IIT Madras

ONLINE DEGREE

Statistics for Data Science - 1

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Indian Institute of Technology Madras

Lecture 9.5

Variance of a Random Variable

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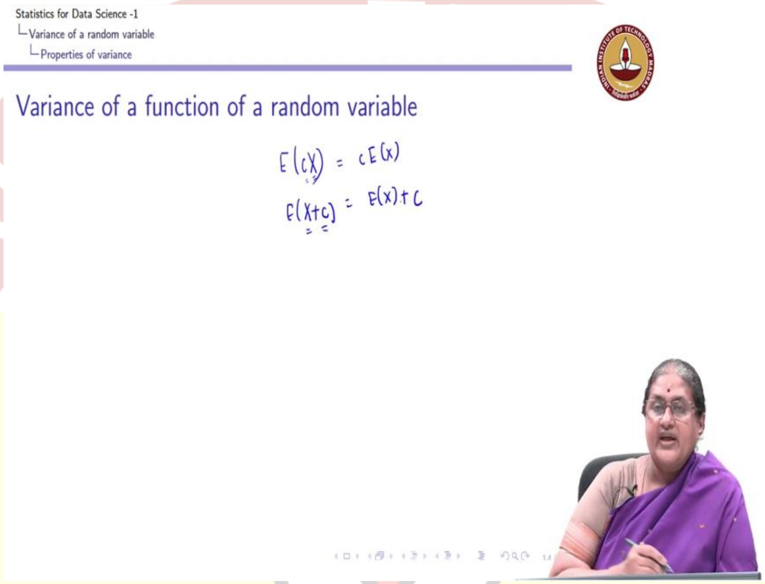

Then next thing which we saw is let us see a few properties of the variance of a random variable. Recall when we wanted to look at properties of expectation, we want we saw what is $E(cX)$ and what was $E(c + X)$ variable that is what would happen to the expectation if I multiply my random variable with a constant and if I add a constant to my random variable if you recall these were the properties when I looked at the expected value of a random variable. Now, let us continue with the same exercise for variance of a random variable.

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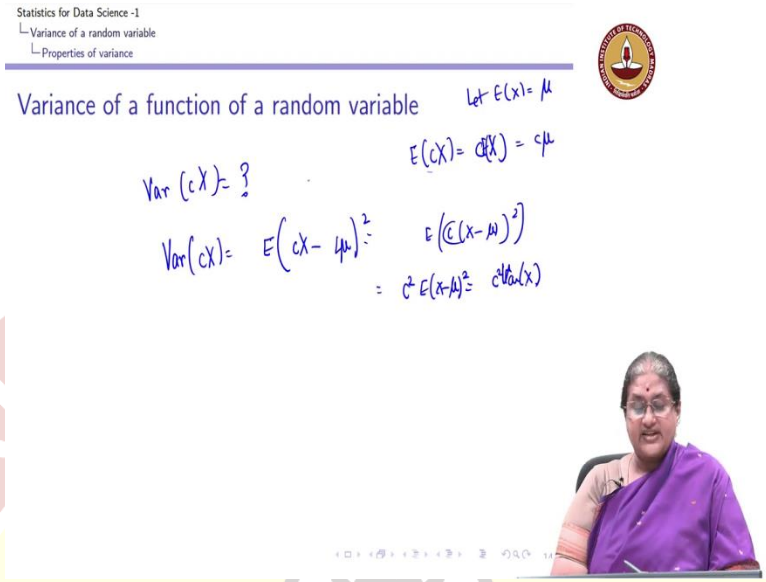

So, the question is, suppose I have X is a random variable and I am multiplying X with the constant I want to know what this variance of X. Recall $E(c\mu) = cE(\mu)$. So, from my first principles I know $Var(cX) = E(cX - c\mu)^2$, which is nothing but $E(c(X - \mu))^2$which is going to be $c^2E(X - \mu)^2$ which is going to be $c^2Var(X)$.

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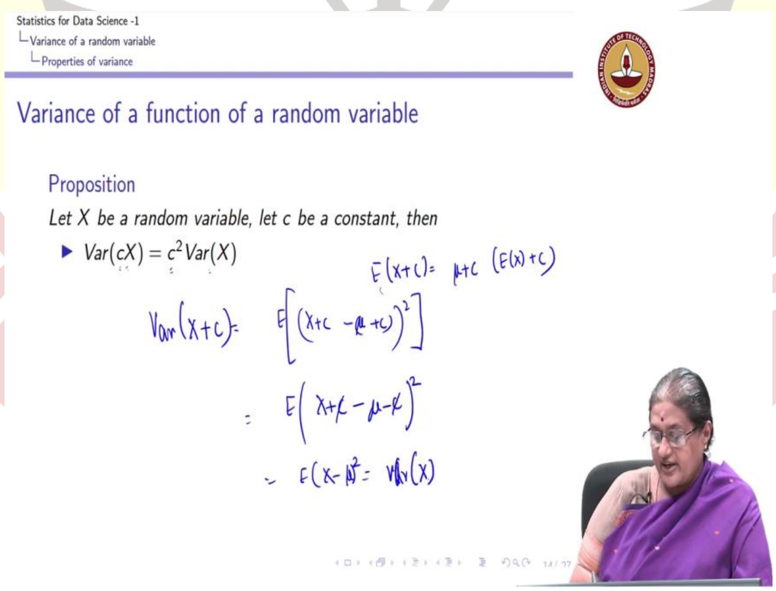

So, we can choose and we can see that if X is a random variable and c is a constant then $Var(cX) = c^2Var(X)$. Now, what would happen to $Var(X + c)$, again recall $E(X + c) = \mu +$

$c = E(X) + c$. So, $Var(X + c) = E((X + c) - (\mu + c))^2$. So, this is going to be $E(X + c - \mu - c)^2$ these two cancel out which is $E(X - \mu)^2 = Var(X)$.

