Enhanced Portfolio Optimization

Lasse Heje Pedersen, Abhilash Babu, and Ari Levine.

Financial Analysts Journal, 2021

解读者：屠雪永

2022.03.21

1. Introduction-- Motivation

- Mean-variance optimization(MVO) often produces large and unintuitive bets that perform poorly in practice (Michaud 1989)
- Perhaps as a result, many investors skip optimization altogether
    - value (HML), size (SMB), and momentum (UMD)
- Optimization should be a big help

1. Introduction-- Questions

- Why does standard optimization perform so poorly?
  - noise in the estimation of risk and expected return in the least principal component
- Is there a better way to use the information contained in estimated risks, correlations, and expected returns?
  - down-weight these problem portfolios
    —enhanced portfolio optimization (**EPO**)
  - unifies a broad range of existing methods
- If so, how much is performance improved?
  - the EPO method improves industry momentum and time series momentum performance

1. Introduction-- Related Literature

- **Improving the variance-covariance** estimate using shrinkage, factor models, or random matrix theory.
  - (Ledoit and Wolf, 2003; Elton, Gruber, andSpitzer, 2006. Fan, Fan, and Lv, 2008. e.g., Ledoit and Wolf 2004, 2012, 2017, Karoui 2008, and Bun, Bouchaud, and Potters 2017)
- **Expected returns**
  - Black and Litterman (1992)
- The literature on robust optimization
  - (Fabozzi, Huang,and Zhou, 2010; Raponi, Uppal, and Zaffaroni ,2020)
- Regularize regressions
  - (Ao, Li, and Zheng,2019; Kozak, Nagel, and Santosh,2020)

1. Introduction-- Framework

Identifying the problem with standard optimization

Principal components

Addressing the problem

Shrinking correlations: The Simple EPO

Anchoring expected returns: A Bayesian approach

Anchoring expected returns: Robust optimization

A unified approach

EPO in Practice

1. Introduction-- Contribution

- We develop a more general form of EPO
- One of our theoretical contributions is to unify and demystify these seemingly different frameworks

2. Identifying the problem with standard optimization

- A. Standard mean-variance optimization
  - The investor's future wealth

$$W = W_0\left(1 + r^f + x'r\right)$$

  - Maximize her mean-variance utility

$$E(W|s) - \frac{\overline{\gamma}}{2} Var(W|s) = W_0\left(1 + r^f + x's - \frac{\gamma}{2} x'\Sigma x\right)$$

  - To pick her optimal portfolio x

$$\max_x \left(x's - \frac{\gamma}{2} x'\Sigma x\right)$$

  - Standard mean-variance optimal portfolio

$$x^{MVO} = \frac{1}{\gamma}\Sigma^{-1}s$$

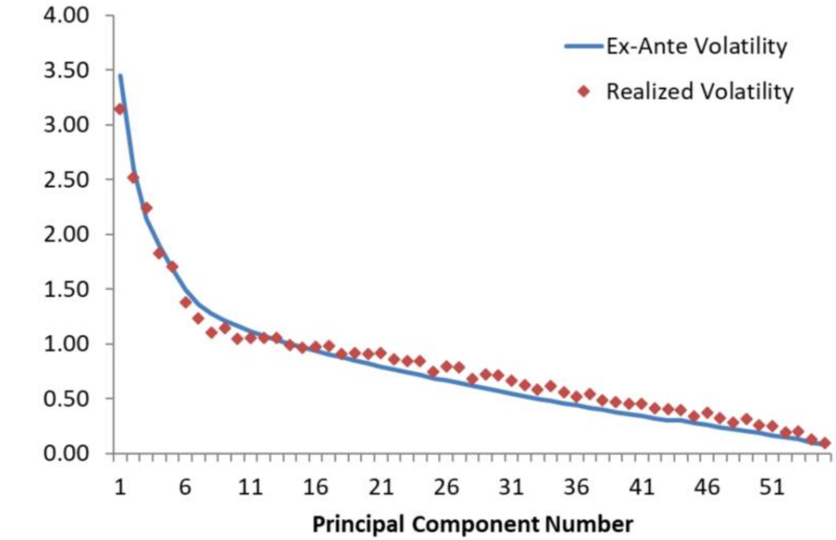
2. Identifying the problem with standard optimization

