Representation of Curves & Surfaces

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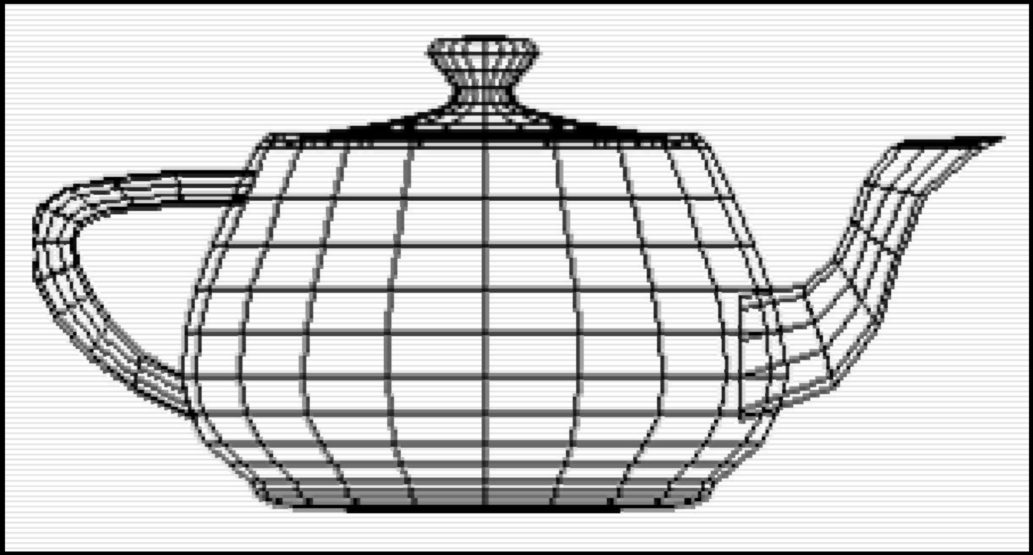
The Teapot

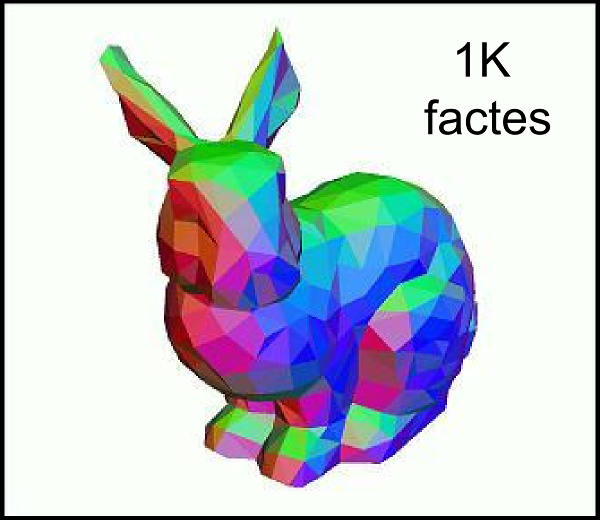


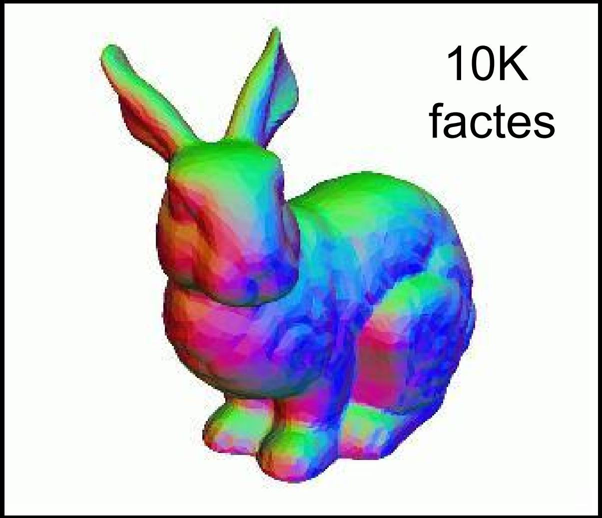
Representing Polygon Meshes

- Explicit representation
- By a list of vertex coordinates

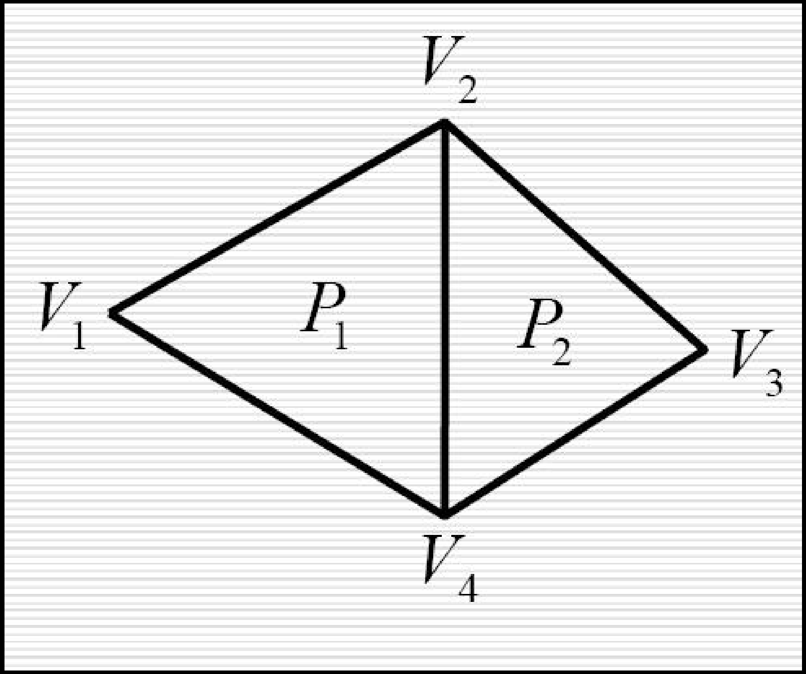
$$P = ((x_1, y_1, z_1), (x_2, y_2, z_2), \ldots, (x_n, y_n, z_n))$$

