CHAPTER 1 : SET THEORY.

A set is a collection of objects which are known as elements or members. Elements of a set can be anything, such as numbers, lines, students, fishes and even sets. We can think of a set as a box that contains elements i.e. objects inside the box. Just as a box may be empty, a set can be empty as well. We denote the empty set as ϕ. i.e. ϕ is the set with no elements in it. A set with a single element is called a singleton set. Note that, a singleton set is different from the element it contains, just as a box containing a hat is different from the hat.

Notation ~~###~~.

Usually, we shall use upper case letters to label sets and lower case letters to label elements in a set. For example, let A be a set and let x be an object. Then, we write

i) $x \in A$ if x is an element of A

ii) $x \notin A$ if x is not an element of A.

Section 1.1. Basic terminologies

Since a set is defined by its elements, we write a set by ~~defin~~ listing / defining its elements. This is usually done in two ways

i) **ROSTER METHOD:** In this method, we write a set by listing all its elements.

i) $A = \{2, 3, 4, 5, 6\}$

ii) $B = \{1, 2, \ldots\ldots, 100\}$

iii) $C = \{$ set of all BS-MS students in IISER-K, set of all books in the IISER-K library $\}$.

iv) $\mathbb{N} = \{1, 2, 3, \ldots\ldots\}$

v) $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$

Not all sets can be written as rosters. This brings us to the second method.

ii) **SET-BUILDER METHOD:** In this method, we describe the set by describing the properties of its elements ONLY satisfied by them.

i) A = set of real numbers lying between 0 and 1
$= \{x \in \mathbb{R} \mid 0 < x < 1\}$.

ii) $\mathbb{Q} = \{ p/q : p \in \cancel{\mathbb{N}}, q \in \mathbb{N} \}$
$= \{ p/q : p \in \mathbb{Z}, q \in \mathbb{N}, \gcd(p, q) = 1 \}$.

~~iii) $S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$~~.

iii) Let S be the set of all three letter words in English uppercase alphabet. Therefore, S consists of 26^3 elements. It is very cumbersome to write the set using the roaster method. However using the set-builder method, we can write S as

$$S = \{ xyz \mid x, y, z \in \{A, B, \ldots, Z\} \}.$$

Definition 1.1.1 (EMPTY SET)

The set that has no element is called the empty set. We denote the empty set as ϕ. Furthermore, we call a set A non-empty, i.e. $A \neq \phi$, if A has at least one element in it.

Remark 1.1.2. Note that, ϕ is not the same as $\{\phi\}$. $\{\phi\}$ is a singleton set containing ϕ whereas ϕ has no element.

Definition 1.1.3 (SUBSET)

Let A, B be two sets. We say that A is a subset of B, denoted by $A \subseteq B$, if every element of A is also an element of B.

Definition 1.1.4 (EQUALITY OF SETS)

Let A, B be two sets. We say that A and B are equal, denoted by $A = B$, if $A \subseteq B$ and $B \subseteq A$. i.e. every element of A is also an element of B and vice versa.

Definition 1.1.5 (PROPER SUBSET)

Let A, B be two sets. We say that A is a proper subset of B, denoted by $A \subset B$ or $A \subsetneq B$, if $A \subseteq B$ and $A \neq B$. In other words, A is a proper subset of B, if every element of A is also an element of B and there is an element of B that is not in A.

Example 1.1.6.

i) Let us write $A := \{1, 2\}$, $B = \{x \in \mathbb{R} \mid x^2 - 3x + 2 = 0\}$.
$C = \{x \in \mathbb{R} \mid x^3 - 3x^2 + 2x = 0\}$.

Then, $A = B$, $A \subsetneq C$, $B \subsetneq C$.

ii) $\mathbb{N} \subsetneq \mathbb{Z}$ iii) $\mathbb{Z} \subsetneq \mathbb{Q}$.

Section 1.2. Operations on Sets.

In this section, we introduce various set operations that allow us generate new sets.

Definition 1.2.1.

Let A, B be two sets. Then

i) UNION: the union of A and B, denoted by $A \cup B$, is defined as

$$\boxed{A \cup B := \{x : x \in A \text{ or } x \in B\}}$$

ii) INTERSECTION: the intersection of A and B, denoted by $A \cap B$, is defined as

$$\boxed{A \cap B := \{x : x \in A \text{ and } x \in B\}}$$

iii) DIFFERENCE: the difference of B from A, denoted by $A \setminus B$, is defined as

$$\boxed{A \setminus B := \{x : x \in A \text{ and } x \notin B\}}$$

iv) SYMMETRIC DIFFERENCE: the symmetric difference of A and B, denoted by $A \triangle B$, is defined as

$$\boxed{A \triangle B := (A \setminus B) \cup (B \setminus A).}$$

Example 1.2.2

Let $A := \{1,2,3,4,5\}$, $B := \{1,4,6,9\}$.

