INDIAN INSTITUTE OF TECHNOLOGY MADRAS

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IIT Madras

ONLINE DEGREE

Mathematics for Data Science 2
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Lecture 12
Computing areas

Hello and welcome to the maths 2 component of the online BSc program on Data Science and Programming. In this video we are going to talk about computing areas So, this is a recollection of some of the things that you have learnt probably in school and towards the end of the video we will try and look at shapes that are, maybe a little more involved with the idea that we will try to evolve some method to compact the area of those shapes.

And in the next videos, that will set the tempo for the next videos where we will see a more detailed method to do the same thing and which will be lead us to what is called a theory of integration.

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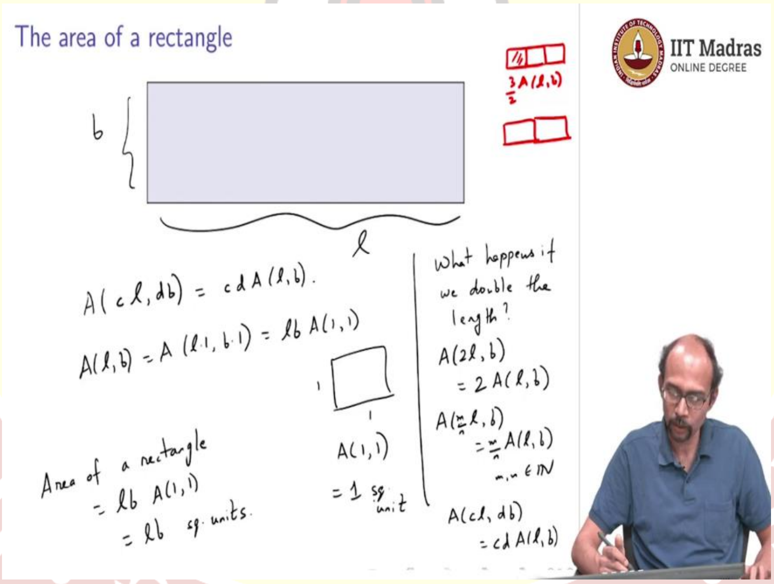

So, let us start by computing areas of some very basic figures that we know. So, the area of a rectangle so here is a rectangle, so rectangle is described by a length and its breadth. So, if you know its length and you know its breadth, then the rectangle is unique, meaning any two rectangles with this length and this breadth has the same area. In particular, you can super impose them on each other, if you do it carefully, then you will get the same shape. So, they in particular have the same area.

So, now how do I compute to the area of a rectangle? So, of course, this is a formula that everyone remembers or knowns from school. So, let us give names to length and the breadth.

So, suppose this is the length and this is the breadth. Now I want to compute what is the area of this rectangle and as we just noted, the area of the rectangle is the same as the area of keeping track of the length and the breadth. So, the area is a function of the length and the breadth that much we understand.

So, I will call that function $A(l, b)$. Of course, you all know what the function is, I am sure. But we are going to try and understand why we get that formula. Fine. So, now let us ask what happens if you double the length? So, if you double the length, so what happens if we double the length? Well, I think you can imagine what happens. We have two, I mean you can divide it in the middle, so we will have two such rectangles and then each of them will have the same length and breadth namely l and b respectively.

So, the area will be, for each of them will be same. So, the total area will be two times the area of the original rectangle. So, what, in other words what we are saying is, $A(2l, b)$ that is what we mean by 2 times the length. So, if you double your length and you maintain your breadth, then the area is twice the area that you would get with just l and b. This is a very clear, you can clearly see this.

Maybe if it is unclear, what I am saying is if this is l and b. So, this is $2l$ and b, they both have the same area, so the total area is 2 times area of just one of them. So, the same argument holds if you do 3 times, 4 times, 5 times, 20 times, 100 times, etc. So, if you do, in other words instead of double, you do $n \times l$, you get $n \times A(l, b)$, where n is an integer or let us say n is a natural number since negatives areas, I think we do not want to talk about right now.