- Pointers to a vertex list
- Pointers to an edge list





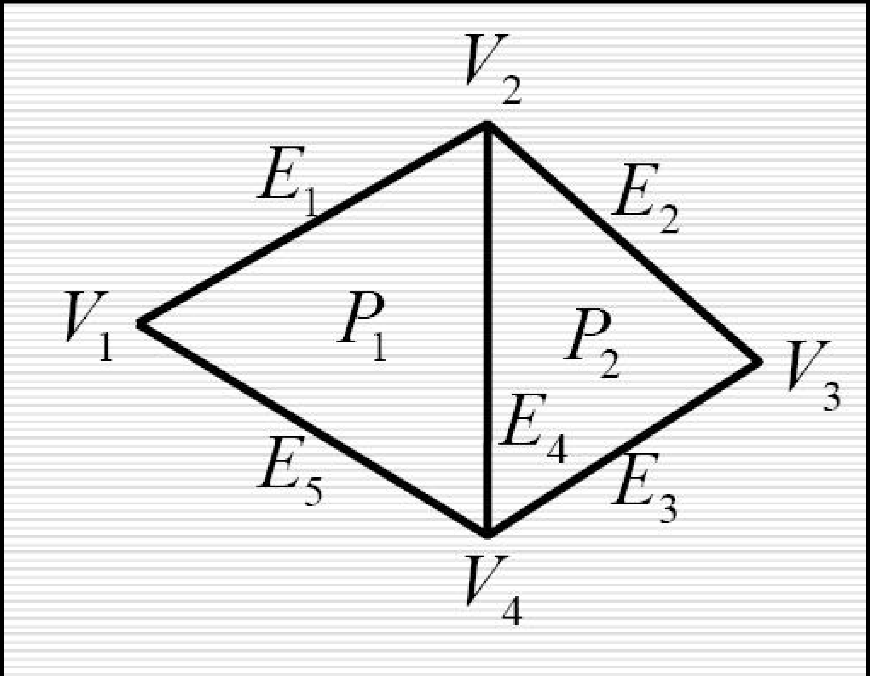
Pointers to A Vertex List



$$V = (V_1, V_2, V_3, V_4)$$
$$= ((x_1, y_1, z_1), \ldots, (x_4, y_4, z_4))$$

$$P_1 = (1,2,4)$$
$$P_2 = (4,2,3)$$

Pointers to An Edge List



$$V = (V_1, V_2, V_3, V_4)$$

$$= ((x_1, y_1, z_1), \ldots, (x_4, y_4, z_4))$$

$$E_1 = (V_1, V_2, P_1, \lambda)$$

$$E_2 = (V_2, V_3, P_2, \lambda)$$

$$E_3 = (V_3, V_4, P_2, \lambda)$$

$$E_4 = (V_4, V_2, P_1, P_2)$$

$$E_5 = (V_4, V_1, P_1, \lambda)$$

$$P_1 = (E_1, E_4, E_5)$$

$$P_2 = (E_2, E_3, E_4)$$

Parametric Cubic Curves

- The cubic polynomials that define a curve segment $Q(t) = [x(t) \quad y(t) \quad z(t)]^T$ are of the form:

$$x(t) = a_x t^3 + b_x t^2 + c_x t + d_x$$

$$y(t) = a_y t^3 + b_y t^2 + c_y t + d_y$$

$$z(t) = a_z t^3 + b_z t^2 + c_z t + d_{z,} \quad 0 \le t \le 1$$

Parametric Cubic Curves

- The curve segment can be rewritten as

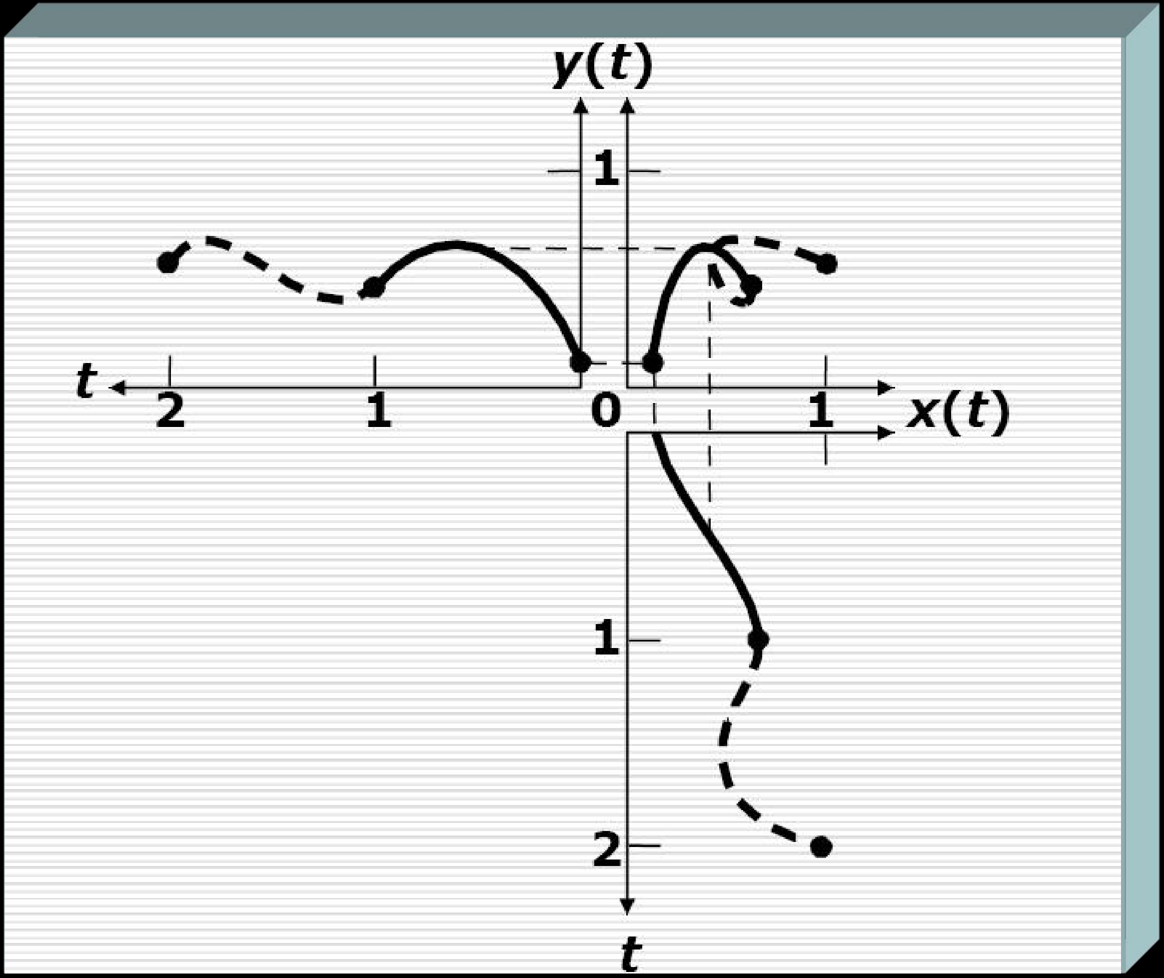
$$Q(t) = [x(t) \quad y(t) \quad z(t)]^T = C \cdot T$$

