MA 1101 : Mathematics I

Problem 1.

(i) We prove $A \cup B = B \cup A$ by first showing that $A \cup B \subseteq B \cup A$, then showing that $B \cup A \subseteq A \cup B$. Consider

$$\begin{aligned} x \in A \cup B &\implies (x \in A) \text{ or } (x \in B) \\ &\implies (x \in B) \text{ or } (x \in A) \\ &\implies x \in B \cup A, \end{aligned}$$

which proves that $A \cup B \subseteq B \cup A$. Similarly, consider

$$\begin{aligned} x \in B \cup A &\implies (x \in B) \text{ or } (x \in A) \\ &\implies (x \in A) \text{ or } (x \in B) \\ &\implies x \in A \cup B. \end{aligned}$$

This proves that $B \cup A \subseteq A \cup B$. Therefore, $A \cup B = B \cup A$.

(ii) We prove $(A \cup B) \cup C = A \cup (B \cup C)$ by first showing that $(A \cup B) \cup C \subseteq A \cup (B \cup C)$, then showing that $A \cup (B \cup C) \subseteq (A \cup B) \cup C$.

Consider

$$\begin{aligned} x \in (A \cup B) \cup C &\implies (x \in A \cup B) \text{ or } (x \in C) \\ &\implies ((x \in A) \text{ or } (x \in B)) \text{ or } (x \in C) \\ &\implies (x \in A) \text{ or } (x \in B) \text{ or } (x \in C) \\ &\implies (x \in A) \text{ or } ((x \in B) \text{ or } (x \in C)) \\ &\implies (x \in A) \text{ or } (x \in B \cup C) \\ &\implies x \in A \cup (B \cup C), \end{aligned}$$

which shows $(A \cup B) \cup C \subseteq A \cup (B \cup C)$. Similarly, consider

$$\begin{aligned} x \in A \cup (B \cup C) &\implies (x \in A) \text{ or } (x \in B \cup C) \\ &\implies (x \in A) \text{ or } ((x \in B) \text{ or } (x \in C)) \\ &\implies (x \in A) \text{ or } (x \in B) \text{ or } (x \in C) \\ &\implies ((x \in A) \text{ or } (x \in B)) \text{ or } (x \in C) \\ &\implies (x \in A \cup B) \text{ or } (x \in C) \\ &\implies x \in (A \cup B) \cup C. \end{aligned}$$

This proves that $A \cup (B \cup C) \subseteq (A \cup B) \cup C$. Therefore, $(A \cup B) \cup C = A \cup (B \cup C)$.

(iii) Let us suppose that $A \subseteq B$. We show that $A \cup B = B$, we first show that $A \cup B \subseteq B$, then show that $B \subseteq A \cup B$.
Consider

$$\begin{aligned} x \in (A \cup B) &\implies (x \in A) \text{ or } (x \in B) \\ &\implies (x \in B) \text{ or } (x \in B) && \text{(Using } A \subseteq B\text{)} \\ &\implies x \in B. \end{aligned}$$

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This proves that $A \cup B \subseteq B$. Similarly, consider

$$x \in B \implies (x \in A) \text{ or } (x \in B) \implies x \in A \cup B$$

which proves that $B \subseteq A \cup B$.

Conversely, let us suppose that $A \cup B = B$. Consider

$$x \in A \implies (x \in A) \text{ or } (x \in B) \implies x \in A \cup B \implies x \in B, \qquad \text{(Using } A \cup B = B)$$

which shows that $A \subseteq B$.

(xi) Let us begin by noting that the symmetric difference is commutative, i.e. $X \Delta Y = Y \Delta X$. This is easy to see, as

$$X \Delta Y = (X \setminus Y) \cup (Y \setminus X) = (Y \setminus X) \cup (X \setminus Y) = Y \Delta X.$$

Let $U := A \cup B \cup C$ be the universal set. Note that

$$A \Delta B = (A \setminus B) \cup (B \setminus A) = (A \cap B^{\mathsf{c}}) \cup (B \cap A^{\mathsf{c}}). \tag{1}$$

