INDIAN INSTITUTE OF TECHNOLOGY MADRAS

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IIT Madras

ONLINE DEGREE

Mathematics for Data Science 2
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Lecture No. 15
Computing Areas Using Integrals

Hello and welcome to the maths 2 component of the online BSc program on data science and programming. The idea behind this lecture is that we have now reached a place where we know how to compute an integral, definite integral. And we know that a definite integral represents area namely it represents what is colloquially called area under the curve. Of course the curve could be below the x axis. So, we have to interpret that suitably. So, we know how to do that. And now we will use this to compute areas.

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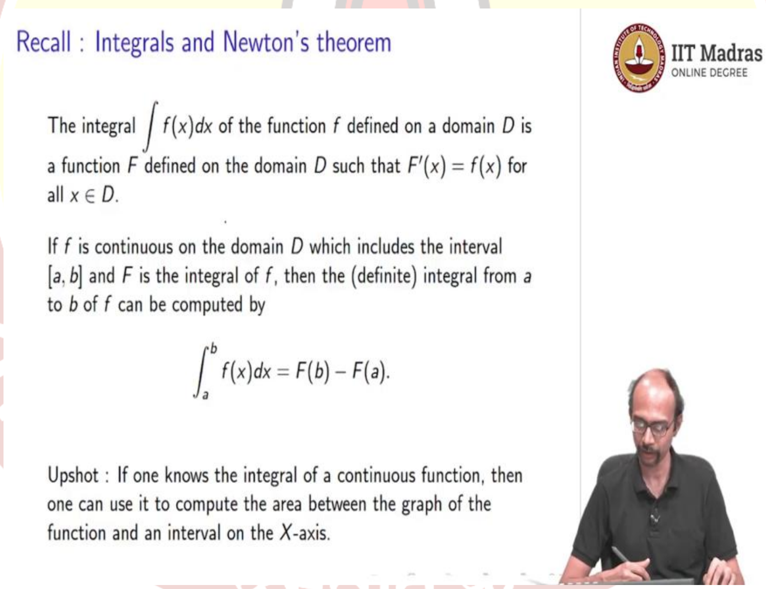

So, let us recall first what we have done so far in the series on integrals. So, we have of course defined the definite integral from a to b of a function f. Which was that we took Riemann sums and then we took the limit as the Riemann sums the underlying partitions on the Riemann sums became finer and finer. And we hope that this limit exists and that was called the integral. And the integral was representing as I said the area under the curve.

But this was a rather difficult approach to take to compute the integral. Because computing limits is hard that is what we have understood from the videos on limits. So, we always try to bypass computing limits by either making good hypothesis on functions or by proving theorems that allow us to compute them in different ways. So, Newton's theorem was what we studied in the previous video. And Newton's theorem allowed us to compute these integrals via the indefinite integral or in other words the anti-derivative. So, it linked up derivation and integration.

So, let us go through the statement the integral which is denoted as $\int f(x)dx$ of the function f defined on a domain D is a function F defined on the domain D. Such that $F'(x) = f(x)$ for all x in D. So, this $\int f(x)dx = F(x)$. That is what the statement is saying. And what is the defining property of F(x)? When you differentiate F(x) you get f(x). So, it is the anti-derivative which we prefer calling the integral.

Now I am saying the anti-derivative or the integral because any two anti-derivatives differ only by a constant. So, we know that up to constant it will be the same function. Now Newton's theorem said if f is continuous the integral has two parts but I am stating the part that we really need in order to compute integrals. If f is continuous on the domain D which includes the interval (a, b) so now all the action is taking place on this interval (a, b). So, as we have you must have observed in the video so far the domain D is just a placeholder.

So, if f is continuous on the domain D which includes the interval (a, b) and F is the integral of f then $\int f(x)dx$ can be computed as $F(b) - F(a)$. So, this was one part of Newton's theorem. And this gave us a handle on how to compute integrals. Namely you find the indefinite integral or the anti-derivative of your function f and then evaluate at b and a and take the difference of the two. So, this was the method we saw a few examples in the previous video.

