Cauchy Interlacing Theorem and Eigen value constraint

Abstract—**The problem of reconstructing the topology of the passive-resistive network using limited boundary measurements and spectral constraints is considered. The boundary measurements, i.e., boundary terminal voltages and boundary terminal currents, are used to build a resistance distance matrix. The partially known resistance distance matrix along with spectral constraints are formulated as the linear matrix inequalities (LMI). The solution to the LMI uncovers the electric network topology and the edge resistor values. Numerical simulation establishes the effectiveness of the proposed strategy.**

I. Introduction

Electrical network topology reconstruction involves identification of both electrical network structure and edge resistor values. This area of research has seen significant interest among researchers due to its applications in a wide range of areas, such as system biology, geology, medical imaging, chemical engineering, and power networks.Two primary objectives that are broadly considered in electrical network topology reconstruction are: i) to find the conductivity distribution of the electrical network using the measurements of voltages and current at the boundary nodes, and ii) to find the conductance of each edge in electrical network. In this paper, we propose a strategy to estimate the resistivity distribution and edge resistance values simultaneously using limited boundary voltage and current measurements.

Consider a connected circular planar passive-resistive electrical network inside a black box. The interior of the box consists of conductors joining the n exposed boundary terminals, labelled $\{1, 2, ..., n\} = [n]$. The boundary terminals correspond to vertices, and the conductors are the edges of a graph G. The forward problem, unlike the inverse problem, presumes that the graph G and the conductance $\gamma(\sigma)$ of each edge σ in G are known. When a voltage V_b is imposed on the boundary terminals, the resulting current ϕ at the boundary nodes is called the network response. The linear map $\Lambda = \Lambda_\gamma$, which transforms the boundary voltage V_b to boundary current ϕ, is called the response map.

II. PRELIMINARIES AND PROBLEM FORMULATION

Consider a circular planar passive-resistive electrical network $\Gamma = (G, \gamma)$. A finite, simple graph with a boundary is $G = (V_B, E)$, E = the set of edges, and V_B is the set of boundary nodes, together with a function $\gamma : G \to \mathbb{R}^+$. The conductivity function γ assign to each edge $\sigma \in E$ a number $\gamma(\sigma)$ known as the conductance of σ. A circular planar graph corresponding to the circular planar electrical network is a graph G with a boundary embedded on a disc D in the plane. The boundary nodes lie on circle C that bounds D and the rest of G inside D. The boundary nodes V_B can be labeled as $[n] = \{1, \ldots, n\}$ in clockwise circular order around C. We define a distance metric on the circularly ordered $[n]$; this distance is equivalent to effective electrical resistance between nodes i and j, and is called the resistance distance r_{ij} between nodes i and j. The resistance distance metric for the network Γ is a symmetric matrix R_D, with $R_D(i, j) = r_{ij}$ and $R_D(i, i) = 0\ \forall\ i$ and $j \in [n]$. Given a graph G, various matrices can be associated with the graph. The Laplacian matrix L corresponding to any graph G is a symmetric $n \times n$ matrix, L(G), defined as follows:

$\mathrm{L}_{ij} = \frac{-1}{w_{ij}}$, if $i \neq j$ and the vertices i and j are adjacent,
$\mathrm{L}_{ij} = 0$, if $i \neq j$ and the vertices v_i and v_j are not adjacent,
$\mathrm{L}_{ij} = \sum_{i, i \neq j} w_{ij}, i = j$

Let us consider an unknown circular planar passive resistive electrical network as shown in Fig.1. The symbol $\circ$ on the boundary terminals $[n]$ represents terminals that cannot be used for experiments, while the symbol $\bullet$ on the boundary terminals represents terminals that can be used for experiments. Let $\circ_{[n]}$ represent the set of terminals unavailable for experiments, and $\bullet_{[n]}$ represent the set of terminals available for experiments.