Then

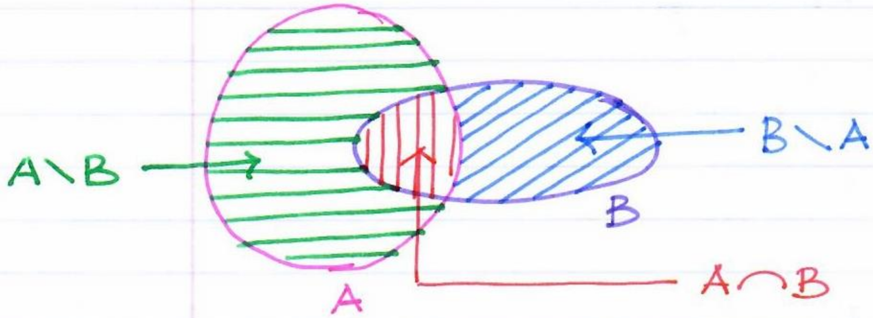
i) $A \cup B = \{1,2,3,4,5,6,9\}$, v) $A \Delta B = \{2,3,5,6,9\}$

ii) $A \cap B = \{1,4\}$

iii) $A \setminus B = \{2,3,5\}$

iv) $B \setminus A = \{6,9\}$

Remark 1.2.3 i) The aforementioned operations have the following interpretation in the Venn diagram



ii) Note that $A \Delta B = B \Delta A$, hence the name symmetric difference.

Theorem 1.2.4

Let A, B, C be three sets. Then,

i) Associativity: $A \cup (B \cup C) = (A \cup B) \cup C$, $A \cap (B \cap C) = (A \cap B) \cap C$

ii) Distributivity: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

iii) Commutativity: $A \cup B = B \cup A$, $A \cap B = B \cap A$

Proof: Exercise.

Definition 1.2.5 (DISJOINT SETS)

Let A, B be two sets. We say that A, B are disjoint if $A \cap B = \emptyset$. i.e. A, B have no element in common.

Definition 1.2.6 (COMPLEMENT)

Let X be the "universal set" and let $A \subseteq X$. The complement of A (relative to X) is defined as
$$A^c := X \setminus A = \{x \in X \mid x \notin A\}.$$

Remark 1.2.7 It is important to define the universal set unambiguously in order to be able to talk about the complement of a set.

Remark 1.2.7½. Note that $(A^c)^c = A$.

Theorem 1.2.8 (DE MORGAN'S LAW)

Let X be the universal set and let $A, B \subseteq X$. Then,

i) $(A \cup B)^c = A^c \cap B^c$

ii) $(A \cap B)^c = A^c \cup B^c$

PROOF: i) Let $x \in (A \cup B)^c := X \setminus (A \cup B)$. Then, $x \in X$ and $x \notin A \cup B$. Therefore, $x \notin A$ and $x \notin B \Rightarrow x \in X \setminus A$ and $x \in X \setminus B \Rightarrow x \in A^c$ and $x \in B^c \Rightarrow x \in A^c \cap B^c$. Therefore, $(A \cup B)^c \subseteq A^c \cap B^c$.

Conversely, let $y \in A^c \cap B^c$ Then, $y \in X$, $y \notin A$ and $y \notin B$. Therefore, $y \notin A \cup B$. Hence, $y \in (A \cup B)^c$
$\Rightarrow A^c \cap B^c \subseteq (A \cup B)^c$.

$$\Rightarrow \quad (A \cup B)^c = A^c \cap B^c$$

ii) Using i) we have, $(A^c \cup B^c)^c = (A^c)^c \cap (B^c)^c$
$$\Rightarrow (A^c \cup B^c)^c = A \cap B \Rightarrow \left[(A^c \cup B^c)^c\right]^c = (A \cap B)^c$$
$$\Rightarrow A^c \cup B^c = (A \cap B)^c.$$

(Proved.

Definition 1.2.9 (POWER SET)

Let A be a set. The power set of A, denoted by $\mathcal{P}(A)$, is the set of subsets of A.

Remark 1.2.10

i) The power set of A is also denoted by 2^A.

ii) Note that, $\mathcal{P}(A) \neq \emptyset$ for any set A, as $\emptyset \in \mathcal{P}(A)$.

ii) Let $n \in \mathbb{N}$ and let A be set of n elements. Then, $\mathcal{P}(A)$ has exactly 2^n elements (EXERCISE).

Example 1.2.11

i) Let $A = \phi$. Then $\mathcal{P}(A) = \{\phi\}$.

ii) Let $A = \{\phi\}$. Then, $\mathcal{P}(A) = \{\phi, \{\phi\}\} = \{\phi, A\}$.

iii) Let $A = \{a, b\}$. Then,

$$\mathcal{P}(A) = \{\phi, \{a\}, \{b\}, A\}.$$

We now introduce the notion of the cartesian product. Let us begin with the def^n of an ordered pair.

~~Let~~ Def^n 1.2.12 (INFORMAL) (ORDERED PAIR)

Let A, B be non-empty sets and let $a \in A, b \in B$. The ordered pair (a,b) is a notation specifying a and b, in that order.

Remark 1.2.13 The definition is unsatisfactory because it is only descriptive and based on an intuitive understanding of order.

We shall now give a formal def^n of the notion of an ordered pair that captures the main property of an ordered pair, which is

$$(a,b) = (c,d) \text{ if and only if } a = c \text{ and } b = d.$$

There are several ways in which the notion of an ordered pair can be formally defined. We state the one that is most commonly used. The definition is due to Kuratowski (1921).

Def^n 1.2.14 (ORDERED PAIR)

Let A, B be non-empty sets and let $a \in A, b \in B$. The ordered pair (a,b) is defined as

$$\boxed{(a,b) := \{\{a\}, \{a,b\}\}.}$$

Lemma 1.2.15

Let A, B be non-empty sets & let $a, x \in A$, $b, y \in B$. ~~satisfy~~

Then, $(a,b)=(x,y)$ if and only if $a=x$ and $b=y$.

