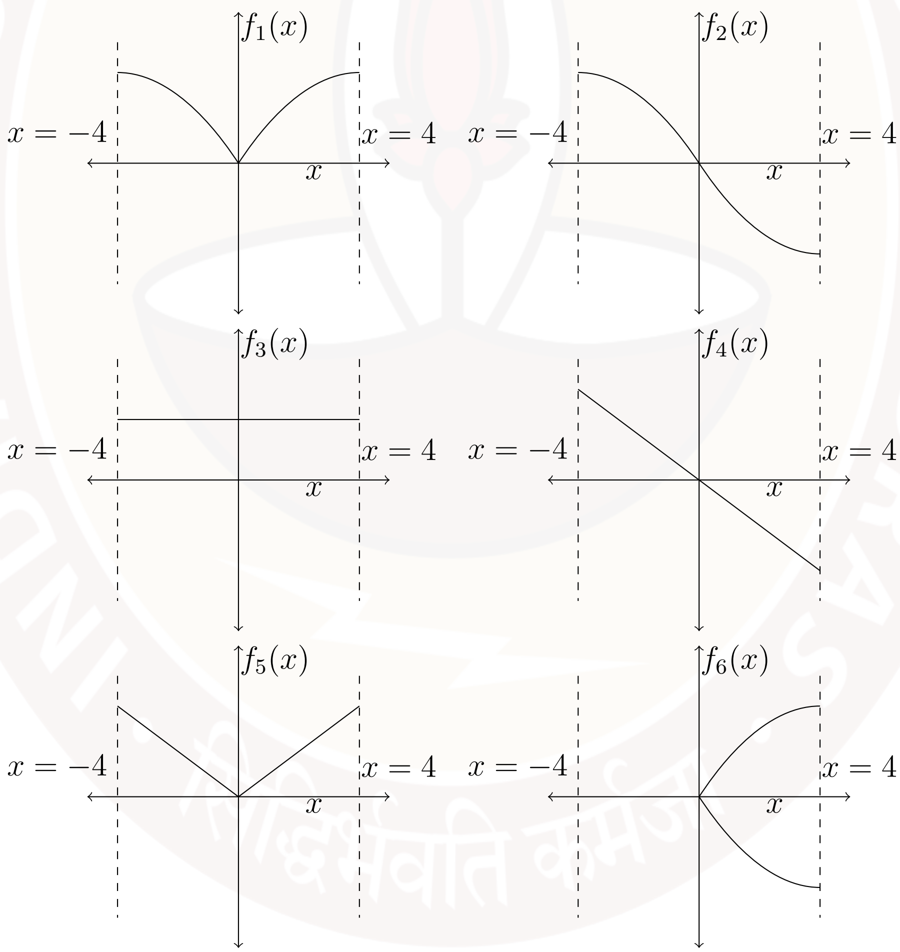
Week - 7
Practice assignment Solution
Exponential Functions
Mathematics for Data Science - 1

NOTE:
There are some questions which have functions with discrete valued domains (such as month or year). For simplicity, we treat them as continuous functions.

1 Multiple Choice Questions (MCQ):

Answer the questions 1, 2, and 3 based on the given graphs.



Domain for each one is $[-4, 4]$.

1. Choose the correct option.
   - ◯ f_3 is not a function.
   - ◯ **f_6 is not a function.**
   - ◯ f_5 is not a function.
   - ◯ All of the above are functions.

   Solution:

   Vertical line test fails only for f_6 and therefore $f_6(x)$ is not a function.

2. Choose the correct option.
   - ◯ f_1 and f_3 are one-one functions in the given domain.
   - ◯ **f_2 and f_4 are one-one functions in the given domain.**
   - ◯ f_3 and f_5 are one-one functions in the given domain.
   - ◯ f_5 is one-one function in the given domain.

   Solution:

   The function f_2 and f_4 are strictly decreasing function in the domain [-4, 4], therefore these are one to one functions.

   Or

   The functions f_2 and f_4 are the only functions which satisfy the conditions of horizontal and vertical line tests in the domain [-4, 4], therefore these are one to one functions.

