Topology Reconstruction of a Resistive Network with Limited Boundary Measurements

Shivanagouda Biradar and Deepak U. Patil

***Abstract*— We consider the problem of reconstructing all possible topologies of the circular planar passive-resistive network with only 1Ω resistances, housed inside a black box, with limited boundary measurements. The reconstruction problem is an inverse problem and, in general, has no unique solution. The limitedly available boundary measurements are used to construct a partially known resistance distance matrix. The partially known resistance distance matrix is then related to the unknown Laplacian matrix, resulting in many nonlinear multivariate polynomials. A method is proposed to reconstruct the network topology and edge resistor values simultaneously using the Gröbner basis. Numerical simulation establishes the effectiveness of the proposed strategy.**

resistor network, topology reconstruction, boundary measurements, Gröbner basis

I. Introduction

Electrical network topology reconstruction is an ill-posed problem and, in general, has many solutions. Network topology reconstruction involves simultaneously identifying electrical network structure and edge resistor values. This area of research has seen significant interest among researchers due to its applications in a wide range of areas, such as system biology [1], geology [2], medical imaging [3], and power system networks [4]. Two primary objectives that are broadly considered in electrical network topology reconstruction are *i*) to determine the topology of the electrical network hidden in a black box, using the measurements of voltages and current at the boundary nodes and *ii*) to estimate the edge conductance's in the electrical network [5]. In this paper, we propose a strategy to reconstruct all possible non-unique electrical network topologies with limitedly available boundary measurements using the *Gröbner basis*. We consider electrical networks with 1Ω edge resistances only.

Consider a connected circular planar passive-resistive electrical network, with only 1Ω edge resistances, encased inside a black box. The black box consists of conductors joining the n exposed boundary terminals, labeled $[n] = \{1, 2, ..., n\}$. The conductors are the edges of graph G, and the boundary terminals are the vertices. Unlike the inverse problem, the forward reconstruction problem presumes that the conductivity distribution and edge (σ) conductance's, $\gamma(\sigma)$, of an electrical network is completely known. When boundary terminals are subjected to voltage v_b, the resulting current ϕ at the boundary terminals is referred to as the network response. The network response map $\Lambda = \Lambda_\gamma$ is the linear map that transforms the boundary voltage v_b to the boundary current ϕ.

It has been shown in [5] that if the response matrix (Λ) satisfies the condition of a non-negative circular minor for a circular planar electrical network, with all the boundary terminals available, then we can reconstruct both the topology of the planar network and the edge conductance of the network. In [6], the authors present an approach for calculating the conductor values in a circular planar passive resistor network, using voltages and currents measured at the boundary. Assuming that only the network structure is known and all the boundary terminals are available, a γ-harmonic function on the circular network is defined. Then using the harmonic continuation and Λ, the conductor values are computed. In [7], it is shown that if G is any critical [5] circular planar graph corresponding to the circular planar resistor network with all boundary terminals available for collecting data, the conductor values can be computed using Λ. A similar problem of topology reconstruction is being researched in phylogenetics as seen in [8], and [9]. The electrical and phylogenetic networks are combinatorially identical objects [9]; however, the edge weights are less well understood in phylogenetic networks [10]. Electrical networks are traditionally weighted with conductance, whereas phylogenetic networks have traditionally been weighted with statistical distance measures. However, the notion of genetic distance between taxa (species or individuals) in phylogenetics and the resistance distance between terminals are similar [11]. In [9], assuming that all the boundary terminals and response matrix are already available, the authors find a corresponding split network [12]. The split network yields the bridge structure of the unknown network, which is used to reconstruct the local graph. However, to the best of our knowledge, there are no general methods that consider partially available boundary terminals and list all the possible non-unique electrical networks corresponding to the collected limited boundary data, since there could be several network topologies corresponding to the partially available boundary measurements.

The Gröbner basis [24] is a set of multivariate nonlinear basis polynomials, which allows simple solutions to the otherwise huge multivariate nonlinear polynomials. The Gröbner basis has seen a considerable increase in applications with an increase in the computational capability in recent years; some of the significant applications are automatic theorem proving [17], graph coloring [18], integer programming [19], solving inverse and forward kinematics in robotics applications [20], signal and image processing [21], testing controllability and observability, determination of equilibrium points, computing the domain of attraction [22] and computing time-optimal feedback control [23]. Our work presents a new application of Gröbner basis in characterizing all the circuit topologies

for the limitedly available boundary measurements.
Contribution: This paper makes two main contribution:

- We consider a circular planar resistive network with only 1Ω resistances, and partially available boundary terminals. The partially available boundary terminals leads to partially available boundary measurements. Using this partially available boundary measurement data we estimate the internal structure of the resistive network.
- We pose the partially available boundary measurements as a set of nonlinear multivariate polynomials, say $\mathbf{F}$. We then use the Gröbner basis to find a solution set of $\mathbf{F}$. The solution set is the list of possible topologies corresponding to the partially available boundary measurements.

