Quivers, 3d gauge theories and 3-mfds

Summary workshop: knots, homologies and physics

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Quivers

- Quivers are symmetric matrices

$$P_{C_{ij}}(q;x_1,\cdots,x_N) := \sum_{d_1,\ldots,d_N=0}^{\infty} (-\sqrt{q})^{\sum_{i,j=1}^{N} C_{ij} d_i d_j} \frac{x_1^{d_1} x_2^{d_2} \cdots x_N^{d_N}}{(q;q)_{d_1} (q;q)_{d_2} \cdots (q;q)_{d_N}}$$

- Knots-quivers correspondence

$$\text{Knots} \longrightarrow C_{ij}/\sim$$

Equivalent quivers

Motivation

- We hope to use physics and geometry to understand this correspondence and quivers.

Tools

- 3d N=2 gauge theories: dualities, gauging
- String theories: M-theory/IIB duality, 3d brane webs
- 3-manifolds: surgery, Kirby moves

- We find:

Knots $\leftrightarrow$ Quivers $\leftrightarrow$ 3d N=2 gauge theories $\leftrightarrow$ 3-mfds

3d N=2 plumbing theories

- The vortex part. function some theories can be written as quiver generating functions

$$Z^{\text{1-loop}} Z^{\text{vortex}}_{\mathfrak{a}} = P_{C_{ij}}(x_i)$$

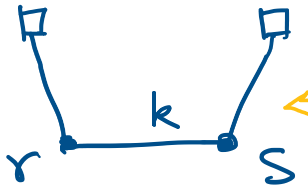
- 3d N=2 theories $U(1) \times \cdots \times U(1) + n\,\Phi_i$

$$K^{\text{eff}}_{ij} = C_{ij}$$ mixed CS levels = quivers

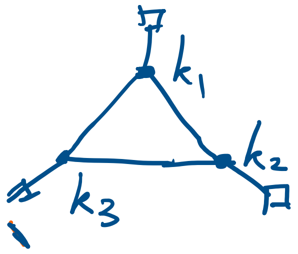
Plumbing graphs

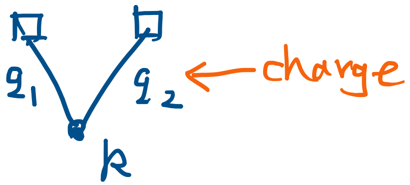
- A new quiver diagram:

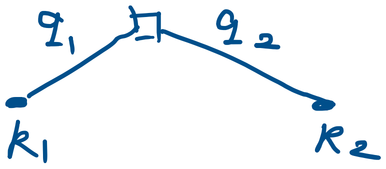
Notation: $\overset{\bullet}{k}$ $U(1)_k$ $\square$ $1\ \Phi$

Quiver theories: 

$$C_{ij} = \begin{bmatrix} r & k \\ k & s \end{bmatrix}$$



Generic theories: 



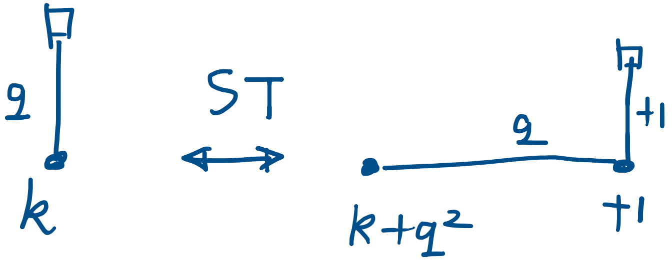
3d dualities

- Gauge the mirror duality -> ST-moves

1 free field <-> U(1) +1 field

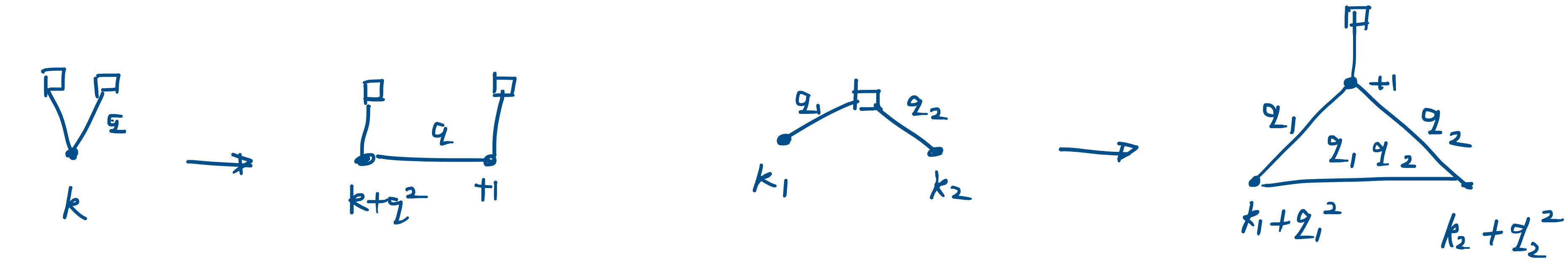
Flavor symmetry $U(1)_F \longleftrightarrow U(1)_T$