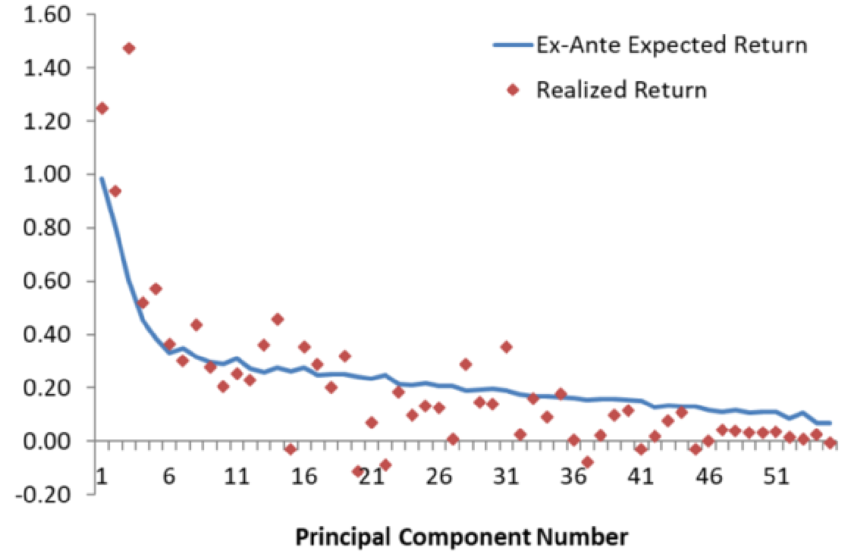
- B. Identifying "problem portfolios"
  - Variance-covariance matrix