So, the upshot is that if one knows the integral of a continuous function by integral of course we mean the indefinite integral. Then one can use it to compute the area between the graph of the function and an interval on the x axis.

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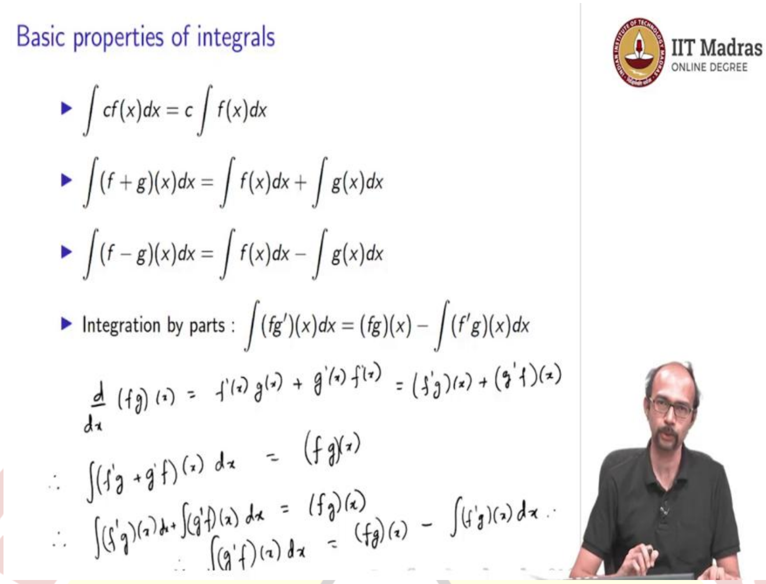

So, let us now study some basic properties of integrals. These are on the lines of properties for derivatives. And the proofs are basically the same as a proof that we had further follow from the properties for the derivative. So, I would strongly suggest that you try to prove them yourself except for the last one which I will do here. So, $\int cf(x)dx = c\int f(x)dx$. So, c here is supposed to be a constant which is a real number. So, that comes out of the integral.

If you take a sum of functions and integrate that then that is the same as taking the individual integrals and then adding them same for the difference. And then finally we have something called integration by parts. So, this is a very important and useful idea. And the proof is actually quite simple.

So, the theorem is or the statement is $\int (fg')(x)dx = (fg)(x) - \int (f'g)(x)dx$. Now let us quickly give a proof of this. So, let us start from $(fg)(x)$ and ask what is the derivative? Now we know by the product rule from derivatives that this is exactly $f'(x)g(x) + f(x)g'(x)$. This is how we define productive functions. Since the derivative of f times g is the function on the right that means the integral of the function on the right is f times g. So, $\int \big((f'g)(x) + (fg')(x)\big)dx = (fg)(x)$ and I think at this point you probably see what is coming next. This is a sum by the second property right here and we can split the sum and write this integral in this form. And now when you move $(f'g)$ to the other side you get exactly what you want. So, therefore we get $\int (fg')(x)dx = (fg)(x) - \int (f'g)(x)dx$. This is exactly the statement that we have. So, this statement is called integration by parts. And this is a very useful although it is an easy trick it is a very useful trick in practice to compute integrals. And we will do an example as we go ahead.

