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Mathematics for Data Science 2
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Lecture No. 08
Computing Derivatives and L'Hopital's Rule

Hello and welcome to the Maths 2 component of the online BSC Program on Data Science and Programming. In this video, we are going to talk about computing derivatives and L'Hopital's Rule. So, this is a continuation of our discussion on the notion of differentiation which we started because we wanted to talk about tangents and we are coming close to understanding how to talk about tangents.

So, before that we are going to talk about how to compute derivatives first and then we are going to see a very nice application of derivatives to computing limits which we used actually in defining derivatives. So, this is kind of an interesting interplay between how the derivative is useful in computing limits.

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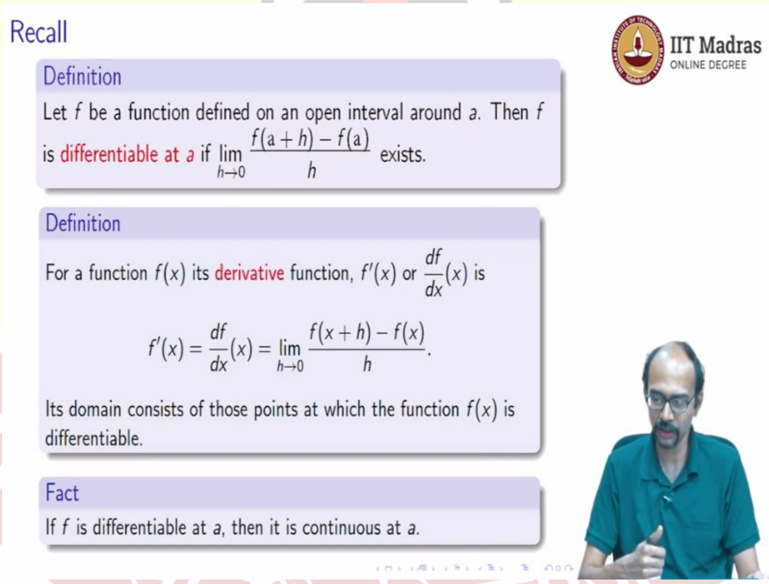

So, let us begin by recalling the notion of what is the derivative. So, let f be a function defined on an open interval around the point a. Then f is differentiable at a, if $\lim_{h\to 0} f\frac{(a+h)-f(a)}{h}$ exists. So, we have seen this definition in a previous video. So, this a, just to jog your memory about that and then we defined this limit to be the derivative. So, at every point for which this limit exists we call it $f'(x)$ or $\frac{df}{dx}(x)$

And then $f'(x)$ or $\frac{df}{dx}(x)$ so this defines a new function and that function is called the derivative function or just the derivative of f. And of course note that this limit may not exist at all points. So, this derivative function has domain which consist of those points at which the function $f(x)$ is differentiable. So, now colloquially one says that f is differentiable to mean that f is differentiable at all points. So, if one does not qualify at which points and just says that f is differentiable then one means that it is differentiable everywhere.

So, in that case the derivative function will be defined on all points for which f itself is defined. So, here is the fact which we indicated last time the proof of. So, if f is differentiable at a then it is continuous at a and it was this fact that allowed us to say that if f is differentiable then we get rid of the two problematic cases that we saw when we are going to talk about the tangent namely the function mod x which had a node and the function floor of x which was the step function. So, it was discontinuous.

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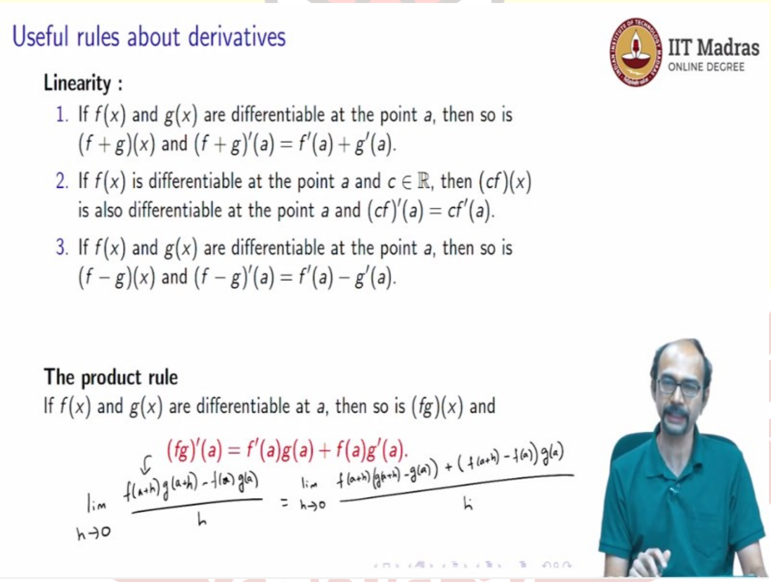