But what about if you multiply it by a rational number? Let us say I take one and half times the length. Well, you can still see that the formula holds good because if this is your original rectangle and if you do one and half, that means what you are doing is you are really taking this half here, you divide this into half and you are talking 3/2. So, one and half is 3/2. And so, this is 3 times the area of this guy, but that guy has area half of the original one. So, it is, so the total area that you got was $\frac{3}{2}A(l, b)$.

So, this same idea the way we did it for one and half will work for any integer. So, we can actually say, not just for a rational number, not just for an integer or natural number, but for any rational number if you multiply this by $\frac{m}{n}$, then this is $\frac{m}{n}A(l, b)$; where m and n are in natural numbers. And again, the same, there is nothing special about length, the same thing will hold for breadth as well.

So, in other words what we get is, if you do $cA(l, db)$, then you will get $cdA(l, b)$. So, this formula holds good for rational number and once such formula starts holding good for rational numbers, we can do something that we have done earlier, namely take limits and by taking limits on both sides, we can get sequence of rational numbers that tends to any real number and from there we can conclude that the area of, this formula holds true for any two real numbers c and d, of course, we are not talking about negative areas right now. So, let us assume c and d are positive real numbers.

So, in other words what we are saying is that if you take any real number, any two real numbers c and d, then this is $cdA(l, b)$. But now I can actually take c and d to be l and b itself. I can take, I can start with l comma b and think of this as $A(1 \times l, 1 \times b)$ and then write this as $lbA(1,1)$ l. And what is this $A(1,1)$? That means that the rectangle with 1 unit as length and 1 unit as breadth which is a square. So, let me draw this better. So, 1 unit and 1 unit. Of course, we have not specified what the units are.

So, now you can see it is reduced, the area of a rectangle has reduced actually to the following formula. So, the area of a rectangle is length times breadth times whatever the area is for a square with 1 unit side. And by convention we take this to be 1. Of course, if you change your units, then there will be a corresponding change. So, that is why our, that is why when we write areas, we have to write in terms of what unit?

So, that is something that we have learnt in school or later it is deployed more in our physics courses and so on. So, but without getting into what those units are, if we assume it is some units, then if we declare that area of this, we will call this 1 square unit, then this is length times breadth times length times breadth square units.

So, I want you to make points, one is that well, we, you probably have not, at least I do not remember having gone through this kind of explanation when I was in school, this was something which was sort of given and it seemed very correct also, I mean that did not seem like any reason to question it, but I, what I am describing now is some kind of a proof. And it also tells you why we have to take these units. If you change your units, then your, then $A\ (1, 1)$ may change and then you will have something else, you will, you may have a constant.

For example, suppose I now suddenly say that well, I am measuring this in terms of centimetres, then this 1, this square of, no, with the one unit that I have drawn, I have to ask how many centimetres it is, it may not be 1 centimetre and then I have to convert it into square centimetre which may not give me 1 square centimetre, but there will be some constant which is what is $A\ (1,1)$ and it will be those many square centimetres. And then I have to just multiply that to $1 \times b$; where l and b are remember still in my original units. So, I hope this discussion made sense. So, we know how to compute the area of a rectangle. Once you have decided on some unit in which to measure our lengths.