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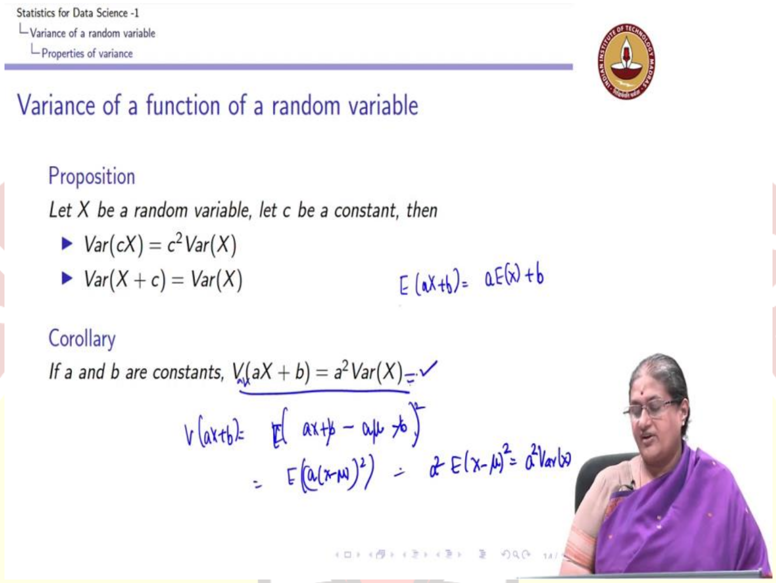

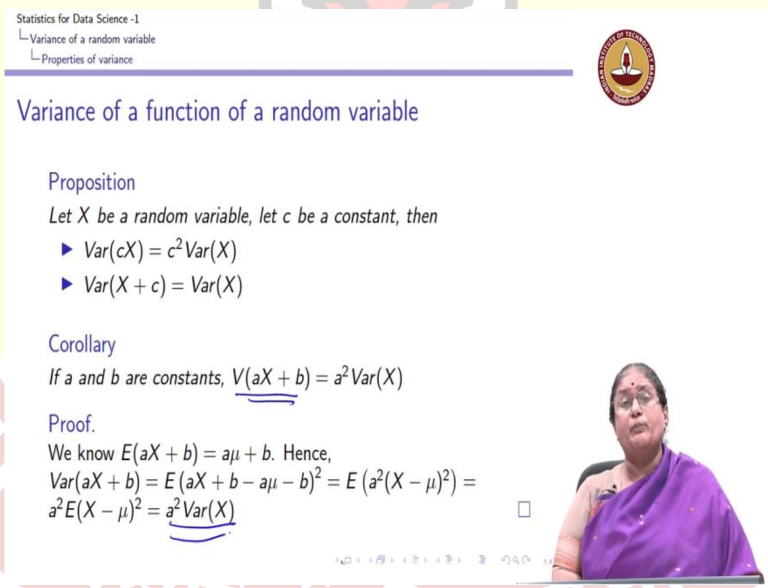

So, variance of a constant times a random variable is c square times variance of X, whereas variance of X plus a constant is the same as variance of X. Again why is this? Remember if I add a constant two variables, so if this is the variability and I add a constant the variance does not change. This is something which if I am adding 1 then the variance does not change the means changes but the variance does not change if I add a constant because it just going to be a shift in my distribution. So, I can generalize this result as a corollary, which was very similar to $E(aX + b) = aE(X) + b$ I get $Var(aX + b) = a^2Var(X)$.

The proof this is also very simple, $Var(aX + b) = E(aX + b - a\mu - b)^2$, which is the expectation of aX plus b and minus b cancel out. So, I get $E(aX - a\mu)^2 = E(a(X - \mu))^2 = a^2E(X - \mu)^2 = a^2Var(X)$, and that is what I have here. Hence, I know that if a and b are constants X $Var(aX + b) = a^2Var(X)$. These are very important properties of variance.

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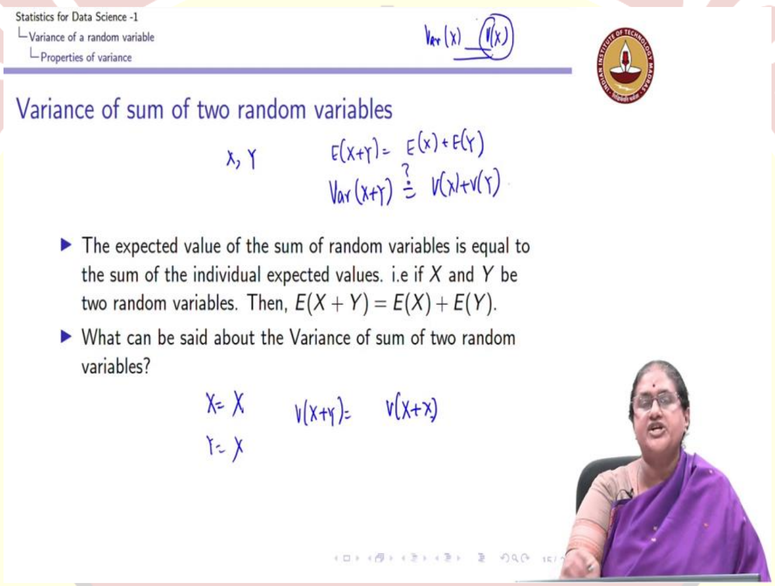