This article is structured as follows. In Section II, we formulate the problem and briefly overview the required mathematical preliminaries on graph theory. In Section III, we propose an algorithm wherein we relate R_D with the Laplacian matrix $\mathcal{L}$. This relation gives rise to a set of multivariate nonlinear polynomials $\mathbf{F}$. The corresponding Gröbner basis is calculated, which is used to calculate all possible non-unique network topologies corresponding to the experimental data on partially available boundary terminals. We use numerical examples to demonstrate the developed methodology in Section IV. Finally, concluding remarks are presented in section V.

II. PRELIMINARIES AND PROBLEM FORMULATION

Consider a circular planar passive-resistive electrical network $\Gamma = (G, \gamma)$. A finite, simple graph with a boundary is $G = (V_B, E)$, E = the set of edges, and V_B is the set of boundary nodes, together with a function $\gamma : G \to \mathbb{R}^+$. The conductivity function γ assign to each edge $\sigma \in E$, a number $\gamma(\sigma)$ known as the conductance of σ, in this paper we assume $\gamma(\sigma) = 1\mho\ \forall\ \sigma \in E$. A circular planar graph corresponding to the circular planar electrical network is a graph G with a boundary embedded on a disc D in the plane. The boundary nodes lie on circle C that bounds D and the rest of G inside D. The boundary nodes V_B can be labeled as $[n] = \{1, \ldots, n\}$, in clockwise circular order around C. We define a distance metric on the circularly ordered $[n]$; this distance is equivalent to effective electrical resistance between nodes i and j, and is called the resistance distance r_{ij} between nodes i and j [16]. The resistance distance metric for the network Γ is a symmetric matrix R_D, with $R_D(i, j) = r_{ij}$ and $R_D(i,i) = 0\ \forall\ i$ and $j \in [n]$. Given a graph G, various matrices can be associated with the graph. The Laplacian matrix $\mathcal{L}$ corresponding to any graph G is a symmetric $n \times n$ matrix $\mathcal{L}(G)$, defined as follows:

$$[\mathcal{L}(G)]_{ij} = \mathcal{L}_{ij} = \begin{cases} -\gamma(ij) & i \neq j\ \&\ ij \in E, \\ 0 & i \neq j\ \&\ ij \notin E, \\ d_i & i = j, \end{cases}$$

where d_i is the degree of the node i. Since $\sum_{i=1}^{n} \mathcal{L}_{ij} = \sum_{j=1}^{n} \mathcal{L}_{ij} = 0$ for any graph G, we have $det(\mathcal{L}(G)) = 0$.

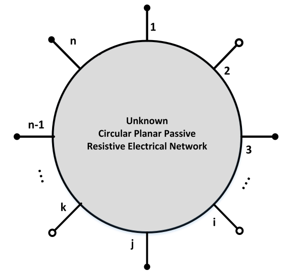


Fig. 1. Unknown circular planar graph G.

The submatrix $\mathcal{L}[i]$ of $\mathcal{L}$ is obtained by deleting i^{th} row and i^{th} column from the Laplacian matrix $\mathcal{L}$. The submatrix obtained by deleting i^{th} and j^{th} rows and i^{th} and j^{th} columns of the Laplacian matrix $\mathcal{L}(G)$, is denoted by $\mathcal{L}[i, j]\ \forall i, j \in [n]$. For any connected graph G, we can relate R_D and the Laplacian matrix $\mathcal{L}(G)$ using the following theorem:

Theorem 1. *[13] Let $G = (V_B, E)$ be a connected graph on n vertices, $n \geq 3$, corresponding to the electrical network $\Gamma = (G, \gamma)$. Let $\mathcal{L}[i]$ and $\mathcal{L}[i, j]$ be sub-matrix of Laplacian matix $\mathcal{L}(G)$. Then, the resistance distance r_{ij} between two terminal i and j, $\forall\ 1 \leq i \neq j \leq n$, is given as:*

$$r_{ij} = \frac{det\left(\mathcal{L}\left[i, j\right]\right)}{det\left(\mathcal{L}\left[i\right]\right)}. \tag{1}$$

Our objective is to reconstruct all possible network topologies corresponding to limitedly available boundary terminals. Let us proceed with a connected passive-resistive electrical network Γ encased in a black box, as shown in Figure 1. The symbol $\circ$ on the boundary terminals $[n]$ represents terminals that cannot be used for experiments, while the symbol $\bullet$ on the boundary terminals represents terminals that can be used for experiments. Let $\mathcal{U}$ represent the set of terminals unavailable for experiments, and $\mathcal{A}$ represent the set of terminals available for experiments. The boundary terminals in $\mathcal{A}$ are used to conduct experiments. Let us consider terminals $k, m \in \mathcal{A}$, we apply a voltage v_{km} across terminals k and m and the resulting boundary current i_k is noted, as shown in Figure 2. Using this measurement we calculate the resistance distance $r_{km} = \frac{v_{km}}{i_k}$. Similar experiments are conducted on terminals in $\mathcal{A}$ to collect all possible boundary measurement data. We denote the

