Topology Reconstruction of a Circular Planar Resistor Network

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***Abstract*— We consider the problem of reconstructing all possible topologies with their edge resistance values of an unknown circular planar passive resistive network, whose response matrix is known apriori. The reconstruction problem is an inverse problem and, in general, has no unique solution. The response matrix is used to deduce a set of all possible k-connections in the unknown circular planar network. A reduced set of k-connections is derived, then, corresponding to each k-connection, we generate all possible graph permutations. The graph permutations are then used to generate several candidate planar graphs using union and edge deletion operations. A method is proposed wherein, the candidate planar graphs are posed as a set of non-linear multivariate polynomials. We then use the Gröbner basis to simultaneously reconstruct all possible topologies and the edge resistance values of an electrical network enclosed inside a black box. Numerical simulation establishes the effectiveness of the proposed strategy.**

I. INTRODUCTION

The network topology reconstruction problem involves simultaneous identification of an electrical network structure and the underlying edge resistor values of an unknown circular planar electrical network contained in the black box, using the available measurements. The network reconstruction problem is ill-posed and, in general, has multiple non-unique solutions. This area of research has witnessed significant interest among researchers due to its widespread application in areas such as system biology [1], geology [2], medical imaging [3], power system networks [4], phylogenetics [5] and reconfiguration of VLSI array's [6]. Two principal objectives addressed in electrical network topology reconstruction are $i)$ to determine the topology of a circular planar electrical network enclosed inside a black box, using the boundary voltage and current measurements, and $ii)$ to estimate the edge conductance values of an electrical network [7]. In this paper, we propose a novel strategy to reconstruct all possible electrical network topologies and edge resistance values simultaneously using graph-based methods and the *Gröbner basis*.

Consider a black box with n exposed boundary terminals enclosing a circular planar passive resistive electrical network. The n exposed boundary terminal are labelled $[n] = \{1, 2, ..., n\}$, and are connected through d interior nodes (placed inside the black box), labelled $[d] = \{n+1, n+2, ..., n+d\}$. The connections between the nodes form the edges, the boundary terminal and the interior nodes are the vertices of the graph G. The boundary terminals are used to conduct experiments and collect boundary data. Unlike the inverse problem, the forward problem assumes that network topology and the edge (σ) conductances, $\gamma(\sigma)$, are known. A response matrix Λ is defined on an electrical network that maps the applied boundary voltages v_b to response boundary currents ϕ_b.

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Related work. Electrical network reconstruction is a challenging problem and has invoked much interest among researchers in physics, VLSI, geology, electrical impedance tomography, and control communities. Existing studies on electrical network reconstruction relate the response matrix Λ to the possible k-connections in unknown circular planar resistive network, which helps realize the structure of the network. In [7], it has been shown that the response matrix (Λ) corresponding to a circular planar electrical network with positive circular minors enumerates all possible k-connections between the $2k$ boundary terminals. In [8], the authors present an approach for computing the values of the conductances in a circular planar passive resistor network, using boundary voltages and currents. The network structure is assumed to be known and is, in particular, a circular network. A γ-harmonic function is defined on the circular network. The process of harmonic continuation, along with the response matrix Λ, is used to compute the conductances in a circular network. In [9], it is shown that for any critical [5] circular planar graph G, the conductance values can be computed using the response matrix Λ. The paper also gives an algebraic description of the set of all response matrices corresponding to the boundary data obtained from the circular planar network.

The network reconstruction problem is also being widely studied in phylogenetics [10] and [11]. The reconstruction problem in phylogenetics investigates the evolutionary relationships based on morphological and physiological characteristics [12]. The electrical and phylogenetic networks are combinatorially similar objects [11]; however, in phylogenetic networks, the edge weights are not well understood as in electrical networks. The edges in electrical networks are normally weighted using the conductances, while the relationship between two nodes in phylogenetic networks is weighted with a statistical distance measure. The notion of statistical distance measure in phylogeny and resistance distance in electrical networks are similar [11]. In [11], with the assumption that the phylogenetic response matrix is available, the authors uncover a split network, which yields a bridge structure corresponding to the unknown phylogenetic network. This bridge structure helps reconstruct the local evolutionary relationships.

