Quivers, 3d gauge theories and 3-mfds

Summary workshop: knots, homologies and physics

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Quivers

- Quivers are symmetric matrices

$$P_{C_{ij}}(q;x_1,\cdots,x_N) := \sum_{d_1,\ldots,d_N=0}^{\infty} (-\sqrt{q})^{\sum_{i,j=1}^{N} C_{ij}d_id_j} \frac{x_1^{d_1}x_2^{d_2}\cdots x_N^{d_N}}{(q;q)_{d_1}(q;q)_{d_2}\cdots(q;q)_{d_N}} .$$

- Knots-quivers correspondence

$$\text{Knots} \longrightarrow C_{ij}/\sim$$

Equivalent quivers

Motivation

- We hope to use physics and geometry to understand this correspondence and quivers.

Tools

- 3d N=2 gauge theories: dualities, gauging
- String theories: M-theory/IIB duality, brane webs
- 3-manifolds: surgery, Kirby moves

- We find:

Knots $\leftrightarrow$ Quivers $\leftrightarrow$ 3d N=2 gauge theories $\leftrightarrow$ 3 mfds

3d plumbing theories

- The vortex part. function can be written as quiver generating functions

$$Z^{\text{1-loop}} Z^{\text{vortex}}_{\mathfrak{a}} = P_{C_{ij}}(x_i)$$

- 3d N=2 theories $U(1) \times \cdots \times U(1) + n\,\Phi_i$

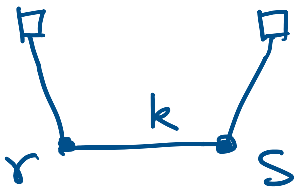
$$K^{\text{eff}}_{ij} = C_{ij}$$ mixed CS levels = quivers

Plumbing graphs

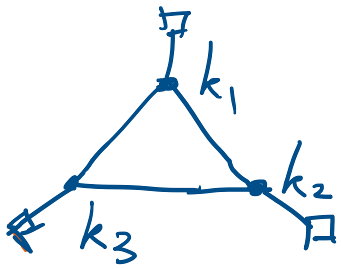
- A new quiver diagram:

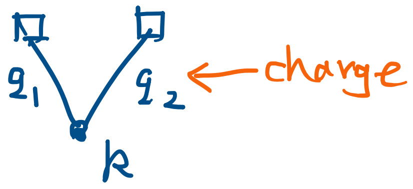
Notation: $\bullet_k$ $\quad U(1)_k \qquad \square \quad 1\ \Phi$

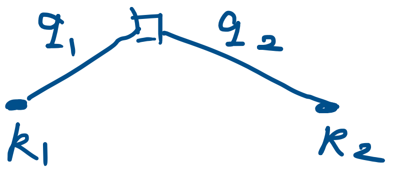
Examples:



$$C_{ij} = \begin{bmatrix} r & k \\ k & s \end{bmatrix}$$







3d dualities

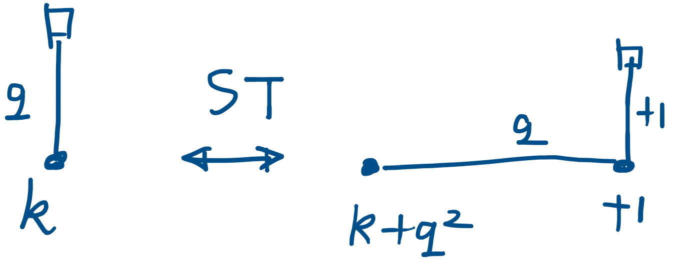
- Gauge the mirror duality -> ST-moves

Free field <-> U(1) +1 field

Flavor symmetry

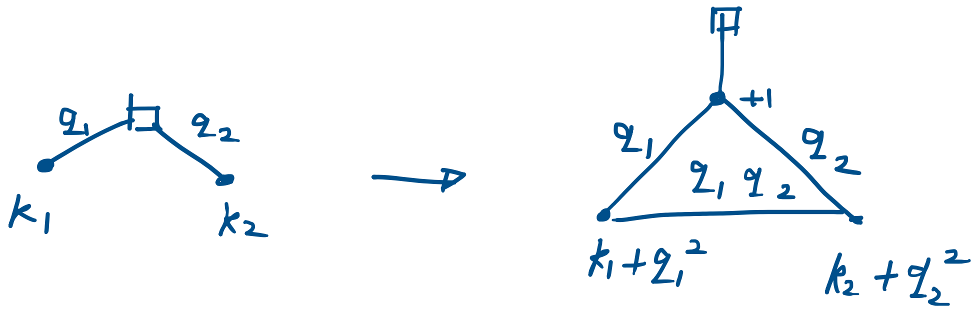
$$U(1)_F \leftrightarrow U(1)_T$$

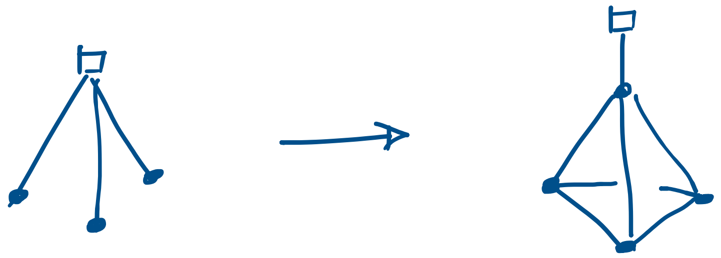
Gauge the U(1)



ST-moves: application

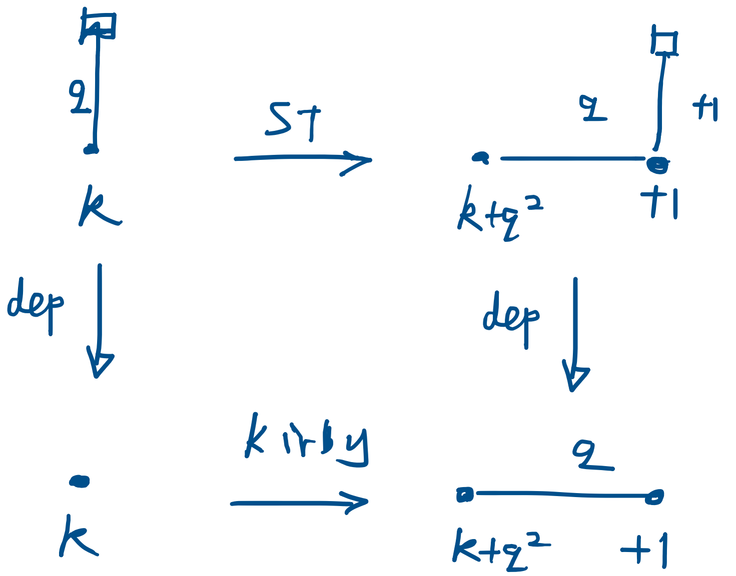
- Examples





Decoupling

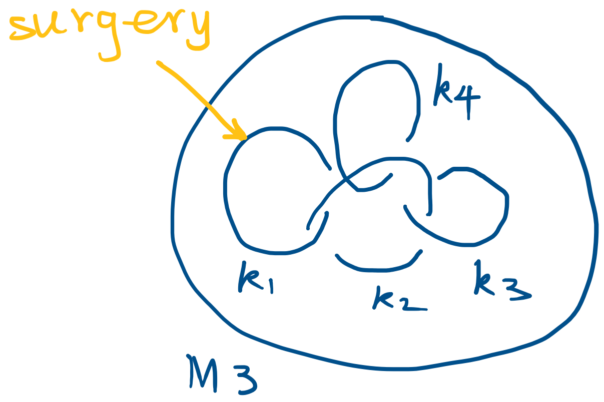
- After decoupling the matter, ST-moves reduce to Kirby moves



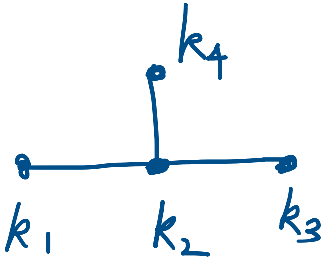
- Why is it a Kirby move?