So, let us now talk about useful rules about derivatives. So, remember that we have had this discussion before about limits and about continuity as well. So, since the derivative is defined as in terms of limits it is natural that the rules that apply to limits are going to help us to get to rules about derivatives. So, here is the first rule so derivatives are linear so linearity. What does that mean?

If $f(x)$ and $g(x)$ are differentiable at that point a then so is f plus g of x and the derivative at a is the derivative of f at a plus the derivative of g at a which means $(f+g)'(x)=f'(x)+g(x)$. So, we saw something similar for limits and it follows in the straight forward way from there. If f(x) is differentiable at point a and we have a constant so some real number and we multiply this function f by the constant c, then cf(x) is a new function then this function is also going to be differentiable at the point a and c times f that function when you take its derivatives at a it will turn out that this is just $cf'(a)$.

Again, this is a very direct consequence of what happens for limits and then you can put the first two together and by taking c equals minus 1 you can obtain that if f and g are differentiable at the point a, then so is the function f minus g of x and indeed the derivative of that function at a is $f'(a)-g'(a)$. So, this is the statement about linearity.

Let us talk about the product rule. So, what happens if you have a product of functions? This is where it starts getting a little different from what we have seen before. If you remember for continuity and so on and just the limit. If the limit existed then they got multiplied, but here remember we are not taking limits of the functions themselves, what we are taking limit of is $\frac{f(x+h)-f(x)}{h}$. So, this is a much more trickier limit.

So, if you wanted to take the limit of the product so now here we have two functions f and g which are differentiable at a. So, then what happens is that f times g is also differentiable at a. And in fact the product the derivative of the product at a is $f'(a)g(a)+f(a)g'(a)$. So, this is a little different from what we saw. So, let me try to give you a brief idea of why it works out like this. So, if you want to compute this left hand side, this left hand side is given by $\lim_{h\to 0} f\frac{(a+h)g(a+h)-f(a)g(a)}{h}$, that is what this things going to be.

But now, I know individually that these are differentiable so what do you do you take a you add and subtract something in the middle. So, you do something like $\lim_{h\to 0}\frac{f(a+h)g(a+h)-g(a)+f(a+h)-f(a)g(a)}{h}$ and now you can break this into two separate limits and then when you take the limit you get exactly what you have here.

So, this is the idea behind this product rule. So, in fact I have given you a more or less complete proof of that as well with some small missing pieces if at all. So, whether or not you understood the proof, do remember the identity we will see examples of this in a few slides, but do remember the identity. So, this is about computation how do you compute the derivative of product?

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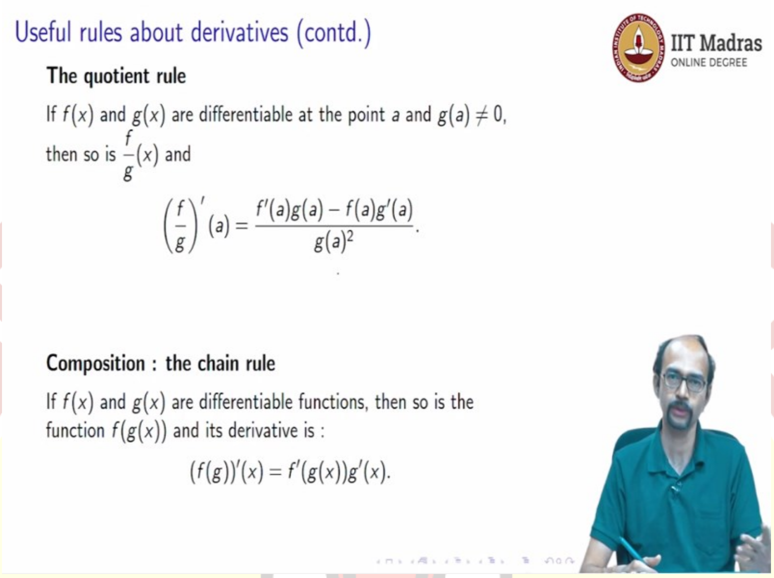

Let us see more rules. So, let us look at the quotient rule. So, now suppose you have two functions f and g both are differentiable at the point a and further g (a) is nonzero otherwise of course, it may be difficult to even consider what the function is at a of course it need not be defined at a, but in this case we would like that at least g is defined at a and is nonzero. So, then we can get the product.

