INDIAN INSTITUTE OF TECHNOLOGY MADRAS

सिद्धिर्भवति कर्मजा

IIT Madras

ONLINE DEGREE

Statistics for Data Science – 1
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Lecture No. 5.3
Permutations and Combinations – Permutations: Distinct Objects

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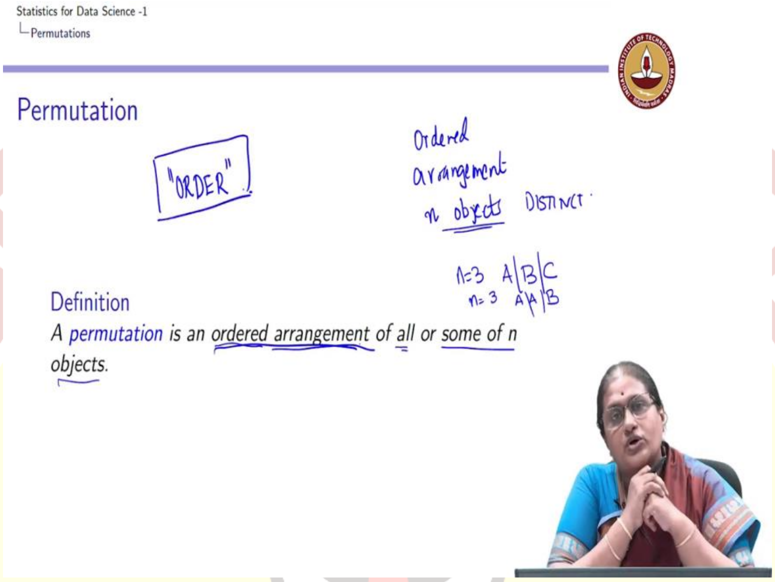

We now introduce what is a Permutation. Permutations and Combinations are extremely important when we are going to learn about probability. So, what is, we formally define what is a permutation? We can formally define what is a permutation as an ordered arrangement of all or some of n objects. I want to know you all to notice something very carefully. We have 3, qualified the statement with 3 important words. The first word is ordered, the second is arrangement, and the third is n objects.

Now, for, now, we are going that assume that these n objects are distinct objects. Now, what do I mean by distinct object? For example, if I take A, B, and C, these could be 3 people. When I am referring to 3 people, they are distinct people. So, first we are going to understand, and then we will start looking at what would happen if they are not distinct. For example, I can again, so n = 3 here, A, B, C are distinct. I can again take n = 3 objects or I can have an A, A and B.

It could be a case that I have, 2 red balls and 1 blue ball. So here, I do not have all the 3 of them distinct, but if I had a red, blue and a yellow ball, all of them were distinct. So, now, a permutation is an ordered arrangement. So, the key we need to notice here is this idea of order. And this is what defines what is a permutation.

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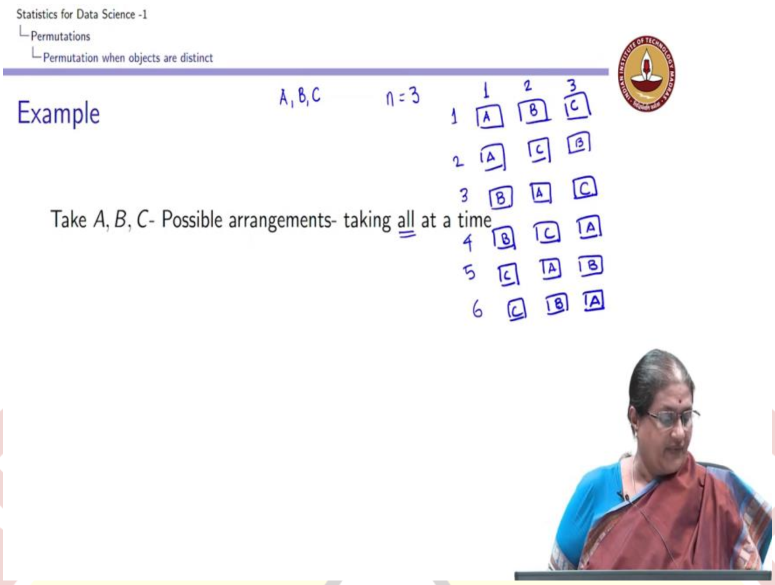

So, what is a permutation? Let us start with a very simple example, where we will extend what we learned about the fundamental principle of counting. So, what is this example? So, let us look at A, B and C. When we look at A, B and C, the number of objects or number of articles or number of things, whatever we want to refer to it, n equal to 3, because I have a A, I have a B, and I have a C. Now, let us look at all possible arrangements taking all at a time. In a sense, I want to see what are the all the possible arrangements of this A, B and C when I take all of them at a time.

So, the way I can look at it is, the another way to look at it is, I have 3 boxes, and I am looking at filling these 3 boxes with this A, B, C. And I want to know how I can do it, how many ways I can do this. So, you can start simply, intuitively. If I put a A here, let me fix the first box with an A. In the second box, I could have a B or a C. So, A, B, C, and A, C, B are possible ways of filling this box.

The another way I can look at it is, I can start with a B. Similarly, the second box and the third box. Second box could be a A, in which I can have the third box, only a C. Now, I am allow, I am not allowing the same repetition. When do I say I am allowing a repetition? For example, if I have, if I want to pick up some balls, I pick it up, I note the number and put it back. I am not allowing that. I am not allowing the repetitions in this case.

Similarly, if I have a B, I could have a C and a A. If I have the first, if my first blank is a C, first box is a C, the second could be a A, the third could be a B. The first as a C, the second could be a B and third could be a A. So, I can enumerate the number of possible arrangements when I am taking all the 3 alphabets here, A, B, C, which are distinct. If I count the number of possible arrangements, I have a 1, 2, 3, 4, 5 and 6.

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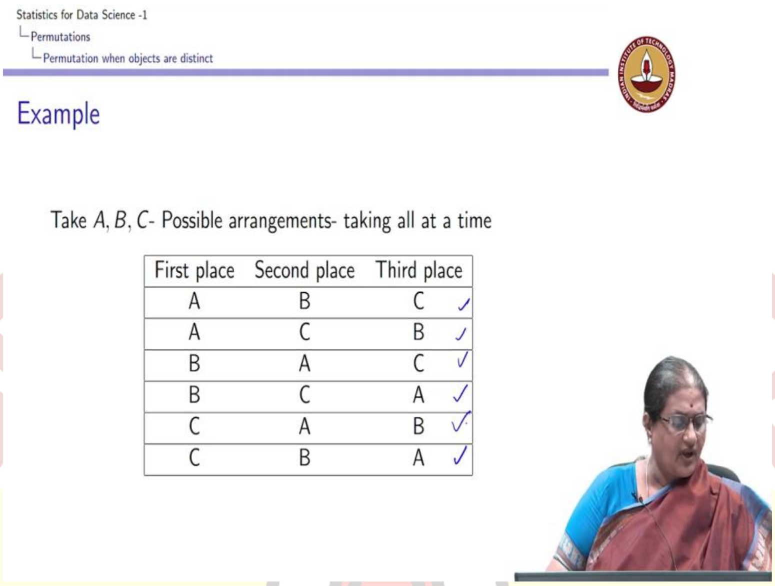

So, the total number of possible arrangements that are possible here are 6. So, ABC, ACB, BAC, BCA, CAB and CBA. Now, let us look at a slightly, so if you go back and see in the definition, I said it is an ordered arrangements of all or some. So, it is not necessary that I should take all the 3 of them together. So, I am interested, the second thing I am interested in knowing is, what are the possible arrangements of this ABC if I only take 2 of them?

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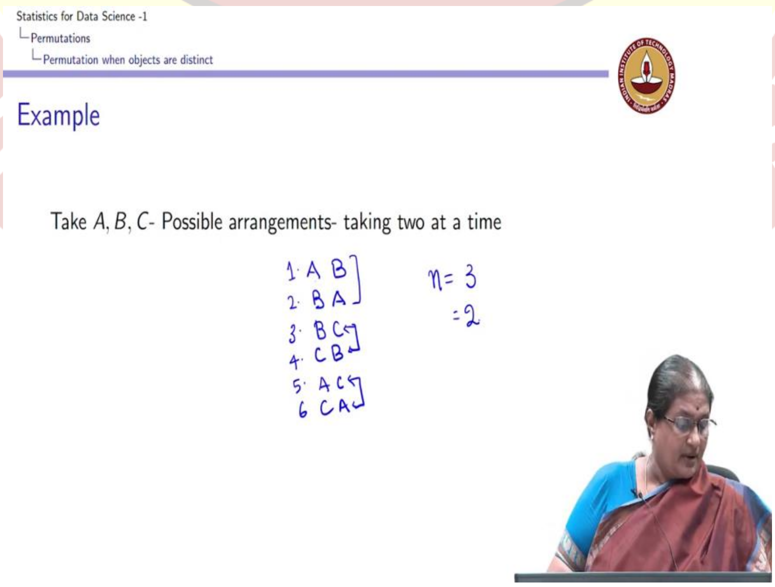

That is I am choosing 2 out of A,B, and C. How many ways can I choose it, we will come to this in a quite a time and we are going to talk about combinations. But I can look at A, B and C, taking 2 at a time, I can choose A and B. And within A and B, I can arrange them as AB or BA. Similarly, I can choose B and C. Within B and C, CB, because this ordering is different from this ordering. And I can look at AC and CA. So, when I am taking out of n = 3, I am taking 2 at a time, I see, again the number of possible arrangements are 1, 2, 3, 4, 5 and 6. In this case, both the first answer and second answer are the same.