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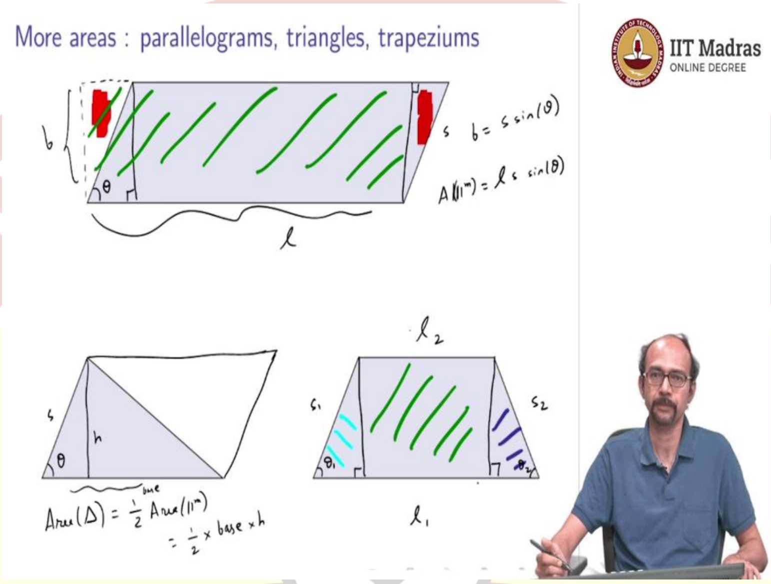

And once we know how to measure rectangles, we can go on and ask well what happens to other figures. So, for example, parallelograms or triangles or trapeziums. So, let us look at the parallelogram. Well, here, what do we do, we have a trick. So, what we do is we observe that if you drop this perpendicular here, so you drop this perpendicular here and you put in this perpendicular here, so this maybe is not so well drawn.

Well, you can transport this part, this triangle on the right to the left. So, then you can just put it in here, again not very well drawn, but I hope you get the idea. This is, this, so this part whatever it is, this part here is exactly this part here, although the picture may not match that, that is my bad, my drawing is not particularly good. But I hope you get the idea. And then we can, well, from here we can describe what is the area of the parallelogram because now we have converted the (parallel), the area of a parallelogram computation into the area of a rectangle computation. So, what we have got now is a rectangle.

So, in other words, if you take this part here, so this part here not including the right triangle, but now including the left triangle. This is a rectangle and the area of the parallelogram is equal to the area of the rectangle. So, now the only question is what is the length of this rectangle and what is the breadth of this rectangle. The length is clearly the same as the length of this side. How do we get the breadth?

So, this is the breadth here. So, this is the breadth which is the same as the length of this perpendicular that we have dropped here or this perpendicular that has been drawn here. So, typically what, so either that may be given to you or what may be given is this angle theta and the length of this side here or the length of this side here. So, if we call the length of this side as s maybe. So, then we know from trigonometry now that the length of the opposite side is s time sin theta. So, $b = s \times sin\theta$ and so the area is $l \times s \times sin\theta$ because area is equal to lb. Fine.

So, that is the computation of the parallelogram and then how about the triangle? Well, this is another trick. So, for the triangle we will complete this to a parallelogram by drawing this part here and now we know exactly what, how to compute this because the two areas; now we have two triangles and both of them have the same area. So, the total area is 2 times the area of the triangle.

So, in other words the area of the triangle is half times the area of the parallelogram that we have drawn and now we know how to compute the area of the parallelogram, it is half times base which was called l over there. So, this is the base times height and of course, sometimes you may not be given height instead you may be given let us say this side and this angle and then you have to again go to taking $s \times sin\theta$. So, but this is a familiar argument, so this is half times base times height. Fine.

So, now let us look at the trapezium. So, a trapezium is maybe a little more complicated. So, here we can break it up in to, well, again let us drop these perpendiculars and once you drop these perpendiculars, well we have broken it up into three figures. All 3 of which we know how to compute the area of, so there is a square over here, this is a rectangle over here and then there is a triangle over here and then we have a third triangle over here, sorry, second triangle over here.

And for each of these we can compute what is the area because we know how to compute the area of rectangles and triangles and the only thing that we will need to know now is what are the lengths of the sides in question. So that, so the first question is how do we describe a trapezium? So, if we know the total length, so if I know the length of this side, if I know the length of this side, and then let us say I know this angle and this angle, maybe I, so I, the possible things that we know are these ones.