So, if you have $\frac{f}{g}(x)$, then the derivative of f by g at a is this rather complicated looking formula $\frac{f'(a)g(a)-f(a)g'(a)}{g(a)^2}$. So, again the idea is somewhat similar to what I showed in the previous slide for the product, but you have to manipulate it a little bit more. I will encourage those of you who are more mathematical inclined to try out writing out the limit and seeing how to juggle the expression around.

It is not particularly difficult, but you will need a little bit of finesse in juggling the equations around, expressions around rather. So, again whether or not you understand the proof you should know how to apply it. So, if you have two functions f and g you take the quotient f by g, then the derivative is given by this formula and of course since g(a) is in the denominator g(a) better be nonzero.

So, this is a final rule we will talk about, the composition rule which is also called the chain rule and why it is called the chain rule we will see towards the end of this series of lectures because we will see the same thing in higher dimensions and that will explain the chain rule. So, if f and g are differentiable functions let us assume they are defined on the entire real line and they are differentiable wherever they are defined and you can compose them.

So, you have f (g(x)) I want to find out its derivative, then it is given by this formula: $(f(g))'(x)=f'(g(x))g'(x)$. So, this looks a little complicated, but we will see examples of what does this mean. So, if you are bothered by at this point by any of these rules and what they mean I will, I want to show you examples in the next slides about how to apply this rules.

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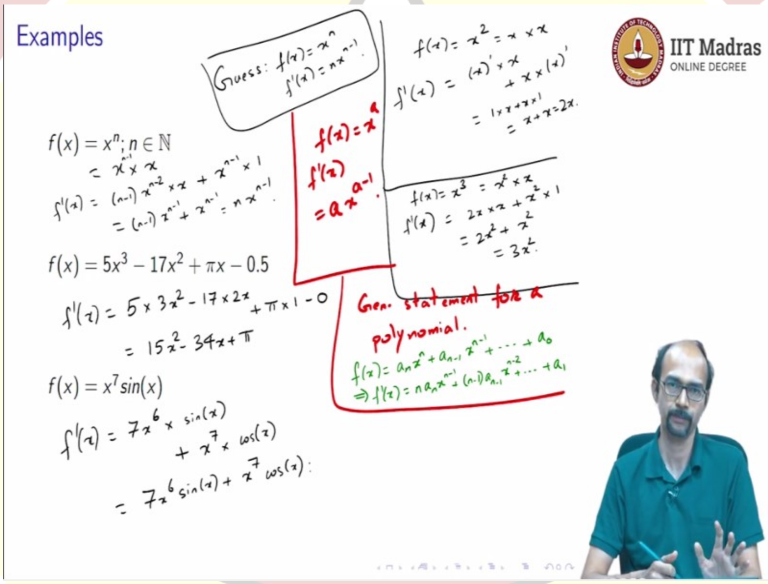

So, let us start by the baby examples. So, if you have $f(x)=x^n$, where n is a natural number. Now I think we have already seen how to get this limit, how to get the derivative of this from first principles, but here I want to use the product rule and try to do the same thing. Ok so, we will assume of course that we know what happens for f(x) = x. So for f(x) = x, we know that the derivative is 1.

So, now suppose you have x^2 let me do a case of $f(x)=x^2$. So, here you can write this as $f(x)=x^2$ and now if you want $f'(x)$, you get it $f'(x)=xx'+x'x$. So, derivative of x is 1 so $f'(x)=2x$.

So, if you feel if you think carefully I think you can see some pattern, but let me do one more example and from there we can establish the pattern clearly. So, now let us do $f(x)=x^3$. So, here I can write this as $f(x)=x^2x$, so if I want the derivative what do I do. I take the derivative of x square which is 2x multiplied by this function which is x and add x square which is this function times the derivative of x which is 1.