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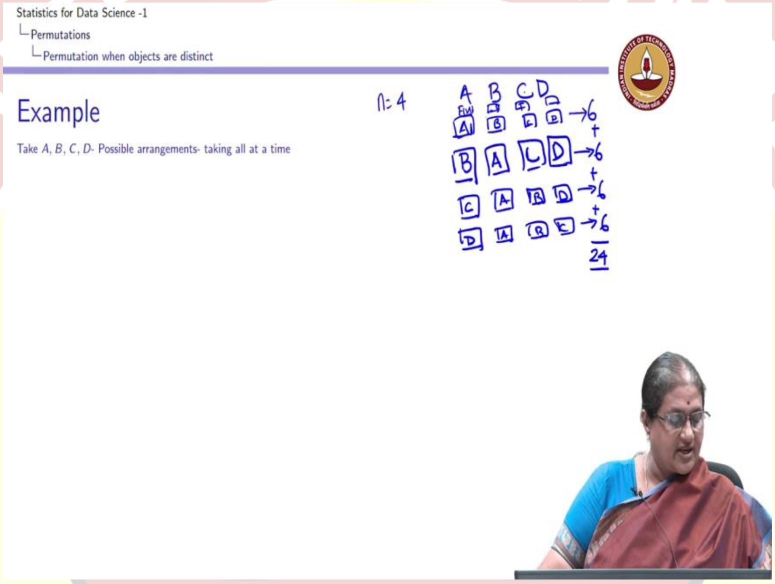

But then after this, we will, let us look at another example where I am looking at now, n = 4. And I have A, B, C, and D. Now, one way to look at it as you can fix A in my first place. And I know that B, C, D, if I look at B, C, D from a earlier example, I know if I have 3 places, the second, third and fourth, and within, between BCD, the way I can fill these 3 places with B, C and D are 6. So, if I fix A in my first place, I get 6 ways of getting hold of different possible arrangements. Is it clear?

So, the second thing, if I fix B, I can do the similar thing. I have a A, C and D to choose from for the 3 places. And I also know again that there are 6 ways of doing this. Similarly, if I fix a C, I can choose from A, B and D. And again, there are 6 ways of doing this. If I fix D as the first place, I have A, B and C. And there are again 6 ways of doing this, giving me 6 + 6 + 6 + 6, a total number of 24 arrangements, when n = 4. I can list all the arrangements in that following way.

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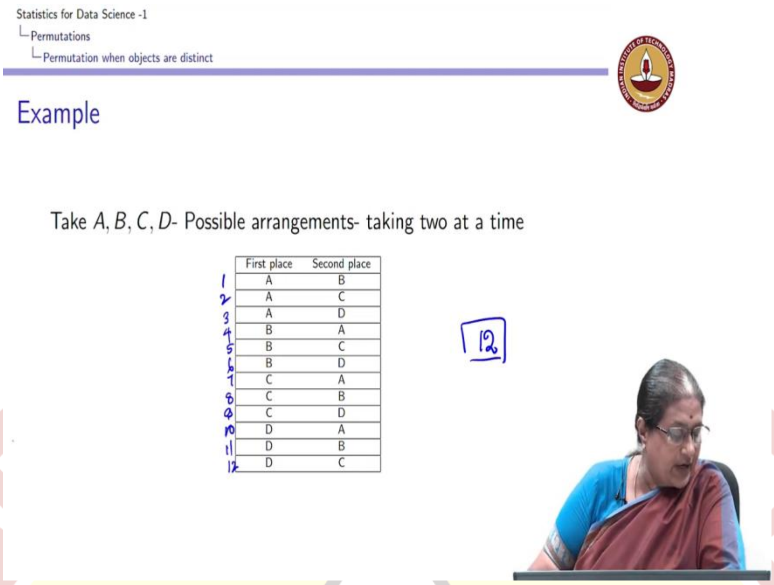

So, I have 6 from A, 6 from B. I can, again I can show you that I have. So, these are the 6 from A, I have 6 from B. I have 6 from C and 6 from D. And the total number of possible arrangements, the total number of possible arrangements in this case, you can see is 6 + 6 +6 + 6. I have 24 possible arrangements of A, B, C, D taking all at a time.

Now let us look at the same thing. Now if I am taking looking at the same possible arrangements A, B, C, D but I am now taking 2 at a time. Now again, if I have if I take A as my first object, the second could be a B, C or D. So, it could be AB, AC or AD. But that AB, I can have another arrangement BA, CA and DA. So again, with B, I could either have a A, but I already have listed that here. So, BC, BD, which will give me CB and DB. I have CD and DC.

So, the total number of arrangements I have, taking 2 at a time are AB, AC, AD, BA, BC, BD, CA, CB, CD, DA, DB, and DC, I have 12 arrangements of taking 2 at a time. Now, let us formalize this example to define what is a permutation?

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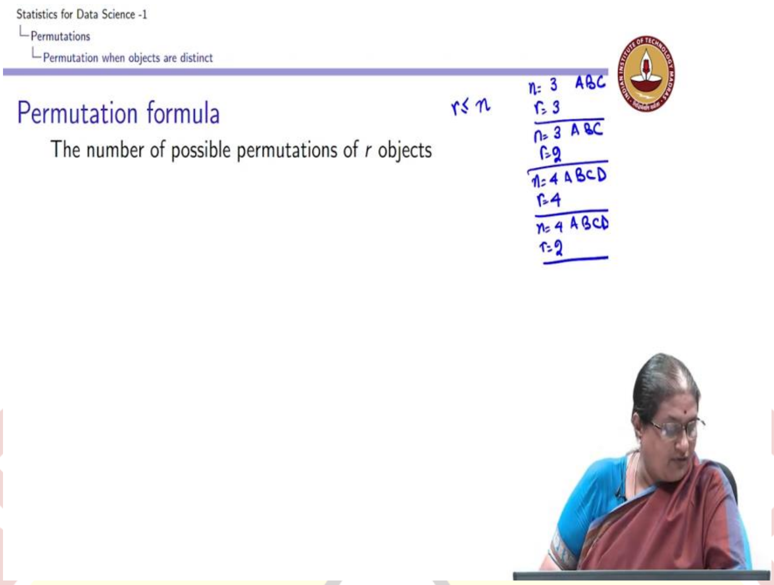

So, we are now ready to have the definition of a permutation. I have n objects of this, I am choosing r object. So, r ≤n. In my first example, when n was 3, I looked at ABC, I took at all at a time for my r was also 3. The second thing I took n = 3 again, ABC, but I was looking at 2 at a time, so my r was 2.

My third example, n was 4. I looked at ABCD, I again looked at all at a time, r was for 4. My fourth example, I had a ABCD, I again looked at 2 at a time, my r was 2. So, the question I am asking now is, how many possible permutations, these arrangements are also referred to as permutations. So, the question we are asking is, how many possible permutations are of r objects from a collection of n distinct?

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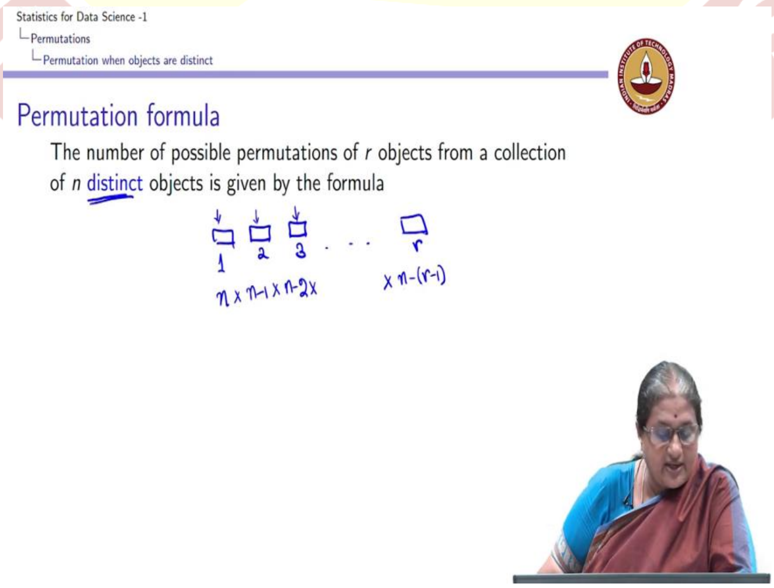

It is very important for us to understand that we are having it as distinct objects is given by the formula. So now, let us put in r blocks, 1, 2, 3. So, this is my first block, this is my second block, this is my third block, I can go so forth to r block. The number of ways, I can fill in this first block is n base. Now, once I have chosen from these n distinct objects, I have chosen 1 object to fill in this block. I do not have that available with me for the remaining. So, number of ways, I can fill in the second is n - 1.

So now, I have, the first and second are filled with 2 objects from these n objects. So, third, I can fill in n - 2 possible ways. So forth I can go, the rth, I can fill in n – (r – 1) possible ways. So, the first one can be filled in n way, second in n - 1 ways, third in n - 2 ways and the rth can be filled in n – (r – 1) ways.

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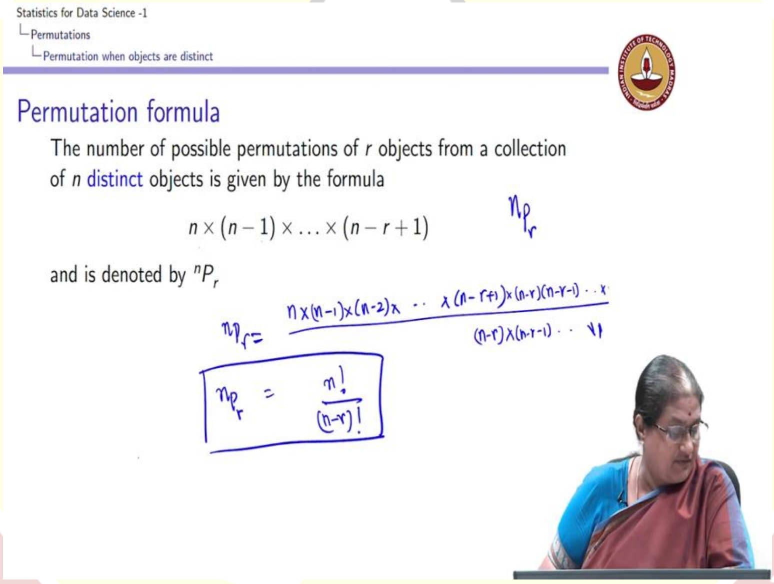

Again, using the fundamental multiplication rule of counting, I know the total number of ways I can fill, these r boxes n(n - 1) (n - 2) ..(n - r + 1), which can be written formally as n(n - 1) (n - 2) ..(n - r + 1). It is denoted by ${}^n_rP$. The way I can express this is, the collection from a collection of n objects, I am choosing r objects, this is given by ${}^n_rP$.

