Learning with Gaussian Processes using GPy

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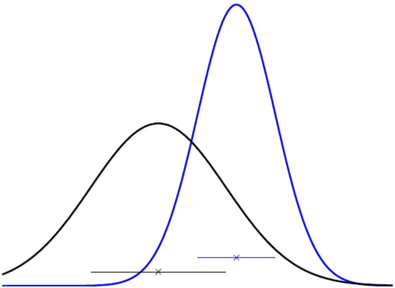
Supervised Learning: Ubiquitous questions

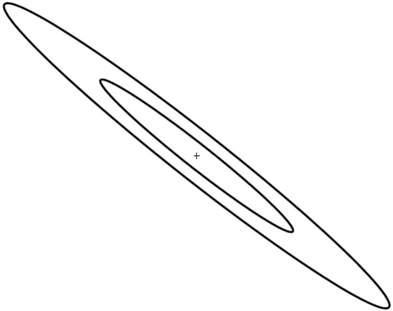
- Model fitting
  - How to fit parameters?
  - How to handle overfitting?
- Model selection
  - Which model best represents data?
  - How sure can I be?
- Interpretation
  - What is the accuracy of predictions?
  - Can I trust predictions under model uncertainity?

Gaussian Processes provides framework to address these issues.

Outline

Gaussian Distribution

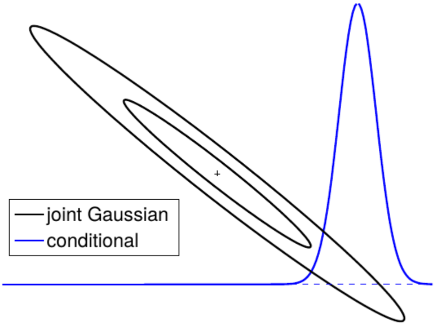


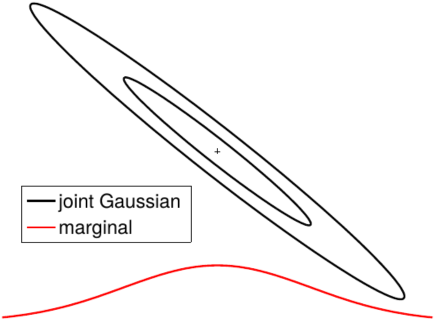


$$p(\mathbf{x}|\boldsymbol{\mu}, \boldsymbol{\Sigma}) = \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}) = (2\pi)^{-D/2}|\boldsymbol{\Sigma}|^{-1/2} \exp\left(-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu})\right)$$

$\boldsymbol{\mu}$: mean vector, $\boldsymbol{\Sigma}$: covariance matrix

Conditional and Marginal of a Gaussian





Conditional and Marginal of a joint Gaussian is also Gaussian.

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Generalization of a multivariate Gaussian to infinitely many variables.

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$$\mathbf{f} = (f_1, \ldots, f_n)^T \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}), \quad \text{indices } i = 1, \ldots, n$$

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Gaussian process: mean function, $m(x)$, and covariance function $k(x, x')$:

$$f(x) \sim \mathcal{GP}(m(x), k(x, x')), \quad \text{indices: } x$$

Marginalization Property

How can we represent infinite mean vector and infinite covariance matrix?

...luckily saved by *marginalization property*:

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For Gaussians:

$$p(\mathbf{x}, \mathbf{y}) = \mathcal{N}\left(\begin{bmatrix}\mathbf{a} \\ \mathbf{b}\end{bmatrix}, \begin{bmatrix}\mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{C}\end{bmatrix}\right)$$

$$p(\mathbf{x}) = \mathcal{N}(\mathbf{a}, \mathbf{A})$$

Random sampling from Gaussian Process

Considering one dimensional Gaussian process:

$$p(f(x)) \sim \mathcal{GP}\left(m(x) = 0, k(x, x') = \exp\left(-\frac{1}{2}(x - x')^2\right)\right)$$

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Sampling is done by focusing on subset $\mathbf{f} = (f(x_1), f(x_2), \ldots, f(x_n))^T$:

$$\mathbf{f} \sim \mathcal{N}(0, \mathbf{\Sigma}), \text{ where } \mathbf{\Sigma}_{ij} = k(x_i, x_j)$$

Coordinates of $\mathbf{f}$ are plot as a function of corresponding x

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Parametric Model:

- data: $\mathbf{x}, \mathbf{y}$
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Posterior parameter distribution:

$$p(\mathbf{w}|\mathbf{x}, \mathbf{y}, M) = \frac{p(\mathbf{w}|M)p(\mathbf{y}|\mathbf{x}, \mathbf{w}, M)}{p(\mathbf{y}|\mathbf{x}, M)}$$

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Problem: integrals are intractable for most interesting models!