So, $f'(x)=2xx+x^2=3x^2$ and now I think you can see the pattern. So, the guess for us will be that $f'(x)=nx^{n-1}$ and so the guess is f of x is x to the n, f prime x is n times x to the n minus 1. Now, we will prove this, so to proof this the standard route is something called induction which means I will assume we know this for n minus 1 and then I will use it for x to the n.

So, I will write $f(x)=x^{n-1}x$ keeping with this example here. So, here we have x cube is x square times x. Here we have x to the n x to the n minus 1 times x and now f prime x is going to work out to be by induction we know that x^{n-1} the derivative is $(n-1)x^{n-2}$.

Add the same function x^{n-1} and multiply by the derivative of the second function which is 1 and that gives us $(n-1)x^{n-1}+x^{n-1}$. So, this is nx^{n-1}, which is exactly what we guessed here. So, the derivative of x^n is nx^{n-1} of course here we assume that n is a natural number.

So, here is a fact which I will not prove this you can do from first principles. If you have $f(x)=x^a$, then $f'(x)=ax^{a-1}$. So, this you have to do from first principle. There are some caveats of course once you start allowing a to be negative numbers then you have to be careful about whether it works for x = 0 and so on. So, with caveats this is the general situation. But the point I was trying to make here was how to apply the product rule so I hope that is clear.

Let us now take this polynomial function. Let us find its derivatives and now it is very easy to find this because I know linearity that means for each of these monomials each of these powers and its coefficients I can compute my derivative separately and then just add it or subtract it as per the sign. So, the derivative of $5x^3$ is 5 times the derivative of x^3 because remember constants come out.

So, the derivative of x^3 we just saw is $3x^2$ then minus 17 times the derivative of x square which is 2x plus pi times the derivative of x which is 1 and minus the derivative of the constant function which if you well we actually know that it is 0 that is from first principles or if you want to apply this formula, then you can think of that as x^0 or $0.5x^0$ and if you put into this formula you get $0x^{0-1}$.

Now, this is actually not a correct way of thinking about it, but either way it gives you the number 0 so this is 0. So, the derivative of constant is 0 a+nd that is how we got this. So, we get $15x^2-34x+\pi$. So, in general for a polynomial so let us write down what happens for a polynomial it is a general statement.

So, general statement for a polynomial. So, suppose $f(x)=a_n x^n+a_{n-1}x^{n-1}+\dots+a_0$, then $f(x)=a_n x^n+a_{n-1}x^{n-1}+\dots+a_1$. Why a1, because the constant term gets dropped and you had a1 times x, x the derivative is 1 so you get a1. So, this is a general statement for a polynomial and the proof is exactly on the same lines as we did for this special case. So, I will encourage you to check that.

Now, let us do $f(x)=x^7\sin(x)$. So, for this again you apply the product rule. So, if you do that f '(x) is going to be the derivative of x to the 7 which is 7x to the 7 minus 1 which is 6 times sin of x plus x to the 7 times the derivative of sin x which we know is cosine x is for something we did from first principle again. So, this is $7x^6\sin(x)+x^7\cos(x)$ So, I hope you are getting the hang of this.

So, you can now that if you have a complicated function also for function which a priori looks complicated the derivative are not all that hard to compute. If you know the derivatives of all the functions appearing in the expression of that function. If you know the individual derivatives often you will be able to put them together to get the derivative of the entire function.

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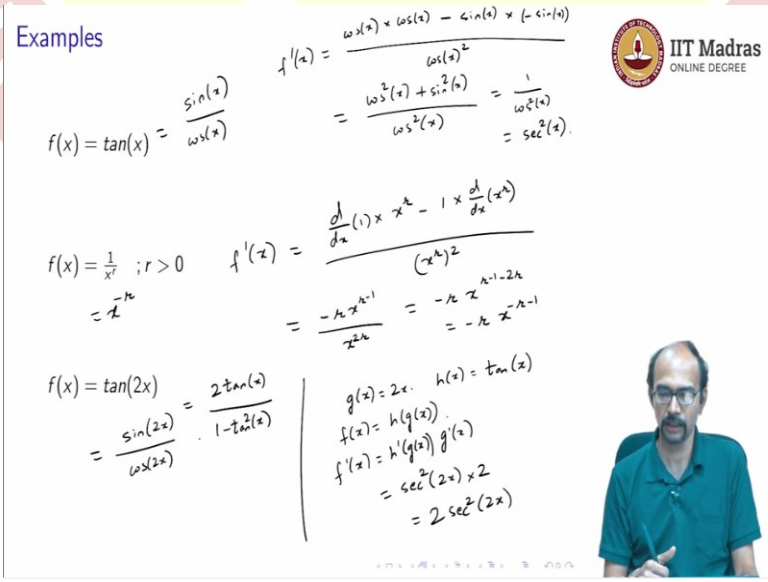