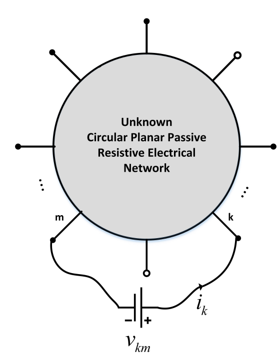


Fig. 2. Calculating resistance distance r_{km}.

set of boundary measurements as $\mathcal{M} = \{(v_{km}, i_k) : k, m \in \mathcal{A}\}$. Using these boundary measurements in $\mathcal{M}$ we calculate all the resistance distances $r_{km} \forall k, m \in \mathcal{A}$. The unknown Laplacian matrix $\mathcal{L}(G) \in \mathbb{R}^{n \times n}$, is expressed as:

$$\begin{bmatrix} l_{11} & -l_{12} & \cdots & -l_{1n} \\ -l_{12} & l_{22} & \cdots & -l_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -l_{1n} & -l_{2n} & \cdots & l_{nn} \end{bmatrix}, \tag{2}$$

$\frac{n(n-1)}{2}$ elements, $l_{12}, \dots, l_{ij}, \dots, l_{(n-1)n}$ in $\mathcal{L}(G)$ are to be computed to reconstruct the network Γ, therefore let $\mathbf{w} = [l_{12} \ \dots \ l_{ij} \ \dots \ l_{(n-1)n}]^T \in \mathbb{F}_2^{\frac{n(n-1)}{2}}$, where $\mathbb{F}_2 = \{0, 1\}$. Let us for a moment draw our attention towards important terminologies related to Gröbner basis. The ideal of the set of polynomials $\mathbf{F}$, denoted as $\langle \mathbf{F} \rangle$, is defined as:

Definition 1 (Ideal of $\mathbf{F}$)**.**

$$\langle \mathbf{F} \rangle = \left\{ \sum_{i=1}^{n} h_i f_i : h_1, \dots h_n \in \mathbb{R}\left[l_{12}, \cdots, l_{(n-1)n}\right], \forall f_i \in \mathbf{F} \right\}$$

We define a solution set $\mathbf{V}$, generated by set of polynomials $\mathbf{F}$ as: definition [Variety of $\mathbf{F}$] $\mathbf{V}(\mathbf{F})$ is called the affine variety generated by the set of polynomials $\mathbf{F}$ and is defined as

$$\mathbf{V}(\mathbf{F}) = \left\{ \mathbf{w} \in \mathbb{F}_2^{\frac{n(n-1)}{2}} : f(\mathbf{w}) = 0, \forall f \in \mathbf{F} \right\}. \tag{3}$$

Let us define few more terms to better understand the number of solutions i.e. $|\mathbf{V}(\mathbf{F})|$. For a fixed term order [22] of polynomial we define initial term of a polynomial as

Definition 2 (Initial Term of a polynomial)**.** Initial term of a polynomial say f, $in_{<}(f)$, is the largest monomial in f.

Corresponding to the $\langle \mathbf{F} \rangle$, we define the initial ideal as

Definition 3 (Initial Ideal of $\langle \mathbf{F} \rangle$)**.**

$$in_{<}(\langle F \rangle) = \langle in_{<}(f) : f \in \langle F \rangle \rangle$$

Initial ideal of $\langle \mathbf{F} \rangle$ is the ideal generated by the initial terms of all $f \in \langle \mathbf{F} \rangle$.

Lastly, we define a standard monomial as

Definition 4 (Standard monomial)**.** A monomial say $x^\alpha = x_1^{a_1} \cdots x_n^{a_n}$, is said to be a standard monomial if $x^\alpha \notin in_{<}(\langle F \rangle)$.

The known resistance distances r_{km} are related to the unknown Laplacian matrix $\mathcal{L}(G)$ as given in Theorem 1:

$$r_{km} = \frac{det(\mathcal{L}[k,m])}{det(\mathcal{L}[k])}, \tag{4}$$

resulting in,

$$r_{km} det(\mathcal{L}[k]) - det(\mathcal{L}[k,m]) = 0, \tag{5}$$

which can be represented as,

$$f_{km} = 0. \tag{6}$$

Let $\mathbf{F}$ be the set of all multivariate polynomials f_{km} i.e. $\mathbf{F} = \{f_{km} : (v_{km}, i_k) \in \mathcal{M}, \forall k, m \in \mathcal{A}\}$. Consider a system of polynomial equations $\mathcal{F}$ constructed using the polynomials in set $\mathbf{F}$, i.e.,

$$\begin{gathered} \mathcal{F}(\mathbf{w}) = 0 \\ \Downarrow \\ f_{km}\left(l_{12} \ \dots \ l_{ij} \ \dots \ l_{(n-1)n}\right) = 0, \forall f_{km} \in \mathbf{F} \end{gathered} . \tag{7}$$

Our objective is to,

$$\begin{gathered} solve \ \ for \ \ \mathbf{w} \\ such \ \ that : \mathcal{F}(\mathbf{w}) = 0 \ \& \ \mathbf{w} \in \mathbb{F}_2^{\frac{n(n-1)}{2}} \end{gathered} . \tag{8}$$

Let $\mathbf{L}$ be a linear transformation that transforms a vector $\mathbf{w} \in \mathbb{F}_2^{\frac{n(n-1)}{2}}$ to a matrix $\mathbf{L_w} \in \mathbb{R}^{n \times n}$.