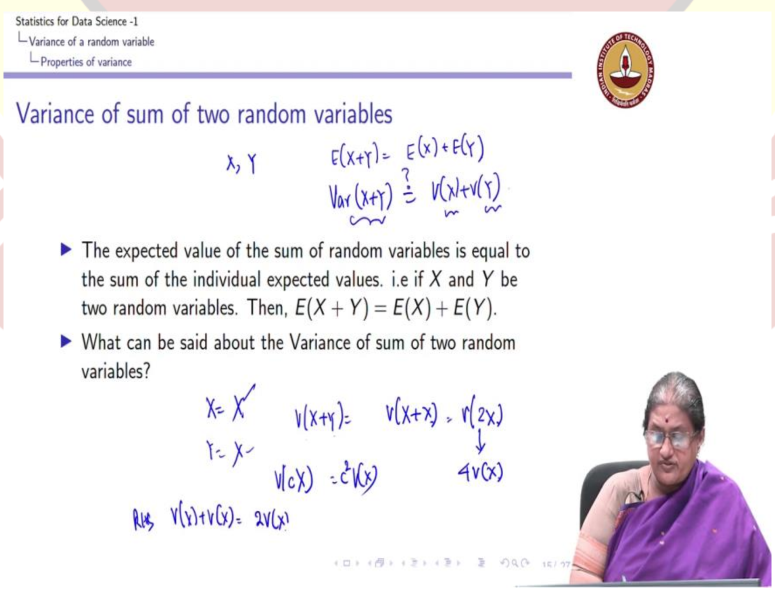

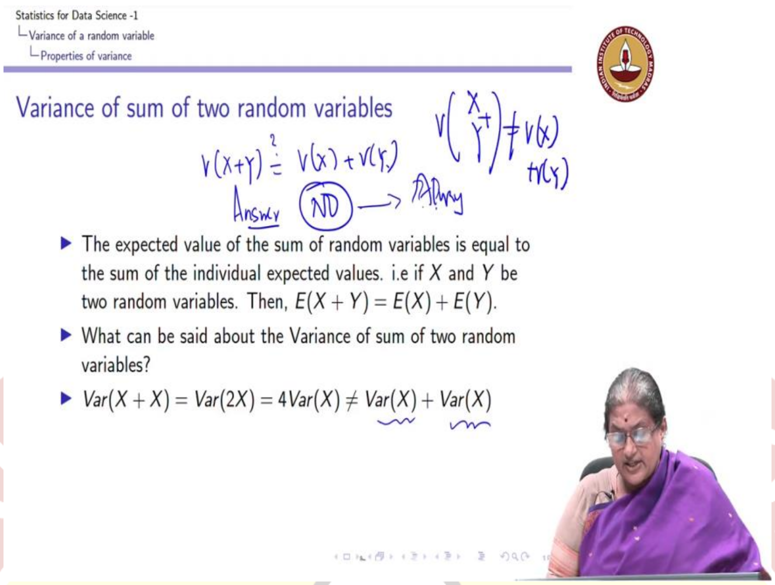

And let us apply this to understand how to get the variance of sum of random variables. Again recall when I had random variables X and Y expectation of X plus Y was the sum of expectation that is expectation of sum was sum of expectations $E(X+Y) = E(X) + E(Y)$. So, the question we are asking is variance of sum will it be equal to the sum of the variances that is the question we are asking. Now, let us look at the following case. Let X be equal to X and Y be also equal to X then I know $V(X+X) = V(2X)$. Let me tell you I am going to write Var(X) and V(X) they mean the same this is just a short hand notation to represent variance of X.

So, V(X)and Va (X) and V(Y) and Var(Y) represent the same quantity, which is variance of a random variable. So, $Var(X+Y) = Var(X+X) = Var(2X)$. We already know $Var(cX) = c^2Var(X)$ hence $Var(2X) = 4Var(X)$. Now, if I wanted to know whether $Var(X+Y) = Var(X) + Var(Y)$, so if I am going to look at the right hand side I get $Var(X) + Var(X) = 2V(X)$, I can see $Var(2X) = 4Var(X) \neq Var(X) + Var(X)$.

So, in general we ask whether $Var(X+Y) = Var(X) + Var(Y)$ the answer is no. It need not be equal and we illustrated it with an example showing that variance of X plus X is not equal to variance of X plus variance of X. So, the natural question is this always true, is this always true that is given two random variables variance of X plus Y is $Var(X+Y) \neq Var(X) + Var(Y)$? Again the answer is no.

