MA 1101 : Mathematics I

Problem 1.
Let $\emptyset \neq D \subseteq \mathbb{R}$, let $c \in D$ and let $f : D \to \mathbb{R}$ be continuous at c with $f(c) > 0$. Show that, there exists $\delta > 0$ such that

$$f(x) > 0, \text{ for all } x \in (c - \delta, c + \delta) \cap D.$$

Problem 2.
Let $\emptyset \neq D \subseteq \mathbb{R}$, let $c \in D$ and let $f, g : D \to \mathbb{R}$ be continuous at c. Show that

(i) $f + g$ is continuous at c.

(ii) For all $\alpha \in \mathbb{R}$, αf is continuous at c.

(iii) fg is continuous at c.

(iv) If $g(c) \neq 0$, $\frac{f}{g}$ is continuous at c.

Problem 3.
Let $I \subseteq \mathbb{R}$ be an open interval, let $c \in I$ and let let $f, g : I \to \mathbb{R}$ be differentiable at c. Show that

(i) $f + g$ is differentiable at c and $(f + g)'(c) = f'(c) + g'(c)$.

(ii) For all $\alpha \in \mathbb{R}$, αf is differentiable at c and $(\alpha f)'(c) = \alpha f'(c)$.

(iii) fg is differentiable at c and $(fg)'(c) = f'(c)g(c) + g'(c)f(c)$.

(iv) If $g(c) \neq 0$, $\frac{f}{g}$ is differentiable at c and $\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{g(c)^2}$.

Problem 4.
Let $a, b \in \mathbb{R}$ be such that $a < b$, let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) with $f(a) = f(b) = 0$. Show that, for all $\alpha \in \mathbb{R}$, there exists a $c \in (a, b)$ (c depends on α) such that

$$f'(c) = \alpha\, f(c).$$

Problem 5.
Let $a, b \in \mathbb{R}$ be such that $a < b$ and $a + b \neq 0$, let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) and let $af(b) = bf(a)$. Prove that, for some $c \in (a, b)$,

$$f'(c) = \frac{f(b) + f(a)}{b + a}.$$

Problem 6.
Let $a, b \in \mathbb{R}$ be such that $a < b$ and $ab > 0$, let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) and let $\frac{1}{f(b)} - \frac{1}{f(a)} = \frac{1}{b} - \frac{1}{a}$. Prove that, for some $c \in (a, b)$,

$$f'(c) = \frac{f(b)f(a)}{b\,a}.$$

Problem 7.
Establish the following inequalities.

(i) $\frac{x}{1+x} < \ln(1 + x) < x$, for all $x > 0$.

(ii) $e^x > 1 + x + \frac{x^2}{2}$, for all $x > 0$.

(iii) $|\sin x - \sin y| \leqslant |x - y|$, for all $x, y \in \mathbb{R}$.