INDIAN INSTITUTE OF TECHNOLOGY MADRAS

सिद्धिर्भवति कर्मजा

IIT Madras

ONLINE DEGREE

Mathematics for Data Science 1
Professor. Neelesh S Upadhye
Department of Mathematics
Indian Institute of Technology, Madras
Lecture No. 52
Logarithmic Functions

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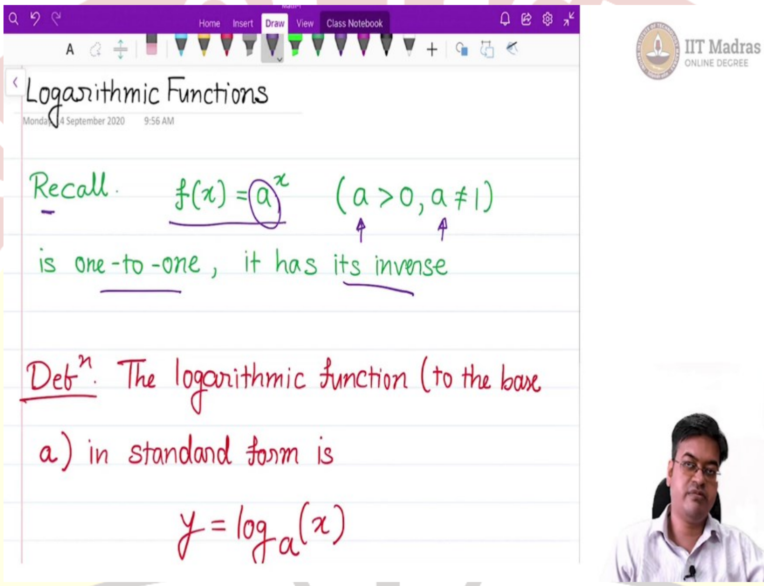

So, in this video we are going to look at the inverse of exponential function. In the last video we have seen the inverse of a general function and we have concluded that if the function is one-to-one, then the finding the inverse of a function is very easy. So, let us focus on inverse of exponential function in this video and see its properties graph or how it is graphed and a various other properties about domain and range of these inverse functions for exponential functions.

So, let us recall our notion of exponential function, we started with a function which is a function will be called as exponential function if it is written in the form $f(x) = a^x$ where there were some conditions on a, for example, a should be greater than 0 and a cannot be equal to 1, a greater than 0 is a typical condition which we need because otherwise we have to deal with complex random, complex variables which is out of scope of this course.

So, we are putting a to b greater than 0 and a≠1 is the condition because if you put a=1, then $f(x) = 1^x$ which is 1 for all of them, so it is not an interesting function to study. So, whenever these conditions are enforced we know that our exponential function $f(x) = a^x$ is one-to-one and because every one-to-one function has the inverse this function also has the inverse, there is nothing special about it. And that inverse we will define as logarithmic function. So, naturally since we are talking about exponential function with base a so we will talk about logarithmic function with base a.

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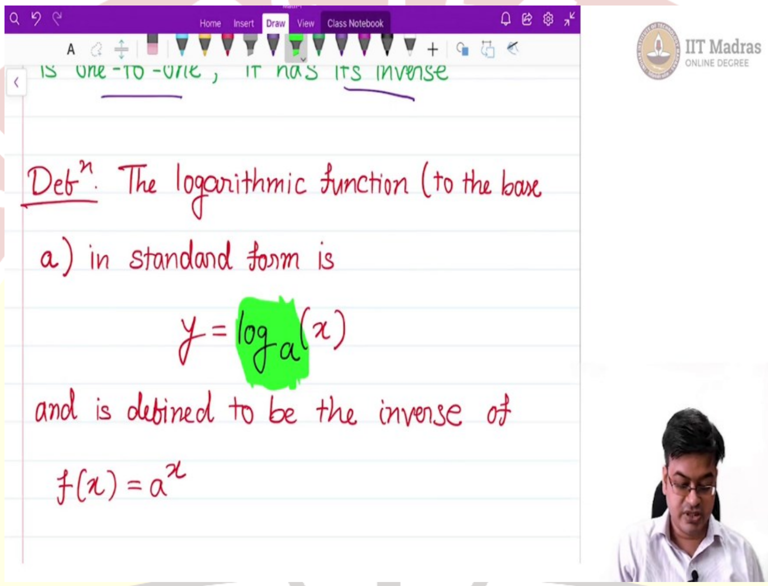

So, here is a definition of a logarithmic function. The definition says that the logarithmic function to the base a in the standard form is given by y=$\log_a x$. So, remember this function is represented by log to the base a and x is the argument of the function, so this is the definition of a function or this is replacing f, $f^{-1}(x)$and then x is the argument and we are plotting it along y axis and is defined to be the inverse of the function $f(x) = a^x$.