- Where

$$T = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix}^T$$

$$C = \begin{bmatrix} a_x & b_x & c_x & d_x \\ a_y & b_y & c_y & d_y \\ a_z & b_z & c_z & d_z \end{bmatrix}$$

Parametric Cubic Curves



Tangent Vector

$$\frac{d}{dt}Q(t) = Q^{'}(t) = [\frac{d}{dt}x(t) \quad \frac{d}{dt}y(t) \quad \frac{d}{dt}z(t)]^T$$

$$= \frac{d}{dt}C \cdot T = C \cdot [3t^2 \quad 2t \quad 1]^T$$

$$= [3a_x t^2 + 2b_x t + c_x \quad 3a_y t^2 + 2b_y t + c_y \quad 3a_z t^2 + 2b_z t + c_z]^T$$

Continuity Between Curve Segments

- *G^0* geometric continuity
  - Two curve segments join together
- *G^1* geometric continuity
  - The directions *(but not necessarily the magnitudes)* of the two segments' tangent vectors are equal at a join point

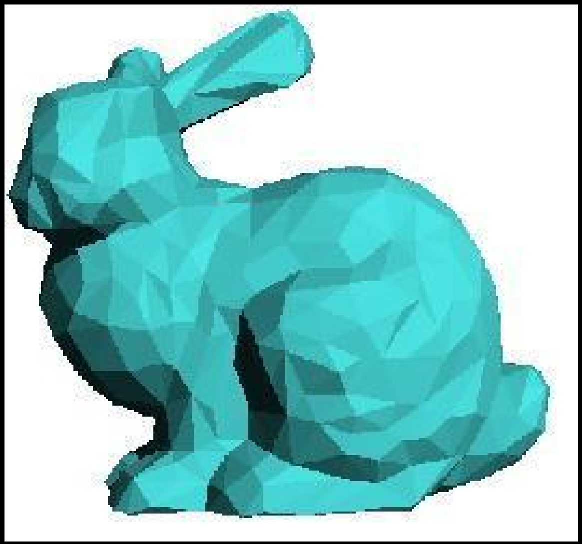
Continuity
Between Curve Segments

- C^1 continuous
  - The tangent vectors of the two cubic curve segments are equal *(both directions and magnitudes)* at the segments' joint point
- C^n continuous
  - The direction and magnitude $d^n / dt^n[Q(t)]$ of through the *n*th derivative are equal at the joint point

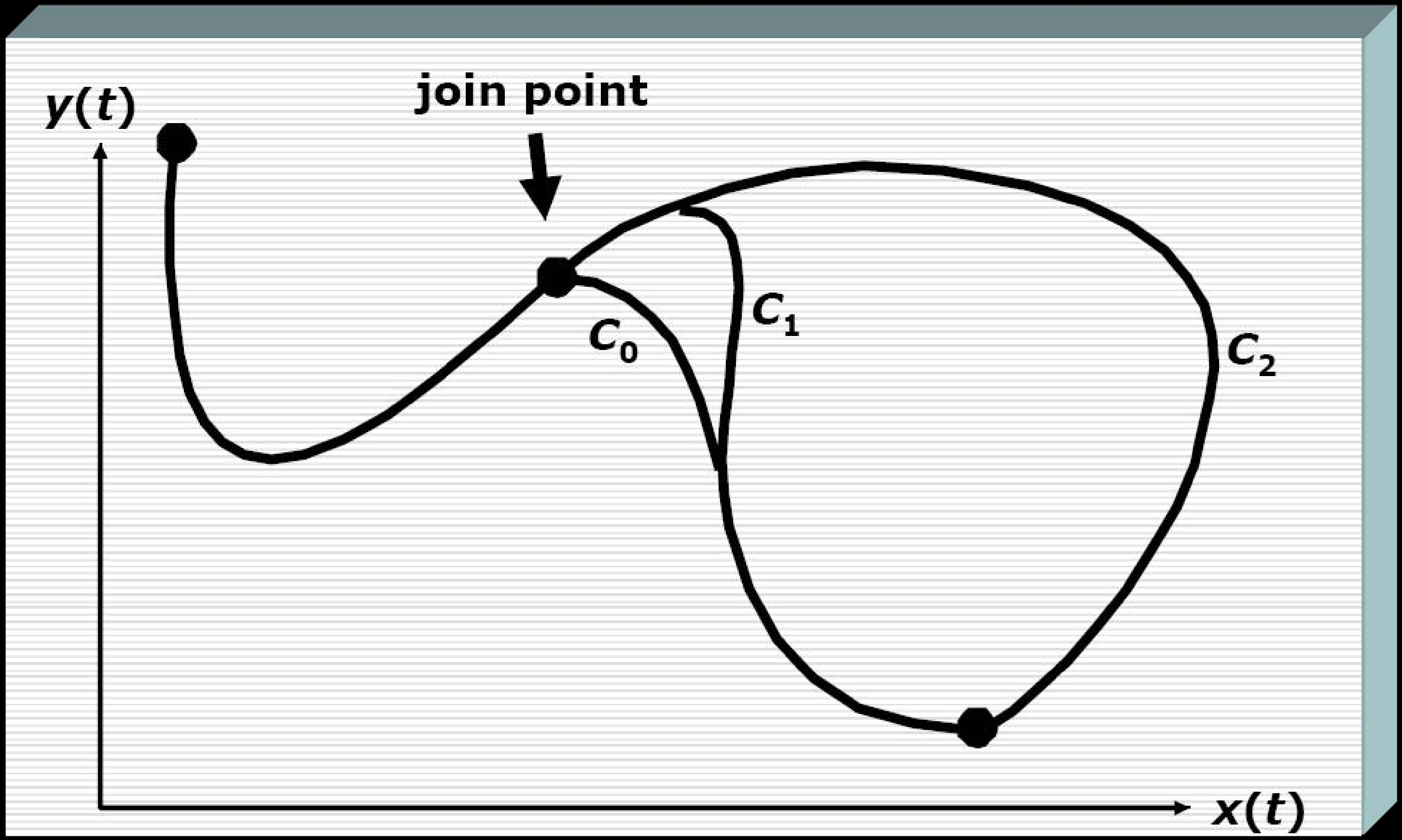
Examples

- C^1 continuous
  - "looks smooth, no facets"