Definition 5. The linear transformation $\mathbf{L} : \mathbb{F}_2^{\frac{n(n-1)}{2}} \to \mathbb{R}^{n \times n}$ is defined as

$$[\mathbf{L_w}]_{ij} = \begin{cases} -w_{i+q_j} & i > j, \\ [\mathbf{L_w}]_{ji} & i < j, \\ \sum_{i \neq j} [\mathbf{L_w}]_{ij} & i = j, \end{cases} \tag{9}$$

where $q_j = -j + \frac{j-1}{2}(2n - j)$. The matrix $\mathbf{L_w}$ satisfies the constraints $[\mathbf{L_w}]_{ij} = [\mathbf{L_w}]_{ji}$ and $\mathbf{1} \cdot [\mathbf{L_w}] = [\mathbf{L_w}] \cdot \mathbf{1} = 0$.

We form a solution set $\mathbf{T}$, corresponding to the objective described in (8), which is defined as:

$$\mathbf{T} = \left\{ \mathbf{L_w} \in \mathbb{R}^{n \times n} : \mathbf{w} \in \mathbf{V}(\mathbf{F}) \ \& \ \mathbf{w} \in \mathbb{F}_2^{\frac{n(n-1)}{2}} \right\}. \tag{10}$$

For the set $\mathbf{F}$, a special polynomial basis can be generated known as Gröbner basis $\hat{\mathbf{F}} \subset \mathbb{R}\left[l_{12}, \cdots, l_{ij}, \cdots, l_{n(n-1)}\right]$, that allows a simple algorithmic solution. The variety of the polynomials generated by $\mathbf{F}$ and $\hat{\mathbf{F}}$ i.e. $\mathbf{V}(\mathbf{F})$ and $\mathbf{V}(\hat{\mathbf{F}})$ respectively, are related and the relation is given using the following lemma,

Lemma 1. *[24] Let* $\mathbf{F}$ *and* $\hat{\mathbf{F}}$ *be a set of polynomials as defined above, then,* $\langle \mathbf{F} \rangle = \langle \hat{\mathbf{F}} \rangle$.

Lemma 2. *[24] Let* $\mathbf{F}$ *and* $\hat{\mathbf{F}}$ *be polynomials in* $\mathbb{R}\left[l_{12}, \cdots, l_{(n-1)n}\right]$*, such that* $\langle \mathbf{F} \rangle = \langle \hat{\mathbf{F}} \rangle$*, then* $\mathbf{V}(\mathbf{F}) = \mathbf{V}(\hat{\mathbf{F}})$.

Let $\hat{\mathcal{F}}(\mathbf{w}) = 0$ be system of polynomial equations formed using the Gröbner basis polynomials $\hat{\mathbf{F}}$. Therefore, the objective in (8) can be reformulated as

$$\begin{gathered} solve \ \ for \ \ \mathbf{w} \\ such \ \ that : \hat{\mathcal{F}}(\mathbf{w}) = 0 \ \& \ \mathbf{w} \in \mathbb{F}_2^{\frac{n(n-1)}{2}} \end{gathered} . \tag{11}$$

Now, corresponding to the reformulated objective in (11), we define a new solution set $\hat{\mathbf{T}}$ as

$$\hat{\mathbf{T}} = \left\{ \mathbf{L_w} \in \mathbb{R}^{n \times n} : \mathbf{w} \in \mathbf{V}(\hat{\mathbf{F}}) \ \& \ \mathbf{w} \in \mathbb{F}_2^{\frac{n(n-1)}{2}} \right\}. \tag{12}$$

The solution sets $\mathbf{T}$ and $\hat{\mathbf{T}}$ are equal from lemma 1 and 2, and $\hat{\mathbf{T}} \subset \mathbf{V}(\hat{\mathbf{F}})$. Therefore, the problem of interest is stated as follows:

Problem: Find the set $\hat{\mathbf{T}}$ corresponding to the Gröbner basis polynomials $\hat{\mathbf{F}}$.

The set $\hat{\mathbf{T}}$ defines all possible combinations of network topologies corresponding to the limitedly available boundary data.