3. Choose the correct option.
   - ◯ f_1 and f_5 are strictly increasing functions in the given domain.
   - ◯ **f_2 and f_4 are strictly decreasing functions in the given domain.**
   - ◯ f_4 and f_5 are strictly decreasing functions in the given domain.
   - ◯ f_5 is strictly increasing function in the given domain.

   Solution:

   A function $f(x)$ is said to be strictly decreasing on a given interval if $f(b) < f(a)$ for all $b > a$, where $a,\ b$ belong to the domain. On the other hand, if $f(b) <= f(a)$ for all $b > a$, then the function is said to be simply decreasing function.
   Clearly from the given graph, f_2 and f_4 are strictly decreasing functions in the domain [-4, 4].

Use the following information for the questions 4 and 5.

Let N_0 be the number of atoms of a radioactive material at the initial stage i.e., at time $t = 0$, and $N(t)$ be the number of atoms of the same radioactive material at a given time t, which is given by the equation $N(t) = N_0 e^{-\lambda t}$, where λ is the decay constant.

4. If at time t_1, the number of atoms reduces to the half of N_0 and at the time t_2 the number of atoms reduces to the one fourth of N_0, then which one of the following equations is correct?

   - ◯ $e^{\frac{t_1}{t_2}} = 2$
   - ◯ $e^{\frac{t_2}{t_1}} = 2$
   - ◯ $\boldsymbol{e^{\lambda(t_2-t_1)} = 2}$
   - ◯ $e^{\lambda(t_1-t_2)} = 2$

   Solution:

   According to the question, at t_1,

   $$N(t) = \frac{N_0}{2}$$

   According to the equation,

   $$N(t) = N_0 \ e^{-\lambda t}$$

   Therefore for $t = t_1$,

   $$\frac{1}{2} \times N_0 = N_0 \ e^{-\lambda t_1}$$

   $$\frac{1}{2} = e^{-\lambda t_1} \tag{1}$$

   It is also given that at t_2, $N = \frac{N_0}{4}$

   $$\frac{1}{4} \times N_0 = N_0 \ e^{-\lambda t_2}$$

   $$\frac{1}{4} = e^{-\lambda t_2} \tag{2}$$

   On dividing (1) by (2) we get,

   $$e^{\lambda(t_2-t_1)} = 2$$

5. If $N_{\frac{1}{\lambda}}$ is the number of atoms at time $t = \frac{1}{\lambda}$, then what is the ratio of N_0 to $N_{\frac{1}{\lambda}}$?

   - ◯ $1 : e$
   - ◯ $\boldsymbol{e : 1}$
   - ◯ $1 : e^{-\lambda}$

◯ $1 : e^{\lambda}$

Solution:

It is given that at $t = \frac{1}{\lambda}$, $N = N'$

$$N' = N_0 \; e^{-\frac{\lambda}{\lambda}}$$

$$N' = \frac{N_0}{e}$$

$$\frac{N_0}{N'} = \frac{e}{1}$$

Therefore,

$N_0 : N' = e : 1$

2 Multiple Select Questions (MSQ):

6. Selvi deposits ₹P in a bank A which provides an interest rate of 10% per year. After 10 years, she withdraws the whole amount from bank A and deposits it in another bank B for n years which provides an interest rate of 12.5% per year. $M_A(x)$ represents the amount in Selvi's account after x years of depositing in bank A. $M_B(y)$ represents the amount in Selvi's account after y years of depositing in bank B. If the interests are compounded yearly, then choose the set of correct options.

