TRANSCENDENTAL FOLIATED SURFACES WITH SLOPE LESS THAN 2

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Abstract. In this paper, we construct a canonical resolution of double coverings of foliated surfaces and provide explicit formulas for the Chern numbers c_1^2, c_2 and χ of the resulting double foliated surface. During these computations, we also establish a theorem concerning the Zariski decomposition of adjoint divisors of the form $K_{\mathcal{F}} + \Delta$, which generalizes McQuillan's theorem in the case where $\Delta = 0$. As an application, we prove that the slope of a double foliated surface of general type is at least 4, provided the original foliation satisfies $c_1^2 \geq 4\chi$ and the ramification divisor avoids the saddle-node singularities. In the final section, we construct examples of transcendental double foliated surfaces with slope 12/7.

1. Introduction

A foliated surface is a pair $(X, \mathcal{F})$, where X is a smooth projective surface and $\mathcal{F}$ is a (holomorphic) foliation on X. A holomorphic foliation $\mathcal{F}$ can be defined by a first-order differential equation $\alpha = 0$, where α is a nonzero rational 1-form on X. We say that $\mathcal{F}$ is *algebraic* if the differential equation admits a rational first integral; otherwise, we call $\mathcal{F}$ *transcendental*.

In 1891, Poincaré, in his works [Poin91a, Poin91b, Poin97], studied the following problem:

Problem 1.1 (Poincaré). Is it possible to decide whether a given differential equation $\alpha = 0$ is *algebraic*?

A similar problem was proposed by Painlevé [Pain74], namely:

Problem 1.2 (Painlevé). Is it possible to decide whether a differential equation $\alpha = 0$ admits a rational first integral of a given genus g?

Lins-Neto [Lin02] showed that the genus is not an invariant of differential equations, by constructing counterexamples.

On the other hand, several birational invariants of fibrations extend naturally to foliations. Examples include the theory of minimal models (Seidenberg [Sei68], Brunella [Bru15]), the canonical divisor $K_{\mathcal{F}}$, the pluri-canonical genera $p_n(\mathcal{F})$, the Kodaira dimension $\kappa(\mathcal{F})$ and numerical Kodaira dimension $\nu(\mathcal{F})$ (McQuillan [MMcQ08], Mendes [Men00]), as well as the Chern numbers $c_1^2(\mathcal{F}), c_2(\mathcal{F})$ and

$$\chi(\mathcal{F}) = \tfrac{1}{12}\big(c_1^2(\mathcal{F}) + c_2(\mathcal{F})\big)$$

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introduced by Tan (see [HLT20] or [LT]).

For a foliation $\mathcal{F}$ of general type, both $c_1^2(\mathcal{F})$ and $\chi(\mathcal{F})$ are positive. Moreover, $\mathcal{F}$ is of general type if and only if $c_1^2(\mathcal{F}) > 0$. This motivates the definition of the *slope* of $\mathcal{F}$ as

$$\lambda(\mathcal{F}) := \frac{c_1^2(\mathcal{F})}{\chi(\mathcal{F})}.$$

By the Noether's equality, one always has $0 < \lambda(\mathcal{F}) \leq 12$. In the algebraic case, when $\mathcal{F}$ comes from a fibration, Xiao [Xiao87] proved that

$$4 - \frac{4}{g} \leq \lambda(\mathcal{F}) \leq 12,$$

where the lower bound is attained precisely for hyperelliptic fibrations, as shown by Konno [KK93].

The method of double coverings plays an important role in the study of algebraic surfaces of general type. For instance, Gang Xiao [Xiao91] investigated hyperelliptic fibrations via double coverings of rational fibrations. In a similar spirit, we study the double covering of a foliated surface $(X, \mathcal{F})$, which leads to what we call a *double foliated surface*, denoted by $(Y, \mathcal{G})$.

Let B be the branch locus of the double cover

$$\pi : (Y, \mathcal{G}) \longrightarrow (X, \mathcal{F}).$$

Without loss of generality, we may assume that B is reduced and can be written as

$$B = B_h + B_v,$$

where B_v consists of the $\mathcal{F}_i$-invariant components of B.

Note that

$$K_{\mathcal{G}} = \pi^*(K_{\mathcal{F}} + \tfrac{1}{2}B_h),$$

and

$$P(\mathcal{G}) = \pi^* P(B_h), \qquad N(\mathcal{G}) = \pi^* N(B_h),$$

where $K_{\mathcal{G}} \equiv P(\mathcal{G}) + N(\mathcal{G})$ and $K_{\mathcal{F}} + \frac{1}{2}B_h$ are the respective Zariski decompositions. By the basic theory of double covers, the data of $(Y, \mathcal{G})$ is completely determined by the triple

$$(X, \mathcal{F}, B = B_h + B_v).$$

1.1. **Canonical resolution of double covers.** We study the canonical resolution of a double covering $(Y, \mathcal{G})$ of a reduced foliated surface $(X, \mathcal{F})$:

$$\begin{array}{ccc} (\widetilde{Y}, \widetilde{\mathcal{G}}) & \longrightarrow & (Y, \mathcal{G}) \\ \big\downarrow \scriptstyle{\widetilde{\pi}} & & \big\downarrow \scriptstyle{\pi} \\ (\widetilde{X}, \widetilde{\mathcal{F}}) & \xrightarrow{\Sigma} & (X, \mathcal{F}), \end{array}$$

where Σ is a sequence of blow-ups, $\widetilde{Y}$ is the normalization of $\widetilde{X} \times_X Y$, and $\widetilde{\mathcal{G}}$ is the induced foliation. To simplify the discussion, we assume that $(X, \mathcal{F})$ is *relatively minimal*, i.e. $\mathcal{F}$ is reduced and X contains no $\mathcal{F}$-exceptional curves. Finally, we construct Σ so that $\widetilde{\mathcal{G}}$ becomes a reduced foliation on the smooth surface $\widetilde{Y}$. This occurs provided that

(i) $\widetilde{B}$ is smooth and reduced;
(ii) $\mathrm{tang}(\widetilde{\mathcal{F}}, \widetilde{B}_h) = 0$ (cf. Proposition 5.2).

We briefly describe the canonical resolution procedure under the assumption that $\nu(\mathcal{F}) \geq 0$. (See §5.4 for further details.)

Step 1. Minimal resolution of $(X, \mathcal{F}, \frac{1}{2}B_h)$:

$$\sigma : (\bar{X}, \bar{\mathcal{F}}, \tfrac{1}{2}\bar{B}_h) \longrightarrow (X, \mathcal{F}, \tfrac{1}{2}B_h),$$

where σ is the minimal resolution satisfying $\operatorname{tang}(\bar{\mathcal{F}}, \bar{B}_h) = 0$, and $\bar{B}_h = \sigma_*^{-1}(B_h)$. We can decompose σ as

$$\sigma = \sigma_1 \cdots \sigma_n,$$

where each σ_i is a blow-up at an *infinitely near singularity* p_i of type $S_{l,m}$, with $l = l(p_i)$ and $m = \operatorname{mult}_{p_i}(B_h)$.

Step 2. Blow-up of the remaining singularities of $\bar{B} = \bar{B}_h + \bar{B}_v$:

$$\sigma' : (\widetilde{X}, \widetilde{\mathcal{F}}) \longrightarrow (\bar{X}, \bar{\mathcal{F}}),$$

where each blow-up center of σ' belongs to one of the following cases:
(i) $q \in \bar{B}_h \cap \bar{B}_v$, or
(ii) $q \in \bar{B}_v \setminus \bar{B}_h$ with $\operatorname{mult}_q(\bar{B}_v) = 2$.

We call $\Sigma = \sigma \cdot \sigma'$ the *canonical resolution* of the double cover $\pi : (Y, \mathcal{G}) \to (X, \mathcal{F})$.

Remark 1.3. In the case where $\nu(\mathcal{F}) = -\infty$, one can run the above procedure after a suitable flip, as explained in Remark 4.2.

1.2. **Formulas for the Chern numbers of double foliated surfaces.** For each *infinitely near singularity* p of type $S_{l,m}$, we introduce its local invariants $T_1(p), T_2(p)$ (cf. §5.6). With these invariants, we obtain the following formulas for the Chern numbers of the double foliation:

Theorem 1.4 (cf. Theorem 5.18)**.** *If $(X, \mathcal{F})$ is relatively minimal with $\nu(\mathcal{F}) \geq 0$, then*

$$c_1^2(\mathcal{G}) = 2c_1^2(\mathcal{F}) + \tfrac{3}{2}K_{\mathcal{F}}B_h + 2N^2 - 2N(B_h)^2 + \sum_{p \in S_{l,m}} T_1(p),$$

$$c_2(\mathcal{G}) = 2c_2(\mathcal{F}) - 2N^2 + 2N(B_h)^2 - \tfrac{3}{2}s(B_v) + \sum_{p \in S_{l,m}} T_2(p) - \ell(\widetilde{\mathcal{G}}),$$

$$\chi(\mathcal{G}) = 2\chi(\mathcal{F}) + \tfrac{1}{8}K_{\mathcal{F}}B_h - \tfrac{1}{8}s(B_v) + \sum_{p \in S_{l,m}} \tfrac{1}{12}\big(T_1(p) + T_2(p)\big) - \tfrac{1}{12}\ell(\widetilde{\mathcal{G}}).$$

Here N is the negative part of the Zariski decomposition of $K_{\mathcal{F}}$, $\ell(\widetilde{\mathcal{G}})$ denotes the number of $\widetilde{\mathcal{G}}$-exceptional curves containing saddle-nodes, and the definition of $s(B_v)$ is given in Definition 5.7.

1.3. **Slope of double foliations.**

Theorem 1.5 (cf. Theorem 5.30)**.** *Under the assumptions of Theorem 1.4, if $c_1^2(\mathcal{F}) \geq 4\chi(\mathcal{F})$ and the ramification divisor B of π avoids the set of saddle-nodes, then*

$$c_1^2(\mathcal{G}) \geq 4\chi(\mathcal{G}).$$

In fact, the authors of [HLT20] proved a special case of the above theorem. In their work, $(X, \mathcal{F})$ is a transcendental Riccati foliated surface with $c_1^2(\mathcal{F}) = \chi(\mathcal{F}) = 0$, and B is a normal crossing divisor containing no $\mathcal{F}$-invariant components and disjoint from the singularities of $\mathcal{F}$.

As a byproduct, we obtain the following proposition.

Proposition 1.6 (cf. Proposition 5.31)**.** *If the foliated surface $(Y, \mathcal{G})$ corresponds to a double elliptic fibration with genus $g \geq 5$, then $c_1^2(\mathcal{G}) \geq 4\chi(\mathcal{G})$.*

In other words, we reprove that the minimal modular slope of a non-isotrivial double elliptic fibration of genus $g \geq 5$ is 4, as shown in [Bar01].

2. Preliminaries

This section reviews foundational definitions and results concerning foliations on smooth projective surfaces, primarily based on [Bru15, MMcQ00, MMcQ08]. We work over the complex number $\mathbb{C}$.

2.1. **Foliations and Singularities.**

Definition 2.1. Let X be a smooth projective surface. A *foliation* $\mathcal{F}$ on X is a saturated invertible subsheaf $T_{\mathcal{F}} \subset T_X$ of the tangent sheaf T_X, i.e., $T_X/T_{\mathcal{F}}$ is torsion free. A *foliated surface* $(X, \mathcal{F})$ consists of a smooth projective surface X together with a foliation $\mathcal{F}$ on X.

Equivalently, a foliation $\mathcal{F}$ on X can be given by an exact sequence:

$$0 \to T_{\mathcal{F}} \to T_X \to \mathcal{I}_Z(N_{\mathcal{F}}) \to 0, \tag{2.1}$$

where $T_{\mathcal{F}}$ (resp. $N_{\mathcal{F}}$) are called the *tangent bundle* and *normal bundle* of $\mathcal{F}$, and $\mathcal{I}_Z$ an ideal sheaf supported on a finite set. $K_{\mathcal{F}} := c_1(T_{\mathcal{F}}^{\vee})$ is called the *canonical divisor* of $\mathcal{F}$.

Definition 2.2. Let $(X, \mathcal{F})$ be a foliated surface. The *singular locus of* $\mathcal{F}$, denoted by $\mathrm{Sing}(\mathcal{F})$, is the set of points $p \in X$ where the quotient sheaf $T_X/T_{\mathcal{F}}$ is not locally free at p. A point $p \in X$ is called a *regular point* of $\mathcal{F}$ if $p \notin \mathrm{Sing}(\mathcal{F})$.

Let p be a singularity of $\mathcal{F}$. Locally, $\mathcal{F}$ is defined by a vector field

$$\nu = A(x, y)\frac{\partial}{\partial x} + B(x, y)\frac{\partial}{\partial y}, \tag{2.2}$$

where $p = (0, 0)$. The two eigenvalues λ_1, λ_2 of the linear part $(D\nu)(p)$ of ν at p are well defined.

Definition 2.3. Let p be a singularity of $\mathcal{F}$ and let λ_1, λ_2 be as above.

(1) The singularity p is called *non-degenerated* if the two eigenvalues λ_1, λ_2 are both nonzero.
(2) The singularity p is called *reduced* if one of the two eigenvalues, say, λ_2, is nonzero and and the quotient $\lambda = \lambda_1/\lambda_2$ is not a positive rational number. In particular, if $\lambda = 0$, we call p a *saddle-node*.

A foliation $\mathcal{F}$ is said to be *reduced* if any singularity of $\mathcal{F}$ is reduced.

Remark 2.4. If p is a saddle-node, by a suitable transition of coordinates, the foliation $\mathcal{F}$ at p can be represented by

$$\nu = \left(x + axy^k + yF(x, y)\right)\frac{\partial}{\partial x} + y^{k+1}\frac{\partial}{\partial y},$$
$$\text{or} \quad \omega = \left(x + axy^k + yF(x, y)\right)\mathrm{d}y - y^{k+1}\mathrm{d}x,$$

where $a \in \mathbb{C}$, $k \in \mathbb{Z}^+$, and F is a holomorphic function which vanishes at $p = (0, 0)$ up to order k. The curve $(y = 0)$ is a separatrix, called *strong separatrix*. If $F = 0$, then the separatrix $(x = 0)$ is called the *weak separatrix*.

Theorem 2.5 (Seidenberg)**.** *Given any foliated surface $(X, \mathcal{F})$, there exists a sequence of blowing-ups $\sigma : X' \to X$, such that the induced foliation $\mathcal{F}'$ on X' is reduced.*

Proof. See [Sei68] or [Bru15, Theorem 1.1]. □

Definition 2.6. Let $(X, \mathcal{F})$ be a foliated surface. A curve $C \subset X$ is said to be *$\mathcal{F}$-exceptional* if the following conditions are satisfied:

(1) C is a smooth rational curve with self-intersection -1;
(2) the contraction of C to a point p yields a foliated surface $(X', \mathcal{F}')$, where p is either a *regular point* or a *reduced singularity* of $\mathcal{F}'$.

A foliated surface $(X, \mathcal{F})$ is called *relatively minimal* if it is reduced and contains no $\mathcal{F}$-invariant curves.

For any foliated surface $(X, \mathcal{F})$, by Theorem 2.5, there exists a relatively minimal foliated surface $(X', \mathcal{F}')$ birationally equivalent to $(X, \mathcal{F})$. We call $(X', \mathcal{F}')$ the *relatively minimal model* of $(X, \mathcal{F})$.

2.2. **Index Theorems.** A curve $C \subseteq X$ is said to be *$\mathcal{F}$-invariant* if the inclusion $T_{\mathcal{F}}|_C \to T_X|_C$ factors through T_C, where T_C is the tangent bundle of C.