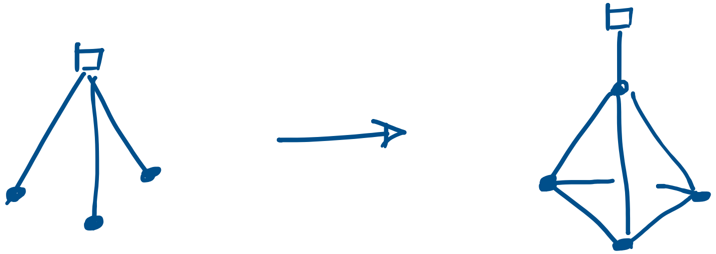
Gauge the U(1)

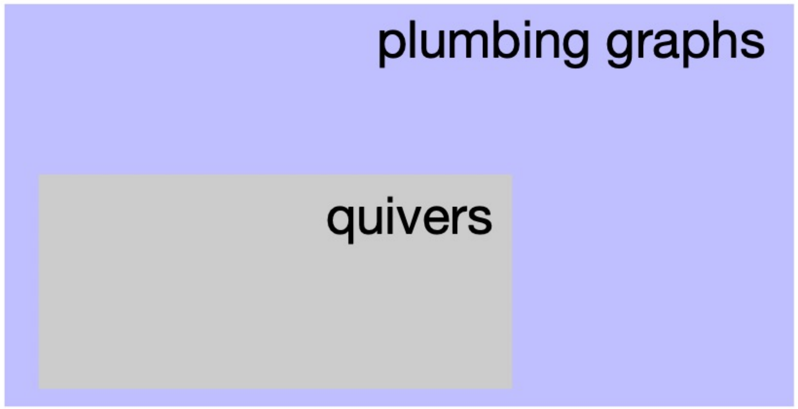


ST-moves: application

- Examples

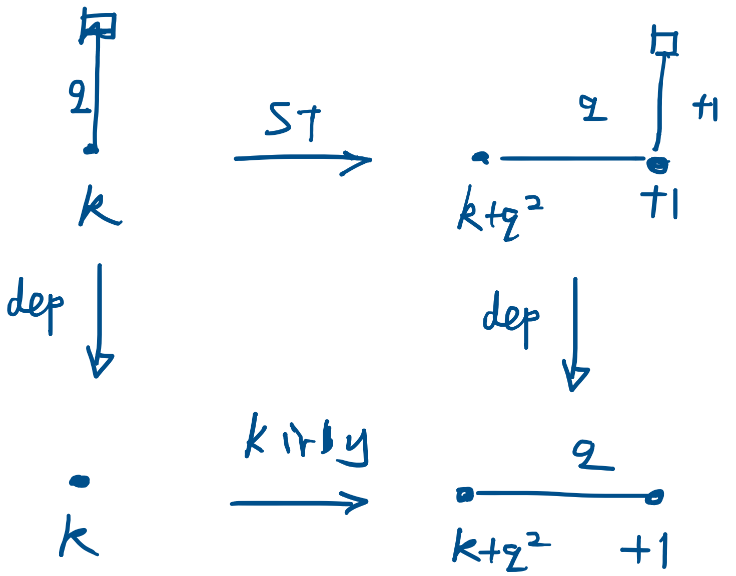






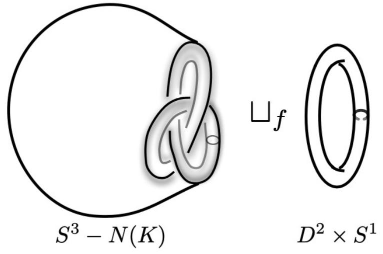
Decoupling

- After decoupling the matter, ST-moves reduce to Kirby moves

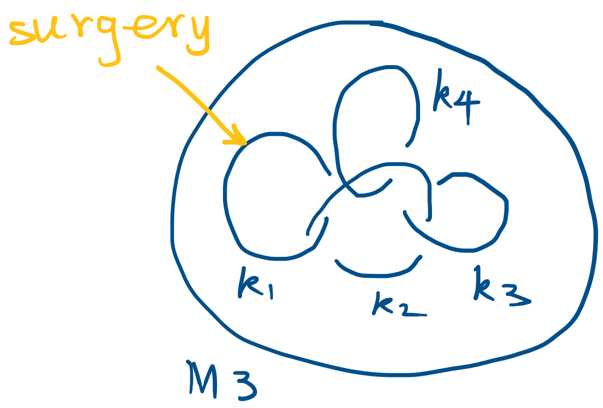


- Why is it a Kirby move?