Closed 3-manifolds, T[M_3] theories

- In Gadde,Gukov, Putrov “Fivebranes and 4-mfds” [1306.4320]. Pure plumbing theories are realized by wraping a single M5-brane on closed three-manifolds.



3d $T[M_3]$:

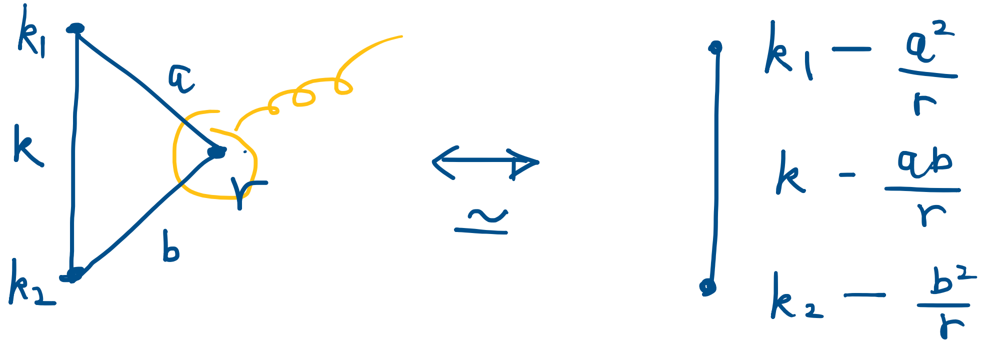


Linking number = CS levels

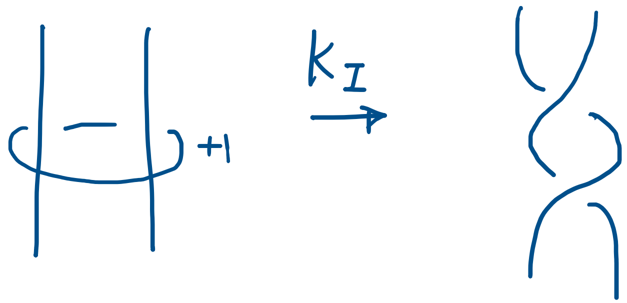
$$L_{ij} = K_{ij}$$

Kirby moves

- Kirby move is integrating in/out gauge nodes U(1)_k

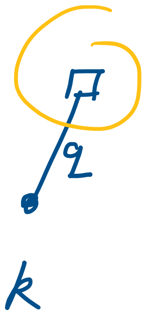


- For 3-mfds, the Kirby move-I is an equivalent surgery



Question: how to add matters?

- Does the matter $\square$ correspondes to some structure on the 3-mfds?

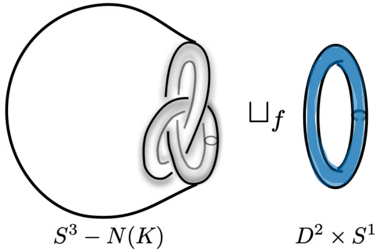


what is this guy ?

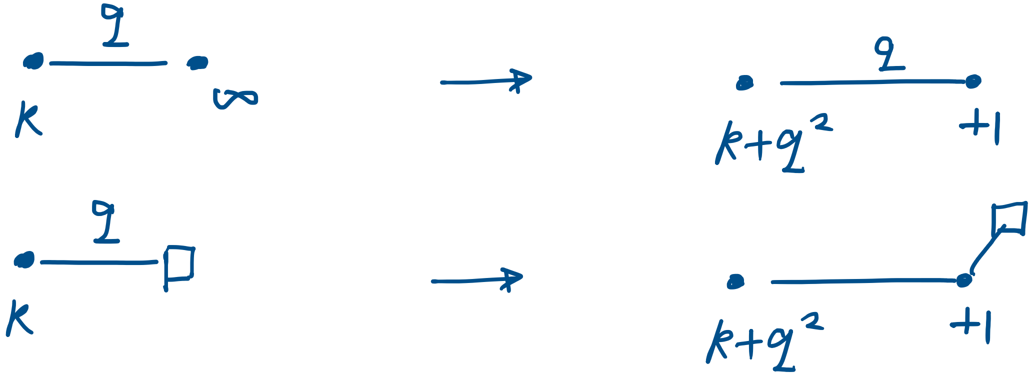
* Is it real ?