So, $f^{-1}(x)$is actually $\log_a x$, is this simple. So, now we need to understand what will be the domain and codomain or range of this function that is an important thing that we need to understand. So, in order to that let us try to devise some rule so that we will have a track of what is exactly happening when we are talking about logarithmic function and how it is related to exponential function.

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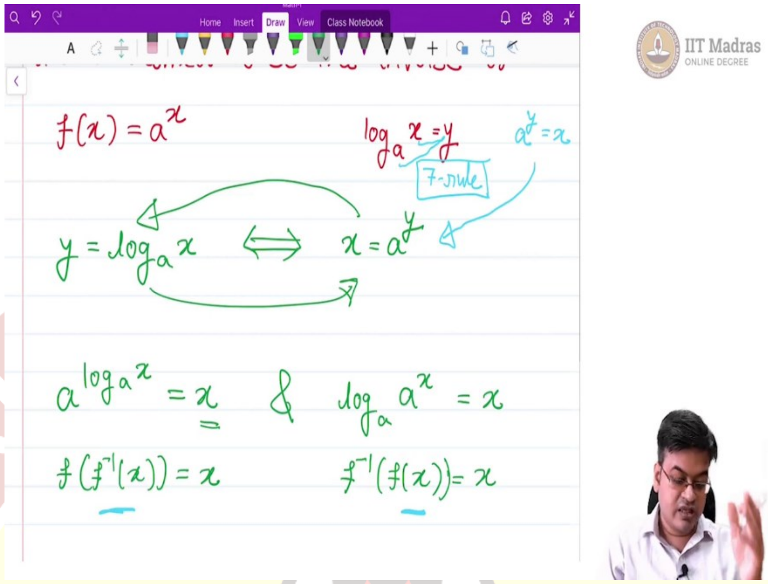

So, there is a one to one correspondence between logarithmic function and an exponential function which is expressed by this relation y=$\log_a x$ if and only if x= a^y or for more precision you can write this as $\log_a x$ =y then you can actually virtually assume this 7 rule that is you start from the base, go to the right hand side and come back that means what we are saying is you start with a, go to the right hand side, that right hand side is raise to the power and that should give you x, that is what this rule is.

So, this is simple technique to remember known as 7 rule. So, you can use this 7 rule to memorize the one-to-one correspondence between log and the exponential function. You can easily see that by definition if I write x= a^y, then I want to know the value of y, I should be able to get it by taking the log of this function x.

So, this is the mathematical definition of our logarithmic function. To make this mathematical definition precise we need to understand some prototypes that is whether this function we have defined it to be the inverse of f but whether this function is actually the inverse of f or not that is what we need to figure out.

So, as stated earlier we can actually check these two rules $f(f^{-1}(x)) = x$ and $f^{-1}(f(x))$. So, what is$f(f^{-1}(x))$? As I mentioned earlier $f^{-1}(x)$ is nothing but $\log_a x$ and f is a^x so you just substitute $a^{f^{-1}(x)}$. What is that? $a^{\log_a x}$. Now, what this should be? You use this one to one correspondence from here to here and here to here and you will get this to be equal to x.

In a similar manner you can apply it to f of x and$f^{-1}(x)$ f inverse, so $f^{-1}(f(x))$ is $\log_a f(x)$ but what is $f(x)$? It is $a^x$and therefore $\log_a a^x = x$.

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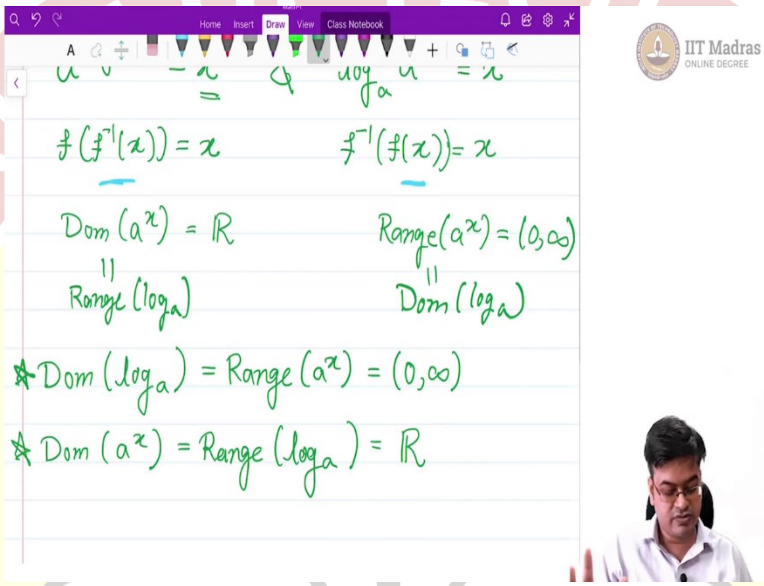

Now, in order to understand this completely I need to understand the domain of log function and range of log function and the range of log, range of exponential function and the domain of exponential function. So, let us understand this particular thing. We have already seen what is the domain of a^x, so we already know domain of a^x because x can be entire real line and then it maps this domain onto the range of a^xthat range cannot take negative values, this is what we have seen when we studied.