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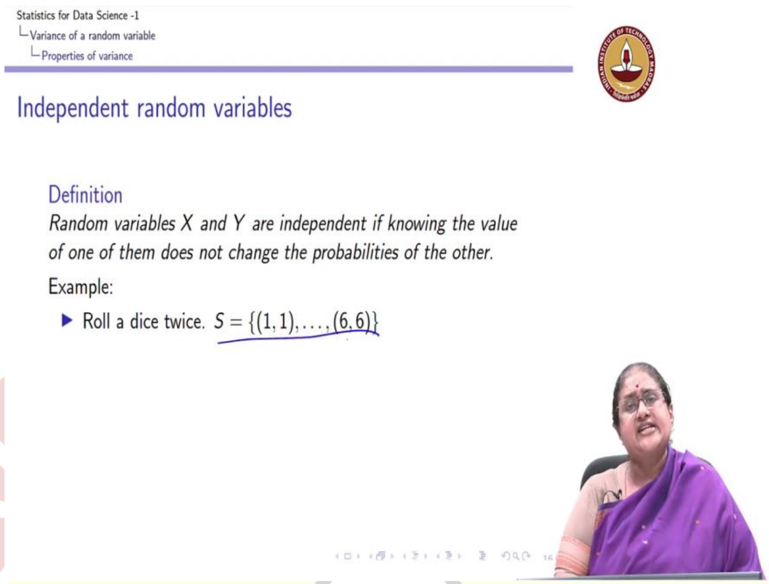
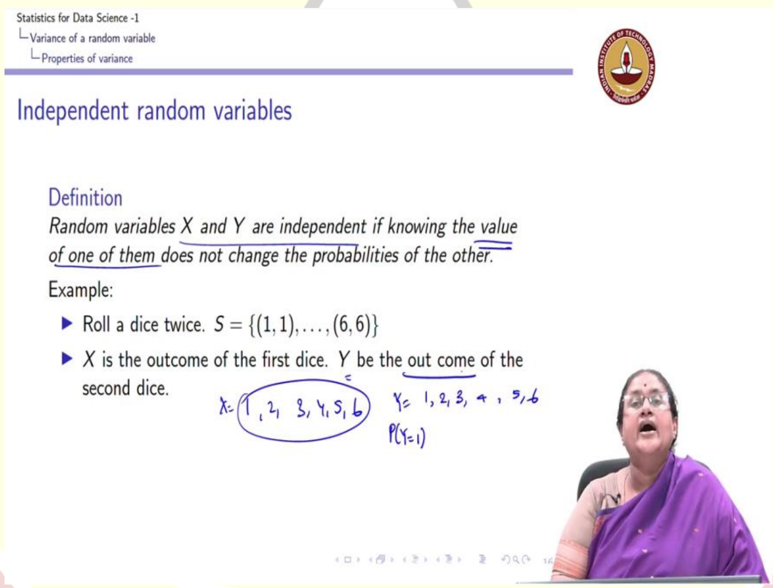
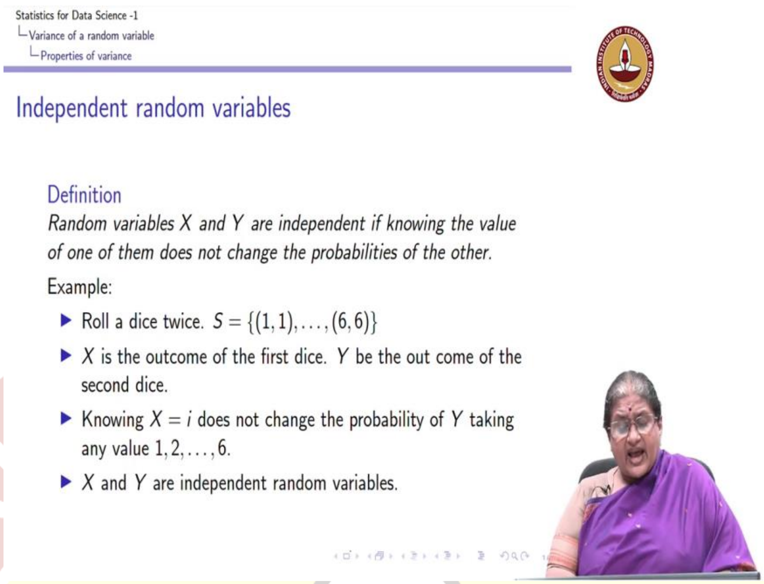

Statistics for Data Science -1
Variance of a random variable
Properties of variance
Independent random variables
Definition
Random variables X and Y are independent if knowing the value of one of them does not change the probabilities of the other.
Example:
Roll a dice twice. $S = \{(1,1), \ldots, (6,6)\}$


Statistics for Data Science -1
Variance of a random variable
Properties of variance
Independent random variables
Definition
Random variables X and Y are independent if knowing the value of one of them does not change the probabilities of the other.
Example:
Roll a dice twice. $S = \{(1,1), \ldots, (6,6)\}$
X is the outcome of the first dice. Y be the out come of the second dice.
$X = 1, 2, 3, 4, 5, 6$
$Y = 1, 2, 3, 4, 5, 6$
$P(Y=1)$



So, we introduced the notion of independent random variables. What do we mean by random variables that are independent? I say that two random variables X and Y are independent if knowing the value of one of them does not change the probability of the other. Let us, look at an example. Again roll a dice twice, I know this is my sample space.

Now, let X be a random variable, which is the outcome of the first dice and let Y be the outcome of the second dice I know X takes the value 1, 2, 3, 4, 5, 6, Y also takes the value 1, 2, 3, 4, 5, 6. Now, X the P(X+1) is independent of what was the outcome of X. So, X and Y are independent because I, given that X has taken some that is given the outcome of the first dice does not impact the outcome of the second where Y is defined as the outcome of the second dice, hence X and Y are independent random variables. Now, why does this become important to us?