Let us do a couple of other functions so far we have not used the other two rules the quotient rule and the chain rule. So, let us do f (x) is tan x. So, of course already tan x, there is warning bells because as we know tan x is not defined at $\frac{-\pi}{2}$ and $\frac{\pi}{2}$ So, when we are doing this derivatives all this make sense only for values where the function is nice there are places where the function is not nice you have to be careful and do it in a more either from first principles or by in a more careful manner.

So, let us do f of x is tangent of x. So, tangent of x as we know is $\frac{\sin x}{\cos x}$ So, let us compute what is the derivative. So, we will apply the quotient rule so under the quotient rule f '(x) is going to be let me recall actually for you what is a quotient rule. Here is the quotient rule so f by g the derivative is $\frac{f'(a)g(a)-f(a)g'(a)}{g(a)^2}$

So, in our example tangent of x, f is sin ,g is cosine so first you are going to differentiate sin multiplied by cosine and then subtract out sin times derivative of cosine and then divide by cosine squared. So, let us do that so you have a prime x the derivative of sin is cos x, times cos x. So, the first cosine came as the derivative of sin x. The second cosine came from the cosine denominator cosine x minus sin of x which is the numerator multiplied by the derivative of the denominator which is minus sin of x.

So, the derivative of cosine x is minus sin x and then you divide by the denominator square and so this gives you $f'(x)=\frac{\cos(x)\cos(x)-\sin(x)(-\sin(x))}{\cos(x)^2}=\frac{\cos(x)^2+\sin(x)^2}{\cos(x)^2}$ and now we are familiar with this identity we have seen this before so this is 1, by $\cos(x)^2+\sin(x)^2=1.$ And $\frac{1}{\cos(x)^2}$ you can write as $sec(x)^2$ this was one of the definitions of secant.

So, what is the upshot? The upshot is that the derivative of the tangent function is the square of secant function. Now as I said of course you have to be careful that at pi by 2 and minus pi by 2 and all odd multiplies of pi by 2 at those values of course cosine is 0 and as a result this expression will not make sense. So, this is valid only for other values. What about $\frac{1}{x^r}$?

So, in the previous slide we talked about x to the power a that was a times x to the power a minus 1 and we showed it for x to the power n where n was a natural number. Now let us use that to, let us assume we know what happens for x to the power r when r is positive and then from there ask what happens for 1 over x to the power r. Again the same caveats as in the tan x example hold. This function is not defined at 0 so we are talking about all values other than 0.

So, what is the derivative here? So, by definition you take the denominator multiply by the derivative of the numerator subtract out, I wrote this in the opposite way let me write it in the way our rule is described. So, take the derivative of the numerator multiplied by the denominator, subtract out the numerator minus the derivative of the denominator and divide by the denominator square.

So, as we know $\frac{d}{dx}1=0$ for a constant the derivative is 0. So, the first term does not contribute and the second term what does it give us? So, in the second term we get rx^{r-1} and comes with a minus sign and then you divide by x to the power 2r and now we can rewrite this in a different way. So, remember that $\frac{1}{x^r}$ can also be written x^{-r}

So, this is $-rx^{r-1-2r}$ which is $-rx^{-r-1}$. So, this also satisfies the same rule as we saw for the natural number that if we have x to the power n the derivative is n times x to the power n minus 1. So, if you have x to the power minus r the derivative is minus r times x to the power -r-1.

And then, let us do tangent of 2x and I am giving this example for a specific reason. One way of doing is it is to do it as $\frac{\sin(2x)}{\cos(2x)}$, of course you can write it like this and then ask what is the derivative of sin 2x etc. what is the derivative of cos(2x). The other way is that we know this in terms of tangent of x. So, $\tan(2x)=\frac{2\tan(x)}{1-\tan^2 x}$ and now we know the derivative for all the terms involved.

