate η_t and $x > 0$. Conversely, for any convolution semigroup $(\eta_t)_{t\geq 0}$ supported by $[0, \infty)$ there exists a unique Bernstein function f such that (16.16) holds.

Proof See, e.g. [94, Theorem 3.9.7]. □

We recall the correspondence between convolution semigroups and Lévy processes.

Theorem 16.3.6 *Let X be a Lévy process, where for each $t \geq 0$ $X(t)$ has law η_t, then $(\eta_t)_{t\geq 0}$ is convolution semigroup.*

Proof See [3, Proposition 1.4.4]. □

The semigroup of subordinated Feller processes can now be characterized.

Theorem 16.3.7 *Let $(T_t)_{t\geq 0}$ be a Feller semigroup on a Banach space $\mathcal{H}$ with generator $\mathcal{A}$, let $f:(0,\infty)\to\mathbb{R}$ be a Bernstein function and let $(\eta_t)_{t\geq 0}$ denote the associated convolution semigroup on $\mathbb{R}$ supported on $[0,\infty)$. Define $T_t^f u$ for $u\in\mathcal{H}$ by the Bochner integral*

$$T_t^f u = \int_0^\infty T_s u\,\eta_t(\mathrm{d}s). \tag{16.17}$$

Then, the integral is well-defined and $(T_t^f)_{t\geq 0}$ is a Feller semigroup on $\mathcal{H}$.

Proof See [94, Theorem 4.3.1 and Corollary 4.3.4]. □

The representation (16.17) of $T_t^f u$ can be used for numerical methods, but it involves the approximation of an integral over a possibly semi-infinite interval, which can be very costly if the integrand is not well-behaved. The generator $\mathcal{A}^f$ of the semigroup $(T_t^f)_{t\geq 0}$ is a PDO and, in the case of a spatially homogeneous semigroup $(T_t)_{t\geq 0}$, its symbol corresponds to a Lévy process X given as follows.

Theorem 16.3.8 *Let $(T_t)_{t\geq 0}$ be a Feller semigroup with generator $\mathcal{A}$ with constant symbol $\psi(\xi)$ and $f(x)$ as in the previous theorem, then the symbol $\psi^f(\xi)$ of the generator $\mathcal{A}^f$ of the semigroup $(T_t^f)_{t\geq 0}$ is given as*

$$\psi^f(\xi) = f(\psi(\xi))\,.$$

Proof See [3, Proposition 1.3.27]. □

The same characterization does not hold in the case of a more general subordinated process. We obtain the following representation for $\mathcal{A}^f$.

Theorem 16.3.9 *Let f and $(T_t)_{t\geq 0}$ be as in Theorem 16.3.7. For all $u\in D(\mathcal{A})$, we have $u\in D(\mathcal{A}^f)$ and*

$$\mathcal{A}^f u = au + b\mathcal{A}u + \int_0^\infty R_\lambda \mathcal{A}u\,\mu(\mathrm{d}\lambda),$$

where R_λ denotes the resolvent of $\mathcal{A}$, i.e. $R_\lambda u = (\lambda - \mathcal{A})^{-1}u$ and a, b and $\mu(ds)$ are defined in Theorem 16.3.5.

Proof See [96, Theorem 2.15]. □

This resolvent representation is of limited use for numerical computation, and we therefore aim at a characterization of the symbol of the PDO $\mathcal{A}^f$. We consider a certain symbol class as in [96]. Let $L(x,D)$ be the differential operator given by

$$L(x,D) = -\sum_{i,j=1}^{d} a_{i,j}(x)\frac{\partial^2}{\partial x_i\partial x_j} + c(x),$$

where $a_{i,j}(x)$, $1 \le i, j \le d$ are continuously differentiable functions such that $a_{i,j}(x) = a_{j,i}(x)$ and

$$\kappa_1 |\xi|^2 \le \sum_{i,j=1}^{d} a_{i,j}(x)\xi_i\xi_j \le \kappa_2 |\xi|^2$$

holds for all $x \in \mathbb{R}^d$ and $\xi \in \mathbb{R}^d$ for some constants $0 < \kappa_1 \le \kappa_2$. Besides, we assume

$$\sum_{i=1}^{d} \frac{\partial a_{i,j}}{\partial x_i} = 0,$$

for any $j = 1, \ldots, d$ and $c(x)$ is a continuous and bounded function satisfying $0 < \underline{c} \le c(x) \le \overline{c} < \infty$. In this situation, we obtain the following representation of $\mathcal{A}^f = f(\mathcal{A})$ for $u \in D(\mathcal{A})$, $\mathcal{A} = L(x, D)$.

$$f(\mathcal{A})u = f(L(x,\xi))u + \int_0^\infty \lambda R_\lambda K_\lambda(x, D)u\mu(\mathrm{d}\lambda), \tag{16.18}$$

where

$$\begin{aligned} K_\lambda(x, D) &= (L(x, D) + \lambda\,\mathrm{id}) \circ q_\lambda(x, D) - \mathrm{id}, \\ q_\lambda(x,\xi) &= \frac{1}{L(x,\xi) + \lambda\xi}, \\ L(x,\xi) &= \sum_{i,j=1}^{d} a_{i,j}\xi_i\xi_j + c(x) \end{aligned}$$

and R_λ denotes the resolvent of $\mathcal{A}$ at λ. We remark that $K_\lambda \equiv 0$ for constant symbols, which proves Theorem 16.3.8. In general, both terms in (16.18) have to be considered. We refer to Carr [34] for a generalization of the VG model.

Remark 16.3.10 The consideration of symbols of the type $a(x,\xi)$, where $a(x,\xi)$ is a Lévy symbol for all $x \in \mathbb{R}$ is therefore in general not equivalent to a construction via subordination, i.e. for $\mathcal{A} = L(x, D)$ with symbol $\psi(x,\xi)$ the symbol of $\mathcal{A}^f$ is not given as $f(\psi(x,\xi))$. It has a more involved structure as described above. This observation was made by [7], where some asymptotic expansion of the difference in terms of the symbol under certain assumptions on the structure of the process was provided.

16.4 Admissible Market Models

We now formulate the requirements for market models which are admissible for our pricing schemes in terms of the marginals and the copula function. These requirements not only ensure existence and uniqueness of a solution of the corresponding pricing problem, but also ensure that the presented FEM based algorithms are feasible.

Definition 16.4.1 We call a d-dimensional Feller process with characteristic triplet $(\gamma(x), Q(x), \nu(x, \mathrm{d}z))$ a time-homogeneous *admissible market model* if it satisfies the following properties.

(i) The vector function $x \mapsto b(x) \in \mathbb{R}^d$ is smooth and bounded.
(ii) The matrix function $x \mapsto Q(x) \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ is smooth and bounded and, for all $x \in \mathbb{R}^d$, the matrix $Q(x)$ is positive semidefinite.
(iii) The jump measure $\nu(x, \mathrm{d}z)$ is constructed from d independent, univariate Feller–Lévy measures with a 1-homogeneous copula function F that fulfills the following estimate: there is a constant $C > 0$ such that for all $u \in (\mathbb{R}\backslash\{0\})^d$ and all $n \in \mathbb{N}_0^d$ it holds

$$\left|\partial^n F(u)\right| \le C^{|n|+1}\, |n|! \min\{|u_1|, \dots, |u_d|\} \prod_{i=1}^{d} |u_i|^{-n_i}\,.$$

(iv) For the marginal densities $\nu_i(x_i, \mathrm{d}z_i) = k_i(x, z)\,\mathrm{d}z$ the mapping $x_i \mapsto \nu_i(x_i, B)$ is smooth for all $B \in \mathcal{B}(\mathbb{R})$.
(v) There exist univariate Lévy kernels $\overline{k}_i(z)$, $i = 1, \dots, d$, with semi-heavy tails, i.e. which satisfy

$$\overline{k}_i(z) \le C \begin{cases} e^{-\beta^-|z|}, & z < -1, \\ e^{-\beta^+ z}, & z > 1, \end{cases} \tag{16.19}$$

for some constants $C > 0$, $\beta^- > 0$ and $\beta^+ > 1$. These Lévy kernels satisfy the following estimates

$$0 \le \nu_i(x_i, B) \le \int_B \overline{k}_i(z)\,\mathrm{d}z \quad \forall x_i \in \mathbb{R}, \quad B \in \mathcal{B}(\mathbb{R}),\ i = 1, \dots, d.$$

(vi) Besides, we require the following estimates on the derivatives of $k_i(x, z)$

$$\left|\partial_x^n k_i(x, z)\right| \le C^{n+1} n!\, |z|^{-Y_i(x)-\delta n-1}\,,$$
$$\left|\partial_z^n k_i(x, z)\right| \le C^{n+1} n!\, |z|^{-Y_i(x)-n-1}\,,$$

for any $\delta \in (0, 1)$, for all $0 \ne z, x \in \mathbb{R}$ and $\overline{Y} < 2$ as well as $\underline{Y} > 0$, $Y_i(x) = \widehat{Y}_i + \widetilde{Y}_i(x)$, $\widehat{Y}_i \in \mathbb{R}_+$ and $\widetilde{Y}_i(x) \in \mathcal{S}(\mathbb{R})$, $i = 1, \dots, d$.
(vii) Finally, we require F^0 to be a 1-homogeneous Lévy copula and $k_i^0(x_i, z_i)$ to be $Y_i(x_i)$-stable densities with tail integrals $U_i^0(x_i, z_i)$, $i = 1, \dots, d$ such that

$$k_i(x, z) \ge C k_i^0(x, z), \quad \forall 0 < |z| < 1, \forall x \in \mathbb{R}, \quad i = 1, \dots, d,$$
$$\partial_1 \cdots \partial_n F(U(x, z)) \ge C \partial_1 \cdots \partial_n F^0(U^0(x, z)) \quad \forall 0 < |z| < 1,$$

for some constant $C > 0$.

Remark 16.4.2 Note that the smoothness assumptions on $Y_i(x_i)$, $i = 1, \dots, d$, are necessary in order to obtain symbols as given in Definition 16.2.2. We confine the discussion to such symbols in order to use the results of [137] that rely on pseudodifferential calculus for symbols of variable order, cf. [86, 104]. The derivation of similar results for symbols with lower regularity is open to our knowledge.

In the following lemma, we characterize the symbol classes of admissible market model. This is crucial for the well-posedness of the pricing equation as discussed in Sect. 16.5.

Lemma 16.4.3 *The symbol $\psi(x,\xi)$ of a time-homogeneous admissible market model given as*

$$\psi(x,\xi) = -i\langle b(x),\xi\rangle + \frac{1}{2}\langle \xi, Q(x)\xi\rangle + \int_{0\neq z\in\mathbb{R}^d}\Big(1-e^{i\langle z,\xi\rangle}+i\langle z,\xi\rangle\Big)\,\nu(x,\mathrm{d}z)$$

is contained in the following symbol classes:

$$\begin{cases} \psi(x,\xi)\in S^2_{1,\delta} & \text{for } Q(x)\geq Q_0>0,\\ \psi(x,\xi)\in S^{\mathbf{Y}(x)}_{1,\delta} & \text{for } Q=0,\ \gamma=0,\\ \psi(x,\xi)\in S^{2\widetilde{\mathbf{m}}(x)}_{1,\delta} & \text{for } Q=0,\ \gamma\neq 0,\end{cases}$$

where $\delta\in(0,1)$ and $\widetilde{m}_i(x_i)=\frac{\max(Y_i(x_i),1)}{2}$, $i=1,\dots,d$.

Proof We have, analogous to [134, Proposition 3.5],

$$\forall\xi\in\mathbb{R}^d:\quad \int_{\mathbb{R}^d}\Big(1-e^{i\langle z,\xi\rangle}+i\langle z,\xi\rangle\Big)\,\nu(x,\mathrm{d}z)\leq C_1\sum_{i=1}^d|\xi_i|^{Y_i(x_i)},$$

for some positive constant $C_1, C_2, C_3 > 0$. The following estimate holds for the diffusion and the drift component:

$$\forall\xi\in\mathbb{R}^d:\quad \left|\frac{1}{2}\langle\xi, Q(x)\xi\rangle\right|\leq C_2\sum_{i=1}^d|\xi_i|^2 \text{ and } |i\langle b(x),\xi\rangle|\leq C_3\sum_{i=1}^d|\xi_i|. \qquad \square$$

Remark 16.4.4 The partially degenerate case $Q\neq 0$, but $Q\not>0$ can be analyzed as in [134, Remark 4.9]. Note that in the case $\gamma\neq 0$ and $Q=0$ additional assumptions on the behavior of $Y_i(x_i)$ at 1 are necessary in order to ensure the smoothness of $\widetilde{m}_i(x_i)$, $i=1,\dots,d$.

The infinitesimal generator $\mathcal{A}$ of a time-homogeneous admissible market model X reads

$$\begin{aligned} \mathcal{A}\varphi(x) &= \mathcal{A}_{\mathrm{Tr}}\varphi(x)+\mathcal{A}_{\mathrm{BS}}\varphi(x)+\mathcal{A}_{\mathrm{J}}\varphi(x),\\ \mathcal{A}_{\mathrm{Tr}}\varphi(x) &= b(x)^\top\nabla\varphi(x),\\ \mathcal{A}_{\mathrm{BS}}\varphi(x) &= \frac{1}{2}\mathrm{tr}\big[\mathcal{Q}(x)D^2\varphi(x)\big],\\ \mathcal{A}_{\mathrm{J}}\varphi(x) &= \int_{\mathbb{R}^d}\Big(\varphi(x+z)-\varphi(x)-z^\top\nabla\varphi(x)\Big)\,\nu(x,\mathrm{d}z), \end{aligned} \tag{16.20}$$

for $\varphi\in C^\infty(\mathbb{R}^d)$.

Note that the (non-constant) symbol of the infinitesimal generator of X does generally not coincide with the characteristic exponent of the process X, due to the spatial inhomogeneity of X. We illustrate the preceding, abstract developments with an example related to the so-called *tempered-stable* class of Lévy processes which were advocated in recent years in the context of financial modeling.