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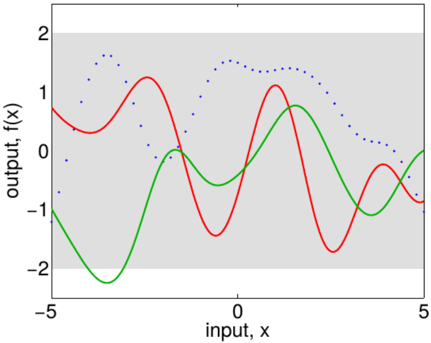
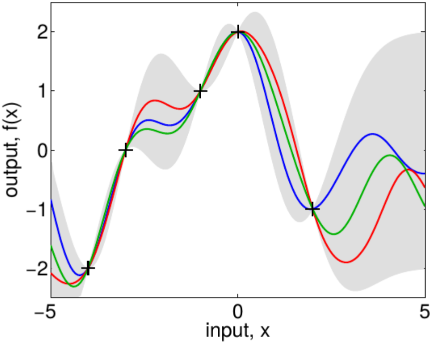
Leading to Gaussian Process Posterior:

$$f(x)|\mathbf{x}, \mathbf{y}, M \sim \mathcal{GP}(m_{\text{post}}(x) = k(x, \mathbf{x})[K(\mathbf{x}, \mathbf{x}) + \sigma^2_{noise} I]^{-1}\mathbf{y},$$
$$k_{\text{post}}(x, x') = k(x, x') - k(x, \mathbf{x})[K(\mathbf{x}, \mathbf{x}) + \sigma^2_{noise} I]^{-1} k(\mathbf{x}, x'))$$

Prior and Posterior for $\mathcal{GP}$ Learning





Gaussian Process Predictive Distribution:

$$p(y^*|x^*, \mathbf{x}, \mathbf{y}) \sim \mathcal{N}(k(x^*, \mathbf{x})[K + \sigma^2_{noise}]^{-1}\mathbf{y},$$
$$k(x*, x*) - k(x^*, \mathbf{x})[K + \sigma^2_{noise}I]^{-1}k(\mathbf{x}, x^*))$$

Optimizing Marginal Likelihood

$$\log p(\mathbf{y}|\mathbf{x}, M) = -\frac{1}{2}\mathbf{y}^T K^{-1}\mathbf{y} - \frac{1}{2}\log|K| - \frac{n}{2}\log(2\pi)$$

is a combination of data fit and complexity penalty terms. Occam's razor is automatic!

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Learning in Gaussian process models involves finding:

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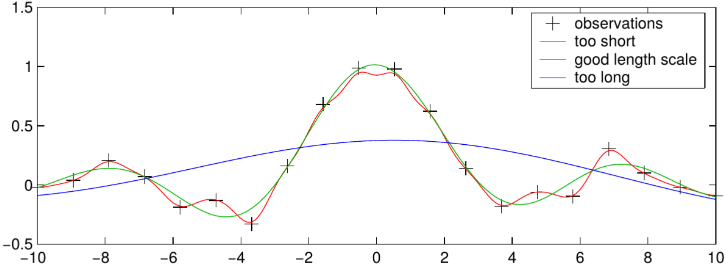
- Unknown hyperparameter values θ

This can be done by optimizing the marginal likielihood:

$$\frac{\partial \log p(\mathbf{y}|\mathbf{x}, \theta, M)}{\partial \theta_j} = \frac{1}{2}\mathbf{y}^T K^{-1}\frac{\partial K}{\partial \theta_j}K^{-1}\mathbf{y} - \frac{1}{2}\mathrm{trace}\left(K^{-1}\frac{\partial K}{\partial \theta_j}\right)$$

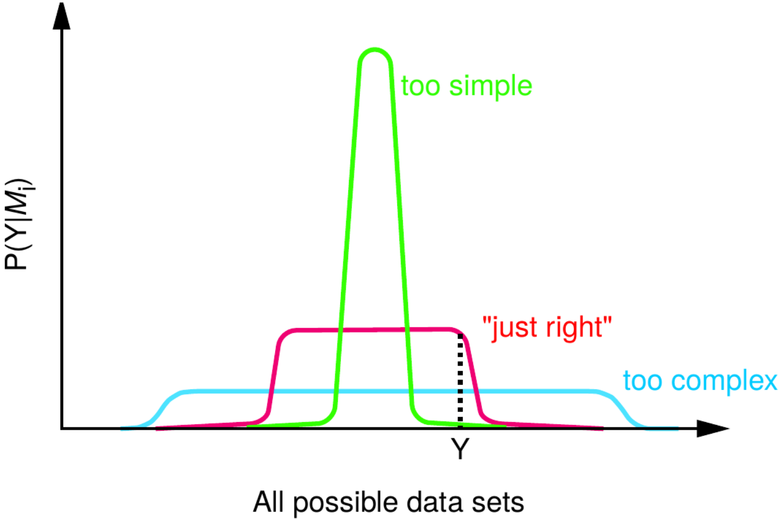
Example: Length Parameter Learning

Covariance function: $k(x, x') = \nu^2 \exp\left(-\frac{(x - x')^2}{2l^2}\right) + \sigma^2_{noise}\delta_{xx'}$



Posterior mean function is plotted for 3 different length scales, green curve maximizes marginal likelihood. Although exact fit for data can be found, marginal likelihood does not favour this!