PROOF: Let us suppose that $(a,b)=(x,y)$. We claim that $a=x$ and $b=y$. We consider two cases.

Case 1. $\boxed{a=b}$

Then, $(a,b)=\{\{a\},\{a,b\}\} = \{\{a\},\{a\}\} = \{\{a\}\}$.

Therefore,

$$\{x,y\} \in \{\{x\},\{x,y\}\} = (x,y) = (a,b) = \{\{a\}\},$$

which implies that $\{x,y\}=\{a\}$. Hence, $x=y=a$ i.e. $x=a$ & $y=a=b$.

Case 2. $\boxed{a \neq b}$

In this case, we first note that $x \neq y$. For if $x=y$, then $\{a,b\} \in (a,b)=(x,y)=\{\{x\}\}$ which implies that $a=b=x$, which contradicts the hypothesis. Therefore, $x \neq y$.

Next, as $\{x\} \in (x,y) = (a,b) = \{\{a\},\{a,b\}\}$, we have $\{x\}=\{a\}$ as $\{x\} \neq \{a,b\}$ because $a \neq b$. Hence, ~~$x=y$~~ $x=a$.

It remains to show that $y=b$. We have $\{x,y\} \in$ ~~$(a,b$~~ $(x,y)=(a,b)$

Therefore, $\{x,y\}=\{a,b\}$ as, $\{x,y\} \neq \{a\}$ because $x \neq y$.

This implies that $y=b$ as $y \neq a$ because $x \neq y$.

Therefore, $x=a$ and $y=b$.

Converse is easy to prove and is left as an exercise. (Proved).

We are now ready to introduce the notion of the cartesian product.

Definition 1.2.16 (CARTESIAN PRODUCT)

Let A, B be two sets. The cartesian product of A and B, denoted by $A \times B$, is defined as

$$\boxed{A \times B := \begin{cases} \{(a,b) \mid a \in A,\ b \in B\} & \text{when, } A \neq \phi, B \neq \phi \\ \phi & \text{when } A = \phi \text{ or } B = \phi \end{cases}}$$

Example 1.2.17

i) Let $A = \{1, 2\}$, $B = \{a, b, c\}$. Then,
$$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}.$$

ii) Let $A = \{a, b\}$. Then,
$$A \times A = \{(a, a), (a, b), (b, a), (b, b)\}.$$

Section 1.3. Relation

We start with the definition.

Definition 1.3.1 (RELATION)

Let $X, Y \neq \phi$. A relation between X and Y is a subset $R \subseteq X \times Y$. If $X = Y$, we say that R is a relation on X. Given a relation $R \subseteq X \times Y$, and given $x \in X$, $y \in Y$, we say that x is R-related to y if $(x, y) \in R$. In this case, we write xRy.

Example 1.3.2

i) Let $X := \{a, b, c\}$, $Y := \{1, 2, 3\}$ and let
$$R := \{(a, 1), (a, 2), (b, 3), (c, 1)\}.$$
Then

i) $aR1$, $aR2$, $bR3$, $cR1$.

ii) $a \not{R} 3$, $b \not{R} 1$, $b \not{R} 2$, $c \not{R} 2$, $c \not{R} 3$.

ii) Let $X = Y$ be the set of all human beings. Let us define
$$R := \{(x, y) \in X \times X \mid x \text{ is a father of } y\}.$$
Then R defines a relation ~~between~~ on X. Note that, xRy if and only if x is a father of y.

iii) Let $X = Y =: \mathbb{N}$. We define a relation R on $\mathbb{N}$ as follows
$R =: \{(m, n) \in \mathbb{N} \mid n \text{ is a multiple of } m\}$. In other words, for $m, n \in \mathbb{N}$, we have mRn if and only if $n = km$, for some $k \in \mathbb{N}$.

Definition 1.3.3

Let $X \neq \phi$ and let R be a relation on X i.e. $R \subseteq X \times X$. Then,

i) REFLEXIVE: R is said to be REFLEXIVE if xRx for all $x \in X$.

ii) SYMMETRIC: R is said to be SYMMETRIC if for all $x, y \in X$, $xRy \Rightarrow yRx$.

iii) TRANSITIVE: R is said to be TRANSITIVE if for all $x, y, z \in X$ xRy and $yRz \Rightarrow xRz$.

Example 1.3.4

i) The relation defined in Example ii) not reflexive, symmetric and transitive.

ii) The relation defined in Example iii) is reflexive, transitive but not symmetric.

iii) Let $X = Y = \mathbb{R}$ and let $R := \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid xy = 0\}$. Then. R is symmetric but not reflexive and transitive.

iv) Let $X = \{a, b, c\}$ and $R := \{(a, b), (b, a), (a, a), (b, b)\}$. Then R is symmetric, transitive but not reflexive.

v) Let $X = Y = \mathbb{R}$ and let us define.

$$R_1 := \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x \geq y\}$$
$$R_2 := \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x > y\}.$$

Then, i) R_1 is reflexive, transitive but not symmetric.
ii) R_2 is transitive but not reflexive and symmetric.

vi) Equality Relation: Let $X \neq \phi$. Let us define

$$\Delta(X) := \{(x, y) \in X \times X \mid x = y\} = \{(x, x) : x \in X\}.$$

Then, $\Delta(X)$ is reflexive, symmetric and transitive.

vii) Congruence Modulo n: Let $X = Y = \mathbb{Z}$ and let $n \in \mathbb{N}$. Let us define $\equiv_n := \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid n \text{ divides } (b - a)\}$

Then, $\equiv_n$ is reflexive, symmetric and transitive

Definition 1.3.5 (EQUIVALENCE RELATION)

Let $X \neq \phi$. A relation $\sim$ on X is said to be an EQUIVALENCE relation on X if $\sim$ is reflexive, symmetric and transitive. Furthermore, for $x \in X$,

$$[x] := \{ y \in X \mid y \sim x \}$$

is said to be the equivalence class of x for $\sim$. If $a \in [x]$, we say that a is a representative of $[x]$.