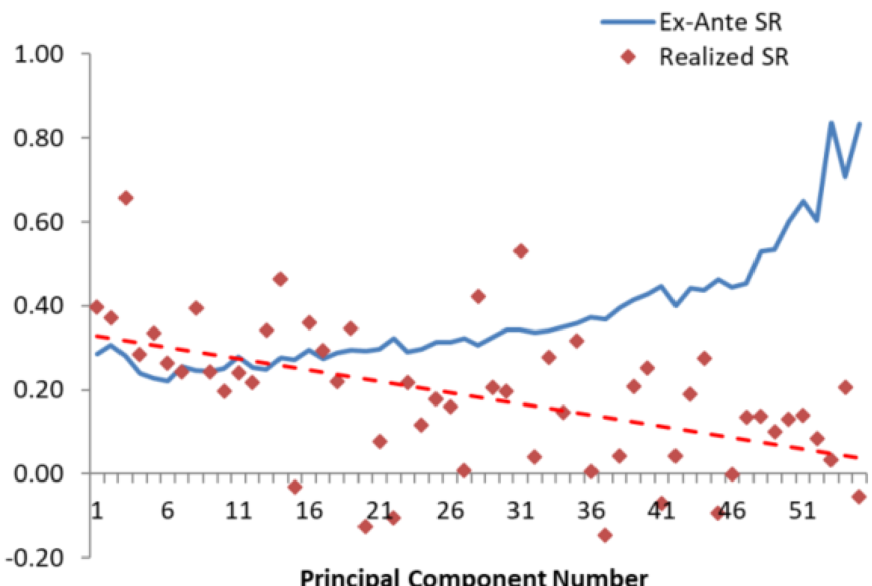
$$\Sigma = \sigma\Omega\sigma$$

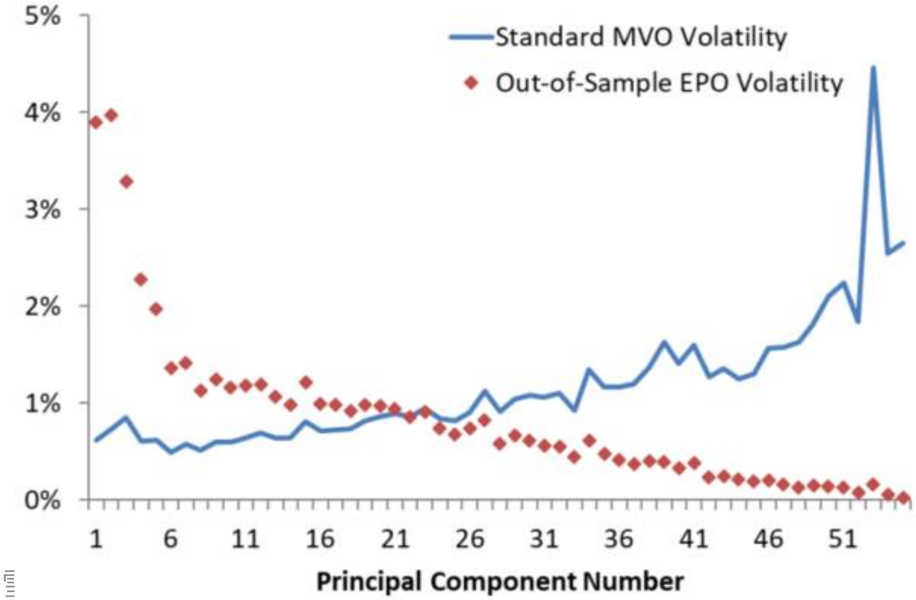
  - The correlation matrix Ω, asset volatilities $\sigma = \mathrm{diag}(\sqrt{\Sigma^{11}}, \ldots, \sqrt{\Sigma^{nn}})$
  - The first principal component：
    - Maximizes the function $h'\Omega h$ subject to $h'h = 1$
    - Maximizes the variance $h'\Omega h$ of any portfolio h
  - The second principal component：
    - Maximizes the same function $h'\Omega h$ being independent of the first
  - The **last principal components** give trouble to the standard mean-variance optimization

2. Identifying the problem with standard optimization









2. Identifying the problem with standard optimization

- B. Identifying “problem portfolios”
  - The eigen-decomposition of the correlation matrix

$$\Omega = PDP^{-1}$$

$$P^{-1} = P'$$

  - P :Its columns are the **principal components** (eigenvectors)
  - D :diagonal matrix of **the variances** of each principal component (eigenvalues)

  - Principal component portfolios have realized excess returns $P'\sigma^{-1}r$
    - expected excess returns $s^p = P'\sigma^{-1}s$

  - The portfolio optimization problem

$$\Sigma = \sigma\Omega\sigma$$

$$\Omega = PDP^{-1}$$

$$x's - \frac{\gamma}{2}x'\Sigma x = (P'\sigma x)'s^p - \frac{\gamma}{2}(P'\sigma x)'D(P'\sigma x) = z's^p - \frac{\gamma}{2}z'Dz$$

$$z^{MVO} = \frac{1}{\gamma}D^{-1}s^p$$

$$\underbrace{z_i^{MVO}}_{\substack{\text{notional}\\ \text{position in}\\ \text{portfolio } i}} = \underbrace{\frac{1}{\gamma}\underbrace{\frac{s_i^P}{\sqrt{D_i}}}_{\substack{\text{Sharpe}\\ \text{ratio of}\\ \text{portfolio } i}}}_{\substack{\text{desired}\\ \text{volatility for}\\ \text{portfolio } i}} \underbrace{\frac{1}{\sqrt{D_i}}}_{\substack{\text{leverage}\\ \text{needed to}\\ \text{achieve a}\\ \text{volatility of 1}\\ \text{for portfolio } i}}$$