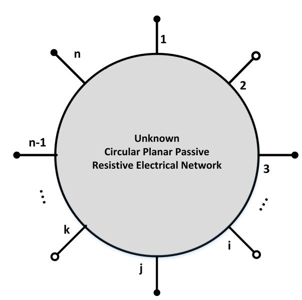

Fig. 1. Unknown circular planar graph G.

Suppose we want to find the resistance distance r_{km} between the available terminals k and $m \in \bullet_{[n]}$; we apply a voltage v_{km}, which results in the boundary current i_k and hence $r_{km} = \frac{v_{km}}{i_k}$, as shown in Fig-2.

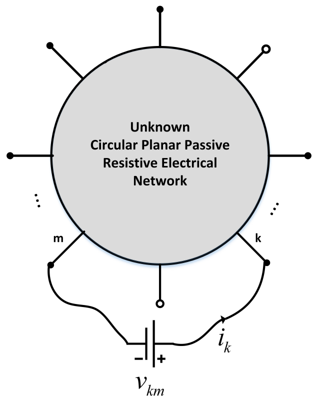

Fig. 2. Calculating resistance distance r_{km}.

With such similar experiments on the terminals k and $m \in \bullet_{[n]}$, we measure all the resistance distances r_{km}.
The resistance distance matrix R_D formed using the measurements $r_{km}, \forall k, m \in \bullet_{[n]}$ is partially known, i.e. only some of the elements of the matrix R_D are known. Using the available resistance distance measurement r_{km}, we relate the known r_{km} to Moore-Penrose pseudoinverse of Laplacian matrix, X,

$$r_{km} = x_{kk} + x_{mm} - 2x_{km}. \tag{1}$$

x_{ij} is the $(ij)^{th}$ element of the $n \times n$ unknown symmetric matrix X. The resistance distance matrix of the network is given as:

$$R_D = J\bar{X} + \bar{X}J - 2X \tag{2}$$

$\bar{X}$ is a diagonal matrix containing the diagonal elements of the matrix X. The Moore-Penrose pseudoinverse of the Laplacian matrix is:

$$X = \left[L + \frac{1}{n}J\right]^{-1} \tag{3}$$

The sub-matrix $R_D\left[r_1, r_2, \cdots, r_{|\circ_{[n]}|}\right], \forall r_1, r_2, \cdots, r_{|\circ_{[n]}|} \in \circ_{[n]}$ is obtained by deleting the $r_1, r_2, \cdots, r_{|\circ_{[n]}|}$ row and column of the matrix R_D.

Let the cardinality of $\bullet_{[n]}$, $\left|\bullet_{[n]}\right| = b$, then we have $\frac{b(b-1)}{2}$ system of linear equations in elements x_{ij} of X. We define the system of linear equation as :

$$Ax = b. \tag{4}$$

In (4), $A \in \mathbb{R}^{\frac{b(b-1)}{2} \times \frac{n(n-1)}{2}}$, $x \in \mathbb{R}^{\frac{n(n-1)}{2}}$. The system of linear equation in (4) is an underdetermined system. Hence the system of linear equation is either inconsistent or has infinitely many solution. The infinitely many solution of (4) is given as:

$$x = x_p + Qx_{hh}, \tag{5}$$

$x_p \in \mathbb{R}^{\frac{n(n-1)}{2}}$ is the particular solution, $Q \in \mathbb{R}^{\frac{n(n-1)}{2} \times nul(A)}$, $nul(A)$ is nullity of A and $x_{hh} = \left[x_{hh1}, \cdots, x_{hhnul(A)}\right]^T \in \mathbb{R}^{nul(A)}$. We define a linear transformation $T : \mathbb{R}^{\frac{n(n-1)}{2}} \to \mathbb{R}^{n \times n}$, such that $T(x) = X$. Hence we have,

$$T(x) = X = T\left(x_p\right) + T\left(Q(:,1)\right)x_{hh1} + \cdots + T\left(Q(:,nul(A))\right)x_{hhnul(A)}. \tag{6}$$

In (6), $Q(:,i) \in \mathbb{R}^{\frac{n(n-1)}{2}}$ is i^{th} column vector in Q. We design x_{hh} such that a proper X

III. Properties of X

We will study the properties of X, which will help in forming LMI constraints. We know that

$$X = \left[L + \frac{1}{n}J\right]^{-1},$$

Laplacian matrix L is positive semidefinite matrix.