So, it was 0 to ∞, so this should be clear before going to the range of log function. So, if at all the logarithmic function is to be defined, this if you recollect this should become domain of log to the base a and this should become the range of log to the base a, so the this is the crux of the definition of inverse. So, when this is satisfied you are done.

So, essentially your log function will be defined from 0 to ∞ to real line. That means in the domain it cannot have negative values, it cannot have 0 as well and in the range it will have the entire real line that is what is written here in this case that is domain of log to the base a is actually range of

a^x which is 0, ∞ and domain of a^x is actually the range of log to the base a which is real line, the entire real line.

These are the two important points which will help you in understanding the domains of the functions which are derived from these functions that is logarithmic functions or exponential functions. So, these, all these things you should always remember the valid ranges and domains of the function. So, this completes our verification that logarithm function the way we have defined is actually an inverse of exponential function. Once the verification is complete let us dwell more and find the domain of the derived functions, derived, by derived functions means composition of basic logarithmic function.

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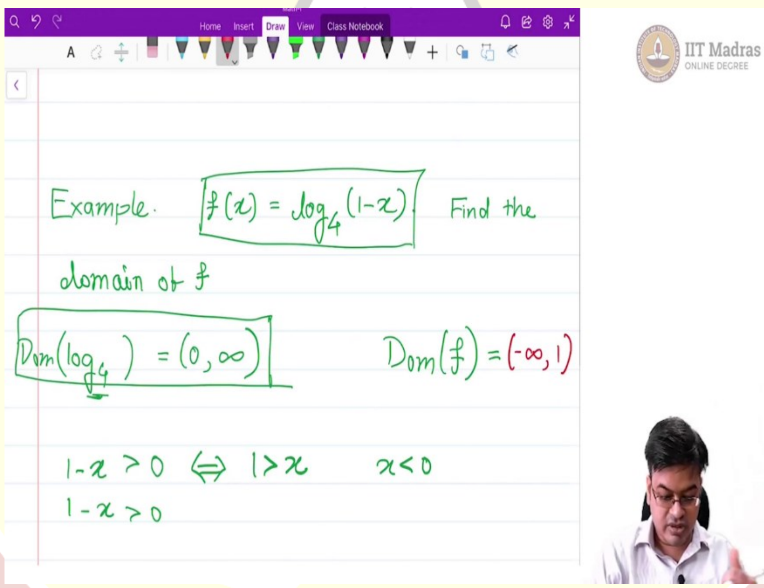

For example, let us take an example of f x which is log to the base 4 of 1 -x. Now, log to the base 4 is actually a function which has a domain. What is the domain of this function? The domain of this function is actually 0 to ∞. Now, that means the argument that is supplied to this function log to the base 4 cannot be 0, or it cannot be a negative value. So, based on this understanding from the definition of our log function you can look at this function which is f of x and look at the argument of the function 1 -x.

According to this definition 1 -x must be strictly greater than 0. This will happen if and only if my 1>x, 1>x and because 1 -x needs to be greater than 0 can x be less than 0, if you look at x to be less than 0, 1 -x will actually be greater than 0. So, the only condition that we require over here is my function should be defined that is domain of this function f should be equal to, it cannot include 1, 1 to, it is not 1 to ∞, this is how we commit mistakes.

So, domain of f is x should always be less than 1 that means the domain of this function should be here -∞ to 1 and it cannot go beyond 1 this is what our understanding is about this function. Now, let us go and enhance our understanding in finding the domain of a function which is slightly more complicated than this function.

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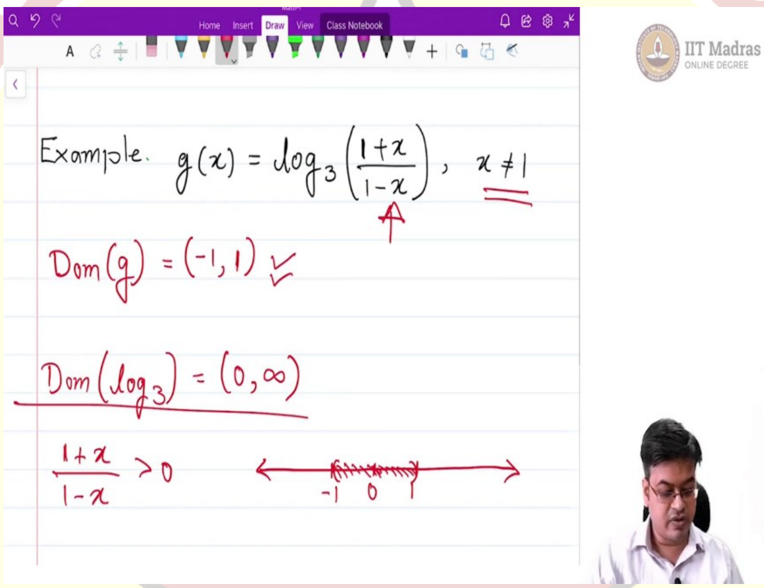