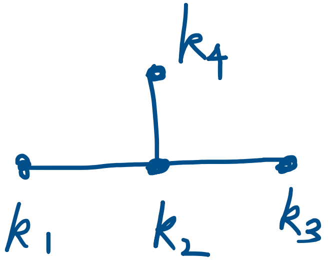
Closed 3-manifolds, T[M_3] theories



- In Gadde, Gukov, Putrov “Fivebranes and 4-mfds” [1306.4320]. Pure plumbing theories are realized by wraping a single M5-brane on closed three-manifolds.



3d T[M3] :

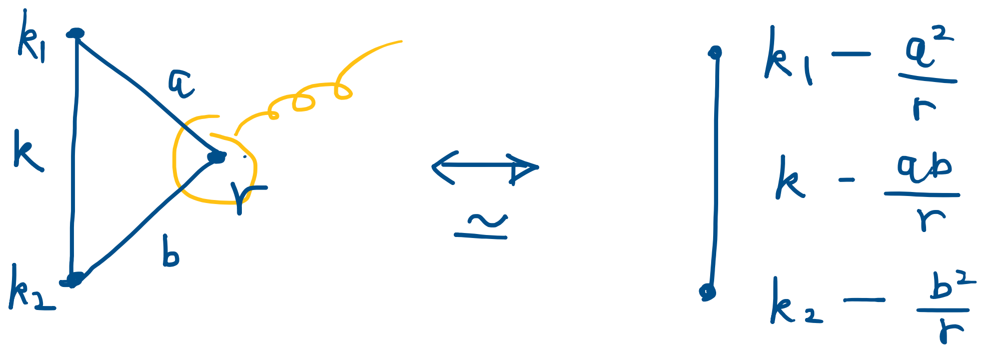


Linking number = CS levels

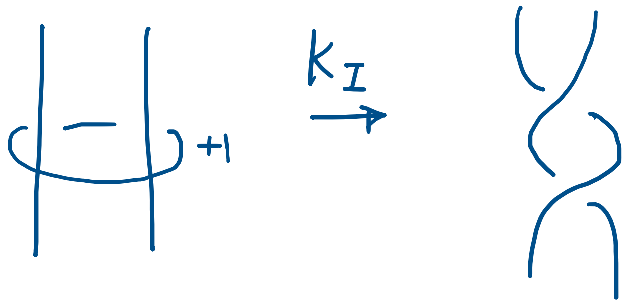
$$L_{ij} = K_{ij}$$

Kirby moves

- Kirby moves are integrating in/out gauge nodes U(1)_k:

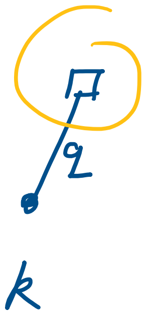


- For 3-mfds, the Kirby-I move is an equivalent surgery.



Question: how to add matters?

- Does the matter $\square$ corresponde to some structure on the 3-mfds?

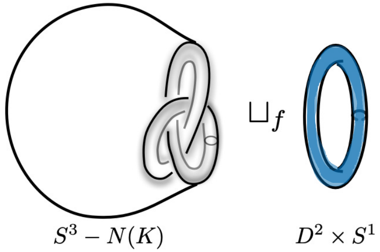


← what is this guy ?