2.2.1. *Non-invariant curves.* For a non-$\mathcal{F}$-invariant curve C and $p \in C$, we can define the *tangency order* of $\mathcal{F}$ along C at p to be

$$\operatorname{tang}(\mathcal{F}, C, p) := \dim_{\mathbb{C}} \frac{\mathcal{O}_p}{\langle f, \nu(f)\rangle},$$

where ν is the local generator of $\mathcal{F}$ at p and $f = 0$ is the local equation of C. We define

$$\operatorname{tang}(\mathcal{F}, C) := \sum_{p\in C} \operatorname{tang}(\mathcal{F}, C, p).$$

Lemma 2.7. *We have*

$$\operatorname{tang}(\mathcal{F}, C) = K_{\mathcal{F}} \cdot C + C^2 \, (\geq 0). \tag{2.3}$$

Proof. See [Bru15, Proposition 2.2]. □

2.2.2. *Invariant curves.* Let C be an $\mathcal{F}$-invariant curve on X, and let $p \in C$ be a point. In a neighborhood of p, the foliation $\mathcal{F}$ is defined by a 1-form ω, and C is locally given by the equation $f = 0$. Since C is $\mathcal{F}$-invariant, we may write

$$g\omega = hdf + f\eta,$$

where η is a holomorphic 1-form and g, h are holomorphic functions defined around p, with h and f are coprime. We define

$$\mathrm{Z}(\mathcal{F}, C, p) := \text{vanishing order of } \left.\frac{h}{g}\right|_C \text{ at } p \tag{2.4}$$

$$\mathrm{CS}(\mathcal{F}, C, p) := \operatorname{Res}_p \left\{ -\left.\frac{\eta}{h}\right|_C \right\}. \tag{2.5}$$

By definition, $\mathrm{Z}(\mathcal{F}, C, p) = \mathrm{CS}(\mathcal{F}, C, p) = 0$ if p is not a singularity of $\mathcal{F}$. If p is a reduced singularity of $\mathcal{F}$, then $\mathrm{Z}(\mathcal{F}, C, p) \geq 0$. Let

$$\mathrm{Z}(\mathcal{F}, C) := \sum_{p\in C} \mathrm{Z}(\mathcal{F}, C, p), \quad \mathrm{CS}(\mathcal{F}, C) := \sum_{p\in C} \mathrm{CS}(\mathcal{F}, C, p).$$

Proposition 2.8. *Let C be an $\mathcal{F}$-invariant curve on X. Then*

$$\mathrm{Z}(\mathcal{F}, C) = K_{\mathcal{F}} \cdot C + 2 - 2p_a(C), \tag{2.6}$$

$$\mathrm{CS}(\mathcal{F}, C) = C^2, \tag{2.7}$$

where $p_a(C)$ denotes the arithmetic genus of C. The second equality above is called the Camacho-Sad formula.

Proof. See [Bru15, Proposition 2.3 and Theorem 3.2]. □

Next, we recall the separatrix theorem.

Theorem 2.9 (Separatrix theorem)**.** *Let $\mathcal{F}$ be a foliation on a smooth projective surface X and let $C \subset X$ be a connected compact $\mathcal{F}$-invariant curve such that:*

(i) *All the singularities of $\mathcal{F}$ on C are reduced (in particular, C has only normal crossing singularities);*

(ii) *If $C_1, \cdots, C_n$ are the irreducible components of C, then the intersection matrix $(C_iC_j)_{1\leq i,j\leq n}$ is negative definite and the dual graph Γ is a tree.*

Then there exists at least one point $p \in C \cap \mathrm{Sing}(\mathcal{F})$ and a separatrix through p not contained in C.

Proof. See [Bru15, Theorem 3.4]. □

2.3. **Chern numbers of foliations.** For a given complex number a, we define

$$\beta(a) := \begin{cases} \dfrac{\gcd(m,n)^2}{mn}, & \text{if } a = \frac{m}{n} \in \mathbb{Q} \setminus \{0\} \\ 0, & \text{others.} \end{cases}$$

For a reduced singularity p of $\mathcal{F}$, we define

$$\beta_p(\mathcal{F}) := \beta(-\lambda_p),$$
$$\chi_p(\mathcal{F}) := -\frac{1}{12}\left(\mathrm{BB}(\mathcal{F}, p) + m_p(\mathcal{F}) - \beta_p(\mathcal{F})\right),$$

where $\mathrm{BB}_p(\mathcal{F})$ is the Baum-Bott index of $\mathcal{F}$ at p (cf. [Bru15, Ch.3, Sec.1]). In particular, if $\lambda_p \neq 0$, then

$$\chi_p(\mathcal{F}) = \frac{1}{12}\left(\lambda_p + \frac{1}{\lambda_p} - \beta_p(\mathcal{F})\right) - \frac{1}{4}.$$

Definition 2.10 ([LT])**.** Suppose $(X, \mathcal{F})$ is a *relatively minimal* foliated surface. If $\nu(\mathcal{F}) \geq 0$, we define the following three Chern numbers:

$$\begin{aligned} c_1^2(\mathcal{F}) &= K_{\mathcal{F}}^2 + \sum_{p\in N} \beta_p(\mathcal{F}), \\ c_2(\mathcal{F}) &= \sum_{p\notin N} \beta_p(\mathcal{F}), \\ \chi(\mathcal{F}) &= \chi(\mathcal{O}_X) + \frac{1}{4}K_{\mathcal{F}} \cdot N_{\mathcal{F}} + \sum_{p} \chi_p(\mathcal{F}), \end{aligned} \tag{2.8}$$

where N is the negative part of the pseudo-effective divisor $K_{\mathcal{F}} = P + N$. If $\nu(\mathcal{F}) = -\infty$, then we define $c_1^2(\mathcal{F}) = c_2(\mathcal{F}) = \chi(\mathcal{F}) = 0$.

Proposition 2.11 ([LT])**.** *Suppose $(X, \mathcal{F})$ is a minimal foliated surface. Then we have*

(1) *$c_1^2(\mathcal{F}), c_2(\mathcal{F}), \chi(\mathcal{F})$ are non-negative rational numbers.*

(2) *$c_1^2(\mathcal{F}), c_2(\mathcal{F}), \chi(\mathcal{F})$ are birational invariants.*
(3) *If $K_{\mathcal{F}}$ has a Zariski decomposition with positive part P, then $c_1^2(\mathcal{F}) = P^2$.*
(4) *(Noether's equality) $c_1^2(\mathcal{F}) + c_2(\mathcal{F}) = 12\chi(\mathcal{F})$.*
(5) *If $\mathcal{F}$ is birationally equivalent to a fibration $f : X \longrightarrow C$, then*

$$c_1^2(\mathcal{F}) = \kappa(f), \quad c_2(\mathcal{F}) = \delta(f), \quad \chi(\mathcal{F}) = \lambda(f),$$

where $\kappa(f), \delta(f), \lambda(f)$ are modular invariants of f.

By the formulas above, we can define the Chern numbers of any foliated surface $(X, \mathcal{F})$ by its *relatively minimal model*. More precisely, if $(X', \mathcal{F}')$ is the *relatively minimal model* of $(X, \mathcal{F})$, then we define the Chern numbers of $\mathcal{F}$ as

$$c_1^2(\mathcal{F}) := c_1^2(\mathcal{F}'), \quad c_2(\mathcal{F}) := c_2(\mathcal{F}'), \quad \chi(\mathcal{F}) := \chi(\mathcal{F}').$$

Definition 2.12. For any redcued foliated surface $(X, \mathcal{F})$, we define

$$\ell(\mathcal{F}) := K_{\mathcal{F}}^2 + \sum_{p \in X} \beta_p(\mathcal{F}) - 12\chi(\mathcal{F}). \tag{2.9}$$

Lemma 2.13. $\ell(\mathcal{F}) \geq 0$. *In particular, $\ell(\mathcal{F}) = 0$ if and only if there is no $\mathcal{F}$-exceptional curve which is the strong separatrix of a saddle-node.*

Proof. Let $(X', \mathcal{F}')$ be the minimal model of $(X, \mathcal{F})$ with $\sigma : (X, \mathcal{F}) \longrightarrow (X', \mathcal{F}')$. By the Noether formula, we have

$$\chi(\mathcal{F}) = \chi(\mathcal{F}') = \frac{1}{12}\left[K_{\mathcal{F}'}^2 + \sum_{p \in X'} \beta_p(\mathcal{F}')\right].$$

Then we can see $\ell(\mathcal{F})$ denote the number of the exceptional curves in σ, which are the strong separatrix of saddle-nodes. Next is clear. □

3. The Zariski decomposition of $K_{\mathcal{F}} + \Delta$

Let $\mathcal{F}$ be a reduced foliation on a smooth surface X, and let

$$\Delta = \sum_{i=1}^{l} a_i C_i$$

be an effective $\mathbb{Q}$-divisor, where each coefficient $a_i \in \left[\frac{1}{2}, 1\right]$ and each component C_i is not $\mathcal{F}$-invariant. We assume that $K_{\mathcal{F}} + \Delta$ is pseudo-effective, and denote its Zariski decomposition by

$$K_{\mathcal{F}} + \Delta = P(\Delta) + N(\Delta), \tag{3.1}$$

where $P(\Delta)$ is the positive part and $N(\Delta)$ is the negative part. For instance, if $\nu(\mathcal{F}) \geq 0$, i.e. $K_{\mathcal{F}}$ is pseudo-effective, then $K_{\mathcal{F}} + \Delta$ is automatically pseudo-effective.

Our main goal in this section is to describe the construction of the negative part $N(\Delta)$ of the adjoint divisor $K_{\mathcal{F}} + \Delta$. To achieve this goal, we first introduce the notion of $(\Delta, \mathcal{F})$-chains.

3.1. $(\Delta,\mathcal{F})$**-chains.**

Definition 3.1. Let $\mathcal{F}$ be a foliation on a surface X. A compact curve $\Theta\subset X$ is called an *$\mathcal{F}$-chain* if

(1) Θ is a Hirzebruch-Jung string, $\Theta=\Gamma_1+\cdots+\Gamma_r$;
(2) each irreducible component Γ_j is $\mathcal{F}$-invariant;
(3) $\mathrm{Sing}(\mathcal{F})\cap\Theta$ are reduced and non-degenerated;
(4) $\mathrm{Z}(\mathcal{F},\Gamma_1)=1$ and $\mathrm{Z}(\mathcal{F},\Gamma_i)=0$ for all $i\geq 2$.

We say an $\mathcal{F}$-chain is *maximal*, if it cannot be contained in another $\mathcal{F}$-chain. (See [Bru15, Definition 8.1])

Now let $\Theta=\Gamma_1+\cdots+\Gamma_r$ be an $\mathcal{F}$-chain. Since the intersection matrix $(\Gamma_i\cdot\Gamma_j)_{1\leq i,j\leq r}$ is negative definite, there exists a unique effective $\mathbb{Q}$-divisor

$$M(\Theta)=\sum_{i=1}^{r}\gamma_i\Gamma_i \tag{3.2}$$

supported on Θ, such that

$$M(\Theta)\Gamma_1=-1,\quad M(\Theta)\Gamma_i=0,\quad \text{for } i\geq 2.$$

By a straightforward computation, we have:

$$\gamma_i=\frac{\lambda_i}{n},\quad \text{for }\ i=1,\cdots,r, \tag{3.3}$$

where

$$n=[e_1,\cdots,e_r]>\lambda_1=[e_2,\cdots,e_r]>\cdots>\lambda_r=1,$$

with $e_i=-\Gamma_i^2$, and $[e_1,\cdots,e_r]$ denotes the determinant of the intersection matrix $(-\Gamma_i\cdot\Gamma_j)_{1\leq i,j\leq r}$. In particular, for any irreducible $\mathcal{F}$-invariant curve C meeting transversely with Θ ($C\neq\Gamma_i$ for all i's), we have

$$M(\Theta)C=\gamma_rC\Gamma_r=\gamma_r\leq\frac{1}{2}. \tag{3.4}$$

Lemma 3.2 ([LT])**.** *We have*

$$M(\Theta)^2=-\frac{\lambda_1}{n}=-\sum_{p\in\Theta}\beta_p(\mathcal{F}), \tag{3.5}$$

where the sum runs over all singularities p of $\mathcal{F}$ contained in Θ.

Definition 3.3. A *$(\Delta,\mathcal{F})$-chain* $\Theta=\Gamma_1+\cdots+\Gamma_r$ is an $\mathcal{F}$-chain satisfying that

$$\Delta\Gamma_1<1,\quad \Delta\Gamma_i=0,\quad \text{for } i\geq 2.$$

We define $\theta:=\Delta\cdot\Gamma_1$ as the *multiplicity* of Θ with respect to Δ.

By the assumption of Δ, we see that $\theta\in[\frac{1}{2},1)\cup\{0\}$.
Similarly, for a $(\Delta,\mathcal{F})$-chain Θ, there exists uniquely an effective $\mathbb{Q}$-divisor

$$M(\Delta,\Theta)=\sum_{i=1}^{r}\gamma_i\Gamma_i \tag{3.6}$$

such that $M(\Delta,\Theta)\Gamma_i=(K_{\mathcal{F}}+\Delta)\Gamma_i$ for all i. It is clear that

$$M(\Delta,\Theta)=(1-\theta)M(\Theta), \tag{3.7}$$

where $\theta=\Delta\Gamma_1$.

Lemma 3.4. *Let $\Theta = \sum_{i=1}^{r} \Gamma_r$ be a $(\Delta, \mathcal{F})$-chain. Then*

$$M(\Delta, \Theta) \cdot \Delta = \sum_{p \in \Theta} \theta(1-\theta)\beta_p(\mathcal{F}),$$

where $\theta = \Delta\Theta = \Delta\Gamma_1$ is the multiplicity of Θ w.r.t. Δ (cf. Definition 3.3).

Proof.

$$\begin{aligned} M(\Delta, \Theta)\Delta &= (1-\theta)M(\Theta)\Delta = (1-\theta)\gamma_1 \cdot \Gamma_1\Delta = \theta(1-\theta)\gamma_1 \\ &= \sum_{p \in \Theta} \theta(1-\theta)\beta_p(\mathcal{F}) \qquad \text{(by Lemma 3.2)}. \end{aligned}$$

□

Lemma 3.5. *Let Θ be a $(\Delta, \mathcal{F})$-chain and let C_i be an irreducible component of Δ. Then*

$$\theta \cdot M(\Theta) \cdot C_i \;=\; a_i \cdot M(\Theta) \cdot C_i \;\geq\; \tfrac{1}{2} M(\Theta) \cdot C_i.$$

In particular, $\theta = a_i$ whenever $C_i \cdot \Gamma_1 > 0$.

3.2. Zariski decomposition of $K_{\mathcal{F}} + \Delta$.

Definition 3.6. An $\mathcal{F}$-exceptional curve E is called $(\Delta, \mathcal{F})$-*exceptional* if either

- $K_{\mathcal{F}}E = -1$ and $\Delta E < 1$, or
- $K_{\mathcal{F}}E = \Delta E = 0$.

A $(\Delta, \mathcal{F})$-exceptional curve E is said to be *of type H–J* if there exists a $(K_{\mathcal{F}}+\Delta)$-non-positive birational contraction between smooth surfaces

$$f : (X, \mathcal{F}) \to (X', \mathcal{F}')$$

contracting E to a point $p = f(E)$ such that p is either a regular point of $\mathcal{F}'$ or lies on a $(\Delta', \mathcal{F}')$-chain, where $\Delta' = f_*\Delta$.

Theorem 3.7. *Let $\mathcal{F}$ be a reduced foliation on a smooth projective surface X, and let*

$$\Delta = \sum_{i=1}^{l} a_i C_i,$$

where each C_i is a non-$\mathcal{F}$-invariant curve and $a_i \in \left[\frac{1}{2}, 1\right]$ for all i. Suppose that $K_{\mathcal{F}} + \Delta$ is pseudo-effective with Zariski decomposition

$$K_{\mathcal{F}} + \Delta = P(\Delta) + N(\Delta).$$

If there is no $(\Delta, \mathcal{F})$-exceptional curve of type H-J on X, then the support of $N(\Delta)$ is a disjoint union of all maximal $(\Delta, \mathcal{F})$-chains on X.

More precisely, let $\{\Theta_1, \ldots, \Theta_s\}$ be the set of all maximal $(\Delta, \mathcal{F})$-chains on X. Then

$$N(\Delta) = \sum_{i=1}^{s} M(\Delta, \Theta_i).$$

Proof. Let $\mathfrak{S}$ denote the set of non-$\mathcal{F}$-invariant curves D contained in $N(\Delta)$, and we set

$$T = \sum_{D \in \mathfrak{S}} \alpha_D D, \quad \alpha_D = \begin{cases} 1, & \text{if } D \not\subset \Delta, \\ 1 - a_i, & \text{if } D = C_i \subset \Delta \end{cases} \tag{3.8}$$

Note that $a_i \in [\frac{1}{2}, 1]$, so $T \geq 0$. Let $\{\Theta'_1, \cdots, \Theta'_t\}$ be the set of all maximal $(\Delta + T, \mathcal{F})$-chains, where $t \leq r$ and we assume $\Theta'_i \subset \Theta_i$ for $i = 1, \cdots, t$. It is clear that each Θ'_i is disjoint from $\mathfrak{S}$, by the definition of a $(\Delta + T, \mathcal{F})$-chain. Set

$$V := \sum_{i=1}^{t} M(\Delta + T, \Theta'_i).$$

Note that if $T = 0$, then $t = s$, $\Theta'_i = \Theta_i$, and so $V = \sum\limits_{i=1}^{s} M(\Delta, \Theta_i)$.