Now, let us look at this n into, now let us look at n into n minus 1 into n minus 2. So, I know nPr from the fundamental theorem of counting is given by this expression. Now, can I simplify this expression? If you look at it, if I can multiply my numerator with n minus r into n minus r minus 1 up to 1, I divide both numerator and denominator with the same quantity. The numerator now is nothing but n factorial. The denominator is nothing but n minus r factorial. So, I have a simplified expression for my nPr, which is n factorial by n minus r factorial.

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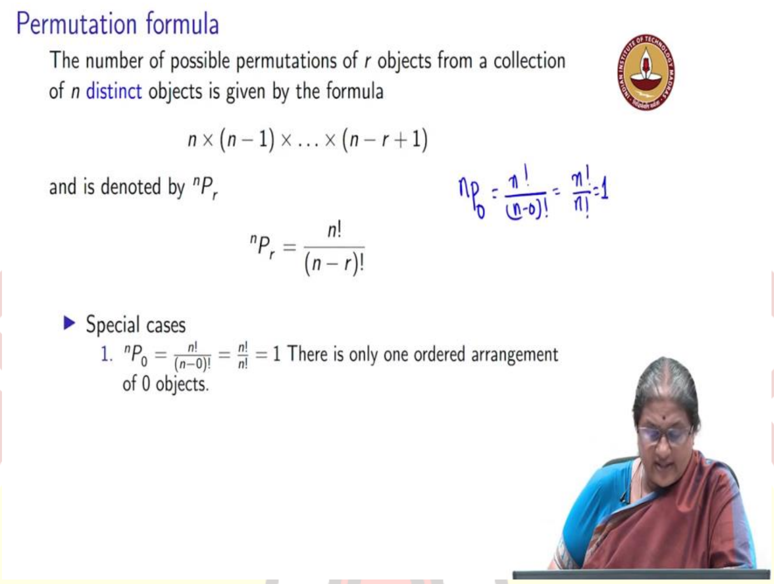

So, nPr is n factorial by n minus r factorial. Now, going back to the examples we have just seen, we can see that there are certain special cases of this formula. What are the special cases of this formula? What is nP0? The special cases of this formula, if you look at nP0, I apply the same formula, it is n factorial by n minus 0 factorial, n minus 0 is again n. So, I have n factorial by n factorial, which is 1. So, what does this mean? It means that when I am looking at no object in nP0, I have only 1 ordered arrangement of 0 objects. Now, if I am looking at the second special case, what is the second special case?

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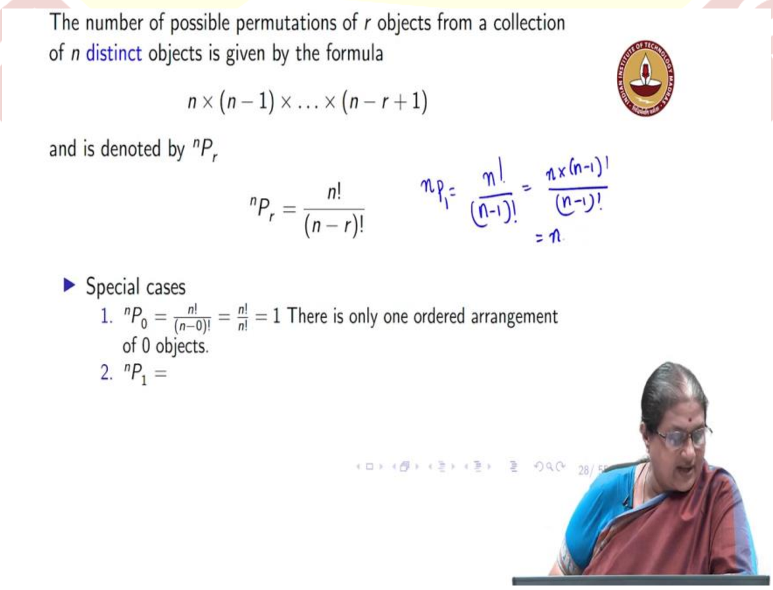

The second special case is, let me look at, nP1, what is nP1? Again, nP1 equal to n factorial by n minus 1 factorial. Now, I can write this numerator as n into n minus 1 factorial. i divided by n minus 1 factorial, so I have n as the answer. Now, nP1 is n, which is, this also can be viewed as the number of ways I can fill in 1 blank when I can choose from n objects.

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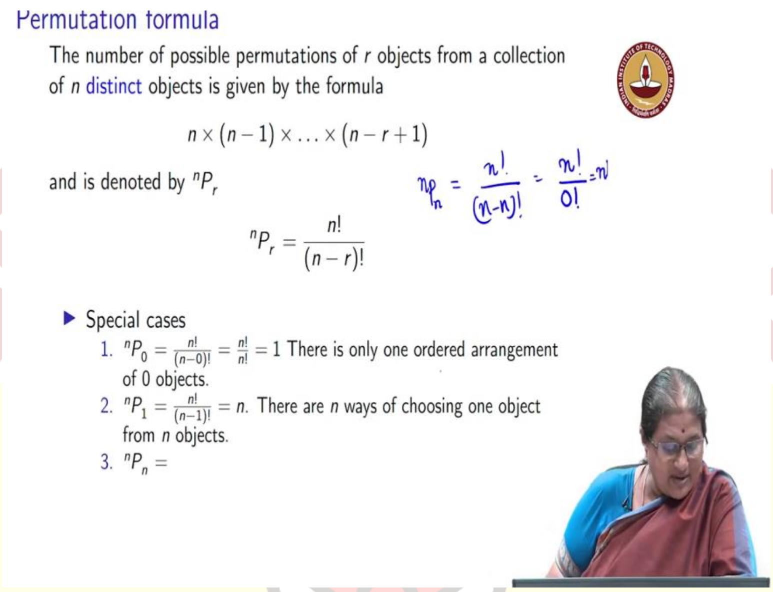

So, nP1 is n. There are n ways of choosing 1 objects from available and distinct objects. The third thing I want, we want to see is what is nPn. nPn is again number of ways of choosing n objects from an available n objects. So, nPn, if we apply the formula again, is n factorial by n minus n factorial, which is n factorial by 0 factorial. We already know, 0 factorial is 1, which is n factorial. So, nPn is the same as n factorial, which we have, we can see that this actually is what we got when we applied the fundamental theorem of multiplication, theorem of counting.

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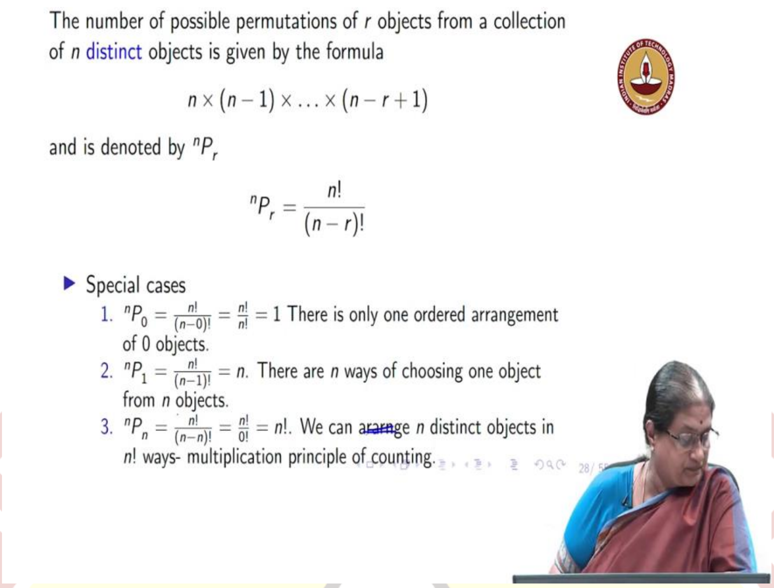

So, I have nPn equal to n factorial, where I am arranging the n distinct objects in n factorial ways.

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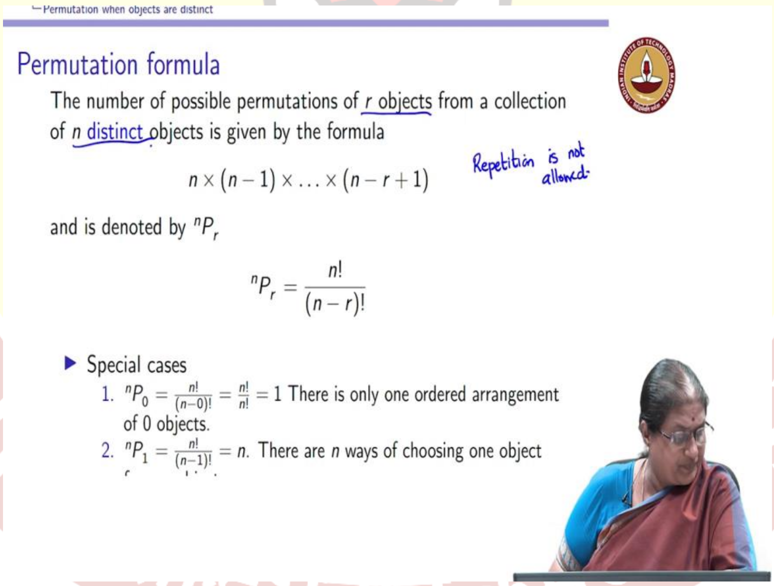

So, this is the main permutation formula, which tells me the possible permutations of r objects from n distinct objects. The key thing to remember here is repetition is not allowed. That is a key thing too. So, in that case, I have nPr is n factorial by n minus r factorial.

