

Week 1 - graded assignment solution

(The numbers in your assignment may differ)

1) let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers.

let $a_n = 1$ for all n and $b_n = -1$ for all n .

$\{a_n\}$ converges to 1, $\{b_n\}$ converges to -1

$$a_n + b_n = 0 \quad \text{for all } n.$$

$\{a_n + b_n\}$ converges to 0.

Hence option 1 is not correct.

let $\{a_n\}$ be an increasing sequence.

$$a_n = n \quad \text{for all } n.$$

$$(-1)^n a_n = (-1)^n n.$$

Hence $\{(-1)^n a_n\}$ is not a decreasing sequence in this case.

Hence option 2 is not correct.

If $\{a_n\} \rightarrow a$, $\{b_n\} \rightarrow b$

then, $\{a_n b_n\} \rightarrow ab$.

If both a and b are non-zero, then ab must be non-zero.

Hence option 3 is correct.

If $\{a_n\} \rightarrow a$ and $\{b_n\} \rightarrow a$

then $\{a_n - b_n\} \rightarrow a - a = 0$.

Hence option 4 is correct.

Let $a_n = \begin{cases} n & \text{when } n = \text{odd.} \\ 1 & \text{when } n = \text{even.} \end{cases}$

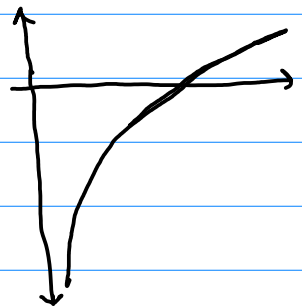
Hence, a_n is a divergent sequence.

But $\{a_{2n}\}$ is a constant sequence $\{1\}$,

which is obviously a convergent subsequence of $\{a_n\}$.

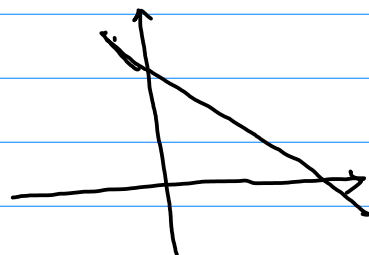
Hence option 5 is not correct.

2) i) $f(x) = 3 \ln x - 2$ d) logarithmic function 2)

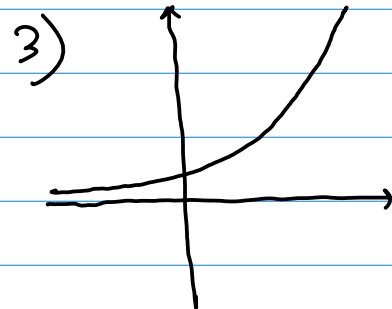


ii) $f(x) = 10^{-4}x$ c) linear function

4)



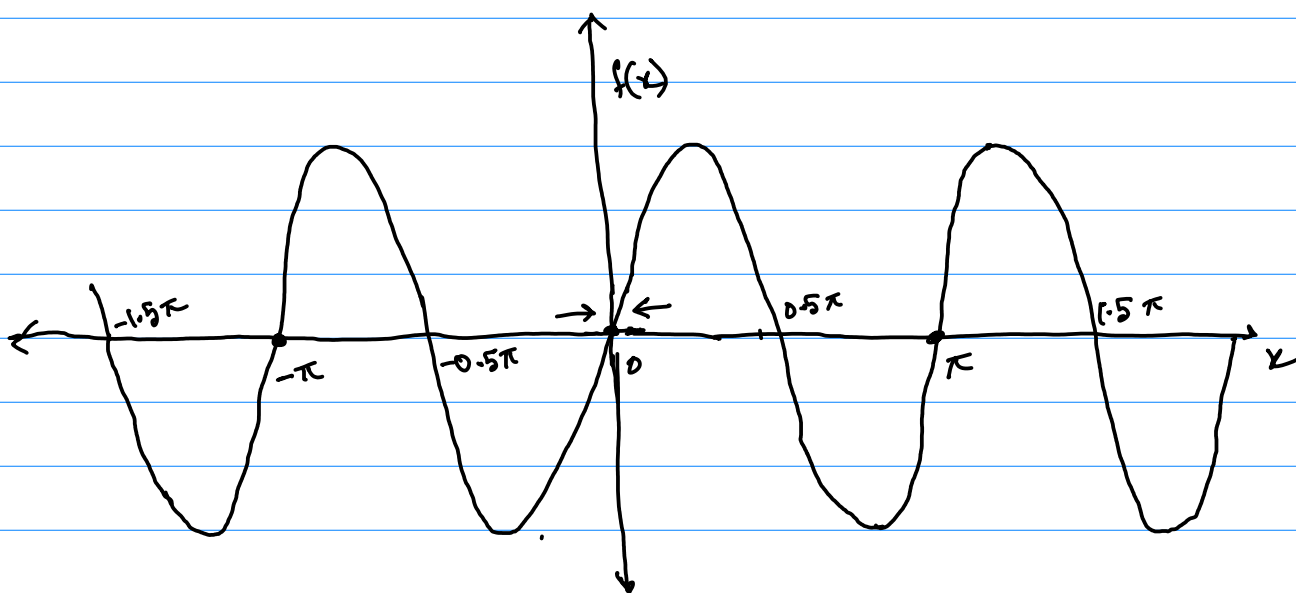
iii) $f(x) = 2^x + 7$ a) exponential function