Claim. $(N(\Delta) - V) \cdot \Gamma = 0$ for any irreducible component Γ of V.

Indeed, assume without loss of generality that Γ lies in $\Theta := \Theta'_1 = \Gamma_1 + \cdots + \Gamma_r$, with multiplicity $\theta = \theta_1 = (\Delta + T)\Gamma_1 = \Delta\Gamma_1$. Then a direct computation yields

$$(V \cdot \Gamma,\ \Delta \cdot \Gamma,\ K_{\mathcal{F}} \cdot \Gamma) = \begin{cases} (\theta - 1,\ \theta,\ -1), & \text{if } \Gamma = \Gamma_1, \\ (0, 0, 0), & \text{otherwise.} \end{cases}$$

Therefore,

$$(N(\Delta) - V) \cdot \Gamma = (K_{\mathcal{F}} + \Delta - V) \cdot \Gamma = 0.$$

By [Luc01, Lemma 14.15], this claim implies that $N(\Delta) - V \geq 0$. Therefore, to conclude the proof, it remains to show that

$$M := N(\Delta) - V + T \equiv 0.$$

We proceed in steps.

Step 1. For any $C \in \mathfrak{S}$, we claim that $M \cdot C \geq 0$.

Since V is supported away from $\mathfrak{S}$, we have $V \cdot C = 0$. Also, $C^2 < 0$. Thus

$$M \cdot C = (K_{\mathcal{F}} + \Delta + T) \cdot C \geq K_{\mathcal{F}} \cdot C + \Delta \cdot C + \alpha_C C^2.$$

If $C \not\subset \Delta$, then $\alpha_C = 1$ and $\Delta \cdot C \geq 0$. If $C \subset \Delta$, say $C = C_1$, then $\alpha_C = 1 - a_1$ and $\Delta C \geq a_1 C^2$. In both cases, we have $\Delta \cdot C + \alpha_C C^2 \geq C^2$, hence

$$M \cdot C \geq K_{\mathcal{F}} \cdot C + C^2 = \operatorname{tang}(\mathcal{F}, C) \geq 0.$$

Step 2. For any component C of V, we claim that $M \cdot C = 0$.

By the claim above and the fact that $TC = 0$ from the definition of V, we have

$$M \cdot C = (N(\Delta) - V) \cdot C = 0.$$

Step 3. Suppose that $M > 0$. Since $\operatorname{Supp}(M) \subset \operatorname{Supp}(N(\Delta))$, we have $M^2 < 0$. So there exists an irreducible component C of M such that $M \cdot C < 0$.

By Step 1 and 2, we must have $C \notin \mathfrak{S} \cup \operatorname{Supp}(V)$. Thus C is an $\mathcal{F}$-invariant component of $N(\Delta)$ which is not contained in V. We will show that such a curve C cannot exist.

Assume C meets transversely the $(\Delta{+}T, \mathcal{F})$-chains $\Theta'_1, \ldots, \Theta'_k$ at points $p_1, \ldots, p_k$ respectively, and contains h other singularities $q_1, \ldots, q_h$ of $\mathcal{F}$ through which there exist separatrices not contained in $C+\Theta_1'+\cdots+\Theta_k'$. By the separatrix theorem ([Bru15], Theorem 3.4), we have $h\geq 1$. Then

$$V\cdot C=\sum_{i=1}^{k} M(\Delta+T,\Theta_i')\cdot C\leq \frac{k}{2}\quad \big(\text{by (3.4) and (3.7)}\big),$$

and

$$K_{\mathcal{F}}\cdot C=\mathrm{Z}(\mathcal{F},C)-\chi(C)\geq h+k-2+2p_a(C).$$

Therefore,

$$0>M\cdot C=(K_{\mathcal{F}}+\Delta-V+T)\cdot C\geq h+\frac{k}{2}-2+2p_a(C)+(\Delta+T)\cdot C.$$

If $T\cdot C>0$, then by (3.8), we have $(\Delta+T)\cdot C\geq 1$, hence $M\cdot C\geq 0$, a contradiction. So $T\cdot C=0$. Similarly, we must have $p_a(C)=0$, $h=1$, and $k\leq 1$. Then

$$\frac{k}{2}-\Delta\cdot C\geq V\cdot C-\Delta\cdot C>K_{\mathcal{F}}\cdot C\geq k-1.$$

This leads to two possibilities:

(i) $k=0$, $K_{\mathcal{F}}\cdot C=-1$, and $\Delta\cdot C<1$;
(ii) $k=1$, $K_{\mathcal{F}}\cdot C=0$, and $\Delta\cdot C<\frac{1}{2}$ (so $\Delta\cdot C=0$).

In both cases, all singularities of $\mathcal{F}$ on C are non-degenerate. Under our assumption, $C^2<-1$, since otherwise C would be a $(\Delta,\mathcal{F})$-exceptional curve of type H-J, contradicting the hypothesis. Hence, C must be either a $(\Delta+T,\mathcal{F})$-chain or the union $\Theta_i'+C$ is a $(\Delta+T,\mathcal{F})$-chain for some i, contradicting the maximality of the Θ_i''s.

We conclude that such a curve C does not exist, so $M=0$, as desired. □

Corollary 3.8. *Suppose $\nu(\mathcal{F})\geq 0$ and there is no $(\Delta,\mathcal{F})$-exceptional curve of type H-J over X. Then for $\mu=\min\{a_i\}_{i=1}^{l}$,*

$$\big[(1-\mu)N-N(\Delta)\big]\Delta\geq 0. \tag{3.9}$$

In particular,

$$\big[\frac{1}{2}N-N(\Delta)\big]\Delta\geq 0. \tag{3.10}$$

Proof. Using the notations in the proof of Theorem 3.7, for any irreducible component C of Δ, say $C=C_1$,

$$\begin{aligned}\big[(1-\mu)N-N(\Delta)\big]C&=(1-\mu)NC-(1-a_1)\sum_{i=1}^{s}M(\Theta_i)C\quad(\text{Lemma 3.5})\\&\geq(1-\mu)\left(N-\sum_{i=1}^{s}M(\Theta_i)\right)C\geq 0.\end{aligned}$$

□

Corollary 3.9. *If $\nu(\mathcal{F})\geq 0$ and there is no $(\Delta,\mathcal{F})$-exceptional curve of type H-J over X, then*

$$\big(N+N(\Delta)\big)\Delta+N^2-N(\Delta)^2=P(\Delta)N\geq 0. \tag{3.11}$$

Proof.

$$
\begin{aligned}
&\big(N+N(\Delta)\big)\Delta+N^2-N(\Delta)^2=(\Delta+N-N(\Delta))(N+N(\Delta))\\
=&P(\Delta)(N+N(\Delta))=P(\Delta)N\geq 0.
\end{aligned}
$$

□

Corollary 3.10. *If $\nu(\mathcal{F})\geq 0$ and there is no $(\Delta,\mathcal{F})$-exceptional curve of type H-J over X, then*

$$(2-\mu)N\Delta+N^2-N(\Delta)^2\geq 0, \tag{3.12}$$

where $\mu=\min\{a_i\}_{i=1}^{l}\geq\frac{1}{2}$. In particular,

$$\frac{3}{2}N\Delta+N^2-N(\Delta)^2\geq 0. \tag{3.13}$$

Corollary 3.11. *Under the assumptions and notations in Theorem 3.7, one has*

$$
\begin{aligned}
N(\Delta)^2&=-\sum_{i=1}^{s}\sum_{p\in\Theta_i}(1-\theta_i)^2\beta_p(\mathcal{F}),\\
N(\Delta)\Delta&=\sum_{i=1}^{s}\sum_{p\in\Theta_i}\theta_i(1-\theta_i)\beta_p(\mathcal{F}).
\end{aligned}
$$

Definition 3.12 ([MMcQ08], Definition I.1.5)**.** Let $(X,\mathcal{F},\Delta)$ be a foliated triple and $f:X'\to X$ be a proper birational morphism. For any divisor E on X', we define the *discrepancy* of $(\mathcal{F},\Delta)$ along E to be

$$a(E,\mathcal{F},\Delta)=\mathrm{ord}_E\left(K_{\mathcal{F}'}+\Delta'-f^*(K_{\mathcal{F}}+\Delta)\right),$$

where $\Delta'=f_*^{-1}\Delta$ denotes the strict transform of Δ. We say $(X,\mathcal{F},\Delta)$ is *canonical* if $a(E,\mathcal{F},\Delta)\geq 0$ for every f-exceptional divisor E over X'.

Let $\mathcal{F}$ be a redcued foliation over a smooth surface X and Δ be as above.

Lemma 3.13. *$(X,\mathcal{F},\Delta)$ is canonical if and only if* $\mathrm{tang}(\mathcal{F},\Delta_{\mathrm{red}})=0$*.*

4. Minimal resolution of $(\Delta,\mathcal{F})$

If $\mathrm{tang}(\mathcal{F},\Delta_{\mathrm{red}})>0$, then there exists a birational morphism

$$f:(X',\mathcal{F}',\Delta')\to(X,\mathcal{F},\Delta),$$

where $\Delta'=f_*^{-1}\Delta$ denotes the strict transform of Δ, such that $\mathrm{tang}(\mathcal{F}',\Delta'_{\mathrm{red}})=0$. If f is minimal with this property, we call f a *minimal resolution* of $(\Delta,\mathcal{F})$.

In this section, we study how the Zariski decomposition of $K_{\mathcal{F}}+\Delta$ behaves under the minimal resolution f of $(\Delta,\mathcal{F})$. More precisely, we analyze the relation between the negative parts $N(\Delta')$ and $N(\Delta)$.

For this purpose, we first examine the behavior of $N(\Delta)$ under a blow-up.

4.1. **Change under a blow-up.** Let $p \in X$ be a point such that

$$t_p := \mathrm{tang}(\mathcal{F}, \Delta_{\mathrm{red}}, p) > 0.$$

Consider the blow-up at p:

$$\sigma : (X', \mathcal{F}', \Delta', E) \to (X, \mathcal{F}, \Delta, p),$$

where E denotes the exceptional divisor and $\Delta' = \sigma_*^{-1}\Delta$ is the strict transform of Δ. Recall that

$$K_{\mathcal{F}'} + \Delta' = \sigma^*(K_{\mathcal{F}} + \Delta) + \big(1 - l(p) - m_p(\Delta)\big)E,$$

where $\Delta = \sum_i a_i C_i$ and $m_p(\Delta) := \sum_i a_i \cdot \mathrm{mult}_p(C_i)$.

For convenience, we assume that there are no $(\Delta, \mathcal{F})$-exceptional curves of type H–J on X.

We will distinguish the following three cases:

(i) $t_p > 0$, $l(p) = 0$, and $m_p(\Delta) \geq 1$;
(ii) $t_p > 0$, $l(p) = 1$, and $m_p(\Delta) > 0$;
(iii) $t_p > 0$, $l(p) = 0$, and $0 < m_p(\Delta) < 1$.

The following discussion shows that these three cases are sufficient for our purposes.

Lemma 4.1. *Suppose $t_p > 0$, $l(p) = 0$ and $m_p(\Delta) \geq 1$, and there exists no $(\Delta, \mathcal{F})$-exceptional curves of type H-J on X. Then*

(1) *p does not lie on $N(\Delta)$.*
(2) *There exists no $(\Delta', \mathcal{F}')$-exceptional curve of type H–J on X' unless p lies on a smooth rational $\mathcal{F}$-invariant curve C such that*

$$\Delta C \leq m_p(\Delta) + 1, \qquad C^2 = 0, \qquad C \cap \mathrm{Sing}(\mathcal{F}) = \emptyset.$$

In this case, $\mathcal{F}$ corresponds to a rational fibration $X \to B$ with smooth fiber C.
(3) *We have*

$$N(\Delta') = \begin{cases} \sigma^* N(\Delta), & \text{if } \nu(\mathcal{F}) \geq 0, \\ \sigma^* N(\Delta) + \big(1 + m_p(\Delta) - \Delta C\big)\,\bar{C}, & \text{if } \nu(\mathcal{F}) = -\infty, \end{cases} \tag{4.1}$$

where $\bar{C} = f_*^{-1}(C)$ denotes the strict transform of C.

Proof. (1) follows directly from Definition 3.3.

(2) Suppose that $\bar{C}$ is a $(\Delta', \mathcal{F}')$-exceptional curve of type H–J. Since $K_{\mathcal{F}'}E = -1$ and $\Delta' E = m_p(\Delta) \geq 1$, it follows that E is not $(\Delta, \mathcal{F})$-exceptional. Hence $\bar{C} \neq E$. Let $C = \sigma_*(\bar{C})$ be the image of $\bar{C}$ under σ.

If $p \notin C$, then $\bar{C}$ is a $(\Delta', \mathcal{F}')$-exceptional curve of type H-J if and only if C is a $(\Delta, \mathcal{F})$-exceptional curve of type H-J, contradicting our assumption. Thus $p \in C$, and we set $q = \bar{C} \cap E$. Since $l(p) = 0$ and $\bar{C}^2 = -1$, we have

$$\{q\} = \bar{C} \cap \mathrm{Sing}(\mathcal{F}') = E \cap \mathrm{Sing}(\mathcal{F}').$$

In this case,

$$C^2 = \bar{C}^2 + 1 = 0, \qquad C \cap \mathrm{Sing}(\mathcal{F}) = \emptyset.$$

We claim that this implies $\nu(\mathcal{F}) = -\infty$. Indeed, if $\nu(\mathcal{F}) \geq 0$, then writing $K_{\mathcal{F}} = P + N$ as its Zariski decomposition, we obtain $CN = 0$, hence

$$K_{\mathcal{F}} C = PC + NC = PC \geq 0.$$

On the other hand,

$$K_{\mathcal{F}}C = \mathrm{Z}(\mathcal{F}, C) - \chi(C) = -2 < 0,$$

a contradiction.

Therefore, $\mathcal{F}$ corresponds to a rational fibration $g : X \to B$, with C a smooth fibre of g. Moreover, from

$$0 \geq (K_{\mathcal{F}'} + \Delta')\bar{C} = (K_{\mathcal{F}} + \Delta)C - m_p(\Delta) + 1 = \Delta C - m_p(\Delta) - 1,$$

we deduce $\Delta C \leq m_p(\Delta) + 1$.

(3) If $\nu(\mathcal{F}) \geq 0$, this is a direct consequence of (2) and Theorem 3.7. Next, assume $\nu(\mathcal{F}) = -\infty$. Consider the contraction

$$\sigma' : (X', \mathcal{F}') \longrightarrow (X'', \mathcal{F}'')$$

obtained by contracting $\bar{C}$ as in (2). It is clear that there exist no $(\Delta'', \mathcal{F}'')$-exceptional curves of type H–J on X'', and that all maximal $(X'', \mathcal{F}'')$-chains correspond bijectively to the maximal $(X, \mathcal{F})$-chains. Thus,

$$\sigma'^* N(\Delta'') = \sigma^* N(\Delta).$$

Since $(K_{\mathcal{F}'} + \Delta')\bar{C} = \Delta C - m_p(\Delta) - 1 \leq 0$, we see

$$K_{\mathcal{F}'} + \Delta' = K_{\mathcal{F}''} + \Delta'' + \big(1 + m_p(\Delta) - \Delta C\big) \cdot \bar{C}.$$

Therefore,

$$N(\Delta') = \sigma'^* N(\Delta'') + \big(1 + m_p(\Delta) - \Delta C\big)\bar{C} = \sigma^* N(\Delta) + \big(1 + m_p(\Delta) - \Delta C\big)\bar{C}.$$

□

Remark 4.2. For the exceptional case in Lemma 4.1 (2), we can reduce it to a simpler case using a flip trick as follows:

$$\begin{tikzcd} & (X', \mathcal{F}') \arrow[dl, "\sigma"'] \arrow[dr, "\sigma'"] & \\ (X, \mathcal{F}) & & (X'', \mathcal{F}''), \end{tikzcd}$$

where σ is as above and σ' is the blow-up contracting $\bar{C}$ to $\hat{p} = \sigma(C)$. It is clear that $l(\hat{p}) = 0$ and

$$1 - l(\hat{p}) - m_{\hat{p}}(\Delta'') = -(K_{\mathcal{F}'} + \Delta')\bar{C} \geq 0,$$

so $m_{\hat{p}}(\Delta'') \leq 1$. On the other hand, E is not $(\Delta', \mathcal{F}')$-exceptional.

