***Abstract*—We consider the problem of reconstructing all possible topologies with edge resistance values of the circular planar passive-resistive network, corresponding to limited boundary measurements. The reconstruction problem is an inverse problem and, in general, has no unique solution. The limitedly available boundary measurements are used to build a resistance distance matrix. The partially known resistance distance matrix is represented as a set of polynomials. A method is proposed to reconstruct the network topology and edge resistor values simultaneously using the Gröbner basis. Numerical simulation establishes the effectiveness of the proposed strategy.**

***Index Terms*—resistor network, topology reconstruction, boundary measurements, Gröbner basis**

I. Introduction

Electrical network topology reconstruction is an ill-posed problem and, in general, has many solutions. Network topology reconstruction involves simultaneously identifying electrical network structure and edge resistor values. This area of research has seen significant interest among researchers due to its applications in a wide range of areas, such as system biology [1], geology [2], medical imaging [3], and power system networks [4]. Two primary objectives that are broadly considered in electrical network topology reconstruction are: i) to determine the conductivity distribution of the electrical network using the measurements of voltages and current at the boundary nodes and ii) to find the conductance of each edge in the electrical network [5]. In this paper, we propose a strategy to reconstruct all possible non-unique electrical network topologies and edge resistance values simultaneously, corresponding to limitedly available boundary measurements using the ***Gröbner basis***.

Consider a connected circular planar passive-resistive electrical network inside a black box. The interior of the box consists of conductors joining the n exposed boundary terminals, labelled $\{1, 2, ..., n\} = [n]$. The boundary terminals correspond to vertices, and the conductors are the edges of a graph G. The forward problem, unlike the inverse problem, presumes that the graph G and the conductance $\gamma(\sigma)$ of each edge σ in G are known. When a voltage V_b is imposed on the boundary terminals, the resulting current ϕ at the boundary nodes is called the network response. The linear map $\Lambda = \Lambda_\gamma$, which transforms the boundary voltage V_b to boundary current ϕ, is called the response map.

It has been shown in [5] that if the response matrix (Λ) satisfies the condition of a non-negative circular minor for a circular planar electrical network, with all the boundary terminals available, then we can reconstruct both the topology of the planar network and the edge conductance of the network. In [6], the authors present an approach for calculating the conductor values in a circular planar passive resistor network, using voltages and currents measured at the boundary. Assuming that only the network structure is known and all the boundary terminals are available, a γ-harmonic function on the circular network is defined. Then using the harmonic continuation and Λ, the conductor values are computed. In [7], it is shown that if G is any critical [5] circular planar graph corresponding to the circular planar resistor network with all boundary terminals available for collecting data, the conductor values can be computed using Λ. Recent work in phylogenetics has also concentrated on similar reconstruction problems, as seen in [8] and [9]. Phylogenetic and electrical networks are combinatorially identical objects [9]; however, the edge weights are less well understood in phylogenetic networks [10]. Electrical networks have been traditionally weighted with conductance, whereas phylogenetic networks have traditionally been weighted with statistical distance measures. However, there is a straightforward analogy between the two paradigms: the genetic distance between two existing taxa (species or individuals) can be compared to the resistance distance [11] between two exposed terminals. In [9], assuming that all the boundary terminals and response matrix are already available, the authors find a corresponding split network [12]. The split network yields the bridge structure of the unknown network, which is used to reconstruct the local graph. However, to the best of our knowledge, there are no general methods that consider partially available boundary terminals and fail to list all the possible non-unique electrical networks corresponding to the collected limited boundary data. In this article, we consider a circular planar passive-resistive network with no interior nodes; we show that the limitedly available data collected from the experiments done on the exposed boundary terminals can be used to design a resistance distance matrix R_D. The unknowns in the R_D can be posed as a set of multivariate polynomials F. The solution set of F contains all possible topologies. Hence, we use Gróbner basis to find solution set of F and a list of all possible solutions to the Gröbner basis is all possible non-unique electrical network topologies with their edge conductance values.

This article is structured as follows. In Section II, we formulate the problem and briefly overview the required mathematical preliminaries on graph theory. In Section III, we propose an algorithm wherein we relate R_D with the Laplacian matrix L. This relation gives rise to a set of polynomials F. The corresponding Göbner basis is calculated, which is used to calculate all possible non-unique network topologies corresponding to the experimental data on partially available boundary terminals. We use numerical examples to demonstrate the developed methodology in Section IV. Finally, concluding remarks are presented in section V.