* How to geometrically realize it ?

Rational equivalent surgery



- The identical surgery, and rational equivalent surgery



looks similar

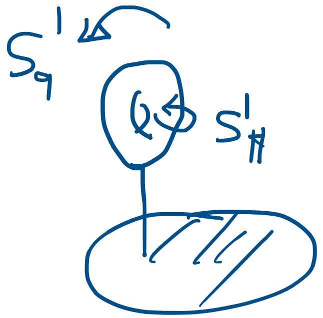
An observation

- Is the matter an circle for identical surgery?

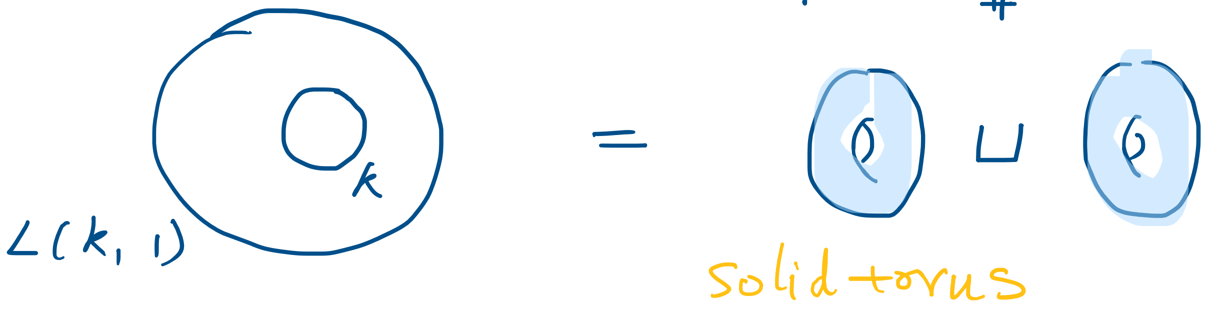
- However, the identical circle can be ignored on 3-mfds and is not physical, while the matter field is physical.
- So, we should do something to make it physical.
- Before that, let us revisit the GGP’s construction using string theory.

Revisit GGP's construction

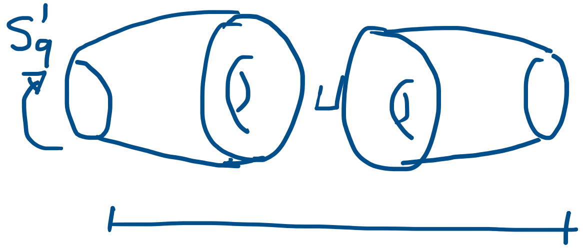
- Lens space L(k,1) in M-theory should be elliptically fibered:



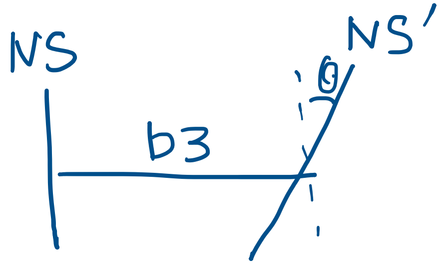
$$T^2 = S^1_q \times S^1_\#$$



- Putting a M5-brane on it duals to a brane web of U(1)_k



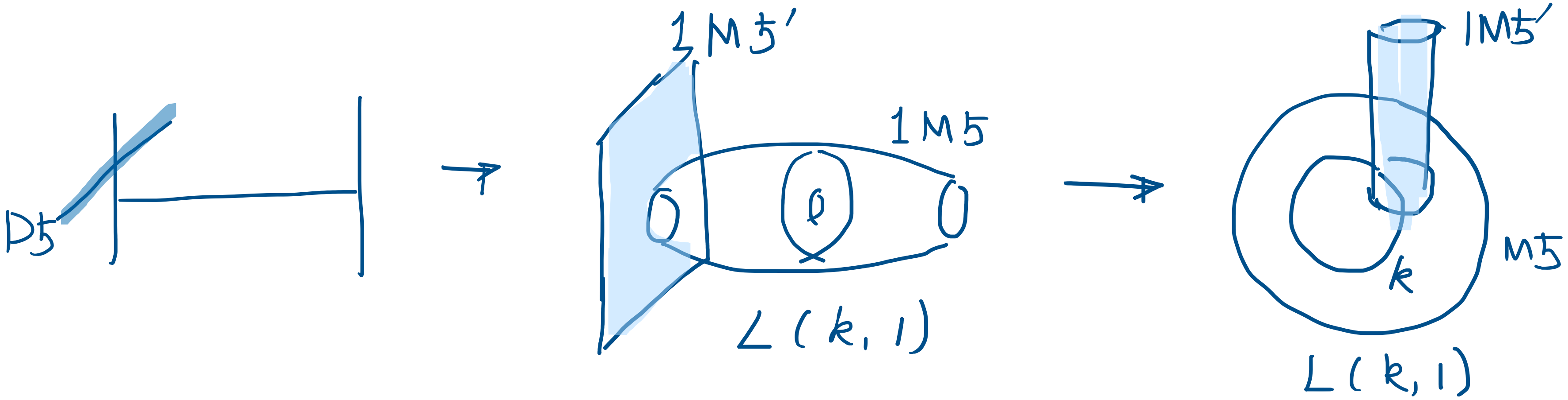
$M \longleftrightarrow IIB$

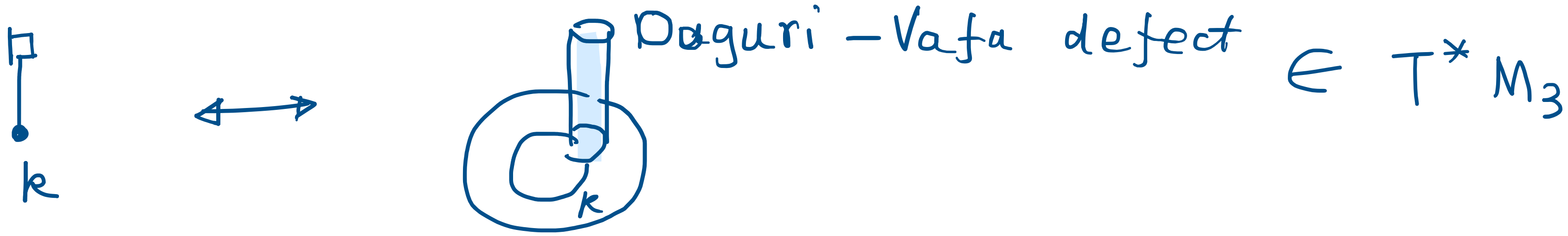


$k = \tan\theta$

OV defect -> matter

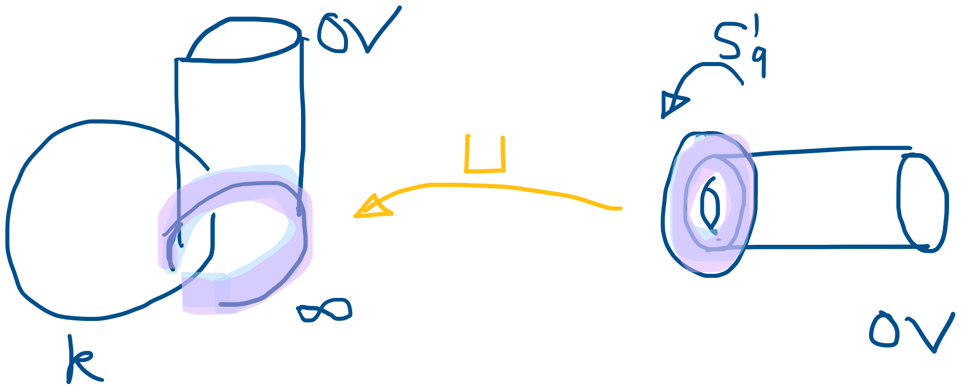
- Adding D5-branes lead to matters





Adding a 1 M5-brane on OV defect in the cotangle bundle realizes a matter field.