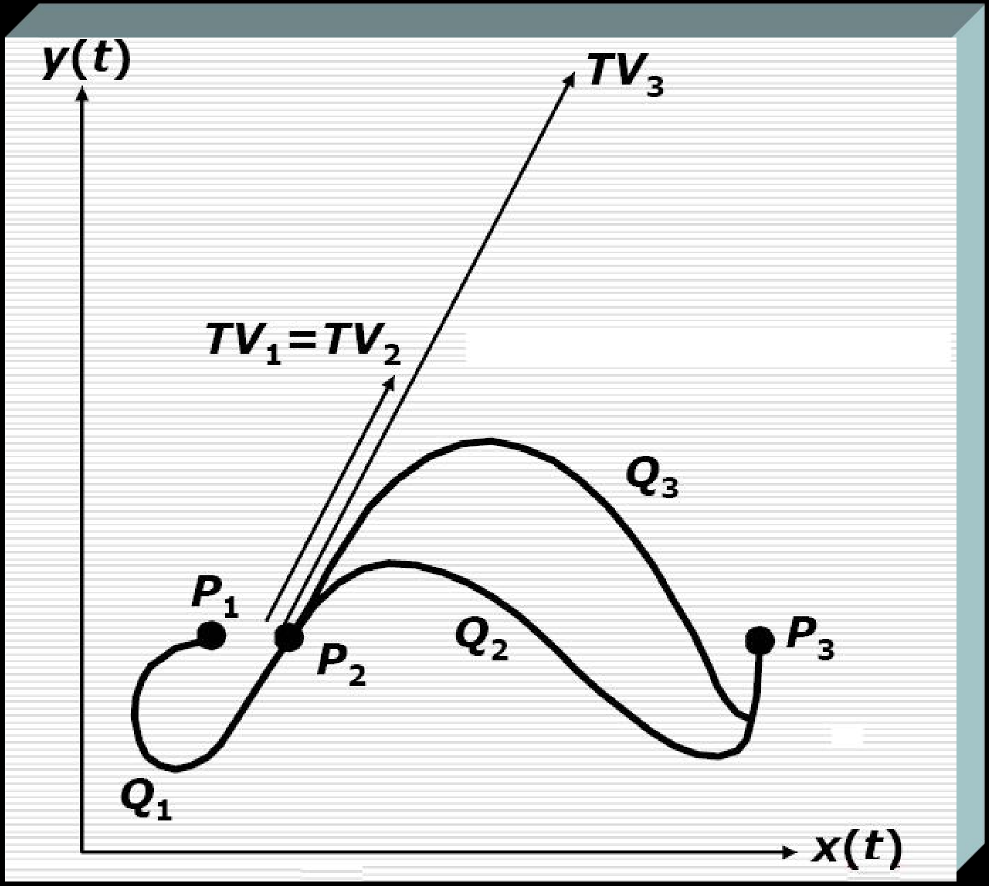
- C^2 continuous
  - Actually important for shading



Continuity
Between Curve Segments



Continuity
Between Curve Segments



Three Types of Parametric Cubic Curves

- Hermite Curves
  - Defined by two **endpoints** and two endpoint **tangent vectors**
- Bézier Curves
  - Defined by two **endpoints** and two **control points** which control the endpoint' **tangent vectors**
- Splines
  - Defined by four **control points**

Parametric Cubic Curves

- Representation: $Q(t) = C \cdot T$
- Rewrite the coefficient matrix as $C = G \cdot M$
  - where *M* is a 4x4 **basis matrix**, *G* is called the **geometry matrix**
  - So

$$Q(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \underbrace{\begin{bmatrix} G_1 & G_2 & G_3 & G_4 \end{bmatrix}}_{\text{4 endpoints or tangent vectors}} \begin{bmatrix} m_{11} & m_{21} & m_{31} & m_{41} \\ m_{12} & m_{22} & m_{32} & m_{42} \\ m_{13} & m_{23} & m_{33} & m_{43} \\ m_{14} & m_{24} & m_{34} & m_{44} \end{bmatrix} \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

Parametric Cubic Curves

$$Q(t) = G \cdot M \cdot T = G \cdot B$$

– Where $B = M \cdot T$ is called the **blending functions**

Hermite Curves

- Given the endpoints P_1 and P_4 and tangent vectors at them R_1 and R_4
- What is
  - **Hermite basis matrix** M_H
  - **Hermite geometry vector** G_H
  - **Hermite blending functions** B_H

- By definition

  $G_H = [P_1 \ \ P_4 \ \ R_1 \ \ R_4]$

Hermite Curves

- Since

$$Q(0) = P_1 = G_H \cdot M_H \cdot \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T$$

$$Q(1) = P_4 = G_H \cdot M_H \cdot \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T$$

$$Q^{'}(0) = R_1 = G_H \cdot M_H \cdot \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}^T$$

$$Q^{'}(1) = R_4 = G_H \cdot M_H \cdot \begin{bmatrix} 3 & 2 & 1 & 0 \end{bmatrix}^T$$

$$G_H = \begin{bmatrix} P_1 & P_4 & R_1 & R_4 \end{bmatrix} = G_H \cdot M_H \cdot \begin{bmatrix} 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

Hermite Curves

- So

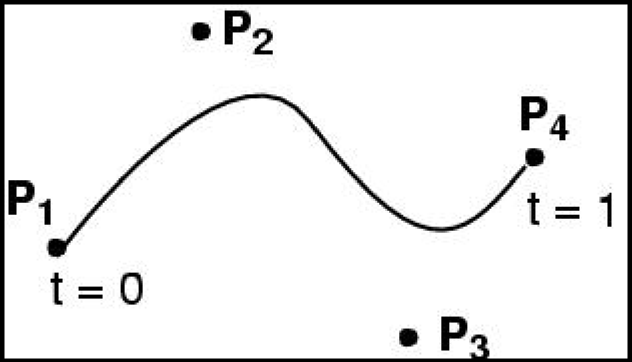
$$M_H = \begin{bmatrix} 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & -3 & 0 & 1 \\ -2 & 3 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$

- And $Q(t) = G_H \cdot M_H \cdot T = G_H \cdot B_H$

$$B_H = \begin{bmatrix} 2t^3 - 3t^2 + 1 & -2t^3 + 3t^2 & t^3 - 2t^2 + t & t^3 - t^2 \end{bmatrix}^T$$

Bézier Curves

- Given the endpoints P_1 and P_4 and two control points P_2 and P_3 which determine the endpoints' tangent vectors, such that



$$R_1 = Q'(0) = 3(P_2 - P_1)$$
$$R_4 = Q'(1) = 3(P_4 - P_3)$$

- What is
  - **Bézier basis matrix** M_B
  - **Bézier geometry vector** G_B
  - **Bézier blending functions** B_B