   - ○ $\boldsymbol{M_A(x)}$ **is an one-one function of** $\boldsymbol{x}$**, for** $\boldsymbol{x \in (0, 10)}$**.**
   - ○ $\boldsymbol{M_B(y)}$ **is an one-one function of** $\boldsymbol{y}$.
   - ○ $M_A(12) = P \times 1.1^{12}$
   - ○ $\boldsymbol{M_A(12) = 0}$
   - ○ $\boldsymbol{M_A(x)}$ **is a strictly increasing function of** $\boldsymbol{x}$**, for** $\boldsymbol{x \in (0, 10)}$**.**
   - ○ $M_B(y)$ is a decreasing function of y.
   - ○ $\boldsymbol{M_B(n) = (P \times 1.1^{10}) \times (1.125)^n}$
   - ○ $M_B(n) = (P \times 1.1^n) \times (1.125)^{10}$

Solution:

When the principal amount P is compounded annually, the amount M after q years is given by

$$M = P \times \left(1 + \frac{\text{Interest rate}}{100}\right)^q$$

Amount $M_A(x)$ after x years in bank A will be

$$M_A(x) = P \times \left(1 + \frac{10}{100}\right)^x$$

So after 10 years the amount $M_A(10)$ will be

$$M_A(10) = P \times (1.1)^{10}$$

As Selvi has withdrawn all the amounts from bank A after 10 years so amount left in bank A after 12 years will be $M_A(12) = 0$.

After 10 years the new principal amount $P \times (1.1)^{10}$ is deposited in another bank B, so for any years y the amount will be $M_B(y)$ which is given by

$$M_B(y) = P \times (1.1)^{10} \times \left(1 + \frac{12.5}{100}\right)^y$$

So for n years

$$M_B(n) = P \times (1.1)^{10} \times (1.125)^n$$

Clearly $M_A(x)$ and $M_B(y)$ are strictly increasing functions therefore both are one-to-one functions of x and y respectively.

Use the following information for questions 7 and 8.

There are two offers in a shop. In the first offer, the discount in total payable amount is $M(n)\%$ if the number of products bought at a time is n. The second offer involves a discount of ₹1000 on the total payable amount. If Geeta shops of ₹15,000, then answer the following questions.

7. If the total payable amounts after applying the first and second offers (one at a time) are represented by the functions $f(n)$ and $g(n)$ respectively and the total payable amount after applying both the offers together is represented by $T(n)$, then choose the set of correct options.

   - ◯ $f(n) = (100 - M(n)) \times 15000$ and $g(n) = 14000$
   - ◯ $f(n) = (100 - M(n)) \times 1500$ and $g(n) = (100 - M(n)) \times 15000 - 1000$
   - ◯ $\boldsymbol{f(n) = (100 - M(n)) \times 150}$ **and** $\boldsymbol{g(n) = 14000}$
   - ◯ $T(n) = (100 - M(n)) \times 15000$ is the total payable amount when the first offer is applied after the second.
   - ◯ $\boldsymbol{T(n) = (100 - M(n)) \times 140}$ **is the total payable amount when the first offer is applied after the second.**
   - ◯ $\boldsymbol{T(n) = (100 - M(n)) \times 150 - 1000}$ **is the total payable amount when the second offer is applied after the first.**

   Solution:

   It is given that total payable amount without any offer is ₹15,000. Then, total payable amount after first offer is

$$f(n) = \frac{(100 - M(n))}{100} \times 15,000 = (100 - M(n)) \times 150$$

   And total payable amount if second offer is applied will be

$$g(n) = 15,000 - 1000 = ₹14,000.$$

   Now, total payable amount when the first offer is applied after the second will be

$$T(n) = \frac{100 - M(n)}{100} \times g(n)$$

$$T(n) = \frac{(100 - M(n))}{100} \times 14000 = (100 - M(n)) \times 140$$

And total payable amount when the second offer is applied after the first will be

$$T(n) = f(n) - 1000$$

$$T(n) = \frac{(100 - M(n))}{100} \times 15000 - 1000 = (100 - M(n)) \times 150 - 1000$$

8. If Geeta is allowed to use the offer in any sequence and $M(n) = -n^2 + 18n - 72$, where $n \in \{6, 7, 8, 9\}$, then choose the set of correct options which minimizes the total payable amount.

   ◯ Total payable amount is same irrespective of the order in which the offers are applied.