Finally we write

$$X/\sim := \{ [x] : x \in X \}$$

to denote the set of all equivalence classes of X.

Remark 1.3.6

i) Note that, $[x] \neq \phi$, for all $x \in X$. This is because $x \in [x]$ as $\sim$ is reflexive.

ii) Note that, $[a] = [x]$ if $a \in [x]$. Indeed if $y \in [a]$. Then, $y \sim a$. Since $a \sim x$, we have $y \sim x$. $y \in [x]$. Therefore, $[a] \subseteq [x]$. Similarly, one can show that $[x] \subseteq [a]$. Therefore, $[a] = [x]$.

Example 1.3.7

i) Equality Relation: Let $X \neq \phi$ and let

$$\Delta(X) := \{ (x,y) \in X \times X \mid x = y \}.$$

Then, $\Delta(X)$ is an equivalence relation on X. Furthermore, for all $x \in X$,

$$[x] = \{ y \in X \mid y \sim x \} = \{ y \in X \mid y = x \} = \{x\}.$$

Therefore, $\{x\}$ is the equivalence class of x, for all $x \in X$., and

$$X/\sim = \{ \{x\} : x \in X \}.$$

ii) Congruence Modulo Relation: Let $n \in \mathbb{N}$. We define the relation $\equiv_n$ on $\mathbb{Z}$ as follows. We say that, for all $a, b \in \mathbb{Z}$.

$$\boxed{a \equiv_n b \text{ if and only if } n \text{ divides } (b-a)}.$$

Then, $\equiv_n$ is an equivalence relation on $\mathbb{Z}$. Let us try to find the equivalence classes. Let $a \in \mathbb{Z}$. We claim that

$$\boxed{[a] = \{a + kn \mid k \in \mathbb{Z}\}. \quad (*)}$$

To prove $(*)$, let us ~~write~~ $S := \{a + kn \mid k \in \mathbb{Z}\}$. Let $b \in [a]$. Then, ~~n divides~~ $a \equiv_n b$. Therefore, n divides $b - a$, which implies that, for some $k \in \mathbb{Z}$, $(b-a) = kn \Rightarrow b = a + kn \in S$. Hence, $[a] \subseteq S$. Conversely, if $b \in S$. Then, $b = a + tn$, for some $t \in \mathbb{Z}$ which implies that, n divides $b - a$. i.e. $a \equiv_n b$, i.e. $b \in [a]$. Therefore, $S \subseteq [a]$. Hence, $S = [a]$. which proves $(*)$.

We now find the distinct equivalence classes. We claim that

$$\boxed{\mathbb{Z}/\equiv_n = \{[0], \ldots\ldots, [n-1]\}} \quad (*_1).$$

We prove $(*_1)$ in two steps. In the first step, we claim that

<u>STEP 1</u>. for all $0 \leq r < s \leq n-1$,

$$[s] \cap [r] = \phi.$$

If not, using Lemma 1.3.8, $[s] = [r]$ i.e. n divides $s - r$. ~~As $0 < s - r$~~ This is a contradiction as $0 < s - r \leq n-1$. Therefore, $[s] \cap [r] = \phi$. This proves the first step. In the next step, we claim that,

<u>STEP 2</u>. Let $a \in \mathbb{Z}$. Then, for some $r \in \{0, \ldots, n-1\}$,

$$[a] = [r].$$

Indeed, using division algorithm, we find $q \in \mathbb{Z}$ and $r \in \{0, .., n-1\}$ such that

$$a = qn + r.$$

Therefore, n divides $a-r$ i.e. $a \equiv_n r$. Therefore, $[a] = [r]$. This proves the step 2.

Hence, we have proved that

$$\mathbb{Z}/\equiv_n = \{[0], \ldots, [n-1]\}.$$

iii) Let us define a relation $\sim$ on $\mathbb{R}$ as follows. We define,

$$\sim := \{(x,y) \in \mathbb{R} \times \mathbb{R} \mid xy > 0\} \cup \{(0,0)\}.$$

Then, $\sim$ is an equivalence relation on $\mathbb{R}$. Let $x \in \mathbb{R}$. Note that,

$$[x] = \begin{cases} (0, \infty) & \text{if } x > 0, \\ \{0\} & \text{if } x = 0, \\ (-\infty, 0) & \text{if } x < 0. \end{cases}$$

Therefore,

$$\mathbb{R}/\sim = \{(-\infty, 0), \{0\}, (0, \infty)\}.$$

Lemma 1.3.8

Let $X \neq \emptyset$, let $\sim$ be an equivalence relation on X and let $x, y \in X$. Then, exactly one of the following holds

i) $[x] = [y]$.

ii) $[x] \cap [y] = \emptyset$

PROOF: If ii) holds, clearly i) does not hold. Let us suppose that ii) does not hold. We shall show that i) holds.