Therefore, by this flip trick, the exceptional case can be reduced to the situation where $l(p) = 0$ and $0 < m_p(\Delta) \leq 1$, where:

(i) If $m_p(\Delta) = 1$, then there exist no $(X', \mathcal{F}')$-exceptional curves of type H–J, and we have $N(\Delta') = \sigma^* N(\Delta)$.

(ii) If $0 < m_p(\Delta) < 1$, this case will be treated in Lemma 4.4.

Lemma 4.3. *Suppose $t_p > 0$, $l(p) = 1$, and $m_p(\Delta) > 0$, and assume that there exists no $(\Delta, \mathcal{F})$-exceptional curve of type H-J on X. Then*

(1) *There exists no $(\Delta', \mathcal{F}')$-exceptional curve of type H-J on X'.*

(2) *We have $N(\Delta') = \sigma^* N(\Delta)$ unless one of the following exceptional cases occurs:*

(e1) $p = \Delta \cap \Gamma$, *where* $\Theta = \Gamma$ *is a maximal* $(\Delta, \mathcal{F})$*-chain.*

(e2) $p = \Delta \cap \Gamma$, *where* Γ *is a smooth rational* $\mathcal{F}$*-invariant curve containing a unique non-degenerate reduced singularity* p *of* $\mathcal{F}$*, such that*

$$1 \leq \Delta \cdot \Gamma < m_p(\Delta) + 1.$$

(e3) $p = \Delta \cap \Gamma$, *where* Γ *is a smooth rational* $\mathcal{F}$*-invariant curve containing two non-degenerate reduced singularities* p, q *of* $\mathcal{F}$*, such that*

  (i) Γ *meets a maximal* $(\Delta, \mathcal{F})$*-chain* $\Theta = \Gamma_1 + \cdots + \Gamma_r$ *at* q*, and*

  (ii) $\Delta \cdot \Gamma = m_p(\Delta)$.

(3) *We have*

$$N(\Delta') = \begin{cases} \sigma^*\left(N(\Delta) - \frac{1-\Delta\Gamma}{-\Gamma^2}\Gamma\right) + \frac{1}{1-\Gamma^2}\bar{\Gamma}, & \text{if } p \text{ belongs to (e1);} \\ \sigma^* N(\Delta) + \frac{1+m_p(\Delta)-\Delta\Gamma}{1-\Gamma^2}\bar{\Gamma}, & \text{if } p \text{ belongs to (e2);} \\ \sigma^*\left(N(\Delta) - M(\Delta, \Theta)\right) + M(\Delta', \Theta + \bar{\Gamma}), & \text{if } p \text{ belongs to (e3);} \\ \sigma^* N(\Delta), & \text{others,} \end{cases}$$

where $\bar{\Gamma} = \sigma_*^{-1}(\Gamma)$ *is the strict transform of* Γ.

Proof. The proof of (1) is analogous to that of Lemma 4.1(2). Assertions (2) and (3) follow from a direct computation together with Theorem 3.7. □

For case (iii), namely when $t_p > 0$, $l(p) = 0$, and $0 < m_p(\Delta) < 1$ (so that $m_p(\Delta_{\rm red}) = 1$), we consider the following sequence of blow-ups:

$$\sigma: \ (X', \mathcal{F}') = (X_n, \mathcal{F}_n) \xrightarrow{\sigma_n} (X_{n-1}, \mathcal{F}_{n-1}) \xrightarrow{\sigma_{n-1}} \cdots \xrightarrow{\sigma_1} (X_0, \mathcal{F}_0) = (X, \mathcal{F}),$$

where $n = t_p + 1 \geq 2$, each σ_i denotes the blow-up at q_i with exceptional divisor $E_i = \sigma_i^{-1}(q_i)$, and

$$q_1 = p, \qquad q_k = E_{k-1} \cap \Delta_{k-1}, \quad k = 2, \ldots, n.$$

Note that $l(q_1) = 0$ and $l(q_i) = 1$ for $i = 2, \cdots, n$.

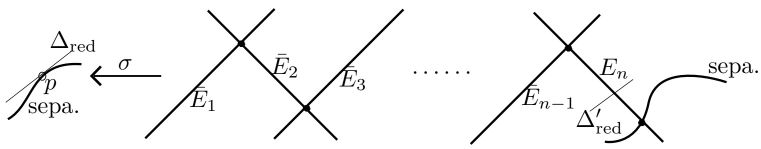

FIGURE 1. $E_n^2 = -1, \bar{E}_i^2 = -2\,(i < n)$.

Lemma 4.4. *Suppose* $t_p > 0$, $l(p) = 0$, *and* $0 < m_p(\Delta) < 1$ *(so that* $m_p(\Delta_{\rm red}) = 1$*), and assume that there exists no* $(\Delta, \mathcal{F})$*-exceptional curve of type H-J on* X*. Then*

(1) p *does not lie on* $N(\Delta)$*, and for any* $q \in \sigma^{-1}(p)$ *we have*

$$t_q = \mathrm{tang}(\mathcal{F}', \Delta_{\rm red}, q) = 0.$$

(2) *There exists no* $(\Delta', \mathcal{F}')$*-exceptional curve of type H-J on* X'.

(3) *The divisor*

$$\Theta = \bar{E}_1 + \cdots + \bar{E}_{t_p}$$

is a new maximal $(\Delta', \mathcal{F}')$*-chain with*

$$M(\Delta', \Theta) = \sum_{i=1}^{t_p} \frac{t_p + 1 - i}{t_p + 1}\, \bar{E}_i.$$

In particular,

$$N(\Delta') = \sigma^* N(\Delta) + \sum_{i=1}^{t_p} \frac{t_p + 1 - i}{t_p + 1}\, \bar{E}_i.$$

Proof. The proofs of (1) and (2) are analogous to those of Lemma 4.1(1)–(2). Assertion (3) follows from a straightforward computation together with Theorem 3.7. □

4.2. **Infinitely near singularities of** $(\Delta, \mathcal{F})$**.** In this section, we consider a sequence of blow-ups over p with $t_p > 0$:

$$\sigma:\ (X', \mathcal{F}') = (X_r, \mathcal{F}_r) \xrightarrow{\sigma_r} (X_{r-1}, \mathcal{F}_{r-1}) \xrightarrow{\sigma_{r-1}} \cdots \xrightarrow{\sigma_1} (X_0, \mathcal{F}_0) = (X, \mathcal{F})$$

where $\sigma = \sigma_1 \circ \cdots \circ \sigma_r$ satisfies the following conditions:

(i) $r \geq 1$, and for any $q \in \sigma^{-1}(p) \cap \Delta'$, the point q is a regular point of $\mathcal{F}$, i.e., $l(q) = 0$.

(ii) σ is minimal with this property.

Definition 4.5. Let q_i (resp. E_i) denote the center (resp. the exceptional divisor) of the blow-up σ_i. We call the sequence

$$(q_r \to q_{r-1} \to \cdots \to q_1 = p)$$

an *infinitely near singularity of* $(\Delta, \mathcal{F})$ of type $S_{l,m}$, where $l = l(p)$ and $m = m_p(\Delta_{\mathrm{red}})$. We refer to the birational morphism σ as a *sequence of blow-ups over this infinitely near singularity.*

Suppose that there exist no $(\Delta, \mathcal{F})$-exceptional curves of type H-J on X. Then:

(1) If $l(p) = 0$ and $0 < m_p(\Delta) < 1$, the birational morphism σ coincides with the one described in Lemma 4.4. In fact, in this case, p is of type $S_{0,1}$.

(2) If $l(p) = 1$ or $l(p) = 0$ with $m_p(\Delta) \geq 1$, each blow-up σ_i in σ is as described in Lemma 4.4 and Lemma 4.3. Here, we avoid the exceptional case in Lemma 4.1 by employing the flip trick (cf. Remark 4.2).

It follows that there exist no $(\Delta', \mathcal{F}')$-exceptional curves of type H–J on X'. By Theorem 3.7, the support of $N(\Delta')$ (resp. $N(\Delta)$) consists of all maximal $(\Delta', \mathcal{F}')$ (resp. $(\Delta, \mathcal{F})$)-chains on X' (resp. X). Let $\Theta_1, \ldots, \Theta_s$ be all maximal $(\Delta, \mathcal{F})$-chains on X.

Case $S_{0,m}$**.** Suppose $l(p) = 0$ with $t_p > 0$.

In this situation, the support of $N(\Delta')$ consists of $\Theta_1, \ldots, \Theta_s$ together with a possible new $(\Delta', \mathcal{F}')$-chain Θ whose first curve is $\bar{E}_1$, where

(i) If $\bar{E}_1 \cdot \Delta' \geq 1$, then no such chain Θ arises. In this case one has $N(\Delta') = \sigma^* N(\Delta)$.

(ii) If $\bar{E}_1 \cdot \Delta' < 1$, then such a chain Θ arises, and its support is contained in $\sigma^{-1}(p)$. In this case

$$N(\Delta') = \sigma^* N(\Delta) + M(\Delta', \Theta).$$

Case $S_{1,m}$. Suppose $l(p) = 1$ with $t_p > 0$.

In this situation, one has $N(\Delta') = \sigma^* N(\Delta)$ unless one of the following exceptional cases occurs:

(E1) $p = \Delta \cap \Gamma$, where $\Theta = \Gamma$ is a maximal $(\Delta, \mathcal{F})$-chain. In this case there exists a maximal $(\Delta', \mathcal{F}')$-chain Θ^+ whose first curve is $\bar{\Gamma}$, and whose remaining components (if any) are contained in $\sigma^{-1}(p)$. In this case

$$N(\Delta') = \sigma^*(N(\Delta) - M(\Delta, \Theta)) + M(\Delta', \Theta^+).$$

(E2) $p = \Delta \cap \Gamma$, where Γ is a smooth rational $\mathcal{F}$-invariant curve containing a unique non-degenerate reduced singularity p of $\mathcal{F}$, such that

$$1 \le \Delta \cdot \Gamma < (\Delta\Gamma)_p + 1.$$

In this case there exists a new maximal $(\Delta', \mathcal{F}')$-chain Θ^+ whose first curve is $\bar{\Gamma}$, and whose remaining components (if any) are also contained in $\sigma^{-1}(p)$. In this case

$$N(\Delta') = \sigma^* N(\Delta) + M(\Delta', \Theta^+).$$

(E3) $p = \Delta \cap \Gamma$, where Γ is a smooth rational $\mathcal{F}$-invariant curve containing two non-degenerate reduced singularities p, q of $\mathcal{F}$, such that Γ meets a maximal $(\Delta, \mathcal{F})$-chain $\Theta = \Gamma_1 + \cdots + \Gamma_r$ at q, and $\Delta \cdot \Gamma = (\Delta\Gamma)_p$.

In this case there exists a maximal $(\Delta', \mathcal{F}')$-chain Θ^+ whose first $r + 1$ curves are $\Theta + \bar{\Gamma}$, and whose remaining components (if any) are contained in $\sigma^{-1}(p)$. In this case

$$N(\Delta') = \sigma^*(N(\Delta) - M(\Delta, \Theta)) + M(\Delta', \Theta^+).$$

Definition 4.6. A curve Γ satisfying one of the conditions (E1)–(E3) is called a *potential curve* of $(\Delta, \mathcal{F})$. More precisely, a curve Γ as in (E1) is called the *first potential curve* of $(\Delta, \mathcal{F})$, while a curve Γ as in (E2) or (E3) is called the *second potential curve* of $(\Delta, \mathcal{F})$.

4.3. **Minimal resolution of $(X, \mathcal{F}, \Delta)$.** We now assume that there are no $(\Delta, \mathcal{F})$-exceptional curves of type H–J and describe our procedure for obtaining the minimal resolution. Consider the sequence of blow-ups

$$(\bar{X}, \bar{\mathcal{F}}) = (X_r, \mathcal{F}_r) \xrightarrow{\sigma_r} (X_{r-1}, \mathcal{F}_{r-1}) \xrightarrow{\sigma_{r-1}} \cdots \xrightarrow{\sigma_1} (X_0, \mathcal{F}_0) = (X, \mathcal{F}),$$

where each σ_i is a sequence of blow-ups over a point $q_i \in (X_{i-1}, \mathcal{F}_{i-1})$ with $t_{q_i} > 0$ (cf. Definition 4.5). Here $\mathcal{F}_{i+1} = \sigma_{i+1}^* \mathcal{F}_i$ denotes the pullback foliation, and $\Delta_{i+1} = \sigma_{i+1*}^{-1} \Delta_i$ the strict transform of Δ_i.

Without loss of generality, we may assume that $l(q_i) = 1$ for $i \le t$ and $l(q_i) = 0$ for $i > t$. Note that

$$\{q_i \mid l(q_i) = 1\} \;=\; \{\, q \in X \mid q \in \Delta,\ l(q) = 1 \,\}.$$

After finitely many such steps, the resulting triple $(\bar{X}, \bar{\mathcal{F}}, \bar{\Delta})$ satisfies:

(1) there are no $(\bar{\Delta}, \bar{\mathcal{F}})$-exceptional curves of type H–J over $\bar{X}$;

(2) for every $q \in \bar{X}$, we have $t_q = 0$, i.e. $(\bar{X}, \bar{\mathcal{F}}, \bar{\Delta})$ is canonical.

We call $\sigma : (\bar{X}, \bar{\mathcal{F}}, \bar{\Delta}) \to (X, \mathcal{F}, \Delta)$ a *minimal resolution* of $(X, \mathcal{F}, \Delta)$.

5. Double covers over foliated surfaces $(X, \mathcal{F})$ with $\nu(\mathcal{F}) \geq 0$

Let $\mathcal{F}$ be a reduced foliation on a smooth surface X with $\nu(\mathcal{F}) \geq 0$. Write the Zariski decomposition of the canonical divisor $K_{\mathcal{F}}$ as

$$K_{\mathcal{F}} \equiv P + N,$$

where P (resp. N) denotes the positive (resp. negative) part.

Let $\pi : Y \to X$ be a double cover branched along a reduced, even effective divisor B, and set $\mathcal{G} := \pi^*\mathcal{F}$. We decompose

$$B = B_v + B_h,$$

where B_v consists of the $\mathcal{F}$-invariant irreducible components of B, and B_h consists of the remaining ones. For convenience, we refer to B_v (resp. B_h) as the *$\mathcal{F}$-invariant part* (resp. *non-$\mathcal{F}$-invariant part*) of B.

It follows that the divisor $K_{\mathcal{F}} + \frac{1}{2}B_h$ is pseudo-effective. We denote its Zariski decomposition by

$$K_{\mathcal{F}} + \tfrac{1}{2}B_h \equiv P(B_h) + N(B_h),$$

where $P(B_h)$ (resp. $N(B_h)$) stands for the positive (resp. negative) part.

5.1. **Double cover with a smooth branch locus.**

In this section, we assume the branch locus B is smooth and reduced.

5.1.1. *Classification of the singularities of $\mathcal{G}$.* Let q be a singularity of $\mathcal{G}$ over Y and $p = \pi(q) \in X$.

(I) Suppose $p \notin B$. Then the $\pi^{-1}(p)$ consists of two reduced singularities of $\mathcal{G}$ which are exactly the copies of q.

(II) Suppose $p \in B_h$.

(1) If p is a regular point of $\mathcal{F}$, then $B, \mathcal{F}, \pi$ can be locally defined by

$$B = (x + y^l = 0), \quad \omega = \mathrm{d}x, \quad \begin{cases} x + y^l = u^2, \\ y = v, \end{cases}$$

where $p = (0,0)$, $l = t_p + 1 \geq 1$ and $\widetilde{B} := (\pi^*B)_{\mathrm{red}} = (u = 0)$. Then around q, $\mathcal{G}$ is locally defined by

$$\widetilde{\omega} = \pi^*(\omega) = \mathrm{d}(u^2 - v^l) = 2u\mathrm{d}u - lv^{l-1}\mathrm{d}v.$$

This implies that q is redcued iff $l \leq 2$ (i.e., $t_p \leq 1$). In particular, q is regular if $t_p = 0$ and $\lambda_q = -1$ if $t_p = 1$.