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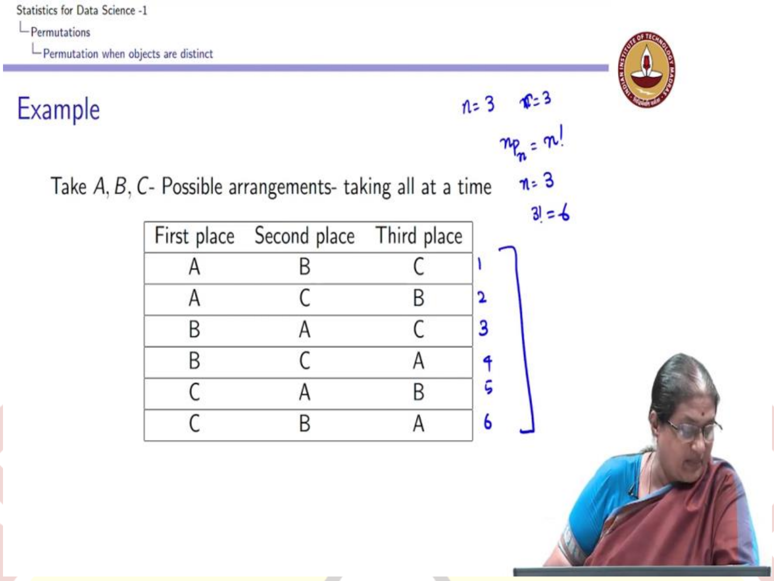

So, let us apply that to what we have seen so far. Now, when, the first example my n was 3, I am taking all at a time, so my r is also equal to 3. So, we can say that this would reduce the nPn, which is n factorial when n equal to 3. I know 3 factorial is 6.

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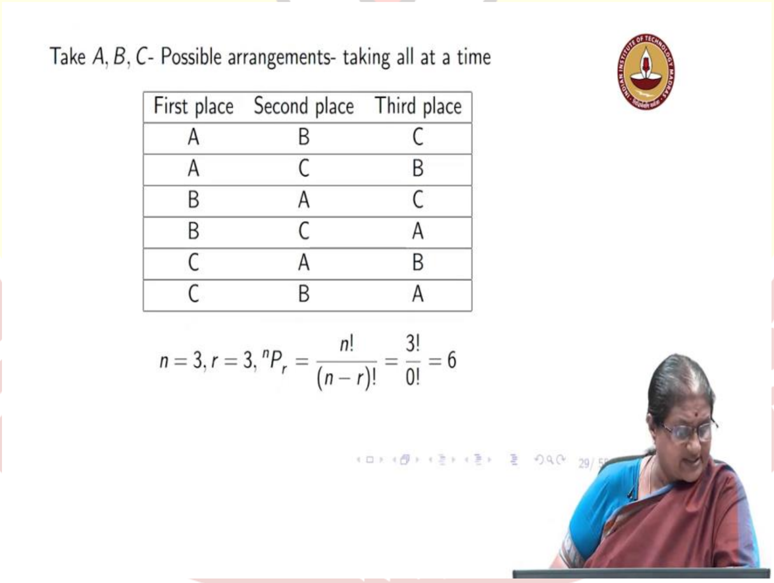

And I have 1, 2, 3, 4, 5, 6, which is the total number of ways in my first example, which we discussed. So, in the first example, I have nPr, which is equal to 6, which is nothing but n factorial. Now, let us look at the second case.

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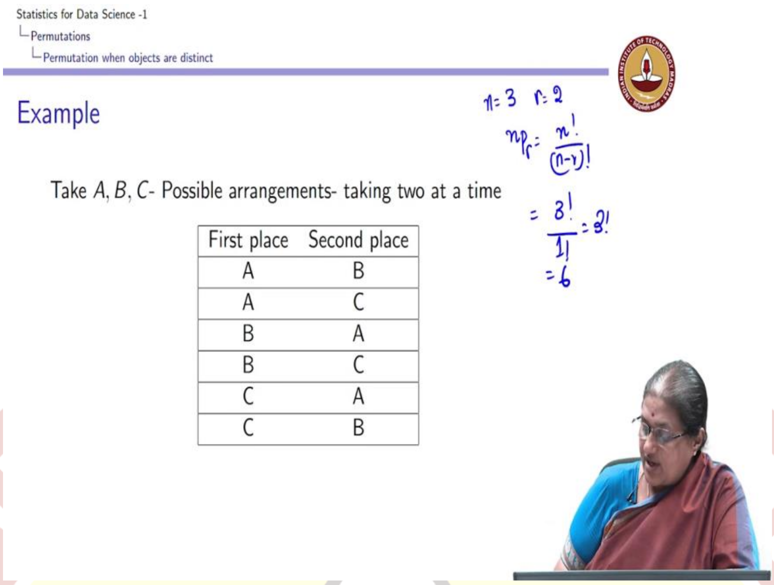

Now in the second case, I am looking at, again I have 3, n equal to 3, but my r equal to 2. So, nPr is n factorial by n minus r factorial, which is 3 factorial by 1 factorial, which is again 3 factorial, which is 6.

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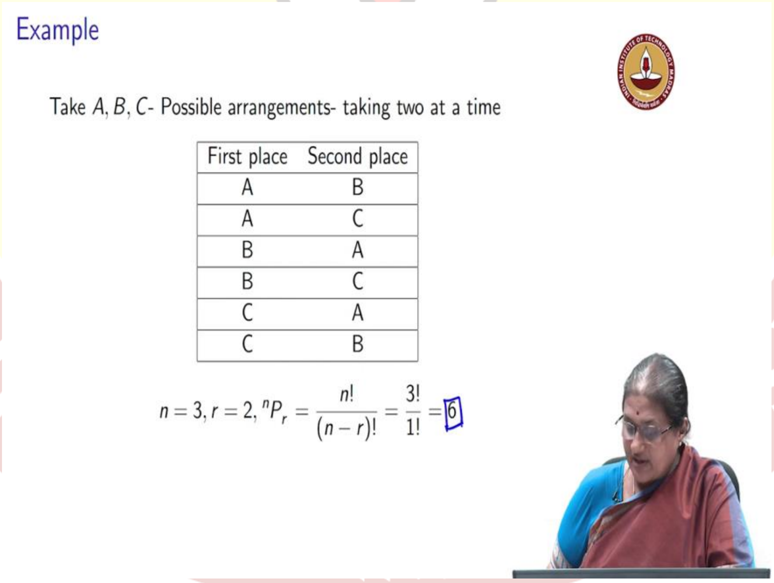

So here, in this case, my n equal to r 3, and r equal to 2, and I have the total number of possible arrangements is again 6. Now let us look at this third example, which we considered.

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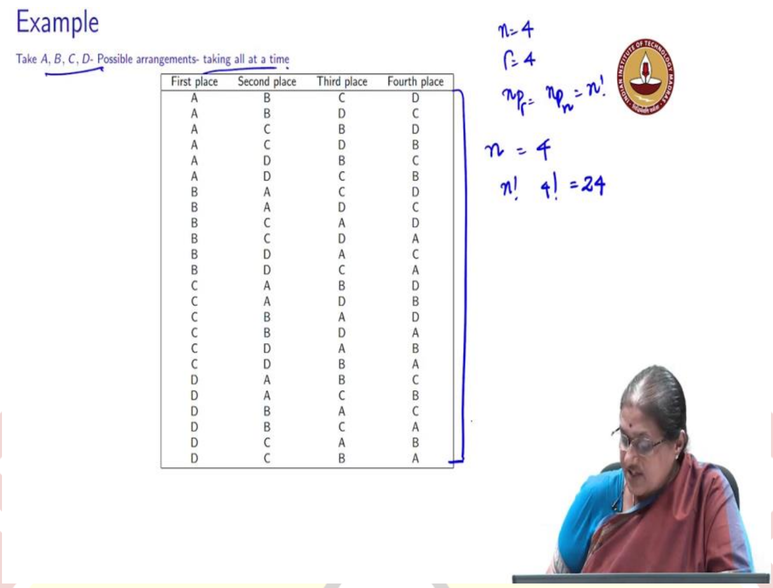

In the third example, I had all possible arrangements of ABCD taking all at a time. So here, my n is equal to 4, my r equal to 4. It is again nPr, which is the same as nPn, which is equal to n factorial, which I know here n equal to 4, n factorial is 4 factorial, which is 24. And I have 24 possible arrangements here.

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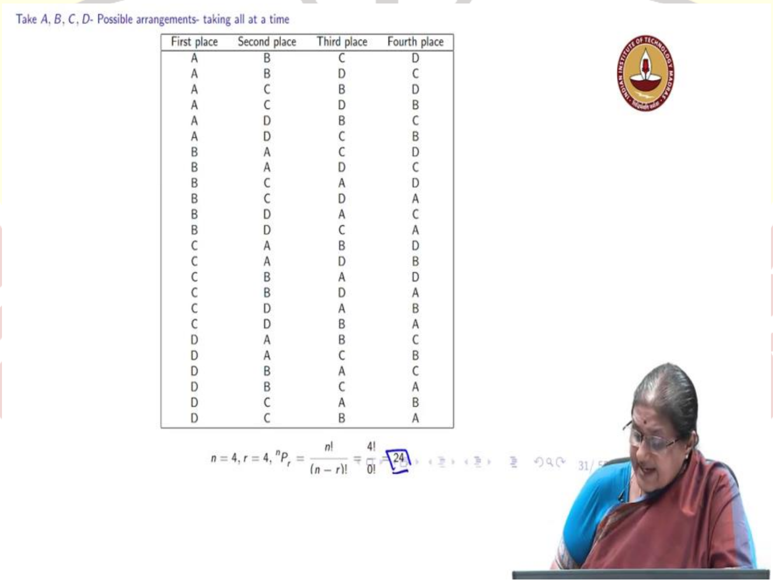

So, you can see that in this case, I have an, I am looking at all at a time, I have 4 factorial, and which is given by 24.