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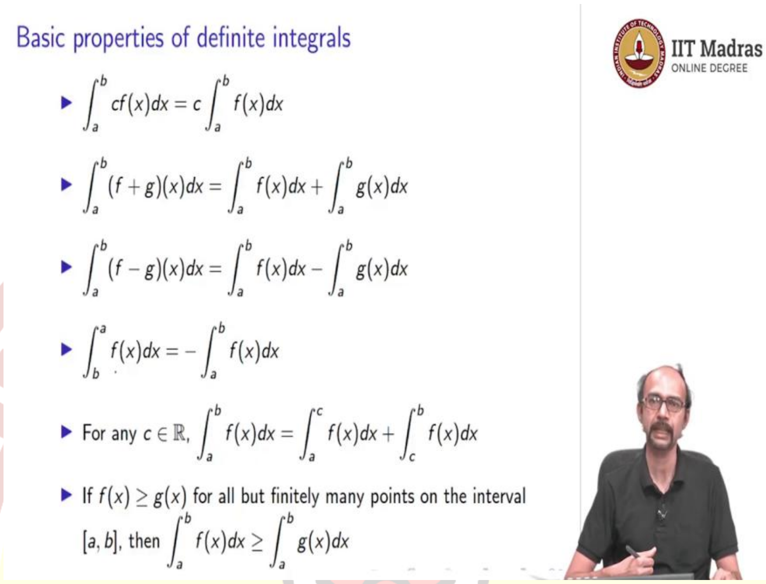

So, those were the properties of some basic properties of indefinite integrals. Now let us talk about definite integrals which is really what we want to compute. So, the first few properties will follow from the properties of the indefinite integrals. So, if you take a constant times f, then you can move the constant out of the definite integral. You can do this either from first principles or from Newton's theorem which will allow you to write both sides as in terms of F and c times F.

Then the sum of functions if you take the definite integral that is the sum of the definite integrals. And the same thing for, if you subtract g from f then take the integral from a to b instead you could first take the integral of f and subtract out the integral of g both from a to b. Now here is something a bit which might you might find strange which is if you integrate from $\int_a^b f(x)dx$ then you get minus of $\int_b^a f(x)dx$.

Now there are two ways of interpreting this. One is to interpret b to a as going backwards in the interval. So, when you do your limit, when you do your Riemann sum then we did the small interval those things would be negative. And then your Riemann sums become negative not negative, but they become you know minus of the usual Riemann sum that you get and hence you pick up this minus sign. So, that is from the definition perspective.

The other is you could call this itself a definition. So, whichever way you prefer the statement seems very natural. $\int_a^b f(x)dx = -\int_b^a f(x)dx$. The right hand side we know and it makes perfect sense. The left hand side as I said you can either define it this way or you can change your definition of the definite integral to include not just intervals.

But sets which start somewhere and end somewhere where the starting or ending point need not be the starting point need not be less than the ending point could be larger also. Either way so this is a fact that you will find useful for any $c \in R$. So, c is a some real number if you integrate $\int_a^b f(x)dx$then you could integrate from a to c and then integrate from c to b and add them up. So, if c is in the middle if c is somewhere in the middle of a and b this seems like a perfectly natural statement.

But if c is not in the middle then you have to interpret this statement in view in the light of the previous statement here. So, if c is less than a for example then this is essentially saying that you will when you take the sum on the right you will subtract out c to a. But then you will also add a to c when you add the second one. So, if c is less than a or c is bigger than b you have to interpret this suitably in terms of what we have above and final property on this slide.

So, if f(x) is greater than equal to g(x) for all but finitely many points on the interval (a, b) in fact we can even relax finitely many but then you would have to become a bit more technical. So, if for finite many points all about finitely many points if f(x) is larger than or equal to g(x) then the integral from $\int_a^b f(x)dx$ is also larger than the integral of $\int_a^b g(x)dx$. This again seems a very natural property from the perspective of Riemann sums and from the perspective that the integral computes areas.

So, one upshot of this maybe I should mention. So, if $f(x) = g(x)$ for all our finitely many points then from here we get $\int f(x)dx = \int g(x)dx$. And the main take home from there is that the function need not be continuous if we applied Newton's theorem and so on many places that we saw that we could get nice ways of handling the integral was under the hypothesis that f is continuous.

So, what this is saying is that even if f is not continuous at a few points that still you can treat it as if it is continuous at those points and compute the integral in the same way. So, this is a useful statement in that sense. One further consequence of this statement is that if $f(x) \geq 0$ then the integral $\int_a^b f(x)dx \geq 0$. This is something that we find natural because in that case the integral actually computes the area under the curve. That is where we whatever starting point for integrals was.

