Mathematics for Data Science - 1
Graded Assignment Week 1 Solution
Jan 2022 term
Max marks : 20

1 Multiple Choice Questions (MSQ)

1. Which of the following are rational numbers? (Answer: (a),(b),(c))(1 mark)

   ◯ $3\sqrt{2} + (1 - \sqrt{2})(2 - \sqrt{2})$
   ◯ $(\sqrt{12} + \sqrt{3})(\sqrt{12} - \sqrt{3})$
   ◯ $(\sqrt{12} - \sqrt{3})(\sqrt{12} - \sqrt{3})$
   ◯ $3\sqrt{2} + (1 - \sqrt{2})(1 + \sqrt{2})$

   Solution: In this question we have to find rational numbers from the given options. From the definition we know that a number is a rational if it can be represented as p/q form where p and q are integers. Each of the options are explained below:

   - **Option (a):** $3\sqrt{2} + (1 - \sqrt{2})(2 - \sqrt{2}) = 3\sqrt{2} + 2 - 2\sqrt{2} - \sqrt{2} + 2 = 4$
     On simplification we get 4 which can be written as p/q form. So the given number in this option is a rational number.
   - **Option (b):** $(\sqrt{12} + \sqrt{3})(\sqrt{12} - \sqrt{3}) = 12 + \sqrt{36} - \sqrt{36} - 3 = 9$
     On simplification we get 9 which can be written as p/q form. So the given number in this option is also a rational number.
   - **Option (c):** $(\sqrt{12} - \sqrt{3})(\sqrt{12} - \sqrt{3}) = 12 - \sqrt{36} - \sqrt{36} + 3 = 3$
     On simplification we get 3 which can be written as p/q form. So the given number in this option is also a rational number.
   - **Option (c):** $3\sqrt{2} + (1 - \sqrt{2})(1 + \sqrt{2}) = 3\sqrt{2} + 1 - 2 = 3\sqrt{2} - 1$
     On simplification we get $3\sqrt{2} - 1$ which can not be written as p/q form. So the given number in this option is not a rational number. It is an irrational number.

   So the correct options are (a), (b) and (c).

2. Consider the following sets.

- $S_1 = \{x \mid x \in \mathbb{R},\ 0 < x < 10\}$
- $S_2 = \{x \mid x \in \mathbb{Z},\ -10 \leq x \leq 10\}$
- $S_3 = \{x \mid x \in \mathbb{Q},\ 10 \leq x \leq 15\}$

Choose the set of correct options. (Answer: (a),(d))(3 marks)

○ $(S_2 \cap S_3) \cup (S_3 \cap S_1) = \{10\}$.

○ $(S_2 \setminus S_1) \cap (S_3 \setminus S_1) = \phi$.

○ The cardinality of the set $S_1 \cap S_2$ is 10.

○ The cardinality of the set $(S_1 \cap S_2) \cup (S_2 \cap S_3)$ is 10.

Solution: In this question we have to choose the correct statements from the given options based on the three sets S_1, S_2 and S_3.

- Option (a): $(S_2 \cap S_3) \cup (S_3 \cap S_1) = \{10\}$
  Here, $(S_2 \cap S_3) = \{10\}$ and $(S_3 \cap S_1) = \phi$
  Therefore, $(S_2 \cap S_3) \cup (S_3 \cap S_1) = \{10\}$.
  Hence option (a) is correct.

- Option (b): $(S_2 \setminus S_1) \cap (S_3 \setminus S_1) = \phi$
  Here, $(S_2 \setminus S_1) = \{$-10, -9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 10$\}$, and
  $(S_3 \setminus S_1) = S_3$
  Therefore, $(S_2 \setminus S_1) \cap (S_3 \setminus S_1) = \{10\}$
  Hence option (b) is incorrect.

- Option (c): The cardinality of the set $S_1 \cap S_2$ is 10.
  Here, $(S_1 \cap S_2) = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. So the cardinality of the set $S_1 \cap S_2$ is 9 not 10. Hence option (c) is not correct.

- Option (d): The cardinality of the set $(S_1 \cap S_2) \cup (S_2 \cap S_3)$ is 10.
  Here, $(S_1 \cap S_2) = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, and $(S_2 \cap S_3) = \{10\}$
  Now, $(S_1 \cap S_2) \cup (S_2 \cap S_3) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. So the cardinality of the set $(S_1 \cap S_2) \cup (S_2 \cap S_3)$ is 10.
  Hence option (d) is correct.

So the correct options are (a) and (d).

3. If $a \bmod 3 = 1$ and $b \bmod 3 = 2$, where $a, b \in \mathbb{N}$, then choose the set of correct options. (Answer: Options (b),(c))(2 marks)

   - ◯ $(ab) \bmod 3 = 0$.
   - ◯ $(ab) \bmod 3 = 2$.
   - ◯ $(a+b) \bmod 3 = 0$.
   - ◯ $(a+b) \bmod 2 = 1$.