The Laplacian matrix L of the graph G. Its eigenvalues and eigenvectors are referred to as the Laplacian eigenvalues and Laplacian eigenvectors of G. These will be denoted by $\mu_1, \mu_2, \ldots, \mu_n$ and $U_1, U_2, \ldots, U_n$, respectively, so that the equalities

$$\mathbf{L}U_i = \mu_i U_i \tag{7}$$

are satisfied for $i = 1, 2, \ldots, n$. We label the Laplacian eigenvalues so that

$$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n.$$

Then, μ_n is always equal to zero, whereas μ_{n-1} differs from zero if and only if the underlying graph G is connected. Consequently, the Laplacian matrix of any graph is singular, and its inverse does not exist. (We are interested in graphs corresponding to the electrical networks, that necessarily are connected. Therefore in what follows it will be understood that $\mu_{n-1} \neq 0$.)

According to (7), the eigenvectors U_i are n-dimensional column vectors. We choose them to be normalized, real-valued and mutually orthogonal, and denote their components so that $U_i = (u_{i1}, u_{i2}, \ldots, u_{in})^T$.

In the notation just introduced, we have

$$\mathbf{U}\mathbf{L}\mathbf{U}^{\mathrm{t}} = \operatorname{diag}\left[\mu_1, \mu_2, \ldots, \mu_{n-1}, \mu_n\right] \tag{8}$$

where $\mathbf{U} = [u_{ij}]_{n \times n}$ is an orthogonal matrix, i. e.,

$$\mathbf{U}\mathbf{U}^{\mathrm{t}} = \mathbf{U}^{\mathrm{t}}\mathbf{U} = \mathrm{I} \tag{9}$$

A. Spectrum of X

We first show that the matrix $\mathbf{U}$, occurring in (8), diagonalizes also the matrix $\mathbf{J}$. Indeed,

$$\begin{aligned}
\left(\mathbf{U}\mathbf{J}\mathbf{U}^t\right)_{ij} &= \sum_k \sum_\ell (\mathbf{U})_{ik}(\mathbf{J})_{k\ell}\left(\mathbf{U}^{\mathrm{t}}\right)_{\ell j} \\
&= \sum_k \sum_\ell u_{ik} J_{k\ell} u_{j\ell} \\
&= \sum_k \sum_\ell u_{ik} u_{j\ell} \\
&= \left(\sum_k u_{ik}\right)\left(\sum_\ell u_{j\ell}\right) = \sigma\left(U_i\right)\sigma\left(U_j\right)
\end{aligned}$$

$$\left(\mathbf{U}\mathbf{J}\mathbf{U}^{\mathrm{t}}\right)_{ij} = \begin{cases} n & \text{if } i = j = n \\ 0 & \text{otherwise} \end{cases}$$

i.e., $\mathbf{U}\mathbf{J}\mathbf{U} = \operatorname{diag}[0, 0, \ldots, 0, n]$

Theorem 1. *Let G be a graph on n vertices, $n \geq 2$, and let $\mu_1, \mu_2, \ldots, \mu_{n-1}, \mu_n = 0$ be its Laplacian eigenvalues. Then the eigenvalues of the matrix $\mathbf{L} + \frac{1}{n}\mathbf{J}$ are μ_1, $\mu_2, \ldots, \mu_{n-1}, 1$.*