Bézier Curves

- By definition $G_B = \begin{bmatrix} P_1 & P_2 & P_3 & P_4 \end{bmatrix}$
- Then $G_H = \begin{bmatrix} P_1 & P_4 & R_1 & R_4 \end{bmatrix}$

$$= \begin{bmatrix} P_1 & P_2 & P_3 & P_4 \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -3 \\ 0 & 1 & 0 & 3 \end{bmatrix} = G_B \cdot M_{HB}$$

- So

$$Q(t) = G_H \cdot M_H \cdot T = (G_B \cdot M_{HB}) \cdot M_H \cdot T$$

$$= G_B \cdot (M_{HB} \cdot M_H) \cdot T = G_B \cdot M_B \cdot T$$

Bézier Curves

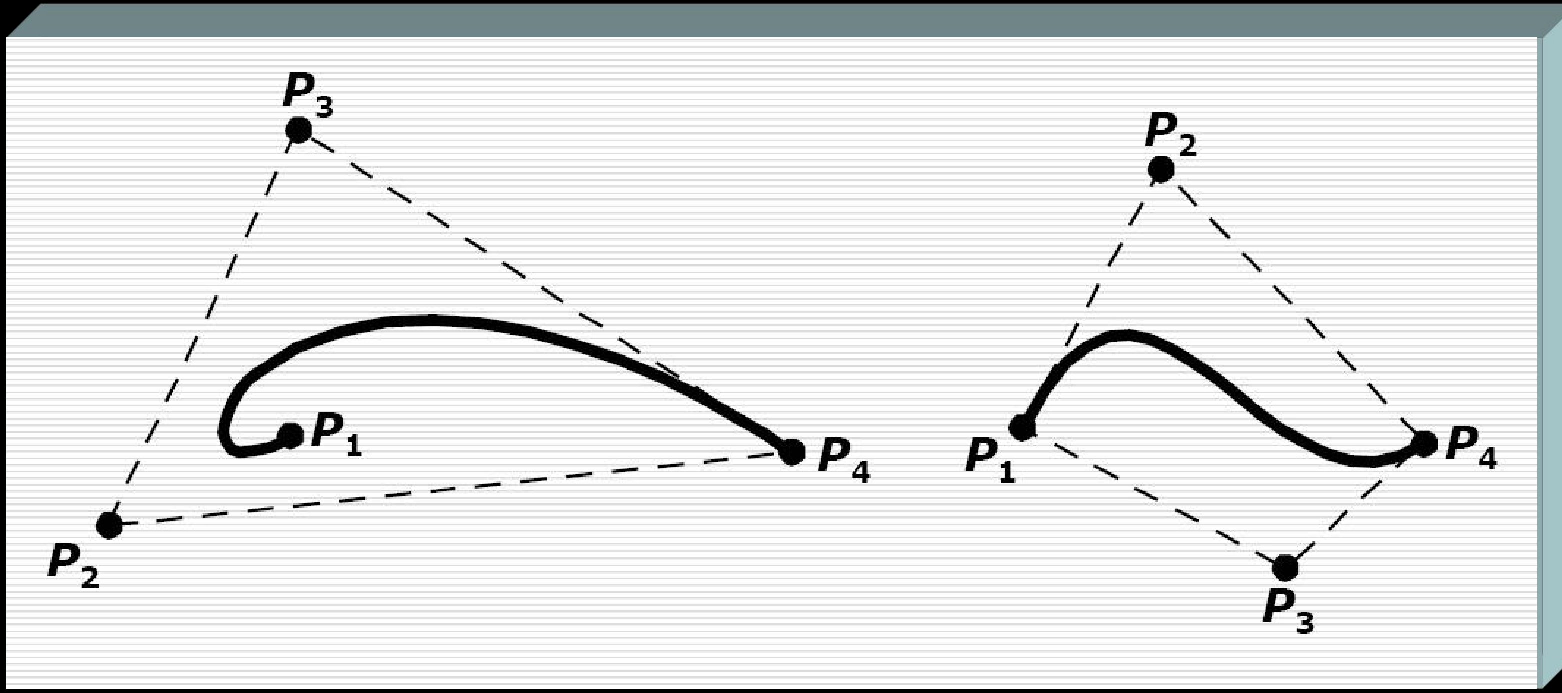
- And

$$M_B = M_{HB} \cdot M_H = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = G_B \cdot M_{HB}$$

$$Q(t) = (1-t)^3 P_1 + 3t(1-t)^2 P_2 + 3t^2(1-t)P_3 + t^3 P_4$$

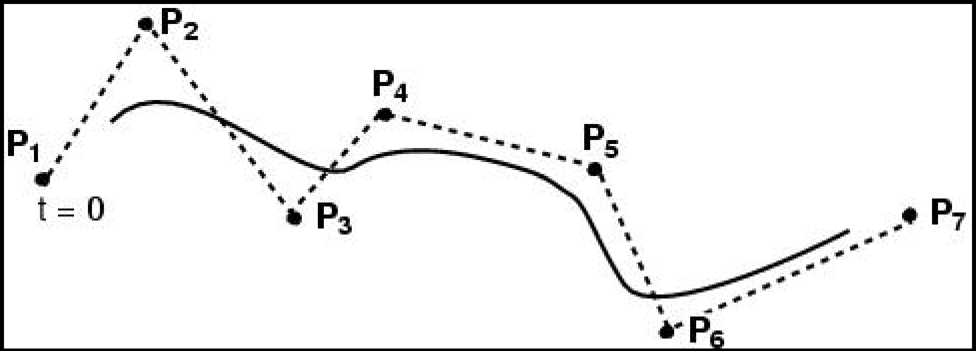
$$BB = \left[(1-t)^3 \quad 3t(1-t)^2 \quad 3t^2(1-t) \quad t^3 \right]^T$$ (Bernstein polynomials)