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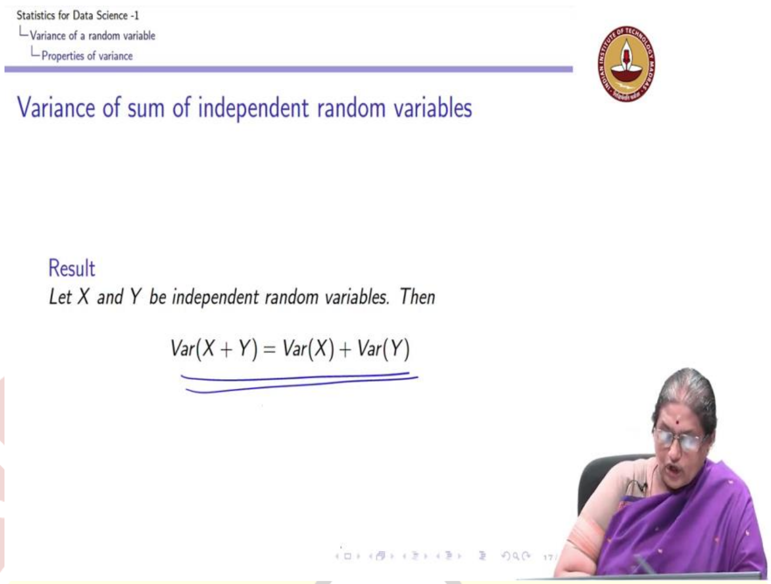

Now, if I have X and Y as independent random variables, then I can show that $Var(X+Y) = Var(X) + Var(Y)$. In other words, variance of X plus Y need not always be equal to the sum of variances. However, if X and Y are independent random variables then the variance of the sum is the sum of the variances.

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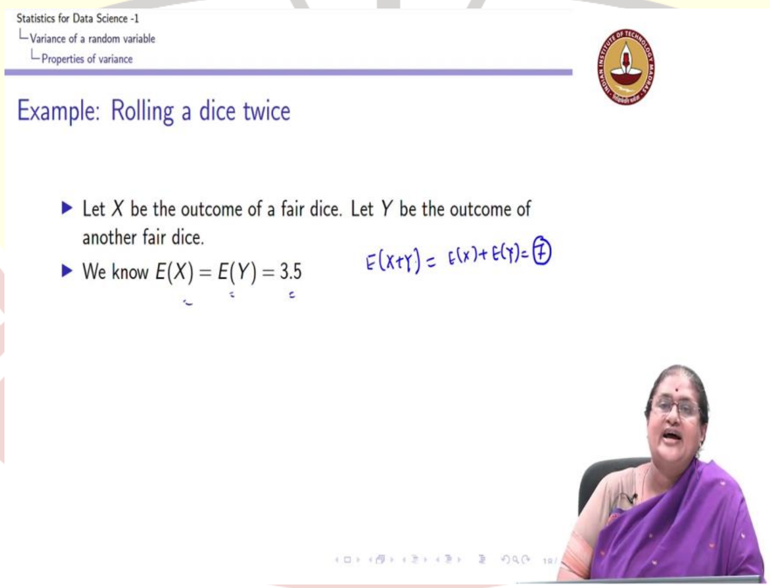


Example: Rolling a dice twice

- Let X be the outcome of a fair dice. Let Y be the outcome of another fair dice.
- We know $E(X) = E(Y) = 3.5$
- $X + Y$ is the sum of outcomes of both the dice rolled together.

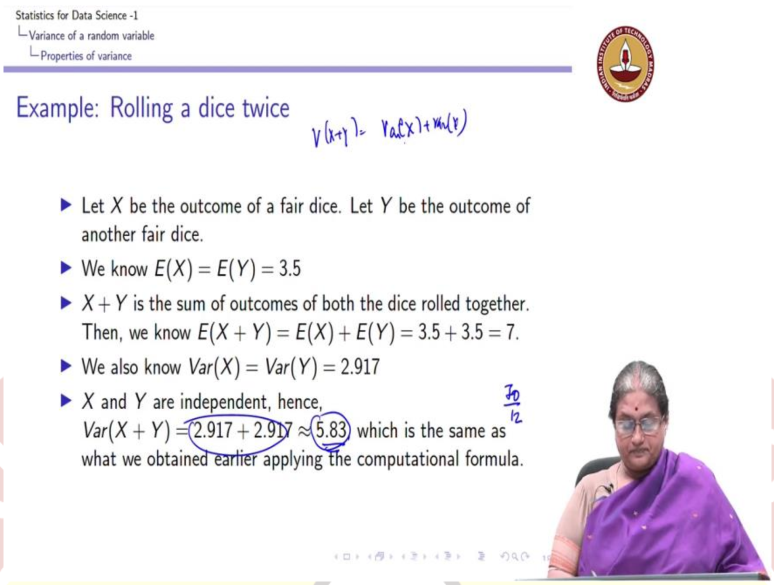
$V(X) = \frac{35}{12}$ $\quad V(Y) = \frac{35}{12}$

$V(X+Y) = \frac{70}{12} =$

Statistics for Data Science -1
└Variance of a random variable
└Properties of variance

Example: Rolling a dice twice

- Let X be the outcome of a fair dice. Let Y be the outcome of another fair dice.
- We know $E(X) = E(Y) = 3.5$
- $X + Y$ is the sum of outcomes of both the dice rolled together. Then, we know $E(X + Y) = E(X) + E(Y) = 3.5 + 3.5 = 7$.
- We also know $Var(X) = Var(Y) = 2.917$



Now, we are going to look at application of this result. Again, let us look at rolling a dice twice again X is the outcome of the first dice and Y is the outcome of the second toss or first fair dice and second fair dice. Now, I now that $E(X) + E(Y) = 3.5$ and we also verified $E(X + Y) = E(X) + E(Y) = 7$ when I am rolling a dice twice you recall that the expectation of the sum of dice was equal to 7.