So, these angles, the lengths above and below and the sides s_1 and s_2. So, all of these data are not necessary to describe a trapezium. So, for example, if I know l_1, l_2 and presumably θ_1, θ_2 and s_1 that should be enough. So, I should presumably be able to find out s_2 or I could drop θ_1 and instead have s_1, s_2 and θ_2. So, one of those I mean, out of these 6 I think 5 are enough to describe your trapezium.

And once we know these 5, anyway once we know these 6, we know all the, we can compute all the rest using trigonometry and then we can write down the area of a trapezium. Of course, there are other ways to do it, but I want to emphasize here that, the main point I want to emphasize here is that, you keep going back to figures that you already know the area of. So, and the basic figure there is the rectangle. So, in all these computations, the main figure you went back to was the rectangle.

So, we know how to compute area of rectangles and then you split your, from there you go to parallelograms and triangles and then for any other thing, essentially split it into rectangles or triangles or parallelograms which you know how to compute because you know how to compute area of rectangles. Fine.

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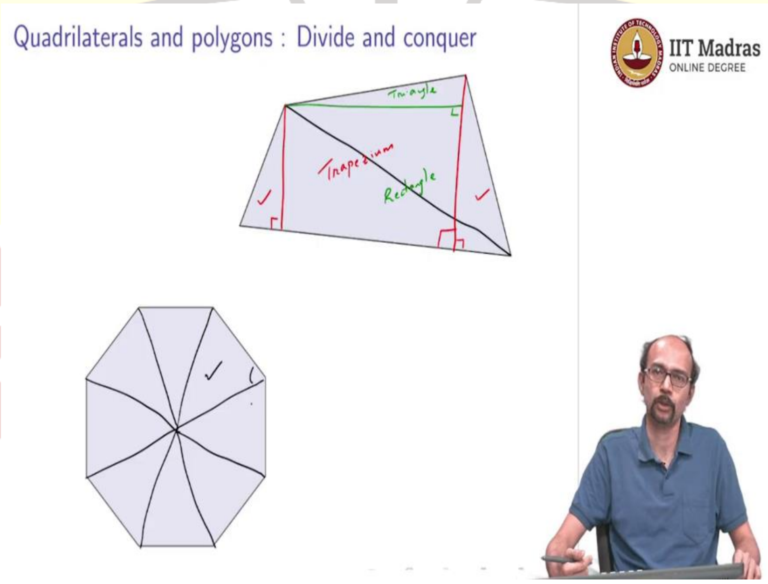

So, with that idea, let us go to more general figures. So, how about quadrilaterals and polygons. So, the idea here is just what I just described, Divide and conquer. So, here is a quadrilateral and here is a polygon. So, this is an octagon. So, for the quadrilateral what do we do? So, one possible thing we can do is, we can draw this diagonal here and now we know two triangles and now with some work we will be able to find out what the area of this of each of these triangles. That is…

So, it needs, we need to know exactly how to describe this quadrilateral first and then once we know that we will be able to find this out by breaking it into two triangles. The other option is break it into maybe drop this perpendicular here, drop this perpendicular here. So, now this part is a triangle; so, triangle. So, we know how to do this, this part is a triangle; we know how to do this. In fact, these are both right angled triangles.

And now what you have in the middle is a trapezium because you have two parallel sides and then these two angles. So, this is a trapezium and so from our previous work, we know how to compute this. So, one option again for trapezium is in this case, it so happens that this angle is 90 degrees, so you could further draw this line here which is again 90 degrees, then this becomes a rectangle and this becomes another triangle. So, I can use this for example to describe my quadrilateral and using this I can compute its area. So, again the idea is the same thing, you divide it into shapes that you understand and the basic shape that we understand is the rectangle.

Now, let us go for the polygon. So, here this is an octagon. So, for the octagon what do we do? Well, let us take the centre of the octagon and one possible way is to draw all these radii and each of these give us triangles. So, we have 8 triangles. In fact, these are all, all these 8 triangles are congruent to each other. So, if we compute the, I mean this is a regular octagon. So, if we compute the area of one triangle, that is enough.