So, our question is to find the domain of this function g. In order to find the domain of this function g, let us first understand what is the domain of the function log to the base 3. Now, this function is defined when the argument given is between 0 to ∞. So, now I want the argument of this function which is this gx to be between 0 to ∞. So, what I should do is I want this 1+x upon 1 -x trapped between 0 to ∞ that means it should be greater than 0. Now, when this can happen?

So, naturally let us split the real line into some parts x is≠1 is already given to you, so x cannot take the value 1, this is a point 0, this is a point 1, let, for safety let us put the point -1 as well here. And now x cannot be equal to 1, so this point is actually deleted, so this point cannot be there. Then, 1 -x should, if 1 -x>0 that means my x<1 the function is defined.

So, I have this in the similar manner -∞ to 1 but let us not go for -∞ because there is in the numerator there is 1+x, so this 1+x, it can become, it can take a negative value when x<-1 and if x<-1 this 1 -x will become positive. So, I have to rule out that part as well. So, this -1 to 1 is rule, -∞ to -1 is ruled out, -1 will give me the value 0 so -1 is also ruled out and therefore I am only left with the interval of this form which is -1 to 1.

So, based on the arguments and based on this domain I know that the domain of this function is valid only between -1 to 1. Now, you may say why not 0? 0 will not cause any problem because if you look at the function, if you substitute x=0 you will get log to the base 3 of 1 which is a positive number and therefore it is well defined. So, the domain of this function is nothing but -1 to 1, this is how we need before trying to solve any problem related to logarithms we need to first verify whether it is, what the problem that we are willing to solve is defined in a proper domain or not.

Most of the times when you try to formulate a problem the problem may not be defined in a proper domain and then solving that problem is a meaningless exercise. So, just to ensure that always your problem is defined in a valid domain. So, this ends the verification of this. Now, let us take one more example which will actually help you in understanding the reversibility of log and exponential function.

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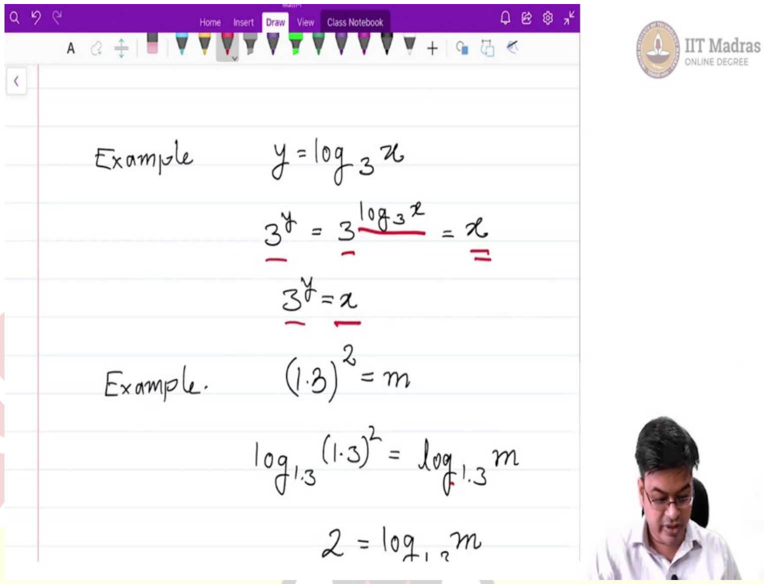

