CRITERIA on RESISTANCE DISTANCE R_D WITH BOUNDARY NODES

Abstract—**The problem of reconstructing the topology of the passive-resistive network using limited boundary measurements and spectral constraints is considered. The boundary measurements, i.e., boundary terminal voltages and boundary terminal currents, are used to build a resistance distance matrix. The partially known resistance distance matrix along with spectral constraints are formulated as the linear matrix inequalities (LMI). The solution to the LMI uncovers the electric network topology and the edge resistor values. Numerical simulation establishes the effectiveness of the proposed strategy.**

I. Introduction

Electrical network topology reconstruction involves identification of both electrical network structure and edge resistor values. This area of research has seen significant interest among researchers due to its applications in a wide range of areas, such as system biology, geology, medical imaging, chemical engineering, and power networks.Two primary objectives that are broadly considered in electrical network topology reconstruction are: i) to find the conductivity distribution of the electrical network using the measurements of voltages and current at the boundary nodes, and ii) to find the conductance of each edge in electrical network. In this paper, we propose a strategy to estimate the resistivity distribution and edge resistance values simultaneously using limited boundary voltage and current measurements.

Consider a connected circular planar passive-resistive electrical network inside a black box. The interior of the box consists of conductors joining the n exposed boundary terminals, labelled $\{1, 2, ..., n\} = [n]$. The boundary terminals correspond to vertices, and the conductors are the edges of a graph G. The forward problem, unlike the inverse problem, presumes that the graph G and the conductance $\gamma(\sigma)$ of each edge σ in G are known. When a voltage V_b is imposed on the boundary terminals, the resulting current ϕ at the boundary nodes is called the network response. The linear map $\Lambda = \Lambda_\gamma$, which transforms the boundary voltage V_b to boundary current ϕ, is called the response map.

II. PRELIMINARIES AND PROBLEM FORMULATION

Consider a circular planar passive-resistive electrical network $\Gamma = (G, \gamma)$. A finite, simple graph with a boundary is $G = (V_B, E)$, E = the set of edges, and V_B is the set of boundary nodes, together with a function $\gamma : G \to \mathbb{R}^+$. The conductivity function γ assign to each edge $\sigma \in E$ a number $\gamma(\sigma)$ known as the conductance of σ. A circular planar graph corresponding to the circular planar electrical network is a graph G with a boundary embedded on a disc D in the plane. The boundary nodes lie on circle C that bounds D and the rest of G inside D. The boundary nodes V_B can be labeled as $[n] = \{1, \ldots, n\}$ in clockwise circular order around C. We define a distance metric on the circularly ordered $[n]$; this distance is equivalent to effective electrical resistance between nodes i and j, and is called the resistance distance r_{ij} between nodes i and j. The resistance distance metric for the network Γ is a symmetric matrix R_D, with $R_D(i, j) = r_{ij}$ and $R_D(i, i) = 0\ \forall\ i$ and $j \in [n]$. Given a graph G, various matrices can be associated with the graph. The Laplacian matrix L corresponding to any graph G is a symmetric $n \times n$ matrix, L(G), defined as follows:

$\mathrm{L}_{ij} = \frac{-1}{w_{ij}}$, if $i \neq j$ and the vertices i and j are adjacent,
$\mathrm{L}_{ij} = 0$, if $i \neq j$ and the vertices v_i and v_j are not adjacent,
$\mathrm{L}_{ij} = \sum_{i,i \neq j} w_{ij}, i = j$

Let us consider an unknown circular planar passive resistive electrical network as shown in Fig.1. The symbol $\circ$ on the boundary terminals $[n]$ represents terminals that cannot be used for experiments, while the symbol $\bullet$ on the boundary terminals represents terminals that can be used for experiments. Let $\circ_{[n]}$ represent the set of terminals unavailable for experiments, and $\bullet_{[n]}$ represent the set of terminals available for experiments.

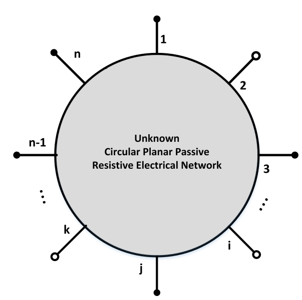

Fig. 1. Unknown circular planar graph G.