Proof. It is sufficient to show that $\mathbf{U}$ *diagonalizes* $\mathbf{L}+\frac{1}{n}\mathbf{J}$ *:*

$$\begin{aligned}\mathbf{U}\left(\mathbf{L}+\frac{1}{n}\mathbf{J}\right)\mathbf{U}^{\mathrm{t}} &= \mathbf{U}\mathbf{L}\mathbf{U}^{\mathbf{T}}+\frac{1}{n}\mathbf{U}\mathbf{J}\mathbf{U}^{t}\\ &= \operatorname{diag}\left[\mu_1,\mu_2,\ldots,\mu_{n-1},0\right]\\ &\quad+\frac{1}{n}\operatorname{diag}[0,0,\ldots,0,n]\\ &= \operatorname{diag}\left[\mu_1,\mu_2,\ldots,\mu_{n-1},1\right]\end{aligned}$$

Theorem 2. *Consider a graph G, which is connected, with the laplacian matrix L, then*
1. The matrix $X=\left(\mathbf{L}+\frac{1}{n}\mathbf{J}\right)^{-1}$ *exists.*
2. The eigenvalues of X are $1/\mu_1, 1/\mu_2, \ldots, 1/\mu_{n-1}, 1$.
3. The eigenvectors of X coincide with the Laplacian eigenvectors $U_1, U_2, \ldots, U_{n-1}, U_n$ *of G.*
4. $x_{ij}=\frac{1}{n}+\sum_{k=1}^{n-1}\frac{1}{\mu_k}u_{ki}u_{kj}$ *and, in particular,*

$$x_{ii}=\frac{1}{n}+\sum_{k=1}^{n-1}\frac{u_{ki}^2}{\mu_k}>0.$$

5.

$$\sum_{i=1}^{n}x_{ii}=1+\sum_{k=1}^{n-1}\frac{1}{\mu_k}$$

From $Theorem 3$ we can say that X is positive definite matrix. Hence

$$\begin{gathered}X\succ 0\\ T(x)\succ 0\\ T\left(x_p\right)+T\left(Q(:,1)\right)x_{hh1}+\cdots+T\left(Q(:,nul(A))\right)x_{hhnul(A)}\succ 0\end{gathered} \tag{10}$$

We design the vector $x=[x_{hh1},x_{hh2},\ldots,x_{hhnul(A)}]^T$, such that X is positive definite. This constraint alone will not be able to construct the X. We impose eigen value constraint on x using the submatrix of R_D. Submatrix of R_D gives us some important conclusion on eigen value of the resistance distance matrix R_D

IV. Eigen Value Interlacing Theorem

Let us consider resistance distance matrix R_D, since only some of the above terminals are available. The resistance distance matrix is hence partially available. The submatrix $R_D\left([\bullet_{[n]}],[\bullet_{[n]}]\right)$ obtained by deleting the rows and columns in set $\bullet_{[n]}$. The eigen values of the R_D and $R_D\left([\bullet_{[n]}],[\bullet_{[n]}]\right)$ interlace. The following eigen value interlacing theorem give the relation between R_D and $R_D\left([\bullet_{[n]}],[\bullet_{[n]}]\right)$:

Theorem 3. *(Eigenvalue Interlacing Theorem) Suppose* $R_D\in\mathbb{R}^{n\times n}$ *is symmetric. Let* $R_D\left([\bullet_{[n]}],[\bullet_{[n]}]\right)$ *be a principal submatrix. Suppose* R_D *has eigenvalues* $\lambda_1\le\cdots\le\lambda_n$ *and* $R_D\left([\bullet_{[n]}],[\bullet_{[n]}]\right)$ *has eigenvalues* $\beta_1\le\cdots\le\beta_m$*. Then*

$$\lambda_k\le\beta_k\le\lambda_{k+n-m}\quad for\quad k=1,\cdots,m$$

And if $m=n-1$

$$\lambda_1\le\beta_1\le\lambda_2\le\beta_2\le\cdots\le\beta_{n-1}\le\lambda_n$$

.

These inequalities are used as constraints in terms of $x=[x_{hh1},x_{hh2},\ldots,x_{hhnul(A)}]^T$.