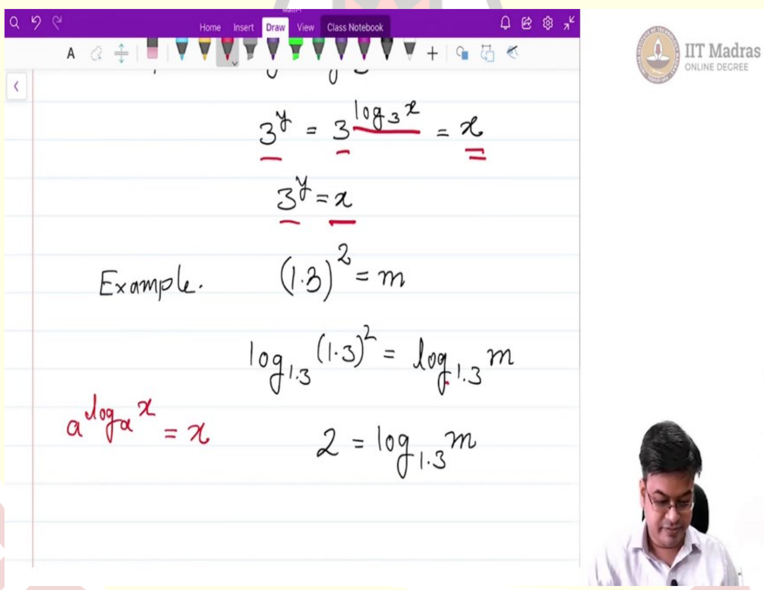

So, here is an example where we are actually demonstrating the reversibility of a log function or the inverse of a log function. So, y=log to the base 3 of x. We assume that everything is well defined and this x belongs to 0 to ∞. In that case this y will belong to the real line and if I want to write 3^ythen I will write y as $33^{\log_3 x}$.

By definition, by definition this function is the inverse of the log function. Therefore, you will get this to be equal to x and therefore your 3^y =x. Now, how this helps in your calculations? Suppose, you know some number $1.3^2 = m$ and you want to identify this m. Then you can actually take the log of this function, log of this function which is the inverse of this and which will be equal to log to the base 1.3 of m and if you equate these two what you get here is 2 being equal to log to the base 1.3 of m.

Why is it so? Because 1.3 square we have taken the log so this is like a^x and you are simplifying it. So, a^x, $a^{\log_a x}$ is actually x. So, you will get the number 2 naturally. So, this is how the log thing helps.

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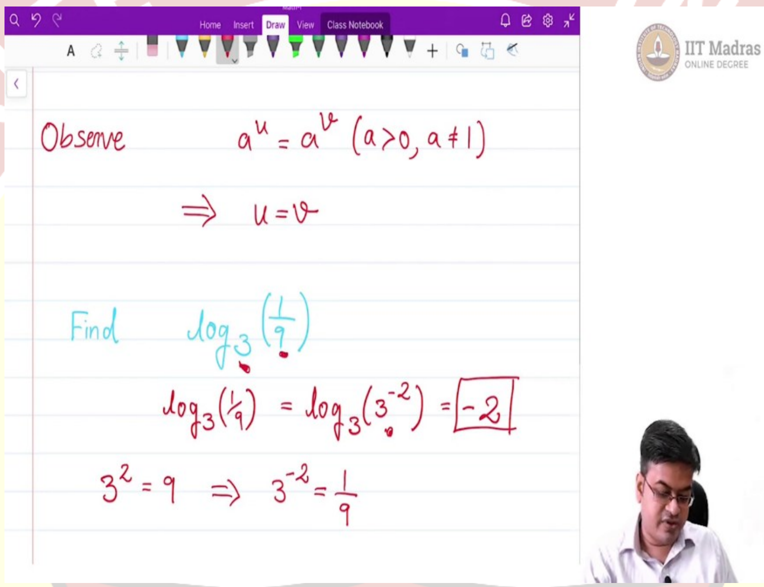

And here what the, the fact that we have used is $a^u = a^v \; for \; a > 0$ and $a \neq 1$ implies $u = v$. If you use this fact and you are asked to find the log to the base 3 of 1 by 9, then you can easily find. Let us see how. So, you start with log to the base 3 of 1 by 9. Now, you look at this 9 and 3. If you look at 3 square that will give you 9 isn't it and that also implies3^{-2} will give me $\frac{1}{9}$. So, I will simply use the fact that $\log_3 3^{-2} = \frac{1}{9}$.

So, but this is an inverse function, this is like 3 raise this particular thing is like 3^x, $\log_3 3^x$is again going to be x, so you will get -2 to be the answer, there this is how you can solve some problems very easily when you can identify the base is actually multiple of this particular argument. So, this is the use of log we will deal with it in more detail when we will solve the problems on logarithms. Now, for a moment we have identified what is the inverse function of our exponential function, it is logarithmic function to the same base as exponential function.

