# Statistics for Data Science -1

## Lecture 9.2: Expectation of a Random Variable

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# Learning objectives

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4. Cumulative distribution function, graphs, and examples.
5. Expectation and variance of a random variable.

# Expectation of a random variable

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  *: A winning of $-x$ indicates a loss of $x$ amount.
- Question: Would you play this game?

# Simulating the game

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- First, roll the dice 100 times. Observe the outcomes. They are summarised in the table.

| | | | | |
|---|---|---|---|---|
| 1 | 5 | 4 | 6 | 4 |
| 6 | 4 | 3 | 1 | 5 |
| 5 | 2 | 1 | 6 | 2 |
| 6 | 5 | 4 | 5 | 5 |
| 2 | 4 | 2 | 2 | 5 |
| 4 | 1 | 3 | 4 | 5 |
| 6 | 1 | 5 | 2 | 5 |
| 1 | 6 | 6 | 5 | 2 |
| 3 | 6 | 5 | 6 | 3 |
| 5 | 3 | 2 | 5 | 4 |
| 3 | 3 | 5 | 4 | 4 |
| 5 | 1 | 2 | 3 | 4 |
| 3 | 2 | 1 | 1 | 6 |
| 1 | 3 | 4 | 3 | 4 |
| 4 | 4 | 5 | 6 | 1 |
| 3 | 3 | 5 | 1 | 1 |
| 1 | 6 | 5 | 4 | 3 |
| 5 | 1 | 4 | 6 | 5 |
| 4 | 4 | 4 | 3 | 6 |
| 5 | 5 | 4 | 1 | 3 |

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- This is close to what we got as the average winning for 1000 rolls of the dice.

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- The Expectation of a random variable can be considered the "long-run-average" value of the random variable in repeated independent observations.
- Lets apply the definition to the examples we have considered before

## Rolling a dice once

- Random experiment: Roll a dice once.
- Sample space: $S = \{1, 2, 3, 4, 5, 6\}$
- Random variable $X$ is the outcome of the roll.
- The probability distribution is given by

| $X$ | 1 | 2 | 3 | 4 | 5 | 6 |
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- Does this mean that if we roll a dice once, should we expect the outcome to be 3.5?
- NO!!-the expected value tells us is what we would expect the average of a large number of rolls to be in the long run.

# Summary of the rolling dice simulation

| | 100 rolls | | 1000 rolls | | |
|---|---|---|---|---|---|
| Outcome | Freq | Rel. Freq | Freq | Rel. Freq | Probability |
| 1 | 16 | 0.16 | 177 | 0.177 | 0.166667 |
| 2 | 10 | 0.1 | 177 | 0.177 | 0.166667 |
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- Notice that average of the rolls need not be exactly 3.5.
- However, we can expect it to be close to 3.5.
- The expected value of $X$ is a theoretical average.

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- $X$ is a random variable which is defined as sum of outcomes
- Probability mass function

| $X$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $P(X = x_i)$ | $\frac{1}{36}$ | $\frac{2}{36}$ | $\frac{3}{36}$ | $\frac{4}{36}$ | $\frac{5}{36}$ | $\frac{6}{36}$ | $\frac{5}{36}$ | $\frac{4}{36}$ | $\frac{3}{36}$ | $\frac{2}{36}$ | $\frac{1}{36}$ |

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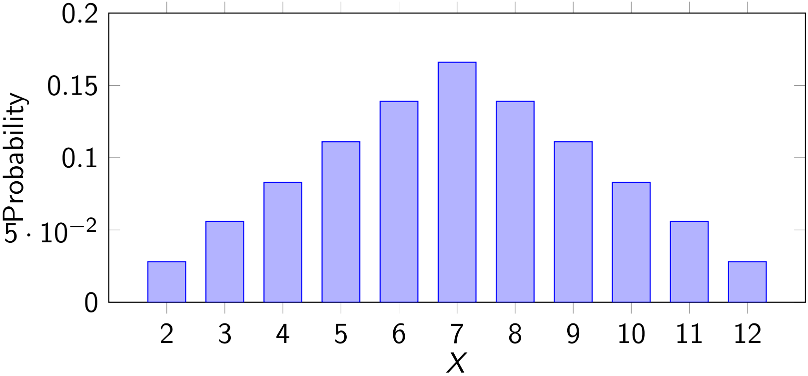
- If I rolled two dice a large number of times, what can I expect the average of the sum of the outcomes to be?

$$E(X) = 2 \times \frac{1}{36} + 3 \times \frac{2}{36} + \ldots + 11 \times \frac{2}{36} + 12 \times \frac{1}{36} = 7$$

- Interpretation: When two dice are rolled over and over for a long time, the mean sum of the two dice is 7.


0.2
0.15
0.1
$5 \cdot 10^{-2}$
0
Probability
2
3
4
5
6
7
8
9
10
11
12
$X$

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$$\frac{(0 \times 1) + (1 \times 3) + (2 \times 3) + (3 \times 1)}{8} = \frac{3}{2}$$

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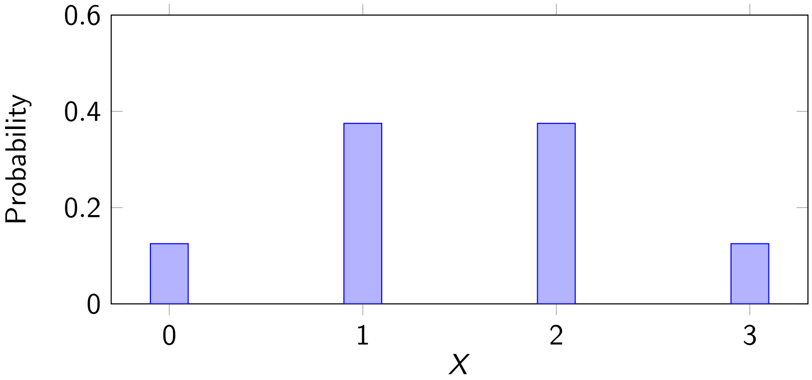
- Interpretation: When a coin is tossed three times over and over for a long time, the mean number of heads in the three tosses is 1.5.


0.6
0.4
0.2
0
Probability
0
1
2
3
X

# Bernoulli random variable

- A random variable that takes on either the value 1 or 0 is called a Bernoulli random variable.
- Let $X$ be a Bernoulli random variable that takes on the value 1 with probability p.
- The probability distribution of the random variable is

| $X$ | 0 | 1 |
|---|---|---|
| $P(X = x_i)$ | $1 - p$ | $p$ |

- Expected value of a Bernoulli random varaible:

$$E(X) = 0 \times (1 - p) + 1 \times p = p$$

# Discrete uniform random variable

- Let $X$ be a random variable that is equally likely to takes any of the values $1, 2, \ldots, n$
- Probability mass function

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- $E(X) = \sum_{i=1}^{n} x_i p(x_i) = \dfrac{(1 \times 1) + (2 \times 1) + \ldots + (n \times 1)}{n} = \dfrac{n(n+1)}{2 \times n} = \dfrac{(n+1)}{2}$

# Section summary

- Notion of expectation
- Bernoulli and Discrete uniform random variable.