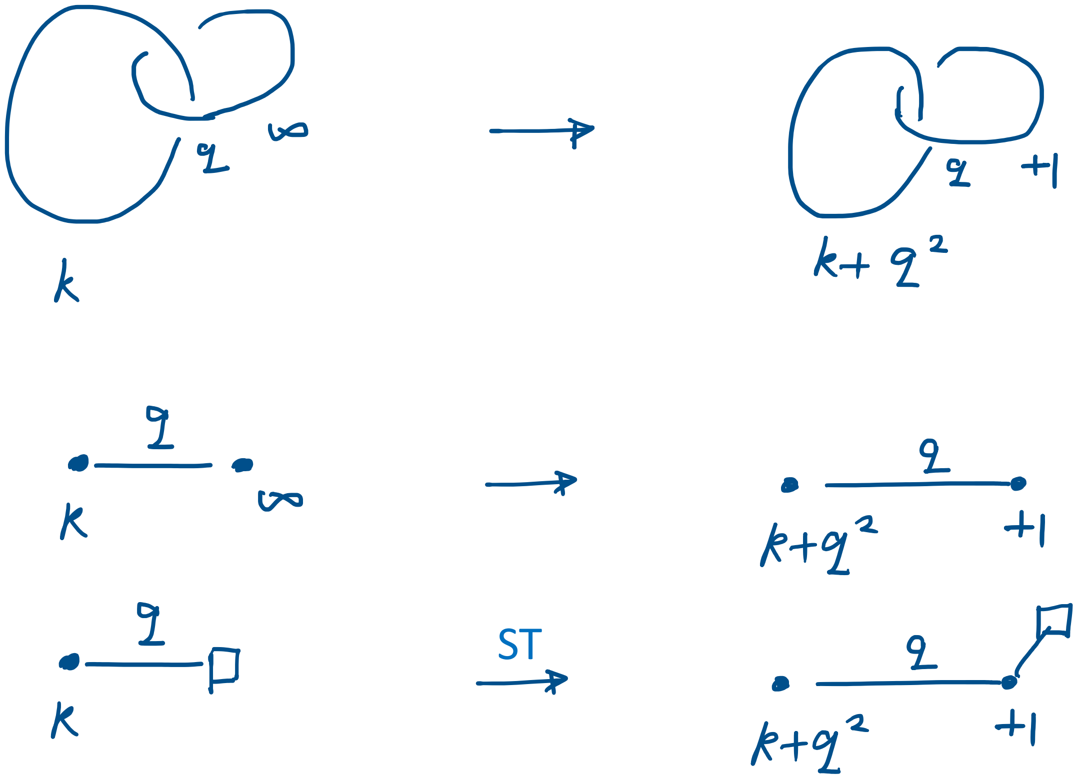
* Is it real ?

* How to geometrically realize it ?

Rational equivalent surgery



- The identical surgery, and rational equivalent surgery



} looks similar 

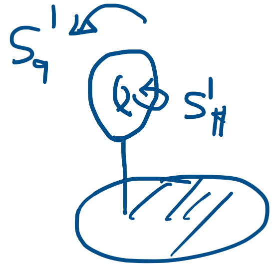
An observation

- Is the matter an circle for identical surgery?

$$\square \xleftrightarrow{?} \bigcirc_{\infty}$$

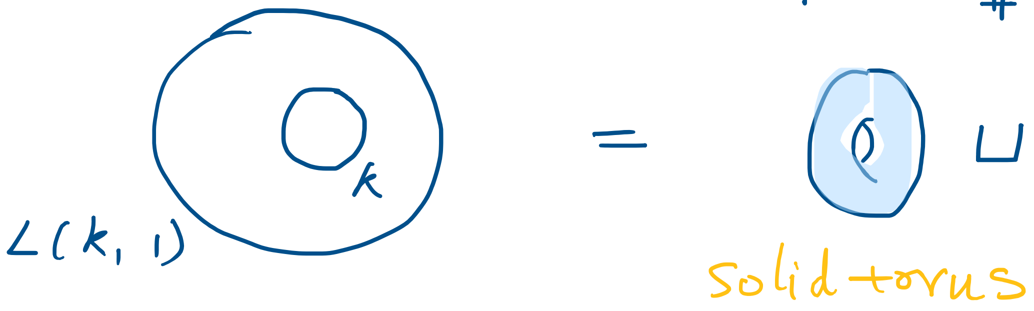
- However, the identical circle can be ignored on 3-mfds and is not physical, while the matter field is physical.
- So, we should do something to make it physical.
- Before that, let us revisit the GGP's construction using string theory.