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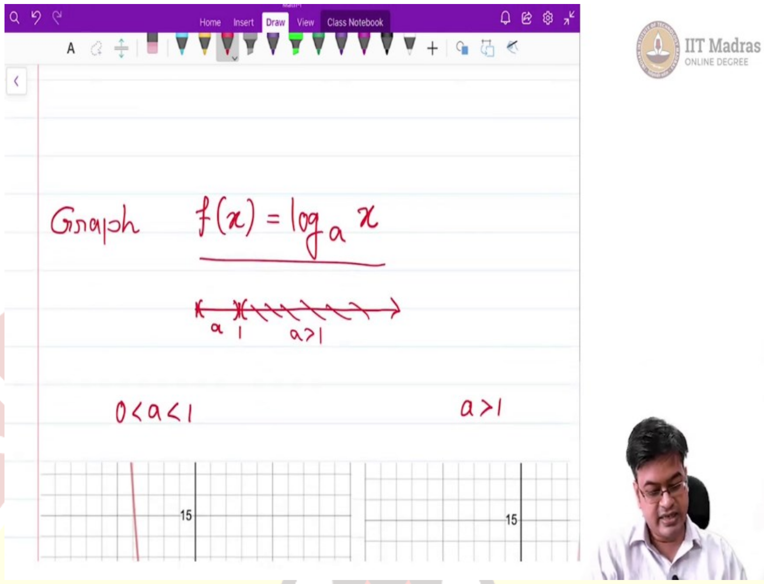

Let us try to look at the graph of the inverse function that is graph of f x equal to log to the base a of x. How will it look like? If you remember the graphs of exponential functions, the graphs of exponential functions were having two discriminations, like if you take a the line from 0 to ∞, then there was some split at 1 and from 0 to 1 when there is, the value of a lies in 0 to 1, the graph was different and from this side onwards that is a>1 the graph was different.

So, let us first imagine those graphs and let us recollect from the previous video what was the interpretation of the graph of the inverse function. If you recollect from the previous video, the graph of the inverse function is nothing but the reflection of the original function f along the line y=x or the mirror image of the function.

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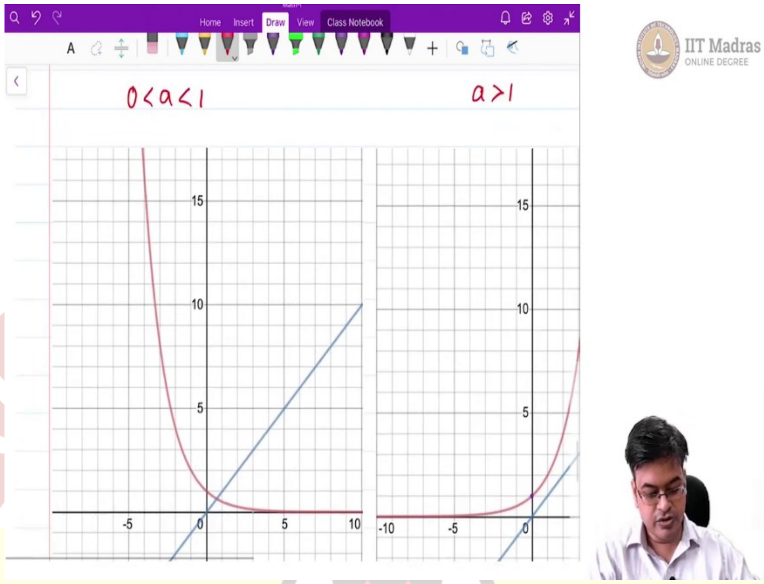