II. PRELIMINARIES AND PROBLEM FORMULATION

Consider a circular planar passive-resistive electrical network $\Gamma = (G, \gamma)$. A finite, simple graph with a boundary is $G = (V_B, E)$, $E =$ the set of edges, and V_B is the set of boundary nodes, together with a function $\gamma : G \to \mathbb{R}^+$. The conductivity function γ assign to each edge $\sigma \in E$ a number $\gamma(\sigma)$ known as the conductance of σ. A circular planar graph corresponding to the circular planar electrical network is a graph G with a boundary embedded on a disc D in the plane. The boundary nodes lie on circle C that bounds D and the rest of G inside D. The boundary nodes V_B can be labeled as $[n] = \{1, \ldots, n\}$ in clockwise circular order around C. We define a distance metric on the circularly ordered $[n]$; this distance is equivalent to effective electrical resistance between nodes i and j, and is called the resistance distance r_{ij} between nodes i and j. The resistance distance metric for the network Γ is a symmetric matrix R_D, with $R_D(i, j) = r_{ij}$ and $R_D(i, i) = 0$ $\forall$ i and $j \in [n]$. Given a graph G, various matrices can be associated with the graph. The Laplacian matrix L corresponding to any graph G is a symmetric $n \times n$ matrix, L(G), defined as follows:

$\mathrm{L}_{ij} = -1$, if $i \neq j$ and the vertices i and j are adjacent,

$\mathrm{L}_{ij} = 0$, if $i \neq j$ and the vertices v_i and v_j are not adjacent,

$\mathrm{L}_{ij} = d_i$, if $i = j$,

here d_i is the degree of the node i. Since $\sum_{i=1}^{n} \mathrm{L}_{ij} = \sum_{j=1}^{n} \mathrm{L}_{ij} = 0$ for any graph G, we have $det(\mathrm{L}(G)) = 0$. The submatrix obtained by deleting the i^{th} row and the i^{th} column from the Laplacian matrix L is denoted as L$[i]$. The submatrix obtained by deleting the i^{th} and j^{th} rows and the i^{th} and j^{th} columns of the Laplacian matrix L will be denoted by L$[i, j]$ $\forall i, j \in [n]$. For any connected graph G with n vertices, $n \geq 3$, and $1 \leq i \neq j \leq n$, we can relate R_D and the Laplacian matrix L [13] as:

Theorem 1. *Let $G = (V_B, E)$ be a connected graph on n vertices, $n \geq 3$, $\forall$ $1 \leq i \neq j \leq n$. Let $L(i)$ and $L(i, j)$ be sub-matrix of Laplacian matix L, as defined as above. Then*

$$r_{ij} = \frac{\det(\mathrm{L}[i, j])}{\det(\mathrm{L}[i])}. \tag{1}$$

Our objective is to reconstruct all possible network topologies corresponding to limitedly available boundary terminals. Let us proceed with a connected passive-resistive electrical network inside a black box, as shown in Fig-1.

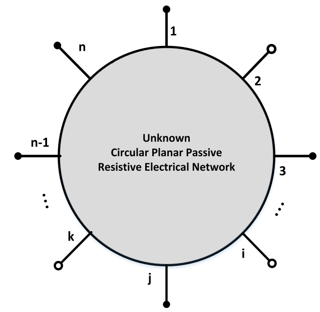


Fig. 1. Unknown circular planar graph G.

The symbol $\circ$ on the boundary terminals $[n]$ represents terminals that cannot be used for experiments, while the symbol $\bullet$ on the boundary terminals represents terminals that can be used for experiments. Let $\circ_{[n]}$ represent the set of terminals unavailable for experiments, and $\bullet_{[n]}$ represent the set of terminals available for experiments. Suppose we want to find the resistance distance r_{km} between the available terminals k and $m \in \bullet_{[n]}$; we apply a voltage v_{km}, which results in the boundary current i_k and hence $r_{km} = \frac{v_{km}}{i_k}$, as shown in Fig-2.

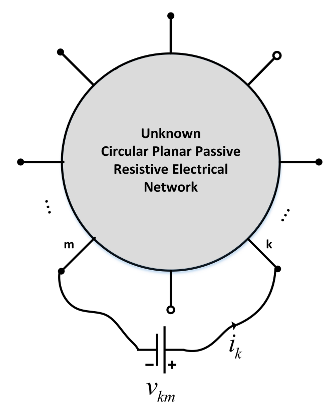


Fig. 2. Calculating resistance distance r_{km}.

