IIT Madras

ONLINE DEGREE

Mathematics for Data Science 1
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Week - 01
Lecture – 10
Set versus Collections

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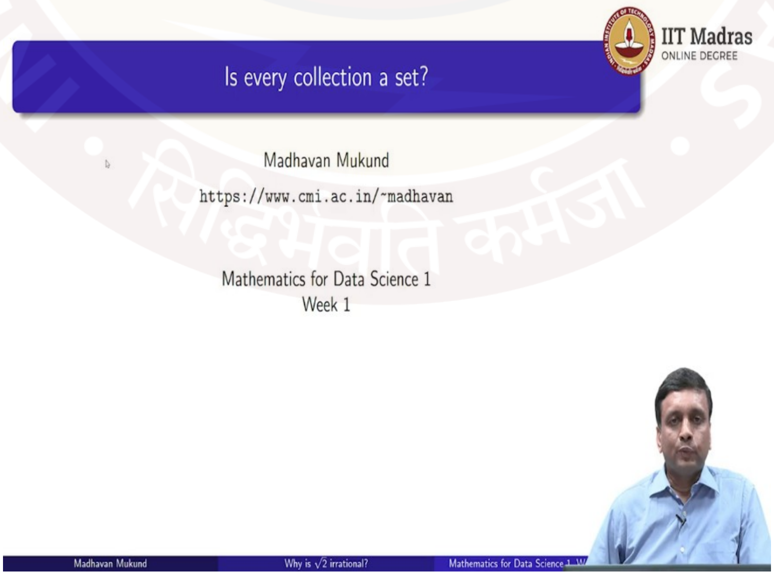

So, we have looked at sets, and we said that a set loosely speaking is a collection of items. And then we made some remarks in that lecture that not everything can be thought of with a set. So, let us ask whether every collection is in fact a set, and if not, why not?

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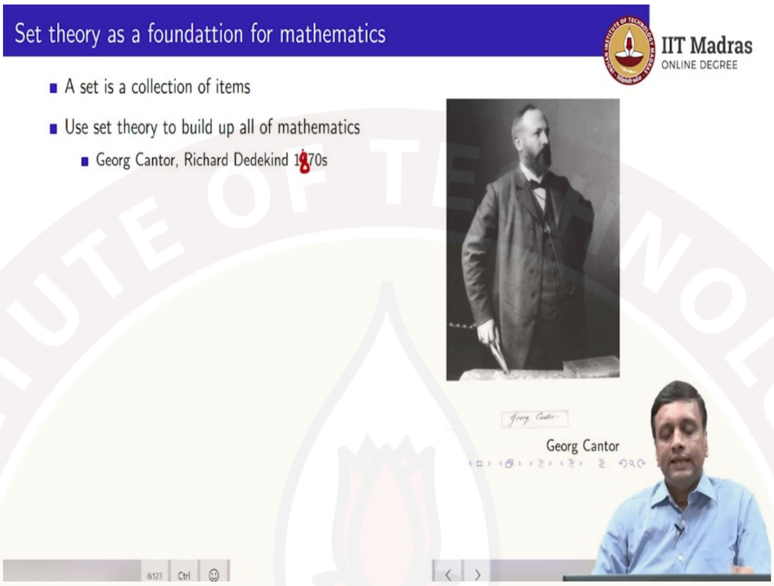

So, as we said a set is a collection of items. And when set theory was investigated formally starting from the late 1800s, the idea was to make set theory a foundation of mathematics. So, let us try to briefly understand what that means. So, we wanted to the mathematicians of the time wanted to start off with very basic things and build up all of mathematics from that, and they felt that set theory was a good place to start.

So, some of the mathematicians who are involved in this was Georg Cantor and Richard Dedekind from the 1870. So, this is a mistake, this is not the 1970s of course, but the 1870s.

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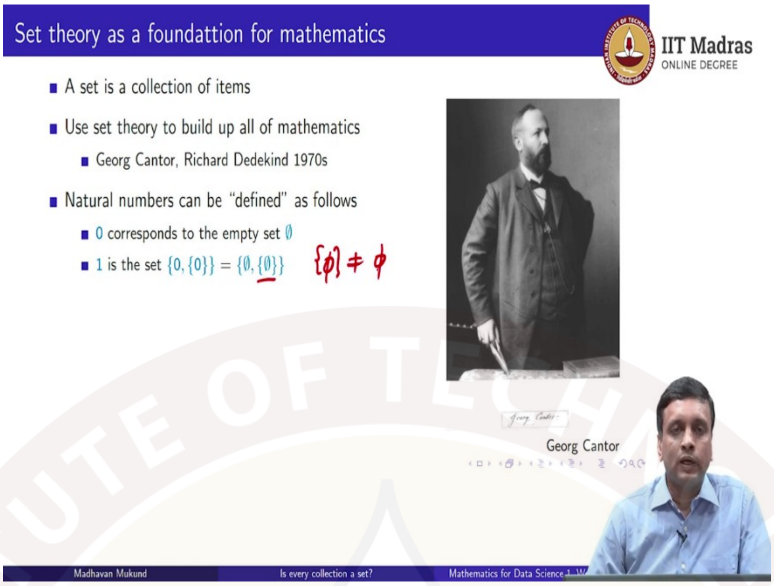