In general, of course, if your polygon is not a regular polygon, the same kind of idea might work, but you will have to compute each individually. So, now the question is how do I compute the area of each triangle? Well, again that is something that is very doable from the given data. So, I would not explain this in detail, I will encourage you instead to use your trigonometry and try to ask yourself what is the area of this triangle.

So, I know each, for each triangle how to compute the area based on the fact that is a regular octagon. So, I, that you should tell me what is this angle here and once I know this angle, I should be able to compute what the area of this is. And then the area of this octagon will be 8 times that. So, in general, if you have an n-gon, you will be able to use this idea to compute the area of the n-gon, regular n-gon.

And if you have a general polygon, meaning it is not regular, you can use the same idea but then you will have to be more careful in computing the area of the triangles. So, each of them you will have to do individually. But in any case, you can see that the idea, the general principle still holds, namely that you divide this figure into figures that you understand well which basically means rectangles or triangles and then try to compute their areas.

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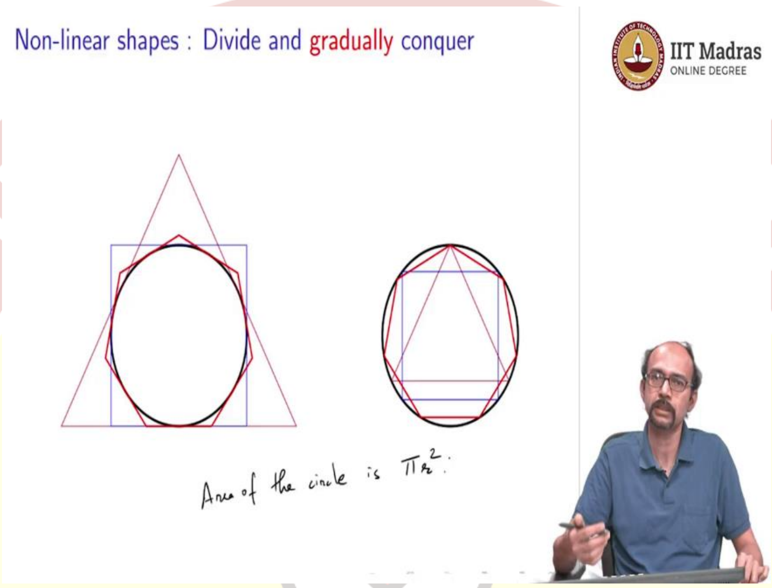

So, let us finally come to non-linear shapes. So, so far whatever shapes we described, they were based on line. So, everything was a straight line meaning, the sides were straight lines and we could make use of the fact that well I can divide this figure nicely into rectangles and triangles or other shapes which are also described by the joining of lines. What if we have a circle?

So, here is a circle and how do I try to do this? So, this is in fact, of course, all of us know what is the area of a circle. I think this is something we have learnt in school. But this was a question which actually rose antiquity. So, this was studied long time ago and one possible solution was given by someone named Archimedes. So, I think we have all heard of Archimedes.

And what Archimedes did was, he started drawing triangles and then the square and then maybe more complicated polygons, regular polygons. So, I think this is a septagon meaning it has 7 sides and so on. And he kept drawing these and as you can see as you increase the number of sides, it starts approximating your circle better and better. And now we know from our previous what we have done earlier, how to compute the area of a regular n-gon. So, using this, he could say that the area of circle cannot exceed this, so he could give a an upper bound on the area of the circle.

And similarly, what he did was he started drawing things inside. So, here is a triangle drawn inside, here is a square drawn inside, here is a septagon drawn inside and you can go on and Archimedes actually gave explicit formulae for these polygons, you can do this, it is not terribly hard. And then, what he said is well, in the limit I kind of know what happens to this; in the limit I know what happens to this and it turns out they are the same that means, the area of the circle must be this and what was that this, so we know that the area of the circle is πr^2 square, where r is the radius.