iv) $f(x) = x^2 - 4x + 4$ b) quadratic function



3)



$\lim_{x \rightarrow 0^+} f(x) = 0 = \lim_{x \rightarrow 0^-} f(x)$. Hence Option 2 is correct.

At $x = \pi$ and $x = -\pi$ there are no sharp corners at the given curve. So, option 5 is correct.

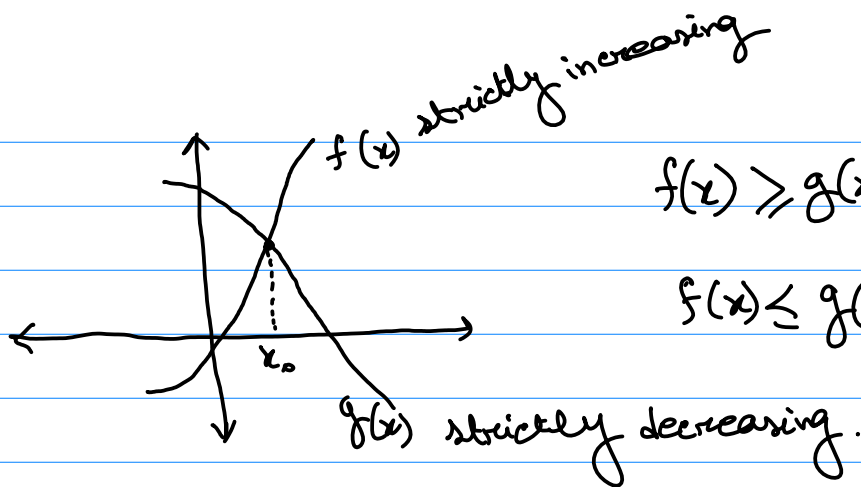
In the interval $[-0.5\pi, 0.5\pi]$ the function is oscillatory (neither monotonically increasing nor monotonically decreasing).

$$4) \lim_{x \rightarrow 0} \frac{\log(1+x)}{\sin x} = \lim_{x \rightarrow 0} \frac{\frac{\log(1+x)}{x}}{\frac{\sin x}{x}} = \frac{\lim_{x \rightarrow 0} \frac{\log(1+x)}{x}}{\lim_{x \rightarrow 0} \frac{\sin x}{x}} = \frac{1}{1} = 1.$$

$$\lim_{x \rightarrow 0} \frac{\sin 5x}{x} = \lim_{x \rightarrow 0} \frac{\sin 5x}{5x} \times 5 = 5 \lim_{x \rightarrow 0} \frac{\sin 5x}{5x} = 5 \times 1 = 5$$

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{e^{x/2} - 1}{\sin 2x} &= \lim_{x \rightarrow 0} \frac{\frac{e^{x/2} - 1}{x/2} \times \frac{x}{2}}{\frac{\sin 2x}{2x} \times 2x} \\ &= \lim_{x \rightarrow 0} \frac{\frac{e^{x/2} - 1}{x/2}}{\frac{\sin 2x}{2x}} \times \frac{1}{4} \\ &= 1 \times \frac{1}{4} = \frac{1}{4} \end{aligned}$$

5)



$$f(x) \geq g(x) \quad \text{for all } x \geq x_0.$$

$$f(x) \leq g(x) \quad \text{for } x \leq x_0.$$

We have, $f(x_0) = g(x_0)$. But for any $x > x_0$, $f(x)$ and $g(x)$ will never intersect. So, Option 2 is incorrect.