Example 16.4.5 (Feller–CGMY) We consider a d-dimensional Feller process with Clayton Lévy copula

$$F(u_1, \dots, u_d) = 2^{2-d} \left(\sum_{i=1}^{d} |u_i|^{\vartheta} \right)^{-\frac{1}{\vartheta}} \left(\rho \mathbb{1}_{\{u_1,\dots,u_d \ge 0\}} - (1-\rho) \mathbb{1}_{\{u_1,\dots,u_d \le 0\}} \right),$$

where $\vartheta > 0$, $\rho \in [0, 1]$ together with CGMY-type densities

$$k_i(x, z) = C(x) \left(\frac{e^{-\beta_i^-(x)|z|}}{|z|^{1+Y_i(x)}} \mathbb{1}_{\{z<0\}} + \frac{e^{-\beta_i^+(x)|z|}}{|z|^{1+Y_i(x)}} \mathbb{1}_{\{z>0\}} \right),$$

with smooth and bounded functions $C(x) > 0$, $\beta_i^-(x) > 0$, $\beta_i^+(x) > 1$ and $0 < \underline{Y}_i < Y_i(x) \le \overline{Y}_i < 2$, $Y_i(x)$ sufficiently smooth, for $i = 1, \dots, d$. We assume the Gaussian component $Q(x)$ to be positive semidefinite, smooth and bounded. The drift $\gamma(x)$ is assumed to be smooth and bounded. It is easy to see that this market model satisfies properties (i), (ii), (iv)–(vi) of the above definition. Properties (iii) and (vii) follow analogously to the proof of [163, Proposition 2.3.7].

16.5 Variational Formulation

We first prove a sector condition for admissible market models, which is crucial for the proof of well-posedness of the pricing equation. Subsequently, well-posedness results for certain types of pricing equations are given. These equations are of parabolic type, with the highest order operator being the diffusion or jump part of the generator of the market model.

16.5.1 Sector Condition

The sector condition for the symbol $\psi(x, \xi)$ of a Feller process X is one of the main ingredients for proving well-posedness of the initial boundary value problems for the PIDEs arising in option pricing problems. The sector condition reads:

$$\exists C > 0 \quad \text{s.t. } \forall x, \xi \in \mathbb{R}^d : \qquad \Re\, \psi(x, \xi) + 1 \ge C \langle \xi \rangle^{2\mathbf{m}(x)}. \qquad (16.21)$$

Theorem 16.5.1 *Let X be an admissible time-homogeneous market model process taking values in $\mathbb{R}^d$ with characteristic triplet $(b(x), Q(x), k(x, z)\,\mathrm{d}z)$. Then, there* *exists a constant* $C > 0$ *such that for all* $x \in \mathbb{R}^d$ *and* $\|\xi\|_\infty$ *sufficiently large*

$$\Re\psi(x,\xi) \geq C \sum_{j=1}^{d} |\xi|^{Y_j(x_j)}, \tag{16.22}$$

where $Y_j(x_j) = 2$ *in the case* $Q_0 \geq Q > 0$.

Proof The proof mainly follows the arguments of [163, Proposition 2.4.3]. We refer to [135, Theorem 5.1.6]. □

16.5.2 Well-Posedness

For an admissible time-homogeneous market model X, we can derive a PDO and PIDE representation and prove well-posedness of the weak formulation of the problem on a bounded domain. Due to no arbitrage considerations, we require the considered processes to be martingales under a pricing measure $\mathbb{Q}$. This requirement can be expressed in terms of the characteristic triplet.

Lemma 16.5.2 *Let* X *be an admissible time-homogeneous market model with characteristic triplet* $(b(x), Q(x), \nu(x, \mathrm{d}z))$ *and semigroup* $(T_t)_{t\geq 0}$ *further let* $T_t(e^{x_j}) < \infty$ *hold for* $t \geq 0$, $j = 1, \ldots, d$. *Then,* e^{X_j} *is a* $\mathbb{Q}$*-martingale with respect to the canonical filtration of* X *if and only if*

$$\frac{Q_{jj}(x)^2}{2} + b_j(x) + \int_{0\neq y\in\mathbb{R}} (e^{z_j} - 1 - z_j)\nu_j(x, \mathrm{d}z_j) = 0 \quad \forall x \in \mathbb{R}. \tag{16.23}$$

Proof This is a direct consequence of [61, Sect. 3]. □

Remark 16.5.3 Note that without the assumption of finiteness of exponential moments of the processes X_j, the processes e^{X_j}, $j = 1, \ldots, d$ would generally only be local martingales. For Lévy processes, exponential decay of the jump measure implies the existence of exponential moments, cf. [143, Theorem 25.3]. This is not obvious for general Feller processes. Recently, Knopova and Schilling have proved in [106] the finiteness of exponential moments for a certain class of Feller processes assuming exponential decay of the density of the jump measure.

We are now able to derive a PDO and PIDE for option prices. Let the stochastic process X be an admissible time-homogeneous market model with generator $\mathcal{A}$ and let be g be sufficiently smooth and $\mathcal{V} = H^1(\mathbb{R}^d)$ for diffusion market models, $\mathcal{V} = H^{\mathbf{m}}(\mathbb{R}^d)$, $\mathbf{m} = [Y_1/2, \ldots, Y_d/2] \in (0,1)^d$ for general space and time-homogeneous models and $\mathcal{V} = H^{\mathbf{m}(x)}(\mathbb{R}^d)$ as in Definition 16.2.2, with $\mathbf{m}(x) = [Y_1(x_1)/2, \ldots, Y_d(x_d)/2]$, for time-homogeneous admissible market models considered here. Then, we obtain formally from semigroup theory and from the representation $u(t,x) = T_t(g) = e^{-r(T-t)}\mathbb{E}[g(X_t)|X_0 = x]$ by differentiation in t the following backward PIDE

$$\partial_t u + \mathcal{A}u - ru = 0 \quad \text{in } (0,T) \times \mathbb{R}^d, \tag{16.24}$$

$$u(T) = g \quad \text{in } \mathbb{R}^d\,. \tag{16.25}$$

We set $t_0 = 0$ for notational convenience. Testing with a function $v \in \mathcal{V}$ and transforming to time-to-maturity, we end up with the following parabolic evolution problem: find $u \in L^2((0,T);\mathcal{V}) \cap H^1((0,T);\mathcal{V}^*)$ s.t. for all $v \in \mathcal{V}$ and a.e. $t \in [0,T]$ the following holds:

$$\langle \partial_t u, v\rangle_{\mathcal{V}^* \times \mathcal{V}} + a(u,v) = 0, \qquad u(0) = g, \tag{16.26}$$

where *the bilinear form* $a(\varphi,\phi) = \langle -\mathcal{A}\varphi, \phi\rangle_{\mathcal{V}^*,\mathcal{V}} + r(\varphi,\phi)_{L^2(\mathbb{R}^d)}$ *is closely related to the Dirichlet form of the stochastic process* X. Although in option pricing, only the homogeneous parabolic problem (16.26) arises, the inhomogeneous equation (16.27) is useful in many applications. We mention only the computation of the time-value of an option, or the computation of quadratic hedging strategies and the corresponding hedging error. Thus, we consider the nonhomogeneous analogue of the above equation. The general problem reads: Find $u \in L^2((0,T);\mathcal{V}) \cap H^1((0,T);\mathcal{V}^*)$ s.t.

$$\langle \partial_t u, v\rangle_{\mathcal{V}^* \times \mathcal{V}} + a(u,v) = \langle f, v\rangle_{\mathcal{V}^* \times \mathcal{V}} \quad \text{in } (0,T),\ \forall v \in \mathcal{V} \qquad u(0) = g \tag{16.27}$$

for some $f \in L^2((0,T);\mathcal{V}^*)$. Now we consider the localization of the unbounded problem to a bounded domain G. For any function u with support in a bounded domain $G \subset \mathbb{R}^d$, we denote by $\widetilde{u}$ the zero extensions of u to $G^c = \mathbb{R}^d \backslash \overline{G}$ and define $\mathcal{A}_G(u) = \mathcal{A}(\widetilde{u})$. The variational formulation of the pricing equation on a bounded domain $G \subset \mathbb{R}^d$ reads: Find $u \in L^2((0,T);\mathcal{V}_G) \cap H^1((0,T);(\mathcal{V}_G)^*)$ s.t. for all $v \in \mathcal{V}_G$ and a.e. $t \in [0,T]$ the following holds:

$$\langle \partial_t u, v\rangle_{\mathcal{V}_G^* \times \mathcal{V}_G} + a_G(u,v) = \langle f, v\rangle_{\mathcal{V}_G^* \times \mathcal{V}_G}, \tag{16.28}$$

$$u(0) = g|_G, \tag{16.29}$$

where $a_G(u,v) := a(\widetilde{u},\widetilde{v})$. The spaces $\mathcal{V}_G =: \{v \in L^2(G) : \widetilde{v} \in \mathcal{V}\}$ consist of functions which vanish in a weak sense on ∂G. A comparable result for general Feller processes does not appear to be available, yet.

Remark 16.5.4 Formulation (16.28)–(16.29) naturally arises for payoffs with finite support such as digital or (double) barrier options. The truncation to a bounded domain can thus be interpreted economically as the approximation of a standard derivative contract by a corresponding barrier option on the same market model.

Existence and uniqueness of weak solutions of (16.28)–(16.29) follows from continuity of the bilinear form $a_G(\cdot,\cdot)$ and a Gårding inequality which follows from the Theorem 16.5.5.

Theorem 16.5.5 *Let the generator* $\mathcal{A}(x,D) \in \Psi^{2\mathbf{m}(x)}_{\rho,\delta}$ *be a pseudodifferential operator of variable order* $2\mathbf{m}(x)$, $0 < m_i(x_i) < 1$, $i = 1,\dots,d$ *with* $\mathbf{m}(x) =$

$(m_1(x_1), \ldots, m_d(x_d))$ *and symbol* $\psi(x, \xi) \in S^{2\mathbf{m}(x)}_{\rho,\delta}$ *for some* $0 < \delta < \rho \leq 1$ *for which there exists* $C > 0$ *with*

$$\Re\,\psi(x, \xi) + 1 \geq C\langle\xi\rangle^{2\mathbf{m}(x)} \quad \forall x, \xi \in \mathbb{R}^d. \tag{16.30}$$

Then, $\mathcal{A}(x, D) \in \Psi^{2\mathbf{m}(x)}_{\rho,\delta}$ *satisfies a Gårding inequality in the variable order space* $\widetilde{H}^{\mathbf{m}(x)}(G)$*: There are constants* $C_1 > 0$ *and* $C_2 \geq 0$ *such that*

$$\forall u \in \widetilde{H}^{\mathbf{m}(x)}(G): \quad \Re a(u, u) \geq C_1\|u\|^2_{\widetilde{H}^{\mathbf{m}(x)}(G)} - C_2\|u\|^2_{L^2(G)}, \tag{16.31}$$

and

$$\exists\lambda > 0 \quad \text{such that } \mathcal{A}(x, D) + \lambda I : \widetilde{H}^{\mathbf{m}(x)}(G) \to H^{-\mathbf{m}(x)}(G) \tag{16.32}$$

is boundedly invertible, for $a(u, v) = \langle\mathcal{A}u, v\rangle_{H^{-\mathbf{m}(x)}(G), \widetilde{H}^{\mathbf{m}(x)}(G)}$, $u, v \in \widetilde{H}^{\mathbf{m}(x)}(G)$.

Proof The proof follows along the lines of the proof of [137, Theorem 5], where the case $d = 1$ was treated. □

Note that in the case of an admissible time-homogeneous market model, $\Re a_G(u, u) = a_G(u, u)$ holds for $u \in \mathcal{V}_G$ and $a_G(\cdot, \cdot)$ as in (16.28).

Theorem 16.5.6 *The problem* (16.28)–(16.29) *for an admissible time-homogeneous market model* X *with symbol* $\psi(x, \xi)$ *with initial condition* $g \in \mathcal{H} = L^2(\mathbb{R}^d)$ *and* $\underline{Y} \geq 1$, $Q = 0$ *or* $Q \geq Q_0 > 0$ *has a unique solution.*

Proof We obtain from Lemma 16.4.3

$$\psi(x, \xi) \in S^{\mathbf{Y}(x)}_{1,\delta} \quad \text{for } \underline{Y} \geq 1,\ Q = 0,$$

$$\psi(x, \xi) \in S^{2}_{1,\delta} \quad \text{for } Q \geq Q_0 > 0.$$

Theorem 16.5.1 implies

$$\Re\psi(x, \xi) + 1 \geq C\langle\xi\rangle^{\mathbf{Y}(x)} \quad \text{for } \underline{Y} \geq 1,\ Q = 0, \tag{16.33}$$

$$\Re\psi(x, \xi) + 1 \geq C\langle\xi\rangle^{2} \quad \text{for } Q \geq Q_0 > 0, \tag{16.34}$$

for all $x, \xi \in \mathbb{R}^d$. An application of Theorem 16.5.5 implies the claimed result. □

16.6 Numerical Examples

In this section the implementation of numerical solution methods for the Kolmogorov equations for admissible market models with inhomogeneous jump measures using the techniques described above is discussed. We assume the risk-neutral dynamics of the underlying asset to be given by

$$S(t) = S(0)e^{rt + X(t)},$$

where X is a Feller process with characteristic triple $(b(x), Q(x), k(x,z)\mathrm{d}z)$ under a risk neutral measure $\mathbb{Q}$ such that e^X is a martingale with respect to the canonical filtration of X. In the following we set $r = 0$ for notational convenience. We consider Feller processes X that are admissible time-homogeneous market models. In the following we consider a special family of Feller processes to confirm the theoretical results of the previous chapters.

Example 16.6.1 We consider a CGMY-type Feller process with jump kernel

$$k(x,z) = C \begin{cases} e^{-\beta^+ z} z^{-1-Y(x)}, & z > 0, \\ e^{-\beta^- |z|} |z|^{-1-Y(x)}, & z < 0, \end{cases} \quad Y(x) = k e^{-x^2} + 0.5.$$

This process has no Gaussian component and the drift $\gamma(x)$ is chosen according to (16.23).

We also consider the following family of processes that do not satisfy the conditions of the theory developed above, since the variable order is assumed to be Lipschitz continuous only.