III. Reconstruction Algorithm

Consider a circular planar passive-resistive electrical network $\Gamma = (G, \gamma)$, with only 1Ω edge resistances, inside a black box. The basic idea of the algorithm is to relate the available resistance distance measurements $\{r_{km} : \forall k, m \in \mathcal{A}\}$ to the unknown Laplacian matrix $\mathcal{L}(G)$ using the expression in (4). We rewrite our objective again for the sake of completeness, i.e.

$$\begin{array}{c} solve \;\; for \;\; \mathbf{w} \\ such \;\; that : \hat{\mathcal{F}}(\mathbf{w}) = 0 \;\&\; \mathbf{w} \in \mathbb{F}_2^{\frac{n(n-1)}{2}} \end{array} . \tag{13}$$

A detailed explanation of the algorithm to solve the objective (8) is given in the following flow chart,

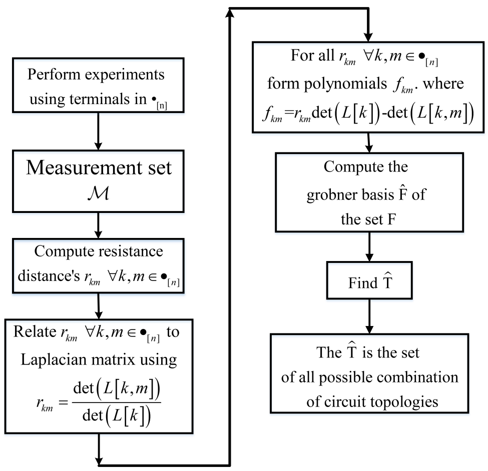


Fig. 3. Reconstruction algorithm.

There are various algorithms for the computation of the Gröbner basis. In this paper, we use the Buchberger's algorithm, which is briefly introduced below,

Algorithm 1 Buchberger's Algorithm

```
Input: F
Output: F̂
F̂ ← F
do
    G′ ← F̂
    ∀{f1, f2} ∈ G′, f1 ≠ f2
    r := S(f1, f2)^G′   (reduction modulo G′)
        if r ≠ 0
            F̂ ← F̂ ∪ {r}
        else
            Break;
        end
While F̂ = G′
Return F̂
```

for more information on the algorithm refer to [24].

A. Solutions of the System of Polynomial Equations

Let us consider $|\mathcal{A}| = b$, and the $|\mathcal{M}|$ be the number of boundary measurements. We have $\frac{n(n-1)}{2}$ unknowns (elements of vector $\mathbf{w}$) which takes values in $\mathbb{F}_2$. We examine two cases i.e. $b < n$ and $b = n$ for studying the solutions of the system of polynomial equations, say $\mathcal{F}(\cdot) = 0$. For

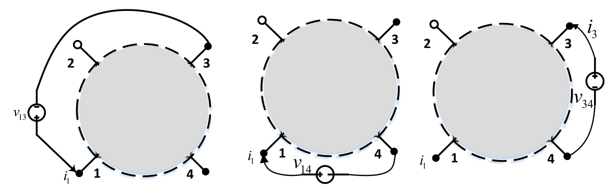
- Case1: $b < n$, we get $|\mathcal{M}| < \frac{n(n-1)}{2}$. Therefore, the number of equations are less than number of unknowns.
- Case2: $b = n$, we get $|\mathcal{M}| = \frac{n(n-1)}{2}$ and hence number of equations are equal to the number of unknowns.

The $\mathcal{F}(\mathbf{w}) = 0$ is always consistent with trivial solution $\mathbf{w} = 0$, apart from the trivial solution, in both the cases, $\mathcal{F}(\mathbf{w}) = 0$ must have finite multiple solutions. Since, there exist multiple topologies which satisfies all the resistance distances $\{r_{km} : k, m \in \mathcal{A} \;\&\; \gamma(\sigma) = 1 \;\forall\; \sigma \in E\}$, hence from Theorem 1 the system of polynomial equations $\{r_{km}det(\mathcal{L}[k]) - det(\mathcal{L}[k,m]) = 0 \;:\; k, m \in \mathcal{A}\}$ must also satisfy multiple topologies. Therefore, there must exist finite multiple solutions, consequently $|\mathbf{T}(\mathbf{F_b})| < \infty$, $|\mathbf{T}(\mathbf{F_n})| < \infty$ and $\mathbf{T}(\mathbf{F_n}) \subset \mathbf{T}(\mathbf{F_b})$. Where $\mathbf{T}(\mathbf{F_b}) = \left\{\mathbf{L_w} \in \mathbb{R}^{n \times n} : \mathbf{w} \in \mathbf{V}(\mathbf{F_b}) \;\&\; \mathbf{w} \in \mathbb{F}_2^{\frac{n(n-1)}{2}}\right\}$, $\mathbf{T}(\mathbf{F_n}) = \left\{\mathbf{L_w} \in \mathbb{R}^{n \times n} : \mathbf{w} \in \mathbf{V}(\mathbf{F_n}) \;\&\; \mathbf{w} \in \mathbb{F}_2^{\frac{n(n-1)}{2}}\right\}$, $\mathbf{F_b} = \{f_{km} : (v_{km}, i_k) \in \mathcal{M}, \forall k, m \in \mathcal{A} \;\&\; |\mathcal{A}| = b\}$ and $\mathbf{F_n}$ is defined similarly with $|\mathcal{A}| = n$ i.e. all the terminals are available.