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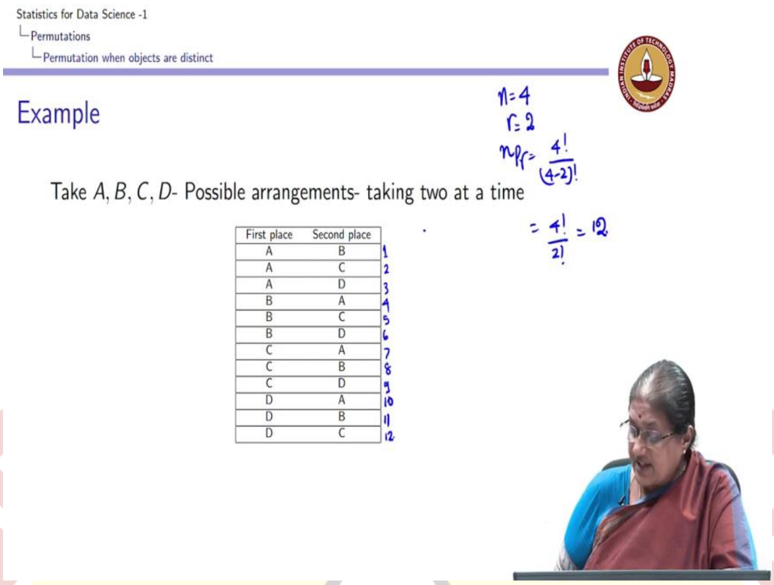

The last example, which we considered was, again ABCD taking 2 at a time again. Again my n is 4, my r is 2, my nPr equal to 4 factorial by 4 minus 2 factorial, which is 4 factorial by 2 factorial, I have 12 as the answer, and I can see that matches with what I have here 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12. I had 12 possible arrangements, which is nothing but what is my nPr in that case, 12.

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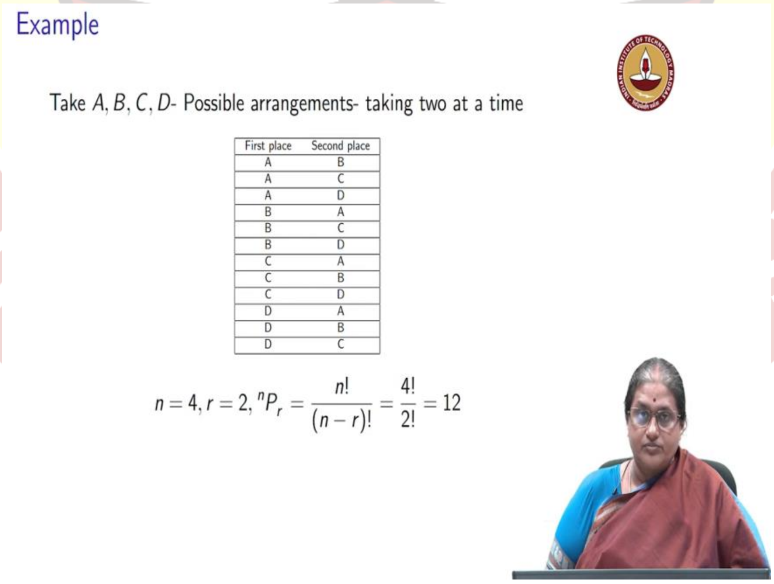

So, this is the first permutation formula, which is obtained when I have n objects, n distinct objects, and I am taking r from these n distinct objects, I am not allowing repetition.

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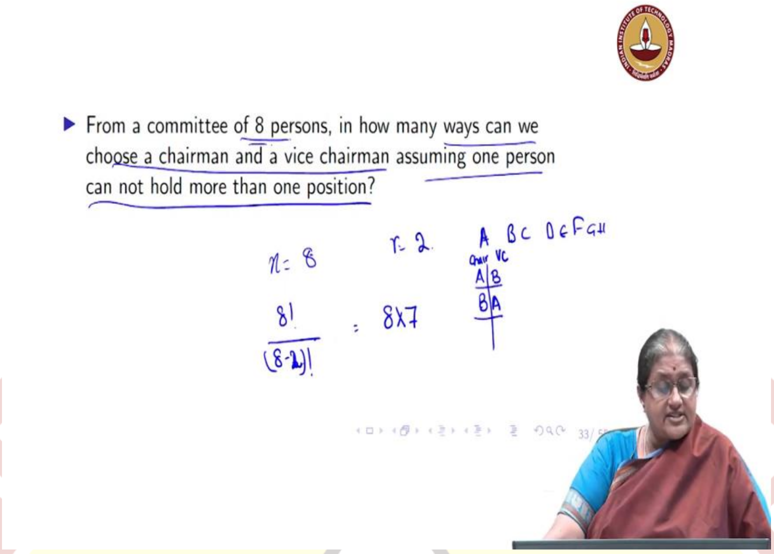

Let us look at a few applications of this formula. Suppose I want to form a committee of 8 people. The question is, how many ways can we choose a chairman and a vice chairman, assuming 1 person cannot hold more than 1 position? So, the first part, which we notice is 1 person cannot hold more than 1 position reflects the non-repetitive behaviour of our, whatever requirement which we have.

From a committee of n (persons), 8 persons, so this n equal to 8. I want to see how many ways can we choose a chairman and vice chairman. So, the r equal to 2, how many ways that is also something which I can, this one, so I need to know whether a chairman, whether if I have A, B, C, D, E, F, G, H, these are the 8 people. A could be a chairman, B could be vice chairman, B could be chairman, A could be vice chairman. So, these 2 orders are different. So, this is a chairperson, this is a vice chairperson. So, these 2 orders are different. So, applying the formula, I have 8 factorial by 8 minus 2 factorial, which is 8 into 7.

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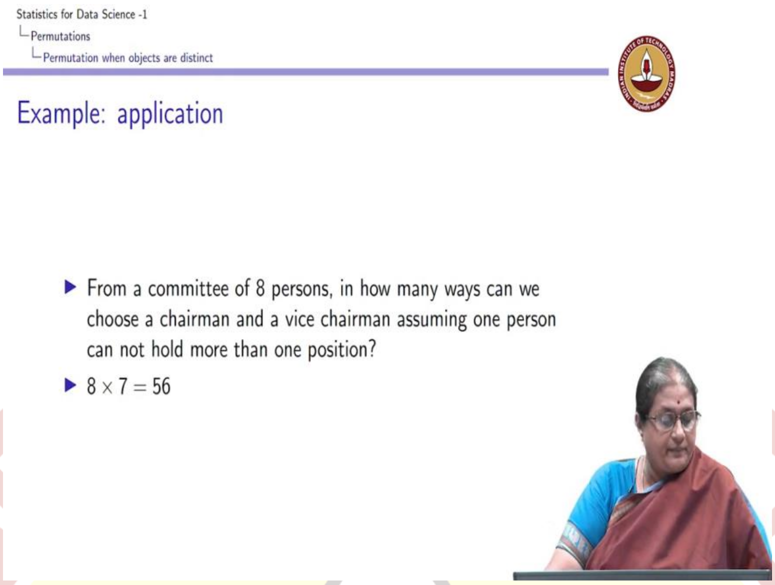

So, the total number of ways I can do this is 8 into 7, which is 56 ways.

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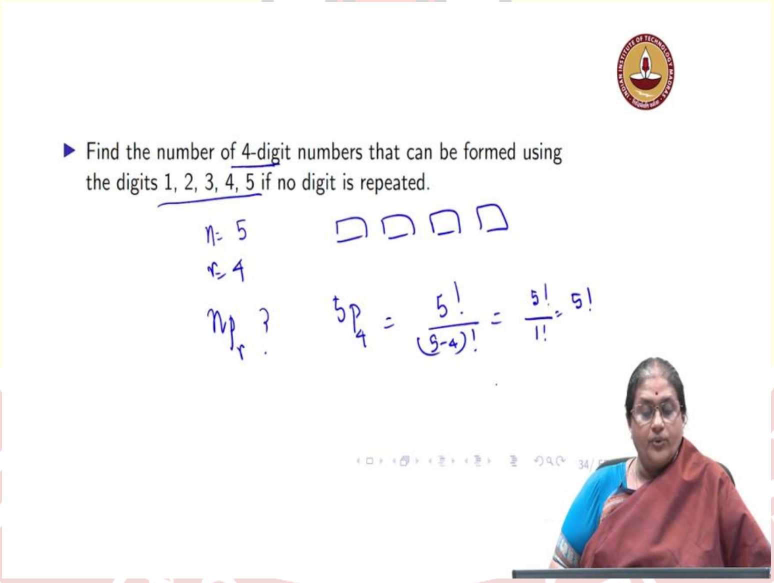

Now, let us look at the next application. In the next application, I am looking at the number of 4-digit numbers that can be formed using digits 1, 2, 3, 4, 5. Again, no digit is repeated. So, my n in this case is 5, I am choosing from 5 distinct objects, I am looking at number of 4-digit numbers, so I have four 4 blanks. So, my r equal to 4. And I am asking the question, what is nPr again? It is 5p4, which is nothing but 5 factorial to 5 minus four factorial, which is same as 5 factorial by 1 factorial, which is 5 factorial.