Example 16.6.2 We consider again a CGMY-type Feller process with jump kernel

$$k(x,z) = C \begin{cases} e^{-\beta^+ z} z^{-1-Y(x)}, & z > 0, \\ e^{-\beta^- |z|} |z|^{-1-Y(x)}, & z < 0, \end{cases}$$

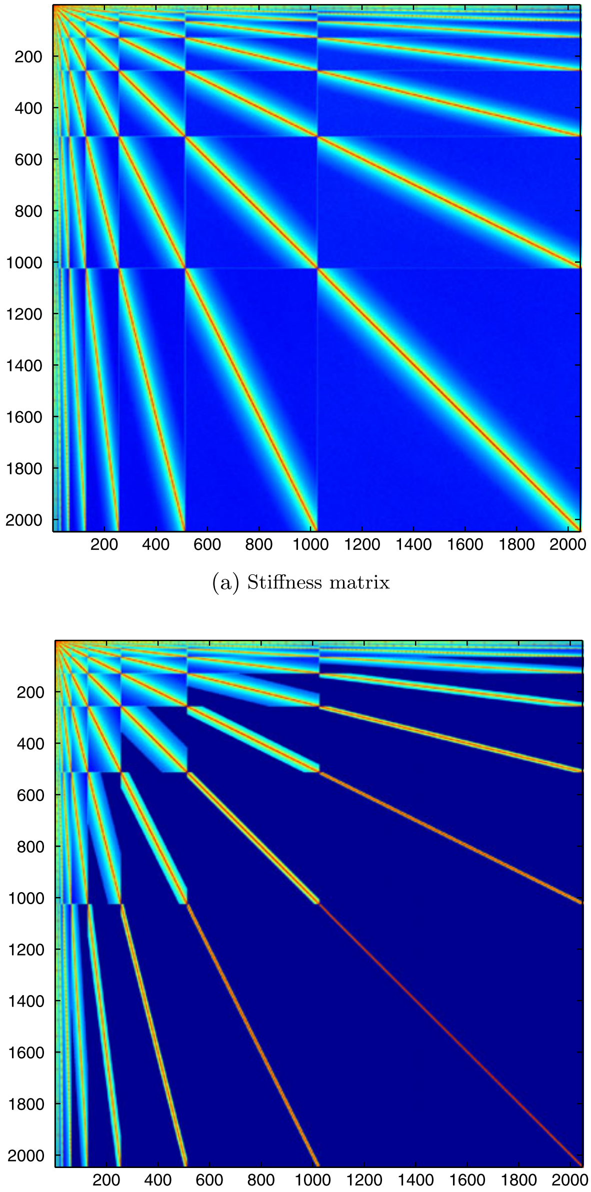
$$Y(x) = 0.5 + k \begin{cases} 0.4x, & 0.25 > x > 0, \\ 0.8x - 0.1, & 0.5 > x \geq 0.25, \\ -0.4x + 0.5, & 0.75 > x \geq 0.5, \\ -0.8x + 0.8, & 1 > x \geq 0.75, \\ 0, & \text{else}. \end{cases}$$

This process has no Gaussian component and the drift $b(x)$ is chosen according to (16.23).

In Fig. 16.1 the stiffness matrix for the process in Example 16.6.1 is depicted. Note that the uncompressed stiffness matrix is densely populated, but structurally very similar to the matrix in the Black–Scholes model, compare with Fig. 16.2.

The assembly of the stiffness matrix is carried out using numerical quadratures. Standard quadratures, such as the Gauss quadrature, yield unsatisfactory results due to the weak singularity of the integrand. We therefore employ tensorized composite Gauss quadratures for the computation of the matrix entries, see [138, Sect. 7.2], [163, Chap. 5] and [149]. Using such an approach we obtain quadrature rules that converge exponentially with respect to the number of quadrature points even for weakly singular kernels. The condition numbers of the preconditioned stiffness matrices have to be uniformly bounded in the number of levels due to arguments similar to Sect. 12.2.3, we refer to [135, Chap. 5] for details. A parameter study for

Fig. 16.1 Stiffness matrices for the pure jump case with CGMY-type Lévy kernel ($Y(x) = 1.25e^{-x^2} + 0.5$)

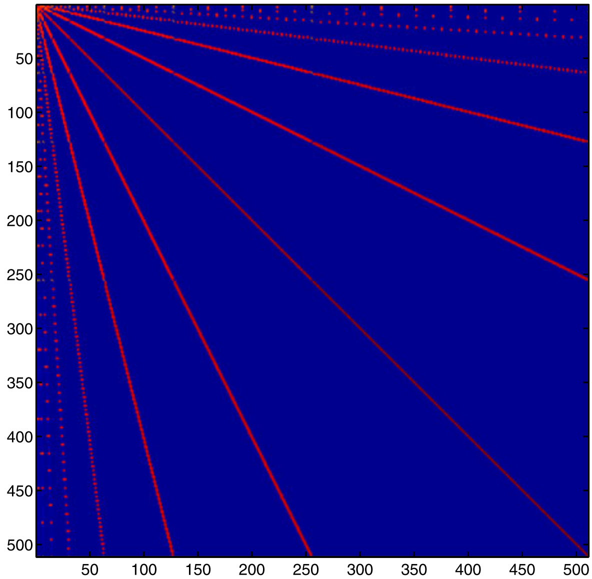


(a) Stiffness matrix

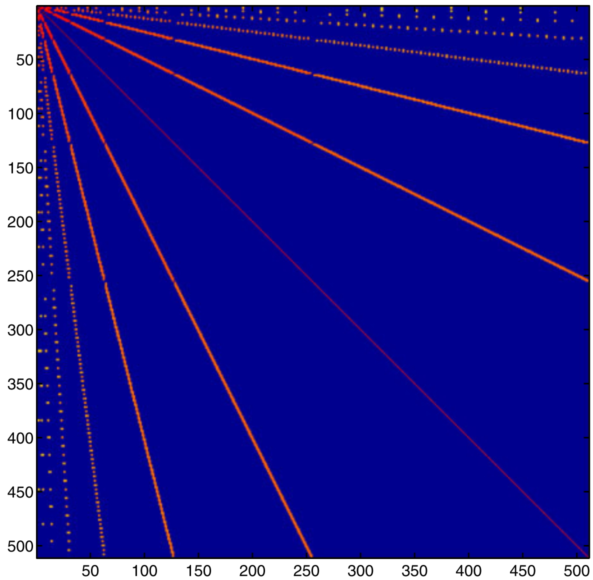
(b) Compressed stiffness matrix

various choices of k in Example 16.6.1 and Example 16.6.2 is shown in Fig. 16.3. The condition numbers are uniformly bounded and of order 10^1, although the theoretical results only apply to Example 16.6.1. For variable orders with $1.95 \leq \overline{Y}$ we obtain condition numbers of order 10^2. Note that the condition numbers are not only influenced by the order of the singularity of the jump kernel at $z = 0$, but also

Fig. 16.2 Stiffness and Mass matrices for the Black–Scholes model with $\sigma = 0.3$ and $r = 0$



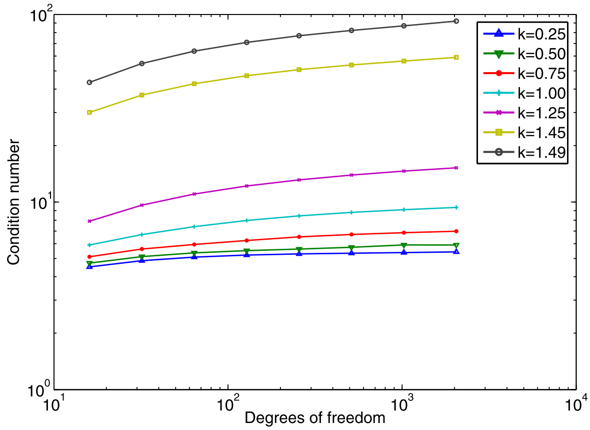
(a) Stiffness matrix



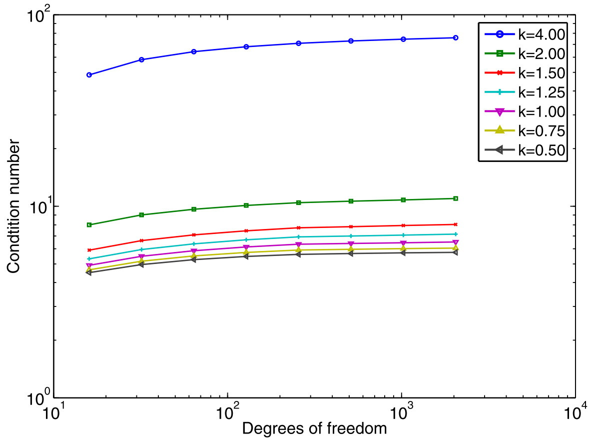
(b) Mass matrix

by the rates of exponential decay β^+ and β^-. Fast decaying tails, i.e., large β^+ and β^- may lead to larger constants. Figure 16.4 shows the price of a European put option for several Lévy processes and for one Feller process. In the Feller case

Fig. 16.3 Condition numbers for different levels and choices of k



(a) Example 16.6.1



(b) Example 16.6.2

we choose the CMGY parametrization with state-dependent parameter Y given by $Y(x) = 0.8e^{-x^2} + 0.1$ in Example 16.6.1. For the Lévy models we choose constant $Y \in \{0.1, 0.5, 0.7, 0.8, 0.9\}$. In all cases we set $C = 1$, $\beta^+ = \beta^- = 10$ and use truncation parameters $a = -3$, $b = 3$ in log-moneyness coordinates. The prices in the Feller model are significantly different from the prices in the different Lévy models. This can be explained by the ability of the Feller model to account for different tail behavior for different states of the process, which is not possible using Lévy processes. Figure 16.5 shows the prices of American put option for a Feller process and for several Lévy models. To enforce the constraint, we use a Lagrangian multiplier approach as described in [92, 93], see Chap. 5. The parameters were chosen as above.

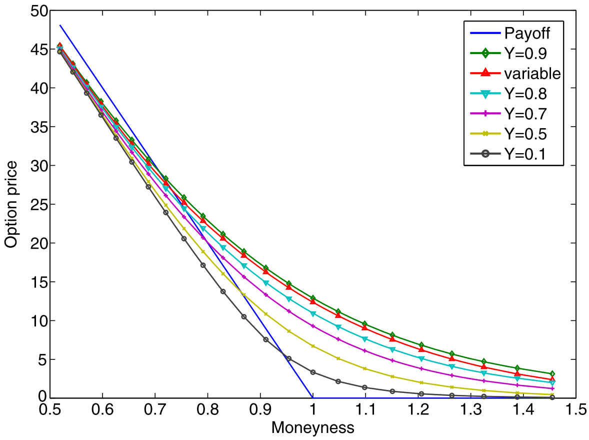


Fig. 16.4 Option prices for several models for a European put option with $T = 1$ and $K = 100$

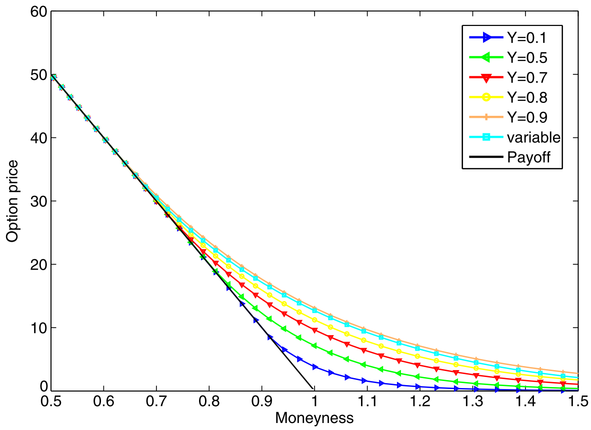


Fig. 16.5 Option prices for several models for an American put option with $T = 1$ and $K = 100$

16.7 Further Reading

Schneider et al. [137] consider the well-posedness and discretization of pricing equations on variable order spaces. For the multidimensional setting we refer to Reichmann and Schwab [138] and Reichmann [135]. Extensions of Lévy type market models involving local speed functions have also been considered by Carr et al. [35]. A generalized NIG model was proposed by Barndorff-Nielsen and Levendorskiĭ [7].

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Appendix A
Elliptic Variational Inequalities

This appendix gives more information on elliptic variational inequalities. Some definitions and results from functional analysis are summarized in Sects. A.1 and A.2. Existence and uniqueness results are discussed in Sect. A.3.

A.1 Hilbert Spaces

Let $\mathcal{H}$ be a vector space over $\mathbb{R}$. An *inner product* (u, v) is a bilinear form from $\mathcal{H} \times \mathcal{H} \to \mathbb{R}$ which is

$$\text{symmetric: } \forall u, v \in \mathcal{H} \quad (u, v) = (v, u),$$

$$\text{positive definite: } \forall u \in \mathcal{H} \;\; (u, u) \geq 0, \;\; (u, u) = 0 \Longleftrightarrow u = 0 .$$

Each inner product $(\cdot, \cdot)$ on $\mathcal{H}$ induces a norm on $\mathcal{H}$ via

$$\|v\| := (v, v)^{\frac{1}{2}} \qquad \forall v \in \mathcal{H},$$

and satisfies the *Cauchy–Schwarz inequality*:

$$\forall u, v \in \mathcal{H} : \; |(u, v)| \leq \|u\| \; \|v\| . \tag{A.1}$$

Moreover, there holds the parallelogram law

$$\left\| \frac{u+v}{2} \right\|^2 + \left\| \frac{u-v}{2} \right\|^2 = \frac{1}{2} \left(\|u\|^2 + \|v\|^2 \right) \quad \forall u, v \in \mathcal{H} . \tag{A.2}$$

Definition A.1.1 A vector space $\mathcal{H}$ is a *Hilbert space* if it is endowed with an inner product $(\cdot, \cdot)$ and if it is *complete* with respect to the norm $\|u\| = (u, u)^{\frac{1}{2}}$.

A subset $\mathcal{K} \subset \mathcal{H}$ is convex, if

$$\forall u, v \in \mathcal{K} : \; \{\lambda u + (1 - \lambda)\, v : 0 < \lambda < 1\} \subset \mathcal{K} . \tag{A.3}$$

N. Hilber et al., *Computational Methods for Quantitative Finance*, Springer Finance,
DOI 10.1007/978-3-642-35401-4,

Theorem A.1.2 (Projection onto closed convex subsets) *Let $\mathcal{H}$ be a Hilbert space and $\mathcal{K} \subset \mathcal{H}$ be a closed and convex subset. Then, for every $f \in \mathcal{H}$ there exists a unique $u \in \mathcal{K}$ such that*

$$\|f - u\| = \min_{v \in \mathcal{K}} \|f - v\| . \tag{A.4}$$

Moreover, u is characterized by the variational inequality: *find*

$$u \in \mathcal{K}: \ (f - u, v - u) \leq 0 \quad \forall v \in \mathcal{K} . \tag{A.5}$$

We define $u = P_{\mathcal{K}} f$ as the projection of f onto $\mathcal{K}$.