Furthermore, as $A \Delta B = (A \cup B) \setminus (A \cap B) = (A \cup B) \cap (A \cap B)^{\mathsf{c}}$, it follows from De Morgan's Law that

$$(A \Delta B)^{\mathsf{c}} = (A \cap B) \cup (A \cup B)^{\mathsf{c}} = (A \cap B) \cup (A^{\mathsf{c}} \cap B^{\mathsf{c}}). \tag{2}$$

We now begin the proof of associativity. Note that

$$\boxed{(A \Delta B) \Delta C = [(A \Delta B) \cap C^{\mathsf{c}}] \cup [C \cap (A \Delta B)^{\mathsf{c}}]\,.} \tag{3}$$

It follows from Equations (1) and (2), using De Morgan's Law, that

$$(A \Delta B) \cap C^{\mathsf{c}} = [(A \cap B^{\mathsf{c}}) \cup (B \cap A^{\mathsf{c}})] \cap C^{\mathsf{c}} = (A \cap B^{\mathsf{c}} \cap C^{\mathsf{c}}) \cup (B \cap A^{\mathsf{c}} \cap C^{\mathsf{c}}) = [A \setminus (B \cup C)] \cup [B \setminus (C \cup A)], \tag{4}$$

and

$$C \cap (A \Delta B)^{\mathsf{c}} = C \cap [(A \cap B) \cup (A^{\mathsf{c}} \cap B^{\mathsf{c}})] = (A \cap B \cap C) \cup (C \cap (A^{\mathsf{c}} \cap B^{\mathsf{c}})) = (A \cap B \cap C) \cup [C \setminus (A \cup B)]. \tag{5}$$

Therefore, plugging Equations (4), (5) into Equation (3), we see that

$$\boxed{(A \Delta B) \Delta C = (A \cap B \cap C) \cup [A \setminus (B \cup C))] \cup [B \setminus (C \cup A)] \cup [C \setminus (A \cup B)].} \tag{6}$$

On noting that the Equation (6) is symmetric with respect to A, B and C, we deduce that

$$(A \Delta B) \Delta C = (B \Delta C) \Delta A = A \Delta (B \Delta C),$$

where, in the second step, we used the commutative property of the symmetric difference. This concludes the proof.

(xii) When $B = C$, evidently $A \Delta B = A \Delta C$. Conversely, let us assume that $A \Delta B = A \Delta C$. Then, $A \Delta (A \Delta B) = A \Delta (A \Delta C)$. As the symmetric difference is associative, we have

$$(A \Delta A) \Delta B = A \Delta (A \Delta B) = A \Delta (A \Delta C) = (A \Delta A) \Delta C. \tag{7}$$

Since$A \Delta A = \emptyset$, $\emptyset \Delta B = B$ and $\emptyset \Delta C = C$, it follows from Equation (7) that $B = C$, which proves the claim.

Problem 2.

(iii) If $A = \emptyset$ or $B \setminus C = \emptyset$ both sides are equal to $\emptyset$. Let us assume that $A, B \setminus C \neq \emptyset$. We prove $A \times (B \setminus C) = (A \times B) \setminus (A \times C)$ by showing that each side is a subset of the other.

Let us first claim that $A \times (B \setminus C) \subseteq (A \times B) \setminus (A \times C)$. To see this, let $(x, y) \in A \times (B \setminus C)$. Then, we have $x \in A$ and $y \in B \setminus C$, i.e. $y \in B$ and $y \notin C$. This implies that $(x, y) \in A \times B$ and $(x, y) \notin A \times C$. Therefore, $(x, y) \in (A \times B) \setminus (A \times C)$, which proves $A \times (B \setminus C) \subseteq (A \times B) \setminus (A \times C)$.

To prove the reverse inclusion $(A \times B) \setminus (A \times C) \subseteq A \times (B \setminus C)$, we proceed as follows. Let $(x, y) \in (A \times B) \setminus (A \times C)$. Then, we have $(x, y) \in A \times B$ and $(x, y) \notin A \times C$, which implies that $x \in A$, $y \in B$ and $y \notin C$, i.e. $y \in B \setminus C$. Hence, $(x, y) \in A \times (B \setminus C)$ from where it follows that $(A \times B) \setminus (A \times C) \subseteq A \times (B \setminus C)$.

Therefore, we conclude that $A \times (B \setminus C) = (A \times B) \setminus (A \times C)$.