So, one is aspect of this foundational nature of set theory is for insists how do you generate numbers if you have only sets. So, one of the things that you need if you start with set theory is the empty set. So, you have it for free. So, what they said is that 0 can be thought of as the empty set.

So, we are going to use sets to represent numbers, and we are going to use the empty set to stand for 0. So, what is 1? Well, 1 is a set that consists of 0 and the set containing 0; in other words it is a set containing the empty set, and the set containing empty set. So, remember that the set containing empty set this is not the same as this right. The empty set has no elements; the set containing the empty set has one element.

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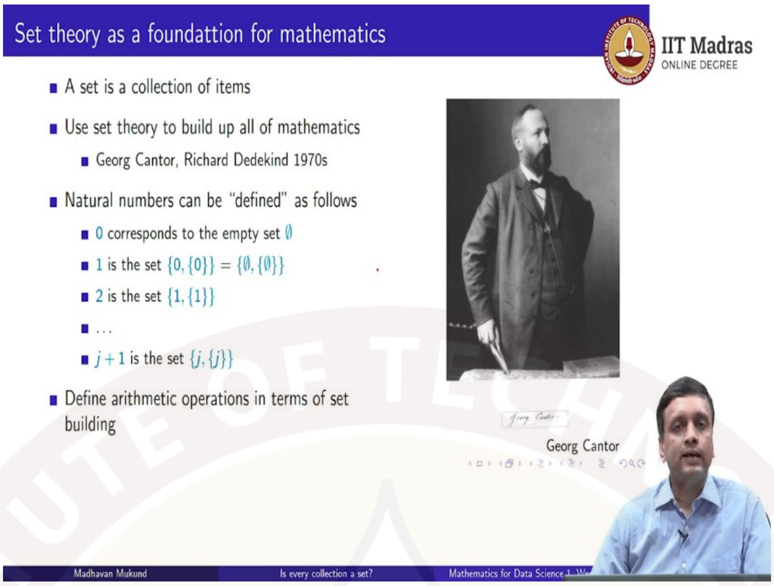

Similarly, 2 would be the set which contains 1 in the representation above, and the set containing 1. So, it is a bit tedious to write out. So, I have not expanded it. But you just take the expression for one in terms of the empty set replace it twice, and you get the number 2. And in this way for any number j plus 1, you can get it from the number j by taking the representation of j adding the set containing the representative j putting it into a new set.

So, these are the natural numbers as expressed using sets starting from the empty set. And then you can actually define set theoretic ways of combining these two define, the addition of two numbers and this format to get a new number which is the sum and the product and so on. So, this is what it means to use something like set theory as a foundation of mathematics.

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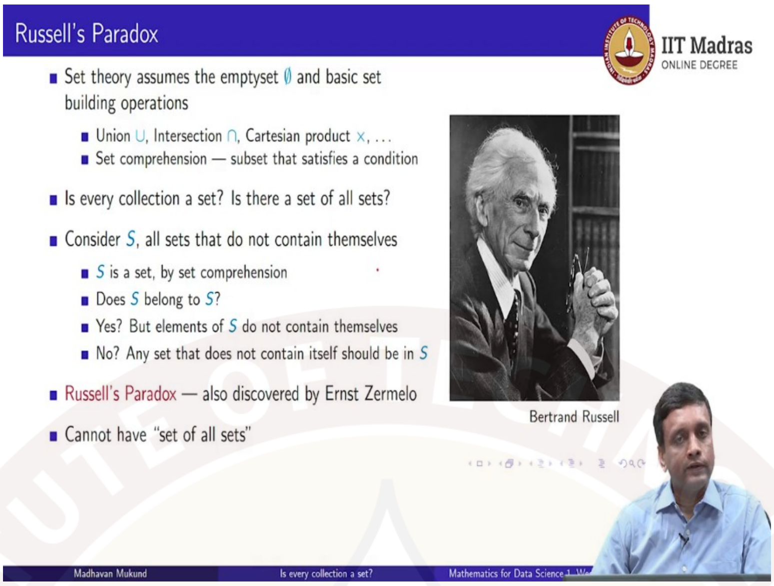

So, basically set theory assumes that you have the empty set, and then you have basic set building operations. For instance, you can take the union of sets, you can take the intersection of sets, you can take the Cartesian product which we saw when we were looking at relations. And you can of course to set comprehension which is that you can take some elements from a set which satisfy a condition and build a subset.