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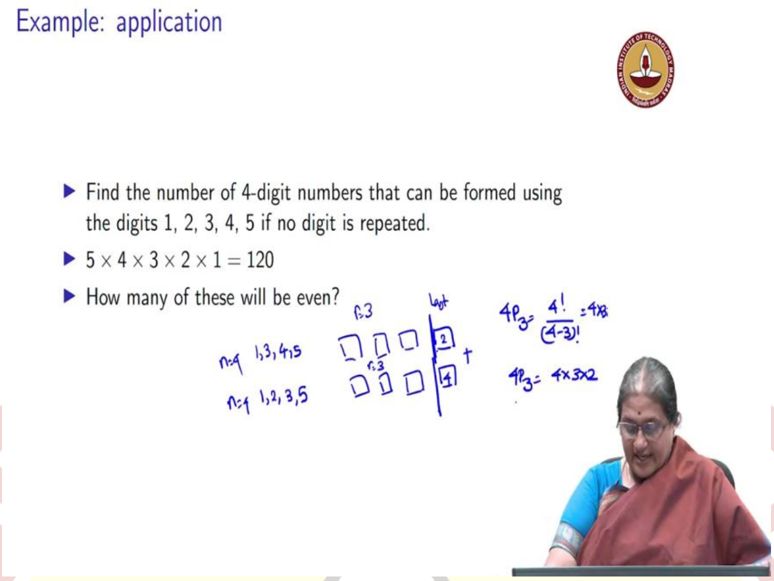

And I know that 5 factorial is equal to my 5 into 4 into 3 into 2 into 1, so I have a result, which is 120 ways of creating these 4-digit numbers or 4-digit codes, whatever we want. Let us look at an example. Let us slightly modify the application, which we looked. So earlier, I said find the number of 4-digit numbers, which can be formed using these digits. But now suppose I am saying that I want to know how many even numbers can be formed. I know if I want to count the number of even numbers, the last thing can either be a 2 or a 4. I am fixing the last digit here. So, if I fix the last digit, I have 3 blank spaces that are available.

Now, once I fixed the last digit, these, now my r becomes 3, and I can choose from the remaining. So, if it is 2, and r is 3, I have only remaining 1, 3, 4, and 5, from which I can choose the remaining 3 numbers. And that can be done in 4p3 ways, which is 4 factorial by 4 minus 3 factorial, which is 4 into 3 into 2 ways. Similarly, if I fix a 4, I have again r equal to 3 blanks, and I have 1, 2, 3 and 5, my n equal to 4. Again, I can choose it in 4p3 ways, which is again 4 into 3 into 2 ways.

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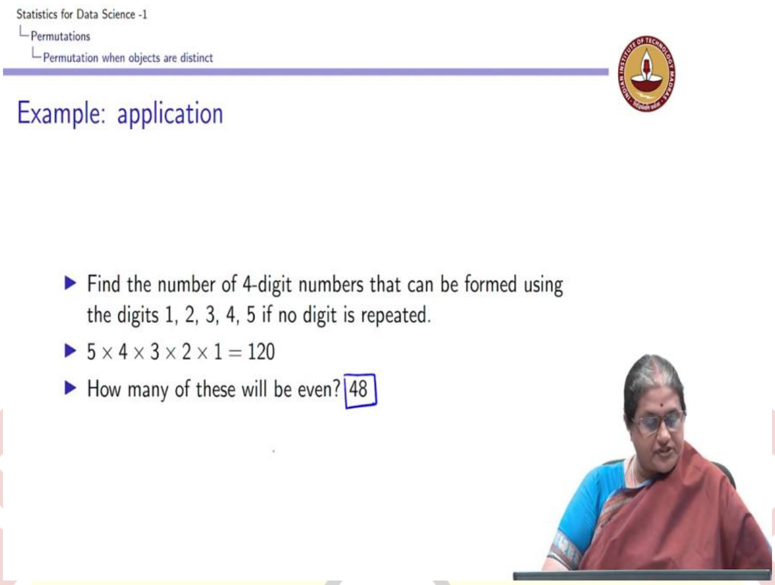

So, if I add these 2 up, I see that the total number of ways is 24 plus 24, which is 48. So, you can see that we can use this simple formula of permutation to count the number of arrangements that can arise in various situations.

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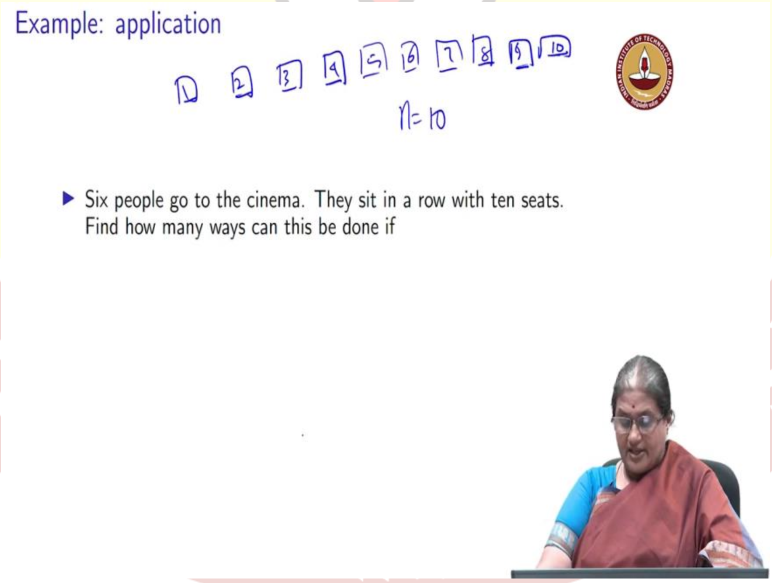

Moving forward, let us look at what would happen, another example, let us look at another application. Suppose I have 6 people who have gone to a cinema. In cinemas again, lets again remember that what we are talking about here is arranging people in a line, in a linear order. So again, if you look at cinema or stadium, typically people sit in a row. So, I go to a cinema, I have 10 seats. So, these are the 10 seats, I can number my seats as 1, 2, 3, 4, 5, 6, 7, 8, 9 and

10. These are the 10 seats that are available to me. So, 6 people go to a cinema. So, my n equal to 10. 6 people go to a cinema.

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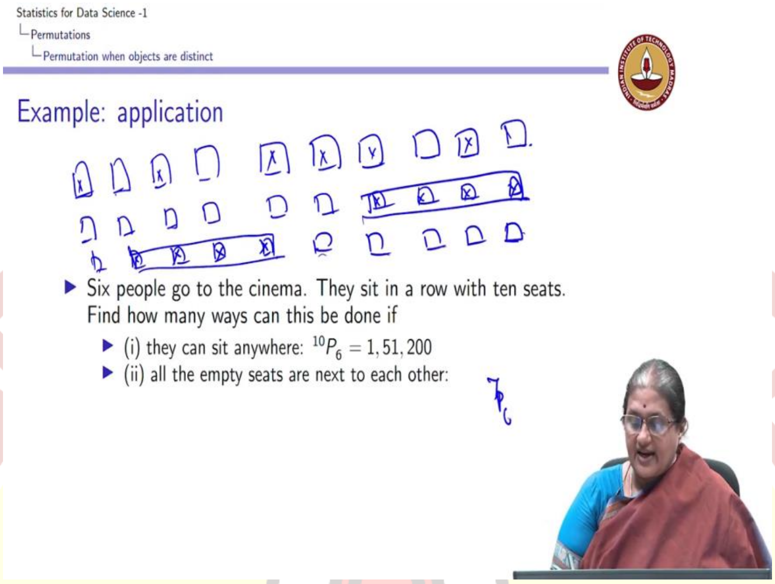

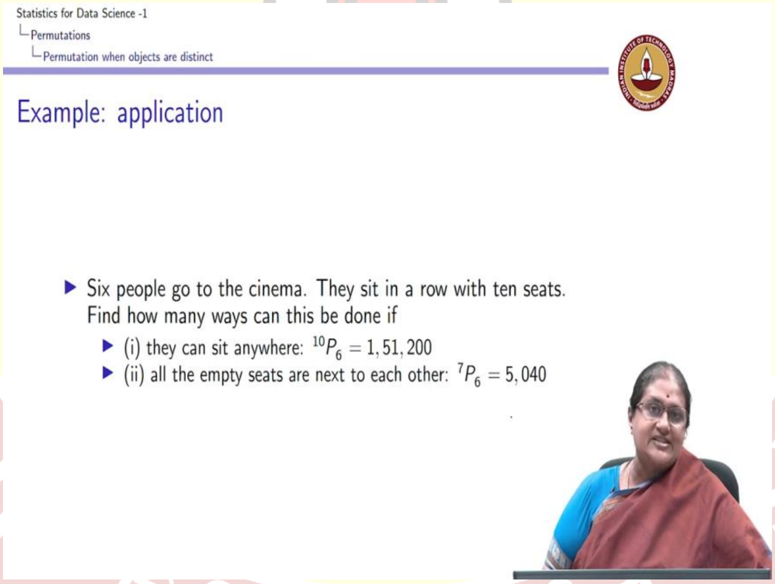

So, the question, the first question we are asking here is, how can I makes the 6 people sit in this row? I have 10 available seats, how many ways can they do if my people can sit anywhere? So, if these 6 people can sit anywhere, I know that n equal to 10, my r equal to 6 and number of arrangements applying my formula is 10p6.mSo, the first answer is, well if I can sit anywhere, the answer is 10p6, these are the number of ways 6 people can sit in available 10 seats.

But now, suppose I am giving another thing is all the empty seats are next to each other. How do we solve this problem? For example, I cannot have a situation of this kind. I again have the 10 seats, I cannot have a person occupying this seat, this seat 1, 2, 3, 4, 5, 6. So, these empty seats are not next to each other, whereas this is permitted. So, this is permitted for me. I can have all the 4 empty seats next to each other, or even this is permitted.

What we need to understand here is these four empty seats form a block. And I can consider these 4 empty seats as a distinct object in it, because I am not allowing them to move. Now, if these 4 empty seats are considered as 1 distinct object, then the total number of distinct objects are 1, 2, 3, 4, 5, 6 and 7. I need to see how many arrangements can be done, that is this. So now, I have 7 places available to me. I want to set these 6 people in these 7 places, and that can be done in 7p6 ways.

So, these are application of this simple, these are applications of the simple formula of arranging r objects out of n distinct objects when repetition is not allowed.