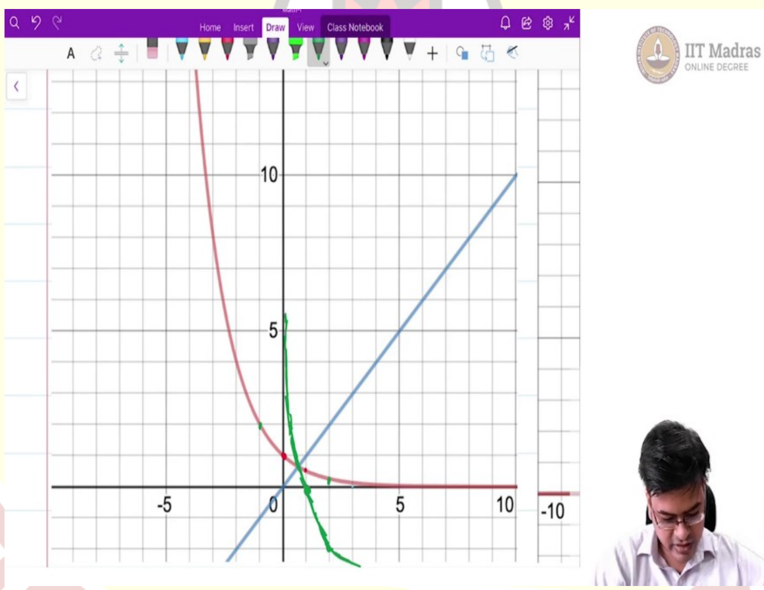

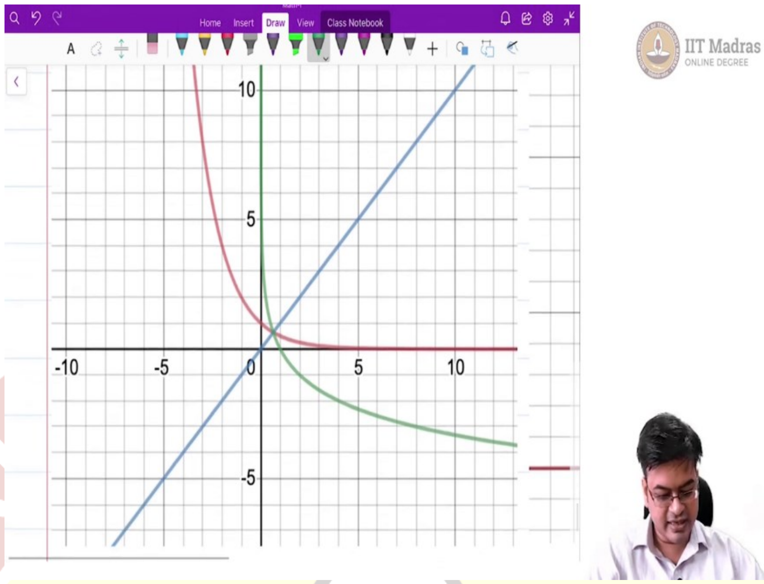

So, let us look at the exponential function first when $0<a<1$ and $a>1$. So, this is the graph when $0<a<1$. Now, I have made it big enough so that you can understand better and the blue line is the line $y=x$. Now, if I want to translate the mirror image of this function how will I translate?

Let us take one point so let us take a point 0, 1 over here, the translation of that point will be 1, 0 over here and then take this point over here, I should not draw any point here because it may confuse you so the translation of that point in this zone is a point over here and a point over here, and similarly you go on translating and connect the two lines.

For example, here if I go on translating this point then the translation will actually go to some place over here and if you take one more point over here then the translation will actually go to the other quadrant which is 2 units below this and over here. So, the graph of this function will actually look something like this, it will pass through the same nodal point and it will pass through this and then on y axis it will be very flat, very close to the y axis and so on, so this is how the graph of the function will look like because it is a mirror image.

So, this is how it will look like, it is not an asymptote but because the graph paper is over I am not able to draw. In a similar manner this is the case when $0<a<1$. So, I have drawn the graph in the next sheet which is a green line you can see this green line actually matches with this green line, I have slightly shifted the graph paper in order to have a better visibility.

Now, you can actually see this is the original function, this is the new inverse function and this is the line y=x, so you can see the correspondence of the inverse function with respect to the original function, all this is possible because our function is one-to-one.

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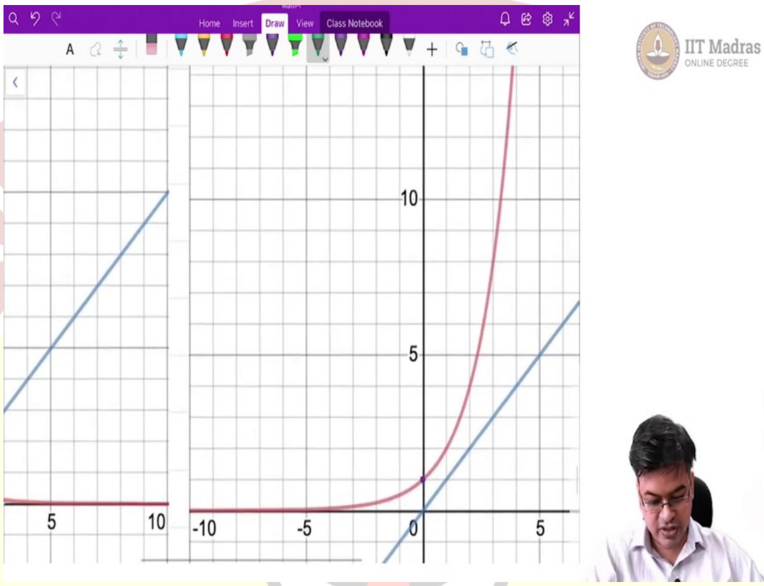