Remarks: The number of solutions in $\mathbf{V}(\cdot)$ i.e. $|\mathbf{V}(\cdot)|$ depends on the number of standard monomials. Let $\widehat{\mathbf{F}}_b$ be the Gröbner basis corresponding to the set $\mathbf{F}_b$, the initial ideal of $\widehat{\mathbf{F}}_b$ is represented as $in_{\prec}\left(\left\langle \widehat{\mathbf{F}}_b \right\rangle\right)$ and let the set of the standard monomials corresponding to $in_{\prec}\left(\left\langle \widehat{\mathbf{F}}_b \right\rangle\right)$ be $\mathcal{S}_m$. The $|\mathcal{S}_m|$ is finite if every variable l_{ij} in $\mathbf{w}$ appears as some positive integer power in $in_{\prec}(\langle \mathbf{F}_b \rangle)$, and the $\mathbf{V}\left(\widehat{\mathbf{F}}_b\right)$ is finite if $|\mathcal{S}_m|$ is finite, consequently $|\mathbf{T}(\mathbf{F_b})| < \infty$.

IV. Numerical Examples

We consider two examples. Firstly, let us consider an electrical network $\Gamma_1 = (G_1, \gamma)$ with four boundary nodes labeled as $[n_1] = \{1, 2, 3, 4\}$. For graph G_1, we define $\mathcal{A}_1 = \{1, 3, 4\}$. Experiments are conducted on the exposed boundary terminals $\mathcal{A}_1$ as shown in Figure 4.



$v_{13}(V)$	$i_1(A)$	$r_{13}(\Omega)$
5	5	1

$v_{14}(V)$	$i_1(A)$	$r_{14}(\Omega)$
5	6.666	0.75

$v_{34}(V)$	$i_3(A)$	$r_{34}(\Omega)$
5	6.666	0.75

Fig. 4. Experiments on the exposed boundary terminals.

Therefore, we have the measurement set $\mathcal{M} = \{(5, 5), (5, 6.666), (5, 6.666)\}$, using $\mathcal{M}$ we calculate all the resistance distances r_{13}, r_{14} and r_{34}. The original electrical network inside a black box is shown in Figure 5. We relate the known resistance distances to the Laplacian matrix $\mathcal{L}(G_1)$ using Theorem 1. The polynomials corresponding to resistance distances r_{13}, r_{14} and r_{34} are f_{13}, f_{14} and f_{34} respectively, given as:

$$f_{13} = r_{13}det(\mathcal{L}[1]) - det(\mathcal{L}[1, 3]) \tag{14}$$

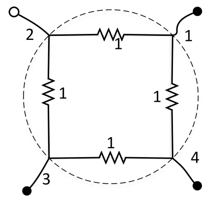


Fig. 5. Original network Γ_1.

$$f_{14} = r_{14} det(\mathcal{L}[1]) - det(\mathcal{L}[1,4]) \tag{15}$$

$$f_{34} = r_{34} det(\mathcal{L}[3]) - det(\mathcal{L}[3,4]) \tag{16}$$

. The polynomials f_{13}, f_{14} & f_{34} are given in the appendix. The Laplacian matrix $\mathcal{L}(G_1)$, is defined as:

$$\mathcal{L}(G_1) = \begin{bmatrix} l_{11} & -l_{12} & -l_{13} & -l_{14} \\ -l_{12} & l_{22} & -l_{23} & -l_{24} \\ -l_{13} & -l_{23} & l_{13}+l_{23}+l_{34} & -l_{34} \\ -l_{14} & -l_{24} & -l_{34} & l_{44} \end{bmatrix}. \tag{17}$$

Let the set of multivariate polynomials be $\mathbf{F} = \{f_{13}, f_{14}, f_{34}\}$. The unknowns l_{12}, l_{13}, l_{14}, l_{23}, l_{24}, l_{34} are to be computed to get the reconstructed network Γ_1. Therefore, let $\mathbf{w} = [l_{12}\ l_{13}\ l_{14}\ l_{23}\ l_{24}\ l_{34}]^T$ and hence, we define our objective as:

$$\begin{aligned} &solve\ \ for\ \ \mathbf{w} = [l_{12}\ l_{13}\ l_{14}\ l_{23}\ l_{24}\ l_{34}]^T \\ &such\ \ that : \mathbf{w} \in \mathbf{V}(\mathbf{F})\ \&\ \mathbf{w} \in \mathbb{F}_2^{\frac{n(n-1)}{2}} \end{aligned} . \tag{18}$$

Corresponding to $\mathbf{F}$, we compute the Gröbner basis $\hat{\mathbf{F}}$ using the Buchberger's algorithm [14]. Therefore, our objective translates to

$$\begin{aligned} &solve\ \ for\ \ \mathbf{w} = [l_{12}\ l_{13}\ l_{14}\ l_{23}\ l_{24}\ l_{34}]^T \\ &such\ \ that : \mathbf{w} \in \mathbf{V}\left(\hat{\mathbf{F}}\right)\ \&\ \mathbf{w} \in \mathbb{F}_2^{\frac{n(n-1)}{2}} \end{aligned} . \tag{19}$$

We use Matlab's symbolic toolbox [25] to implement the reconstruction algorithm. The solution set $\hat{\mathbf{T}}$ is a set of possible combinations of network topologies, as shown in figure matrix Figure 6.