Proof The set $\{\|f - v\| : v \in \mathcal{K}\}$ is bounded from below by 0, hence $d := \inf_{v \in \mathcal{K}} \|f - v\| \geq 0$, exists. Let $(v_n)_n \subset \mathcal{K}$ denote a minimizing sequence, i.e. $d_n := \|f - v_n\| \to d$ as $n \to \infty$. Then $(v_n)_n$ is Cauchy, since by (A.2) with $f - v_n$ in place of u and with $f - v_m$ in place of v it holds

$$\frac{1}{2}\,(d_n^2 + d_m^2) = \left\| f - \frac{v_n + v_m}{2} \right\|^2 + \left\| \frac{v_n - v_m}{2} \right\|^2 .$$

Since $\mathcal{K}$ is convex, $(v_n + v_m)/2 \in \mathcal{K}$ and hence $\|f - (v_n + v_m)/2\| \geq d$. Therefore,

$$\left\| \frac{v_n - v_m}{2} \right\|^2 \leq \frac{1}{2}\,(d_n^2 + d_m^2) - d^2 \to 0 \text{ as } n, m \to \infty,$$

and hence $(v_n)_n$ is Cauchy. Since $\mathcal{H}$ is complete, there exists a unique $u = \lim_{n\to\infty} v_n$, and since $\mathcal{K}$ is closed, $u \in \mathcal{K}$. Since

$$\begin{aligned} d = \inf\{\|f - v\| : v \in \mathcal{K}\} \leq \ & \|f - u\| = \|f - v_n + v_n - u\| \\ \leq \ & d_n + \|u - v_n\| \xrightarrow[n\to\infty]{} d , \end{aligned}$$

u satisfies (A.4).

To show (A.5), let $u \in \mathcal{K}$ verify (A.4) and let $w \in \mathcal{K}$ be arbitrary. Then $\{(1 - \lambda)u + \lambda w : 0 < \lambda < 1\} \subset \mathcal{K}$ and hence

$$\|f - u\| \leq \|f - [(1 - \lambda)u + \lambda w]\| = \|(f - u) - \lambda(w - u)\| ,$$

so that for $0 \leq \lambda \leq 1$ holds

$$\|f - u\|^2 \leq \|f - u\|^2 - 2\lambda(f - u, w - u) + \lambda^2 \|w - u\|^2 ,$$

from where we find that

$$\forall w \in \mathcal{K}, \quad \forall 0 \leq \lambda \leq 1: \quad 2(f - u, w - u) \leq \lambda \|w - u\|^2 .$$

Choosing here $\lambda = 0$ implies (A.5).

Conversely, let u satisfy (A.5). Then, for all $v \in \mathcal{K}$,

$$\|u - f\|^2 - \|v - f\|^2 = 2(f - u, v - u) - \|u - v\|^2 \leq 0$$

which implies (A.4).

To show that u in (A.4) is unique, we assume that there is $u' \neq u$, $u' \in \mathcal{K}$ such that $d = \|f - u\| = \|f - u'\|$. Then

$$\begin{aligned}\|u-u'\|^2 &= \|(u-f)-(u'-f)\|^2 = \|u-f\|^2 + 2(u-f,u'-f) + \|u'-f\|^2 \\ &= 2d^2 + 2(f-u,u'-u+u-f) \\ &= 2d^2 + 2(f-u,u'-u) - 2(f-u,f-u) \\ &= 2d^2 + 2(f-u,u'-u) - 2\|f-u\|^2 = 2(f-u,u'-u) \le 0\,,\end{aligned}$$

hence $u' = u$, a contradiction. □

We establish some properties of $P_{\mathcal{K}}$.

Theorem A.1.3 *For any $\mathcal{K} \subset \mathcal{H}$ closed and convex, the following holds*:

$$\forall f_1, f_2 \in \mathcal{H} : \|P_{\mathcal{K}}\, f_1 - P_{\mathcal{K}}\, f_2\| \le \|f_1 - f_2\|\,. \tag{A.6}$$

Proof Let $u_1 = P_{\mathcal{K}}\, f_1,\ u_2 = P_{\mathcal{K}}\, f_2$. By (A.5),

$$\forall v \in \mathcal{K} :\ (f_1 - u_1, v - u_1) \le 0\,, \tag{A.7}$$

$$\forall v \in \mathcal{K} :\ (f_2 - u_2, v - u_2) \le 0\,. \tag{A.8}$$

Insert $v = u_2$ into (A.7) and $v = u_1$ into (A.8). Then adding, we get $\|u_1 - u_2\|^2 \le (f_1 - f_2, u_1 - u_2) \le \|f_1 - f_2\|\, \|u_1 - u_2\|$. □

A.2 Dual of a Hilbert Space

A linear map $u^* : \mathcal{H} \to \mathbb{R}$ is continuous if and only if

$$\|u^*\|_{\mathcal{H}^*} := \sup_{u \neq 0} \left\{ \frac{|u^*(u)|}{\|u\|} : u \in \mathcal{H} \right\} < \infty\,. \tag{A.9}$$

Such u^* are called continuous, linear functionals. The set of all continuous, linear functionals on $\mathcal{H}$ is denoted by $\mathcal{H}^*$. It is a normed, linear space with norm $\|u^*\|_{\mathcal{H}^*}$ in (A.9), and called the *dual of* $\mathcal{H}$.

Theorem A.2.1 (Riesz representation theorem) *For every $u^* \in \mathcal{H}^*$ there exists a unique $u \in \mathcal{H}$ such that*

$$\forall v \in \mathcal{H} :\ u^*(u) = (u, v) \tag{A.10}$$

and

$$\|u^*\|_{\mathcal{H}^*} = \|u\|\,. \tag{A.11}$$

Proof Since $u^* \in \mathcal{H}^*$, the set $\mathcal{N} := \{v \in \mathcal{H} : u^*(v) = 0\}$ is a closed linear subspace of $\mathcal{H}$. If $\mathcal{N} = \mathcal{H}$, $u^* \equiv 0$, and we take $u = 0$.

Assume now $\mathcal{N} \neq \mathcal{H}$. Then, we claim that there exists $g \in \mathcal{H} \backslash \mathcal{N}$ such that

$$\|g\| = 1, \quad \forall w \in \mathcal{N} :\ (g, w) = 0\,. \tag{A.12}$$

To this end, let $g_0 \in \mathcal{H}\backslash\mathcal{N}$ and let $g_1 := P_{\mathcal{N}}\, g_0 \neq g_0$. Then $g := \|g_0 - g_1\|^{-1} \times (g_0 - g_1)$ satisfies (A.12).

Next, every $v \in \mathcal{H}$ can be written as $v = \lambda g + w$ with $\lambda \in \mathbb{R}$ and $w \in \mathcal{N}$: put, for example,

$$\lambda = \frac{u^*(v)}{u^*(g)}, \quad w = v - \lambda g\,.$$

Then, we get $0 = (g, w) = (g, v - \lambda g)$, i.e. $(g, v) = \lambda = u^*(v)/u^*(g)$. Hence $u := u^*(g)g$ satisfies

$$\begin{aligned}\forall v \in \mathcal{H}:\ u^*(v) &= u^*(\lambda g + w) = \lambda u^*(g) + u^*(w) = \lambda u^*(g)\\ &= (g, v)u^*(g) = (u, v)\,,\end{aligned}$$

i.e. (A.10) and, by (A.1),

$$\|u^*\|_{\mathcal{H}^*} = \sup_{v\in\mathcal{H}} \frac{|u^*(v)|}{\|v\|} = \sup_{v\in\mathcal{H}} \frac{|(u, v)|}{\|v\|} \leq \|u\|$$

and

$$\|u\| = \frac{(u, u)}{\|u\|} = \frac{|u^*(u)|}{\|u\|} \leq \sup_{v\in\mathcal{H}} \frac{|u^*(v)|}{\|v\|} = \|u^*\|_{\mathcal{H}^*}\,.$$

□

Remark A.2.2 The preceding result shows that $\mathcal{H}^*$ is isomorphic to $\mathcal{H}$, and the map $u^* \to u$ is, by (A.11), an isometry. One therefore *often, but not always* identifies $\mathcal{H}^*$ with $\mathcal{H}$.

In the parabolic setting, we often have the following. Let $\mathcal{V} \subset \mathcal{H}$ be a subspace which is dense in $\mathcal{H}$ and equipped with a norm $\|\cdot\|_{\mathcal{V}}$ such that the canonical embedding $i\colon \mathcal{V} \to \mathcal{H}$ is continuous:

$$\forall v \in \mathcal{V}: \|v\| \leq C\, \|v\|_{\mathcal{V}}\,. \tag{A.13}$$

We identify $\mathcal{H}$ and $\mathcal{H}^*$. Then one can embed $\mathcal{H}$ into $\mathcal{V}^*$ as follows: given $u \in \mathcal{H}$, the map $u_u^*\colon \mathcal{V} \ni v \to (u, v)$ is in $\mathcal{H}^*$ and, due to

$$|u^*(v)| = |(u, v)| \leq \|u\|\, \|v\| \leq C\, \|u\|\, \|v\|_{\mathcal{V}} \quad \forall v \in \mathcal{V}\,,$$

also in $\mathcal{V}^*$. We denote by $T : u \to u_u^*$ the mapping with

$$\forall u \in \mathcal{H} \quad \forall v \in \mathcal{V}:\ (Tu)(v) = (u, v)\,.$$

Then $T : \mathcal{H} \to \mathcal{V}^*$ satisfies:

(i) $\|Tu\|_{\mathcal{V}^*} = \sup_{w\in\mathcal{V}} \frac{|(Tu)(w)|}{\|w\|_{\mathcal{V}}} = \sup_{w\in\mathcal{V}} \frac{|(u,v)|}{\|w\|_{\mathcal{V}}} \leq C\, \|u\|$,
(ii) T injective,
(iii) $T(\mathcal{H})$ is dense in $\mathcal{V}^*$.

With T we can embed $\mathcal{H}$ into $\mathcal{V}^*$ and get the triple

$$\mathcal{V} \hookrightarrow \mathcal{H} = \mathcal{H}^* \hookrightarrow \mathcal{V}^*\,, \tag{A.14}$$

where the canonical injections are continuous and dense, and where $\mathcal{H}$ is called a 'pivot space'. Note that, in (A.14), one *cannot* identify $\mathcal{V}$ and $\mathcal{V}^*$ any more.

A.3 Theorems of Stampacchia and Lax–Milgram

The theorems of Stampacchia and Lax–Milgram are useful existence results for stationary problems. Let $\mathcal{V}$ be a Hilbert space with a norm $\|\cdot\|_{\mathcal{V}}$ and innerproduct $\langle\cdot,\cdot\rangle_{\mathcal{V}}$.

Definition A.3.1 A bilinear form $b(\cdot,\cdot):\mathcal{V}\times\mathcal{V}\to\mathbb{R}$ is

(i) *continuous*, if there exists $C_1>0$ such that

$$\forall u,v\in\mathcal{V}:\ |b(u,v)|\le C_1\|u\|_{\mathcal{V}}\,\|v\|_{\mathcal{V}}\,, \tag{A.15}$$

(ii) *coercive*, if there exists $C_2>0$ such that

$$\forall u\in\mathcal{V}:\ b(u,u)\ge C_2\|u\|^2_{\mathcal{V}}\,. \tag{A.16}$$

Theorem A.3.2 *Let $b(\cdot,\cdot):\mathcal{V}\times\mathcal{V}\to\mathbb{R}$ be a continuous and coercive bilinear form on $\mathcal{V}$, $\emptyset\ne\mathcal{K}\subset\mathcal{V}$ be closed and convex. Then, for any $\ell\in\mathcal{V}^*$ there exists a unique $u\in\mathcal{K}$ solution of the variational inequality*

$$\begin{aligned}&\textit{Find } u\in\mathcal{K} \textit{ such that}\\ &b(u,v-u)\ge\ell(v-u)\quad\forall v\in\mathcal{K}\,.\end{aligned} \tag{A.17}$$

The proof uses

Theorem A.3.3 (Banach's fixed point theorem) *Let X be a complete metric space and $\mathcal{S}:X\to X$ a mapping with*

$$d(\mathcal{S}v_1,\mathcal{S}v_2)\le\kappa d(v_1,v_2)\quad\forall v_1,v_2\in\mathcal{S} \tag{A.18}$$

for some $\kappa<1$. Then, the problem

$$\textit{Find } u\in X \textit{ such that } u=\mathcal{S}u \tag{A.19}$$

admits a unique solution.

Proof of Theorem A.3.2 Given $\ell\in\mathcal{V}^*$, there exists, by Theorem A.2.1, a unique $f\in\mathcal{V}$ such that

$$\forall v\in\mathcal{V}:\ \ell(v)=\langle f,v\rangle_{\mathcal{V}}\,.$$

For every fixed $u\in\mathcal{V}$, the map $v\longmapsto b(u,v)$ is in $\mathcal{V}^*$ and there is a unique representative in $\mathcal{V}$, denoted by Bu, such that $b(u,v)=\langle Bu,v\rangle_{\mathcal{V}},\ \ \forall v\in\mathcal{V}$. The operator $B:\mathcal{V}\to\mathcal{V}$ is linear and, by (A.15), (A.16),

$$\|Bu\|_{\mathcal{V}}\le C_1\,\|u\|_{\mathcal{V}}\quad\forall u\in\mathcal{V}\,, \tag{A.20}$$

$$\langle Bu,u\rangle_{\mathcal{V}}\ge C_2\,\|u\|^2_{\mathcal{V}}\quad\forall u\in\mathcal{V}\,. \tag{A.21}$$

Hence, (A.17) may be rewritten as

$$\begin{aligned}&\text{Find } u\in\mathcal{K} \text{ such that}\\ &\langle Bu,v-u\rangle_{\mathcal{V}}\ge\langle f,v-u\rangle_{\mathcal{V}}\quad\forall v\in\mathcal{K}\,.\end{aligned} \tag{A.22}$$

Let $\rho > 0$ be a parameter. Then (A.22) is equivalent to

$$\begin{aligned} &\text{Find } u \in \mathcal{K} \text{ such that} \\ &\langle \rho f - \rho B u + u - u, v - u \rangle_{\mathcal{V}} \le 0 \quad \forall v \in \mathcal{K}, \end{aligned} \tag{A.23}$$

or to the fixed point problem

$$u = P_{\mathcal{K}}(\rho f - \rho B u + u) \text{ in } \mathcal{K}. \tag{A.24}$$

Define, for $v \in \mathcal{K}$, $\mathcal{S}v = P_{\mathcal{K}}(v - \rho(Bv - f))$. Then, by (A.6), for every $v_1, v_2 \in \mathcal{K}$,

$$\|\mathcal{S}v_1 - \mathcal{S}v_2\|_{\mathcal{V}} \le \|(v_1 - v_2) - \rho B(v_1 - v_2)\|_{\mathcal{V}},$$

and hence

$$\begin{aligned} \|\mathcal{S}v_1 - \mathcal{S}v_2\|_{\mathcal{V}}^2 &\le \|v_1 - v_2\|_{\mathcal{V}}^2 - 2\rho\langle B(v_1 - v_2), v_1 - v_2\rangle_{\mathcal{V}} + \rho^2\|B(v_1 - v_2)\|_{\mathcal{V}}^2 \\ &\le \|v_1 - v_2\|_{\mathcal{V}}^2(1 - 2\rho\, C_2 + \rho^2\, C_1^2). \end{aligned}$$

We fix $\rho > 0$ such that $\kappa := (1 - 2\rho\, C_2 + \rho^2\, C_1^2)^{\frac{1}{2}} < 1$. Then $\mathcal{S}$ is a contraction on $\mathcal{K}$ and (A.24) has a unique solution. □

Remark A.3.4 The proof is constructive—the contraction $\mathcal{S}$ converges with rate $\kappa < 1$. The rate is maximal, if

$$\rho = \rho_* = \arg\min\{1 - 2\rho\, C_2 + \rho^2\, C_1^2\} = C_2/C_1^2, \tag{A.25}$$

then

$$\kappa = \kappa_* = (1 - C_2^2/C_1^2)^{\frac{1}{2}} < 1. \tag{A.26}$$

In particular, if $v^{(0)} \in \mathcal{K}$ is arbitrary, the sequence

$$v^{(i+1)} = \mathcal{S}v^{(i)} = P_{\mathcal{K}}\big(v^{(i)} - \rho(Bv^{(i)} - f)\big) \tag{A.27}$$

converges to $u \in \mathcal{K}$, the solution of (A.17), and

$$\|u - v^{(i)}\|_{\mathcal{V}} \le C\,\kappa_*^i\,\|u - v^{(0)}\|_{\mathcal{V}}, \;\; i \ge 0. \tag{A.28}$$

Corollary A.3.5 *If $\mathcal{M} \subseteq \mathcal{V}$ is a closed, linear subspace, the problem*

$$\textit{Find } u \in \mathcal{M} \textit{ such that} \quad b(u, v) \ge \ell(v) \quad \forall v \in \mathcal{M}$$

admits a unique solution.