V. Convex Characterizations of Upper and Lower Eigenvalue Constraints

We present a scheme to enforce constraints on the resistance distance matrix spectrum by treating $x=[x_{hh1},x_{hh2},\ldots,x_{hhnul(A)}]^T$ decision variables. Let λ_i be the i^{th}-smallest eigenvalue of the R_D, whose eigenvalues are ordered from least to greatest. Given $m\in\{2,\ldots,n\}$ and $\underline{\lambda}_m>0$, the lower eigenvalue bound assignment problem is to guarantee $\lambda_m\ge\lambda_m$. Likewise, given $p\in\{2,\ldots,n\}$ and $\bar{\lambda}_p>0$, the upper eigenvalue bound assignment problem is to guarantee $\lambda_p\le\bar{\lambda}_p$. Our goal is to achieve individual upper and lower bounds for several R_D matrix eigenvalues simultaneously. We show how these bounds can be recast as linear matrix inequality constraints that can be applied using semidefinite programming.

Lemma 4. *Let* $A\in\mathbb{R}^{r\times q}, B\in\mathbb{R}^{q\times r}$*, and* $r\ge q$*. Then AB and BA have* m *identical eigenvalues with AB having* $r-q$ *additional eigenvalues at zero.*

Lemma 5. *If* $A\in\mathbb{R}^{n\times n}$ *is symmetric and if* $x^TAx\ge 0$ *for all vectors* $x\in\mathbb{R}^n$ *in a k-dimensional subspace, then A has at least k nonnegative eigenvalues.*

Let $\lambda_k(R_D)$ denote the k-th smallest eigenvalue of R_D. Given $a,b\le n$, define the sets of indices $\{m_i\}_{i=1}^a$ and $\{p_i\}_{i=1}^b$ with each $2\le m_i\le n$ and $2\le p_j\le n$ an integer contained in $\{2,n\}$. Define the sets of positive scalars $\left\{\underline{\lambda}_{m_i}\right\}_{i=1}^a$ and $\left\{\bar{\lambda}_{p_j}\right\}_{j=1}^b$. We wish to see if there exist $x=[x_{hh1},x_{hh2},\ldots,x_{hhnul(A)}]^T$ that satisfy the constraints in the following problem:

$$\begin{array}{ll}\text{Find} & x=[x_{hh1},x_{hh2},\ldots,x_{hhnul(A)}]^T\\ \text{subject to} & \lambda_{m_i}(R_D)\ge\underline{\lambda}_{m_i}, i=1,\ldots,a\\ & \lambda_{p_j}(R_D)\le\bar{\lambda}_{p_j}, j=1,\ldots,b.\end{array} \tag{11}$$

Our goal is to find vector $x=[x_{hh1},x_{hh2},\ldots,x_{hhnul(A)}]^T$, respectively, to assign individual lower and upper bounds for several eigenvalues of R_D simultaneously.

A. Bounding Eigenvalues From Below

Given $m<n$ and $\underline{\lambda}_m>0$, we wish to design vector $x=[x_{hh1},x_{hh2},\ldots,x_{hhnul(A)}]^T$, respectively, such that $\lambda_m(R_D)\ge\underline{\lambda}_m$. However, when the graph optimization problem imposes upper eigenvalue constraints as in (1) or objective functions, this approach would likely be infeasible. In contrast, our results make it possible to apply several upper and lower eigenvalue bounds at once. To begin, we construct a linear matrix inequality enforcing the eigenvalue constraint, making use of the following lemma:

Lemma 6. *Suppose that* $m<n, Q_m\in R^{n\times(n-m+1)}$ *is a full column rank matrix whose columns are orthogonal, and S is a symmetric matrix. If* $Q_m^TSQ_m\ge 0$*, then* $\lambda_m(S)\ge 0$.