Convex Hull

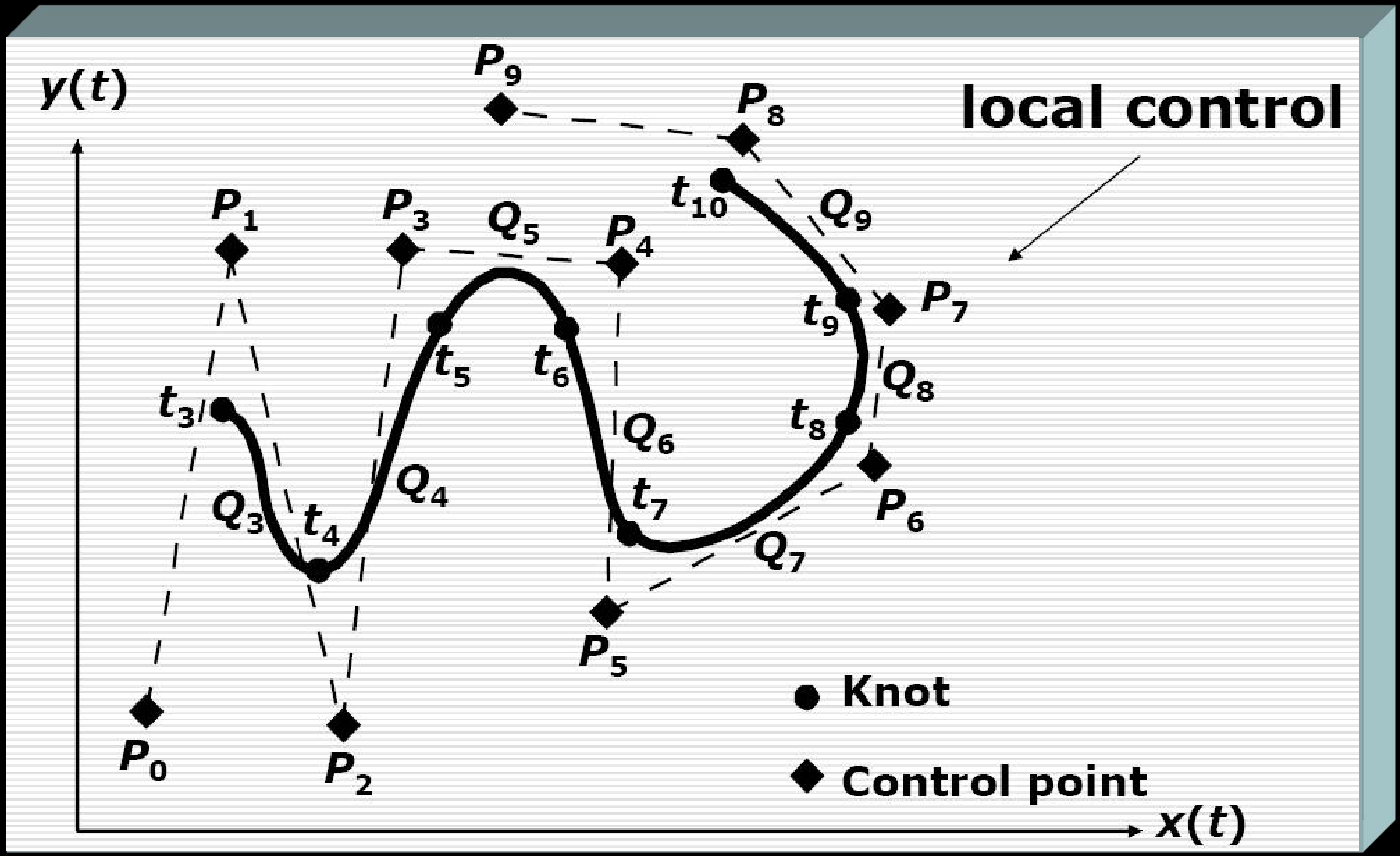


Spline

- The polynomial coefficients for natural cubic splines are dependent on all *n* control points
  - Has one more degree of continuity than is inherent in the Hermite and Bézier forms
  - Moving any one control point affects the entire curve
  - The computation time needed to invert the matrix can interfere with rapid interactive reshaping of a curve



B-Spline



local control
$y(t)$
$x(t)$
P_0
P_1
P_2
P_3
P_4
P_5
P_6
P_7
P_8
P_9
t_3
t_4
t_5
t_6
t_7
t_8
t_9
t_{10}
Q_3
Q_4
Q_5
Q_6
Q_7
Q_8
Q_9
Knot
Control point

Uniform NonRational B-Splines

- Cubic B-Spline
- Has $m+1$ control points $P_0, P_1, \ldots P_m, m \geq 3$
- Has m-2 cubic polynomial curve segments $Q_3, Q_4, \ldots, Q_m$
- Uniform
  - The knots are spaced at equal intervals of the parameter t
- Non-rational
  - Not rational cubic polynomial curves

Uniform NonRational B-Splines

- Curve segment is $Q_i(t)$ defined by points $P_{i-3}, P_{i-2}, P_{i-1}, P_i$
- B-Spline geometry matrix

$$G_{BSi} = \begin{bmatrix} P_{i-3} & P_{i-2} & P_{i-1} & P_i \end{bmatrix}, 3 \le i \le m$$

- If $T_i = \begin{bmatrix} (t - t_i)^3 & (t - t_i)^2 & (t - t_i) & 1 \end{bmatrix}^T$
- Then $Q_i(t) = G_{Bsi} M_{Bs} T_i, t_i \le t \le t_{i+1}$

Uniform NonRational B-Splines

- So B-Spline basis matrix

$$M_{Bs} = \frac{1}{6}\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 0 & 4 \\ -3 & 3 & 3 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

- B-Spline blending functions

$$B_{Bs} = \frac{1}{6}\begin{bmatrix} (1-t)^3 & 3t^3 - 6t^2 + 4 & -3t^3 + 3t^2 + 3t + 1 & t^3 \end{bmatrix}^T$$

Uniform NonRational B-Splines

- The **knot-value sequence** is a nondecreasing sequence
- Allow **multiple knot** and the number of identical parameter is the **multiplicity**
  - Ex. (0,0,0,0,1,1,2,3,4,4,5,5,5,5)
- So

$$Q_i(t) = P_{i-3} \cdot B_{i-3,4}(t) + P_{i-2} \cdot B_{i-2,4}(t) + P_{i-1} \cdot B_{i-1,4}(t) + P_i \cdot B_{i,4}(t)$$

NON-Uniform NonRational B-Splines

- Where $B_{i,j}(t)$ is j th-order blending function for weighting control point P_i

$$B_{i,1}(t) = \begin{cases} 1, & t_i \le t \le t_{i+1} \\ 0, & otherwise \end{cases}$$

$$B_{i,2}(t) = \frac{t - t_i}{t_{i+1} - t_i} B_{i,1}(t) + \frac{t_{i+2} - t}{t_{i+2} - t_{i+1}} B_{i+1,1}(t)$$

$$B_{i,3}(t) = \frac{t - t_i}{t_{i+2} - t_i} B_{i,2}(t) + \frac{t_{i+3} - t}{t_{i+3} - t_{i+1}} B_{i+1,2}(t)$$

$$B_{i,4}(t) = \frac{t - t_i}{t_{i+3} - t_i} B_{i,3}(t) + \frac{t_{i+4} - t}{t_{i+4} - t_{i+1}} B_{i+1,3}(t)$$