Since ii) does not hold, $[x] \cap [y] \neq \phi$. Let $z \in [x] \cap [y]$. Then, $z \sim x$ and $z \sim y$. Using the symmetry and transitivity of $\sim$, we have $x \sim y$. Therefore, $x \in [y]$. Using ii) of Remark 1.3.6 $[x] = [y]$. i.e. i) holds. This proves the lemma. (Proved).

Definition 1.3.9 (PARTITION OF A SET).
Let $X \neq \phi$. A partition of X is a collection of pairwise disjoint non-empty subsets of X whose union is X. In other words, a collection $\{A_i | i \in I\}$ of subsets of X is said to be a partition of X if

i) $A_i \neq \phi$, for all $i \in I$,
ii) $A_i \cap A_j = \phi$, for all $i, j \in I$ with $i \neq j$; and
iii) $X = \bigcup_{i \in I} A_i$.

Example 1.3.10 Let $X = \mathbb{R}$. Then,
i) $\{\{x\} : x \in \mathbb{R}\}$ is a partition of $\mathbb{R}$.
ii) $\{(-\infty, 0), [0, \infty)\}$ is a partition of $\mathbb{R}$.
iii) $\{(-\infty, 0), \{0\}, (0, \infty)\}$ is a partition of $\mathbb{R}$.
iv) $\{\mathbb{Q}, \mathbb{R} \setminus \mathbb{Q}\}$ is a partition of $\mathbb{R}$.

The importance of equivalence relations lies in the following theorem.

Theorem 1.3.11 (EQUIVALENCE RELATION & PARTITION).
Let $X \neq \phi$. ~~Then~~.
i) If $\sim$ is an equivalence relation on X, then $X/\sim$ i.e. the collection of distinct equivalence classes of X forms a partition of X.

ii) Conversely, if $\mathcal{F} = \{A_i | i \in I\}$ is a partition of X, then there exists an equivalence relation $\sim_{\mathcal{F}}$ on X such that

$$X/\sim_{\mathcal{F}} = \mathcal{F} = \{A_i \mid i \in I\}.$$

PROOF: Let $\{[x_i] : i \in I\}$ be the set of all distinct equivalence classes of $\sim$. Then,

$$X/\sim = \{[x_i] : i \in I\}.$$

As ~~$X/\sim = \{[x] : x \in X\}$~~, $X = \bigcup_{x \in X} [x]$, it follows that $X = \bigcup_{i \in I} [x_i]$.

Using Lemma 1.3.8, $[x_i] \cap [x_j] = \phi$ for all $i, j \in I$ with $i \neq j$. Therefore, $\{[x_i] : i \in I\}$ forms a partition of X.

Conversely, let us suppose that $\mathcal{F} = \{A_i \mid i \in I\}$ is a partition of X. Define a relation $\sim_{\mathcal{F}}$ on X as follows. For all $x, y \in X$, we say that

$$x \sim_{\mathcal{F}} y \text{ if and only if for some } i \in I,\ x, y \in A_i.$$

Then, $\sim_{\mathcal{F}}$ is clearly reflexive and symmetric. To see that $\sim_{\mathcal{F}}$ is transitive, let $x \sim_{\mathcal{F}} y$ and $y \sim_{\mathcal{F}} z$. Then, for ~~$i, j \in I$~~, some $i, j \in I$,

$$x, y \in A_i \text{ and } y, z \in A_j$$

Therefore, $y \in A_i \cap A_j$. Since A_i's are pairwise disjoint, it follows that $i = j$. Hence, $x, z \in A_i$ which implies that $x \sim_{\mathcal{F}} z$. Therefore, $\sim_{\mathcal{F}}$ is an equivalence relation.

Let $x \in X$. Then, there exists exactly one $i \in I$ such that $x \in A_i$. Note that

$$[x] = \{y \in X \mid y \sim_{\mathcal{F}} x\} = \{y \in X \mid y \in A_i\} = A_i.$$

Therefore, $X/\sim_{\mathcal{F}} = \mathcal{F} = \{A_i \mid i \in I\}$, which proves the theorem.

(Proved).

Section 1.4 Functions

We begin with the definition.

Definition 1.4.1 (FUNCTION)

Let $X, Y \neq \phi$. A relation $R \subseteq X \times Y$ between X and Y is said to be a function if for every $x \in X$, there exists a unique $y_x \in Y$ such that $x R y_x$ or $(x, y_x) \in R$. Furthermore, we say that X is the domain of R and Y the codomain of Y.

Example 1.4.2

Let $X := \{a, b, c\}$, $Y := \{1, 2\}$. Then.

i) $R_1 := \{(a, 1), (b, 2), (c, 2)\}$ is a function.
ii) $R_2 := \{(a, 1), (b, 1), (c, 1)\}$ is a function.
iii) $R_3 := \{(a, 1), (a, 2), (b, 1), (c, 2)\}$ is not a function
iv) $R_4 := \{(a, 1), (b, 2)\}$ is not a function.

Notation 1.4.3

Let $X, Y \neq \phi$ and let $R \subseteq X \times Y$ be a function. We write $R : X \rightarrow Y$ to denote the function. For each $x \in X$, we write $R(x)$ to denote the unique $y_x \in Y$ for which $(x, y_x) \in R$.