(2) If p is a singularity of $\mathcal{F}$, then $l(p) = 1$. Consider the blow-up over p:

$$\sigma : (X', \mathcal{F}', E) \to (X, \mathcal{F}, p).$$

We have

$$K_{\mathcal{F}'} + \tfrac{1}{2}B'_h = \sigma^*\big(K_{\mathcal{F}} + \tfrac{1}{2}B_h\big) - \tfrac{1}{2}E,$$

where $B'_h = \sigma_*^{-1}B_h$ denotes the strict transform of B_h. This shows that $K_{\mathcal{F}} + \frac{1}{2}B_h$ is not canonical at p (cf. Definition 3.12); hence $K_{\mathcal{G}} =$

$\pi^*(K_{\mathcal{F}} + \frac{1}{2}B_h)$ is not canonical at q. Therefore, q is a non-reduced singularity of $\mathcal{G}$.

(III) Suppose that $p \in B_v$. By choosing suitable local coordinates, we may assume that B_v (resp. π) is defined by $x = 0$ (resp. $z^2 = x$), and that $\mathcal{F}$ (resp. $\mathcal{G}$) is represented by a 1-form ω (resp. $\widetilde{\omega}$) near $p = (0,0)$ (resp. $q = (0,0)$), occurring in one of the following cases:

(1) p is a regular point of $\mathcal{F}$:

$$\omega = \mathrm{d}x, \qquad \widetilde{\omega} = \mathrm{d}z.$$

(2) p is a non-degenerate singularity of $\mathcal{F}$:

$$\omega = \lambda y \,\mathrm{d}x + x\,\mathrm{d}y, \qquad \widetilde{\omega} = 2\lambda y\,\mathrm{d}z + z\,\mathrm{d}y.$$

(3) p is a saddle-node of $\mathcal{F}$ whose strong separatrix is contained in B_v:

$$\begin{aligned}\omega &= \big(y(1+ax^k) + x\,o(k)\big)\,\mathrm{d}x - x^{k+1}\,\mathrm{d}y,\\ \widetilde{\omega} &= 2\big(y(1+az^{2k}) + z^2 o(k)\big)\,\mathrm{d}z - z^{2k+1}\,\mathrm{d}y.\end{aligned}$$

(4) p is a saddle-node of $\mathcal{F}$ whose weak separatrix is contained in B_v:

$$\begin{aligned}\omega &= x\big(1+ay^k + o(k)\big)\,\mathrm{d}y - y^{k+1}\,\mathrm{d}x,\\ \widetilde{\omega} &= z\big(1+ay^k + o(k)\big)\,\mathrm{d}y - 2y^{k+1}\,\mathrm{d}z.\end{aligned}$$

Therefore, in cases (2)-(4), the point q is a reduced singularity of $\mathcal{G}$, while in case (1) it is a regular point.

Thus, we obtain the following proposition.

Proposition 5.1. *With the notation and assumptions as above, the foliation $\mathcal{G}$ is reduced on the smooth surface Y if and only if the following conditions hold:*

(i) *the branch locus B is smooth and reduced;*
(ii) *for any point $p \in B_h$, one has $p \notin \mathrm{Sing}(\mathcal{F})$ and*

$$t_p := \mathrm{tang}(\mathcal{F}, B_h, p) \leq 1.$$

For later computations, we record the following immediate corollary.

Proposition 5.2. *The foliation $\mathcal{G}$ is reduced on the smooth surface Y if*

(i) *the branch locus B is smooth and reduced;*
(ii) $\mathrm{tang}(\mathcal{F}, B_h) = 0$.

Under the assumptions of Proposition 5.2, the (reduced) singularities q of $\mathcal{G}$ can be divided into the following cases:

(A) $p = \pi(q)$ is a singularity of $\mathcal{F}$ outside the branch locus B;
(B) $p = \pi(q)$ is a singularity of $\mathcal{F}$ lying on B_v.

($\mathfrak{B}_1$) p is non-degenerate with $\mathrm{CS}(\mathcal{F}, B_v, p) = -\frac{n}{m}$, where $\gcd(m,n) = 1$ and n is odd.

($\mathfrak{B}_2$) All other cases.

By the discussion above, the following lemma is immediate.

Lemma 5.3. *For any point $p \in X$, we have*

$$\sum_{q\in\pi^{-1}(p)} \beta_q(\mathcal{G}) = \begin{cases} \frac{1}{2}\beta_p(\mathcal{F}), & \text{if } p \in \mathfrak{B}_1, \\ 2\beta_p(\mathcal{F}), & \text{otherwise.} \end{cases}$$

Moreover,

$$\sum_{q\in \operatorname{Sing}\mathcal{G}} \beta_q(\mathcal{G}) = \sum_{p\in \operatorname{Sing}\mathcal{F}} 2\beta_p(\mathcal{F}) - \sum_{p\in\mathfrak{B}_1} \tfrac{3}{2}\beta_p(\mathcal{F}).$$

Under the assumption of Proposition 5.2, we obtain the following specialized formulas for the Chern numbers of the double foliation $\mathcal{G}$.

Theorem 5.4. *If $\nu(\mathcal{F}) \geq 0$, then*

$$c_1^2(\mathcal{G}) = 2c_1^2(\mathcal{F}) + \tfrac{3}{2}K_{\mathcal{F}}B_h + 2N^2 - 2N(B_h)^2,$$

$$\chi(\mathcal{G}) = 2\chi(\mathcal{F}) + \tfrac{1}{8}K_{\mathcal{F}}B_h - \sum_{p\in\mathfrak{B}_1} \tfrac{1}{8}\beta_p(\mathcal{F}) + \tfrac{1}{6}\ell(\mathcal{F}) - \tfrac{1}{12}\ell(\mathcal{G}),$$

$$c_2(\mathcal{G}) = 2c_2(\mathcal{F}) - 2N^2 + 2N(B_h)^2 - \sum_{p\in\mathfrak{B}_1} \tfrac{3}{2}\beta_p(\mathcal{F}) + 2\ell(\mathcal{F}) - \ell(\mathcal{G}),$$

where $\ell(\mathcal{G})$ (resp. $\ell(\mathcal{F})$) denotes the number of $\mathcal{G}$- (resp. $\mathcal{F}$-)exceptional curves containing saddle-nodes.

Proof. It is straightforward to see that $K_{\mathcal{G}}$ admits a Zariski decomposition $K_{\mathcal{G}} = P(\mathcal{G}) + N(\mathcal{G})$, where

$$P(\mathcal{G}) = \pi^*P(B_h), \qquad N(\mathcal{G}) = \pi^*N(B_h).$$

Hence,

$$c_1^2(\mathcal{G}) = P(\mathcal{G})^2 = 2P(B_h)^2 = 2\big(K_{\mathcal{F}} + \tfrac{1}{2}B_h\big)^2 - 2N(B_h)^2.$$

Since $\operatorname{tang}(\mathcal{F}, B_h) = K_{\mathcal{F}}B_h + B_h^2 = 0$ and $P^2 = c_1^2(\mathcal{F})$, we have

$$c_1^2(\mathcal{G}) = 2c_1^2(\mathcal{F}) + \tfrac{3}{2}K_{\mathcal{F}}B_h + 2N^2 - 2N(B_h)^2.$$

Similarly,

$$K_{\mathcal{G}}^2 = 2\left(K_{\mathcal{F}} + \tfrac{1}{2}B_h\right)^2 = 2K_{\mathcal{F}}^2 + \tfrac{3}{2}K_{\mathcal{F}}B_h.$$

By Lemma 5.3, we obtain

$$\sum_{q\in\operatorname{Sing}\mathcal{G}} \beta_q(\mathcal{G}) = \sum_{p\in\operatorname{Sing}\mathcal{F}} 2\beta_p(\mathcal{F}) - \sum_{p\in\mathfrak{B}_1} \tfrac{3}{2}\beta_p(\mathcal{F}).$$

From the formula

$$12\chi(\mathcal{G}) = K_{\mathcal{G}}^2 + \sum_{q\in\operatorname{Sing}\mathcal{G}} \beta_q(\mathcal{G}) - \ell(\mathcal{G}),$$

and the above relations, we deduce

$$\begin{aligned} 12\chi(\mathcal{G}) &= 2K_{\mathcal{F}}^2 + \tfrac{3}{2}K_{\mathcal{F}}B_h + \sum_{p\in\operatorname{Sing}\mathcal{F}} 2\beta_p(\mathcal{F}) - \sum_{p\in\mathfrak{B}_1} \tfrac{3}{2}\beta_p(\mathcal{F}) - \ell(\mathcal{G}) \\ &= 24\chi(\mathcal{F}) + \tfrac{3}{2}K_{\mathcal{F}}B_h - \sum_{p\in\mathfrak{B}_1} \tfrac{3}{2}\beta_p(\mathcal{F}) + 2\ell(\mathcal{F}) - \ell(\mathcal{G}). \end{aligned}$$

Finally, the formula for $c_2(\mathcal{G})$ follows from Noether's formula $12\chi(\mathcal{G}) = c_1^2(\mathcal{G}) + c_2(\mathcal{G})$. □

5.2. **The indexes** $\beta_q(B_v), s(B_v)$**.** In this section, we assume that there is no $(\frac{1}{2}B_h, \mathcal{F})$-exceptional curve of type H-J on X. By Theorem 3.7, we have

$$N(B_h) = \sum_{i=1}^{s} M(\tfrac{1}{2}B_h, \Theta_i) = \sum_{i=1}^{s}(1-\theta_i)M(\Theta_i),$$

where $\{\Theta_1, \cdots, \Theta_s\}$ is the set of all maxiaml $(\frac{1}{2}B_h, \mathcal{F})$-chain and $\theta_i = \frac{1}{2}B_h\Theta_i$ for all i.

Definition 5.5. For any $q \in \mathrm{Sing}\mathcal{F}$, we define $\beta_q(B_v)$ as

$$\beta_q(B_v) := \begin{cases} \beta_q(\mathcal{F}), & \text{if } q \in \mathfrak{D}, \\ 4/3, & \text{if } q \in B_v \setminus B_h \text{ with } m_q(B_v) = 2,\ \lambda_q = 0, \\ 0, & \text{others}, \end{cases} \tag{5.1}$$

where $\mathfrak{D}$ is set of singularities q of $\mathcal{F}$ contained in $B_v \setminus B_h$, satisfying $\lambda_q = -\frac{n}{m} \in \mathbb{Q}^-, \gcd(m,n) = 1$ and

(i) for $m_q(B_v) = 1$, $\mathrm{CS}(\mathcal{F}, B_v, q) = -\frac{n}{m} \in \mathbb{Q}^-$, n is odd,

(ii) for $m_q(B_v) = 2$, $m+n$ is odd.

(Here note that $m_q(B_v) \leq 2$ since $\mathcal{F}$ is reduced.)

Remark 5.6. Suppose $q \in \mathfrak{D}$ with $m_q(B_v) = 2$ and $\lambda_p = -n/m$. Consider a blow-up over p:

$$\sigma : (X', \mathcal{F}', E) \to (X, \mathcal{F}, q).$$

Let $B' = B'_v + B'_h$ is the induced branch locus. We see B' is smooth at $\sigma^{-1}(q)$ and set

$$B' \cap \sigma^{-1}(q) \cap \mathrm{Sing}\mathcal{F}' = B'_v \cap \sigma^{-1}(q) = \{\, p_1,\ p_2 \,\}.$$

Then we see

$$\mathrm{CS}(\mathcal{F}', B_v, p_1) = -\frac{m+n}{n}, \qquad \mathrm{CS}(\mathcal{F}', B_v, p_1) = -\frac{m+n}{n}.$$

Since $\gcd(m+n, n) = \gcd(m,n) = 1$ and $m+n$ is odd, we get

$$\beta_q(B_v) = \beta_{p_1}(B'_v) + \beta_{p_2}(B'_v).$$

Definition 5.7. We define $s(B_v)$ as

$$s(B_v) := \sum_{p \in \mathrm{Sing}\mathcal{F}} \beta_p(B_v) \tag{5.2}$$

If moreover the assumption of Proposition 5.2 holds, it is clear that

$$s(B_v) = \sum_{p \in \mathfrak{B}_1} \beta_p(\mathcal{F}). \tag{5.3}$$

Lemma 5.8. *For a maximal $(\frac{1}{2}B_h, \mathcal{F})$-chain $\Theta = \Gamma_1 + \cdots + \Gamma_r$,*

$$\sum_{q \in \Theta} \beta_q(B_v) = \begin{cases} \sum\limits_{q \in \Theta} \beta_q(\mathcal{F}), & \textit{if } B_h\Gamma_1 = 1 \textit{ and } B_h \cap \Theta \cap \mathrm{Sing}\mathcal{F} = \emptyset; \\ 0, & \textit{others.} \end{cases} \tag{5.4}$$

Proof. If $B_h \cap \Gamma_1 \cap \operatorname{Sing}\mathcal{F} \neq \emptyset$, then $r = 1$ and $\Theta \cap \operatorname{Sing}\mathcal{F} = \{p_1\}$ for $p_1 = \Gamma_1 \cap B_h$. Hence

$$\sum_{q\in\Theta} \beta_q(B_v) = \beta_{p_1}(B_v) = 0.$$

Next we assume $B_h \cap \Gamma_1 \cap \operatorname{Sing}\mathcal{F} = \emptyset$. Consider a sequence of blow-ups

$$\sigma : (X', \mathcal{F}') \to (X, \mathcal{F}),$$

whose centers are the points $q \in \Theta \cap \operatorname{Sing}\mathcal{F}$ with $m_q(B_v) = 2$. Let $\Theta' = (\sigma^*\Theta)_{\mathrm{red}}$ and write

$$\Theta' = D_1 + \cdots + D_l,$$

which has a similar construction to an $\mathcal{F}'$-chain, except that we allow $D_j^2 = -1$ for some j. We set $e_j = -D_j^2$ and note that $D_1 = \bar{\Gamma}_1$. Let $p_i = D_i \cap D_{i+1}$ for $i = 1, \ldots, l-1$ and let p_l be the other singularity of $\mathcal{F}'$ lying on D_l. Then it suffices to prove

$$\sum_{i=1}^{l} \beta_{p_i}(B'_v) = \begin{cases} \sum_{i=1}^{l} \beta(-\lambda_{p_i}), & \text{if } B'_h D_1 = 1, \\ 0, & \text{if } B'_h D_1 = 0. \end{cases}$$

(1) Suppose $B'_h D_1 = 0$ and $D_1 \not\subset B'_v$. Then $D_1, \ldots, D_l \not\subset B'_v$ and $B'_v D_l = 0$, since $B' = B'_h + B'_v$ is a reduced and even divisor. Hence $\beta_{p_i}(B'_v) = 0$ for all $i = 1, \ldots, l$.

(2) Suppose $B'_h D_1 = 0$ and $D_1 \subset B'_v$. Then $D_1, D_3, \ldots, D_{2[\frac{l-1}{2}]+1} \subset B'_v$ and $e_{2i+1} \equiv 0 \pmod 2$. In this case, $p_1, \ldots, p_l \in B'_v$ and $m_{p_i}(B'_v) = 1$ for all i. We set $\mathrm{CS}(\mathcal{F}', B'_v, p_i) = -\frac{n_i}{m_i}$ with $\gcd(n_i, m_i) = 1$.

Since $e_1 \equiv 0 \pmod 2$, we have $n_1 \equiv 0 \pmod 2$ and $m_1 = 1$. By the C-S formula for D_2:

$$\mathrm{CS}(\mathcal{F}', D_2, p_1) + \mathrm{CS}(\mathcal{F}', D_2, p_2) = D_2^2 = -e_2,$$

we obtain

$$e_2 = \frac{m_1}{n_1} + \frac{m_2}{n_2},$$

so that $n_2 m_1 = n_1(n_2 e_2 - m_2)$. Since n_1 is even and m_1 is odd, we see n_2 is even. Similarly, by the C-S formula for D_3, we obtain

$$e_3 = \frac{n_2}{m_2} + \frac{n_3}{m_3},$$

so that $n_3 m_2 = e_3 m_2 m_3 - m_3 n_2$. Since e_3 and n_2 are even while m_2 is odd, n_3 is even. By induction, $n_1, \ldots, n_l$ are all even, hence $\beta_{p_i}(B'_v) = 0$ for all i.