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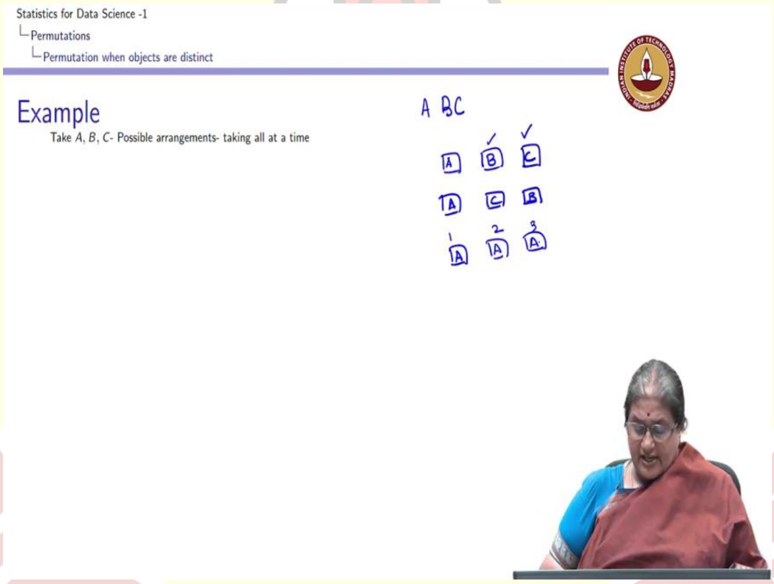

Now, suppose, let us go to the next case, where I want to know about what is that number of permutations or where a number of arrangements when the objects are not distinct, when I am looking at repetitions, so when I am looking at the same (example). So now, let us move forward to understand what is the number of possible arrangements again taking all at a time, but now I am allowing repetition. So, let us look at the first example. I looked at ABC.

Now, I did not allow repetition earlier, so when I looked at the 3 boxes to be filled, once I fill in A in one of the boxes, I do not have A available to fill in B or C. So, only either B or C could get into the second box. So, I had a A, so I had a A with BC or CB. But suppose I am allowing repetitions, so if I fill A in my first box, A is available for the second box, A is available for the third box also. So, AAA is a valid arrangement, because I am allowing repetition. I am not saying that once I have put a particular thing in a box, that is not available for filling up the other 2 boxes.

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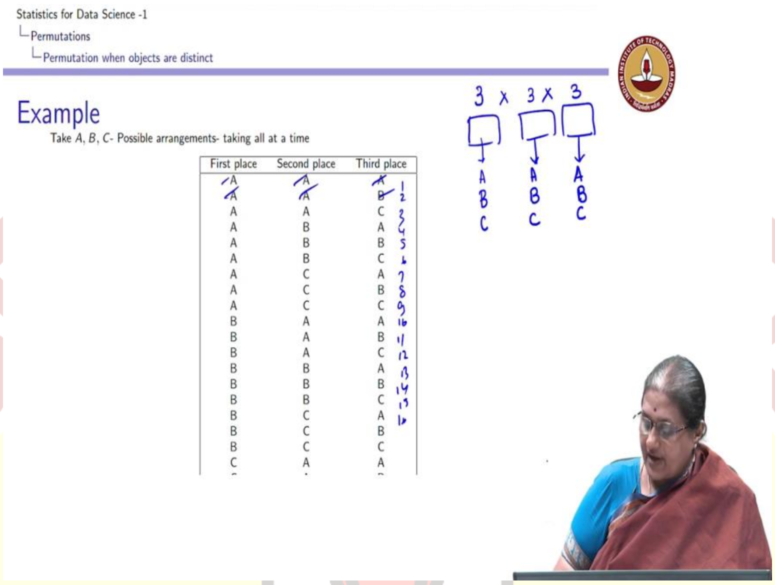

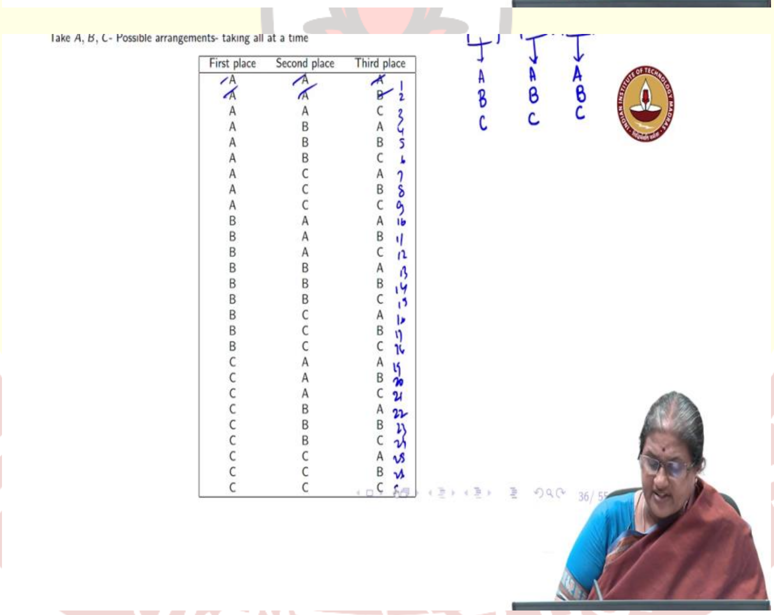

So, if you look at this case, so I am looking at the number of possible arrangements. So, AAA is a possible arrangement, AA with B is a possible arrangement. So, the number of possible arrangements, now if any, the first box can be filled with A or B or C, the second box can be filled with A or B or C, the third box can be filled with a A or B or C. So, there are 3 choices available to fill this box A, there are 3 choices available to fill the box, second box, there are 3 choices available to fill the third box, because all the choices are available for all the boxes.

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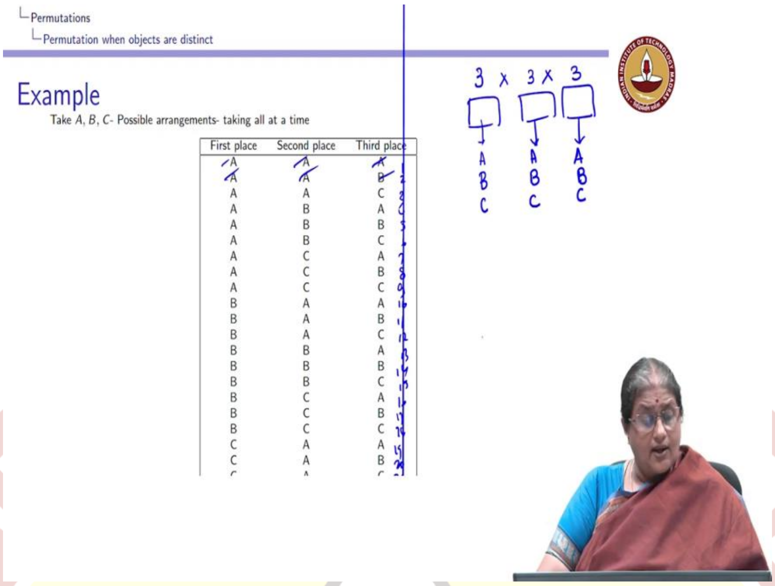

So, the total number of ways I can fill all the 3 boxes together are 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, and you can see that, that goes all the way up to, and you can see that, that 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26 and 27, I have up to 27.

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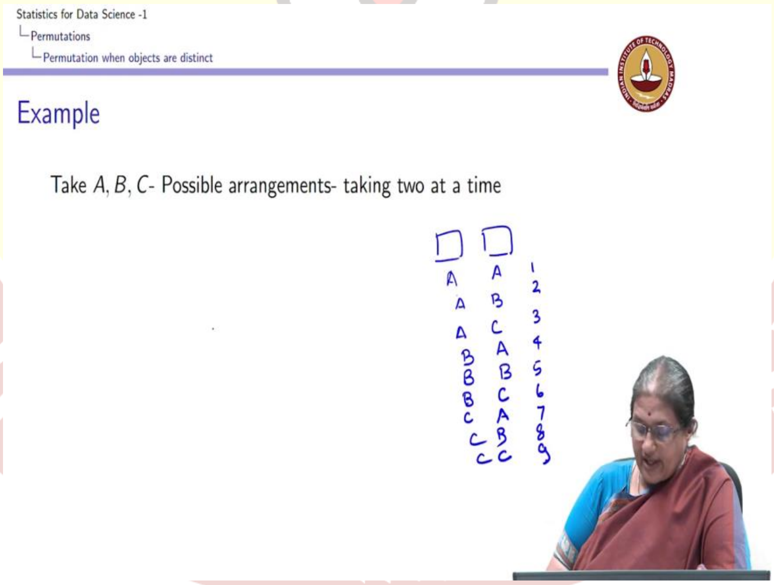

So, let us look at the second example. Now, suppose I am looking at the same possible arrangements but now I am taking 2 at a time. So, I have 2 boxes. The first box can be an A, the second box can be an A, so AA. First is an A, second can be a B. First is an A, second can be a C. Similarly, first is a B, second can be an A. First as a B, second can be a B. First is a B, I have a BC. C goes with an A, B or C, I am allowing repetition, so you can see the total number of possible arrangements possible in this case is 9.

