1 Base model

Variables
x_{id}: [binary] equal to 1 if the traveller is in city i on day d.
y_i: [binary] equal to 1 if the traveller ever visits city i.
m_{ijd}: [binary] equal to 1 if the traveller moves from city i to city j on day d.

Parameters
D: the total number of days.
$\mathcal{C}$: the set of cities available for visiting.
μ_{ij}: the cost to move from city i to city j.
δ_i: the daily cost of city i.
α: the minimum number of days allowed in a city.
ω: the maximum number of days allowed in a city.

$$\text{Min} \sum_{d=1}^{D}\sum_{i\in\mathcal{C}}\delta_i \cdot x_{id} + \sum_{d=1}^{D-1}\sum_{i\in\mathcal{C}}\sum_{j\in\mathcal{C}}\mu_{ij} \cdot m_{ijd}$$

$$\text{s.t.} \quad \sum_{d=1}^{D} x_{id} \geq \alpha \cdot y_i \qquad \forall i \in \mathcal{C} \tag{1}$$

$$\sum_{d=1}^{D} x_{id} \leq \omega \cdot y_i \qquad \forall i \in \mathcal{C} \tag{2}$$

$$x_{id} + x_{j(d+1)} \leq 1 + m_{ijd} \qquad \forall i,j \in \mathcal{C}, d = 1..D-1 \tag{3}$$

$$\sum_{i\in\mathcal{C}} x_{id} = 1 \qquad d = 1..D \tag{4}$$

$$\sum_{i\in\mathcal{C}}\sum_{j\in\mathcal{C}} m_{ijd} = 1 \qquad d = 1..D-1 \tag{5}$$

$$\sum_{d=1}^{D}\sum_{j\in\mathcal{C}} m_{ijd} \leq 1 + \sum_{d=1}^{D} m_{iid} \qquad \forall i \in \mathcal{C} \tag{6}$$

$$x_{j1} + \sum_{d=1}^{D}\sum_{i\in\mathcal{C}} m_{ijd} \leq 1 + \sum_{d=1}^{D} m_{jjd} \qquad \forall j \in \mathcal{C} \tag{7}$$

$$x_{id}, y_i, m_{ijd} \in \{0,1\} \qquad \forall i,j \in \mathcal{C}, d = 1..D \tag{8}$$

$$\tag{9}$$

2 Decaying Enjoyment

Assuming that the enjoyment of staying in a city decays with the number of days stayed. We add the following to the model.

Variables
s_{ij}: [binary] equal to 1 if traveller stays in city i for j days.

Parameters
u_{ij}: The total utility of staying in city i for j days.

Then we no longer need (1) and (2) since we hope that the traveller naturally leaves the city once the marginal enjoyment is low enough.

Then the new model is

$$\text{Max} \sum_{i \in \mathcal{C}} \sum_{j=1}^{D} s_{ij} * u_{ij}$$

$$\text{s.t.} \quad \sum_{j=1}^{D} s_{ij} = y_i \qquad \forall i \in \mathcal{C} \qquad (1)$$

$$s_{ij} \cdot j \leq \sum_{d=1}^{D} x_{id} \qquad \forall i \in \mathcal{C}, j = 1...D \qquad (2)$$

$$x_{id} + x_{j(d+1)} \leq 1 + m_{ijd} \qquad \forall i, j \in \mathcal{C}, d = 1..D-1 \qquad (3)$$

$$\sum_{i \in \mathcal{C}} x_{id} = 1 \qquad d = 1..D \qquad (4)$$

$$\sum_{i \in \mathcal{C}} \sum_{j \in \mathcal{C}} m_{ijd} = 1 \qquad d = 1..D-1 \qquad (5)$$

$$\sum_{d=1}^{D} \sum_{j \in \mathcal{C}} m_{ijd} \leq 1 + \sum_{d=1}^{D} m_{iid} \qquad \forall i \in \mathcal{C} \qquad (6)$$

$$x_{j1} + \sum_{d=1}^{D} \sum_{i \in \mathcal{C}} m_{ijd} \leq 1 + \sum_{d=1}^{D} m_{jjd} \qquad \forall j \in \mathcal{C} \qquad (7)$$

$$x_{id}, y_i, m_{ijd} \in \{0, 1\} \qquad \forall i, j \in \mathcal{C}, d = 1..D \qquad (8)$$

$$(9)$$

With this we find the optimal enjoyment possible for the traveller irrespective of cost, then we optimise again to minimise cost at the optimal enjoyment.

$$\text{Min} \sum_{d=1}^{D} \sum_{i \in \mathcal{C}} \delta_i \cdot x_{id} + \sum_{d=1}^{D-1} \sum_{i \in \mathcal{C}} \sum_{j \in \mathcal{C}} \mu_{ij} \cdot m_{ijd}$$

s.t.