Revisit GGP's construction



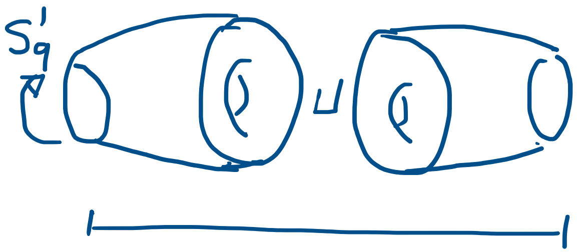
- Lens space L(k,1) in M-theory should be elliptically fibered:

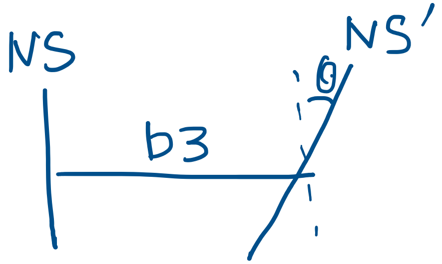
$$T^2 = S^1_q \times S^1_\sharp$$



11d	branes	0	1	2	3	4	5	6	9	$\sharp$
		$S^1\times\mathbb{R}^2$			N_{345}			$I_6\times T^2_{9\sharp}$		
M-theory	N_c M5	0	1	2				6	9_A	$\sharp$
IIA	N_c D4	0	1	2				6	9_A	
IIB	N_c D3	0	1	2				6		
IIA	D0									$\sharp$
IIA	D6	0	1	2	3	4	5		9_A	
IIB	D5 $\xrightarrow{S}$ NS5	0	1	2	3	4	5			
M-theory	M5$''$	0	1	2	3	4			9_A	
IIA	NS5$''$	0	1	2	3	4			9_A	
IIB	NS5$''$ $\xrightarrow{S}$ D5	0	1	2	3	4			9_B	
M-theory	M2	0					5		9_A	
IIB	D1 $\xrightarrow{S}$ F1	0					5			