In networked dynamical systems, the network structure of interconnected dynamical systems is sometimes unknown to us, as seen in system biology [14], power system networks [15], and water distribution networks [16]. Various properties of interconnected dynamical systems like consensus convergence rate, robustness, and controllability depend on its topology [17], [18], [19]. Hence, the information on network structure is critical for analyzing complex dynamical systems. There has been a considerable research on the problem of topology reconstruction of networked dynamical systems using input-output data [20], [21], [22], [23].

The Gröbner basis [24], [25], [26] is a set of nonlinear multivariate polynomials; which allows simple algorithmic solutions to otherwise huge system of nonlinear multivariate polynomial equations. With the increase in computational capabilities in recent years, there has been a significant increase in applications of Gröbner basis. Some of the significant applications are automatic theorem proving [27], graph coloring [28], Integer programming [29], solving inverse and forward kinematics in robotics applications [30], signal and image processing [31], testing controllability and observability, determination of equilibrium points, computing the domain of attraction [32] and computing time-optimal feedback control [33]. Our work presents a new application of Gröbner basis in characterizing all possible circuit topologies corresponding to a known response matrix Λ.

Contributions. This paper makes three main contributions,

- First, we derive a reduced k-connection set from the set of all possible k-connections obtained using the response matrix Λ. The reduced set of k-connections fully describes all possible k-connections.
- Second, we provide a procedure for obtaining candidate planar graphs from the set of reduced k-connections. Here, each k-connection generates a set of graph permutations. These graph permutations are combined using edge deletion and graph union operations to generate candidate planar graphs.
- Finally, each of the generated candidate planar graphs is posed as a set of nonlinear multivariate polynomials. Gröbner basis is then computed corresponding to the set of nonlinear multivariate polynomials. The edge conductance values are the solutions to the Gröbner basis.

Hence, the network topology and edge conductance value corresponding to the response matrix Λ are recovered.

Mathematical Notations. The set $\mathbb{R}^+$ denote the positive real numbers and $\mathbb{Z}_{\geq}0$ is the set of non-negative natural numbers. Cardinality of the set is denoted as $|\cdot|$. An arc of a circle is represented by $\widehat{ab}$. $\varphi^m(l) = \underbrace{(\varphi \circ \varphi \circ \cdots \circ \varphi)}_{m\text{times}}(l)$ is a composition function on l. Let $R = \{a_1, a_2, \ldots, a_r\}$ be row indices and $D = \{b_1, b_2, \ldots, b_s\}$ column indices, the submatrix $\Lambda(R; D)$ is formed from row indices set R and column indices set D.

II. Preliminaries and Problem Setup

A. Preliminaries:

In this paper, we characterize the set of all possible electrical network topologies with their corresponding edge resistance values using the response matrix Λ. Towards this objective, consider a passive resistive circular planar electrical network $\Gamma = (\mathcal{G}, \gamma)$ housed inside a black box with n exposed boundary terminals. A weighted, undirected, simple graph is a triple $\mathcal{G} = (\mathcal{V}_\mathcal{I}, \mathcal{V}_\mathcal{B}, \mathcal{E})$, where $\mathcal{V}_\mathcal{B}$ is the set of boundary nodes and $|\mathcal{V}_\mathcal{B}| = n$. The n boundary nodes are the n exposed boundary terminals. $\mathcal{V}_\mathcal{I}$ is the set of interior nodes housed entirely inside the black box, and $|\mathcal{V}_\mathcal{I}| = d$. $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is the set of undirected edges, where $\mathcal{V} = \mathcal{V}_\mathcal{B} \cup \mathcal{V}_\mathcal{I}$. A circular planar graph $\mathcal{G}$ of Γ is a graph with boundary embedded in disc D on the plane, such that boundary nodes lie on the circle C which bounds D, rest of Γ is entirely inside D [10]. The boundary nodes in $\mathcal{V}_\mathcal{B}$ are placed in circular order in clockwise direction on the boundary circle C. The boundary nodes say $r_1, r_2, ..., r_m$ on C, are in circular order, if

- the points $r_2, \ldots, r_{m-1}$ lie on the arc $\widehat{r_1 r_m}$ on C,
- the order $r_1 < r_2 < \ldots < r_m$ is induced by the angles of the arc, measured clockwise from r_1.