We already computed the variance of X in a roll of a single dice. Now, you recall again X is a uniform distribution, $Var(X) = \frac{35}{12}, Var(Y) = 35/12$ hence $Var(X + Y) = 70/12$ and you can verify that $Var(X + Y) = Var(X) + Var(Y)$, which is almost equal to 5.83 which you can check is 70/12. So, you can verify that this is what we got applying the computational formula, this is what we got by applying the fact that variance of X plus Y equal to variance of X plus variance of Y when X and Y are independent random variables.

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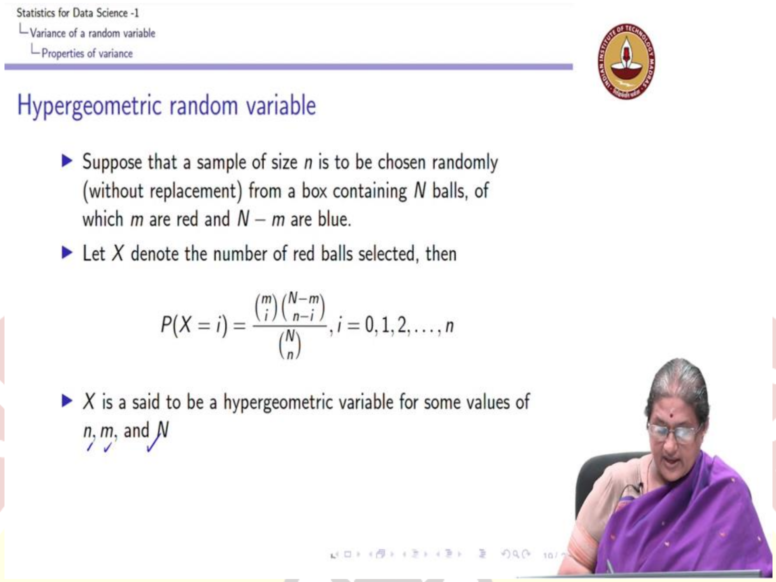

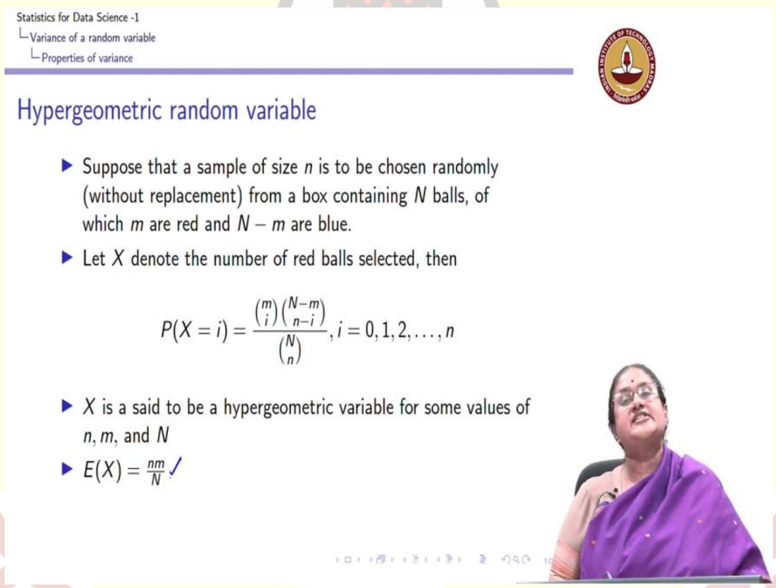

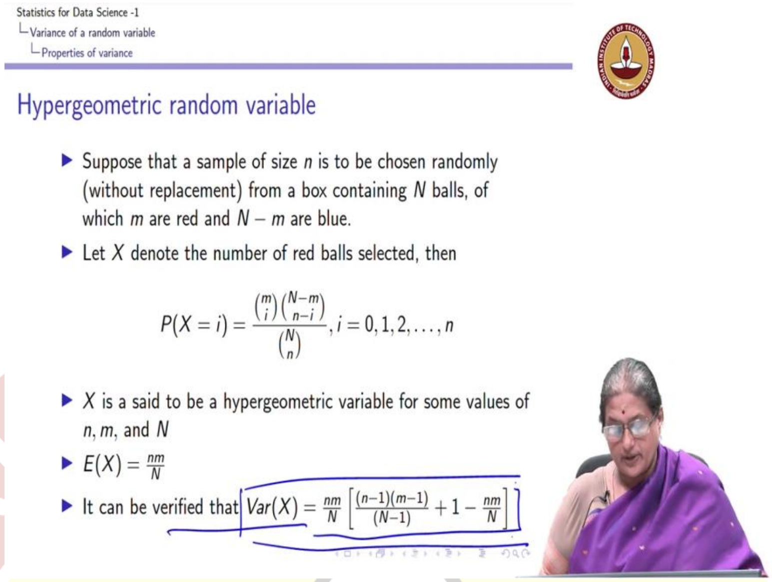

Now, let us look at the case of a hyper geometric random variable. We already introduced the probability mass function. We also check that if I have a hyper geometric variable with parameters n, m and N we already established $E(X) = \frac{nm}{N}$ we can verify that $Var(X) = \frac{nm}{N}\left[\frac{(n-1)(m-1)}{(N-1)} + 1 - \frac{nm}{N}\right]$ given by this quantity the proof of this would be discussed in a tutorial.