3. Addressing the problem: Enhanced portfolio optimization

- A. Shrinking correlations: The Simple EPO

  - The problem :the **estimated variances are likely to be too low for the safest portfolios** (and too high for the riskiest ones).
  - An easy fix is to **shrink their estimated variances toward their average**.
    - The **average variance :**1 (correlation matrix)

  - The modified risks of the principal components $\Omega = PDP^{-1}$

$$\widetilde{D} = (1-\theta)D + \theta I$$

    - where $\theta \in [0,1]$ is the degree of shrinkage, I is the identity matrix
  - Correlation matrix

$$\widetilde{\Omega} = P\widetilde{D}P' = P\big((1-\theta)D + \theta I\big)P' = (1-\theta)\Omega + \theta I$$

  - Variance-covariance matrix $\tilde{\Sigma} = \sigma\widetilde{\Omega}\sigma$
  - Enhanced portfolio optimization $EPO^{s} = \frac{1}{\gamma}\tilde{\Sigma}^{-1}s$

3. Addressing the problem: Enhanced portfolio optimization

- B. Anchoring expected returns: A Bayesian approach

  - The investor observes a vector of signals $s = \mu + \epsilon$
    - true (unobserved) expected return vector μ
    - $\epsilon \sim N(0, \Lambda)$
  - The **investor's prior beliefs** about μ
    - $\mu = \gamma\Sigma a + \eta$
    - $\eta \sim N(0, \tau\Sigma)$
    - an "anchor portfolio" a

$$x = \frac{1}{\gamma}\Sigma^{-1}\mu = a$$

  - **Proposition 1.** *In this Bayesian model, the investor's expected return given the observed signal is*

$$E(\mu|s) = \Sigma(\tau\Sigma + \Lambda)^{-1}(\tau s + \gamma\Lambda a)$$

and the solution to the enhanced portfolio optimization problem is

$$x = \frac{1}{\gamma}(\tau\Sigma + \Lambda)^{-1}(\tau s + \gamma\Lambda a)$$

3. Addressing the problem: Enhanced portfolio optimization

- C. Anchoring expected returns: Robust optimization

  - Address noise in expected returns is to use robust optimization

$$\max_{x}\min_{\mu}\left((x-a)'\mu-\frac{\gamma}{2}x'\Sigma x\right)\text{ s.t. }\ \mu\in\{\bar{\mu}\mid(\bar{\mu}-s)'\Lambda^{-1}(\bar{\mu}-s)\le c^2\}$$

  - **Proposition 2.** *The solution to the robust portfolio optimization problem is:*

$$x=\frac{1}{\gamma}(\tau\Sigma+\Lambda)^{-1}(\tau s+\gamma\Lambda a)$$

3. Addressing the problem: Enhanced portfolio optimization

- D. Putting optimization to work: The simple EPO and the anchored EPO

- The general EPO solution

$$EPO = \frac{1}{\gamma}\left(\tau\tilde{\Sigma} + \Lambda\right)^{-1}(\tau s + \gamma\Lambda a)$$

- The EPO solution can be written as

$$EPO(w) = {\Sigma_w}^{-1}\left([1-w]\frac{1}{\gamma}s + wVa\right)$$

  - With $\Sigma_w = [1-w]\tilde{\Sigma} + wV = \sigma\{[1-w]\tilde{\Omega} + wI\}\sigma$