So, this is a very beautiful argument to get this area and it was known to Archimedes, well a long time ago before, I think before 0 BC. There is a one, I mean two things to take away here. One is that again we have somehow reduced the idea to computing things in terms of rectangles because we reduced to polygons which we will reduced to for example, we could reduce to triangles which we can reduce, which we reduced from rectangles or we could directly reduce this to rectangles; that is one.

And the other is that to really do this well, meaning to get this area clearly, you should allow your n-gon to be really, really close to your circle which means you should allow your n to tend to infinity. So, underline this again there is a notion of a limit. So, these are the two ideas that we have to keep in mind.

One is dividing so that is what is written here, divided into maybe rectangles or figures we understand and then gradually conquer. Previously we could just divide into rectangles or shapes we understand because they were linear and we could directly conquer it. Here we have to further allow us as a limit.

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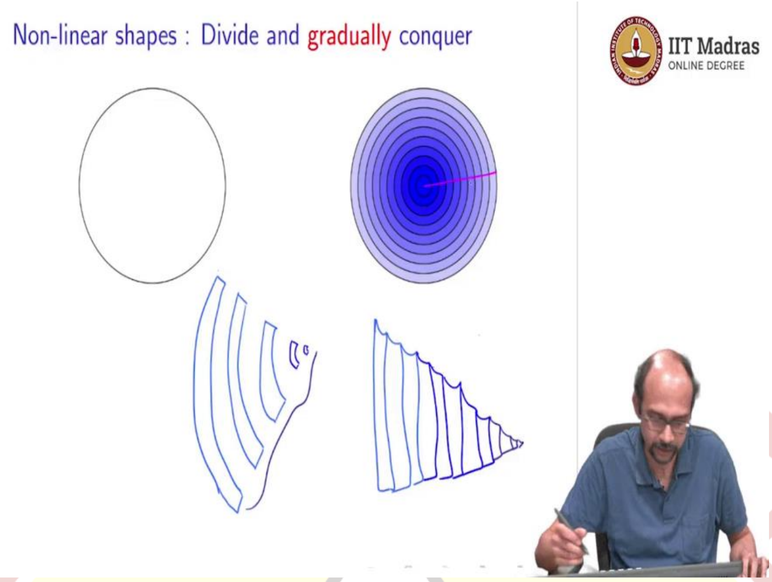

So, here is another way of doing the same thing and this is closure to what we are going to do next, so I want to show you this as well. So, here is your circle again and what have we done? We have broken it up into, we have drawn some smaller circles, so we have drawn circles which are radially shrinking and now what we will do is we will cut this over here. So, maybe draw this. So, I will take some radius here and cut along this.

And then, once I cut along this radius, I will fold it, I will unfold my each of my strips. So, I will each of these I will make into a strip and then how do my strips, how are my strips are going to look like? So, my outer strip will look like this, once I unravel it, and then each, I have each of my inner strips and there is more. Maybe I will draw 1 or 2 more with a darker pen and so on. So, I have not drawn all of them, but I hope you get the idea.

And then what do I do? I make this bottom thing, I make it into a line, so it is arranged like this, so I make it straight and then I make this the curve over here straight. So, I straighten out this part here and I straighten out my sides and I maintain, at all times I maintain the area. So, I do not change the area. Of course, this is going to deform the top. So, the top here is a line, instead I will get something maybe like this and then the next strip will be maybe continuing here something like this and then something like this and I hope you get the idea.

So, if we continue, we will get darker strips like this, then the darkest one which are inside. So, it seems like you can, this figure of course, that top thing is not looking nice, but it does look somewhat rectangle. So, we can harness this idea.