But I am going to do a slightly different way so I am going to use the composition rule. So, what did the composition rule say? So, the composition rule told us that if you have function which you can compose so here we can write this function as so you take g (x) to be 2x and let us say h(x) to be tan(x) then f (x) is h(g(x)) and now I can apply my composition rule.

So, if you apply the composition rule you get $f'(x)=h'(g(x))g'(x)$. So, let us see what that gives us? That gives us h' we have just computed so tangent of x the derivative is $sec(x)^2$ So, this is $sec^2 g(x)$, but what is g(x)? 2x and then you multiply by g’(x). What is g*x) is 2x so g’(x) is 2. So, you get $2sec^2(2x)$.

And you can check using any of these forms $\frac{2\tan(x)}{1-\tan^2(x)}$ or $\frac{\sin(2x)}{\cos(2x)}$ that indeed you get the same answer I will encourage you to do that. So, in general what happens if you have a composed function you start differentiating from the one which is on the outside and keep whatever is inside intact, then you move inside take its derivative multiply that this is the rule.

So, I hope this explains what the chain rule is or what the composition rule is. So, we will of course see plenty of examples in the tutorials and exercise sheets and assignment sheets practice problems. So, I hope you will be very comfortable with taking one variable derivatives very shortly. Fine.

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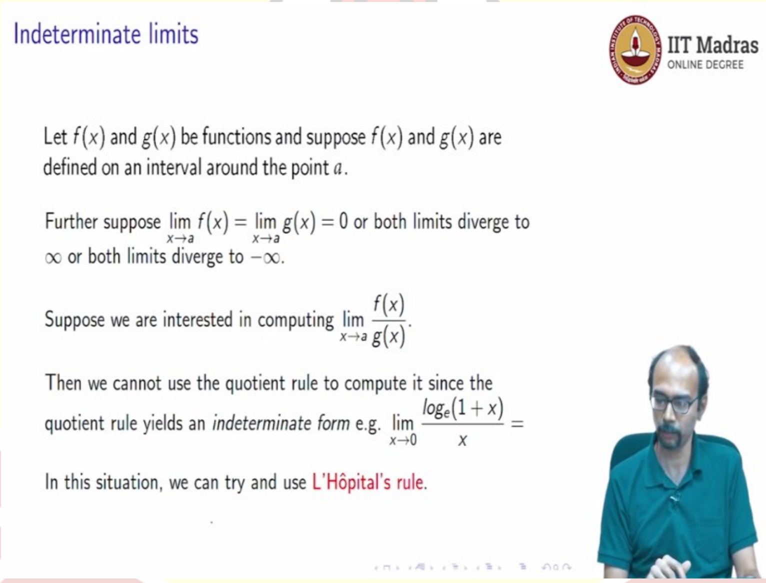

So, now let us talk about the case of indeterminate limits. So, this is an example this is a use of derivatives. So, why are derivatives useful of course we are going to see a much better use than this namely related to tangents, but this is also going to help us in some very interesting limits. So, what is an indeterminate limit? So, suppose f(x) and g(x) are two functions defined on an interval around the point a.

Further, suppose that these limits as you go towards a so $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ or both limits diverge to infinity or both limits diverge to minus infinity, one of these cases occurs either both limits are 0 or both limits are infinity or both limits are minus infinity. Suppose, we are interested in computing $\lim_{x\to a}\frac{f(x)}{g(x)}$.

Now, we have seen that in case both of the limits exists the numerator and denominator and the denominator limit is nonzero, then you just take those limits and the limit of f(x)/g(x) is the quotient of the individual limits.

But if you have that the denominator is 0 and the numerator is 0, then we do not really know what happens. If you know only the denominator is 0, but the numerator is limit is nonzero then we know that the limit does not exist, but if both of them are 0, then we do not know what to do or if both of them are infinity then we do not know what to do

So, in that case we cannot use the quotient rule. So, since the quotient rule yields what is called an indeterminate form. So, what is an indeterminate form that means the form 0/0 when applied individually or the form $\frac{\infty}{\infty}$ when you applied individually. So, here is an example which we have not completed so $\lim_{x\to 0}\frac{\log_e(1+x)}{x}$.