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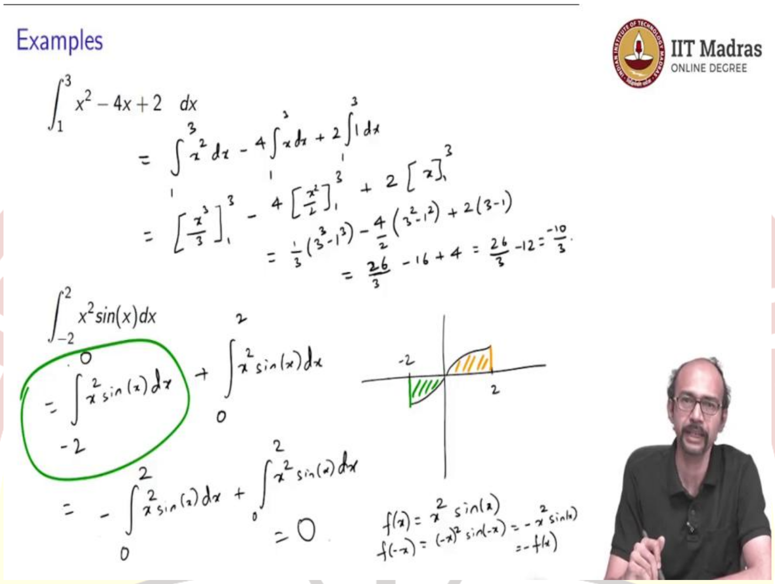

Fine so let us do a couple of examples to set of ideas. So, here is the integral $\int_1^3 x^2 - 4x + 2 \ dx$. So, I can break this $\int_1^3 x^2 dx - 4\int_1^3 x dx + 2\int_1^3 1 dx$. And then I can compute this because I know for each of these how to get the antiderivative.

So, for the first one the anti-derivative is x^3. So, the integral of x square is $x^3/3$ and we want to evaluated it from 1 to 3. Then we have $x^2/2$. Again limits from 1 to 3 and then x from 1 to 3 multiplied by the 2 outside. And now we substitute. So, the first term gives us 1 third times 3 cube minus 1 cube the second term gives us 4 times I can take my half out. So, 4 by 2 times 3 squared minus 1 squared and the third term gives us 2 times 3 minus 1.

And now if we compute this, we get -10/3 and that is our answer. So, I hope it is clear how to compute these integrals. And how do I know each of these, how did I get from step 2 to step 3?

That is because by observation I know that $\int x^n dx = \frac{x^{n+1}}{n+1}$ .So, if you do not remember these basic integrals, you could either try to sort of work out from the derivatives how to do them or just look up tables if you find it confusing. Let us look at the second example. So, in the second example we have $\int_{-2}^{2} x^2 \sin x \, dx$.

And this certainly no obvious way of calculating this integral based on whatever we have done so far. Because for this we need to know either the integral of $x^2 \sin x$. Where we can compute this but it is a little bit of a trick. And you can do it using what is called integration by parts which we will see in a few slides. And other than that we really do not know. So, what is the anti-derivative of $x^2 \sin x$? To get it directly would require some imagination some playing around with the terms etcetera.

So, how do we do this integral? When the trick here is to observe what happens to observe the limits of integration and what kind of function $x^2 \sin x$ is. So, if we plot $x^2 \sin x$ how does it look like when so if f(x) is $x^2 \sin x$ this is really the main point. If you look at f(-x), $x^2 \sin(-x)$. And $\sin(-x) = \sin x$.

What does that mean? That means for example $f(-1) = -f(1), f(-2) = -f(2)$. If $x^2 \sin x$ look something like this. So, there is a symmetry about this only thing is the symmetry is on the negative side.