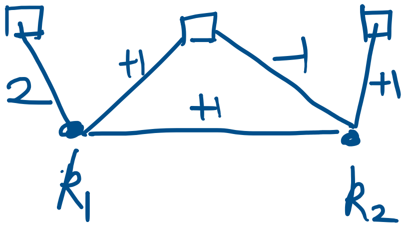
- The neighbourd of the intersection is always an idnetical surgery circle:

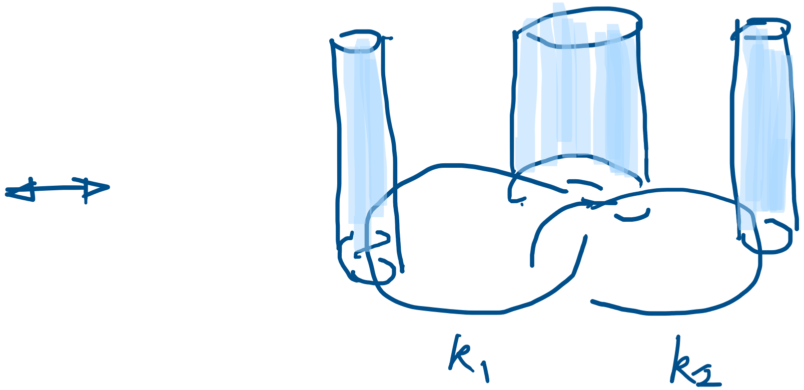


		$S^1\times\mathbb{R}^2$			N_{345}			$I_6\times T^2_{9\sharp}$		
11d	branes	0	1	2	3	4	5	6	9	♯
M-theory	N_c M5	0	1	2				6	9_A	♯
IIA	N_c D4	0	1	2				6	9_A	
IIB	N_c D3	0	1	2				6		
IIA	D0									♯
IIA	D6	0	1	2	3	4	5		9_A	
IIB	D5 $\xrightarrow{S}$ NS5	0	1	2	3	4	5			
M-theory	M5″	0	1	2	3	4			9_A	
IIA	NS5″	0	1	2	3	4			9_A	
IIB	NS5″ $\xrightarrow{S}$ D5	0	1	2	3	4			9_B	
M-theory	M2	0					5		9_A	
IIB	D1 $\xrightarrow{S}$ F1	0					5			

- The matter circle/intersection has to be S^1_q

- Example:



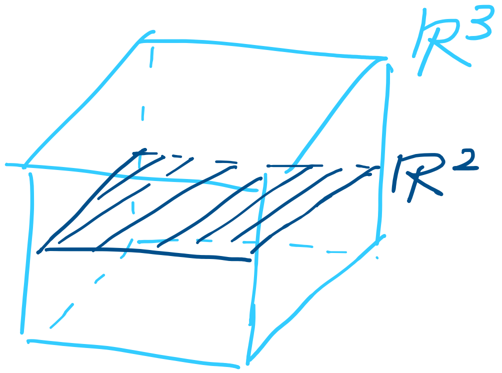


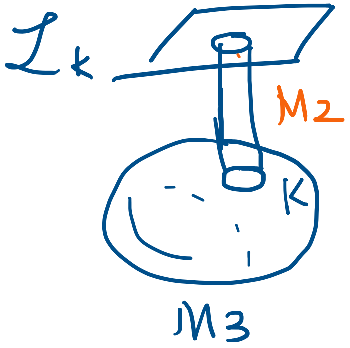
The OV construction

- A point to clarify: the OV defect does really interect with the 3-mfds.

$$\mathcal{L}_K \in \underbrace{T^*M_3}_{= M_3 \times \mathbb{R}^3}, \qquad \mathcal{L}_K = K \times \mathbb{R}^2, \quad \mathbb{R}^2 \subset \mathbb{R}^3$$