- Putting a M5-brane on it duals to a 3d brane web of U(1)_k

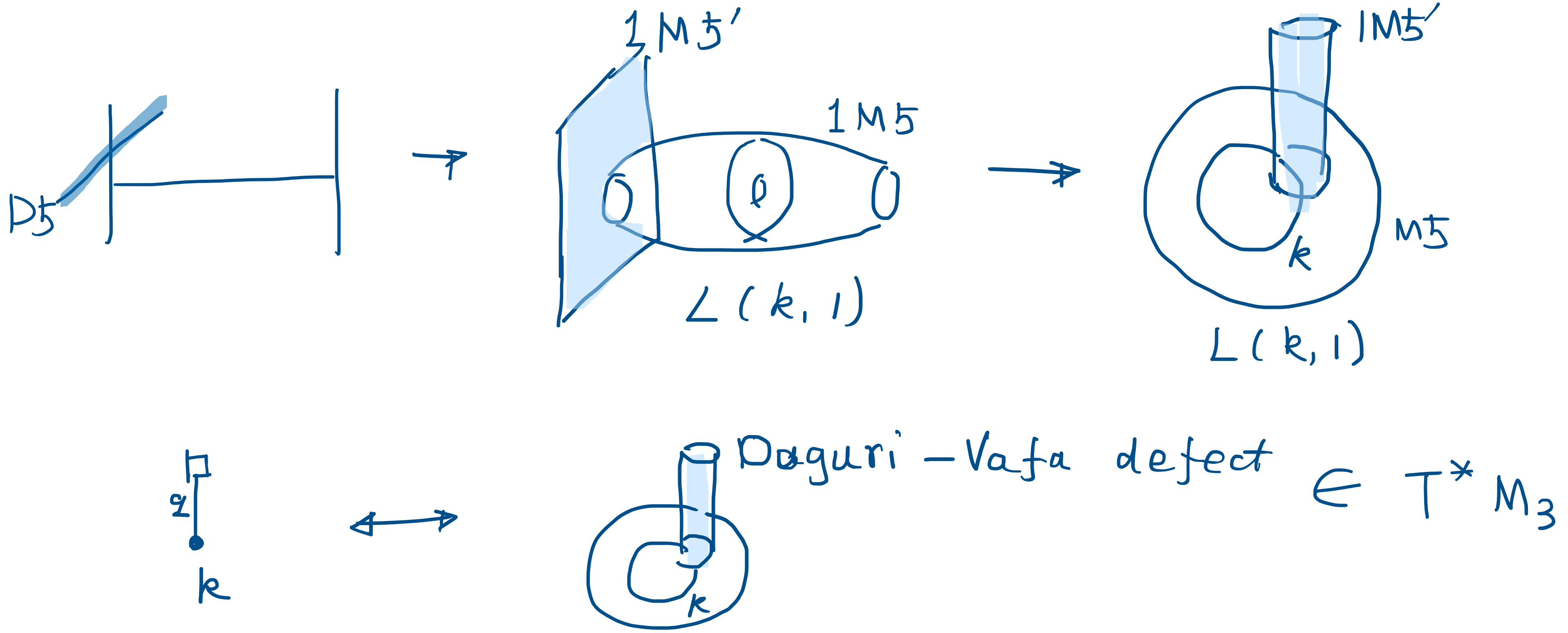




$$k = \tan\theta$$

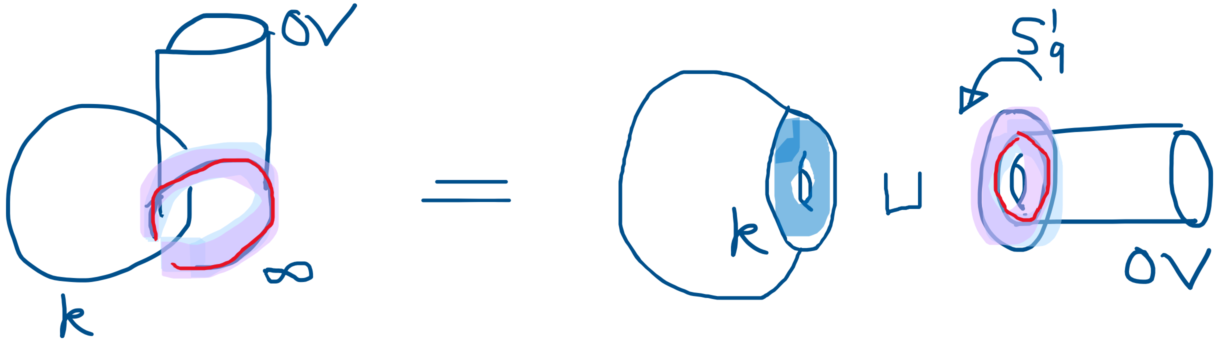
OV defect -> matter

- Adding D5-branes lead to matters



Adding a 1 M5-brane on OV defect in the cotangle bundle realizes a matter field.

- The neighbourd of the intersection is always an idnetical surgery circle:



- The matter circle/intersection has to be S^1_q

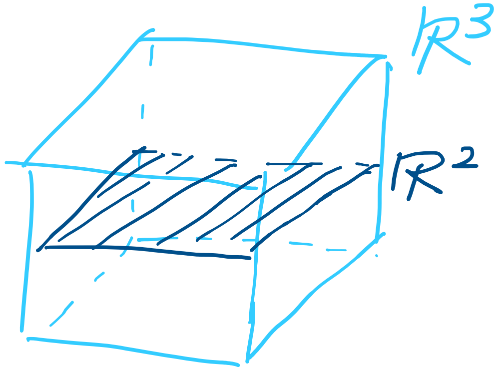
- Example:

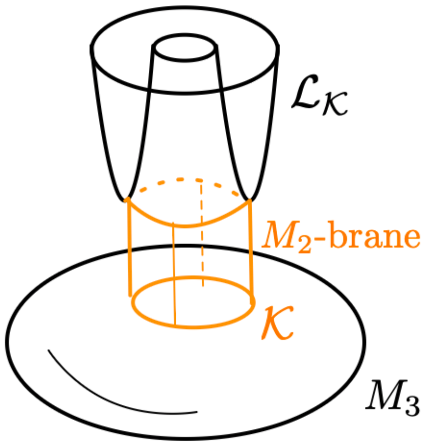
The Ooguri-Vafa construction

- A point to clarify: the OV-defect/brane does really interect with the 3-mfds.

$$\mathcal{L}_K \in T^* M_3 \,, \qquad \mathcal{L}_K = K \times \mathbb{R}^2$$

$$T^* M_3 = M_3 \times \underset{\substack{\uparrow \\ \text{fiber}}}{\mathbb{R}^3} \,, \qquad K \subset M_3 \,, \quad \mathbb{R}^2 \subset \mathbb{R}^3$$





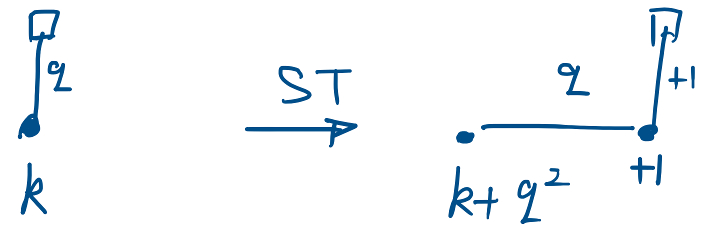
M2-brane : $K \times I$

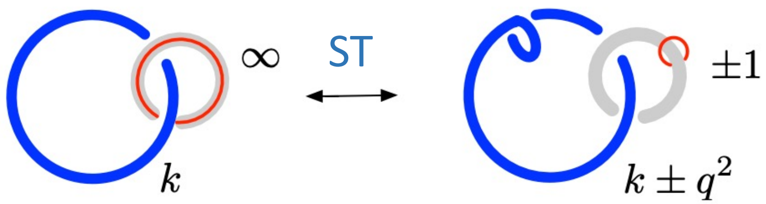
* The M2-brane is a cylinder, and only when it is massless, the L_K and M_3 could kiss each other:

$$\mathcal{L}_K \cap M_3 = K$$

ST-move and 3-mfds

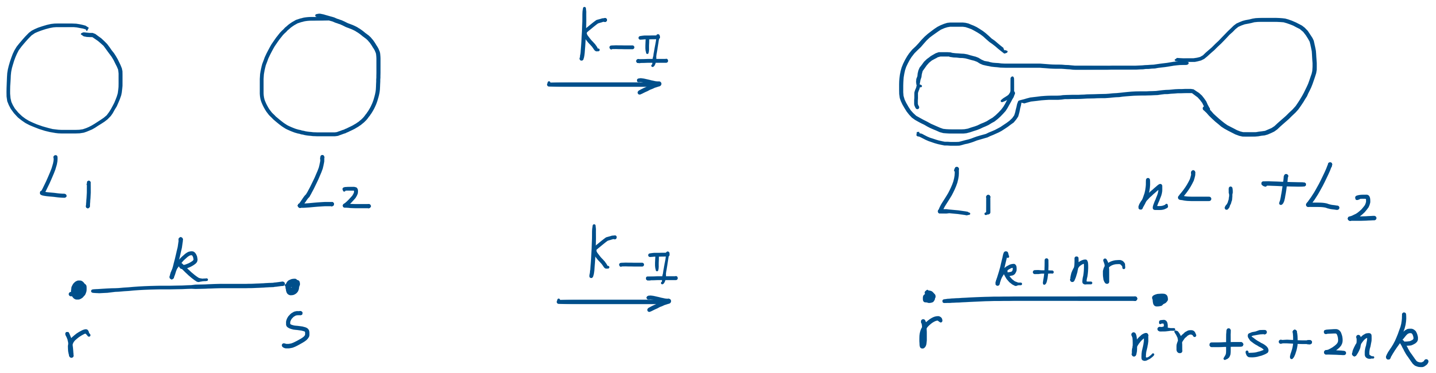
- ST-move is a particular Kirby-I move with an OV-defect/brane:



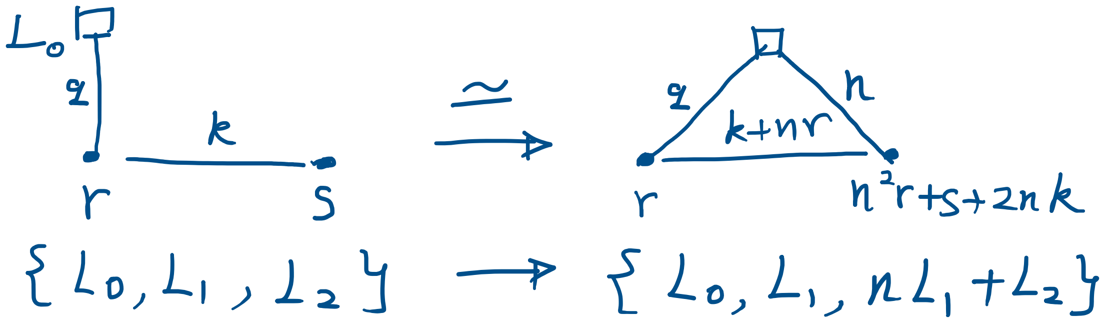


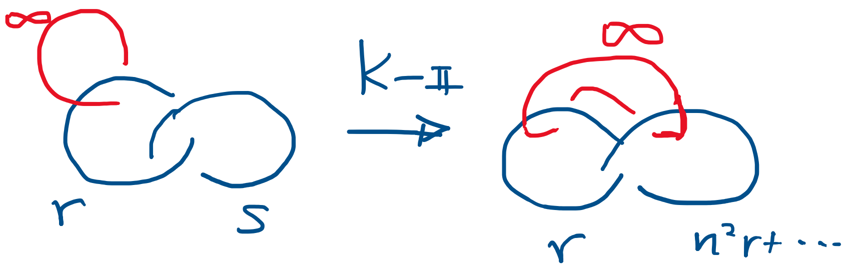
Kirby-II: handle-slides

- Kirby-II is a connected sum of surgery circles (gauge circles):