Appendix B
Parabolic Variational Inequalities

To formulate the parabolic variational inequalities (PVIs) for pricing European and American contracts, we require suitable function spaces.

Let $\mathcal{V}, \mathcal{H}$ be Hilbert spaces with norms $\|\cdot\|_{\mathcal{V}}$ and $\|\cdot\|_{\mathcal{H}}$, respectively. As in Appendix A, we assume that $\mathcal{V} \hookrightarrow \mathcal{H}$ with dense injection and identity $\mathcal{H}$ with its dual $\mathcal{H}^*$, so that we obtain the so-called evolution triple with dense injections,

$$\mathcal{V} \hookrightarrow \mathcal{H} \cong \mathcal{H}^* \hookrightarrow \mathcal{V}^* . \tag{B.1}$$

On the time interval $J := (a, b) \subset \mathbb{R}$, we introduce the spaces of "sum" and of "intersection" type by

$$S(a, b) := L^1(J; \mathcal{H}) + L^2(J; \mathcal{V}^*), \tag{B.2}$$

$$I(a, b) := L^2(J; \mathcal{V}) \cap L^\infty(J; \mathcal{H}^*) . \tag{B.3}$$

The present results on existence and time discretization of abstract parabolic evolution variational inequalities are based on [5].

B.1 Weak Formulation of PVI's

In the triple (B.1), we are given a linear, possibly non-selfadjoint operator $\mathcal{A} : \mathcal{V} \to \mathcal{V}^*$ with the associated bilinear form

$$a(u, v) := \langle \mathcal{A}u, v\rangle_{\mathcal{V}^*,\mathcal{V}} : \mathcal{V} \times \mathcal{V} \to \mathbb{R},$$

where $\langle \cdot, \cdot\rangle_{\mathcal{V}^*,\mathcal{V}}$ denotes the extension of the $\mathcal{H}$ innerproduct to $\mathcal{V}^* \times \mathcal{V}$ by continuity. We assume that there are $\alpha, \beta > 0$

$$\forall u, v \in \mathcal{V} : |a(u, v)| \le \beta \|u\|_{\mathcal{V}} \|v\|_{\mathcal{V}}, \tag{B.4a}$$

$$\forall u \in \mathcal{V} : a(u, u) + \lambda \|u\|^2_{\mathcal{H}} \ge \alpha \|u\|^2_{\mathcal{V}} , \tag{B.4b}$$

for some $\lambda \ge 0$.

A parabolic variational inequality (PVI) in strong form is given by a set

$$\emptyset \ne \mathcal{K} \subset \mathcal{V}, \text{ closed and convex with } 0 \in \mathcal{K} , \tag{B.5}$$

N. Hilber et al., *Computational Methods for Quantitative Finance*, Springer Finance,
DOI 10.1007/978-3-642-35401-4,

of admissible prices, a payoff $u_0 \in \mathcal{H}$, a time horizon $T \le \infty$ and a source term $f : J \to \mathcal{V}^*$, $J = (0, T)$. Then, the PVI reads

$$\begin{aligned} &u : J \to \mathcal{V} \text{ such that } u(t) \in \mathcal{K} \text{ a.e. } t \in J \\ &\langle u' + \mathcal{A}u(t) - f(t), u(t) - v\rangle_{\mathcal{V}^*,\mathcal{V}} \le 0 \quad \forall v \in \mathcal{K}, \text{ a.e. in } J, \\ &u(0) = u_0. \end{aligned} \tag{B.6}$$

Existence results for the PVI (B.6) requires a weak formulation: to state it, we assume in (B.6)

$$u' \in L^2(J;\mathcal{V}), \qquad u(t) \in \mathcal{K} \text{ a.e. } t \in J\,, \tag{B.7a}$$

$$v, v' \in L^2(J;\mathcal{V}), \qquad v(t) \in \mathcal{K} \text{ a.e. } t \in J\,. \tag{B.7b}$$

Then a solution u in (B.6) is such that for all v in (B.7a), (B.7b)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\,\|u(t) - v(t)\|^2_{\mathcal{H}}\right) + \langle v'(t) + \mathcal{A}u(t) - f(t),\, u(t) - v(t)\rangle_{\mathcal{V}^*,\mathcal{V}} \le 0\,. \tag{B.8}$$

This remains valid even for u which do not satisfy (B.7a). The *weak form of* (B.6) *is thus: given*

$$u_0 \in \overline{\mathcal{K}}^{\|\cdot\|_{\mathcal{H}}},\ 0 < T \le \infty,\ f \in L^2(0, T; \mathcal{V}^*)\,, \tag{B.9a}$$

and $\mathcal{K}$ *satisfying* (B.5), $\mathcal{A} : \mathcal{V} \to \mathcal{V}^*$ *satisfying* (B.4a), (B.4b) for $\lambda = 0$, find

$$u \in L^2(0, T; \mathcal{V}) \textit{ such that } u(t) \in \mathcal{K} \text{ a.e. } t \in J\,, \tag{B.9b}$$

and such that the function $\Theta(t)$*, given by*

$$\Theta(t) := \frac{1}{2}\|u(t) - v(t)\|^2_{\mathcal{H}} + \int_0^t \langle v'(\tau) + \mathcal{A}u(\tau) - f(\tau), u(\tau) - v(\tau)\rangle_{\mathcal{V}^*,\mathcal{V}}\mathrm{d}\tau \tag{B.9c}$$

satisfies

$$\Theta'(t) \le 0 \text{ in } J \tag{B.9d}$$

in the sense of distributions and

$$\Theta(t) \le \frac{1}{2}\,\|u_0 - v(0)\|^2_{\mathcal{H}} \text{ in } J \tag{B.9e}$$

for all v *satisfying* (B.7b).

Remark B.1.1 If $u_0 \in \mathcal{H}$ is given, then u_0 in (B.9a) and (B.9e) must be replaced by $\mathcal{P}u_0 = P_{\overline{\mathcal{K}}^{\|\circ\|_{\mathcal{H}}}} u_0$, its projection (cf. Theorem A.3) onto the closure of $\mathcal{K}$ in $\mathcal{H}$ (otherwise, there may be multiple solutions of (B.9a)–(B.9e)), i.e. by

$$\mathcal{P}u_0 \in \overline{\mathcal{K}}^{\|\circ\|_{\mathcal{H}}} : (u_0 - \mathcal{P}u_0, v - \mathcal{P}u_0) \le 0 \quad \forall v \in \overline{\mathcal{K}}^{\|\circ\|_{\mathcal{H}}}\,.$$

Remark B.1.2 If $\mathcal{K}$ is a closed, convex cone with vertex 0, (B.9b) splits into

$$\langle u'(t) + \mathcal{A}u(t) - f(t), v\rangle_{\mathcal{V}^*,\mathcal{V}} \ge 0 \quad \forall v \in \mathcal{K}$$

and

$$\langle u'(t) + \mathcal{A}u(t) - f(t), u\rangle_{\mathcal{V}^*,\mathcal{V}} = 0\,.$$

If $\mathcal{K} \subset \mathcal{V}$ is a subspace, (B.9b) becomes

$$\langle u'(t) + \mathcal{A}u(t) - f(t), v\rangle_{\mathcal{V}^*,\mathcal{V}} = 0 \quad \forall v \in \mathcal{K}\,.$$

Remark B.1.3 In the weak formulation (B.9a)–(B.9e), we assumed (B.4b) with $\alpha > 0$, $\lambda = 0$. If (B.4b) holds only with $\lambda > 0$, the function $\Theta(t)$ in the weak formulation (B.9a)–(B.9e) has to be replaced by

$$\begin{aligned}\Theta_\lambda(t) := \frac{1}{2}\,\|e^{-\lambda t}(u(t) - v(t))\|^2_{\mathcal{H}} \\ + \int_0^t \big\langle e^{-\lambda\tau}(v'(\tau) + \mathcal{A}u(\tau) + \lambda u(\tau) - \lambda v(\tau) - f(\tau)), \\ e^{-\lambda\tau}(u(\tau) - v(\tau))\big\rangle_{\mathcal{V}^*,\mathcal{V}}\mathrm{d}\tau\,;\end{aligned}$$

the spaces $L^2(J;\mathcal{V})$ and $L^2(J;\mathcal{V}^*)$ must be replaced by spaces with weight $\exp(-\lambda\tau)$ or, if $T = \infty$, by $L^2_{\mathrm{loc}}(0,\infty;\mathcal{V})$, etc. Then, (B.9a)–(B.9e) and all what follows will apply also to the case when (B.4b) holds only with $\lambda > 0$.

B.2 Existence

To show the existence of solutions to the PVI (B.9a)–(B.9e), we semidiscretize (B.6) in time as follows: given $k > 0$ with $k = T/M$ if $T < \infty$, we define

$$J_{km} := [mk, (m+1)k]\,, \tag{B.10}$$

$$u_{k,0} = \mathcal{P}u_0, \quad f_{km} := \frac{1}{k}\int_{J_{km}} f(\tau)\,\mathrm{d}\tau \tag{B.11}$$

and replace (B.6), (B.9a)–(B.9e) by the sequence of elliptic variational inequalities: for $m = 0, 1, 2, \ldots$, find

$$u_{k,m+1} \in \mathcal{K} \tag{B.12a}$$

such that

$$\langle u_{k,m+1} - u_{k,m} + k\mathcal{A}u_{k,m+1} - f_{k,m},\, u_{k,m+1} - v\rangle_{\mathcal{V}^*,\mathcal{V}} \le 0 \tag{B.12b}$$

for all $v \in \mathcal{K}$.

By (B.4a), (B.4b) with $\lambda = 0$ and by Theorem A.8, the EVI (B.12b) admits a unique solution $u_{k,m+1}$; hence $\{u_{k,m}\}_{m=0}^M$ ($M = \infty$ if $T = \infty$) is well-defined.

With $\{u_{k,m}\}_{m=0}^\infty$ we associate a function $U_k(t) \in C^0(\overline{J}, \mathcal{V})$ by

$$U_k(t)\big|_{J_{k,m}} \text{ is linear on } J_{k,m},\ m = 0, 1, 2, \ldots \tag{B.13a}$$

$$U_k(mk) = u_{k,m+1}\,. \tag{B.13b}$$

Remark B.2.1 In (B.13b), $u_{k,m+1}$ is needed, as by (B.11) $u_{k,0} = u_0 \notin \mathcal{V}$ for general $u_0 \in \overline{\mathcal{K}}^{\|\circ\|_{\mathcal{H}}}$; the choice $U_k(mk) = u_{k,m}$ in (B.13b) would then imply $U_k(t) \notin \mathcal{V}$ for $0 < t < k$.

For every $k > 0$, U_k is well-defined.

Theorem B.2.2

(i) *The mapping* $T_k : \{u_0, f\} \mapsto U_k(t)$ *is bounded and Lipschitz-continuous from* $\mathcal{H} \times L^2(J; \mathcal{V}^*) \to L^2(J; \mathcal{V})$, *uniformly in* k. *For any* $\{u_0, f\} \in \mathcal{H} \times L^2(0, T; \mathcal{V}^*)$, *the family* $\{U_k\}_{k>0}$ *is Cauchy in* $L^2(0, T; \mathcal{V})$ *and its limit* $u \in L^2(J, \mathcal{V})$ *is the unique weak solution of the PVI* (B.9a)–(B.9e), *which satisfies* (B.8) *and, moreover,*

$$u \in C^0(\overline{J}, \mathcal{H}) \,. \tag{B.14}$$

(ii) *If, moreover,* $f \in S(0, T)$ (*cf.* (B.2)), *then* T_k *is bounded and Lipschitz-continuous from* (*cf.* (B.3))

$$T_k : \mathcal{H} \times S(0, T) \to I(0, T)$$

and $\{U_k\}_{k>0}$ *is Cauchy in* $I(0, T)$.

(iii) *Finally, assuming* $f = g + h$ *with*

$$g \in BV(J; \mathcal{H}), \quad h \in H^1(J; \mathcal{V}^*) \,, \tag{B.15a}$$

$$\mathcal{P}u_0 \in \mathcal{K}, \quad A\mathcal{P}_{u_0} - h(0) \in \mathcal{H}, \tag{B.15b}$$

also $\{U_k'\}_{k>0}$ *is uniformly bounded in* $I(0, T)$, *and* $u = \lim_{k\to 0} U_k$ *satisfies* $u' \in I(0, T)$ *and the second line in* (B.6).