(3) Suppose $B'_h D_1 = 1$ and $D_1 \not\subset B'_v$. Then $D_2, D_4, \ldots, D_{2[\frac{l}{2}]} \subset B'_v$ and $e_{2i} \equiv 0 \pmod 2$. In this case, $p_1, \ldots, p_l \in B'_v$ and $m_{p_i}(B'_v) = 1$ for all i. We set $\mathrm{CS}(\mathcal{F}', B'_v, p_i) = -\frac{n_i}{m_i}$ with $\gcd(n_i, m_i) = 1$. In particular, we see $n_1 = 1$ and $m_1 = e_1$. By the C-S formula for D_2, we obtain

$$e_2 = \frac{n_1}{m_1} + \frac{n_2}{m_2},$$

so that $n_2 m_1 = m_2(m_1 e_2 - n_1)$. Since e_2 is even, n_1 is odd and $\gcd(m_2, n_2) = 1$, we see n_2 is odd. Similarly, by the C-S formula for D_3, we obtain

$$e_3 = \frac{m_2}{n_2} + \frac{m_3}{n_3},$$

so that $m_3 n_2 = n_3(n_2 e_2 - m_2)$. Since n_2 is odd and $\gcd(m_3, n_2) = 1$, we see n_3 is odd. By induction, $n_1, \ldots, n_l$ are all odd, hence $\beta_{p_i}(B'_v) = \beta(-\lambda_{p_i})$ for all i.

(4) Suppose $B'_h D_1 = 1$ and $D_1 \subset B'_v$. Then $D_1, D_3, \dots, D_{2[\frac{l-1}{2}]+1} \subset B'_v$ and $e_1 \equiv 1 \pmod 2$, $e_{2i+1} \equiv 0 \pmod 2$. In this case, $p_1, \dots, p_l \in B'_v$ and $m_{p_i}(B'_v) = 1$. We set $\mathrm{CS}(\mathcal{F}', B'_v, p_i) = -\frac{n_i}{m_i}$ with $\gcd(n_i, m_i) = 1$. In particular, we see $m_1 = 1$ and $n_1 = e_1$ is odd. By the C-S formula for D_2, we obtain

$$e_2 = \frac{m_1}{n_1} + \frac{m_2}{n_2},$$

so that $n_1 m_2 = n_2(n_1 e_2 - m_1)$. Since n_1 is odd and $\gcd(m_2, n_2) = 1$, we see n_2 is odd. Similarly, by the C-S formula for D_3, we obtain

$$e_3 = \frac{n_2}{m_2} + \frac{n_3}{m_3},$$

so that $m_2 n_3 = m_3(m_2 e_3 - n_2)$. Since e_3 is even, n_2 is odd and $\gcd(m_3, n_3) = 1$, we see n_3 is odd. By induction, $n_1, \dots, n_l$ are all odd, hence $\beta_{p_i}(B'_v) = \beta(-\lambda_{p_i})$ for all i. $\square$

Corollary 5.9. *For a maximal $(\frac{1}{2}B_h, \mathcal{F})$-chain $\Theta = \Gamma_1 + \cdots + \Gamma_r$,*

$$\sum_{q \in \Theta} \beta_q(B_v) = \begin{cases} 0, & \text{if } B_h \cap \Theta \cap \mathrm{Sing}\mathcal{F} \neq \emptyset \\ 2M(\frac{1}{2}B_h, \Theta) \cdot B_h, & \text{others.} \end{cases}$$

Proof. By Lemma 3.4, we have

$$2M\left(\tfrac{1}{2}B_h, \Theta\right) \cdot B_h = 4\theta(1-\theta) \sum_{q \in \Theta} \beta_q(\mathcal{F}) = \begin{cases} 0, & \text{if } B_h \Gamma_1 = 0, \\ \sum_{q \in \Theta} \beta_q(\mathcal{F}), & \text{if } B_h \Gamma_1 = 1, \end{cases}$$

where $\theta = \frac{1}{2} B_h \Gamma_1$. The remaining assertion follows directly from Lemma 5.8. $\square$

Then we have the following proposition.

Proposition 5.10. *If $B_h \cap N(B_h) \cap \mathrm{Sing}\mathcal{F} = \emptyset$, then*

$$s(B_v) \geq 2N(B_h)B_h.$$

5.3. **Infinitely near singularities of $(\frac{1}{2}B_h, \mathcal{F})$.** In this section, we assume that there is no $(B_h, \mathcal{F})$-exceptional curve of type H–J over $(X, \mathcal{F})$. We consider an infinitely near singularity p of $(\frac{1}{2}B_h, \mathcal{F})$ of type $S_{l,m}$, together with the following sequence of blow-ups:

$$\sigma: \ (X', \mathcal{F}', B'_h) = (X_r, \mathcal{F}_r, B_{r,h}) \xrightarrow{\sigma_r} X_{r-1} \xrightarrow{\sigma_{r-1}} \cdots \xrightarrow{\sigma_1} (X_0, \mathcal{F}_0, B_{0,h}) = (X, \mathcal{F}, B_h)$$

where:

(i) $B_{i,h}$ denotes the strict transform of B_h, i.e.,

$$B_{i,h} = (\sigma_1 \circ \cdots \circ \sigma_i)_*^{-1}(B_h).$$

(ii) Let q_i (resp. E_i) be the blow-up center (resp. exceptional divisor) of σ_i. Then $q_1 = p$, and for $i \geq 2$ we have

$$q_i \in B_{i-1,h} \cap \mathrm{Sing}(\mathcal{F}_{i-1}) \cap (\sigma_1 \circ \cdots \circ \sigma_{i-1})^{-1}(p).$$

(iii) Every point $q \in B'_h \cap \sigma^{-1}(p)$ is a regular point of $\mathcal{F}'$, i.e., $l(q) = 0$.

In §4.2, we have seen that there is no $(\frac{1}{2}B'_h, \mathcal{F}')$-exceptional curve of type H–J on X'. Next, we will compute how certain invariants change under the morphism σ.

Since $l(q_i) = 1$ for all $i \geq 2$, we see

$$K_{\mathcal{F}'} \cdot B'_h = K_{\mathcal{F}} \cdot B_h + m(1 - l), \tag{5.5}$$

where $m := m + p(B_h)$ and $l := l(p) = l(q_1)$.

Proposition 5.11. *Suppose $p \in S_{0,m}$. Then*

$$N(B'_h) = \sigma^* N(B_h),$$

unless

$$\bar{E}_1 \cdot B'_h \;=\; E_1 \cdot B_{1,h} - (E_1 \cdot B_{1,h})_{q_2} \;\leq 1. \tag{5.6}$$

In particular, if (5.6) holds, then there exists a maximal $(B'_h, \mathcal{F}')$-chain Θ_p contained in $\sigma^{-1}(p)$, whose first component is $\Gamma_1 = \bar{E}_1$ and whose multiplicity (cf. Definition 3.3) is

$$\theta_p = \frac{1}{2}\big(B_{1,h} \cdot E_1 - (B_{1,h} \cdot E_1)_{q_2}\big).$$

Proof. This follows directly from the discussion in §4.2. □

For convenience, we divide $S_{0,m}$-singularities into the following two cases:

$(S^{\theta}_{0,m})$ $p \in S_{0,m}$ with $\theta := \theta_p \in \{0, \frac{1}{2}\}$.

$(S^{*}_{0,m})$ $p \in S_{0,m}$ with $\theta_p \geq \frac{1}{2}$.

Corollary 5.12. *Suppose $p \in S_{0,m}$. Then we have*

$$N(B'_h) = \sigma^* N(B_h) + (1 - \theta_p) M(\Theta_p),$$

where $M(\Theta_p) = 0$ if Θ_p does not exist. Moreover,

$$N(B_h)^2 - N(B'_h)^2 = \begin{cases} (1-\theta_p)^2 \beta_p^-, & \text{if } p \in S^{\theta}_{0,m}, \\ 0, & \text{if } p \in S^{*}_{0,m}, \end{cases} \tag{5.7}$$

where $\beta_p^- := \sum_{q \in \sigma^{-1}(p) \cap \Theta_p} \beta_q(\mathcal{F}')$.

Proof. Clear. □

Proposition 5.13. *Suppose $p \in S_{1,m}$. Then $N(B'_h) = \sigma^* N(B_h)$ unless there is a potential curve Γ of $(\frac{1}{2}B_h, \mathcal{F})$ (cf. Definition 4.6) passing through p. Moreover, if $N(B'_h) \neq \sigma^* N(B_h)$, then there is a unique maximal $(\frac{1}{2}B'_h, \mathcal{F}')$-chain, say Θ_p (with multiplicity θ_p), which is not the pullback of a maxiaml $(\frac{1}{2}B_h, \mathcal{F})$-chain.*

Proof. This follows directly from the discussion in §4.2. □

For convenience, we divide $S_{1,m}$-singularities into the following three cases:

$(S^{\mathrm{I,e}}_{1,m})$ p lies on a first potential curve Γ with $l(p) = 1$ and $e = -\Gamma^2$.

$(S^{\mathrm{II},\theta}_{1,m})$ p lies on a second potential curve Γ with $l(p) = 1$ and

$$\theta := \theta_p = \frac{1}{2}\big(B_h\Gamma - (B_h\Gamma)_p\big) \in \{0, \frac{1}{2}\}.$$

$(S^{*}_{1,m})$ All other cases in $(S_{1,m})$.

5.3.1. *Zariski index $\alpha(p)$.*

Definition 5.14. We define the *Zariski index $\alpha(p)$* of p with respect to σ as

$$\alpha(p) := \begin{cases} \dfrac{3e-1}{4e(e+1)}\,(>0), & \text{if } p \in S_{1,m}^{\mathrm{I},e}, \\ (1-\theta_p)^2\,\beta_p^-, & \text{if } p \in S_{1,m}^{\mathrm{II},\theta} \cup S_{0,m}^{\theta}, \\ 0, & \text{if } p \in S_{1,m}^{*} \cup S_{0,m}^{*}, \end{cases} \tag{5.8}$$

where

$$\beta_p^- := \sum_{q\in\sigma^{-1}(p)\cap\Theta_p} \beta_q(\mathcal{F}'). \tag{5.9}$$

Proposition 5.15. *For $p \in (S_{l,m})$, we have*

$$0 \le \alpha(p) = N(B_h)^2 - N(B_h')^2 < 1.$$

Proof. This follows directly from explicit computation and the discussion in §4.2. □

5.3.2. *Other indexes: $s_0(p), s(p)$.*

Definition 5.16. For any $p \in S_{l,m}$, define

$$s_0(p) := \#\big\{\text{ blow-up points in the morphism } \sigma \text{ that are saddle-nodes}\big\}, \tag{5.10}$$

$$s(p) := \sum_{q\in\sigma^{-1}(p)} \beta_q(B_v'), \tag{5.11}$$

where $\beta_q(B_v')$ is as defined in Definition 5.5.

5.4. **Canonical resolution.** Recall the double cover over a foliated surface

$$\pi : (Y, \mathcal{G}) \longrightarrow (X, \mathcal{F}),$$

where we assume $\mathcal{F}$ is a relatively minimal foliation with $\nu(\mathcal{F}) \ge 0$ and the (redcued) ramification divisor is $B = B_h + B_v$. Consider the following commutative diagram:

$$\begin{tikzcd} (\widetilde{Y}, \widetilde{\mathcal{G}}) \arrow[r] \arrow[d, "\widetilde{\pi}"] & (\bar{Y}, \bar{\mathcal{F}}) \arrow[r] \arrow[d, "\bar{\pi}"] & (Y, \mathcal{G}) \arrow[d, "\pi"] \\ (\widetilde{X}, \widetilde{\mathcal{F}}) \arrow[r, "\sigma'"] & (\bar{X}, \bar{\mathcal{F}}) \arrow[r, "\sigma"] & (X, \mathcal{F}) \end{tikzcd}$$

Here:

(1) The morphism σ is the minimal resolution of $(X, \mathcal{F}, \frac{1}{2}B_h)$ as in §4.3. The surface $\bar{Y}$ is the normalization of $\bar{X} \times_X Y$, $\bar{\mathcal{F}}$ is the induced foliation, and the branch divisor $\bar{B}$ can be written as

$$\bar{B} = \bar{B}_v + \bar{B}_h,$$

where $\bar{B}_h = \sigma_*^{-1} B_h$ denotes the strict transform of B_h.

Moreover, σ can be decomposed as

$$\sigma = \sigma_1 \cdots \sigma_n,$$

where each $\sigma_i : (X_i, \mathcal{F}_i, B_{i,h}) \to (X_{i-1}, \mathcal{F}_{i-1}, B_{i-1,h})$ is a sequence of blow-ups over an infinitely near singularity p_i of the pair $(\frac{1}{2}B_{i-1,h}, \mathcal{F}_{i-1})$ (cf. §5.3).

(2) The morphism σ' is a sequence of blow-ups such that each blow-up point of σ satisfies either $q \in \bar{B}_v \cap \bar{B}_h$ or $q \in \bar{B}_v \setminus \bar{B}_h$ with $m_q(\bar{B}_v) = 2$. The pair $(\widetilde{Y}, \widetilde{\mathcal{G}})$ denotes the induced double cover together with its foliation, and its branch divisor can be written as
$$\widetilde{B} = \widetilde{B}_v + \widetilde{B}_h,$$
where $\widetilde{B}_h = \sigma'^{-1}_* \bar{B}_h$ is the strict transform of $\bar{B}_h$.

It is clear that $\widetilde{B}$ satisfies the assumptions of Proposition 5.2, namely, that $\widetilde{B}$ is reduced and smooth, and
$$\operatorname{tang}(\widetilde{\mathcal{F}}, \widetilde{B}_h) = 0.$$

Definition 5.17. We call the composition $\Sigma := \sigma \circ \sigma'$ the *canonical resolution* of the double cover π.

In fact, to compute the Chern numbers of the double foliation $\mathcal{G}$, it suffices to consider the morphism σ. More precisely, we have the following theorem.

Theorem 5.18. *Under the above notations, we have*

(5.12)
$$\begin{cases} c_1^2(\mathcal{G}) = 2c_1^2(\mathcal{F}) + \dfrac{3}{2}K_{\mathcal{F}}B_h + 2N^2 - 2N(B_h)^2 + \displaystyle\sum_{p \in S_{l,m}} T_1(p), \\ c_2(\mathcal{G}) = 2c_2(\mathcal{F}) - 2N^2 + 2N(B_h)^2 - \dfrac{3}{2}s(B_v) + \displaystyle\sum_{p \in S_{l,m}} T_2(p) - \ell(\widetilde{\mathcal{G}}), \\ \chi(\mathcal{G}) = 2\chi(\mathcal{F}) + \dfrac{1}{8}K_{\mathcal{F}}B_h - \dfrac{1}{8}s(B_v) + \displaystyle\sum_{p \in S_{l,m}} \dfrac{1}{12}(T_1(p) + T_2(p)) - \dfrac{1}{12}\ell(\widetilde{\mathcal{G}}), \end{cases}$$

where
$$T_1(p) = (1 - l(p))\frac{3m(p) - 4}{2} + 2\alpha(p),$$
$$T_2(p) = 2(1 - l(p))^2 - 2\alpha(p) + 2s_0(p) - \frac{3}{2}s(p).$$

Proof. Observe that
$$\sum_{p \in S_{l,m}} T_1(p) = \Big[\frac{3}{2}K_{\mathcal{F}}B_h + 2N^2 - 2N(B_h)^2\Big] - \Big[\frac{3}{2}K_{\bar{\mathcal{F}}}\bar{B}_h + 2\bar{N}^2 - 2N(\bar{B}_h)^2\Big],$$
$$\sum_{p \in S_{l,m}} T_2(p) = \Big[-2\bar{N}^2 + 2N(\bar{B}_h)^2 - \frac{3}{2}s(\bar{B}_v) + 2\ell(\mathcal{F}_i)\Big] - \Big[-2N^2 + 2N(B_h)^2 - \frac{3}{2}s(B_v) + 2\ell(\mathcal{F})\Big].$$

Hence it suffices to prove
$$c_1^2(\mathcal{G}) = 2c_1^2(\mathcal{F}) + \frac{3}{2}K_{\bar{\mathcal{F}}}\bar{B}_h + 2\bar{N}^2 - 2N(\bar{B}_h)^2,$$
$$c_2(\mathcal{G}) = 2c_2(\mathcal{F}) - 2\bar{N}^2 + 2N(\bar{B}_h)^2 - \frac{3}{2}s(\bar{B}_v) + 2\ell(\bar{\mathcal{F}}) - \ell(\widetilde{\mathcal{G}}).$$