So, if you try to do this from the quotient rule then what do we get? So, remember that $\lim_{x\to 0}\log_e(1+x)$ log is a very nice function it is a continuous function so this limit you can just substitute so you will get $\log_e(1+0)$ which is log (1) which is 0. So, this numerator limit is 0 denominator limit is clearly 0 and so I have a 0/0 situation. So, this is a 0/0 situation.

And so I cannot use the quotient rule so how do I solve limit of this form? Now we have seen this particular limit we can do by using some tricks, but we are going to try and use something else called the L'Hopital's Rule so it is pronounced L'Hopital's so it is L with an apostrophe and Ho with a symbol this is French and it is called L'Hopital so the H is silent and actually this word after the L is the French word for Hospital.

So, this is someone's name it is a family name and L'Hopital was a French mathematician and this rule is quite unfortunately named after him although it is quite likely that it was discovered by his mentor whom he possibly paid off to get in named after himself. So, his mentor was someone in Bernoulli the name might be familiar from the statistics courses.

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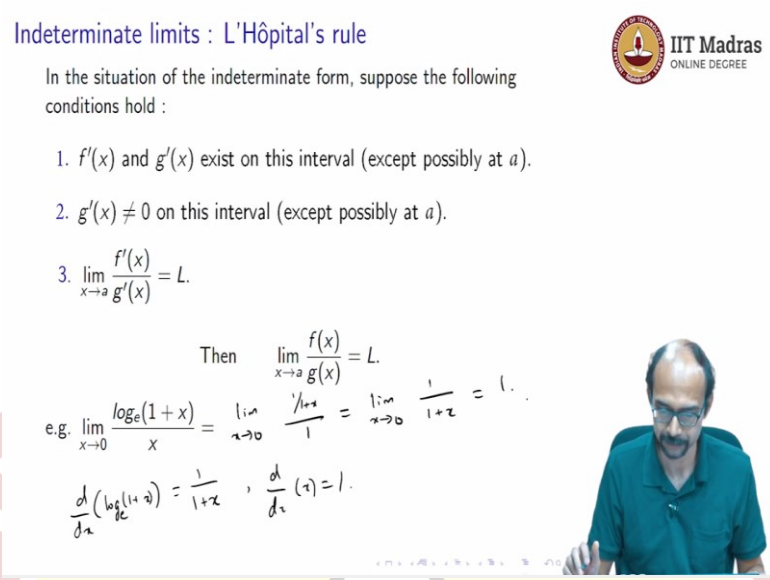

So, what is the L'Hopital Rule? So, in the situation of the indeterminate form suppose the following conditions hold f ' and g ' exists on this interval. So, which interval that same interval on which f and g exists except possibly at a, g prime x is nonzero on this interval except possibly at a and $\lim_{x \to a} \frac{f'(x)}{g'(x)} = L.$ This is extremely important that the limit of the quotient of the derivatives does exist.

Then, $\lim_{x \to a} \frac{f(x)}{g(x)} = L.$ This seems like somewhat of a miracle. So, what is the rule saying? The rule is saying that $\lim_{x \to a} \frac{f(x)}{g(x)}$ you cannot apply the quotient rule because you have an indeterminate form, but then what you can do is if this if both the derivative exists and satisfy these properties and the quotient of the derivatives exists, then the $\lim_{x \to a} \frac{f(x)}{g(x)}$ exists and is actually the limit of the quotient of the derivatives. So, it is the same L.

Let us do this example now. So, we have the indeterminate form so now let us apply L'Hopital's Rule. So, by L'Hopital's Rule you can differentiate the numerator and the denominator so what is the derivative of the numerator? So, the derivative of the numerator is $\frac{1}{1+x}$ this was something that we saw in the previous videos we actually saw a natural logarithm of x.

But you can use the composition rule to get this. It is a straightforward consequence. So, I will encourage you to check that and what is the derivative of the denominator? It is 1 and so if you look at $\frac{1}{1+x}$ divided by 1 you get $\frac{1}{1+x}$ and when you take the limit as x tends to 0 that limit certainly does exist and what is that limit? Well, you can really substitute because for that limit it is a continuous function at 0 no problem. So, you get $\frac{1}{1+0}$ which is 1.