So, if you have this to be -2 and this is to what is basically saying is if you split up this integral into two parts $\int_{-2}^{0} x^2 \sin x \, dx + \int_{0}^{2} x^2 \sin x \; dx$. We know we can split it like this. That is one of the rules that we saw. What does the first part compute the first part compute, this area but with a negative sign and what does the second part compute. The second part computes this but with a positive sign. And clearly these areas are the same. But the first one which is this one comes with a minus sign.

So, in other words, we are saying that this term is equal to $-\int_{0}^{2} x^2 \sin x \, dx$. And then when you add it to the same thing of course you get 0. So, the answer to this integral is 0. And here the trick is to observe that your function has this form f(-x) =-f(x). So, this always this applies to all functions which satisfy this and when the integral has limits which are symmetric about 0.

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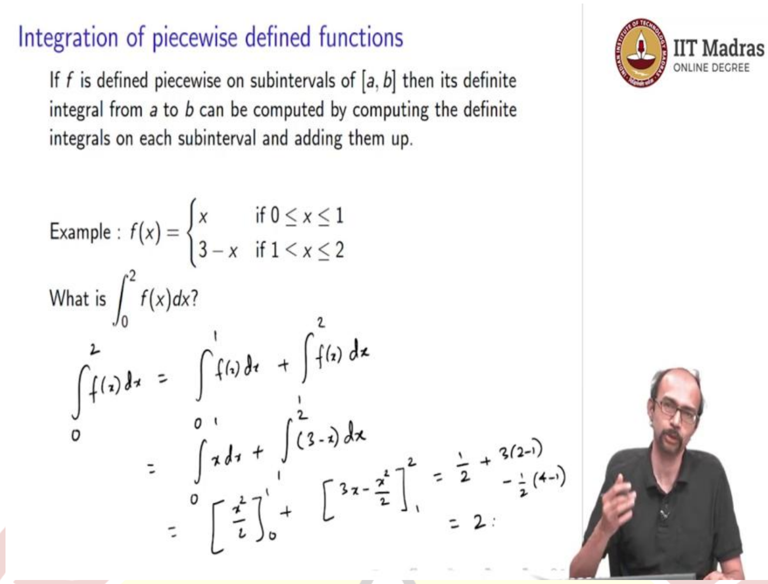

So, let us do a couple of other properties. So, integration of piecewise defined functions. So, if a function is defined on certain pieces so f is defined piecewise on sub intervals of $[a, b]$. Then it is a definite integral from a to b and can be computed by computing the definite integrals on each sub interval and adding them up. So, let us do an example so suppose we have f(x) defined as shown.

What is the integral $\int_0^2 f(x)dx$? So, to compute this we will break it up into each part on which it is defined. So, it is $\int_0^1 f(x)dx + \int_1^2 f(x)dx$. But from 0 to 1 f(x) is exactly x and from 1 to 2 this is 3-x. Now notice that this function is actually not continuous at the point 1 because the left limit is 1, whereas the right limit is 2. So, this function is not continuous at the point 1 but that is okay. If it is not continuous at a few points we can still do our integrals no problem.

So, we just do them as we usually do. So, the answer is 2.

So, I hope the idea is clear that if you integrate piecewise defined functions you have to integrate them on each sub interval they are defined on and then add up to get the total integral.

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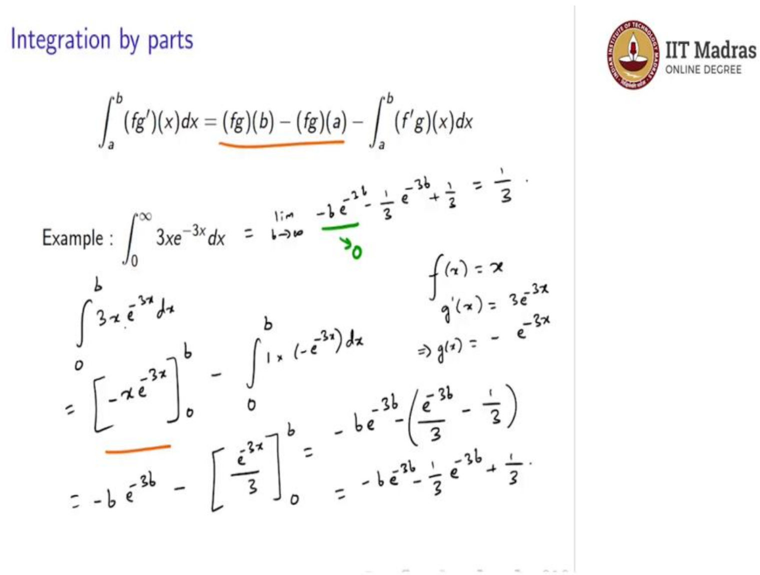