Knot Multiplicity & Continuity

- Since $Q(t_i)$ is within the convex hull of P_{i-3}, P_{i-2} and P_{i-1}
- If $t_i = t_{i+1}, Q(t_i)$ is within the convex hull of P_{i-3}, P_{i-2}, and P_{i-1} and the convex hull of P_{i-2}, P_{i-1} and P_{i-1}, so it will lie on $\overline{P_{i-2}P_{i-1}}$
- If $t_i = t_{i+1} = t_{i+2}, Q(t_i)$ will lie on P_{i-1}
- If $t_i = t_{i+1} = t_{i+2} = t_{i+3}, Q(t_i)$ will lie on both P_{i-1} and P_i, and the curve becomes broken

Knot Multiplicity & Continuity

- Multiplicity 1 : C^2 continuity
- Multiplicity 2 : C^1 continuity
- Multiplicity 3 : C^0 continuity
- Multiplicity 4 : no continuity

NURBS:
NonUniform Rational B-Splines

- Rational
- $x(t), y(t),$ and $z(t)$ are defined as the ratio of two cubic polynomials
- Rational cubic polynomial curve segments are ratios of polynomials

$$x(t) = \frac{X(t)}{W(t)} \qquad y(t) = \frac{Y(t)}{W(t)} \qquad z(t) = \frac{Z(t)}{W(t)}$$

- Can be Bézier, Hermite, or B-Splines

Parametric Bi-Cubic Surfaces

- Parametric cubic curves are $Q(t) = G \cdot M \cdot T$
- So, parametric bi-cubic surfaces are $Q(t) = G \cdot M \cdot S$
- If we allow the points in G to vary in 3D along some path, then

$$Q(s,t) \quad = \begin{bmatrix} G_1(t) & G_2(t) & G_3(t) & G_4(t) \end{bmatrix} \cdot M \cdot S$$

- Since $G_i(t)$ are cubics:
- $G_i(t) = G_i \cdot M \cdot T$, where $G_i = \begin{bmatrix} g_{i1} & g_{i2} & g_{i3} & g_{i4} \end{bmatrix}$

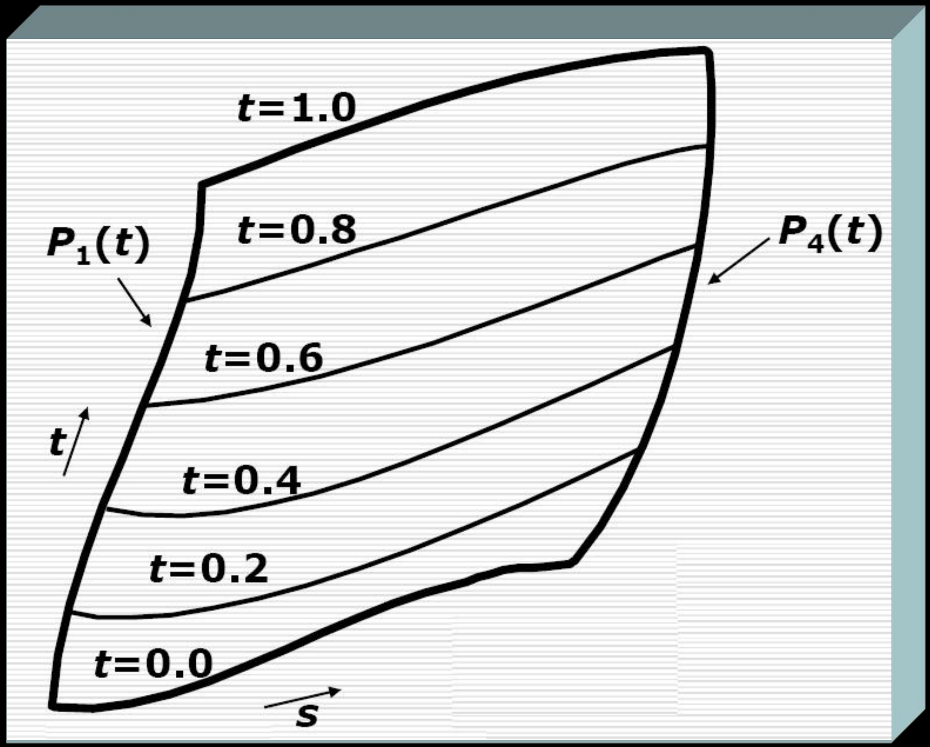
Parametric Bi-Cubic Surfaces

- So

$$Q(s,t) = T^T \cdot M^T \cdot \begin{bmatrix} g_{11} & g_{32} & g_{31} & g_{41} \\ g_{12} & g_{22} & g_{32} & g_{42} \\ g_{13} & g_{23} & g_{33} & g_{43} \\ g_{14} & g_{24} & g_{34} & g_{44} \end{bmatrix} \cdot M \cdot S$$

$$= T^T \cdot M^T \cdot G \cdot M \cdot S, \quad 0 \le s,t \le 1$$

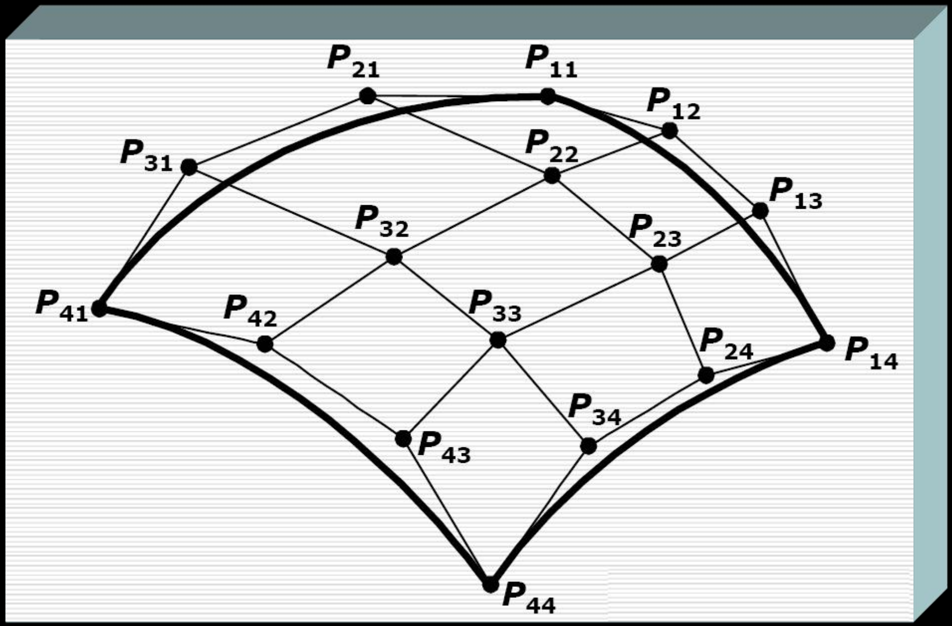
Hermite Surfaces



$$Q(s,t) = T^T \cdot M_H^T \cdot G_H \cdot M_H \cdot S$$

$$\begin{bmatrix} P_1(t) & P_4(t) & R_1(t) & R_4(t) \end{bmatrix} = G_H \cdot M_H \cdot T$$