In this notation, we can write the functions in Example 1.4.2 as follows.

i) $R_1 : X \longrightarrow Y$ is defined as $R_1(a) = 1$, $R_1(b) = 2$, $R_1(c) = 2$.
ii) $R_2 : X \longrightarrow Y$ is defined as $R_2(a) = 1$, $R_2(b) = 1$, $R_2(c) = 1$.

Definition 1.4.4

Let $X, Y \neq \phi$ and let $f : X \rightarrow Y$. Then, we say that

i) X is the domain of f.
ii) Y is the codomain of f.
iii) For each $x \in X$, $f(x)$ is the image of x.
iv) $f(X) = \{f(x) : x \in X\}$ is the range of f.

Remark 1.4.5

Note that $f(X) \subseteq Y$.

$n \mid b-a$
$c \mid b-b$
(a,b)
(b,c)

Example 1.4.6

i) CONSTANT FUNCTION: Let $X, Y \neq \phi$ and let $y_0 \in Y$. We define $f : X \to Y$ by

$$f(x) = y_0 \ , \ \text{for all } x \in X.$$

Note that $f(X) = \{y_0\}$.

ii) IDENTITY FUNCTION: Let $X \neq \phi$. We define $Id_X : X \to X$ as

$$Id_X(x) = x, \ \text{for all } x \in X.$$

Note that, $Id_X(X) = X$.

iii) INCLUSION FUNCTION: Let $X, Y \neq \phi$ with $X \subseteq Y$. The function $i : X \to Y$ defined as

$$i(x) = x \ , \ \text{for all } x \in X,$$

is called the inclusion function

iv) MODULUS FUNCTION: Define $|\cdot| : \mathbb{R} \to \mathbb{R}$ by

$$|x| := \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

v) POLYNOMIAL FUNCTION: Let $n \in \mathbb{N} \cup \{0\}$ and let $a_0, \ldots, a_n \in \mathbb{R}$ with $a_n \neq 0$. Let us define $p : \mathbb{R} \to \mathbb{R}$ by

$$p(x) = \sum_{j=0}^{n} a_j x^j = (a_0 + a_1 x + \cdots + a_n x^n, \ \text{for all } x \in \mathbb{R}.$$

We say that p is a polynomial function of degree n.

Definition 1.4.7 (Equality of functions)

Let $X, Y, A, B \neq \phi$ and let $f : X \to Y$, $g : A \to B$ be two functions. We say that f is equal to g, denoted by $f = g$, if

i) $X = A$,

ii) $Y = B$, and

iii) $f(x) = g(x)$, for all $x \in X$

Example 1.4.8

i) Define $f : \mathbb{N} \to \mathbb{Z}$, $g : \mathbb{Z} \to \mathbb{Z}$ by

$$f(n) := n, \quad \text{for all } n \in \mathbb{N}$$
$$g(m) := m, \quad \text{for all } m \in \mathbb{Z}.$$

Then, even though $f(n) = g(n) = n$, for all $n \in \mathbb{N}$, ~~$f \neq g$~~ we have $f \neq g$ as domains of f and g are different

ii) Define $f: \mathbb{R} \longrightarrow \mathbb{R}$, $g: \mathbb{R} \longrightarrow [0, \infty)$ by
$$f(x) = g(x) = x^2, \text{ for all } x \in \mathbb{R}.$$

Then, $f \neq g$ as codomains of f and g are different. Note that $f(\mathbb{R}) = g(\mathbb{R}) = [0, \infty)$.

Definition 1.4.9

Let $X, Y \neq \phi$ and let $f: X \to Y$. We say that

i) One-one function: f is one-one / injective, if for all $x_1, x_2 \in X$,
$$f(x_1) \neq f(x_2) \text{ whenever } x_1 \neq x_2$$

ii) Onto function: f is onto / surjective if $f(X) = Y$. i.e. for each $y \in Y$, there exists a $x \in X$ such that $f(x) = y$.

iii) bijective function: f is bijective if f is both one-one and onto.

Remark 1.4.10

Let $f: \mathbb{R} \to \mathbb{R}$. Then,

i) f is onto if the graph of f intersects every horizontal ~~line~~ line at least once.

ii) f is one-one if the graph of f intersects every horizontal line atmost once.

iii) f is bijective if the graph of f intersects every horizontal line exactly once.

Example 1.4.11

i) Let $f: (0,1) \to \mathbb{R}$ be defined as
$$f(x) = \frac{1}{x}, \text{ for all } x \in (0,1).$$

Then, f is one-one but ~~onto~~ not onto. Note that $f((0,1)) = (1, \infty)$.

ii) Let $f: \mathbb{R} \to \mathbb{R}$ be defined as
$$f(x) := x^3, \text{ for all } x \in \mathbb{R}.$$
Then, f is a bijection.

iii) Let $f: \mathbb{R} \to \mathbb{R}$, $g: \mathbb{R} \to [0, \infty)$, $h: [0, \infty) \to [0, \infty)$ be defined as
$$f(x) := x^2, \text{ for all } x \in \mathbb{R},$$
$$g(x) := x^2, \text{ for all } x \in \mathbb{R}.$$
$$h(x) := x^2, \text{ for all } x \in [0, \infty).$$

Then, f is neither one-one nor onto, g is onto but not one-one and h is a bijection.

iv) Let $a, b \in \mathbb{R}$, with $a \neq 0$. and let $f: \mathbb{R} \to \mathbb{R}$ be defined as
$$f(x) := ax + b, \text{ for all } x \in \mathbb{R}.$$
Then, f is a bijection.

v) Let $X, Y \neq \phi$ and let $y_0 \in Y$. Define $f: X \to Y$ by.
$$f(x) = y_0, \text{ for all } x \in X.$$
Then
a) f is one-one if and only if X is singleton
b) f is onto if and only if Y is singleton
c) f is bijective if and only if X, Y are singleton.