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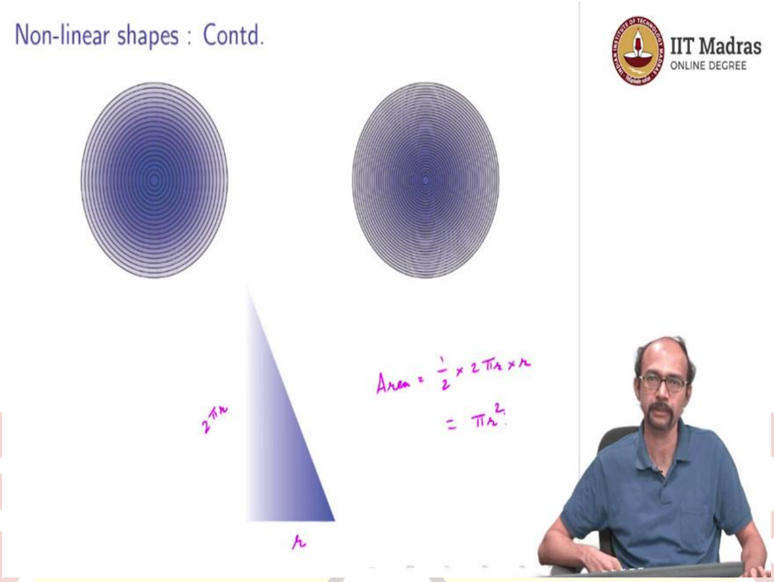

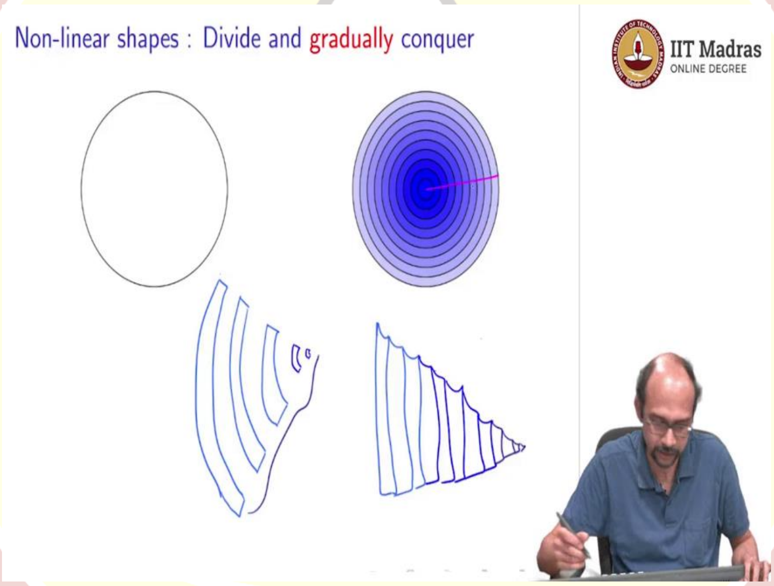

And maybe what we can do is we can divide this into smaller and smaller strips. So, you increase the number of circles that you have drawn. So, here is a larger number and then here is an even larger number and as you go ahead and ahead and you make your strips thinner and thinner, you can see that that top thing that was causing a problem is going to become very, very, very tiny. So, it will, you can almost ignore the curve over there.

And what you may get is something like a triangle and what would be the length and height of this triangle? What will be the base here? So, the base here, let us go back and see what the base was, so the base was exactly the length of this thing which is the length of this pink thing here which is the radius. So, that is the base here. So, the base is the radius. And what is the height?

So, the height is going to be the length of the longest circle which in other words is the circumference of this circle. And we know that the circumference of the circle is $2\pi r$. So, what do we get? That means, we get area is $\frac{1}{2} \times 2\pi r \times r$ and what do we have? We have πr^2.

So, what is the point, I hope you see how I got this in the limit and the point here is that this idea of breaking it up into thinner and thinner things and then estimating them by rectangles seems like a good idea and indeed this is exactly what we are going to use in our next video when we compute what is called the integral or rather, we define what is called the integral. Thank you.