B.3 Proof of the Existence Result

We prove the existence result by establishing a priori estimates for time-semidiscrete approximate solutions in (B.12a), (B.12b). We start by analyzing (B.12a), (B.12b) for $m = 0$, $k > 0$ and write u_1 for $u_{k,1}$ and f_0 in place of $f_{k,0}$ in (B.11). We *assume* that $f \in S(0, T)$, i.e.

$$f = g + h, \;\; g \in L^1(J; \mathcal{H}), \;\; h \in L^2(J; \mathcal{V}^*) \,,$$

and set

$$g_0 := \frac{1}{k} \int_0^k g(\tau)\, \mathrm{d}\tau, \;\; h_0 := \frac{1}{k} \int_0^k h(\tau)\, \mathrm{d}\tau \,. \tag{B.16}$$

Then, (B.12a), (B.12b) reads: find $u_1 \in \mathcal{K}$ such that for all $v \in \mathcal{K}$:

$$\langle u_1 + k\mathcal{A}u_1, u_1 - v\rangle_{\mathcal{V}^*,\mathcal{V}} \le (G_0, u_1 - v) + \sqrt{k}\langle H_0, u_1 - v\rangle_{\mathcal{V}^*,\mathcal{V}} \tag{B.17}$$

where

$$G_0 := u_0 + kg_0 \in \mathcal{H}, \;\; H_0 := \sqrt{k}\, h_0 \in \mathcal{V}^* \,. \tag{B.18}$$

Note that, as $k \to 0$,

$$G_0 \to u_0 \in \mathcal{H}, \quad H_0 \to 0 \text{ in } \mathcal{V}^* \text{ strongly}. \tag{B.19}$$

Since $0 \in \mathcal{K}$, we may choose in (B.17) $v = 0$ and get

$$\|u_1\|_{\mathcal{H}}^2 + k\|u_1\|_{\mathcal{V}}^2 \le C\,(\|G_0\|_{\mathcal{H}}^2 + \|H_0\|_*^2), \tag{B.20}$$

where $\|\cdot\|_*$ denotes the norm in $\mathcal{V}^*$. We claim that

$$\|u_1 - \mathcal{P}u_0\|_{\mathcal{H}}^2 + k\|u_1\|_{\mathcal{V}}^2 \to 0 \text{ as } k \to 0. \tag{B.21}$$

To prove (B.21), we note that $u_1 \in \mathcal{K}$. Also, by the definition of $\mathcal{P}$ in Remark B.1.1, we have

$$(u_0 - \mathcal{P}u_0, u_1 - \mathcal{P}u_0) \le 0,$$

hence

$$\|u_1 - \mathcal{P}u_0\|_{\mathcal{H}}^2 = (u_1 - \mathcal{P}u_0, u_1 - \mathcal{P}u_0) \le (u_1 - u_0, u_1 - \mathcal{P}u_0). \tag{B.22}$$

By (B.4b) with $\lambda = 0$, (B.9c) and (B.9e), (B.17), (B.22), it follows for every $v \in \mathcal{K}$

$$\begin{aligned}
\|u_1 - \mathcal{P}u_0\|_{\mathcal{H}}^2 + \alpha k\|u_1 - v\|_{\mathcal{V}}^2 \;&\le\; (u_1 - u_0, u_1 - \mathcal{P}u_0)\\
&\quad + k\langle \mathcal{A}(u_1 - v), u_1 - v\rangle_{\mathcal{V}^*,\mathcal{V}}\\
&\le\; (u_1 - u_0, v - \mathcal{P}u_0)\\
&\quad + k\langle u_1 - u_0 + k\mathcal{A}u_1, u_1 - v\rangle_{\mathcal{V}^*,\mathcal{V}}\\
&\quad - k\langle \mathcal{A}v, u_1 - v\rangle_{\mathcal{V}^*,\mathcal{V}}\\
&\overset{(B.17)}{\le}\; \|u_1 - u_0\|_{\mathcal{H}}\,\|v - \mathcal{P}u_0\|_{\mathcal{H}}\\
&\quad + (G_0 - u_0, u_1 - v)\\
&\quad + \sqrt{k}\,\langle H_0 - \sqrt{k}\mathcal{A}v, u_1 - v\rangle_{\mathcal{V}^*,\mathcal{V}}.
\end{aligned}$$

Therefore, it holds for all $v \in \mathcal{K}$ that

$$\begin{aligned}
&\|u_1 - \mathcal{P}u_0\|_{\mathcal{H}}^2 + k\|u_1 - u_0\|_{\mathcal{V}}^2\\
&\quad \le C\Big\{\|u_1 - u_0\|_{\mathcal{H}}\,\|v - \mathcal{P}u_0\|_{\mathcal{H}}\\
&\qquad + \Big|(G_0 - u_0, u_1 - v) + \sqrt{k}\big\langle H_0 - \sqrt{k}\mathcal{A}v, u_1 - v\big\rangle_{\mathcal{V}^*,\mathcal{V}}\Big|\Big\}.
\end{aligned} \tag{B.23}$$

Since $\mathcal{P}u_0 \in \overline{\mathcal{K}}^{\|\cdot\|_{\mathcal{H}}}$, there is a sequence $\{v_k\}_{k>0} \subset \mathcal{K}$ such that, as $k \to 0$, $k\|v_k\|_{\mathcal{V}}^2 \to 0$, and $\|v_k - \mathcal{P}u_0\|_{\mathcal{H}} \to 0$. We choose in (B.23) $v = v_k$ and obtain, after passing to the limit and using (B.19), (B.20), the assertion (B.21). We have

Lemma B.3.1 *For any fixed $n > 0$, there exists $C_n > 0$ such that, as $k \to 0$,*

$$\|u_{k,n} - \mathcal{P}u_0\|_{\mathcal{H}}^2 + k\|u_{k,n}\|_{\mathcal{V}}^2 \le C_n\big(\|u_0\|_{\mathcal{H}}^2 + \|f\|_{S(0,nk)}^2\big). \tag{B.24}$$

Moreover, as $k \to 0$,

$$\|u_{k,n} - \mathcal{P}u_0\|_{\mathcal{H}}^2 + k\|u_{k,n}\|_{\mathcal{V}}^2 \to 0 \text{ (non-uniformly in } n\text{)}. \tag{B.25}$$

Proof We proceed by induction on n. Inequality (B.24) for $n = 1$ follows from (B.20), (B.21), if we note that the right hand side of (B.20) is bounded by $C\,\{\|u_0\|^2_{\mathcal{H}} + \|g\|^2_{L^1(0,k;\mathcal{H})} + \|h\|^2_{L^2(0,k;\mathcal{V}^*)}\}$ for $f \in S(0, T)$.

Assume now the assertion (B.24) is true for some n. Then, arguing as in the proof of (B.20), (B.21), we obtain (B.24) for $n + 1$. □

Next, we address the case (iii) in Theorem B.2.2.

Lemma B.3.2 *If* (B.15a), (B.15b) *holds, as* $k \to 0$ *we have*

$$\|u_1 - \mathcal{P}u_0\|^2_{\mathcal{H}} + k\|u_1 - \mathcal{P}u_0\|^2_{\mathcal{V}} \le Ck^2. \tag{B.26}$$

Proof We use (B.18) and insert $v = \mathcal{P}u_0 \in \overline{\mathcal{K}}^{\|\cdot\|_{\mathcal{H}}}$ into (B.23) to obtain

$$\begin{aligned}\|u_1 - \mathcal{P}u_0\|^2_{\mathcal{H}} + k\|u_1 - \mathcal{P}u_0\|^2_{\mathcal{V}} \le C\Big\{& k\|u_1 - \mathcal{P}u_0\|_{\mathcal{H}} \int_0^k \|g(\tau)\|_{\mathcal{H}}\,\mathrm{d}\tau \\ &+ k\big|(h(0) - \mathcal{A}\mathcal{P}u_0, u_1 - \mathcal{P}u_0)\big| \\ &+ |u_1 - \mathcal{P}u_0|_{\mathcal{V}} \int_0^k \|h(\tau) - h(0)\|_*\,\mathrm{d}\tau\Big\}.\end{aligned} \tag{B.27}$$

The assertion (B.26) follows from (B.27) with the estimates

$$\int_0^k \|g(\tau)\|_{\mathcal{H}}\,\mathrm{d}\tau \le k\,\|g\|_{L^\infty(0,k;\mathcal{H})}\,,$$

$$\int_0^k \|h(\tau) - h(0)\|_*\,\mathrm{d}\tau \le C\,k^{3/2}\,\|h'\|_{L^2(0,k;\mathcal{V}^*)}\,.$$ □

Remark B.3.3 Inserting $v = u_{k,1}$ into (B.12b), we obtain by similar arguments

$$\begin{aligned}&\|u_{k,2} - u_{k,1}\|^2_{\mathcal{H}} + k\|u_{k,2} - u_{k,1}\|^2_{\mathcal{V}} \\ &\quad \le (f_{k,1} - k\mathcal{A}u_{k,1},\, u_{k,2} - u_{k,1}) \\ &\quad \le Ck^2\Big\{\sup_{0<\tau<2k} \|g(\tau)\|^2_{\mathcal{H}} + \|h(0) - \mathcal{A}\mathcal{P}u_0\|^2_{\mathcal{H}} + \int_0^{2k} \|h'(\tau)\|^2_*\,\mathrm{d}\tau\Big\}.\end{aligned} \tag{B.28}$$

For the proof of Theorem B.2.2 as well as for a priori error estimates, we require the following perturbation results.

Lemma B.3.4 *Assume* $0 \in \mathcal{K}$ *and we are given sequences* $\{w_m\}, \{p_m\}, \{q_m\}$ *which satisfy*

$$w_0 \in \mathcal{H},\, w_{m+1} \in \mathcal{K},\;\; p_m \in \mathcal{H},\, q_m \in \mathcal{V}^* \textit{ for } m \ge 0\,, \tag{B.29}$$

and are such that, for $m \ge 0$, *it holds*

$$\langle w_{m+1} - w_m + k\mathcal{A}w_{m+1}, w_{m+1}\rangle_{\mathcal{V}^*,\mathcal{V}} \le k(p_m + q_m, w_{m+1})\,. \tag{B.30}$$

Define, for $M \ge 1$, *with* α *as in* (B.4b),

$$X_M := \max_{1\le m\le M}\{\|w_m\|_{\mathcal{H}}\}, \qquad Y_M := \Big(\alpha\, k \sum_{m=1}^{M} \|w_m\|_{\mathcal{V}}^2\Big)^{\frac{1}{2}} \tag{B.31a}$$

$$P_M := k \sum_{m=0}^{M-1} \|p_m\|_{\mathcal{H}}, \qquad Q_M := \Big(\|w_0\|_{\mathcal{H}}^2 + \frac{k}{\alpha} \sum_{m=0}^{M-1} \|q_m\|_*^2\Big)^{\frac{1}{2}}. \tag{B.31b}$$

Then it holds

$$\max(X_M, Y_M) \le P_M + (P_M^2 + Q_M^2)^{\frac{1}{2}}. \tag{B.32}$$

Proof By (B.30) and (B.4b) with $\lambda = 0$,

$$\begin{aligned} &2(w_{m+1} - w_m, w_{m+1}) + 2k\alpha\|w_{m+1}\|_{\mathcal{V}}^2 \\ &\quad\le 2k\|p_m\|_{\mathcal{H}}\|w_{m+1}\|_{\mathcal{H}} + 2k\|q_m\|_*\|w_{m+1}\|_{\mathcal{H}} \\ &\quad\le 2k\|p_m\|_{\mathcal{H}}\|w_{m+1}\|_{\mathcal{H}} + \frac{k}{\alpha}\|q_m\|_*^2 + \alpha k\|w_{m+1}\|_{\mathcal{H}}^2. \end{aligned}$$

Using here

$$(w_{m+1} - w_m, w_{m+1}) \ge \|w_{m+1}\|_{\mathcal{H}}^2 - \|w_m\|_{\mathcal{H}}^2 \tag{B.33}$$

and summing from $m = 0, \ldots, M-1$, we get

$$\|w_M\|_{\mathcal{H}}^2 + Y_M^2 \le 2k \sum_{m=0}^{M-1} \|w_{m+1}\|_{\mathcal{H}}\|p_m\|_{\mathcal{H}} + Q_M^2 \le 2X_M P_M + Q_M^2. \tag{B.34}$$

For $M \ge 1$, we infer from (B.34) the bound

$$Y_M^2 \le 2X_M P_M + Q_M^2, \tag{B.35}$$

and also, for $1 \le m \le M$,

$$\|w_m\|_{\mathcal{H}}^2 \le 2X_M P_M + Q_M^2. \tag{B.36}$$

Hence, from (B.36),

$$X_M^2 = \max_{1\le m\le M} \|w_m\|_{\mathcal{H}}^2 \le 2X_M P_M + Q_M^2,$$

and, combining with (B.35), we get (B.32). □

Remark B.3.5 We can also bound the jumps $w_{m+1} - w_m$:

$$\sum_{m=0}^{M-1} \|w_{m+1} - w_m\|_{\mathcal{H}}^2 \le P_M + (P_M^2 + Q_M^2)^{\frac{1}{2}}. \tag{B.37}$$

To obtain this, replace (B.33) by

$$2(v - w, v) = \|v - w\|_{\mathcal{H}}^2 + \|v\|_{\mathcal{H}}^2 - \|w\|_{\mathcal{H}}^2.$$

We also need a continuous analogue of Lemma B.3.4.