- In the presence of the OV defect (or matter):

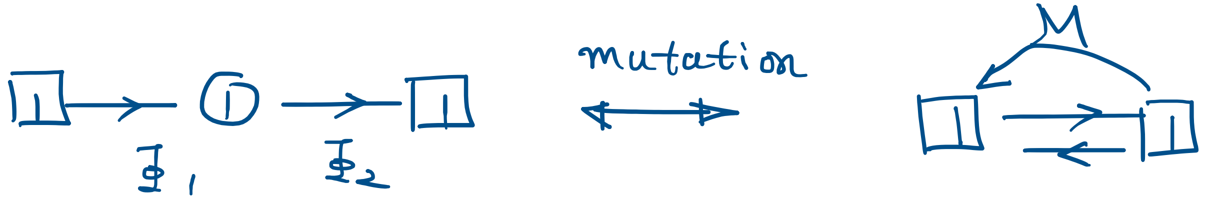




- Kirby-II is the linear sum of scalar fields: $\phi_1' = n\phi_2 + \phi_1 \, , \phi_2' = \phi_2$

Seiberg duality

- SQED-XYZ duality

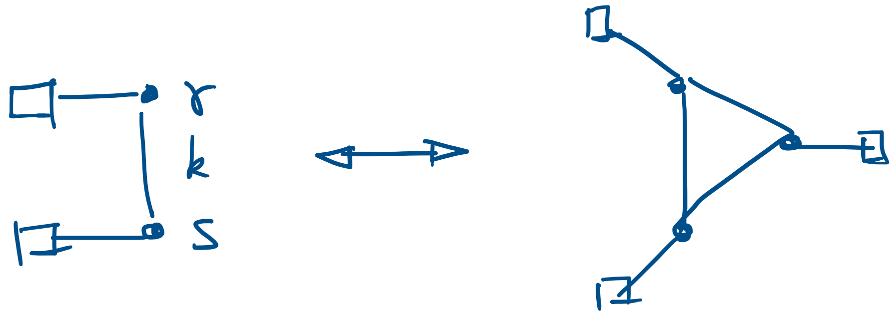


- Superpotential

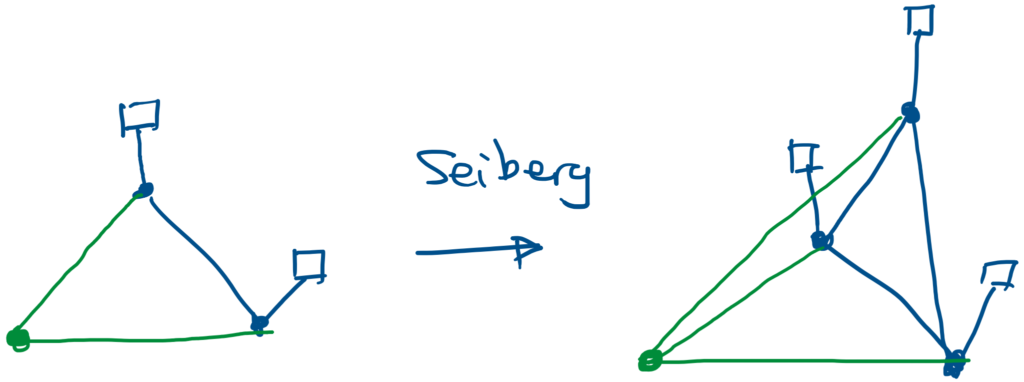
$$W = 0 \qquad\qquad\qquad W = \Phi_1 \Phi_2 M.$$

- Flavor symmetry $U(1)_1 \times U(1)_2$

- Gauging these flavor symmetries leads to unlinking, linking, and other two cases.



- Seiberg dualities is local, so it can couple to external nodes.



- Unfortunately, we have not found the geometrical realization of the Seiberg-duality, or in other words, cubic superpotentials.

Dictionary

Quivers	3d gauge theories	3-mfds
C_{ij}	Mixed CS levels	Linking numbers
Equivalents quivers	Various dualities	Kirby moves w/ OV-branes
$(q,q)_{d_i}$	Matter fields	OV-defects/matter circles
$\sum_{d_i}$	Gauge symmetries	surgery circles/gauge circles

Open questions:

- What is the geometrical realization of Seiberg-dualities?
- It looks that both quivers and knots can be constructed by OV construction. How to direct connect them? The answer may lead to KQ correspondence.
- Non-abelian theories, and F_K invariants

Thank you!