Suppose we want to find the resistance distance r_{km} between the available terminals k and $m \in \bullet_{[n]}$; we apply a voltage v_{km}, which results in the boundary current i_k and hence $r_{km} = \frac{v_{km}}{i_k}$, as shown in Fig-2.

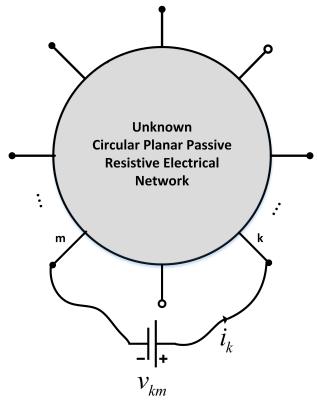

Fig. 2. Calculating resistance distance r_{km}.

With such similar experiments on the terminals k and $m \in \bullet_{[n]}$, we measure all the resistance distances r_{km}.
The resistance distance matrix R_D formed using the measurements $r_{km}, \forall k, m \in \bullet_{[n]}$ is partially known, i.e. only some of the elements of the matrix R_D are known. Using the available resistance distance measurement r_{km}, we relate the known r_{km} to Moore-Penrose pseudoinverse of Laplacian matrix, X,

$$r_{km} = x_{kk} + x_{mm} - 2x_{km}. \tag{1}$$

x_{ij} is the $(ij)^{th}$ element of the $n \times n$ unknown symmetric matrix X. The resistance distance matrix of the network is given as:

$$R_D = J\bar{X} + \bar{X}J - 2X \tag{2}$$

$\bar{X}$ is a diagonal matrix containing the diagonal elements of the matrix X. The Moore-Penrose pseudoinverse of the Laplacian matrix is:

$$X = \left[L + \frac{1}{n}J\right]^{-1} \tag{3}$$

The sub-matrix $R_D\left[r_1, r_2, \cdots, r_{|\circ_{[n]}|}\right], \forall r_1, r_2, \cdots, r_{|\circ_{[n]}|} \in \circ_{[n]}$ is obtained by deleting the $r_1, r_2, \cdots, r_{|\circ_{[n]}|}$ row and column of the matrix R_D.

Let the cardinality of $\bullet_{[n]}$, $\left|\bullet_{[n]}\right| = b$, then we have $\frac{b(b-1)}{2}$ system of linear equations in elements x_{ij} of X. We define the system of linear equation as :

$$Ax = b. \tag{4}$$

In (4), $A \in \mathbb{R}^{\frac{b(b-1)}{2} \times \frac{n(n-1)}{2}}$, $x \in \mathbb{R}^{\frac{n(n-1)}{2}}$. The system of linear equation in (4) is an underdetermined system. Hence the system of linear equation is either inconsistent or has infinitely many solution. The infinitely many solution of (4) is given as:

$$x = x_p + Qx_{hh}, \tag{5}$$

$x_p \in \mathbb{R}^{\frac{n(n-1)}{2}}$ is the particular solution, $Q \in \mathbb{R}^{\frac{n(n-1)}{2} \times nul(A)}$, $nul(A)$ is nullity of A and $x_{hh} = \left[x_{hh1}, \cdots, x_{hhnul(A)}\right]^T \in \mathbb{R}^{nul(A)}$. We define a linear transformation $T : \mathbb{R}^{\frac{n(n-1)}{2}} \to \mathbb{R}^{n \times n}$, such that $T(x) = X$. Hence we have,

$$T(x) = X = T\left(x_p\right) + T\left(Q\left(:,1\right)\right)x_{hh1} + \cdots + T\left(Q\left(:, nul\left(A\right)\right)\right)x_{hhnul(A)}. \tag{6}$$

In (6), $Q(:, i) \in \mathbb{R}^{\frac{n(n-1)}{2}}$ is i^{th} column vector in Q. We design x_{hh} such that a proper X

III. Distance Metric on the Boundary Nodes

We define a distance metric R_D on the boundary nodes $[n] = 1, 2, \ldots, n$ of circular planar passive resistive electrical network $\Gamma = (G, \gamma)$. Let us assume that the response map Γ is assumed to be known. Using Γ, we can decode the k-connection between the set of nodes P and Q as given in lemma 1.