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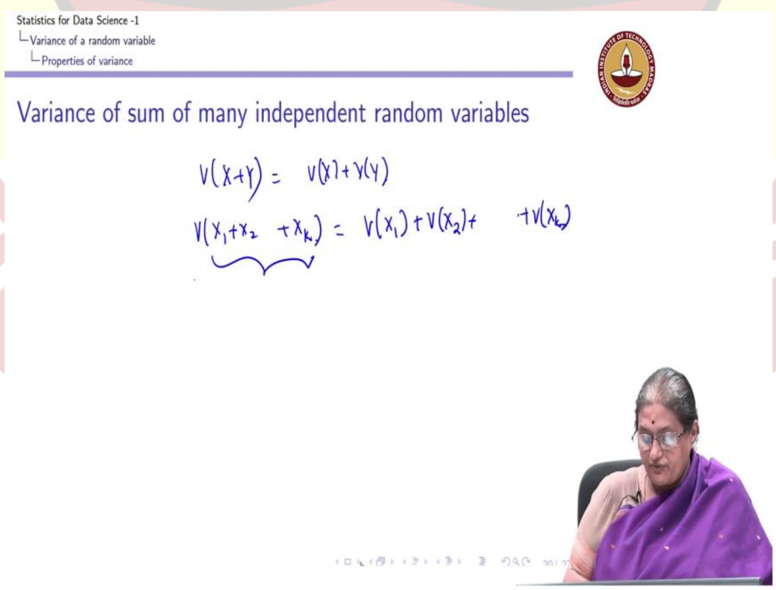

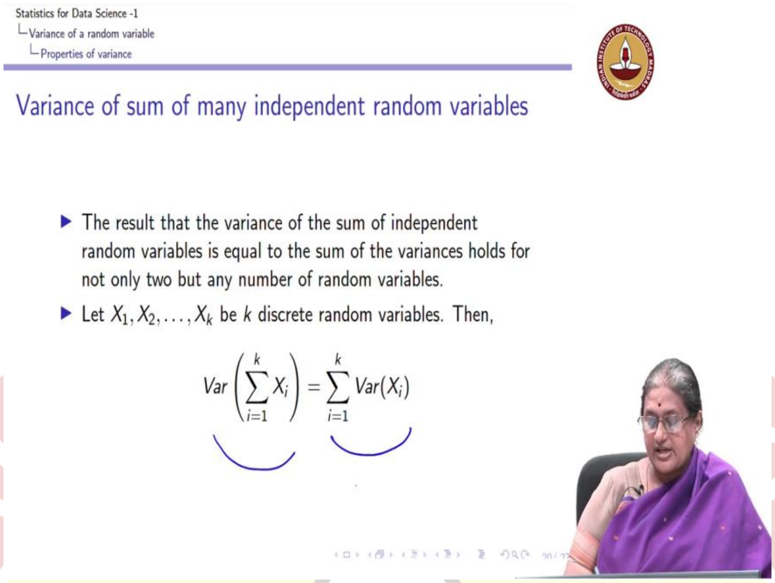

Now, I can extend this property that $Var(X + y) = Var(X) + Var(Y)$ to many independent variable in particular, if I have X_1, X_2 X_k which are k independent random variables, $Var(X_1 + X_2 + \cdots . X_k) = Var(X_1) + Var(X_2) + \cdots + Var(X_k)$. $Var(\sum_{i=1}^{k} X_i) = \sum_{i=1}^{k} Var(X_i)$. I can extend this not only to two but to k variable. So, variance of sum is sum of variances.

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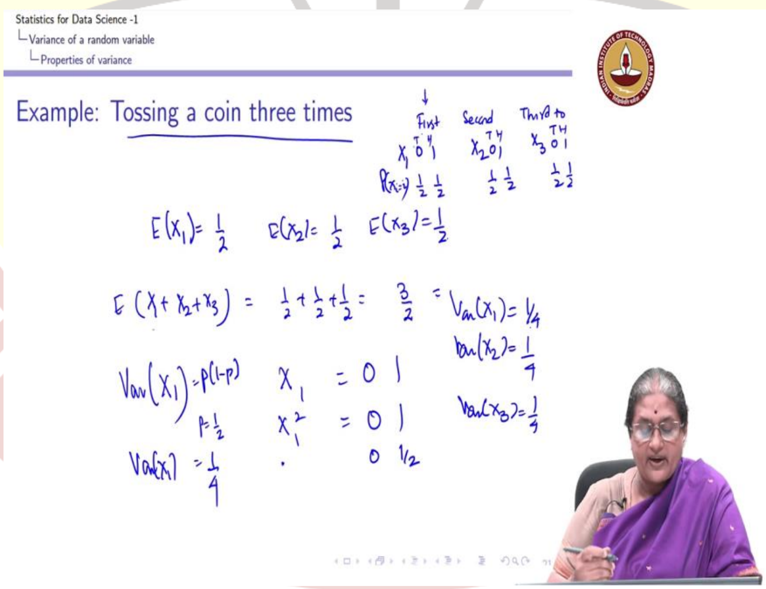