Lemma B.3.6 *Assume* (B.4b) *with* $\lambda = 0$, $\alpha > 0$, *and that we are given* $w : J \to \mathcal{V}$, $s : J \to \mathcal{V}^*$ *and* $r : J \to \mathbb{R}_+$ *with* $I_{\text{loc}}(0, \infty) := L^2_{\text{loc}}([0, \infty); \mathcal{V}) \cap L^\infty_{\text{loc}}([0, \infty); \mathcal{H}^*)$ *and*

$$w, w' \in I_{\text{loc}}(0, \infty),\ w(t) \in \mathcal{K} \text{ for } t > 0\,, \tag{B.38a}$$

$$r(t) \in L^1_{\text{loc}}(0, \infty), \tag{B.38b}$$

$$s(t) = p(t) + q(t);\ \ p(t) \in L^1_{\text{loc}}([0, \infty), \mathcal{H}),\ \ q(t) \in L^2_{\text{loc}}([0, \infty); \mathcal{V}^*)\,, \tag{B.39}$$

and such that

$$\langle w' + \mathcal{A}w, w\rangle_{\mathcal{V}^*,\mathcal{V}} \le \langle s(t), w(t)\rangle_{\mathcal{V}^*,\mathcal{V}} + r(t)\,. \tag{B.40}$$

Define, for $T > 0$ *and for any decomposition* (B.39),

$$X(T) := \|w\|_{L^\infty(J;\mathcal{H})},\ \ Y(T) := \sqrt{\alpha}\,\|w\|_{L^2(J;\mathcal{V})}, \tag{B.41}$$

$$P(T) := \|p\|_{L^1(J;\mathcal{H})},\ \ Q(T) := \Big(\|w(0)\|^2_{\mathcal{H}} + \frac{1}{\alpha}\,\|q\|^2_{L^2(J;\mathcal{V}^*)} + 2R(T)\Big)^{\frac12}, \tag{B.42}$$

$$R(T) := \sup_{0<\theta<T} \int_0^\theta r(\tau)\,\mathrm{d}\tau\,. \tag{B.43}$$

Then it holds

$$\max\{X(T), Y(T)\} \le P(T) + \big(P(T)^2 + Q(T)^2\big)^{\frac12} \tag{B.44}$$

$$\|w\|_{I(0,T)} \le C\big\{\|w(0)\|_{\mathcal{H}} + \|s\|_{S(0,T)} + \sqrt{R(T)}\big\}\,. \tag{B.45}$$

Proof We integrate (B.40) over J and, by (B.4b) with $\lambda = 0$, obtain with (B.39) and $|\langle q, w\rangle_{\mathcal{V}^*,\mathcal{V}}| \le \|q\|_*\|w\|_{\mathcal{V}}$

$$\begin{aligned}\|w(T)\|^2_{\mathcal{H}} &+ 2\alpha \int_0^T \|w(\tau)\|^2_{\mathcal{V}}\,\mathrm{d}\tau\\ &\le \|w(0)\|^2_{\mathcal{H}} + 2\int_0^T \|p(\tau)\|_{\mathcal{H}}\|w(\tau)\|_{\mathcal{H}}\,\mathrm{d}\tau\\ &\quad + 2\int_0^T \|q(\tau)\|_*\|w(\tau)\|_{\mathcal{V}}\,\mathrm{d}\tau + 2R(T)\,.\end{aligned}$$

This implies $\|w(T)\|^2 + Y(T)^2 \le 2X(T)\,P(T) + Q(T)^2$. Now we argue as in the proof of Lemma B.3.4 to obtain (B.44). On the right hand side of (B.44), we may take the infimum over all decompositions (B.39). □

Remark B.3.7 If in (B.4b) $\lambda > 0$, we define $\widetilde{w}(t) = e^{-\lambda t} w(t)$, and $\widetilde{s} = e^{-\lambda t} s$, $\widetilde{p} = e^{-\lambda t} p$, $\widetilde{q} = e^{-\lambda t} q$, $\widetilde{r} = e^{-2\lambda t} r$. The new quantities satisfy

$$\langle \widetilde{w}' + \widetilde{\mathcal{A}}\widetilde{w}, \widetilde{w}\rangle_{\mathcal{V}^*,\mathcal{V}} \le \langle \widetilde{s}, \widetilde{w}\rangle_{\mathcal{V}^*,\mathcal{V}} + \widetilde{r}, \tag{B.46}$$

where $\widetilde{\mathcal{A}} := \mathcal{A} + \lambda I$ satisfies (B.4b) with $\lambda = 0$. Then, (B.45) for the new quantities implies

$$\begin{aligned}&\|e^{-\lambda t} w(t)\|_{I(0,T)}\\ &\quad \le C\Big\{\|w(0)\|_{\mathcal{H}} + \|e^{-\lambda t} s(t)\|_{S(0,T)} + \Big(\sup_{0<\theta<T} \int_0^\theta e^{-2\lambda\tau} r(\tau)\,\mathrm{d}\tau\Big)^{\frac{1}{2}}\Big\}. \end{aligned} \tag{B.47}$$

We apply Lemma B.3.4 to the time-discrete PVI (B.12a), (B.12b). We assume (B.4b) with $\lambda = 0$ and choose

$$p_m = \frac{1}{k}\int_{J_{k,m}} p(\tau)\,\mathrm{d}\tau, \quad q_m = \frac{1}{k}\int_{J_{k,m}} q(\tau)\,\mathrm{d}\tau \tag{B.48}$$

where $s = p + q$ satisfies (B.39). Then, by Lemma B.3.4,

$$P_M \le \|p\|_{L^1(0,Mk;\mathcal{H})}, \quad Q_M \le \|w_0\|_{\mathcal{H}} + \alpha^{-\frac{1}{2}} \|q\|_{L^2(0,Mk;\mathcal{V}^*)}\,.$$

Then (B.32) implies

$$\max\{X_M, Y_M\} \le C\{\|w_0\|_{\mathcal{H}} + \|s\|_{S(0,Mk)}\}\,. \tag{B.49}$$

Remark B.3.8 In (B.49), we assumed (B.4b) with $\lambda = 0$. If $\lambda > 0$, we replace (B.48) by

$$p_m = \frac{\int_{J_{k,m}} e^{-\lambda\tau} p(\tau)\,\mathrm{d}\tau}{\int_{J_{k,m}} e^{-\lambda\tau}\mathrm{d}\tau}\,, \quad q_m = \frac{\int_{J_{k,m}} e^{-\lambda\tau} q(\tau)\,\mathrm{d}\tau}{\int_{J_{k,m}} e^{-\lambda\tau}\mathrm{d}\tau}\,. \tag{B.50}$$

We define for k sufficiently small $\widetilde{w}_m = (1-\lambda k)^m w_m$, and analogously $\widetilde{p}_m$, $\widetilde{q}_m$. Then the same reasoning as in Remark B.3.7 gives for k sufficiently small

$$\max\{\widetilde{X}_M, \widetilde{Y}_M\} \le C\Big\{\|w_0\|_{\mathcal{H}} + \|e^{-\lambda\tau} s(\tau)\|_{S(0,Mk)}\Big\}\,, \tag{B.51}$$

if we use $1/(1-\lambda k)^M \le \exp(-\lambda k^M/(1-\lambda k))$.

We now apply (B.49), (B.51) to obtain a priori estimates for the approximate solution $U_k(t)$ obtained from (B.12a), (B.12b), (B.13a), (B.13b). To this end, we identify the sequence $\{w_m\}$ with the step function $\overline{w}_k(t)$ such that

$$\overline{w}_k(t)\big|_{J_{k,m}} = w_m, \quad m = 1, \ldots, M\,. \tag{B.52}$$

Lemma B.3.9 *Assume* (B.4b) *with* $\lambda = 0$. *Then there exists* $C > 0$ *depending only on* α *such that*

$$\|\overline{w}_k(t+k)\|_{I_{(0,T)}} \le C\,(\|w_0\|_{\mathcal{H}} + \|s(t)\|_{S(0,T)}) \tag{B.53}$$

and

$$\|\overline{w}_k(t+k)\|_{I_{(0,T)}} \le C\,(\|w_1\|_{\mathcal{H}} + \sqrt{k}\|w_1\|_{\mathcal{V}} + \|s(t)\|_{S(0,T+k)})\,. \tag{B.54}$$

Proof Since $T = Mk$, (B.49) and (B.52) give (B.53). To show (B.54), we may assume $T > k$. We apply (B.49) to the shifted sequence $\{w_{m+1}\}_{m=1}^M$. □

We are now in position to give a priori estimates for the sequence of solutions to (B.12a), (B.12b).

Proposition B.3.10 *Define the step-functions* $\{\overline{u}_k(t)\}_{k>0}$ *by*

$$\overline{u}_k(t)\Big|_{J_{k,m}} := u_{k,m}, \quad m = 0, \dots, M-1, \tag{B.55}$$

and analogously $\overline{f}_k(t)$. *Assume* $0 \in \mathcal{K}$ *and* (B.4b) *with* $\lambda = 0, \alpha > 0$. *Then,*

$$\|\overline{u}_k(t+k)\|_{I(0,T)} \le C(\alpha)\left\{\|w_0\|_{\mathcal{H}} + \|\overline{f}_k(t)\|_{S(0,T+k)}\right\}, \tag{B.56}$$

$$\|\overline{u}_k(t+k)\|_{I(0,T)} \le C(\alpha)\left\{\|u_{k,1}\|_{\mathcal{H}} + \sqrt{k}\|u_{k,1}\|_{\mathcal{V}} + \|\overline{f}_k(t)\|_{S(0,T+k)}\right\}. \tag{B.57}$$

Proof We choose $0 = v \in \mathcal{K}$ in (B.12b). Since, for $m > 0$, $u_{k,m} \in \mathcal{K}$ and we have (B.29), (B.30) with $w_m = u_{k,m}$, $f_{k,m} = p_m + q_m$, (B.56) and (B.57) follow then from (B.53), (B.54). □

We also have a bound for $U_k(t)$ in (B.13a), (B.13b).

Lemma B.3.11 *Assume* (B.4b) *with* $\lambda = 0$ *and* $0 \in \mathcal{K}$. *The semi-discretization* (B.11), (B.12a), (B.12b) *of PVI* (B.9a)–(B.9e) *is stable in the sense that for* $T = Mk$

$$\|U_k(t)\|_{I(0,T)} \le C\left\{\|u_0\|_{\mathcal{H}} + \|f(t)\|_{S(0,T+2k)}\right\}. \tag{B.58}$$

Proof For $T = Mk$, we have

$$\|f_k(t)\|_{S(0,T)} \le \|f(t)\|_{S(0,T+k)}; \quad \|U_k(t)\|_{I(0,T)} \le \|\overline{u}_k(t)\|_{I(0,T+k)}.$$

Inserting this into (B.56), we get (B.58). □

Remark B.3.12 Note that we could also obtain an a priori bound for $U_k(t)$ from (B.57): here (B.25) with $n = 1$ would imply

$$\|U_k(t)\|_{I(0,T)} \le C\left\{\|\mathcal{P}u_0\|_{\mathcal{H}} + \|f(t)\|_{S(0,T+2k)}\right\}. \tag{B.59}$$

We are now able to show our first main result. To state it, we define, for $\overline{f}_k(t)$ as in (B.55),

$$E(k,T) := \begin{cases} \|\overline{f}_k(t) - f(t)\|_{S(0,T+2k)} + \|f(t+k) - f(t)\|_{S(0,T+2k)} \\ \quad + \|u_{k,2} - u_{k,1}\|_{\mathcal{H}} + \sqrt{k}\|u_{k,2} - u_{k,1}\|_{\mathcal{V}} + \|u_{k,1} - \mathcal{P}u_{k,0}\|_{\mathcal{H}}. \end{cases} \tag{B.60}$$

Lemma B.3.13 *We have for* $T > 0$, *as* $k = T/M \to 0$,

$$\|kU_k'(t)\|_{I(0,T)} \le C\,E(k,T). \tag{B.61}$$

Proof We note that due to its definition (B.13a), (B.13b), $kU_k'(t)|_{J_{k,m}} = u_{k,m+2} - u_{k,m+1}$. Selecting $v = u_{k,m+2}$ in (B.12b) and also $v = u_{k,m+1}$ in (B.12b) with $m+1$ in place of m gives the two inequalities:

$$\langle u_{k,m+1} - u_{k,m} + k\mathcal{A}u_{k,m+1} - f_{k,m},\, u_{k,m+1} - u_{k,m+2}\rangle_{\mathcal{V}^*,\mathcal{V}} \le 0,$$
$$-\langle u_{k,m+2} - u_{k,m+1} + k\mathcal{A}u_{k,m+2} - f_{k,m+1},\, u_{k,m+1} - u_{k,m+2}\rangle_{\mathcal{V}^*,\mathcal{V}} \le 0.$$

Adding these inequalities together gives an inequality of the type (B.30), where $p_m + q_m = f_{k,m+2} - f_{k,m+1}$, $w_m = u_{k,m+1} - u_{k,m}$, $m \geq 1$. We apply Lemma B.3.9 to $\overline{w}_k(t)$ defined in (B.52), with $s(t) = \overline{f}_k(t+k)$ and $w_0 := u_{k,1} - Pu_0$. Observing that $w_k(t) = kU_k'(t)$, we get (B.61) with (B.60) from (B.53), (B.54). $\square$

From (B.61) the size of $E(k,T)$ as $k \to 0$ for fixed $T > 0$ is important. We have

Lemma B.3.14 *Assume* $0, u_0 \in \overline{\mathcal{K}}^{\|\cdot\|_{\mathcal{H}}}$ *and* (B.11). *Then*

$$E(k,T) = o(1) \text{ as } k \to 0\,, \tag{B.62a}$$

and, for compatible data satisfying (B.15a), (B.15b), *it holds*

$$E(k,T) \leq C\,k \text{ as } k \to 0\,. \tag{B.62b}$$

Proof We show (B.62b). The terms in the second row of the definition (B.60) of $E(k,T)$ can be bounded as in (B.62b), by (B.26), (B.28). The terms in the first row of (B.60) are bound using (B.15a), (B.15b) as follows: we decompose $f = g + h \in S(0,T)$ and define $\overline{g}_k(t)$, $\overline{h}_k(t)$ as in (B.55). Assume $k = T/M$ for $M \in \mathbb{N}$. Then

$$\begin{aligned}
\|h(t+k) - h(t)\|_{L^2(J;\mathcal{V}')} &\leq k\|h'(t)\|_{L^2(0,T+k;\mathcal{V}^*)}\,,\\
\|\overline{h}_k(t) - h(t)\|_{L^2(J;\mathcal{V}')} &\leq k\|h'(t)\|_{L^2(0,T+k;\mathcal{V}^*)}\,,
\end{aligned}$$

as well as

$$\begin{aligned}
\|g(t+k) - g(t)\|_{L^2(J;\mathcal{H})} &= \sum_{m=0}^{M-1} \int_{J_{k,m}} \|g(\tau+k) - g(\tau)\|_{\mathcal{H}}\,\mathrm{d}\tau\\
&= \int_0^k \sum_{m=0}^{M-1} \|g(\tau + (m+1)k) - g(\tau + m\,k)\|_{\mathcal{H}}\,\mathrm{d}\tau\\
&\leq k \sum_{m=0}^{M-1} \mathrm{Var}(g, J_{k,m}; \mathcal{H})\\
&\leq k\mathrm{Var}(g, J; \mathcal{H})\,.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\|\overline{g}_k(t) - g(t)\|_{L^1(J;\mathcal{H})} &= \sum_{m=0}^{M-1} \int_{J_{k,m}} \left\| \frac{1}{k} \int_{J_{k,m}} g(\tau)\,\mathrm{d}\tau - g(t) \right\|_{\mathcal{H}} \mathrm{d}t\\
&\leq \frac{1}{k} \sum_{m=0}^{M-1} \int_{J_{k,m}} \int_{J_{k,m}} \|g(s) - g(t)\|_{\mathcal{H}}\,\mathrm{d}t\,\mathrm{d}s\\
&\leq k \sum_{m=0}^{M-1} \mathrm{Var}(g, J_{k,m}; \mathcal{H})\\
&\leq k\mathrm{Var}(g, J; \mathcal{H})\,.
\end{aligned}$$

This gives (B.62b), if we take the infimum over all decompositions of $f \in S(0, T)$ of the form $f = g + h$ as in (B.15a), provided u_0 satisfies also (B.15b).