(iv) No. Consider the following counterexample.
Let $A = \{0\}, B = \{1\}$. Then,

$$\begin{aligned}
A \times B &= \{(0,1)\}, \\
\mathcal{P}(A \times B) &= \{\emptyset, \{(0,1)\}\}, \\
\mathcal{P}(A) &= \{\emptyset, \{0\}\}, \\
\mathcal{P}(B) &= \{\emptyset, \{1\}\}, \\
\mathcal{P}(A) \times \mathcal{P}(B) &= \{(\emptyset,\emptyset), (\emptyset, \{1\}), (\{0\}, \emptyset), (\{0\}, \{1\})\}.
\end{aligned}$$

(v) Yes. If $A \cap C = \emptyset$ or $B \cap D = \emptyset$, then both sides are equal to $\emptyset$. Let us assume that $A \cap C, B \cap D \neq \emptyset$. This implies that $A, B, C, D \neq \emptyset$.

To show that $(A \cap C) \times (B \cap D) = (A \times B) \cap (C \times D)$, we show that each side is a subset of the other. We show that $(A \times B) \cap (C \times D) \subseteq (A \cap C) \times (B \cap D)$. The reverse inclusion can be proved similarly. Let $(x, y) \in (A \times B) \cap (C \times D)$. Then, $(x, y) \in A \times B$ and $(x, y) \in C \times D$. Now $(x, y) \in A \times B$ implies that $x \in A$ and $y \in B$. Similarly, $(x, y) \in C \times D$ implies that $x \in C$ and $y \in D$. Therefore, $x \in A$ and $x \in C$, and $y \in B$ and $y \in D$ which shows that $x \in A \cap C$ and $y \in B \cap D$ i.e. $(x, y) \in (A \cap C) \times (B \cap D)$. Therefore $(A \times B) \cap (C \times D) \subseteq (A \cap C) \times (B \cap D)$. The reverse inclusion is left as an exercise.

(vi) No. Consider the following counterexample.
Let $A = \{0\}, B = \{1\}, C = \{2\}, D = \{3\}$. Then,

$$\begin{aligned}
A \cup C &= \{0, 2\}, \\
B \cup D &= \{1, 3\}, \\
(A \cup C) \times (B \cup D) &= \{(0,1), (0,3), (2,1), (2,3)\}, \\
A \times B &= \{(0,1)\}, \\
C \times D &= \{(2,3)\}, \\
(A \times B) \cup (C \times D) &= \{(0,1), (2,3)\}.
\end{aligned}$$

Problem 3.

(i) <u>The number of subsets of X is 2^n</u>.

To prove this, note that for each $x \in X$, we can either choose it or leave it aside when forming a subset of X. In other words, each of the n elements in X presents us with 2 choices, giving us a total of 2^n ways of forming subsets of X. Moreover, every subset of X can be formed in this manner.

(ii) <u>There are $2^n - 1$ non-empty subsets of X.</u>

There is precisely one empty subset out of the 2^n subsets of X.

(iii) <u>There are $\frac{1}{2}(3^n + 1)$ ways of choosing two disjoint subsets of X.</u>

For each $x \in X$, we can either place it in one subset, a second subset, or leave it aside. This gives us a total of 3^n ways of forming an ordered pair (A, B) of disjoint subsets A, B of X. However, we are looking for the number of unordered pairs of disjoint subsets. Thus, we have double-counted all cases where $A \neq B$, of which there are $3^n - 1$. The only case that is not doubly counted is when $A = B = \emptyset$. Therefore, the number of ways is $\frac{1}{2}(3^n - 1) + 1 = \frac{1}{2}(3^n + 1)$.

(iv) <u>There are $\frac{1}{2}(3^n - 2^{n+1} + 1)$ ways of choosing two non-empty disjoint subsets of X.</u>

Of the $\frac{1}{2}(3^n + 1)$ ways of choosing two disjoint subsets of X, consider the case where one of them is empty. This means that the other subset is simply an arbitrary subset of X, of which there are 2^n. Removing these from our count leaves precisely all disjoint non-empty pairs of subsets of X. Thus, we have $\frac{1}{2}(3^n + 1) - 2^n = \frac{1}{2}(3^n - 2^{n+1} + 1)$ ways.