   ◯ **She should choose offer one and then offer two i.e., $\boldsymbol{gof(M(n))}$.**

   ◯ She should choose offer two and then offer one i.e. $gof(M(n))$.

   ◯ **If she chooses offer one and then offer two, the minimum payable amount will be ₹12650.**

Solution:

Total payable amount when she choose offer one and then offer two is

$$T_1(n) = (100 - M(n)) \times 150 - 1000$$

It is given that $M(n) = -n^2 + 18n - 72$, so

$$T_1(n) = (100 - (-n^2 + 18n - 72)) \times 150 - 1000$$

On solving we get,

$$T_1(n) = 150n^2 - 2700n + 24800$$

And total payable amount when she chooses offer two and then offer one is

$$T_2(n) = (100 - M(n)) \times 140$$

On substituting $M(n)$ and solving we get,

$$T_2(n) = 140n^2 - 2520n + 24080$$

Since the coefficient of n^2 is positive for both $T_1(n)$ and $T_2(n)$ therefore minimum value i.e., minimum payable amount of these function can be calculated as follows
For $T_1(n)$

$$\text{Vertex}(n) = \frac{-b}{2a} = \frac{-(-2700)}{2 \times 150} = 9$$

The minimum payable amount will be

$$T_1(9) = 150(9)^2 - 2700(9) + 24800 = ₹12{,}650$$

For $T_2(n)$

$$\text{Vertex}(n) = \frac{-b}{2a} = \frac{-(-2520)}{2 \times 140} = 9$$

The minimum payable amount will be

$$T_2(9) = 140(9)^2 - 2520(9) + 24080 = ₹12{,}740$$

Thus if she chooses offer one and then offer two, the minimum payable amount will be ₹12,650.

n	$T_1(n)$ ₹	$T_2(n)$ ₹
6	14000	14000
7	13250	13300
8	12800	12880
9	12650	12740

Table: M1W8PAS-1

From Table: M1W8PAS-1, it is clear that for all the values of n the total payable amount is lower for $T_1(n)$ as compared to $T_2(n)$ therefore she should choose offer one and then offer two.
Note: This can be also identified by plotting the graph for $T_1(n)$ and $T_2(n)$.

3 Numerical Answer Type (NAT):

Use the following information for questions 9-15.
Given two real valued functions $f(x) = \frac{5x+9}{2x}$, $g(y) = \sqrt{y^2 - 9}$. If $h(x) = f(g(x))$, then answer the following questions.

9. If domain of $f(x)$ and $g(x)$ are $(-\infty, m) \cup (m, \infty)$ and $\mathbb{R} \setminus (-n, n)$ respectively, then find the value of $m + n$. [Ans: 3]
Solution:

At $x = 0$ the function $f(x) \to \infty$ or the function is undefined at $x = 0$ thus the domain of $f(x)$ is $\mathbb{R} \setminus 0$.
We can also write the domain as $(-\infty, 0) \cup (0, \infty)$ therefore, $m = 0$.

It is given that $g(y) = \sqrt{y^2 - 9}$ on changing the variable in terms of x we get $g(x) = \sqrt{x^2 - 9}$.

$g(x)$ will be defined when $x^2 - 9 \geq 0$. On solving

$$x^2 \geq 9$$

$$x \geq 3$$

or

$$x \leq -3$$

Thus the domain will be $\mathbb{R} \setminus (-3, 3)$, hence $n = 3$. So, $m + n = 0 + 3 = 3$

10. If range of $f(x)$ and $g(x)$ are $(-\infty, m) \cup (m, \infty)$ and $[n, \infty)$ respectively, then find the value of $2(m + n)$. [Ans: 5]

Solution:
As $f(x)$ is defined everywhere except 0, therefore there will be an asymptote at $x = 0$.
If we draw a graph of $f(x)$:
End behaviour:
As $x \to \infty$, $f(x) \to \frac{5}{2}$.
As $x \to -\infty$, $f(x) \to \frac{5}{2}$.
The end behaviours show that the function has another asymptote at $f(x) = y = \frac{5}{2}$.