With such similar experiments on the terminals k and $m \in \bullet_{[n]}$, we measure all the resistance distances r_{km}. The known resistance distances r_{km} is related to Laplacian matrix L(G) [13],

$$r_{km} = \frac{\det(\mathrm{L}[k, m])}{\det(\mathrm{L}[k])}, \tag{2}$$

resulting in,

$$r_{km} \det(\mathrm{L}[k]) - \det(\mathrm{L}[k, m]) = 0, \tag{3}$$

can be written as,

$$f_{km} = 0. \tag{4}$$

Let F be the set of all such multivariate polynomial f_{km}, corresponding to the measured resistance distances $r_{km}, \forall k, m \in$

$\bullet_{[n]}$. The unknown $n \times n$ Laplacian matrix L, which is to be computed, is defined as:

$$\begin{bmatrix} l_{11} & l_{12} & \cdots & l_{1n} \\ l_{21} & l_{22} & \cdots & l_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ l_{1n} & l_{2n} & \cdots & l_{nn} \end{bmatrix}.$$

Let F be set of polynomials in $\mathbb{R}\left[l_{12}, \cdots, l_{ij}, \cdots, l_{(n-1)n}\right]$. For the set F, a special polynomial basis can be generated known as Gröbner basis $\hat{\mathrm{F}} \in \mathbb{R}\left[l_{12}, \cdots, l_{ij}, \cdots, l_{n(n-1)}\right]$ that allow simple algorithmic solution. Ideal of the set of polynomials F, $\langle F \rangle$, is defined as:

$$\langle F \rangle = \left\{ \sum_{i=1}^{n} h_i f_i \,|h_1, \ldots, h_s \in \mathbb{R}\left[l_{12}, \cdots, l_{(n-1)n}\right] \right\}.$$

Lemma 2. *Let* F *and* $\hat{\mathrm{F}}$ *be set of polynomials as defined above, then,* $\langle F \rangle = \left\langle \hat{F} \right\rangle$.

We define a solution set, V, generated by polynomials in $\hat{\mathrm{F}}$ as:

$$\mathrm{V}\left(\hat{\mathrm{F}}\right) = \left\{ \begin{array}{l} \left(l_{12}, \cdots, l_{ij}, \cdots, l_{(n-1)n}\right) \in \mathbb{R}^{\frac{n(n-1)}{2}} \\ \left|\hat{f}_{ij}\left(l_{12}, \cdots, l_{ij}, \cdots, l_{(n-1)n}\right) = 0, \forall i, j \in \bullet_G, \forall \hat{f}_{ij} \in \hat{\mathrm{F}} \right. \end{array} \right\}, \tag{5}$$

$\mathrm{V}\left(\hat{\mathrm{F}}\right)$ is called affine variety generated by polynomials in set $\hat{\mathrm{F}}$.

Theorem 3. *Let F and* $\hat{F}$ *be polynomials in* $\mathbb{R}\left[l_{12}, \cdots, l_{(n-1)n}\right]$, *such that* $\langle F \rangle = \left\langle \hat{F} \right\rangle$, *then we have* $\mathrm{V}(F) = \mathrm{V}\left(\hat{F}\right)$.

Theorem 4. *Let* $\hat{\mathrm{F}}$ *be the set consisting of Gröbner basis polynomials in* $\mathbb{R}\left[l_{12}, \cdots, l_{(n-1)n}\right]$. *The variety of* $\hat{\mathrm{F}}$, $\mathrm{V}\left(\hat{\mathrm{F}}\right)$, *is the set of all possible topologies.*

Then, the problem of interest can be stated as follows:
Problem: Find the affine variety V corresponding to the set of Gröbner polynomials $\hat{\mathrm{F}}$. The affine variety $\mathrm{V}\left(\hat{\mathrm{F}}\right)$ defines all possible network topologies corresponding to the limitedly available boundary data.