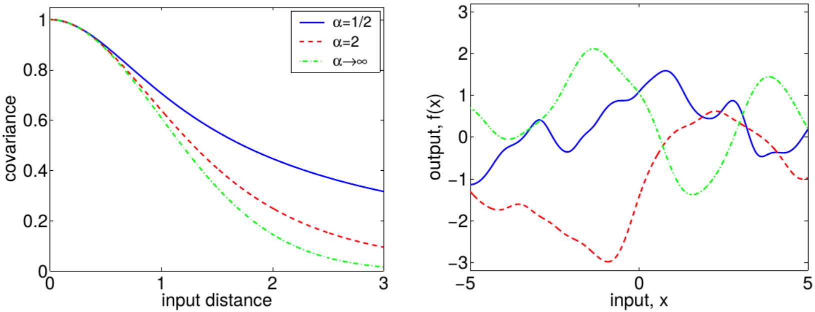
Why does Bayesian Inference work?: Occam's Razor



Rational Quadratic (RQ) Covariance Function

$$k_{RQ}(r) = \left(1 + \frac{r^2}{2\alpha l^2}\right)^{-\alpha}$$

with $\alpha, l > 0$ can be seen as an infinite sum of squared exponential (SE) covariance functions with differen length-scales.



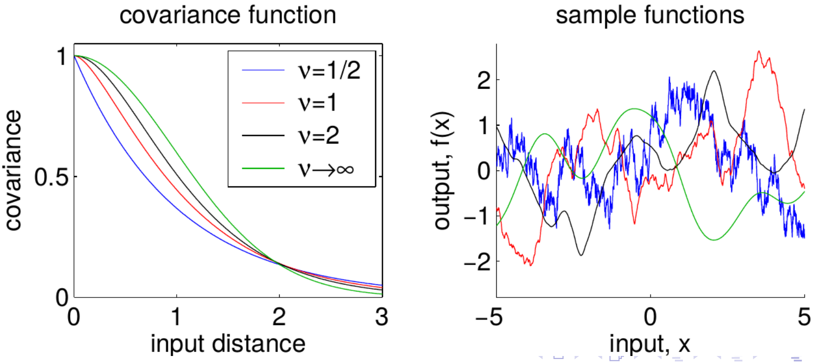
Limit $\alpha \leftarrow \infty$ of the RQ covariance function is SE.

Matern Covariance Function

$$k(x, x') = \frac{1}{\Gamma(\nu)2^{\nu-1}} \left[\frac{\sqrt{2\nu}}{l} |x - x'| \right]^{\nu} K_{\nu} \left(\frac{\sqrt{2\nu l}}{|} x - x'| \right)$$

where K_ν is a Bessel function of order ν, and l is the length scale.

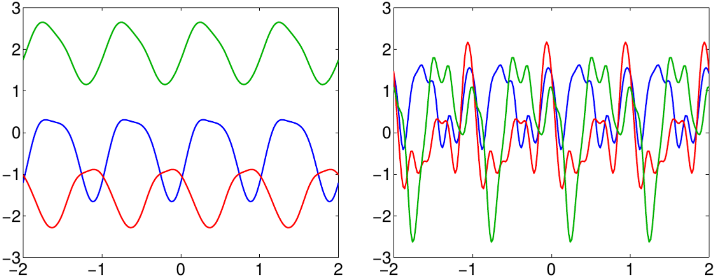
Samples of Matern forms are $\lfloor \nu - 1 \rfloor$ times differentiable.