(1)-(9) from previous model is statisfied and

$$\sum_{i \in \mathcal{C}} \sum_{j=1}^{D} s_{ij} * u_{ij} = (\sum_{i \in \mathcal{C}} \sum_{j=1}^{D} s_{ij} * u_{ij})_{optimal}$$

3 Results

We run the model using a 10% per day rate of decay with the contraint that we stay at least 2 days in any city that we visit. Below is the trip that maximises the utility for 15 days user such conditions. It's noted that this model as it was formulated takes a very long time to run with computations not terminating after 3+ hours, however a trip of 10 days takes only seconds to run and we note that the utility of a trip with more days can only be higher if budget is not constrained.

Also note that since the utility scores are normalised in a way that the maximum is 100, we can also obtain an upper bound to use in the model by adding $100 \cdot (\text{extra days})$ to the optimal solution of the shorter trip. In theory this brings us closer to the convex hull which should aid the model, but in practice the solving times with such constraints were sometimes detrimental. Though tweaking around the theoretical bounds did eventually produce results in reasonable time, there was no observed pattern in how to construct the highly beneficial bounds.

Table 1: Trip plan with optimised decaying utility for 15 days

Days	# of Days	City
1-2	2	Rome
3-4	2	Barcelona
5-6	2	Venice
7-10	3	London
11-12	2	Berlin
13-15	2	Paris

This trip has a total utility of 1219 and cost of \$5195, for comparison we develop a simple heuristic from observations of the behaviour of the MIP model.

Algorithm 1 Decaying utility algorithm

1: Algorithm MaximumUtilityHeuristic
2: **while** DaysLeft > 0 **do**
3:    Find highest utility $City$
4:    **if** $Days_{City} > 0$ **then**
5:      $Days_{City} \leftarrow Days_{City} + 1$
6:      $Utility_{City} \leftarrow Utility_{City} \cdot DecayFactor$
7:      $DaysLeft \leftarrow DaysLeft - 1$
8:    **else**
9:      $Days_{City} \leftarrow Days_{City} + 2$
10:     $Utility_{City} \leftarrow Utility_{City} \cdot DecayFactor^2$
11:     $DaysLeft \leftarrow DaysLeft - 2$
12:    **end if**
13: **end while**
14: $CitiesList \leftarrow \{City : Days_{City} > 0\}$
15: $Trip_1 \leftarrow City$ in $CitiesList$ with lowest cost from Melbourne
16: $CurrentCity \leftarrow Trip_1$
17: **Remove** $CurrentCity$ from $CitiesList$
18: $i \leftarrow 1$
19: **while** $CitiesList$ not empty **do**
20:    $Trip_{i+1} \leftarrow City$ in $CitiesList$ with cheapest flight from $CurrentCity$
21:    $CurrentCity \leftarrow Trip_{i+1}$
22:    **Remove** $CurrentCity$ from $CitiesList$
23:    $i \leftarrow i + 1$
24: **end while**
25: $Trip$ now stores order in which cities are visited and $Days$ stores the days spent in each city.

This is essentially a greedy algorithm which looks for the city with the highest current utility and adds 2 days to its staying duration if it's not already a part of the trip and 1 if it's already being visited. Then we note that flying as little as possible will reduce cost so all days spent in a given city will be spend together, therefore a greedy algorithm is used to find the order in which to visit the cities. Now with a decay factor of 0.9 we obtain the following trip from the heuristic.

Table 2: Trip plan from heuristic for decaying utility for 15 days

Days	# of Days	City
1-3	3	London
4-5	2	Paris
6-7	2	Berlin
8-9	2	Bercelona
10-11	2	Venice
12-13	2	Rome
14-15	2	Madrid

This trip gives an utility of 1096 and total cost fo $5101. Therefore it ends up being slightly cheaper than the solution found by the model but also achieves lower utility. We look at the breakdown of costs.

Table 3: Comparison between MIP and Heuristic solutions

	MIP	Heuristic
Melbourne Travel	$1926	$1928
Europe Travel	$290	$419
Daily Costs	$2979	$2754
Total	$5195	$5101
Utility	1219	1096

Then we see that the heuristic performs reasonably well and the derived utility can be used as a lower bound in the MIP optimisation. We set a budget constraint of $4500 and obtain the following solution with a utility of 1096 at a total cost of $4465.

Table 4: Trip plan from heuristic for decaying utility for 15 days

Days	# of Days	City
1-2	2	Barcelona
3-4	2	Vienna
5-8	4	Berlin
9-11	3	Paris
12-15	4	Amsterdam

Now coincidentally this gives the same utility as the heuristic solution but for a significantly lower cost, this gives a good indication of the strength of the model over the naive heuristic method.