Lemma 1. *Suppose $\Gamma = (G, \gamma)$ is a circular planar resistor network and $(P; Q) = (p_1, \ldots, p_k; q_1, \ldots, q_k)$ is a circular pair of sequences of boundary nodes.*
(a) If $(P; Q)$ are not connected through G, then $\det \Lambda(P; Q) = 0$.
(b) If $(P; Q)$ are connected through G, then $(-1)^k \det \Lambda(P; Q) > 0$.

A metric on $[n]$ can be written as a symmetric matrix R_D. The above lemma 1 brings out an inequality in the metric given as:

Theorem 2. *Suppose $\Gamma = (G, \gamma)$ is a circular planar resistor network. We define a resistance distance metric R_D on boundary nodes $[n]$. For all boundary nodes i, j, k and l, $1 \leq i < j < k < l \leq n$, we have*

$$\begin{gathered} R_D(i,k) + R_D(j,l) \geq R_D(i,j) + R_D(k,l) \\ \text{and} \\ R_D(i,k) + R_D(j,l) \geq R_D(j,k) + R_D(i,l) \end{gathered} \tag{7}$$

Proof. Given a circular, outer planar network N we take the given circular ordering of its leaves and choose any size 4 circular subsequence of the outer nodes: (i, j, k, l).
Then let $Nijkl$ be the circular planar network with the same underlying graph as N, but with the boundary nodes just those four. Let N' be the Kron reduced network of N_{ijkl} and let Γ' be the Kron reduction of Γ with respect to those four nodes. Thus Γ' is a 4×4 response matrix itself. Since the original network is circular planar then the two distinct circular minors of Γ' are non-negative.
Let R'_D be the restriction of R_D to the four chosen nodes (i, j, k, l). Thus R'_D is the resistance matrix corresponding to Γ'. Since R'_D is a restriction we have $R_D\ R_Dij = R_Dij$, and the same for the rest of the respective off-diagonal entries of R_D, R'_D.

If Γ' is non-circular-planar, then it must contain the edges i, k and j, l. In that case N must have contained interior paths from i to k and from $jtol$. However, then N must also contain interior paths from i to j and from k to l. Thus the only non-circular-planar possibility for Γ' is the complete graph on the four nodes i, j, k, l. Thus we consider when Λ is the weighted Laplacian of a complete graph Γ on 4 vertices, with edge weights the corresponding off-diagonal entries of Γ'. The resistances of each edge were chosen for convenience to be p, q, r, x, y, z so that the conductances are the reciprocals. Thus the non-negative circular minors are
$\frac{1}{xq} - \frac{1}{ry} \geq 0$; (this is the circular minor using (i,j;k,1), that is, rows 1,2 and columns 4,3,) $\frac{1}{pz} - \frac{1}{ry} \geq 0$; (this is the circular minor using (j, k; i, l), that is, rows 2,3 and columns 1,4,)

which imply that $ry - pz \geq 0$ and $ry - qx \geq 0$. Next, using either Ohm's law or the pseudoinverse, we calculate: $W_{ik} + W_{jl} - W_{ij} - W_{kl} =$

$$\frac{2qx(ry - pz)}{pqr + pqx + pqz + pry + qrx + prz + qry + pxy + pxz + qxy + pyz + qxz + rxy + qyz + rxz + ryz} \tag{8}$$

which implies $R_D(i, k) + R_D(j, l) - R_D(i, j) - R_D(k, l) \geq 0$,
so
$R_D(i, k) + R_D(jl) \geq R_D(i, j) + R_D(k, l)$.

similarly

$R_D i, k + R_D j, l - R_D j, k - R_D k, l \geq 0$,

so

$R_D i, k + R_D jl \geq R_D i j + R_D k, l$.

$\square$

The above condition is used as an inequality constraint.