Let $T_1(\bar{\mathcal{F}})$ (resp. $T_2(\bar{\mathcal{F}})$) denote the right-hand side of the first (resp. second) equation above. We decompose σ' into successive blow-ups σ_i:

$$\begin{tikzcd} (Y_i, \mathcal{F}_i) \arrow[r] \arrow[d, "\pi"'] & (Y_{i-1}, \mathcal{F}_{i-1}) \arrow[d, "\pi_i"] \\ (X_i, \mathcal{F}_i) \arrow[r, "\sigma_i"'] & (X_{i-1}, \mathcal{F}_{i-1}) \end{tikzcd}$$

where q_i = center of $\sigma_i \in \mathrm{Sing}(\bar{B})$ and π_i is the induced double foliation with branch locus $B_i = B'_{i,h} + B_{i,h}$. Clearly, either $q_i \in \bar{B}_v \cap \bar{B}_h$ or $q_i \in \bar{B}_v \setminus \bar{B}_h$ with $m_{q_i}(\bar{B}_v) = 2$. Consider the three possibilities:

(1) $q_i \in \bar{B}_v \cap \bar{B}_h$, then $l(q_i) = 0$ and

$$K_{\mathcal{F}_i} B_{i,h} = K_{\mathcal{F}_{i-1}} B_{i-1,h} + 1, \quad N_i^2 = N_{i-1}^2 - 1, \quad N(B_{i,h})^2 = N(B_{i-1,h})^2 - \frac{1}{4},$$
$$s(B_{i,v}) = s(B_{i-1,v}) + 1, \quad \ell(\mathcal{F}_i) = \ell(\mathcal{F}_{i-1}).$$

(2) $q_i \in \bar{B}_v \setminus \bar{B}_h$, $m_{q_i}(\bar{B}_v) = 2$, $\lambda_p \neq 0$, then $l(q_i) = 1$ and

$$K_{\mathcal{F}_i} B_{i,h} = K_{\mathcal{F}_{i-1}} B_{i-1,h}, \quad N_i^2 = N_{i-1}^2, \quad N(B_{i,h})^2 = N(B_{i-1,h})^2,$$
$$s(B_{i,v}) = s(B_{i-1,v}), \quad \ell(\mathcal{F}_i) = \ell(\mathcal{F}_{i-1}).$$

(3) $q_i \in \bar{B}_v \setminus \bar{B}_h$, $m_{q_i}(\bar{B}_v) = 2$, $\lambda_p = 0$, then $l(q_i) = 1$ and

$$K_{\mathcal{F}_i} B_{i,h} = K_{\mathcal{F}_{i-1}} B_{i-1,h}, \quad N_i^2 = N_{i-1}^2, \quad N(B_{i,h})^2 = N(B_{i-1,h})^2,$$
$$s(B_{i,v}) = s(B_{i-1,v}) + \frac{4}{3}, \quad \ell(\mathcal{F}_i) = \ell(\mathcal{F}_{i-1}) + 1.$$

In all three cases, we have $T_1(\mathcal{F}_i) = T_1(\mathcal{F}_{i-1})$ and $T_2(\mathcal{F}_i) = T_2(\mathcal{F}_{i-1})$. Hence,

$$T_1(\bar{\mathcal{F}}) = T_1(\widetilde{\mathcal{F}}), \quad T_2(\bar{\mathcal{F}}) = T_2(\widetilde{\mathcal{F}}).$$

By Theorem 5.4, $c_1^2(\mathcal{G}) = T_1(\widetilde{\mathcal{F}})$ and $c_2(\mathcal{G}) = T_2(\widetilde{\mathcal{F}})$, which gives

$$c_1^2(\mathcal{G}) = T_1(\bar{\mathcal{F}}), \quad c_2(\mathcal{G}) = T_2(\bar{\mathcal{F}}).$$

Finally, $\chi(\mathcal{G})$ follows from the Noether formula $12\chi(\mathcal{G}) = c_1^2(\mathcal{G}) + c_2(\mathcal{G})$. □

5.5. **Computation of initial invariants.** In this section, we assume $\mathcal{F}$ is reduced with $\nu(\mathcal{F}) \geq 0$ and there is no $(B_h, \mathcal{F})$-exceptional curve of type H-J over X. We will discuss the positivity of the following two invariants:

$$\begin{aligned} T_1(B, \mathcal{F}) &:= 2c_1^2(\mathcal{F}) + \frac{3}{2} K_{\mathcal{F}} B_h + 2N^2 - 2N(B_h)^2, \\ T_2(B, \mathcal{F}) &:= 2c_2(\mathcal{F}) - 2N^2 + 2N(B_h)^2 - \frac{3}{2} s(B_v) + 2\ell(\mathcal{F}). \end{aligned} \tag{5.13}$$

Proposition 5.19. *$T_1(B, \mathcal{F}) \geq 0$.*

Proof. By Corollary 3.10,

$$T_1(B, \mathcal{F}) \geq \frac{3}{2} N B_h + 2N^2 - 2N(B_h)^2 \geq 0.$$

□

Proposition 5.20. *$T_2(B, \mathcal{F}) \geq 0$, if for any saddle-node $q \in B_v \setminus B_h$, $m_q(B_v) = 1$.*

Proof. We write $s(B_v) = s'(B_v) + s''(B_v) + s'''(B_v)$, where

$$s'(B) = \sum_{q \notin N} \beta_q(B_v), \quad s''(B) = \sum_{q \in N \setminus N(B_h)} \beta_q(B_v), \quad s'''(B) = \sum_{q \in N(B_h)} \beta_q(B_v).$$

By assumption and the definition of $\beta_q(B_v)$, we see $\beta_q(B_v) \leq \beta_q(\mathcal{F})$. So

(I) $2c_2(\mathcal{F}) + 2\ell(\mathcal{F}) - \frac{3}{2} s'(B) \geq 2 \sum_{q \notin N} \beta_q(\mathcal{F}) - \frac{3}{2} \sum_{q \notin N} \beta_q(\mathcal{F}) = \sum_{q \notin N} \frac{1}{2} \beta_q(\mathcal{F}) \geq 0,$

and

$$\sum_{q\in N\setminus N(B_h)} 2\beta_q(\mathcal{F}) - \frac{3}{2}s''(B) \geq \sum_{q\in N\setminus N(B_h)} \frac{1}{2}\beta_q(\mathcal{F}) \geq 0. \tag{II}$$

Recall $N(B_h)^2 = \sum_{i=1}^{s}(1-\theta_i)^2 M(\Theta_i)^2$, where $\Theta_1, \cdots, \Theta_s$ are maximal $(\frac{1}{2}B_h, \mathcal{F})$-chains with $\theta_i = B_h\Theta_i/2$. Next we will show

$$2\sum_{q\in\Theta_i}\beta_q(\mathcal{F}) - 2(1-\theta_i)^2\sum_{q\in\Theta_i}\beta_q(\mathcal{F}) - \frac{3}{2}\sum_{q\in\Theta_i}\beta_q(B_v) \geq 0. \tag{III}$$

This is true, just by Lemma 5.8:

$$\sum_{q\in\Theta_i}\beta_q(B_v)\begin{cases} = 0, & \text{if } \theta_i = 0;\\ \leq \sum\limits_{q\in\Theta_i}\beta_q(\mathcal{F}), & \text{if } \theta_i = \frac{1}{2}.\end{cases}.$$

Now from (I), (II) and (III), $T_2(B, \mathcal{F}) \geq 0$ is clear. □

Proposition 5.21. *If $2c_1^2(\mathcal{F}) \geq c_2(\mathcal{F})$, $\ell(\mathcal{F}) = 0$ and $B_h \cap N(B_h) \cap \mathrm{Sing}\mathcal{F} = \emptyset$, then $2T_1(B, \mathcal{F}) \geq T_2(B, \mathcal{F})$.*

Proof. In this case,

$$\begin{aligned}
&2T_1(B,\mathcal{F}) - T_2(B,\mathcal{F})\\
=&(2c_1^2(\mathcal{F}) - c_2(\mathcal{F})) + 3K_{\mathcal{F}}B_h + 6N^2 - 6N(B_h)^2 + \frac{3}{2}s(B)\\
\geq&3NB_h + 6N^2 - 6N(B_h)^2 + \frac{3}{2}\cdot 2N(B_h)B_h, \quad \text{(Proposition 5.10)}\\
\geq&3\left[(N + N(B_h))B_h + 2N^2 - 2N(B_h)^2\right]\\
\geq&0 \quad \text{(Corollary 3.9)}.
\end{aligned}$$

□

5.6. **Computation of local invariants.** In this section, we assume $\mathcal{F}$ is reduced and there is no $(B_h, \mathcal{F})$-exceptional curve of type H-J over X. We will compute the contribution of the $S_{l,m}$-singularity $p \in B_h$ to $\alpha(p), s_0(p), s(p), T_1(p), T_2(p)$, where

$$\begin{aligned}
T_1(p) &= (1 - l(p))\frac{3m(p) - 4}{2} + 2\alpha(p),\\
T_2(p) &= 2(1 - l(p))^2 - 2\alpha(p) + 2s_0(p) - \frac{3}{2}s(p).
\end{aligned} \tag{5.14}$$

Definition 5.22. We define

$$\Delta(t_p) := \mathrm{tang}(\mathcal{F}, B_h) - \mathrm{tang}(\mathcal{F}', B_h') = m_1(m_1 - 1 + l(p)) + \sum_{i=2}^{s} m_i^2. \tag{5.15}$$

It is clear that $\Delta(t_p) \leq t_p$. If $\Delta(t_p) = t_p$, then there is no more $S_{l,m}$-singularity after p. So the blow-up process of $S_{l,m}$-singularities will continue until the equation $\Delta(t_p) = t_p$ holds.

Lemma 5.23. *If $p \in S_{l,m}$ and p is not a saddle-node, then $2T_1(p) \geq T_2(p)$.*

Proof. In this case, $s_0(p)=0$. So

$$\begin{aligned}2T_1(p)-T_2(p) &= 2\left((1-l(p))\frac{3m-4}{2}+2\alpha(p)\right)-\left(2(1-l(p))^2-2\alpha(p)-\frac{3}{2}s(p)\right)\\ &= (1-l(p))(3m-6+2l(p))+6\alpha(p)+\frac{3}{2}s(p).\end{aligned}$$

If $l(p)=1$ or $l(p)=0$ and $m\geq 2$, $2T_1(p)\geq T_2(p)$ is clear. Next we assume $l(p)=0$ and $m=1$. By Lemma 5.27,

$$T_1(p)=\frac{3t_p-1}{2(t_p+1)},\quad T_2(p)=\frac{3\left(t_p-2\left[\frac{t_p}{2}\right]\right)+1}{2(t_p+1)}.$$

So

$$2T_1(p)-T_2(p)=\frac{3(t_p-1)+6\left[\frac{t_p}{2}\right]}{2(t_p+1)}\geq 0,$$

where $t_p\geq 1$. Then we are done. □

5.6.1. *The case $S_{1,m}$.*

Proposition 5.24. *Suppose $p\in S_{1,m}$, Then $l(p)=1$ and $0\leq\alpha(p)<1$. Hence*

$$T_1(p)=2\alpha(p),\quad T_2(p)=-2\alpha(p)+2s_0(p)-\frac{3}{2}s(p). \tag{5.16}$$

Here

(1) *$T_1(p)\geq 0$ and $T_1(p)=0$ unless p belongs to (e1) or (e2) or (e3).*
(2) *If $\lambda_p=0$, $T_2(p)\geq\frac{1}{2}s_0(p)\geq\frac{1}{2}$, and if $\lambda_p\neq 0$, $0\geq T_2(p)>-\frac{7}{2}\beta_p(\mathcal{F})\geq-\frac{7}{2}$.*

Proof. Clear. □

5.6.2. *The case $S_{0,m}$.*

Proposition 5.25. *Suppose $p\in S_{0,m}$. Then $l(p)=0$, $0\leq\alpha(p)<1$ and*

$$T_1(p)=\frac{3}{2}m(p)-2+2\alpha(p)\geq 0,\quad T_2(p)=2-2\alpha(p)-\frac{3}{2}s(p)\geq 0. \tag{5.17}$$

Proof. The proof of $T_2(p)\geq 0$ is similar to the proof of $T_2(B,\mathcal{F})\geq 0$, see Proposition 5.20. Next we will show $T_1(p)>0$. If $m(p)\geq 2$, then

$$T_1(p)\geq\frac{3}{2}\cdot 2-2=1>0.$$

If $m=1$, by Lemma 5.27,

$$T_1(p)=\frac{3t_p-1}{2(t_p+1)}>0,$$

where $t_p\geq 1$. So we are done. □

Definition 5.26. Suppose $p\in B_h$ is a regular point of $\mathcal{F}$, i.e., $l(p)=0$. Let F denote the separatrix through p. If F (resp. B_h) is locally defined by $g=0$ (resp. $f=0$), then we define

$$\eta_p:=I_p<f,g>.$$

Note that the definition of η_p does depend on the choice of f and g.

Lemma 5.27 ($S_{0,1}$). *Suppose $p \in S_{0,1}$. Then $\Delta(t_p) = t_p \geq 1$, $r = \eta_p = t_p + 1$ and*

$$\beta_p^- = \frac{t_p}{t_p+1}, \quad \theta_p = 0, \quad s(p) = \frac{2\left[\frac{t_p}{2}\right] - t_p + 1}{t_p + 1}.$$

Moreover, $\alpha(p) = \dfrac{t_p}{t_p+1}$ *and*

$$T_1(p) = \frac{3t_p - 1}{2(t_p+1)}, \quad T_2(p) = \frac{3\left(t_p - 2\left[\frac{t_p}{2}\right]\right) + 1}{2(t_p+1)}.$$

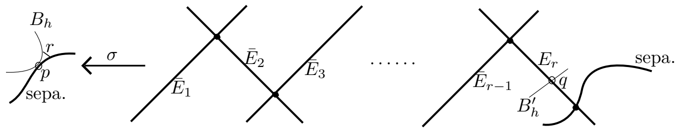

FIGURE 2. $p \in S_m^0$, $m = 1$, $E_r^2 = -1, \bar{E}_i^2 = -2(i < r)$.

Proof. In this case, $r = t_p + 1$, $q_1 = p$ and $q_i = E_{i-1} \cap B_{i-1,h}$ for $i = 2, \cdots, r$, see Figure 2. It is clear that $\Theta_p = \bar{E}_1 + \cdots + \bar{E}_{r-1}$ is a maxiaml $(\frac{1}{2}B_h, \mathcal{F}')$-chain with $\theta_p := \frac{1}{2}\Theta_p B'_h = 0$. So

$$\beta_p^- = \sum_{i=1}^{r-1} \beta(-\lambda_{p^i}) = \sum_{i=1}^{r-1} \frac{1}{i \cdot (i+1)} = 1 - \frac{1}{r} = \frac{r-1}{r} = \frac{t_p}{t_p+1},$$

where $p^i = \bar{E}_i \cap \bar{E}_{i+1}$ for $i = 1, \cdots, r-1$. Let p^r be another singularity of $\mathcal{F}'$ over E_r. Next we compute $s(p)$.

By Lemma 5.8, $\beta_{p^i}(B'_v) = 0$ for $i = 1, \cdots, r-1$. So it suffices to compute $\beta_{p^r}(B'_v)$.

i) If $p \in B_v$, then $E_r \not\subset B'_v$. So $p^r \in B'_v$ with $m_{p^r}(B'_v) = 1$ and
$$\mathrm{CS}(\mathcal{F}', B'_v, p^r) = -r.$$
Thus $\beta_{p^r}(B'_v) = 0$ if r is even and $\beta_{p^r}(B'_v) = \frac{1}{r}$ if r is odd.

ii) If $p \notin B_v$ and r is even, then $E_r \not\subset B'_v$, which implies $p^r \notin B'_v$. So $\beta_{p^r}(B'_v) = 0$.

iii) If $p \notin B_v$ and r is odd, then $E_r \subset B'_v$, which implies $p^r \in B'_v$ with $m_{p^r}(B'_v) = 1$ and
$$\mathrm{CS}(\mathcal{F}', B'_v, p^r) = -\frac{1}{r}.$$
So $\beta_{p^r}(B'_v) = \frac{1}{r}$.

Therefore,

$$s(p) = \beta_{p^r}(B'_v) = \left(2\left[\frac{r-1}{2}\right] + 1 - (r-1)\right)\frac{1}{r} = \frac{2\left[\frac{t_p}{2}\right] - t_p + 1}{t_p+1}.$$

$\square$

For any $p \in S_{0,m}^*$, it is clear that $m \geq 2$. Next we consider a special case $S_{0,m}^{**} := \{p \in S_{0,m}^* \mid m(p) = \eta_p\}$.