$$w = \lambda/(\tau+\lambda) \in [0,1]$$

$$\Lambda = \lambda V$$

$$V = \sigma^2$$

- $s = \mu + \epsilon$
- $\epsilon \sim N(0, \Lambda)$

3. Addressing the problem: Enhanced portfolio optimization

- D. Putting optimization to work: The simple EPO and the anchored EPO

- The EPO solution can be written as

$$EPO(w) = \Sigma_w{}^{-1}\left([1-w]\frac{1}{\gamma}s + wVa\right)$$

- **Simple EPO**: $EPO^s(w) = \frac{1}{\gamma}\Sigma_w^{-1}s$

  with $a = \frac{1}{\gamma}V^{-1}s$

- **Anchored EPO**

$$EPO^a(w) = \Sigma_w{}^{-1}\left([1-w]\frac{\sqrt{a'\tilde{\Sigma}a}}{\sqrt{s'\Sigma_w^{-1}\tilde{\Sigma}\Sigma_w^{-1}s}}s + wVa\right)$$

$$\gamma = \sqrt{s'\Sigma_w^{-1}\tilde{\Sigma}\Sigma_w^{-1}s}/\sqrt{a'\tilde{\Sigma}a}$$

3. Addressing the problem: Enhanced portfolio optimization

- E. A unified approach to optimization

$$EPO = \frac{1}{\gamma}\left(\tau\tilde{\Sigma} + \Lambda\right)^{-1}(\tau s + \gamma\Lambda a)$$

- **Proposition 3.** *The EPO solution (13) is equal to*
  - *a)* ***standard MVO*** *when the estimate of variance has no noise so* $\tilde{\Sigma} = \Sigma$ *and the signal of expected returns has no noise so* $\Lambda = 0$.
  - *b) the anchor when* $\tau = 0$ *as in* ***reverse MVO****.*
  - *c) the* ***Bayesian estimator*** *from Section II.B, which is equivalent to* ***Black-Litterman*** *when the anchor portfolio is the market portfolio, the signal is their "view portfolios", and we assume that the variance-covariance matrix is estimated without error.*
  - *d) the solution to* ***robust optimization*** *with ellipsoidal uncertainty set.*
  - *e) a* ***generalized ridge regression*** *(a form of regularization used in machine learning) of expected returns on the variance-covariance matrix*

Abbreviation	Data Set	Number of Assets	Optimization Method	Risk Model	Return Signal	Start of Data	Start of Backtest
Global 1	Global equities, bonds, FX, and commodities	55	EPO^s	Exponentially-weighted daily volatilties (60-day center-of-mass) and 3-day overlapping correlations (150-day center-of-mass)	TSMOM	1/1/1970	1/1/1985
Global 2	Global equities, bonds, FX, and commodities	55	EPO^s	Risk model from Global 1, where correlations are shrunk 5%	TSMOM	1/1/1970	1/1/1985
Global 3	Global equities, bonds, FX, and commodities	55	EPO^s	Risk model from Global 1, enhanced via random matrix theory	TSMOM	1/1/1970	1/1/1985
Equity 1	49 industry portfolios	49	EPO^s	60 months (equal-weighted), 5% shunk	XSMOM	1/1/1927	1/1/1942
Equity 2	49 industry portfolios	49	EPO^s	40 days (equal-weighted), 5% shunk	XSMOM	1/1/1927	1/1/1942
Equity 3	49 industry portfolios	49	EPO^s	120 days (equal-weighted), 5% shunk	XSMOM	1/1/1927	1/1/1942
Equity 4	49 industry portfolios	49	EPO^s	120 days (equal-weighted), 5% shunk	XSMOM*σ	1/1/1927	1/1/1942
Equity 5	49 industry portfolios	49	EPO^s	120 days (equal-weighted), 5% shunk	XSMOM*σ^2	1/1/1927	1/1/1942
Equity 6	49 industry portfolios	49	EPO^a with anchor= 1/N	60 months (equal-weighted), 5% shunk	XSMOM	1/1/1927	1/1/1942
Equity 7	49 industry portfolios	49	EPO^a with anchor= 1/σ	60 months (equal-weighted), 5% shunk	XSMOM	1/1/1927	1/1/1942
Equity 8	Each industry split in 2 portfolios based on past 12 month return	98	EPO^s	60 months (equal-weighted), 5% shunk	XSMOM	1/1/1927	1/1/1942

$$EPO^s(w) = \frac{1}{\gamma}\Sigma_w^{-1}s$$

$$EPO^a(w) = \Sigma_w{}^{-1}\left([1-w]\frac{\sqrt{a'\tilde{\Sigma}a}}{\sqrt{s'\Sigma_w^{-1}\tilde{\Sigma}\Sigma_w^{-1}s}}s + wVa\right)$$