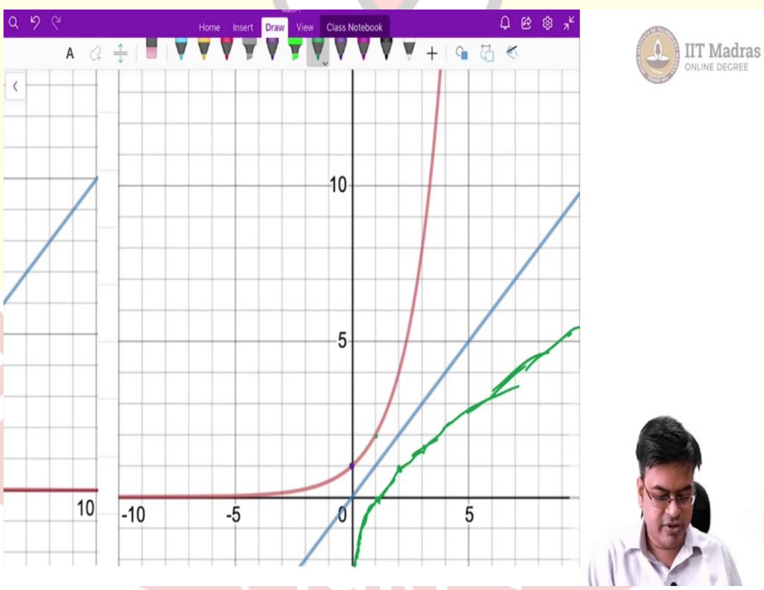

Now, if you look at, again look at the graph of a function where a>1 then this is the graph of a function here there are no overlaps, so it is relatively easy to draw the graph. For example, I can choose this point over here if I go one unit from here I should get something like this here so it is a reflection along x axis, so it will be relatively easy to draw the graph here, this point reflected here that point will be reflected here and then I can draw that, I can join the curve like this and it will be exact mirror image of the original function and it will be going close to this particular function.

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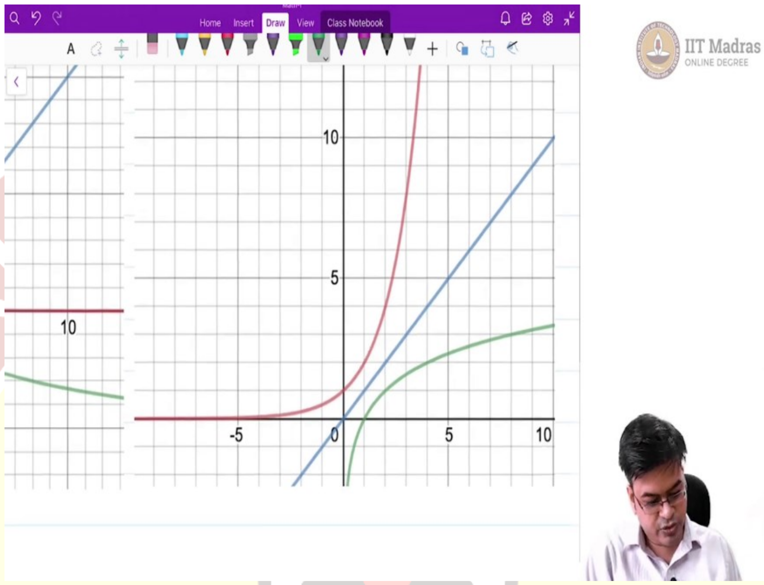

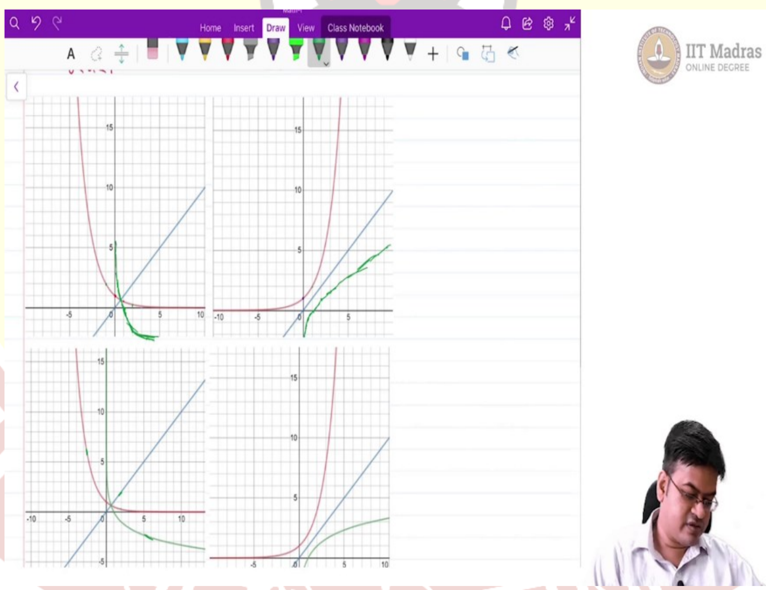

So, roughly this will be the graph, I have drawn the full proof graph on the next graph paper which is here. So, now you can easily visualize the graphs of both the functions, let us zoom out and see all of them together all 4 graphs together. So, these are all 4 graphs handled together, so my graph actually looks like this graph for both the cases, so this is how it is easy to draw the graphs of inverse functions once we know the graph of the original function.

In the next, this is, that is all for this video. In the next video what we will see is we will try to use our knowledge of logarithmic functions and try to see how the formulation of a mathematical problems becomes easy when we consider logarithmic functions, even though there is a limitation that logarithmic function is defined only from 0 to ∞ not on the real line. Thank you.