Periodic Covariance Function

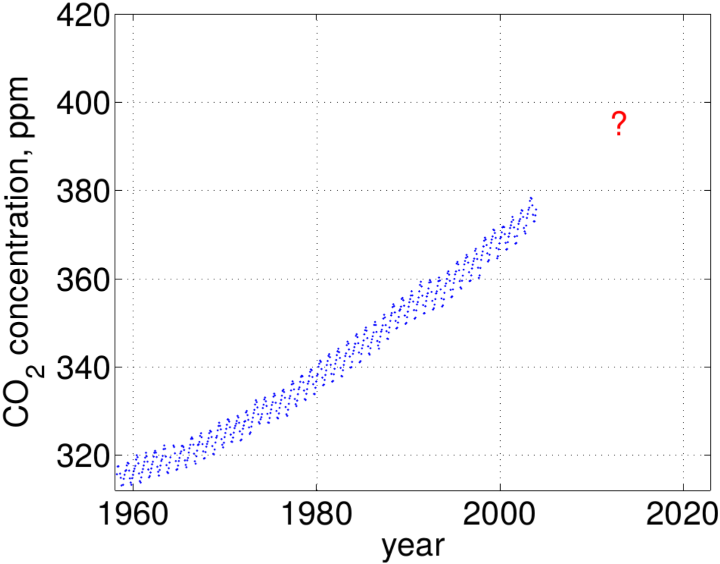
Periodic covariance functions can be obtained by mapping x to $u = (\sin(x), \cos(x))^T$ and combine with SE covariance function:

$$k_{periodic}(x, x') = \exp\left(-\frac{2\sin^2(\pi(x - x'))}{l^2}\right)$$



3 random samples with: left $l > 1$ and right $l < 1$

Prediction Problem



Covariance Functions

- long term smooth trend (squared exponential)

$$k_1(x, x') = \theta_1^2 \exp\left(\frac{(x - x')^2}{\theta_2^2}\right)$$

- seasonal trend (quasi-periodic smooth)

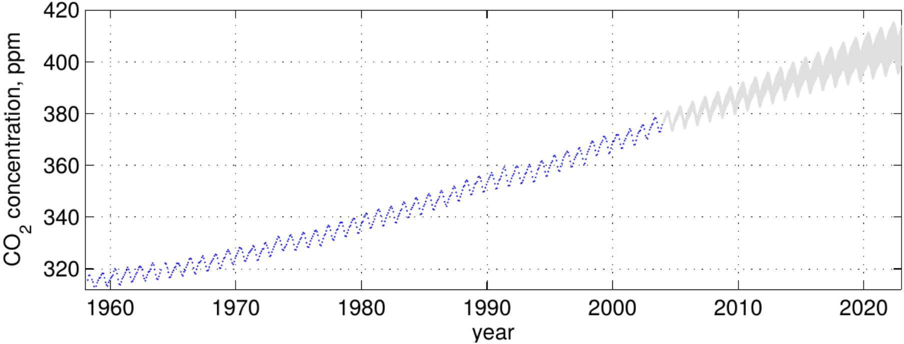
$$k_2(x, x') = \theta_3^2 \exp\left(-\frac{2\sin^2(\pi(x - x'))}{\theta_5^2}\right) \times \exp\left(\frac{(x - x')^2}{2\theta_4^2}\right)$$

- short and medium term anomaly (rational quadratic)

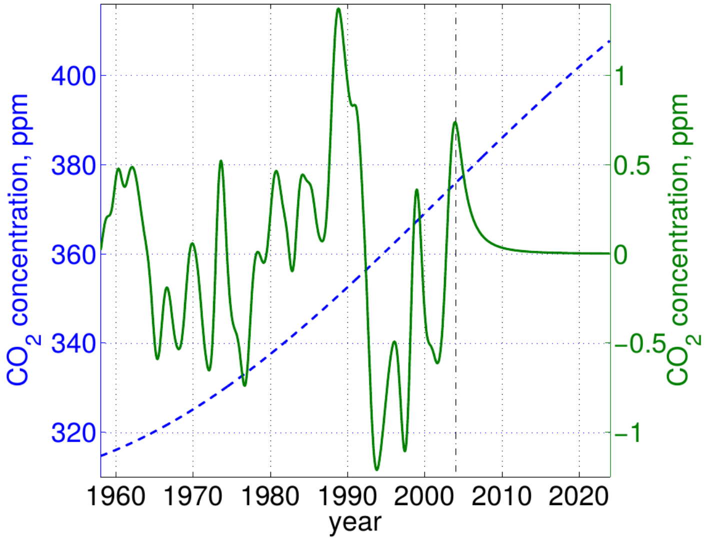
$$k_3(x, x') = \theta_6^2 \left(1 + \frac{(x - x')^2}{2\theta_8\theta_7^2}\right)^{-\theta_8}$$

$$k(x, x') = k_1(x, x') + k_2(x, x') + k_3(x, x') + \text{noise kernel}$$

Carbon Dioxide Predictions



Long and Medium-term Predictions



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Complex non-linear inference problems can be solved by manipulating plain old Gaussian Distributions

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- Predictions are probabilistic
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Scope for research:

- Interesting covariance functions
- Search for efficient approximations and sparse methods
- Application to high-dimensional data (Deep Learning)