Integration by parts. So, we saw this already for the indefinite integral. Now let us apply this for definite integrals. So, if you apply to definite integrals what you get is if you integrate $\int_a^b (fg')(x)dx$ then that is the same $(fg)(b) - (fg)(a) - \int_a^b (f'g)(x)dx$. Here is an interesting example which you may actually see later on maybe in your statistics course.

So, $\int_0^\infty 3xe^{-3x}$. So, I am deliberately taking infinity here to make you comfortable with the idea of integrating till infinity. So, to do this what we have to do is we have to integrate from 0 to b and then take the limit as b tends to infinity. So let us see what this is. So, here $f(x) = x, g(x) = 3e^{-3z}$. So, now let us interpret this in terms of our integration by parts. And the idea is that you should choose f to be such that f' becomes nice. So, that gives us $-xe^{-3x}$ as the first term and $\int -e^{-3x}dx = \frac{e^{-3x}}{3}$ as the second term both evaluated within the limits from 0 to infinity.

So, to get the final result we need to allow b to tend to infinity for this function here.

So, the last thing is a constant so that remains as it is. And for the first two terms note that we have an e^{-}. And for the second term directly goes to 0. For the first term we have it multiplied by b. But now if you remember our video on how fast do functions grow and so on the exponential decay occurs much the exponential beats the polynomial this is what was there. So, e^{-3b} is going to go to 0 much faster than how b is going to increase.

So, in other words this function here is also going to go to 0. So, that is something that we have to know so this is going to go to 0 this tends to 0. So, why because the exponential beats the polynomial this is what you have to keep in mind. So, the first term goes to 0 the second term goes to 0 and so what we get is 1/3. So, you can see that by doing integration by parts, this, we could solve this problem. Now of course there are other ways of solving this problem.

For example, you could be clever and try to guess an integral for this function. In fact we have essentially computed that by integration by parts. So, maybe you could have guessed that function. And that is another way. But slowly it starts becoming difficult to always be clever. Instead you can try and apply these nice results. And keep simplifying the problem. And then hopefully, you can reduce it always to integrals that you know how to compute that is the idea of these techniques.

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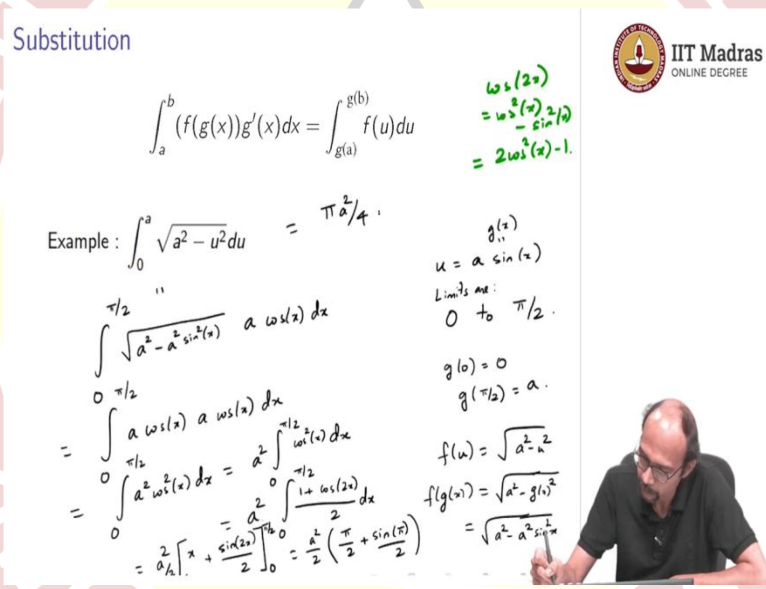