6)

$$a_n = \frac{12n^2}{3n+5} - \frac{4n^2+7}{n+3}$$

$$= \frac{12n^2(n+3) - (4n^2+7)(3n+5)}{(3n+5)(n+3)}$$

$$= \frac{\cancel{12n^3} + 36n^2 - \cancel{12n^3} - 21n - 20n^2 - 35}{3n^2 + 5n + 9n + 15}$$

$$= \frac{16n^2 - 21n - 35}{3n^2 + 14n + 15}$$

$$= \frac{16 - \frac{21}{n} - \frac{35}{n^2}}{3 + \frac{14}{n} + \frac{15}{n^2}}$$

$$\text{as, } n \rightarrow \infty, a_n \rightarrow \frac{16}{3}.$$

$$7) \quad R(\omega) = \frac{50e^\omega}{10 + e^\omega} = \frac{50}{\frac{10}{e^\omega} + 1}$$

$$\text{If } \omega_1 \geq \omega_2 \text{ then, } e^{\omega_1} \geq e^{\omega_2}$$

$$\Rightarrow \frac{10}{e^{\omega_1}} \leq \frac{10}{e^{\omega_2}}$$

$$\Rightarrow \frac{10}{e^{\omega_1}} + 1 \leq \frac{10}{e^{\omega_2}} + 1$$

$$\Rightarrow \frac{50}{\frac{10}{e^{\omega_1}} + 1} \geq \frac{50}{\frac{10}{e^{\omega_2}} + 1}$$

$$\Rightarrow R(\omega_1) \geq R(\omega_2)$$

$R(\omega)$ is increasing function.

$$\lim_{\omega \rightarrow \infty} R(\omega) = 50$$

Hence, the minimum possible value of r such that $R(\omega) < r$, for all ω , where $r \in \mathbb{Z}$, is 50.

$$8) \lim_{n \rightarrow \infty} e^{\sqrt[n]{n!}} \left[\log(1 + 6/n) - \frac{e^{1/n} - 1}{(\sqrt{2\pi n})^{1/n}} \right]$$

$$= \lim_{n \rightarrow \infty} e^{\sqrt[n]{n!}} \left[\frac{\log(1 + 6/n)}{6/n} \times \frac{6}{n} - \frac{\frac{e^{1/n} - 1}{1/n} \times 1/n}{\frac{n (\sqrt{2\pi n})^{1/n}}{(n!)^{1/n}} \cdot \frac{(n!)^{1/n}}{n}} \right]$$

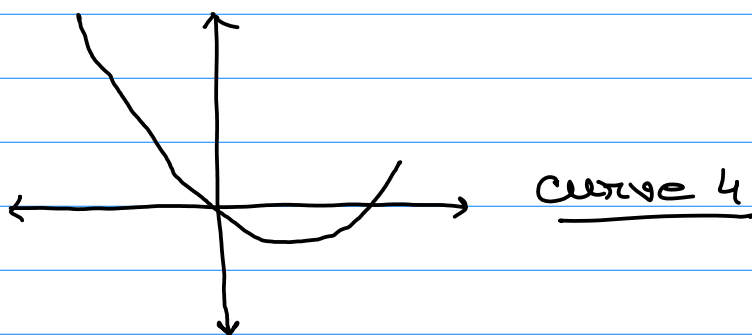
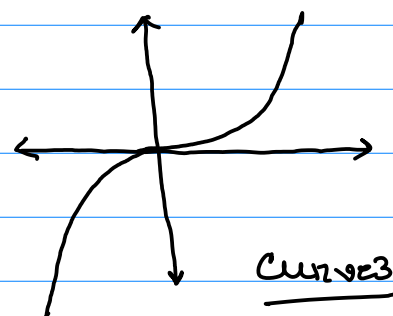
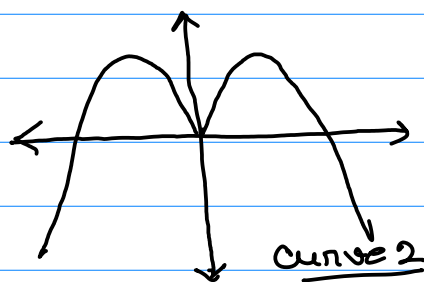
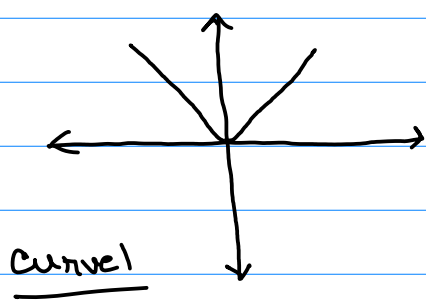
$$= \lim_{n \rightarrow \infty} e^{\sqrt[n]{n!}} \left[\frac{\log(1 + 6/n)}{6/n} \times \frac{6}{\frac{n}{\sqrt[n]{n!}}} - \frac{\frac{e^{1/n} - 1}{1/n}}{n \left(\frac{\sqrt{2\pi n}}{n!} \right)^{1/n}} \times \frac{1/n \cdot \sqrt[n]{n!}}{\frac{\sqrt[n]{n!}}{n}} \right]$$