A conductivity function $\gamma : \mathcal{E} \to \mathbb{R}^+$, assigns to each edge $(i, j) \in \mathcal{E}$ a positive real number $\gamma(i, j) = \gamma_{ij}$, known as the edge conductance. Let us consider two distinct boundary nodes p and q on C. The boundary nodes p and q are said to be connected, $p \overset{\beta}{\leftrightarrow} q$, through $\mathcal{G}$, if there exists a sequence of edges $e_0 = (p, i_1), e_1 = (i_1, i_2), \ldots, e_r = (i_r, q)$ from p to q, such that $i_1, i_2, \ldots, i_r \in \mathcal{V}_\mathcal{I}$. The sequence of edges $e_0, e_1, \ldots, e_r$ form a path $\beta = e_0 e_1 \ldots e_r$. Consider two sequences of boundary nodes $\mathcal{P} = (p_1, p_2, \ldots, p_k)$ and $\mathcal{Q} = (q_1, q_2, \ldots, q_k)$. $\mathcal{P}$ and $\mathcal{Q}$ are said to be connected, $\mathcal{P} \overset{\alpha}{\leftrightarrow} \mathcal{Q}$, through $\mathcal{G}$, if there exist k disjoint paths $\alpha_1, \alpha_2, \ldots, \alpha_k$ in $\mathcal{G}$, such that for each i, the path α_i starts at p_i and terminates at $q_{\tau(i)}$. τ is an element of the permutation group $\mathcal{S}_k$ and is the permutation of the indices $1, ..., k$, and $\alpha = \{\alpha_1, \alpha_2, ..., \alpha_k\}$. The paths $\alpha_1, \alpha_2, \ldots, \alpha_k$ are said to be disjoint if for every $1 \leq l, m \leq k$ and $l \neq m$, paths α_l and α_m have no common vertex. A path β connects one boundary node p to another boundary node q, and hence, called a 1-connection, similarly α is called a k-connection from $\mathcal{P}$ to $\mathcal{Q}$, as shown in Figure. 1. The pair of sequences of boundary nodes $(\mathcal{P}, \mathcal{Q}) = (p_1, p_2, \ldots, p_k; q_1, q_2, \ldots, q_k)$ are considered to be a circular pair if the sequence $(p_1, p_2, \ldots, p_k, q_k, q_{k-1}, \ldots, q_1)$ is in clockwise circular order on boundary circle C. For any circular pair $(\mathcal{P}, \mathcal{Q})$, the connection $\mathcal{P} \overset{\alpha}{\leftrightarrow} \mathcal{Q}$ implies that, for

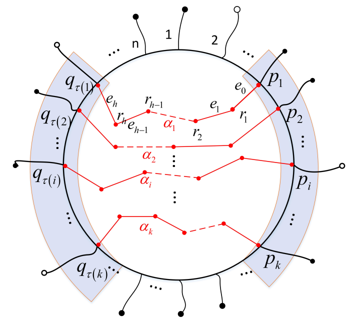

Fig. 1. k-connection between $\mathcal{P}$ and $\mathcal{Q}$

$$1 = a(1,0) \xrightarrow{\varphi} a(1,1) \xrightarrow{\varphi} a(1,2) \xrightarrow{\varphi} \cdots \xrightarrow{\varphi} a(1,n_1) = \tau(1)$$

$$2 = a(2,0) \xrightarrow{\varphi} a(2,1) \xrightarrow{\varphi} a(2,2) \xrightarrow{\varphi} \cdots \xrightarrow{\varphi} a(2,n_2) = \tau(2)$$

$$\cdots$$

$$k = a(k,0) \xrightarrow{\varphi} a(k,1) \xrightarrow{\varphi} a(k,2) \xrightarrow{\varphi} \cdots \xrightarrow{\varphi} a(k,n_k) = \tau(k)$$

Fig. 2. Path diagram of k-connection corresponding to τ and φ

every $l \in \{1, 2, ..., k\}$, $p_l \overset{\alpha_l}{\longleftrightarrow} q_l$ is true. Let us consider the permutation groups $\mathcal{S}_k$ and $\mathcal{S}_d$, and also let a permutation $\tau \in \mathcal{S}_k$ and $\varphi \in \mathcal{S}_d$. τ is the permutation on k interior nodes, and φ is the permutation on d interior nodes. For each $1 \leq l \leq k$ and $0 \leq m \leq n_l$ ($n_l \in \mathbb{Z}_{\geq 0}$), we define $a(l, m) = \varphi^m(l)$ and $\tau(l) = a(l, n_l)$. For fixed τ and each $\varphi \in \mathcal{S}_d$, we get the path diagram as shown in Figure. 2. In the path diagram, $\forall\ 1 \leq l \leq k$, $a(l, 0)$ and $a(l, n_l)$ are the boundary nodes, whereas $a(l, t)\ \forall\ 1 \leq t \leq n_l - 1$ are the interior nodes.