Bézier Surfaces



$$Q(s,t) = T^T \cdot M_{Bs}^T \cdot G_{Bs} \cdot M_{Bs} \cdot S$$

Normals to Surfaces

$$\frac{\partial}{\partial s} Q(s,t) = T^T \cdot M^T \cdot G \cdot M \cdot \frac{\partial}{\partial s} S$$

$$= T^T \cdot M^T \cdot G \cdot M \cdot \begin{bmatrix} 3s^2 & 2s & 1 & 0 \end{bmatrix}^T$$

$$\frac{\partial}{\partial t} Q(s,t) = \frac{\partial}{\partial t}(T^T) \cdot M^T \cdot G \cdot M \cdot S$$

$$= \begin{bmatrix} 3t^2 & 2t & 1 & 0 \end{bmatrix}^T \cdot M^T \cdot G \cdot M \cdot S$$

$$\frac{\partial}{\partial s} Q(s,t) \times \frac{\partial}{\partial t} Q(s,t)$$ normal vector

Quadric Surfaces

- Implicit surface equation

$$f(x,y,z) = ax^2 + by^2 + cz^2 + 2dxy + 2eyz + 2fxz + 2gx + 2hy + 2jz + k = 0$$

- An alternative representation

$$P^T \cdot Q \cdot P = 0$$

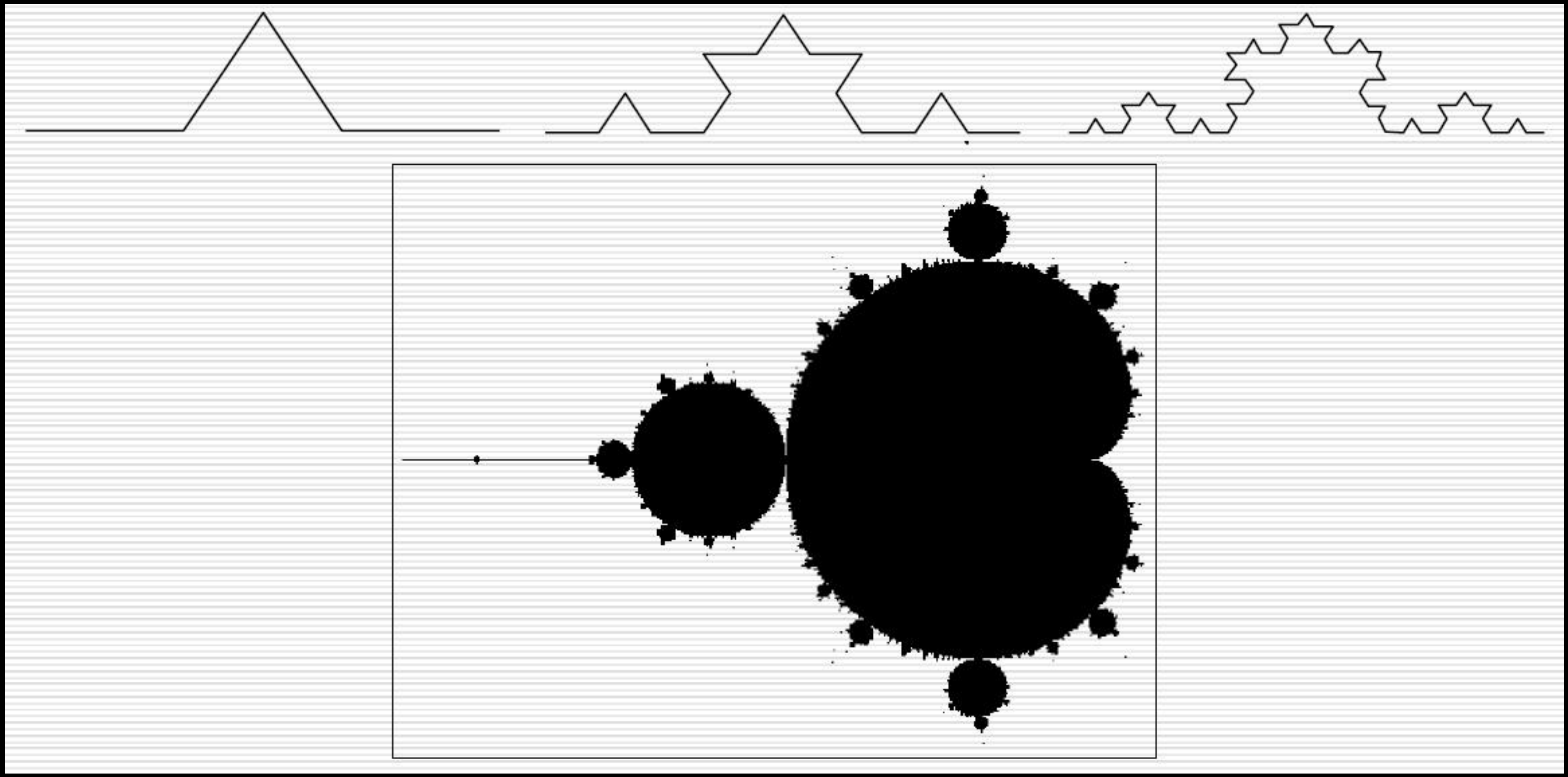
- with

$$Q = \begin{bmatrix} a & d & f & g \\ d & b & e & h \\ f & e & c & j \\ g & h & j & k \end{bmatrix} \qquad P = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Quadric Surfaces

- Advantages:
  - Computing the surface normal
  - Testing whether a point is on the surface
  - Computing z given x and y
  - Calculating intersections of one surface with another

Fractal Models



Grammar-Based Models

- L-grammars
  - A -> AA
  - B -> A[B]AA[B]