MA 1101 : Mathematics I

Problem 1.
Use the (ϵ, δ)-definition to prove the existence or non-existence of the following limits.

(i) $f : \mathbb{R} \to \mathbb{R}$, $f(x) := [x]$; $\lim_{x\to 0} f(x)$.

(ii) $f : \mathbb{R} \to \mathbb{R}$, $f(x) := [x] - \left[\frac{x}{3}\right]$; $\lim_{x\to 0} f(x)$.

(iii) $f : \mathbb{R} \to \mathbb{R}$, $f(x) := \frac{x^3-8}{x-2}$ if $x \neq 2$, 0 at $x = 0$; $\lim_{x\to 2} f(x)$.

(iv) $f : \mathbb{R} \to \mathbb{R}$, $f(x) := x \sin\frac{1}{x}$ if $x \neq 0$, 0 at $x = 0$; $\lim_{x\to 0} f(x)$.

(v) $f : \mathbb{R} \to \mathbb{R}$, $f(x) := \frac{x}{|x|}$ if $x \neq 0$, 0 at $x = 0$;, $\lim_{x\to 0} f(x)$.

Problem 2.
Let $-\infty < a < b < \infty$, let $f, g : (a, b) \to \mathbb{R}$ and let $x_0 \in [a, b]$. Let us suppose that $\lim_{x\to x_0} f(x)$ exists and that $\lim_{x\to x_0} f(x) \neq 0$. Prove that, for some $\delta > 0$,

$$f(x) \neq 0, \text{ for all } x \in (x_0 - \delta, x_0 + \delta)\,,\; x \neq x_0.$$

Problem 3.
Let $-\infty < a < b < \infty$, let $f, g : (a, b) \to \mathbb{R}$ and let $x_0 \in [a, b]$. Let us suppose that $\lim_{x\to x_0} f(x)$, $\lim_{x\to x_0} g(x)$ exist and we write

$$L := \lim_{x\to x_0} f(x),\; M := \lim_{x\to x_0} g(x).$$

Show that

(i) $\lim_{x\to x_0}(f(x) + g(x)) = L + M$.

(ii) For all $\alpha \in \mathbb{R}$, $\lim_{x\to x_0}(\alpha f(x)) = \alpha L$.

(iii) $\lim_{x\to x_0} f(x)g(x) = LM$.

(iv) If $M \neq 0$,

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{L}{M}.$$