Let us consider a circular planar resistive network Γ with n boundary nodes, labelled $\{1, 2, ..., n\}$ in circular order clockwise direction, and d interior nodes labelled $\{n+1, n+2, ..., n+d\}$ arbitrarily. We define the planarity of Γ in disc D as,

Definition 1. The circuit Γ inside the disc D is said to be *planar in* D if

- all the paths are contained in the interior of D bounded by circle C,
- there exists planar embedding of Γ in the interior of D.

We define the current injections $I \in \mathbb{R}^{(n+d)\times 1}$ and the node voltages $V \in \mathbb{R}^{(n+d)\times 1}$. The network variables I and V are related by the current balance equation $I = \mathcal{K}V$, where $\mathcal{K}$ is called the Kirchhoff's matrix. The Kirchhoff's matrix is a symmetric $(n + d) \times (n + d)$ matrix, $\mathcal{K} = [\mathcal{K}_{ij}], 1 \leq i, j \leq n + d$, defined as follows:

$$\mathcal{K} = [\mathcal{K}_{ij}] \begin{cases} = -\gamma_{ij}, & \text{if } (i, j) \in \mathcal{E}, \\ = \sum\limits_{j \in \mathcal{N}(i)} \gamma_{ij}, & \text{if } i = j, \\ = 0, & \text{otherwise.} \end{cases}$$

The current balance equation can be partitioned as a block matrix,

$$\begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_n \\ \hline i_{n+1} \\ i_{n+2} \\ \vdots \\ i_{n+d} \end{bmatrix} = \left[\begin{array}{c|c} \mathcal{K}([n];[n]) & \mathcal{K}([n];[d]) \\ \hline \mathcal{K}([d];[n]) & \mathcal{K}([d];[d]) \end{array}\right] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \\ \hline v_{n+1} \\ v_{n+2} \\ \vdots \\ v_{n+d} \end{bmatrix}. \quad (1)$$

In line with the labelling of nodes, we define $\phi_b = [i_1, i_2, ..., i_n]^T$ as the boundary response current and $v_b = [v_1, v_2, ..., v_n]^T$ is the applied boundary voltages. The boundary voltages v_b is mapped to the response boundary currents ϕ_b through the response matrix Λ and is given as,

$$\phi_b = \Lambda v_b. \quad (2)$$

Λ is given as Schur complement of $\mathcal{K}$ with respect to $\mathcal{K}([d]; [d])$,

$$\Lambda = \mathcal{K}([n]; [n]) - \mathcal{K}([n]; [d])\mathcal{K}([d]; [d])^{-1}\mathcal{K}([d]; [n]). \quad (3)$$

B. Problem Setup:

In this paper we characterize a set of all the Kirchhoff's matrix corresponding to the known response matrix Λ, hence reconstructing the unknown Γ. Consider a problem setup as shown in Figure. 4. Let us consider $l\pi_r$ to be two sequence of l boundary nodes $(\mathcal{P}^{l\pi_r}; \mathcal{Q}^{l\pi_r})$, where $\mathcal{P}^{l\pi_r} = (p_1^{l\pi_r}, p_2^{l\pi_r}, \ldots, p_l^{l\pi_r})$, $\mathcal{Q}^{l\pi_r} = \left(q^{l\pi_r}_{\tau^{l\pi_r}(1)}, q^{l\pi_r}_{\tau^{l\pi_r}(2)}, \ldots, q^{l\pi_r}_{\tau^{l\pi_r}(l)}\right)$ which are circular pair and hence $\tau^{l\pi_r} \in \mathcal{S}_l$ is an identity permutation i.e. $\tau^{l\pi_r}(h) = h$ and $1 \leq p_h^{l\pi_r}, q_h^{l\pi_r} \leq n\ \forall\ 1 \leq h \leq l$. The l connection $l\pi_r$ is said to be l-connected, $\mathcal{P}^{l\pi_r} \overset{\alpha^{l\pi_r}}{\longleftrightarrow} \mathcal{Q}^{l\pi_r}$, i.e. there exist l-disjoint paths $\alpha^{l\pi_r} = \{\alpha_1^{l\pi_r}, \alpha_2^{l\pi_r}, \ldots, \alpha_l^{l\pi_r}\}$, each disjoint path $\alpha_h^{l\pi_r}\ \forall\ 1 \leq h \leq l$ starts at $p_h^{l\pi_r}$ and terminates at $q^{l\pi_r}_{\tau^{l\pi_r}(h)=h}$ using the following result