III. Reconstruction Algorithm

Consider circular planar passive-resistive electrical network $\Gamma = (G, \gamma)$ inside a black box. The set $\bullet_{[n]}$ is as defined. The unknown Laplacian matrix is as defined in (5). The basic idea of the algorithm is to relate the boundary measurements to the Laplacian matrix using the expression in (2). A detailed explanation of the algorithm is given in the following flow chart in Fig. 4,

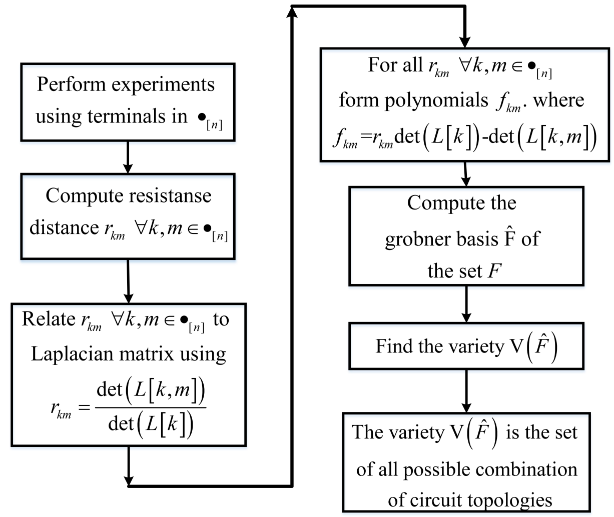
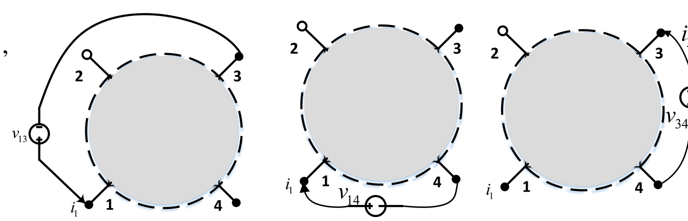


Fig. 3. Reconstruction algorithm.

IV. Numerical Examples

We consider two examples. Firstly, let us consider an electrical network $\Gamma_1 = (G_1, \gamma)$ with four boundary nodes labelled as $[n_1] = \{1, 2, 3, 4\}$. For graph G_1, we define $\bullet_{[n_1]} = \{1, 3, 4\}$. Experiments are conducted on the exposed boundary terminals $\bullet_{[]}$ as shown in Fig. 3.



$v_{13}(V)$	$i_1(A)$	$r_{13}(\Omega)$
5	5	1

$v_{14}(V)$	$i_1(A)$	$r_{14}(\Omega)$
5	6.666	0.75

$v_{34}(V)$	$i_3(A)$	$r_{34}(\Omega)$
5	6.666	0.75

Fig. 4. Experiments on the exposed boundary terminals.

The original electrical network is shown in Fig.5: The

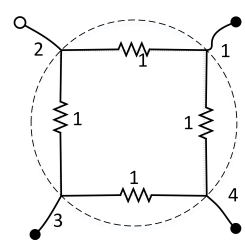


Fig. 5. Original network Γ_1.

polynomials corresponding to resistance distances r_{13}, r_{14} and r_{34} are f_{13}, f_{14} and f_{34} respectively, given as:

$$f_{13} = r_{13} \det(L[1]) - \det(L[1, 3]) \tag{6}$$

$$f_{14} = r_{14} \det(L[1]) - \det(L[1, 4]) \tag{7}$$

$$f_{34} = r_{34} \det(L[3]) - \det(L[3,4]) \tag{8}$$

using (3), and the Laplacian matrix (L) is defined as:

$$L = \begin{bmatrix} l_{12}+l_{13}+l_{14} & -l_{12} & -l_{13} & -l_{14} \\ -l_{12} & l_{12}+l_{23}+l_{24} & -l_{23} & -l_{24} \\ l_{13} & -l_{23} & l_{13}+l_{23}+l_{34} & -l_{34} \\ -l_{14} & -l_{24} & -l_{34} & l_{14}+l_{24}+l_{34} \end{bmatrix} \tag{9}$$

The set of polynomials F in $\mathbb{R}\left[l_{12}, \cdots, l_{ij}, \cdots, l_{n(n-1)}\right]$ is defined as:

$$\mathrm{F} = \{f_{13}, f_{14}, f_{34}\}. \tag{10}$$

We compute the corresponding Gröbner basis $\hat{\mathrm{F}}$ of F using the Buchberger's algorithm [14]. The variety $\mathrm{V}(\hat{\mathrm{F}})$, is a set of all the possible combinations of network topologies, as shown in Fig.6.