So, the last thing we are going to see is something called substitution. So, what this is saying is that if you have an integral $\int_a^b f(g(x))g'(x)dx$. Then that is the same as the integral $\int_{g(a)}^{g(b)} f(u)du$. So, in other words this u is playing the role of g(x). Now although I have written it from left to right so it seems as if the term on the left to be simplified by the term on the right often in practice what you will do is you will go from right to left. So, you will take u and you will substitute u to be g(x).

And here is an example. So, let us look at integral $\int_0^a \sqrt{a^2-u^2}\,du$. So, what is the substitution here? So, the substitution here is so you take, so you put $u = a\sin x$. And then what are the limits? The limits are so limits for u are from 0 to a. So, that means the limits for so the limits are, so the limit for x is going to be 0 to π/2. Because at π/2, u it takes the value a.

So, now utilizing what we have upstairs, we will rewrite this as integral from 0 to π/2. So, what this is saying essentially is that g(0)=0 where g is this function; $a \sin x$ and then what is g(0)? It is $u = a \sin 0$ which is 0, what is g(π/2)? It is $\sin\frac{\pi}{2}$. So, this is a. So, a times 1 is a.

So, you, so this is in the form that we have on the right, and so we will apply the form on the left, we will equate it to whatever is on the left. And so what we will get is $\int_0^{\pi/2} f\big(g(x)\big) acosx\, dx$, so what is f here, f is this function here $\sqrt{a^2 - u^2}$. So, $\sqrt{a^2 - g(x)^2}$, where $g(x) = a \sin x, f\big(g(x)\big) = \sqrt{a^2 - a^2 sin^2 x}$. So, I hope I have completely explained this, if you are seeing this for the first time.

So, there is a trick here, as you can see, namely, I am using a very particular substitution, that is the trick here.

Now, of course, in order to use this, I mean in order for this to be useful, we need to be able to compute this integral after changing it. So, let us see if we can do that. So, you have the integral $\int \sqrt{a^2 - a^2 sin^2 x}\, a\cos x\ dx$. So, you can take a common, so you have a outside, but we know some trigonometric formulae, so $1 - sin^2 x = cos^2 x$. So, we get $acos^2 x$ inside the integral.

And now I have to compute the integral of $acos^2 x$. But again, this is something that you might remember we did in the video on, I mean we used the formula for $cos^2 x$ in the video on limits and the same kind of idea is going to work here.

So, what should I do here? So what I will do here is I will express $cos^2 x$ in terms of $\cos 2x$. So, this is $\int_0^{\pi/2} \frac{1+\cos 2x}{2}$. And now this is an integral that I really know how to compute. So, I will do this last thing rather fast. So, what we get is $\frac{\pi a^2}{4}$.

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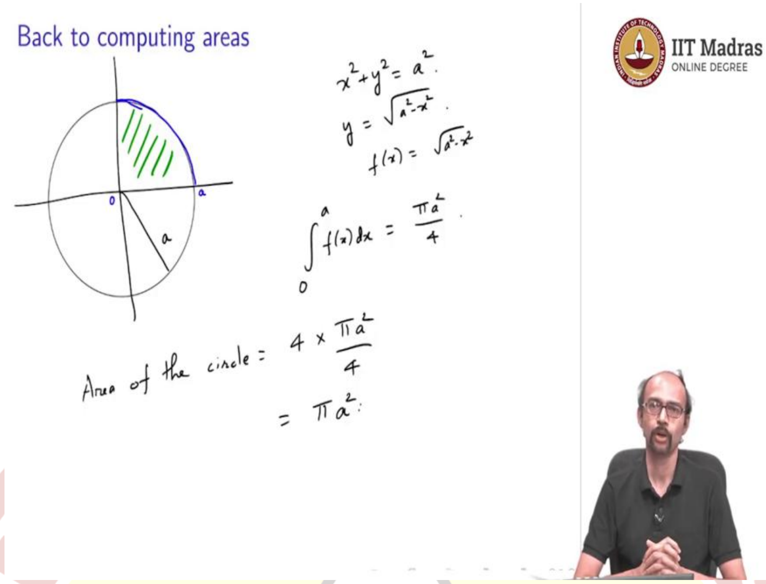