To show (B.62a) for general $f \in S(a, b)$, $u_0 \in \mathcal{H}$, we use (B.21), (B.25) to bound the second row of (B.60), and approximate a general $f \in S(0, T)$ from $\mathrm{BV}(J; \mathcal{H}) + H^1(J; \mathcal{V}^*)$ to bound the first row of (B.60) by $o(1)$ as $k \to 0$. □

We are now ready to prove the existence Theorem B.2.2. To this end, we show that the family $\{U_k(t)\}_{k>0}$ is Cauchy in $I(0, T)$. More precisely, there is $C > 0$ such that

$$\|U_k(t) - U_h(t)\|_{I(0,T)} \leq C\left\{\sqrt{E(k, T)} + \sqrt{E(h, T)}\right\}. \tag{B.63}$$

This and (B.62a), (B.62b) imply that $\{U_k\}_{k>0}$ is Cauchy in $I(0, T)$ and that there is $U = \lim_{k\to 0} U_k(t) \in I(0, T)$ with

$$\|U(t) - U_k(t)\|_{I(0,T)} \leq \sqrt{C E(k, T)}. \tag{B.64}$$

We note that (B.64) with (B.62a), (B.62b) gives an error estimate for (B.12a), (B.12b). To prove (B.63), we recall the definition (B.55) of $\overline{u}_k(t)$ and we also define

$$\widetilde{u}_k(t) = \ell_k(t)\,\overline{u}_k(t) + (1 - \ell_k(t))\,\overline{u}(t + k), \tag{B.65}$$

where

$$\ell_k(t) := m + 1 - t/k \in [0, 1], \quad t \in J_{k,m} = [mk, (m + 1)k]. \tag{B.66}$$

Then it follows from (B.12a), (B.12b) that for all $t \in J$ holds

$$\left\langle \widetilde{u}_k' + A\overline{u}_k(t + k) - \overline{f}_k(t), \overline{u}_k(t + k) - v\right\rangle_{\mathcal{V}^*,\mathcal{V}} \leq 0 \quad \forall v \in \mathcal{K}. \tag{B.67}$$

To prove (B.63), we proceed as follows: we rewrite (B.67) in terms of $U_k(t)$ in (B.13a), (B.13b) with small right hand side. Then (B.63) will be obtained by application of Lemma B.3.6 to $U_k(t) - U_h(t)$.

We have from $0 \leq \ell_k \leq 1$ that in $\mathcal{H}$

$$\|\widetilde{u}_k(t) - \overline{u}_k(t + k)\|_{\mathcal{H}} \leq \|\overline{u}_k(t) - \overline{u}_k(t + k)\|_{\mathcal{H}} = \|k\widetilde{u}_k'(t)\|_{\mathcal{H}}, \tag{B.68}$$

and also in $\mathcal{V}$, and for every $(a, b) \subseteq J$, that

$$\|\widetilde{u}_k(t)\|_{I(a,b)} \leq \|\overline{u}_k(t)\|_{I(a,b)} + \|\overline{u}_k(t + k)\|_{I(a,b)} \leq 2\,\|\overline{u}_k(t)\|_{I(a,b+k)}. \tag{B.69}$$

Lemma B.3.15 *For any $v \in \mathcal{K}$, the following holds*:

$$\begin{aligned} &\langle U_k'(t) + AU_k(t) - \overline{f}_k(t + k),\ U_k(t) - v\rangle_{\mathcal{V}^*,\mathcal{V}} \\ &\quad \leq \beta\|kU_k'(t)\|_{\mathcal{V}}\|U_k(t) - v\|_{\mathcal{V}} \\ &\qquad - \ell_k(t)\left\langle U_k'(t) + \mathcal{A}\overline{u}_k(t + k) - \overline{f}_k(t + k), kU_k'(t)\right\rangle_{\mathcal{V}^*,\mathcal{V}}. \end{aligned} \tag{B.70}$$

Proof We have by (B.4a)

$$\begin{aligned} \langle \mathcal{A}U_k(t), U_k(t) - v\rangle_{\mathcal{V}^*,\mathcal{V}} \leq\ & \langle \mathcal{A}\overline{u}_k(t + k), U_k(t) - v\rangle_{\mathcal{V}^*,\mathcal{V}} \\ & + \langle \mathcal{A}\widetilde{u}_k(t + k) - \mathcal{A}\overline{u}_k(t + k), U_k(t) - v\rangle_{\mathcal{V}^*,\mathcal{V}} \\ \leq\ & \langle \mathcal{A}\overline{u}_k(t + k), U_k(t) - v\rangle_{\mathcal{V}^*,\mathcal{V}} \\ & + \beta\|\widetilde{u}_k(t + k) - \overline{u}_k(t + k)\|_{\mathcal{V}}\|U_k(t) - v\|_{\mathcal{V}} \\ =:\ & \mathrm{I} + \mathrm{II}. \end{aligned}$$

We estimate

$$\mathrm{II} \le M\|U_k'(t)\|_{\mathcal{V}}\|U_k(t) - v\|_{\mathcal{V}}\,,$$

and combine I with the left hand side of (B.70). It then remains to estimate

$$\langle U_k'(t) + \mathcal{A}\overline{u}_k(t+k) - \overline{f}_k(t+k), U_k(t) - v\rangle_{\mathcal{V}^*,\mathcal{V}}\,.$$

To this end, we write

$$\begin{aligned} U_k(t) - v &= U_k(t) - \overline{u}_k(t+k) + \overline{u}_k(t+k) - v \\ &= -\ell_k(t)kU_k'(t) + \overline{u}_k(t+k) - v\,, \end{aligned}$$

and obtain

$$\begin{aligned} &\langle U_k'(t) + \mathcal{A}\overline{u}_k(t+k) - \overline{f}_k(t+k), U_k(t) - v\rangle_{\mathcal{V}^*,\mathcal{V}} \\ &\quad = -\langle U_k'(t) + \mathcal{A}\overline{u}_k(t+k) - \overline{f}_k(t+k), \ell_k(t)kU_k'(t)\rangle_{\mathcal{V}^*,\mathcal{V}} \\ &\qquad + \langle U_k'(t) + \mathcal{A}\overline{u}_k(t+k) - \overline{f}_k(t+k), \overline{u}_k(t+k) - v\rangle_{\mathcal{V}^*,\mathcal{V}} \\ &\quad =: \mathrm{III} + \mathrm{IV}\,. \end{aligned}$$

By (B.65), (B.55) and (B.13a), (B.13b), $U_k'(t) = \widetilde{u}'(t+k)$ and, by (B.67) evaluated at $t+k$, $\mathrm{IV} \le 0$ and III implies (B.70). □

Inspecting the proof, we also have

Corollary B.3.16 *For any $v \in \mathcal{K}$, the following holds*:

$$\begin{aligned} &\langle U_k'(t) + \mathcal{A}U_k(t) - \overline{f}_k(t+k), U_k(t) - v\rangle_{\mathcal{V}^*,\mathcal{V}} \\ &\quad \le \langle s(t), U_k(t) - v\rangle_{\mathcal{V}^*,\mathcal{V}} - \ell_k(t)\langle U_k'(t) + \mathcal{A}\overline{u}_k(t+k) - \overline{f}_k(t+k), kU_k'(t)\rangle_{\mathcal{V}^*,\mathcal{V}} \end{aligned} \tag{B.71}$$

where $s(t)$ satisfies

$$\|s(t)\|_{\mathcal{V}} \le \beta\|kU_k'(t)\|_{\mathcal{V}} \;\; a.e.\; t \in J\,. \tag{B.72}$$

Next, we replace $\overline{f}_k(t+k)$ in the bounds (B.70), (B.71).

Corollary B.3.17 *For any $v(t) \in \mathcal{K}$, a.e. $t \in J$, one has*

$$\begin{aligned} &\langle U_k'(t) + \mathcal{A}U_k - f(t), U_k(t) - v\rangle_{\mathcal{V}^*,\mathcal{V}} \\ &\quad \le \langle s(t), U_k(t) - v\rangle_{\mathcal{V}^*,\mathcal{V}} \\ &\qquad + \ell_k(t)\langle U_k'(t) + \mathcal{A}\overline{u}_k(t+k) - \overline{f}_k(t+k), -kU_k'(t)\rangle_{\mathcal{V}^*,\mathcal{V}} \\ &\qquad + \langle \overline{f}_k(t+k) - f(t), U_k(t) - v\rangle_{\mathcal{V}^*,\mathcal{V}}\,, \end{aligned} \tag{B.73}$$

where $s(t)$ satisfies (B.72).

Proof (B.73) follows from (B.71) by adding and subtracting $f(t)$ on the left hand side of (B.71). □

Lemma B.3.18 *For* $u_0 \in \overline{\mathcal{K}}^{\|\cdot\|_{\mathcal{H}}}$ *and* $f \in S(0,T)$, *and any* $h, k > 0$, *one has*

$$\|U_h(t) - U_k(t)\|_{I(0,T)} \le C\left\{\sqrt{E(h,T)} + \sqrt{E(k,T)}\right\}, \tag{B.74}$$

with $E(h,T)$ *as in Lemma* B.3.14. *In particular,* $\{U_h(t)\}_{h>0}$ *is Cauchy in* $I(0,T)$ *and there exists*

$$u = \lim_{h\to 0} U_h(t) \in I(0,T).$$

Proof We choose in (B.73) $v = U_h(t)$ for some $h > 0$, and then exchange in the resulting inequality the roles of k and h. Adding the resulting two inequalities for the difference $w(t) := U_k(t) - U_h(t)$, we get an inequality of the type considered in Lemma B.3.6, with $s(t)$ replaced by $s(t) + \overline{f}_k(t+k) - f(t)$: To determine $r(t)$ in (B.40), we estimate the last term in the bound (B.73) as follows: using $0 \le \ell_k(t) \le 1$ and

$$\ell_k(t)\langle U_k'(t), -kU_k'(t)\rangle_{\mathcal{V}^*,\mathcal{V}} \le 0, \tag{B.75}$$

we have

$$\begin{aligned} 0 \le r(t) &:= \left|\langle \mathcal{A}\overline{u}_k(t+k) - \overline{f}_k(t+k), -kU_k'(t)\rangle_{\mathcal{V}^*,\mathcal{V}}\right| \\ &\le \left(\beta\|\overline{u}_k(t+k)\|_{\mathcal{V}} + \|\overline{f}_k(t+k)\|_{\mathcal{H}}\right)\|kU_k'(t)\|_{\mathcal{V}}. \end{aligned}$$

Hence, in (B.42),

$$\begin{aligned} R(T) &= \int_0^T r(t)\,\mathrm{d}t \\ &\le \mathrm{Const}\left(\|\overline{u}_k(t+k)\|_{I(0,T)} + \|\overline{f}_k(t)\|_{S(0,T+k)}\right) \cdot \|kU_k'(t)\|_{I(0,T)}. \end{aligned}$$

From (B.56), we get

$$R(T) \le \mathrm{Const}\left\{\|w_0\|_{\mathcal{H}} + \|\overline{f}_k(t)\|_{S(0,T+k)}\right\}\|kU_k'(t)\|_{I(0,T)},$$

and (B.61) gives

$$R(T) \le \mathrm{Const}\left\{\|w_0\|_{\mathcal{H}} + \|\overline{f}_k(t)\|_{S(0,T+k)}\right\}E(k,T). \tag{B.76}$$

To estimate the value of $\|w(0)\|_{\mathcal{H}}$ in (B.76) and in (B.43), we use that, by Lemma B.3.2,

$$\|w(0)\|_{\mathcal{H}} = \|(u_{k,1} - \mathcal{P}u_0) - (u_{h,1} - \mathcal{P}u_0)\|_{\mathcal{H}} \le E(k,T) + E(h,T),$$

which, inserted into Lemma B.3.6, implies the assertion. □

We can now give the proof of Theorem B.2.2(i). Lemma B.3.18 established that $\{u_h\}_{h>0}$ is Cauchy in $I(0,T)$, hence in particular in $L^2(J;\mathcal{V})$ and in $L^\infty(J;\mathcal{H})$. Therefore, $u(t) \in C^0(J;\mathcal{H})$ and $u(0) = \lim_{k\to 0} U_k(0) = \lim_{k\to 0} u_{k,1} = \mathcal{P}u_0 \in \overline{\mathcal{K}}^{\|\cdot\|_{\mathcal{H}}}$, which is the third line in (B.6).

To show that $u(t)$ is a solution of the PVI, pick in (B.73) $v(t)$ satisfying (B.7b), and pass in (B.73) to the limit $k \to 0$, implying the second line of (B.6); since $\mathcal{K}$ is closed in $\mathcal{H}$ and $U_h \to u$ in $L^\infty(J;\mathcal{H})$, we also have the first line of (B.6).

The uniqueness and Theorem B.2(ii) will follow from

Lemma B.3.19 *The map* $T : \{u_0, f\} \to u(t)$ *which is a solution of PVI* (B.6) *is Lipschitz from* $\mathcal{H} \times S(0,T) \to I(0,T)$.

Proof Observe that $\{u_0, f\} \to U_k$ is Lipschitz continuous uniformly in k: let $\{u_0^*, f^*\}$ be a second set of initial data. Then pick $v = u_{k,m+1}$ in (B.12b) and also $v = u_{k,m+1}^*$ in (B.12b). To the difference $w = u_{k,m} - u_{k,m}^*$ we may apply Proposition B.3.10 which gives for the extensions of $u_{k,m}$, $u_{k,m}^*$ as in (B.55)

$$\|\overline{u}_k(t+k) - \overline{u}_k^*(t+k)\|_{I(0,T+k)} \leq C\left\{\|u_{k,1} - u_{k,1}^*\|_{\mathcal{H}} + \|\overline{f}_k(t) - \overline{f}_k^*(t)\|_{S(0,\infty)}\right\}.$$

By (B.68), an analogous estimate uniform in k holds also true for the linear extensions $U_k(t)$, $U_k^*(t)$. □

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N. Hilber et al., *Computational Methods for Quantitative Finance*, Springer Finance,
DOI 10.1007/978-3-642-35401-4, © Springer-Verlag Berlin Heidelberg 2013