   Solution:
   $a \bmod 3 = 1$ can be written as: $3k_1 + 1 = a$, where k_1 is a natural number, and
   $b \bmod 3 = 2$ can be written as: $3k_2 + 2 = b$, where k_2 is a natural number.
   Now we check each options one by one.

   - Option (a): $(ab) \bmod 3 = 0$
     Here, $(ab) = (3k_1 + 1) * (3k_2 + 2) = 3(2k_1 + k_2 + 3k_1k_2) + 2$
     So, $(ab) \bmod 3 = 2$
     Hence option (a) is incorrect.
   - Option (b): $(ab) \bmod 3 = 2$
     This is correct expression. (Please check explanation given in option (a)).
     Hence option (b) is correct.
   - Option (c): $(a+b) \bmod 3 = 0$
     Here, $(a+b) = (3k_1 + 1) + (3k_2 + 2) = 3(k_1 + k_2 + 1)$. So $(a+b)$ has a factor 3.
     Therefore, $(a+b) \bmod 3 = 0$. Hence option (c) is correct.
   - Option (d): $(a+b) \bmod 2 = 1$
     Here, $(a+b) = 2m + 1$, where m is a natural number.
     So, $3(k_1 + k_2 + 1) = 2m + 1$,
     So, $2m = 3(k_1 + k_2) + 2$
     Now, consider case (i) $k_1 + k_2 = 10$, which an even number,
     $2m = 3 * 10 + 2$,
     $m = 16$, which is a natural number. Therefore for this case expression, $(a+b) \bmod 2 = 1$, is correct.
     Now, consider case (ii) $k_1 + k_2 = 15$, which an odd number,
     $2m = 3 * 15 + 2$,
     $m = 47/2$, which is not a natural number. Therefore for this case expression, $(a+b) \bmod 2 = 1$, is incorrect.
     So overall the given expression, $(a+b) \bmod 2 = 1$, is incorrect.

4. Suppose $A = \{a, b, c, d\}$ and $B = \{p, q, r, s\}$ are two sets. Consider the following relations from A to B.

- $R_1 = \{(a, p), (c, r), (d, q)\}$
- $R_2 = \{(a, s), (b, s), (c, p), (d, r)\}$
- $R_3 = \{(a, p), (b, r), (b, s), (d, q)\}$
- $R_4 = \{(a, r), (b, p), (c, q), (d, s)\}$

Which of the following statements are correct? [Answer: (b),(d)](2 marks)

◯ R_2, R_3, and R_4 are functions.

◯ R_2 and R_4 are functions.

◯ R_2 is an injective function.

◯ R_4 is a bijective function.

Solution:
Firstly, investigate every relations and then check options.

- R_1: It is not a function because b is not mapped to any element in the set B.
- R_2: It is also a function but not injective because a and b are mapped to s.
- R_3: It is not a function since b is mapped for two values r and s.
- R_4: It is one-to-one (every element of A is mapped to distinct element of B) and onto function (no any element is left in B). Therefore R_4 is a bijective function.

Now from the options, option (b) and (d) are correct.

5. Let us define a function $f: \mathbb{N} \rightarrow \mathbb{Q}$ as follows,

$$f(n) = \begin{cases} \frac{n-1}{4} & n \text{ is odd} \\ \frac{n+1}{2} & n \text{ is even} \end{cases}$$

Which of the following are true? (Answer: (a),(b))(1 mark)

- ○ f is not one to one.
- ○ f is not onto.
- ○ f is onto.
- ○ f is one to one.

Solution:
For $n = 0, f(0) = 1/2$, and for $n = 3, f(3) = 1/2$. So,
$f(0) = f(3) = 1/2$. Therefore $f(n)$ is not one to one function.

For all values of domain, $\mathbb{N}$, can not be mapped to all elements of co-domain since co-domain is $\mathbb{Q}$. Therefore $f(n)$ is also not onto function.

Hence options (a) and (b) are correct.

2 Numerical Answer Type (NAT)

6. Find the smallest natural number that leaves the remainders 3 and 4 when divided by 5 and 6 respectively. (Answer: 28)(1 mark)

Solution:
Let n be the smallest natural number such that,
$n = 5k_1 + 3$, and $n = 6k_2 + 4$. Where k_1 and k_2 are natural numbers.
So here we create two sets (A and B) for $n = 5k_1 + 3$ (Table 1), and $n = 6k_2 + 4$ (Table 2).

k_1	$n = 5k_1 + 3$ (Set A)
0	3
1	8
2	13
3	18
4	23
5	28
6	33
7	38
8	43
9	48
10	53
11	58
...	...
p	$5p + 3$

Table 1

k_2	$n = 6k_2 + 4$ (Set B)
0	4
1	10
2	16
3	22
4	28
5	34
6	40
7	46
8	52
9	58
10	64
11	68
...	...
q	$6q + 4$

Table 2

Now, we will find the common elements from sets A and B.
$A \cap B = \{28, 58, ...\}$
From the question we have to find the smallest one. As per the question, the smallest number will be 28.