So, now into this picture came Bertrand Russell and he asked whether this would make sense or not. So, here we come back to our fundamental question is every collection a set? In particular he asked can there be a set of all sets? So, remember that sets are objects just like anything else. So, we can collect them together. So, is this collection of all sets in fact a set?

Well, supposing it is a set, then we can do the following. We can apply set comprehension right, and we can pick out some sets from this collection of all sets. So, we will call capital S, the subset of all sets that do not contain themselves. So, this is a subset of this hypothetical set of all sets. So, this capital S is a set because we have applied set comprehension to the set of all sets. So, we have the set of all sets. And among all sets we have pulled out those sets which do not contain themselves. So, this is the condition we have applied, and this is allowed by set comprehension.

Now, the question is does the set that we have constructed belong to itself, does S belong to S? Well, if it does belong to itself, then it does not satisfy its own definition because elements of S should not contain themselves. So, S cannot belong to itself, because if it did it would contradict to way we have pulled out S from the set of all sets. But if it does not belong to itself, then that is also a contradiction, because then S does not belong to S and by the condition that we have applied to pull out sets S must be included in that condition.

So, either way we have a paradox; we have a contradiction. So, S can neither belong to itself nor can it not belong to itself. And this is called Russell's Paradox. He was the first person who published this and made it publicly known, but this was also independently discovered by another well known set theorist of the time called Ernst Zermelo. So, what this really tells us? If you remember our argument is that we made some assumption, and then from that assumption we realized that we have a contradiction or an observed situation.

So, something must be wrong in one of our assumptions. And here it turns out that the assumption that goes wrong in all these is the assumption that there is a set of all sets. If we did not have a set of all sets, we could not have done the set comprehension, and therefore, we would not have reached this observed conclusion.

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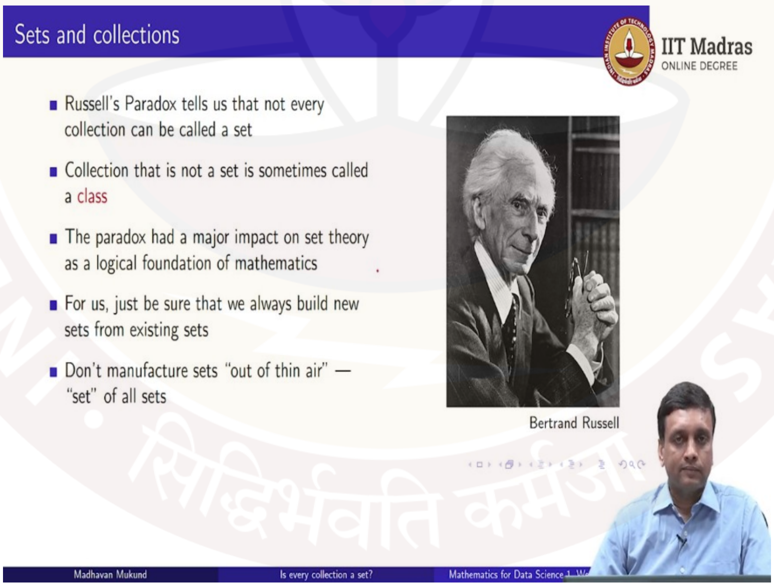

So, what Russell's Paradox really tells us is that, not every collection can be called as set in particular the set of all sets does not exist. So, he went through an exercise of trying to formulate a different version of set theory which he called type theory and so on, but in modern mathematics typically if you are not sure that what you are dealing with is a set then it is safer to just called such a collection a class. So, a class is just a collection of objects which does not have any of the implied properties that you expect from the sets.

So, this paradox as we said came in the context of set theory being used as a foundation of mathematics. And, this seem to casts doubts on whether it could be used at all. So, it had a major impact on this whole mathematical exercise of deriving mathematics from logical foundations which went on into the 20th century which we will not be able to discuss here unfortunately, but it is a fascinating subject in its own right.

For us what we have to be clear about is that whenever we use sets we must make sure that we always start with sets that we have and build new sets from existing sets. So, we can assume that the numbers are sets. So, we have the set of natural numbers, the set of integers, the set of rationals, the set of reals and so on. And, whenever we construct a new set we just have to verify that the set that we started with to construct the new set was already a set.

So, we take a Cartesian product or a union or set comprehension, we always start with old sets and make new sets. So, those old sets must be well-defined. So, in other words, we should not manufacture sets out of thin air such as the set of all sets.