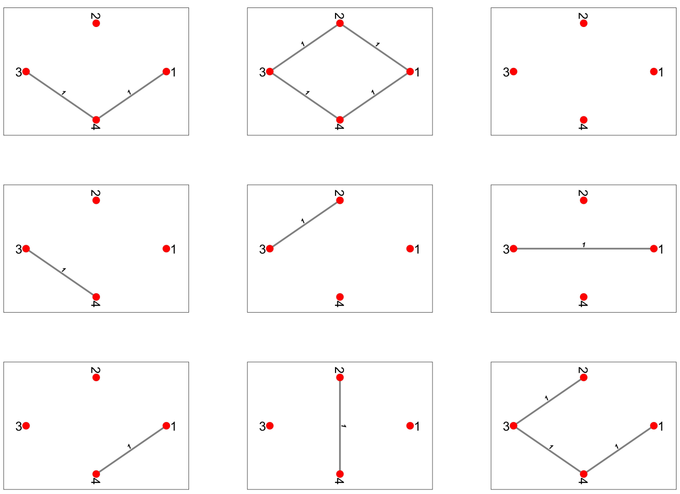


Fig. 6. Variety of Topologies of Γ_1.

The second network from the top left in Fig.6 is the same as the original electrical network. Other networks are not connected, i.e. $\lambda_2(\mathrm{L}(G)) = 0$.

Let us consider second examples, whose original network is shown below in Fig. 7. An electrical network $\Gamma_2 = (G_2, \gamma)$ with five boundary nodes labelled as $[n_2] = \{1, 2, 3, 4, 5\}$. For the network Γ_2, we define $\bullet_{[n_2]} = \{1, 3, 5\}$, i.e. two boundary nodes are not available for experimentation.

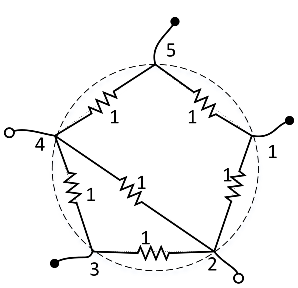


Fig. 7. Original network Γ_2

Experiments are conducted on the exposed terminal $\bullet_{[n_2]}$ and all the resistance distance information are collected. The set of all possible combinations of network topologies corresponding to the resistance distances are shown in Fig.8.

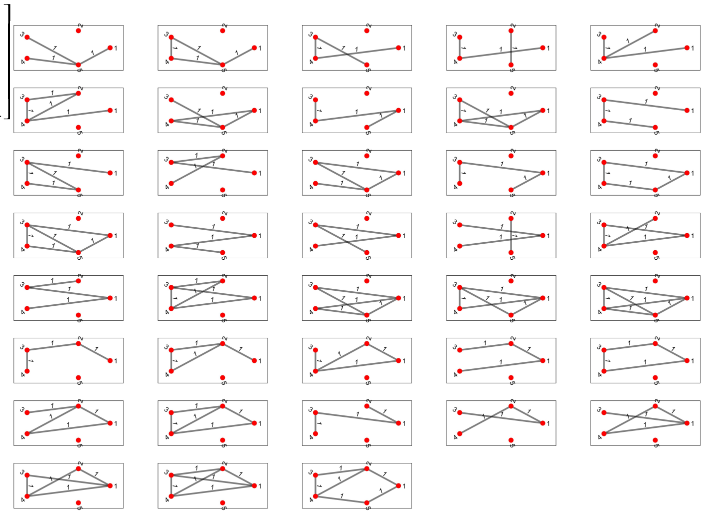


Fig. 8. Variety of topologies of Γ_2

There are many networks which are non-planar and not connected must be discarded in further analysis. The last network in Fig.8 is same as the original electrical network. If connectedness is imposed as a constraint i.e., $\lambda_2(\mathrm{L}(G)) > 0$, then we get only one candidate solution in both examples. If we consider that $|\bullet_{[]}|$ is 2, then the original network can never be constructed.

V. Conclusion

This article uses a Gröbner basis-based technique to get a set of all network topologies corresponding to limitedly available resistance distance data. It is shown that the relation between the resistance distance and the Laplacian matrix generates the set of multivariate nonlinear polynomials, and the variety of the Gröbner basis characterizes all the possible network topologies. The numerical examples demonstrate the effectiveness of the proposed algorithm.

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