Lemma 5.28 ($S_{0,m}^{**}$). *Suppose $p \in S_{0,m}^{**}$. Then $\Delta(t_p) = m(m-1)$, $r = 1$ and*

$$\beta_p^- = 0, \quad s(p) = m - 2\left[\frac{m}{2}\right].$$

Moreover, $\alpha(p) = 0$ and

$$T_1(p) = \frac{3}{2}m - 2, \quad T_2(p) = 2 - \frac{3}{2}m + 3\left[\frac{m}{2}\right].$$

Proof. $r = 1$ and $\beta_p^- = 0$ are clear. Next we compute $s(p)$.

(i) Suppose m is even. If $p \in B_v$, then $m_p(B) = m+1$ is odd and $E_1 \subset B_v'$. So $m_q(B_v') = 2$ and $\lambda_q = -1$ for $q = E_1 \cap \mathrm{Sing}\mathcal{F}$, which implies $\beta_q(B_v') = 0$. If $p \notin B_v$, then $E_1 \not\subset B_v'$. So $q \notin B_v'$ and $\beta_q(B_v') = 0$. Thus in this case that m is even, $s(p) = \beta_q(B_v') = 0$.

(ii) Suppose m is odd. If $p \in B_v$ (resp. $p \notin B_v$), then $E_1 \not\subset B_v'$ (resp. $E_1 \subset B_v'$). They both imply $m_q(B_v') = 1$ and $\lambda_q = -1$. So $\beta_q(B_v') = 1$. Thus in this case that m is odd, $s(p) = \beta_q(B_v') = 1$.

□

5.7. **Slope inequality.** Using the notations in Section 5.4, by blowing up all $S_{1,m}^I$-singularities, it suffices to assume $(X, \mathcal{F})$ is redcued satisfying

- there is no $(B_h, \mathcal{F})$-exceptional curves of type H-J,
- there is no $S_{1,m}^I$-singularities,
- $\ell(\mathcal{F}) = 0$.

So $N(B_h) \cap B_h \cap \mathrm{Sing}\mathcal{F} = \emptyset$. We set

$$F_\lambda(\cdot) := \frac{12-\lambda}{12}T_1(\cdot) - \frac{\lambda}{12}T_2(\cdot).$$

By Proposition 5.21 and Lemma 5.23, we have the following claims.

Lemma 5.29. *Let λ be a positive rational number with $\lambda \le 4$.*

(i) *If $c_1^2(\mathcal{F}) \ge \lambda\chi(\mathcal{F})$, then*

$$F_\lambda(B, \mathcal{F}) := \frac{12-\lambda}{12}T_1(B, \mathcal{F}) - \frac{\lambda}{12}T_2(B, \mathcal{F}) \ge 0.$$

(ii) *For any $p \in S_{l,m}$, if p is not a saddle-node, then*

$$F_\lambda(p) := \frac{12-\lambda}{12}T_1(p) - \frac{\lambda}{12}T_2(p) \ge 0.$$

Thus

$$\begin{aligned} c_1^2(\mathcal{G}) - \lambda\chi(\mathcal{G}) &= 2\left(c_1^2(\mathcal{F}) - \lambda\chi(\mathcal{F})\right) + \frac{12-\lambda}{8}K_{\mathcal{F}}B_h + 2N^2 - 2N(B_h)^2 + \frac{1}{8}s(B_v) \\ &\quad + \sum_{p \in S_{l,m}} \frac{(12-\lambda)T_1(p) - \lambda T_2(p)}{12} + \frac{\lambda}{12}\ell(\widetilde{\mathcal{G}}) \\ &= F_\lambda(B, \mathcal{F}) + \sum_{p \in S_{l,m}} F_\lambda(p) + \frac{\lambda}{12}\ell(\widetilde{\mathcal{G}}). \end{aligned}$$

Theorem 5.30. *Under the notations above. If $c_1^2(\mathcal{G}) \ge 4\chi(\mathcal{G})$ and the branch locus B of π misses the saddle-nodes, then $c_1^2(\mathcal{G}) \ge 4\chi(\mathcal{G})$.*

Proof. Under the assumption above, by Lemma 5.29,

$$F_4(B,\mathcal{F}) \geq 0, \quad F_4(p) \geq 0, \text{ for any } p \in S_{l,m}.$$

So

$$c_1^2(\mathcal{G}) - 4\chi(\mathcal{G}) = F_4(B,\mathcal{F}) + \sum_{p\in S_{l,m}} F_4(p) + \frac{1}{3}\ell(\widetilde{\mathcal{G}}) \geq 0.$$

□

Finally, we consider the case that $(X,\mathcal{F})$ is a relatively minimal elliptic fibration $f : X \longrightarrow C$. In this case, $c_1^2(\mathcal{F}) = 0$ but $\chi(\mathcal{F}) \geq 0$, where $\chi(\mathcal{F}) = 0$ iff f is isotrivial. Since $c_1^2(\mathcal{F}) = \kappa(f)$ and $\chi(\mathcal{F}) = \lambda(f)$ are modular invariants of f, it suffices to assume f is semi-stable. So by ([Bru15], p.22), we have

$$K_{\mathcal{F}} = f^*\big[(f_{*1}\mathcal{O}_X)^\vee\big] = K_f$$

where $\deg(f_{*1}\mathcal{O}_X)^\vee = \chi_f = \chi(\mathcal{F})$. So $K_{\mathcal{F}} \cdot B_h = \chi_f \cdot (B_h \cdot F)$, where F is the general fibre of f. Note that, in this case, $\ell(\mathcal{F}) = \ell(\mathcal{G}) = 0$ and $K_{\mathcal{F}} = K_f$ is nef, which implies $N = N(B_h) = 0$. So by Lemma 5.29 or Lemma 5.23,

$$c_1^2(\mathcal{G}) - 4\chi(\mathcal{G}) = -8\chi(\mathcal{F}) + K_{\mathcal{F}}B_h + \sum_{p\in S_{l,m}} \frac{2T_1(p) - T_2(p)}{3} \geq -8\chi(\mathcal{F}) + K_{\mathcal{F}}B_h.$$

In fact, $(Y,\mathcal{G})$ is a fibration f' of genus g, where $g = g(F')$ for the general fibre F' of f'. (We always call $(Y,\mathcal{G})$ is a *double elliptic fibration.*) Consider the fibers of the two fibrations, and we get a double cover of an elliptic curve with the ramification divisor B. Here we can easily see $\deg B = B_h \cdot F$. So by the Hurwitz's Theorem, we have

$$B_h \cdot F = \deg B = 2g - 2 - 2\cdot(2\cdot 1 - 2) = 2g - 2.$$

So

$$K_{\mathcal{F}}B_h = \chi_f(2g-2).$$

Since f is semi-stable, which implies $\chi(\mathcal{G}) = \chi_f$, we have

$$c_1^2(\mathcal{G}) - 4\chi(\mathcal{G}) \geq -8\chi(\mathcal{F}) + (2g-2)\chi(\mathcal{F}) = (2g-10)\chi(\mathcal{F}).$$

In particular, if $g \geq 5$, then $c_1^2(\mathcal{G}) - 4\chi(\mathcal{G}) \geq 0$.

Proposition 5.31. *If $(Y,\mathcal{G})$ is a double elliptic fibration with $g \geq 5$, then*

$$c_1^2(\mathcal{G}) \geq 4\chi(\mathcal{G}).$$

Note that this result above have proved in [Bar01], where the author considered the slope of bielliptic fibrations, in the sense of relative invariants.

6. Example of foliations with slope $\frac{12}{7}$

Let $X = \mathbb{P}^1 \times \mathbb{P}^1$, let $f : X \to \mathbb{P}^1$ be one of the rulings with a fiber F_0, and let C_0 be a section. Choose a proper coordinate (x,y) nearby $p = (0,0) \in C_0 \cap F_0$ such that C_0 (resp., F_0) is defined by $y = 0$ (resp., $x = 0$).

Example 6.1. Let $\mathcal{F}$ be a foliation on X locally generated by

$$\omega = x^2\mathrm{d}y - y\mathrm{d}x.$$

Let $\pi : (Y,\mathcal{G}) \to (X,\mathcal{F})$ be the double cover locally defined by

$$z^2 = y\big(y + x^{2k}(1+y^2)\big), \quad (k \geq 1).$$

Then we have

$$c_1^2(\mathcal{G}) = 2k, \quad c_2(\mathcal{G}) = 12k, \quad \chi(\mathcal{G}) = \frac{7k}{6}, \quad \lambda(\mathcal{G}) = \frac{12}{7}.$$

Appendix A. Classification of $S_{0,m}$-singularities for $m \leq 3$

A.1. **The case $S_{0,1}$.** We denote by A_0^η the set of points $p \in S_{0,1}$ with $\eta_p = \eta$. By Lemma 5.27, we see $\eta \geq 2$ and we have the following table.

Table 1. $S_{0,1}$.

	$\Delta(t_p)$	$\alpha(p)$	$s(p)$	$T_1(p)$	$T_2(p)$	$t_p = \Delta(t_p)$?
A_0^η	$\eta - 1$	$\frac{\eta-1}{\eta}$	$\frac{2\{\eta/2\}}{\eta}$	$\frac{3\eta-4}{2\eta}$	$\frac{2-3\{\eta/2\}}{\eta}$	Yes

Here we set $\{x\} := x - [x]$.

Remark A.1. It is clear that $\Delta(t_p) \leq t_p$. That the equation $\Delta(t_p) = t_p$ holds means there is no more $S_{l,m}$-singularities after p.

A.2. **The case $S_{0,2}$.** In this case, we divide it into the following 4 cases:

A_1^η. p is a node of B_h with $\eta_p = \eta$.
$A_n^{\eta,I}$. p is a singularity of B_h of type $A_n (n \geq 2)$ with $\eta = \eta_p \leq n-1$.
A_{2k}^{II}. p is a singularity of B_h of type $A_{2k} (k \geq 1)$ with $\eta = 2k+1$.
$A_{2k-1}^{\eta,II}$. p is a singularity of B_h of type $A_{2k-1} (k \geq 2)$ with $\eta \leq 2k$.

Table 2. $S_{0,2}$.

		$\Delta(t_p)$	$\alpha(p)$	$s(p)$	$T_1(p)$	$T_2(p)$	$t_p = \Delta(t_p)$?
A_1^η		η	$\frac{\eta-2}{4\eta-4}$	$\frac{\eta-2+2\{\eta/2\}}{\eta-1}$	$3-\frac{4}{\eta}$	$\frac{2-3\{\eta/2\}}{\eta-1}$	Yes
$A_n^{\eta,I}$		$2\eta-2$	$1-\frac{2}{\eta}$	0	$3-\frac{4}{\eta}$	$\frac{4}{\eta}$	No
A_{2k}^{II}		$4k$	$\frac{2k-1}{2k+1}$	$\frac{2}{2k+1}$	$\frac{6k-1}{2k+1}$	$\frac{1}{2k+1}$	Yes
$A_{4k-1}^{\eta,II}$		$\eta+4k-2$	$\frac{2k-1}{2k}$	$\frac{2\{\eta/2\}}{\eta-2k}$	$\frac{3k-1}{k}$	$\frac{1}{k}-\frac{3\{\eta/2\}}{\eta-2k}$	
$A_{4k+1}^{\eta,II}$	$p \notin B_v$	$\eta+4k$	$\frac{2k}{2k+1}$	$\frac{1}{2k+1}-\frac{1-2\{\eta/2\}}{\eta-2k-1}$	$\frac{6k+1}{2k+1}$	$\frac{1}{4k+2}+\frac{3-6\{\eta/2\}}{2(\eta-2k-1)}$	
	$p \in B_v$			$\frac{1}{2k+1}$		$\frac{1}{4k+2}$	

Proposition A.2. *Suppose $p \in A_n^{\eta,I}$. Then $\eta = 2r$ for some $r \geq 2$.*

(i) *If $n = 2k-1$, then after p, there are $k-r$ $S_{0,2}^{**}$-singularities.*
(ii) *If $n = 2k$, then after p, there are $k-r-1$ $S_{0,2}^{**}$-singularities and one A_0^2-singularity.*

A.3. **The case $S_{0,3}$.** In this case, we divide it into the following cases:

(1) p is a singularity of B_h of type $D_n (n \geq 4)$.

$D_n^{\eta,I}$. the separatrix through p is not tangent to the component of B_h of type A_{n-3} at p.

$D_n^{\eta,II}$. the separatrix through p is tangent to the component of B_h of type A_{n-3} at p with $\eta \leq n-3$.

D_{2k+3}^{III}. the separatrix through p is tangent to the component of B_h of type A_{n-3} at p with $\eta = 2k+2$.

$D_{2k+2}^{\eta,III}$. the separatrix through p is tangent to the component of B_h of type A_{n-3} at p with $\eta \geq 2k+1$.

(2) p is a singularity of B_h of type E_6.

E_6^I. The separatrix through p is not tangent to B_h at p with $\eta = 3$.

E_6^{II}. The separatrix through p is tangent to B_h at p with $\eta = 4$.

(3) p is a singularity of B_h of type E_7.

E_7^I. The separatrix through p is not tangent to B_h at p with $\eta = 3$.

$E_7^{\eta,II}$. The separatrix through p is tangent to B_h at p with $\eta \geq 5$.

(4) p is a singularity of B_h of type E_8.

E_8^I. The separatrix through p is not tangent to B_h at p with $\eta = 3$.

E_8^{II}. The separatrix through p is tangent to B_h at p with $\eta = 5$.

TABLE 3. $S_{0,3}$.

	$\Delta(t_p)$	$\alpha(p)$	$s(p)$	$T_1(p)$	$T_2(p)$	$t_p = \Delta(t_p)$?
$D_n^{\eta,I}$	$\eta+3$	0	$\frac{2\{\eta/2\}}{\eta-2}$	$\frac{5}{2}$	$2-\frac{3\{\eta/2\}}{\eta-2}$	No
$D_n^{\eta,II}$	2η	$\frac{\eta-3}{4\eta-4}$	1	$\frac{3\eta-4}{\eta-1}$	$\frac{1}{\eta-1}$	No
D_{2k+3}^{III}	$4k+4$	$\frac{2k-1}{4(2k+1)}$	$\frac{2k-1}{2k+1}$	$\frac{6k+2}{2k+1}$	$\frac{4}{2k+1}$	Yes
$D_{4k+2}^{\eta,III}$	$\eta+4k+1$	$\frac{2k-1}{8k}$	$1-\frac{1-2\{\eta/2\}}{\eta-2k-1}$	$3-\frac{1}{4k}$	$\frac{1}{4k}+\frac{3(1-2\{\eta/2\})}{2(\eta-2k-1)}$	Yes
$D_{4k}^{\eta,III}$	$\eta+4k-1$	$\frac{k-1}{2(2k-1)}$	$\frac{2k-2}{2k-1}+\frac{2\{\eta/2\}}{\eta-2k}$	$\frac{6k+2}{2k+1}$	$\frac{2}{2k-1}-\frac{3\{\eta/2\}}{\eta-2k}$	Yes
E_6^I, E_7^I, E_8^I	6	0	1	$\frac{5}{2}$	$\frac{1}{2}$	No
E_6^{II}	9	$\frac{1}{4}$	0	3	$\frac{3}{2}$	Yes
$E_7^{\eta,II}$	$\eta+6$	$\frac{1}{3}$	$\frac{2}{3}-\frac{1-2\{\eta/2\}}{\eta-1}$	$\frac{19}{6}$	$\frac{1}{3}+\frac{3(1-2\{\eta/2\})}{2(\eta-1)}$	Yes
E_8^{II}	12	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{33}{10}$	$\frac{3}{10}$	Yes

Proposition A.3. *Suppose $p \in D_n^{\eta,I}$.*

(i) *If $n = 2k+2$, then after p, there are $k-1$ $S_{0,2}^{**}$-singularities.*

(ii) *If* $n = 2k+3$, *then after* p, *there are* $k-1$ $S^{**}_{0,2}$*-singularities and one* A^2_0*-singularity.*

Proposition A.4. *Suppose* $p \in D^{\eta,II}_n$. *Then* $\eta = 2r+1 \leq n-3$.

(i) *If* $n = 2k+2$, *then after* p, *there are* $k-r$ $S^{**}_{0,2}$*-singularities.*

(ii) *If* $n = 2k+3$, *then after* p, *there are* $k-r$ $S^{**}_{0,2}$*-singularities and one* A^2_0*-singularity.*

Proposition A.5. *For* $p \in E^I_i (i = 6, 7, 8)$, p *is just one* $S^{**}_{0,3}$*-singularity. If* $p \in E^I_6$ *(resp.* E^I_7, E^I_8*), then after* p, *there is a singularity of type* A^3_0 *(resp.* A^3_1, A^{II}_2*).*

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