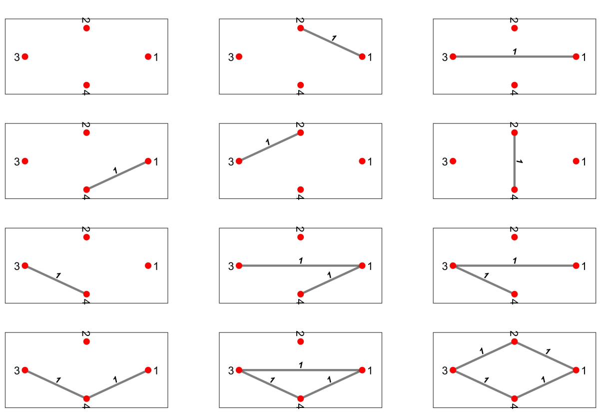


Fig. 6. Group of possible network topologies Γ_1.

The last network $(4,3)$ in figure matrix Figure 6 is the same as the original electrical network as shown in Figure 5. From the set $\hat{\mathbf{T}}$, we see that the Fielder's eigen value's, $\lambda_2(\mathcal{L}(G_1))$, for the networks other than $(4,3)$ in figure matrix Figure 6 is 0 and hence, we arrive at the original network as in Figure 5.

Let us consider the second example, whose original network is shown below in Figure 7. An electrical network $\Gamma_2 = (G_2, \gamma)$ with five boundary nodes labeled as $[n_2] = \{1,2,3,4,5\}$. We define $\mathcal{A}_2 = \{1,3,5\}$ i.e., two boundary nodes are not available for experimentation. Experiments are conducted on the exposed terminal $\mathcal{A}_2$, therefore $\mathcal{M} = \{(5V, 4.23A), (5V, 6.878A), (5V, 4.23A)\}$. We calculate all the resistance distance from $\mathcal{M}$ and compute all the polynomials using (4) to form a set $\mathbf{F}$. We define our objective as:

$$\begin{aligned} &solve\ \ for\ \ \mathbf{w} = [l_{12}\ l_{13}\ l_{14}\ l_{15}\ l_{23}\ l_{24}\ l_{25}\ l_{34}\ l_{35}\ l_{45}]^T \\ &such\ \ that : \mathbf{w} \in \mathbf{V}\left(\hat{\mathbf{F}}\right) \\ &\mathbf{w} \in \mathbb{F}_2^{\frac{n(n-1)}{2}} \end{aligned} , \tag{20}$$

using the set of Gröbner basis polynomial $\hat{\mathbf{F}}$ corresponding to $\mathbf{F}$.

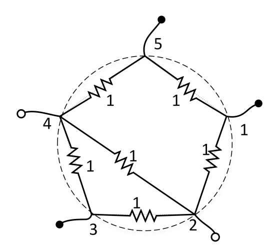


Fig. 7. Original network Γ_2

The solution set $\hat{\mathbf{T}}$ corresponding to (20), is the set of all possible combinations of network topologies corresponding to the resistance distances is shown in figure matrix Figure 8.

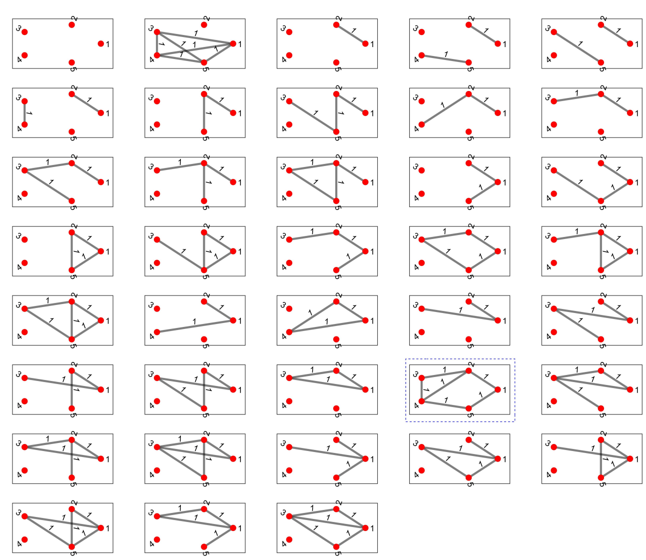


Fig. 8. Group of possible network topologies Γ_2

In Figure 8, the solution in position $(6,4)$ is similar to the original solution as shown in Figure 7. It is observed that the $\lambda_2(\mathcal{L}(G_2))$ of the network topologies in the set $\hat{\mathbf{T}}$ is 0 except the solution in position $(6,4)$ in Figure 8. Hence, the network topology corresponding to the $\mathcal{M}$ is recovered.