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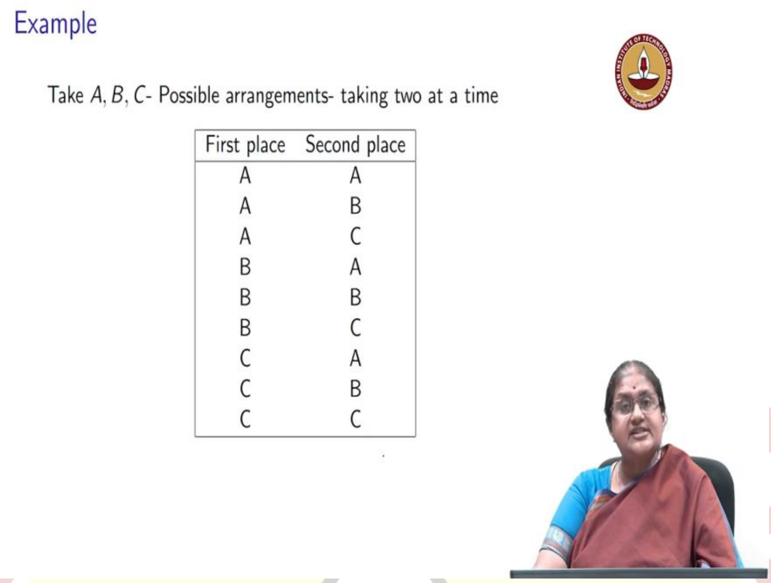

So, we can actually come up with a formula to, so I have 9, if I take 3 objects, 2 at a time, I allow for repetitions, I can see that the total number of possible arrangements is 9 here.

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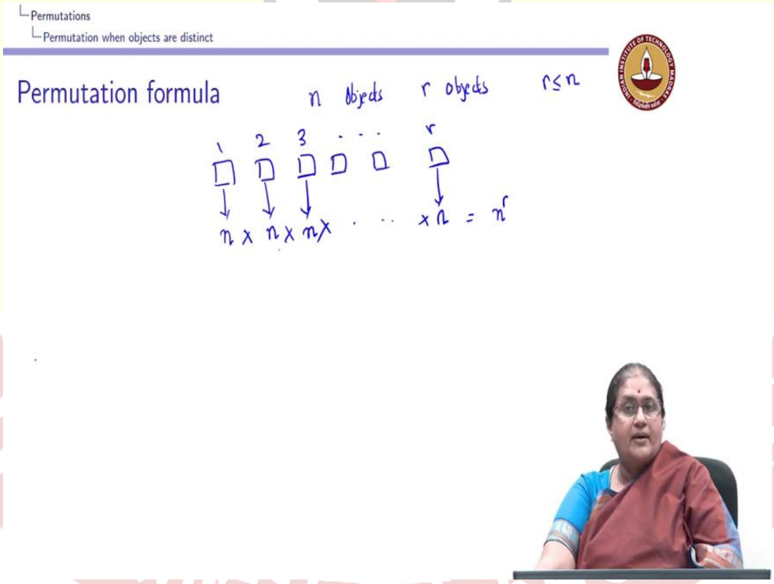

So, let us look at what is the permutation formula. So, I have in this case, n, again I have n distinct objects. I am taking again r objects out of this n distinct objects, r is less than or equal to n. So, how do I, I am allowing repetitions, so if I have r blocks or r blanks, this is my first, this is my second, this is my third and this is my rth block. The first block can be filled with any of the n objects. The second can also be filled with any of the n objects. Third can also be filled with any of the n objects. The rth block can also be filled with any of the n objects.

Again, I apply the multiplication rule of counting. If I apply the multiplication rule of counting, the total number of ways I can fill all these r boxes is nothing but n into n into n r times, which is n to the power of r. So, we can see that this is, this second rule of permutation, where I again have n distinct objects, I am choosing r distinct objects from this n distinct objects, but I am allowing repetition.

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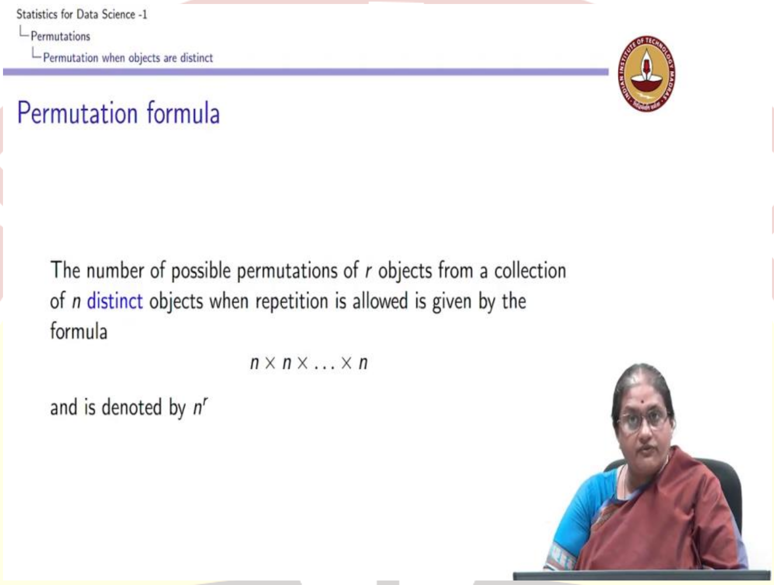

So, how do, what is the formula there? So, if I have r objects from n distinct objects, repetition is allowed. It is given by the formula, the formula is given here, it is n to the power of r, this is what we have seen.

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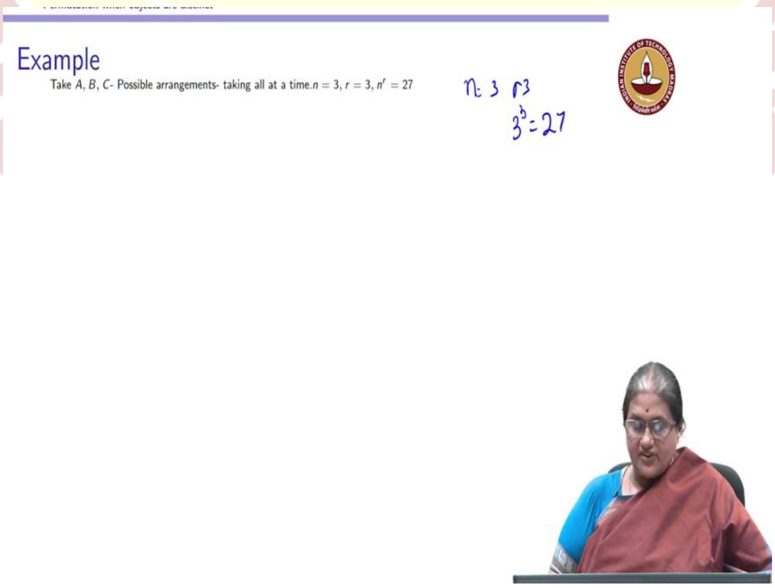

So, in the earlier example, when I had an A,B,C and I am taking all at a time, I had n equal to 3, r is also equal to 3. 3 to the power of 3 is 27.

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And we saw that we had the 27 arrangements, which were possible.

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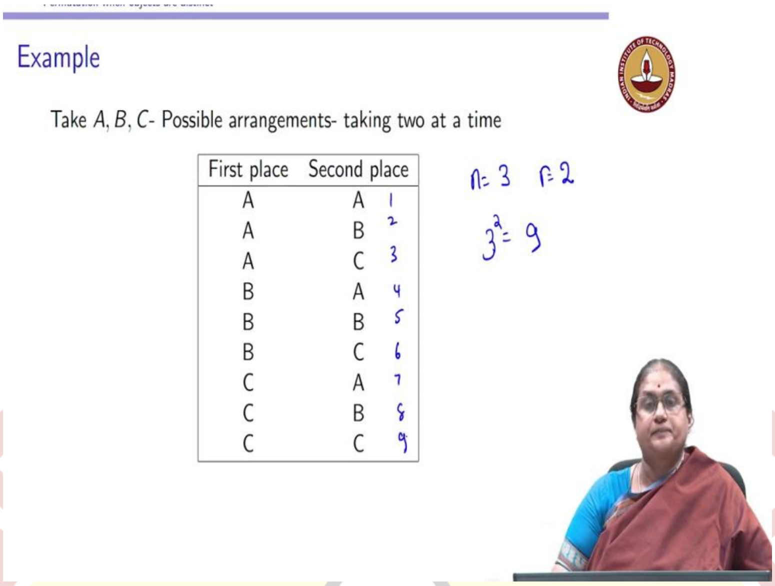

Similarly, when I looked at this case taking 2 at a time, my n equal to 3, r equal to 2, n to the power of r is 3 to the power of 2, which is 9. And we have seen that there are 9 distinct arrangements possible in this case.

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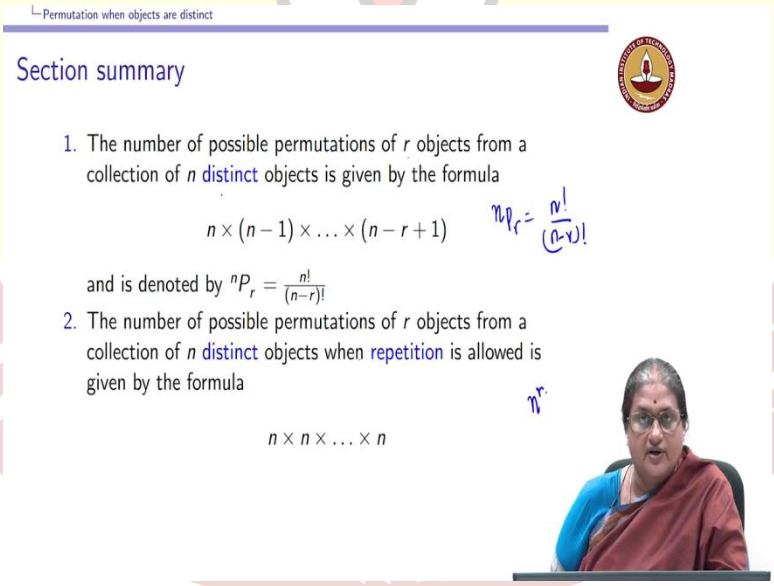

So, let us look at a summary of what we have learned so far in this topic. So, we started with permutations when objects are distinct. We again look at 2 cases in this case. One is when repetitions are not allowed. And the second when repetitions are allowed. When repetitions are not allowed, the number of possible permutations of r objects from a collection of n objects is given by and denoted by the formula nPr, which is n factorial by n minus r factorial. When repetitions are allowed, it is given by n to the power of r. This is what we have learned so far.