Intercept:

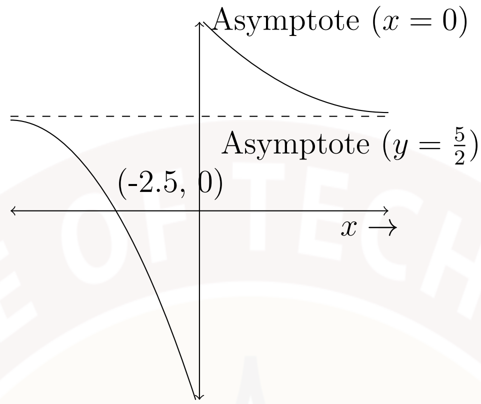
$$f(x) = 0 \implies \frac{5x + 9}{2x} = 0$$

$$x = -\frac{9}{5}$$

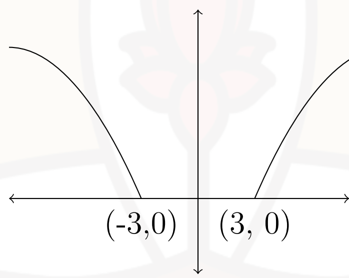
It means $f(x)$ might change the sign at $x = -\frac{9}{5}$.
For $-\infty < x < 0$, $f(x)$ will have value from $-\infty$ to $\frac{5}{2}$.

Similarly for $0 < x < \infty$, $f(x)$ will have value from $\frac{5}{2}$ to ∞.

Therefore the range of $f(x)$ is $(-\infty, \frac{5}{2}) \cup (\frac{5}{2}, \infty)$. A rough diagram of $f(x)$ is shown below.



As $g(x) = \sqrt{x^2 - 9}$ is a positive square root function so it will have only the positive values including zero at $x = 3$ and $x = -3$.
A rough diagram is created using the facts that the $g(x)$ is not defined from (-3, 3) and at $x = 3$ the function gives the value zero. At ∞ the function provides the value ∞. As the quadratic function involved and the $b = 0$ the function will be symmetric about $y-$axis.



Therefore the range will be $[0, \infty)$. Thus $m = 2.5$ and $n = 0$, so,

$$2(m + n) = 2(2.5 + 0) = 5$$

11. If domain of $h(x)$ is $(-\infty, -3) \cup (m, \infty)$, then find the value of m. [Ans: 3]

Solution:
Given,

$$h(x) = f(g(x))$$
$$h(x) = f(\sqrt{x^2 - 9})$$
$$= 2.5 + \frac{4.5}{\sqrt{x^2 - 9}}$$

There are two possibilities when the function is undefined. Firstly when the denominator is zero and secondly when the function in square root provides negative value. It means

$\sqrt{x^2-9} \neq 0$ and $x^2 - 9 \geq 0$.
Combining both the conditions we can say the function is defined only when

$$x^2 - 9 > 0$$

$$x^2 > 9 \implies -3 < x < 3$$

Thus the domain will be $(-\infty, -3) \cup (3, \infty)$, hence $m = 3$.

12. If domain of $f^{-1}(x)$ is $(-\infty, m) \cup (m, \infty)$, then find the value of $2m$. [Ans: 5]

Solution:
Given that $f(x) = \frac{5x+9}{2x}$ let us say $f(x) = y$ so $y = \frac{5x+9}{2x}$ on rearranging,

$$y = \frac{5}{2} + \frac{9}{2x}$$

$$\frac{2y-5}{2} = \frac{9}{2x}$$

$$x = \frac{9}{2y-5}$$

Therefore $f^{-1}(x) = \frac{9}{2x-5}$. This function will be defined when

$$2x - 5 \neq 0$$

$$x \neq \frac{5}{2}$$

The domain of this function is $(-\infty, 2.5) \cup (2.5, \infty)$ thus $m = 2.5$ therefore $2m = 5$

13. If $f^{-1}(5) = 9/m$, then find the value of m. [Ans: 5]

Solution:
$f^{-1}(5) = \frac{9}{2\times5-5} = \frac{9}{5}$, thus $m = 5$.