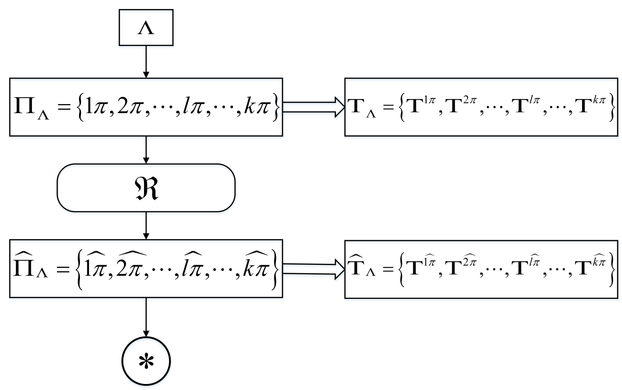

Fig. 3. Problem setup considered in this paper.

Theorem 1. *Let* $\Gamma = (\mathcal{G}, \gamma)$ *be a resistive circular planar network and* $\Lambda \in \mathbb{R}^{n\times n}$ *be the response matrix. Suppose* $\mathcal{P} = (p_1, p_2, \ldots, p_l)$ *and* $\mathcal{Q} = (q_1, q_2, \ldots, q_l)$ *be two sequence of* l *boundary nodes, and the sequence* $(P; Q) = (p_1, p_2, \ldots, p_l; q_1, q_2, \ldots, q_l)$ *is a circular pair.* $\mathcal{P}$ *and* $\mathcal{Q}$ *is said to*

- *l-connected through $\mathcal{G}$: If* $(-1)^l det[\Lambda(P; Q)] \geq 0$,
- *Not l-connected through $\mathcal{G}$: If* $det[\Lambda(P; Q)] = 0$.

We collect all such l-connections using Theorem 1, the set of all l-connections is $l\pi = \{l\pi_1, l\pi_2, \ldots, l\pi_{|l\pi|}\}$, each $l\pi_r \in l\pi$ is associated with a permutation $\tau^{l\pi_r}\ \forall\ 1 \leq r \leq |l\pi|$. Therefore, we construct the set of all permutations $\tau^{l\pi} = \{\tau^{l\pi_1}, \ldots, \tau^{l\pi_r}, \ldots, \tau^{l\pi_{|l\pi|}}\}$ corresponding to $l\pi$. Similarly, we gather all possible connections in a set $\Pi_\Lambda = \{1\pi, \ldots, l\pi, \ldots, k\pi\}$ and all possible permutations corresponding to Π_Λ in set $\mathrm{T}_\Pi = \{\tau^{1\pi}, \ldots, \tau^{l\pi}, \ldots, \tau^{k\pi}\}$.

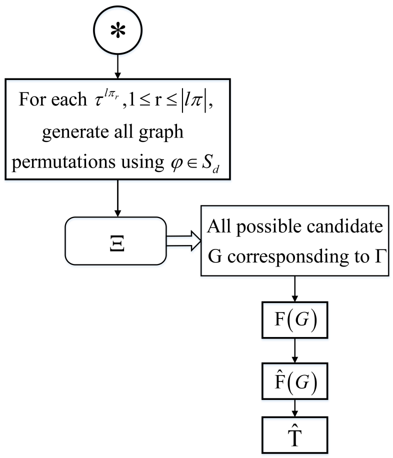

Fig. 4. Problem setup considered in this paper.