Definition 1.4.12 (Composition of Functions)

Let $X, Y, Z \neq \phi$ ~~and~~, let $f: X \to Y$ and let $g: Y \to Z$. The composition of f and g is function $g \circ f: X \to Z$ defined as
$$(g \circ f)(x) := g(f(x)), \text{ for all } x \in X.$$

Example 1.4.13

i) Let $f, g : \mathbb{R} \to \mathbb{R}$ be defined as
$$f(x) := 2x, \quad g(x) := x^3, \text{ for all } x \in \mathbb{R}.$$

Then,
$$(g \circ f)(x) = g(f(x)) = g(2x) = 8x^3, \text{ for all } x \in \mathbb{R}.$$
$$(f \circ g)(x) = f(g(x)) = f(x^3) = 2x^3, \text{ for all } x \in \mathbb{R}.$$

Note that, $g \circ f \neq f \circ g$.

ii) Let $X := \{a, b, c\}$, $Y := \{\alpha, \beta, \gamma\}$, $Z := \{1, 2\}$.
We define $f : X \to Y$, and $g : Y \to Z$ by
$$f(a) := \alpha, \quad f(b) := \beta, \quad f(c) := \gamma.$$
$$g(\alpha) := 1, \quad g(\beta) := 1, \quad g(\gamma) := 2.$$

Then, $g \circ f : X \to Z$ is
$$(g \circ f)(a) = g(f(a)) = g(\alpha) = 1.$$
$$(g \circ f)(b) = g(f(b)) = g(\beta) = 1.$$
$$(g \circ f)(c) = g(f(c)) = g(\gamma) = 2.$$

Definition 1.4.14 (Inverse of a function)

Let $X, Y \neq \phi$ and let $f : X \to Y$ be a bijection. The inverse of f is the function $f^{-1} : Y \to X$ defined as
$$f^{-1}(y) := x, \text{ where } y = f(x), \text{ for all } y \in Y.$$

Remark 1.4.15

Note that, f^{-1} is well-defined because f is both one-one and onto. The ontoness of f is required to define f^{-1} at each point of Y. The one-one of f is required so that y ~~$f^{-1}(y)$ has the unique value~~ gets mapped to exactly one point in X

Example 1.4.16

i) Let $f : \mathbb{R} \to \mathbb{R}$ be defined as

$$f(x) := x^3 \quad \text{for all } x \in \mathbb{R}.$$

Then, $f^{-1} : \mathbb{R} \to \mathbb{R}$ is

$$f^{-1}(y) := y^{1/3}, \quad \text{for all } y \in \mathbb{R}.$$

ii) Let $a, b \in \mathbb{R}$ with $a \neq 0$. and let $f : \mathbb{R} \to \mathbb{R}$ be defined as

$$f(x) := ax + b, \quad \text{for all } x \in \mathbb{R}.$$

Then, $f^{-1} : \mathbb{R} \to \mathbb{R}$ is

$$f^{-1}(y) := \frac{y-b}{a}, \quad \text{for all } y \in \mathbb{R}.$$

iii) Let $X := \{a, b, c\}$, $Y := \{\alpha, \beta, \gamma\}$ and let $f : \cancel{\mathbb{R}} X \to Y$ be defined as

$$f(a) := \alpha, \quad f(b) := \beta, \quad f(c) := \gamma.$$

Then, $f^{-1} : Y \to X$ is

$$f^{-1}(\alpha) := a, \quad f^{-1}(\beta) := b, \quad f^{-1}(\gamma) := c.$$

Theorem 1.4.17

Let $X, Y \neq \emptyset$ and let $f : X \to Y$ be a bijection. Then

$$f^{-1} \circ f = Id_X \text{ on } X, \quad f \circ f^{-1} = Id_Y \text{ on } Y.$$

PROOF: Exercise.

Theorem 1.4.18

Let $X, Y \neq \emptyset$ and let $f : X \to Y$, $g : Y \to X$, $h : Y \to X$ be such that

$$f \circ g = Id_X, \quad h \circ f = Id_X \qquad (*_1).$$

Then, f is a bijection and $g = h = f^{-1}$.

PROOF: We prove the theorem in three steps. In the first step, we show that

STEP 1. f is one-one.

To prove this, let $x_1, x_2 \in X$ be such that $f(x_1) = f(x_2)$.
Then, $(h \circ f)(x_1) = h(f(x_1)) = h(f(x_2)) = (h \circ f)(x_2)$.
Using $(*_1)$,
$x_1 = Id_X(x_1) = (h \circ f)(x_1) = (h \circ f)(x_2) = Id_X(x_2) = x_2$,
which shows that f is one-one. In the next step, we claim that

STEP 2. f is onto.

Indeed, let $y \in Y$. Then, using $(*_1)$ again,

$$f(g(y)) = (f \circ g)(y) = Id_Y(y) = y,$$

which shows that f is onto.

Step 1 and step 2 show that f is a bijection. Hence, $f^{-1}: Y \to X$ exists. In the third step, we claim that

STEP 3. $g = h = f^{-1}$.