•

$$s_t^i = XSMOM_t^i := c_t(r_{t-12,t}^i - \frac{1}{n}\sum_{j=1,\ldots,n} r_{t-12,t}^j)$$

$$s_t^i = 0.1 \times \sigma_t^i \times sign(r_{t-12,t}^i)$$

4. EPO in Practice: Empirical results

- Performance of Optimized TSMOM Portfolios

Portfolio	Global 1	Global 2 (Shrunk)	Global 3 (RMT)
Long Only: 1/N	0.44	0.44	0.44
Long Only: 1/Sigma	0.76	0.76	0.76
TSMOM: Equal Notional Weight	0.74	0.74	0.74
TSMOM: Equal Volatility Weight	1.09	1.09	1.09
EPO^{s} : Out-Of-Sample	1.24	1.24	1.23
$EPO^{s}(w)$: Shrinkage parameter w			
0% (Naïve MVO)	0.87	1.08	1.02
10%	1.15	1.18	1.19
25%	1.24	1.26	1.26
50%	1.31	1.31	1.32
75%	1.32	1.31	1.32
90%	1.26	1.26	1.26
99%	1.13	1.13	1.13
100% (Anchor)	1.09	1.09	1.09

$$x_t^{\text{TSMOM, equal-notional-weighted}} = \frac{1}{n_t}\, sign(r_{t-12,t}^{i})$$

$$x_t^{\text{TSMOM, equal-volatility-weighted}} = \frac{1}{n_t}\frac{40\%}{\sigma_t^{i}}\, sign(r_{t-12,t}^{i})$$

Moskowitz, Ooi, and Pedersen (2012)

4. EPO in Practice: Empirical results

- Alpha of Out-of-Sample EPO for TSMOM

	Dependent Variable				
	EPO			TSMOM	
Alpha	2.48%	2.17%	2.11%	-0.43%	-0.34%
	(3.36)	(2.92)	(2.93)	(-0.57)	(-0.45)
Long Only (1/Sigma)		0.06	0.07		-0.02
		(2.77)	(3.57)		(-0.86)
TSMOM	0.91	0.90			
	(44.82)	(43.77)			
TSMOM(COM)			0.53		
			(26.02)		
TSMOM(EQ)			0.30		
			(14.99)		
TSMOM(FI)			0.34		
			(16.77)		
TSMOM(FX)			0.32		
			(15.69)		
EPO				0.91	0.92
				(44.82)	(43.77)
Information Ratio	0.60	0.53	0.54	-0.10	-0.08
R-Squared	83%	84%	85%	83%	83%

4. EPO in Practice: Empirical results

- Leverage and Turnover of Optimized TSMOM Portfolios

Portfolio	Gross Leverage per 10% Volatility	Annualized Turnover as % of Avg. Gross NAV
Long Only: 1/N	135%	26%
Long Only: 1/Sigma	267%	43%
TSMOM: Equal Notional Weight	167%	153%
TSMOM: Equal Risk	358%	163%
EPO^s : Out-Of-Sample	457%	254%
$EPO^s(w)$: Shrinkage parameter w		
0% (Naïve MVO)	991%	546%
10%	767%	480%
25%	649%	417%
50%	551%	339%
75%	479%	263%
90%	424%	208%
99%	368%	166%
100% (Anchor)	358%	163%

- **Gross leverage** statistics are shown for portfolios ex-post scaled to 10% annualized volatility
- Annualized **turnover** statistics are reported as a percentage of average gross leverage.