The number of possible connections in Γ i.e. $|\Pi_\Lambda|$ is generally large and hence needs to be reduced. We devise a reduction algorithm $\mathcal{R}$, such that $\mathcal{R}:\Pi_\Lambda \to \widehat{\Pi}_\Lambda$, $\widehat{\Pi}_\Lambda$ is the set of reduced connections. Let $\widehat{\Pi}_\Lambda = \left\{\widehat{1\pi},\ldots,\widehat{l\pi},\ldots,\widehat{k\pi}\right\}$, for every $\widehat{l\pi}$, $l \in \{1,\ldots,k\}$, $|\widehat{l\pi}| \le |l\pi|$ is true and corresponding to $\widehat{\Pi}_\Lambda$, we construct a set of permutations $\mathrm{T}_{\widehat{\Pi}} = \left\{\tau^{\widehat{1\pi}},\ldots,\tau^{\widehat{l\pi}},\ldots,\tau^{\widehat{k\pi}}\right\}$. We use the $\widehat{\Pi}_\Lambda$, $\widehat{\mathrm{T}}_{\widehat{\Pi}}$ along with the permutation's $\varphi \in \mathcal{S}_d$ to generates all the possible path diagrams as shown in Figure 2. For better understanding let us consider an example of an unknown circuit Γ with 6 boundary nodes (n) and 2 interior nodes (d) as shown in the Figure 5.

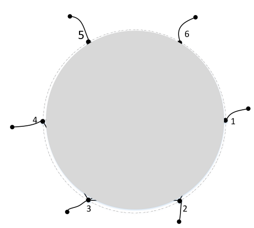

Fig. 5. Γ with 6 boundary nodes and 2 interior nodes.

Let us consider a connected r^{th} 3-connection, $3\pi_r \in 3\pi$, which represents 2-sequences of three boundary nodes $\mathcal{P}^{3\pi_r} = \{p_1^{3\pi_r} = 2, p_2^{3\pi_r} = 3, p_3^{3\pi_r} = 4\}$ and $\mathcal{Q}^{3\pi_r} = \left(q^{3\pi_r}_{\tau^{3\pi_r}(1)} = 1, q^{3\pi_r}_{\tau^{3\pi_r}(2)} = 6, q^{3\pi_r}_{\tau^{3\pi_r}(3)} = 5\right)$, $(\mathcal{P}^{3\pi_r}, \mathcal{Q}^{3\pi_r})$ are circular pairs and hence $\tau^{3\pi_r}$ is an identity permutation. Using $(\mathcal{P}^{3\pi_r}, \mathcal{Q}^{3\pi_r})$, $\tau^{3\pi_r}$ and the permutations $\varphi(\tau^{3\pi_r}) \in \mathcal{S}_d$ we list all possible valid permutations as shown in Figure 6. 7 and 8 are the interior nodes in Γ. A path diagram corresponding to the underlined permutation in Figure 6 is shown in Figure 7. The 3 disjoint paths $\alpha^{3\pi_r} = \left\{\alpha_1^{3\pi_r}, \alpha_2^{3\pi_r}, \alpha_3^{3\pi_r}\right\}$ in Figure 7 connects $\mathcal{P}^{3\pi_r}$ and $\mathcal{Q}^{3\pi_r}$ through $\mathcal{G}$ as shown in a graph permutation in Figure 7. We list all the graph permutations for all possible permutations $\varphi(\tau^{3\pi_r})$ as a set $\mathcal{G}^{3\pi_r} = \left\{\mathcal{G}_1^{3\pi_r}, \mathcal{G}_2^{3\pi_r}, \ldots, \mathcal{G}^{3\pi_r}_{|\mathcal{G}^{3\pi_r}|}\right\}$. Similarly, for r^{th} reduced connection $\widehat{l\pi}_r \in \widehat{l\pi}$ and the corresponding $\tau^{\widehat{l\pi_r}} \in \tau^{\widehat{l\pi}}$ we get a group of graph permutations $\mathcal{G}^{\widehat{l\pi_r}} = \left\{\mathcal{G}_1^{\widehat{l\pi_r}}, \mathcal{G}_2^{\widehat{l\pi_r}}, \ldots, \mathcal{G}^{\widehat{l\pi_r}}_{|\mathcal{G}^{\widehat{l\pi_r}}|}\right\}$, for all the l-connections in the set $\widehat{l\pi}$, we group all the graph permutations to get a set $\mathcal{G}^{\widehat{l\pi}} = \left\{\mathcal{G}^{\widehat{l\pi}_1},\ldots \mathcal{G}^{\widehat{l\pi}_r},\ldots,\mathcal{G}^{\widehat{l\pi}_{|\widehat{l\pi}|}}\right\}$, for all the connections in $\widehat{\Pi}_\Lambda$, we get a set of graph permutations $\mathcal{G}_{\widehat{\Pi}} = \left\{\mathcal{G}^{\widehat{1\pi}},\ldots \mathcal{G}^{\widehat{l\pi}},\ldots,\mathcal{G}^{\widehat{k\pi}}\right\}$.