Indeed, it follows from Theorem 1.4.17 ~~that~~ and $(*_1)$ that

$$g = \underbrace{Id_X \circ g = (f^{-1} \circ f) \circ g}_{\text{Theorem 1.4.17}} = \underbrace{f^{-1} \circ (f \circ g) = f^{-1} \circ Id_Y}_{(*_1)} = f^{-1}$$

$$h = \underbrace{h \circ Id_Y = (h \circ (f \circ f^{-1})}_{\text{Theorem 1.4.17}} = (h \circ f) \circ f^{-1} = \underbrace{Id_X}_{(*_1)} \circ f^{-1} = f^{-1}.$$

Hence, we have proved that $g = h = f^{-1}$.

This proves the theorem. (Proved).

Definition 1.4.19 (Image and Inverse Image)
Let $X, Y \neq \phi$ and let $f: X \to Y$. For $A \subseteq X$, $B \subseteq Y$, we define
i) the image of A, $f(A) := \{f(x) : x \in A\}$
ii) the inverse image of B, $f^{-1}(B) := \{x \in X \mid f(x) \in B\}$.

Remark 1.4.20

i) Note that, $f(A) \subseteq Y$ and $f^{-1}(B) \subseteq X$

ii) Note that, in Definition 1.4.19 (ii), f^{-1} does not stand for the inverse of the function f. ~~as f is not assumed be bijective~~ In fact, the inverse of f need not exist as f is not assumed to be bijective. f^{-1} is just a notation in this definition.

Example 1.4.21

i) Let $f: \mathbb{R} \to \mathbb{R}$ be defined as

$$f(x) := x^2, \text{ for all } x \in \mathbb{R}.$$

Note that f is not bijective. In fact, f is neither one-one nor onto.

a) $f([-1,1]) = [0,1]$, $\quad f([-1,2]) = [0,4]$

$f([-3,2]) = [0,9]$.

b) $f^{-1}([0,1]) = [-1,1]$, $\quad f^{-1}([-1,4]) = f^{-1}([0,4]) = [-2,2]$

$f^{-1}([4,16]) = [-4,-2] \cup [2,4]$

ii) Let $f: \mathbb{R} \to \mathbb{R}$ be defined as

$$f(x) := \begin{cases} 1/x, & \text{if } x \neq 0, \\ 2020, & \text{if } x = 0. \end{cases}$$

Then, $f([0,\infty)) = (0,\infty)$, $\quad f((0,\infty)) = (0,\infty)$.

$f^{-1}(\{2020\}) = \{0, \frac{1}{2020}\}$.

iii) Let $f: \mathbb{R} \to \mathbb{R}$ be defined as

$$f(x) := \sin x, \text{ for all } x \in \mathbb{R}.$$

Then, $f(\mathbb{R}) = [-1,1]$, $\quad f^{-1}([-1,1]) = \mathbb{R}$, $\quad f^{-1}(\{0\}) = \{n\pi \mid n \in \mathbb{Z}\}$.

Theorem 1.4.22

Let $X, Y \neq \phi$, let $f: X \to Y$ ~~and~~, let $A, B \subseteq X$ and let $C, D \subseteq Y$. Then,

i) $f(A \cup B) = f(A) \cup f(B)$, $f(A \cap B) \subseteq f(A) \cap f(B)$.

ii) $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$, $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$.

Remark 1.4.23

Note that, in general, we cannot say that $f(A \cap B) = f(A) \cap f(B)$. The following example is relevant in this context. Let us consider a map $f: X \to Y$ that is not one-one. Then, there exist $x_1, x_2 \in X$ such that $x_1 \neq x_2$ and $f(x_1) = f(x_2)$. Let us define

$$A = \{x_1\} \text{ and } B = \{x_2\}.$$

Then, $\phi = f(A \cap B) \subsetneq f(A) \cap f(B) = \{f(x_1)\}$

PROOF: i) As $A, B \subseteq A \cup B$, we have $f(A), f(B) \subseteq f(A \cup B)$ which implies that $f(A) \cup f(B) \subseteq f(A \cup B)$. Conversely, let $y \in f(A \cup B)$. Then $y = f(x)$ for some $x \in A \cup B$. If $x \in A$, then, $y \in f(A)$. If $x \in B$, Then $y \in f(B)$. Therefore, $y \in f(A) \cup f(B)$. Hence, $f(A \cup B) \subseteq f(A) \cup f(B)$.
Therefore, $f(A \cup B) = f(A) \cup f(B)$.

Since $A \cap B \subseteq A$, we have $f(A \cap B) \subseteq f(A)$. Similarly, we have $f(A \cap B) \subseteq f(B)$. Therefore, $f(A \cap B) \subseteq f(A) \cap f(B)$.

ii) $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$ (Exercise).

Using the similar argument used above, one can show that $f^{-1}(C \cap D) \subseteq f^{-1}(C) \cap f^{-1}(D)$. To prove the reverse inclusion, let $x \in f^{-1}(C) \cap f^{-1}(D)$. Then, $f(x) \in C$ and $f(x) \in D$. Hence, $f(x) \in C \cap D$. which implies that $x \in f^{-1}(C \cap D)$. Therefore, $f^{-1}(C) \cap f^{-1}(D) \subseteq f^{-1}(C \cap D)$.

Hence, $f^{-1}(C) \cap f^{-1}(D) = f^{-1}(C \cap D)$.
This proves the theorem. (Proved)

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