$\mathbb{R}^3 \leftarrow$ fiber





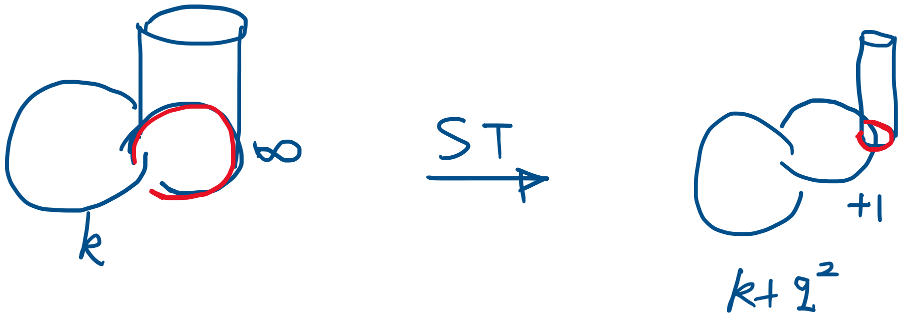
M2-brane : $K \times I$

* The M2-brane is a cylinder, and only when it is massless, the L_K and M_3 will kiss each other:

$$\mathcal{L}_K \cap M_3 = K.$$

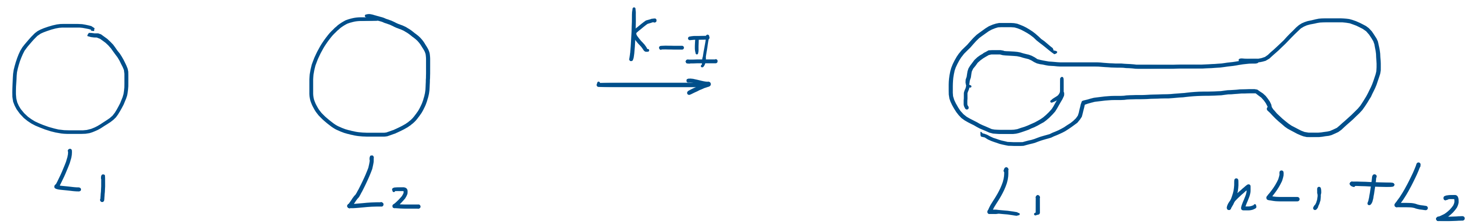
ST-move and 3-mfds

- ST-move is a particular Kirby-I move with a OV defect:

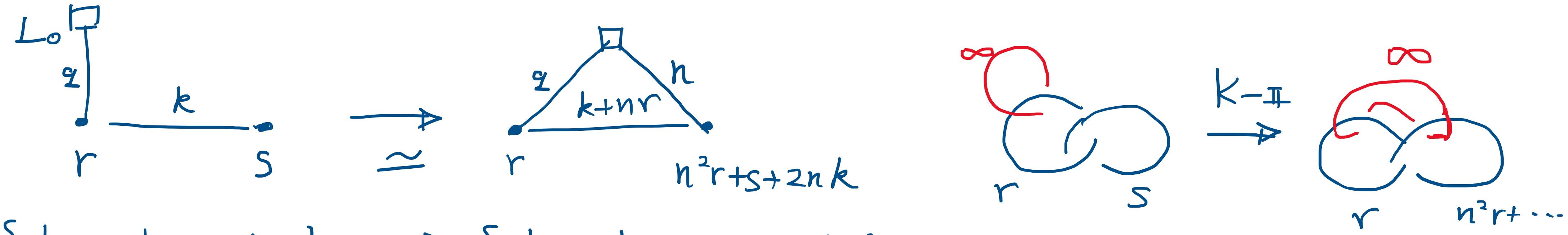


Kirby-II

- Kirby-II is a recombination of circles for Dehn surgeries:



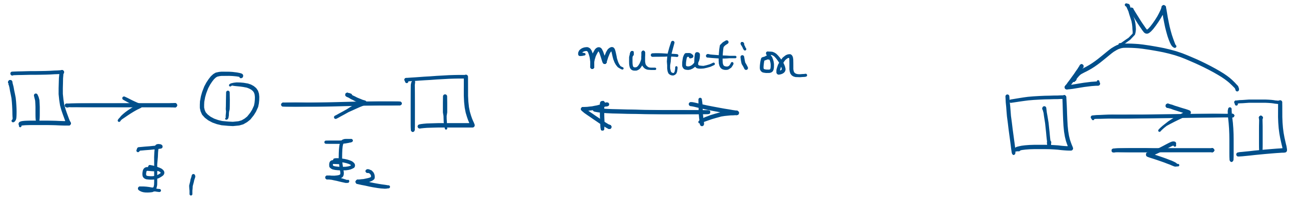
- In the presence of the OV defect (or matter):



$$\{L_0, L_1, L_2\} \to \{L_0, L_1, nL_1+L_2\}$$

Seiberg duality

- SQED-XYZ duality

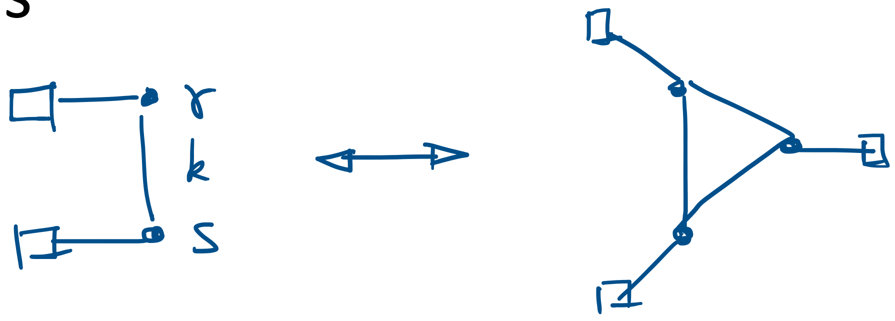


- Superpotential

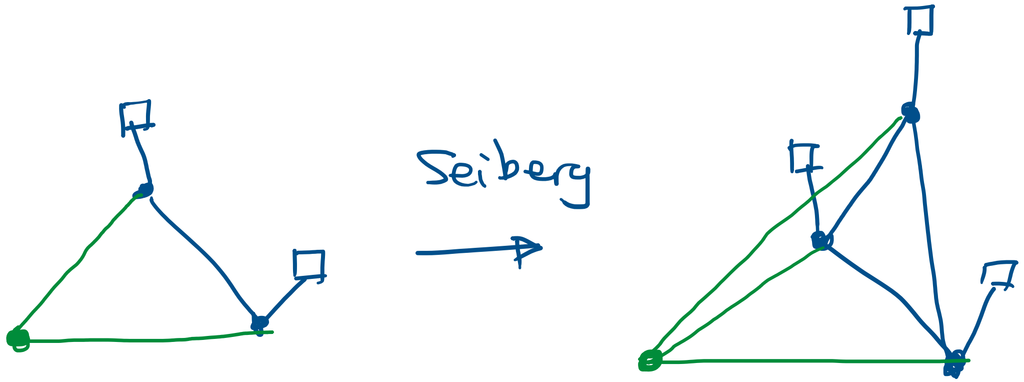
$$W = 0 \qquad\qquad W = \Phi_1 \Phi_2 M.$$

- Flavor symmetry $\quad U(1)_1 \times U(1)_2$

- Gauging these flavor symmetries leads to unlinking, linking, and two other cases



- Seiberg dualities is local, so it could couple to external nodes



- Unfortunately, we have not found the geometric realization of the Seiberg-duality, or in other words, cubic superpotentials.

Dictionary

Quivers	**3d N=2 gauge theories**	**3-mfds**
C_{ij}	Mixed CS levels	Linking numbers
Equivalents quivers	Various dualities	Kirby moves w/ matters
(q,q)_d	matters	OV-defects
d_1,d_2,…,	Gauge symmetry	Surgery circles

Open questions:

- What is the geometrical realization of Seiberg-dualities?
- How to connect quivers and knots using OV constructions? The answer may prove KQ correspondence in an intuitive way.
- Non-abelian theories, and F_K invariants

Thank you!