$p_{\boxed{1}}$	$p_{\boxed{2}}$	$p_{\boxed{3}}$		
1	2	3	7	8
1	2	3	8	7
1	2	8	3	7
		$\vdots$		
$\underline{1}$	$\underline{8}$	$\underline{7}$	$\underline{3}$	$\underline{2}$
		$\vdots$		
8	7	3	2	1

Fig. 6. All the permutations $\varphi \in \mathcal{S}_d$

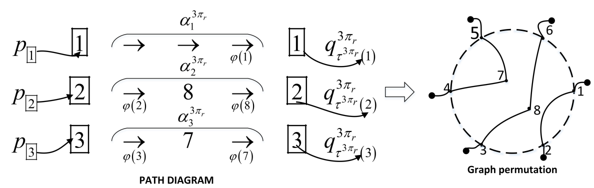

Fig. 7. Path diagram corresponding to the $\varphi(1)$=1, $\varphi(2)$=8, $\varphi(3)$=7, $\varphi(7)$=3, $\varphi(8)$=2 and the corresponding graph permutation.

Now, we use the set of all graph permutations $\mathcal{G}_{\widehat{\Pi}}$ to construct a set of candidate planar graphs $\mathcal{G}^*$. To achieve this we formulate a construction algorithm $\mathscr{C}$, such that $\mathscr{C}:\mathcal{G}_{\widehat{\Pi}} \to \mathcal{G}^*$. Once the set of all candidate planar graph $\mathcal{G}^*$ is obtained, each candidate planar graph $\mathcal{G}_w^* \in \mathcal{G}^*$ $\forall$ $1 \le w \le |\mathcal{G}^*|$ is posed as a set of nonlinear multivariate polynomial, this is done using the equation 3. For instance consider $\mathcal{G}_w^* \in \mathcal{G}^*$, corresponding to known $\mathcal{G}_w^*$ there exist a Kirchhoff's matrix $\mathcal{K}_w^*$ whose structure is known from the known $\mathcal{G}_w^*$. Let $\mathcal{K}_w^*$ be defined as

$$\mathcal{K}_w^* = \left[\begin{array}{c|c} \mathcal{K}_w^*([n];[n]) & \mathcal{K}_w^*([n];[d]) \\ \hline \mathcal{K}_w^*([d];[n]) & \mathcal{K}_w^*([d];[d]) \end{array}\right]. \tag{4}$$

Here, $\mathcal{K}_w^*([n];[n]) \in \mathbb{R}^{n\times n}$ tells us about the connection between n boundary nodes only, $\mathcal{K}_w^*([n];[d]) \in \mathbb{R}^{n\times d}$ stores the information on connection between n boundary nodes and d interior nodes and correspondingly $\mathcal{K}_w^*([d];[d]) \in \mathbb{R}^{d\times d}$ tells us about the internal connection between interior nodes only. The non zero elements in $\mathcal{K}_w^*$ are the unknown's and are to be found. To find out the unknown elements of $\mathcal{K}_w^*$, we relate $\mathcal{K}_w^*$ to the known response matrix Λ using equation 3, we get the following relation,

$$\Lambda = \mathcal{K}_w^*([n];[n]) - \mathcal{K}_w^*([n];[d])\mathcal{K}_w^*([d];[d])^{-1}\mathcal{K}_w^*([d];[n]), \tag{5}$$

Using

$$\mathcal{K}_w^*([d]\,;[d])^{-1} = adj\left(\mathcal{K}_w^*\left([d]\,;[d]\right)\right)\det\left(\mathcal{K}_w^*\left([d]\,;[d]\right)\right)^{-1}$$

in equation 5, we get the following relation

$$\begin{array}{l} \Lambda\det\left(\mathcal{K}_w^*\left([d]\,;[d]\right)\right) \\ +\mathcal{K}_w^*\left([n]\,;[d]\right) adj\left(\mathcal{K}_w^*\left([d]\,;[d]\right)\right)\mathcal{K}_w^*\left([d]\,;[n]\right) \\ -\mathcal{K}_w^*\left([n]\,;[n]\right)\det\left(\mathcal{K}_w^*\left([d]\,;[d]\right)\right) = 0, \end{array} \tag{6}$$

from equation 6 we get $\frac{n(n-1)}{2}$ system of nonlinear polynomial equations.

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