V. Conclusion

This article uses a Gröbner basis-based technique to get a set of all network topologies corresponding to limitedly available resistance distance data. It is shown that the relation between the resistance distance and the Laplacian matrix generates the set of multivariate nonlinear polynomials, and the variety of the Gröbner basis characterizes all the possible network topologies. The numerical examples demonstrate the effectiveness of the proposed algorithm. The proposed algorithm is limited to small networks due to the high worst case computational complexity of the algorithm involving solutions of multivariate nonlinear polynomials. Developing algorithms for reconstructing large passive resistive electrical networks with low computational complexity is a subject of future research.

Appendix

The polynomials corresponding to equation (14) for $r_{13} = 1\Omega$, (15) for $r_{14} = 0.75\Omega$ and (16) for $r_{34} = 0.75\Omega$ are

$$f_{13} = l_{12}\,l_{14} + l_{12}\,l_{24} + l_{14}\,l_{23} + l_{14}\,l_{24} + l_{12}\,l_{34} + l_{23}\,l_{24} + l_{23}\,l_{34} + l_{24}\,l_{34} - l_{12}\,l_{13}\,l_{14} - l_{12}\,l_{13}\,l_{24} - l_{12}\,l_{14}\,l_{23} - l_{13}\,l_{14}\,l_{23} - l_{13}\,l_{14}\,l_{24} - l_{12}\,l_{13}\,l_{34} - l_{12}\,l_{23}\,l_{24} - l_{12}\,l_{14}\,l_{34} - l_{13}\,l_{23}\,l_{24} - l_{14}\,l_{23}\,l_{24} - l_{12}\,l_{23}\,l_{34} - l_{12}\,l_{24}\,l_{34} - l_{13}\,l_{23}\,l_{34} - l_{13}\,l_{24}\,l_{34} - l_{14}\,l_{23}\,l_{34} - l_{14}\,l_{24}\,l_{34}$$

$$f_{14} = l_{12}\,l_{13} + l_{12}\,l_{23} + l_{13}\,l_{23} + l_{13}\,l_{24} + l_{12}\,l_{34} + l_{23}\,l_{24} + l_{23}\,l_{34} + l_{24}\,l_{34} - \frac{3\,l_{12}\,l_{13}\,l_{14}}{4} - \frac{3\,l_{12}\,l_{13}\,l_{24}}{4} - \frac{3\,l_{12}\,l_{14}\,l_{23}}{4} - \frac{3\,l_{13}\,l_{14}\,l_{23}}{4} - \frac{3\,l_{13}\,l_{14}\,l_{24}}{4} - \frac{3\,l_{12}\,l_{13}\,l_{34}}{4} - \frac{3\,l_{12}\,l_{23}\,l_{24}}{4} - \frac{3\,l_{12}\,l_{14}\,l_{34}}{4} - \frac{3\,l_{13}\,l_{23}\,l_{24}}{4} - \frac{3\,l_{14}\,l_{23}\,l_{24}}{4} - \frac{3\,l_{12}\,l_{23}\,l_{34}}{4} - \frac{3\,l_{12}\,l_{24}\,l_{34}}{4} - \frac{3\,l_{13}\,l_{23}\,l_{34}}{4} - \frac{3\,l_{13}\,l_{24}\,l_{34}}{4} - \frac{3\,l_{14}\,l_{23}\,l_{34}}{4} - \frac{3\,l_{14}\,l_{24}\,l_{34}}{4}$$

$$f_{34} = l_{12}\,l_{13} + l_{12}\,l_{14} + l_{12}\,l_{23} + l_{12}\,l_{24} + l_{13}\,l_{23} + l_{13}\,l_{24} + l_{14}\,l_{23} + l_{14}\,l_{24} - \frac{3\,l_{12}\,l_{13}\,l_{14}}{4} - \frac{3\,l_{12}\,l_{13}\,l_{24}}{4} - \frac{3\,l_{12}\,l_{14}\,l_{23}}{4} - \frac{3\,l_{13}\,l_{14}\,l_{23}}{4} - \frac{3\,l_{13}\,l_{14}\,l_{24}}{4} - \frac{3\,l_{12}\,l_{13}\,l_{34}}{4} - \frac{3\,l_{12}\,l_{23}\,l_{24}}{4} - \frac{3\,l_{12}\,l_{14}\,l_{34}}{4} - \frac{3\,l_{13}\,l_{23}\,l_{24}}{4} - \frac{3\,l_{14}\,l_{23}\,l_{24}}{4} - \frac{3\,l_{12}\,l_{23}\,l_{34}}{4} - \frac{3\,l_{12}\,l_{24}\,l_{34}}{4} - \frac{3\,l_{13}\,l_{23}\,l_{34}}{4} - \frac{3\,l_{13}\,l_{24}\,l_{34}}{4} - \frac{3\,l_{14}\,l_{23}\,l_{34}}{4} - \frac{3\,l_{14}\,l_{24}\,l_{34}}{4}$$

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