let $1/n = x$. As $n \rightarrow \infty$, $x \rightarrow 0$

$$= e^{\left[\lim_{x \rightarrow 0} \frac{\log(1 + 6x)}{6x} \times \frac{6}{\lim_{n \rightarrow \infty} \frac{n}{\sqrt[n]{n!}}} - \frac{\lim_{x \rightarrow 0} \frac{e^x - 1}{x}}{\lim_{n \rightarrow \infty} n \left(\frac{\sqrt{2\pi n}}{n!} \right)^{1/n}} \times 1 \right]}$$

$$= e^{\left[1 \times \frac{6}{e} - \frac{1}{e} \times 1 \right]} = 6 - 1 = 5$$

9) Both curve 1 and curve 2 have sharp corners at the origin $(0,0)$. Hence, at the origin these two curves do not have tangents at the origin.



$$10) \quad a_n = \frac{9 + 15 + 21 + \dots + 3(2n-1)}{n^2}$$

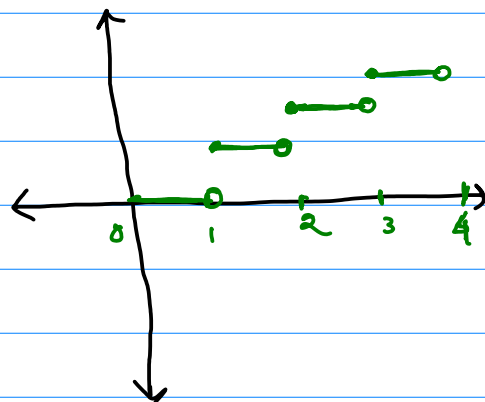
$$= \frac{3(3 + 5 + 7 + \dots + (2n-1))}{n^2}$$

$$= \frac{3(n^2 - 1)}{n^2} = 3\left(1 - \frac{1}{n^2}\right)$$

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} 3\left(1 - \frac{1}{n^2}\right) = 3$$

$$11) \quad 5 \lim_{x \rightarrow 3^+} \lfloor x \rfloor - 3 \lim_{x \rightarrow 1^-} \lfloor x \rfloor$$

$$= 5 \times 3 - 3 \times 0 = 15$$



Comprehension Type Question:

12) Error estimation by Algorithm 1:

$$a_n = \frac{n^2 + 5n}{6n^2 + 1}$$

$$\lim a_n = \lim_{n \rightarrow \infty} \frac{n^2 + 5n}{6n^2 + 1}$$

$$= \lim_{n \rightarrow \infty} \frac{1 + 5/n}{6 + 1/n} = \frac{1}{6} \approx 0.166$$

Error estimation by Algorithm 2:

$$b_n = \frac{1}{8} + (-1)^n \frac{1}{n}$$

$$\lim b_n = \frac{1}{8} + \lim_{n \rightarrow \infty} (-1)^n \cdot \frac{1}{n}$$

$$= \frac{1}{8} + 0 = \frac{1}{8} = 0.125 \quad \left(\text{Ans, } -\frac{1}{n} \leq \frac{(-1)^n}{n} \leq \frac{1}{n} \right)$$

Error in estimation by Algorithm 3:

$$C_n = \frac{e^n + 4}{7e^n}$$

$$\begin{aligned}\lim C_n &= \lim_{n \rightarrow \infty} \frac{e^n + 4}{7e^n} = \lim_{n \rightarrow \infty} \frac{1 + \frac{4}{e^n}}{7} \\ &= \frac{1}{7} \approx 0.143\end{aligned}$$

Maximum error estimation will be given by Algorithm 1.

Minimum error estimation will be given by Algorithm 2.

13) Error in estimation by the new algorithm:

$$\lim (a_n - b_n) = \lim a_n - \lim b_n$$

$$= \frac{1}{6} - \frac{1}{8}$$

$$= \frac{4 - 3}{24} = \frac{1}{24}$$

The error in estimation using the new algorithm is less than the error in estimation using any of the Algorithm 1, Algorithm 2 and Algorithm 3.

14)

$$c'_n = ne^{\frac{1}{8n}} - n$$

$$= \frac{e^{\frac{1}{8n}} - 1}{\frac{1}{8n}} \times \frac{1}{8}$$

$$\lim_{n \rightarrow \infty} c'_n = \lim_{n \rightarrow \infty} \frac{e^{\frac{1}{8n}} - 1}{\frac{1}{8n}} \times \frac{1}{8} = \frac{1}{8} = 0.125$$