7. Suppose $f : \mathbb{N} \longrightarrow \mathbb{N}$ is a function defined by $f(x) = a(x + b)$, where $a, b \in \mathbb{N}$. If f is a bijective function, then find the value of $a + b$. (Answer: 1)(2 marks)

Solution:
If f is a bijective function then it should be one to one and onto function.
Since function f is mapped to $\mathbb{N} \longrightarrow \mathbb{N}$ and an onto function then for some value of x, $f(x) = 0$, since 0 is a natural number. So,
$f(x) = 0 = a(x + b)$,
$ax + ab = 0$
$ab = 0$, $a \neq 0$ (constant function)
Therefore, $b = 0$ Now, $f(x) = ax$. $f(x)$ will be onto function only when $a = 1$ because $f(x)$ should be mapped to all $\mathbb{N}$.
Hence, the value of $a + b = 1$

8. Suppose there are 100 people in a city C. Let X be the set of people who are engineers in the city C, Y be the set of people whose age is above 40 years in the city C, Z be the set of people who has an independent house in the city C. Now, study the following Venn diagram and answer the question given below.

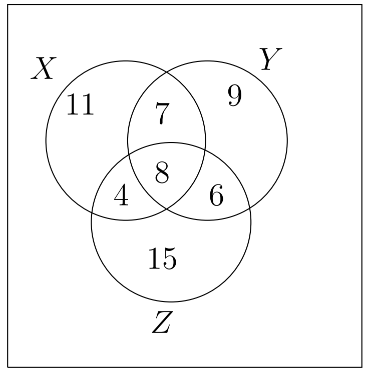

Figure M1T4GA-1

Find the number of engineers who do not have an independent house? (Answer: 18)(2 marks)

Solution:

We have to find here number of engineers who do not have an independent house which means $X \setminus Z$. From the Venn diagram it is pretty straight that the number of elements in set $X \setminus Z$ will be $(11 + 7) = 18$

9. Consider a set $S = \{a \mid a \in \mathbb{N},\ a \leq 18\}$. Let R_1 and R_2 be relations from S to S defined as $R_1 = \{(x, y) \mid x, y \in S,\ y = 3x\}$ and $R_2 = \{(x, y) \mid x, y \in S,\ y = x^2\}$. Find the cardinality of the set $R_1 \setminus (R_1 \cap R_2)$. (3 marks)
   - Answer: 5

   Solution:
   $S = \{a \mid a \in \mathbb{N},\ a \leq 18\}$ then
   $R_1 = \{(0,0), (1,3), (2,6), (3,9), (4,12), (5,15), (6,18)\}$, and
   $R_2 = \{(0,0), (1,1), (2,4), (3,9), (4,16)\}$
   $R_1 \cap R_2 = \{(0,0), (3,9)\}$
   Now we have to find, $R_1 \setminus (R_1 \cap R_2) = \{(1,3), (2,6), (4,12), (5,15), (6,18)\}$
   $|R_1 \setminus (R_1 \cap R_2)| = 5$

10. In a Zoo, there are 6 Bengal white tigers and 6 Bengal royal tigers. Out of these tigers, 5 are males and 10 are either Bengal royal tigers or males. Find the number of female Bengal white tigers in the Zoo. (3 marks)
Answer = 2

Solution:
Let BW = Number of Bengal white tigers,
BR = Number of Bengal royal tigers,
M = Number of male tigers, and
F = Number of female tigers

Now from the given information in the problem,
$BW = 6, BR = 6,$
Then total number of tigers $= BW + BR = 6 + 6 = 12$
$M = 5$, then Number of female tigers $= F = 12 - 5 = 7$
Also it is given, $BR \cup M = 10$
In this question We have to find, $BW \cap F = ?$

From the union formula,
$BR \cup M = BR + M - BR \cap M,$
$10 = 6 + 5 - BR \cap M,$
Hence, $BR \cap M = 1$

Also, $BR = BR \cap M + BR \cap F,$
$6 = 1 + BR \cap F,$
Hence, $BR \cap F = 5$

Also, $F = BW \cap F + BR \cap F,$
$BW \cap F = F - BR \cap F,$
$BW \cap F = 7 - 5,$
Hence, $BW \cap F = 2$