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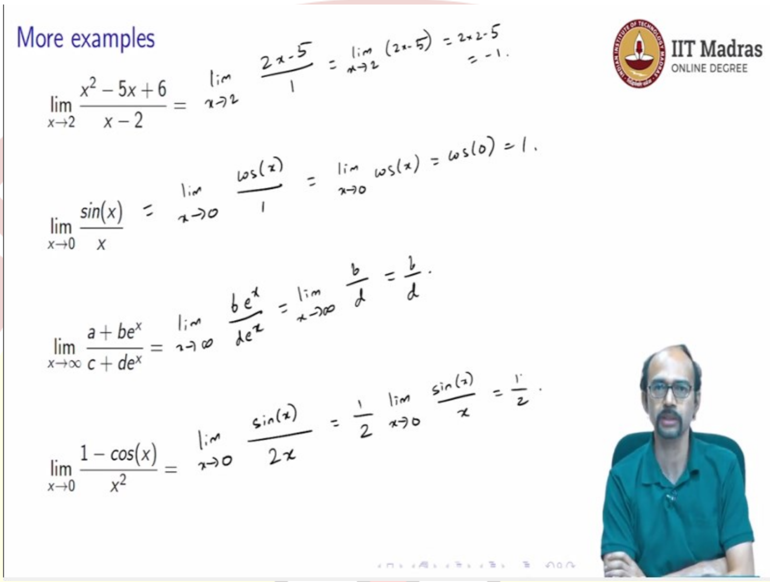

Let us do a couple of other examples. So, I am going to go through these rather fast. So, let us look at $\lim_{x\to 2} \frac{x^2-5x+6}{x-2}$ all of these actually we have done before, but now we are going to use L'Hopital's Rule. Well, this is a 0 by 0 form that we had seen before. So, what can I do? I can differentiate if you differentiate you get $\lim_{x\to 2} \frac{2x-5}{1}$.

And now if you apply the limit well this is just a polynomial function wonderful so what do I get? I get 2*2- 5 so 4-5 is -1. If you remember this is exactly what we got because this function is exactly the function x - 3 except at 2 and then you can just substitute and if you do that you will get a -1. I mean I am doing this fast ideally what we should have done is compute both the derivatives, check those conditions that we have before and then apply, but I know those are going to hold so I will encourage you to check that.

So, I am just going to directly do the calculation without doing the checking. So, now we have $\frac{\sin(x)}{x}$ if you remember this was a fairly complicated proof that this limit is 1. So, instead we can apply L'Hopital you differentiate the numerator you get cosine x differentiate the denominator you get 1. So, this is $\lim_{x \to 0} \cos(x)$ well cosine is a very nice function continuous function you can substitute cosine of 0 which is 1.

Let us do this one. So, this is $\lim_{x \to \infty} \frac{a+be^x}{c+de^x}$ Now, in our previous what I showed you before I have written down that it should be defined in the neighborhood of a where a is a point etc. So, we can allow that point to be infinity although infinity is not a point in the real line, but we can think of it as a point at the end whatever that means of the real line and if all the conditions that we had before instead of interval if they are satisfied for some n which is some you have an n and those conditions are satisfied for all x larger than that n, then we can apply the same rule for this.

So, L'Hopital's Rule applies to this condition and the situation as well. And here what do we get? We get the $\frac{\infty}{\infty}$ form because you have $\lim_{x \to \infty} \frac{a+be^\infty}{c+de^\infty}$ So, this is the infinity by infinity form. So, now what you can do is you can differentiate, if you differentiate you get $\lim_{x \to \infty} \frac{be^x}{de^x}$.

But this is well you can divide this you can knock out the e to the power x so that gives us b by d limit and of course this is a constant. So, the limit is going to be just that constant so this is b/d. So, if you remember how we did this earlier we multiplied everything by e to the power -x and then we got the limit and here the final example. So, this one we did by some trigonometric jugglery because we know that cosine of x we can express in terms of sin (x /2) and so on and we will use that in order to get this answer.

Instead what you can do is you can differentiate. If you differentiate the derivative of the numerator is sin x because for cosine x derivative is -sin x. So, you will get -(-sin x) which is sin x denominator the derivative is 2x we can take this half out and this is the limit that we understand very, very well. So, this is half we have done it right here by using L'Hopital's Rule or from first principles using geometry and that gives us half which is what we got earlier.

So, I hope this example convince you that L'Hopital's Rule is a very nice rule and many nice indeterminate cases can indeed be quickly resolved using this L'Hopital's Rule. So, in this video we have seen some useful ways of computing derivatives and we have seen an application to computing indeterminate limits. Thank you.