Theorem 7. *Theorem 3.2: Let* Q_m *be as in Lemma 5. The constraint*

$$Q_m^T\left(R_D-\underline{\lambda}_mI\right)Q_m\ge 0 \tag{12}$$

implies that $\lambda_m\left(L_g\right)\ge\underline{\lambda}_m$.

Proof: First, we note by Lemma 5 that if (12) holds, then the matrix $R_D - \underline{\lambda}_m I$ has at most $m-1$ negative eigenvalues. We now present a convex feasibility program that enforces the lower eigenvalue bound sufficient linear matrix inequality condition of Theorem 6, as follows:

$$\begin{gathered} \text{Find } x = \left[x_{hh1}, x_{hh2}, \ldots, x_{hhnul(A)}\right]^T \\ \text{subject to} \\ Q_m^T \left(R_D\left(x_{hh1}, x_{hh2}, \ldots, x_{hhnul(A)}\right) - \underline{\lambda}_m I\right) Q_m \geq 0 \end{gathered} \tag{13}$$

Theorem 8. *The inequality $\lambda_m(R_D) \geq \underline{\lambda}_m$ holds if and only if $Q_m^T\left(R_D - \underline{\lambda}_m I\right) Q_m \geq 0$, where $Q_m \in \mathbb{R}^{n\times(n-m+1)}$ is the matrix whose columns are the eigenvectors corresponding to the $n-m+1$ largest eigenvalues of $R_D - \underline{\lambda}_m I$.*

Proof: Considering the projection matrix $Q_m Q_m^T$, it follows that $\left(R_D - \underline{\lambda}_m I\right) Q_m Q_m^T$ must have exclusively non negative eigenvalues. Lemma 4 then implies that $Q_m^T\left(R_D - \underline{\lambda}_m I\right) Q_m \geq 0$

VI. Bounding Eigenvalues From Above

Given $p \leq n$ and $\bar{\lambda}_p \geq 0$, we wish to design vector $x = \left[x_{hh1}, x_{hh2}, \ldots, x_{hhnul(A)}\right]^T$, respectively, such that $\lambda_p\left(L_g\right) \leq \bar{\lambda}_p$. We construct a linear matrix inequality enforcing this eigenvalue constraint. The analysis is similar to that of the previous section, and so the proofs are omitted.

Theorem 9. *Let $U_p \in \mathrm{R}^{n\times p}$ be a full column rank matrix whose columns are orthogonal. The constraint*

$$U_p^T\left(\bar{\lambda}_p M - L_e\right) U_p \geq 0$$

implies that $\lambda_p(L_G) \leq \bar{\lambda}_p$

We now present a convex feasibility program that enforces the upper eigenvalue bound sufficient linear matrix inequality condition of Theorem 9, as follows:

$$\begin{gathered} \text{Find } x = \left[x_{hh1}, x_{hh2}, \ldots, x_{hhnul(A)}\right] \\ \text{subject to} \\ Q_m^T \left(\bar{\lambda}_p I - R_D\left(x_{hh1}, x_{hh2}, \ldots, x_{hhnul(A)}\right)\right) Q_m \geq 0 \end{gathered} \tag{14}$$

As in the case of bounding eigenvalues from below, Theorem 9 provides only a sufficient condition to imply $\lambda_p\left(L_g\right) \leq \bar{\lambda}_p$. The following theorem gives a necessary and sufficient condition enabled by a specific choice of U_p :

Theorem 3.5: The inequality $\lambda_p\left(L_g\right) \leq \bar{\lambda}_p$ holds if and only if $U_p^T\left(\bar{\lambda}_p I - R_D\right) U_p \geq 0$, where $U_p \in \mathrm{R}^{n\times p}$ is the matrix whose columns are the eigen vector corresponding to the p smallest eigen values of $\bar{\lambda}_p I - R_D\left(x_{hh1}, x_{hh2}, \ldots, x_{hhnul(A)}\right)$

VII. Conclusion

References

[1] Asadi, Behrang. Network Reconstruction of Dynamic Biological Systems. University of California, San Diego, 2013.