So finally, let us come back to computing areas. So, this is just a small example, which is a circle. So, we saw two ways of computing the area of the circle. One was by the method of Archimedes, the very beautiful way of taking irregular polygons and then filling it up and then allowing the sides, number of sides to increase; first from inside, then from outside and then bounding it on both sides and that gave us what the area was, I did not do that computation, but I mentioned this. And it is something that those of you who are more interested can try to work out on your own.

The other way I did it was, I cut the circle or the disk rather, into smaller what are called annuli, meaning you take small strips, circular strips and then we unfolded them, then we sort of bent them down, so that it was almost like a rectangle, except that the top was curved. And then we all put them, we put them all together. And then we took the limit, this was sort of an idea that we used to give the intuition for integration. And then we observed that in the limit, this becomes like a triangle and we computed what the area was.

So, both of those methods were very nice. And only thing is that they involve computing some limits, which we kind of did a little bit of a hand wave for. So, now I am going to use integrals to compute this area. So, what I will do is I will draw this line here. So, let us think of this as the x axis. Let us think of this as the y axis. Note that there was no line given to me as such. And all I need is the radius. So, suppose the radius here is, so radius is a, so this is a circle of radius a, then what is the area of the circle? That is what we are trying to find out.

Well now, I can interpret the top so, compute the area of the top part. Or maybe even the part in the first quadrant. So, I will compute the area of this part. How do I do that? Well, I think of this, I think of this part of the circle, this arc here, I can try to think of this as a function. So, this is a function from where to where, this is a function defined on the interval 0 to a. And what is that function? Well, for this we need to write down the parametric equation of the circle.

So, remember that the circle satisfies that $x^2 + y^2 = a^2$, that is all the points which are on the circle. So, all those (x,y) which satisfy this is exactly what defines a circle. So, those ones which are in the first quadrant mean that x is positive and y is positive. And so I can write y as, rewriting this, I can write $y = \sqrt{a^2 - x^2}$. That is a familiar expression. And so, I can look at the function $f(x) = \sqrt{a^2 - x^2}$. And I can ask, what is the integral $\int_0^a f(x)dx$? What is that?

Well, that is exactly this thing in the green portion that we have done here. Why? Because that is the area under the curve where the curve is the graph of this function. But what is the graph? The graph is exactly this blue arc that I have drawn. So, this green portion is exactly going to be computed by integrating f(x). How do I do that? Well, let us go back. This is exactly what I did over here, so I computed root $\int_0^a \sqrt{a^2 - u^2}du$ and got $\frac{\pi a^2}{4}$. So, this is $\frac{\pi a^2}{4}$.

But now the area of the circle is 4 times this area, so it is 4 times $\frac{\pi a^2}{4}$, where remember a is the radius, so this gives us πa^2. So, if you think of your radius as r instead of a, you would have gotten πr^2. The formula that we know and now we have a complete proof of this.

So, I hope this explains to you, how to, one sort of way of how we are going to compute areas using integrals. So, in short, integrals are powerful tool for many things and you are going to come across them in statistics in particular, but they are also useful in more basic geometric operations, in particular in computing areas. And I will strongly urge that you try to compute for example other things like the area of the ellipse or the area under the hyperbola and so on. Thank you.