4. EPO in Practice: Empirical results

- Performance of Optimized Equity Portfolios.

Portfolio	Equity 1	Equity 2	Equity 3	Equity 4	Equity 5	Equity 6	Equity 7	Equity 8
1/N	0.59	0.59	0.59	0.59	0.59	0.59	0.59	0.57
INDMOM	0.63	0.63	0.63	0.63	0.63	0.63	0.63	0.67
MVO (no correlation shrinkage)	0.19	-0.02	0.92	0.84	0.47	0.21	0.21	0.01
EPO : Out-Of-Sample	**0.79**	**0.72**	**0.96**	**0.99**	**0.66**	**0.83**	**0.90**	**0.90**
EPO(w) : In Sample with Shrinkage of *w*								
0% (MVO w/ 5% correlation shrinkage)	0.56	0.82	0.97	0.96	0.66	0.50	0.51	0.60
10%	0.68	0.89	**0.98**	0.99	0.71	0.59	0.60	0.80
25%	0.75	0.92	0.98	**0.99**	**0.72**	0.66	0.67	0.91
50%	0.79	**0.93**	0.96	0.97	0.71	0.72	0.75	**0.98**
75%	**0.80**	0.91	0.93	0.94	0.69	**0.85**	**0.91**	0.98
90%	0.79	0.88	0.89	0.92	0.67	0.83	0.90	0.94
99%	0.73	0.77	0.77	0.91	0.65	0.60	0.63	0.86
100% (Anchor)	0.71	0.73	0.73	0.91	0.63	0.59	0.62	0.81

4. EPO in Practice: Empirical results

- Alpha of EPO for Equity Portfolios.

	Dependent Variable: Out-of-Sample EPO Portfolios							
	Equity 1	Equity 2	Equity 3	Equity 4	Equity 5	Equity 6	Equity 7	Equity 8
Alpha (Annualized)	3.82%	8.09%	7.65%	6.25%	2.50%	1.07%	1.31%	4.40%
	(4.41)	(6.68)	(6.29)	(6.07)	(2.38)	(1.80)	(2.49)	(5.07)
INDMOM	0.78	0.53	0.53	0.69	0.67	0.31	0.33	0.78
	(32.11)	(15.66)	(15.64)	(24.00)	(22.71)	(18.85)	(22.43)	(32.21)
Mkt-RF	0.08	-0.09	-0.07	-0.07	-0.04	0.85	0.91	0.10
	(3.12)	(-2.38)	(-1.95)	(-2.10)	(-1.23)	(45.87)	(55.57)	(3.69)
SMB	-0.06	-0.04	-0.02	-0.04	-0.07	0.16	0.09	-0.05
	(-2.27)	(-1.14)	(-0.66)	(-1.32)	(-2.14)	(9.33)	(5.88)	(-2.04)
HML	-0.01	0.10	0.10	0.04	0.01	0.03	0.05	-0.03
	(-0.44)	(2.04)	(2.12)	(0.94)	(0.23)	(1.40)	(2.31)	(-1.01)
CMA	-0.14	-0.12	-0.12	-0.05	-0.01	-0.02	0.00	-0.10
	(-4.05)	(-2.43)	(-2.35)	(-1.22)	(-0.27)	(-0.84)	(0.12)	(-2.73)
RMW	-0.04	-0.04	-0.04	-0.03	0.01	0.06	0.08	-0.03
	(-1.42)	(-1.04)	(-1.06)	(-0.95)	(0.40)	(3.33)	(5.16)	(-1.10)
Information Ratio	0.63	0.96	0.90	0.87	0.34	0.26	0.36	0.73
R-Squared	64%	29%	28%	49%	46%	83%	87%	64%

V. Conclusion

- We develop a simple and transparent method to make portfolio optimization work in practice.
- EPO improves portfolio performance by accounting for noise in the investor’s estimates of risk and expected return.
- We identify the “problem portfolios” that MVO gives large weight despite their poor performance.