Example: Tossing a coin three times

	First		Second		Third toss	
	T	H	T	H	T	H
	X_1: 0	1	X_2: 0	1	X_3: 0	1
$P(X_i = x)$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

$E(X_1) = \frac{1}{2}$ $\quad E(X_2) = \frac{1}{2}$ $\quad E(X_3) = \frac{1}{2}$

$E(X_1 + X_2 + X_3) = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{3}{2}$

$Var(X_1) = 1/4$

$Var(X_2) = \frac{1}{4}$

$Var(X_3) = \frac{1}{4}$

$Var(X_1 + X_2 + X_3) = Var(X_1) + Var(X_2) + Var(X_3)$

$= \frac{1}{4} + \frac{1}{4} + \frac{1}{4}$

$= 3$

Tossing a coin thrice

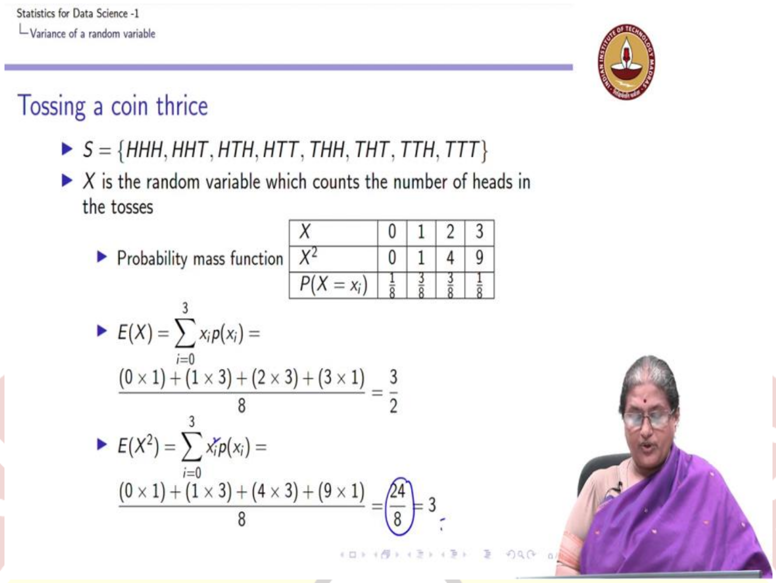
- $S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$
- X is the random variable which counts the number of heads in the tosses
  - Probability mass function

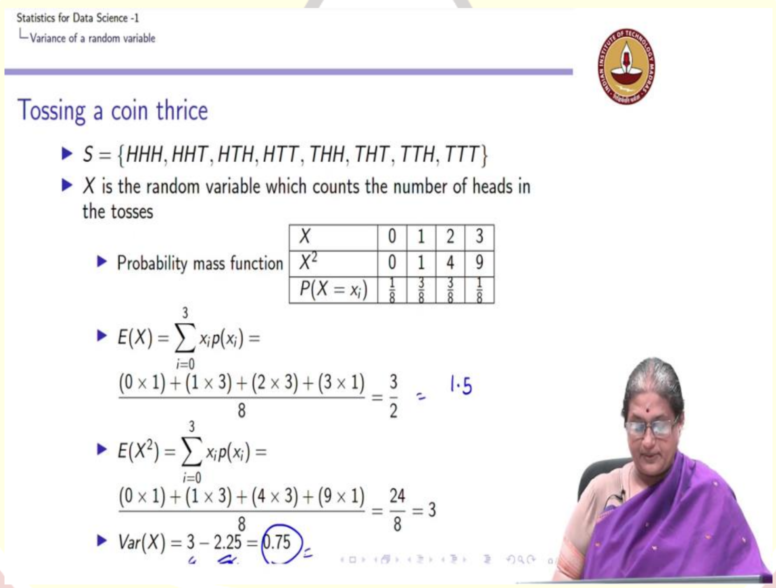
X	0	1	2	3
X^2	0	1	4	9
$P(X = x_i)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

  - $E(X) = \sum_{i=0}^{3} x_i p(x_i) =$

$$\frac{(0 \times 1) + (1 \times 3) + (2 \times 3) + (3 \times 1)}{8} = \frac{3}{2}$$

$$E(X^2) = \frac{(0 \times 1) + (1 \times 3) + (4 \times 3) + (9 \times 1)}{8}$$





Now, how do I apply this formula? Again recall tossing a coin three times is same as noting down the outcomes of a fair toss three times that is I am repeating an experiment three times. So, I have a first toss, I have a second toss, I have a third toss. Let X_1 be my outcome of the first toss, X_2 be the outcome of the second toss, X_3 is outcome of the third toss I know X_1 takes the value 0 or 1 which represents tail or head, X_2 represents tail or head against 0 or 1, X_3 tail or head 0 or 1 since it is a fair coin it takes the value 1/2, 1/2, 1/2, 1/2 and 1/2, 1/2, these are the probabilities with X_i take value i.

We also no expectation of $E(X_1) = E(X_2) = E(X_3) = \frac{1}{2}$, $E(X_1 + X_2 + X_3)$ are the outcomes of my first toss, second toss, and third toss I know they are independent of each other $E(X_1 + X_2 +$

$X_3) = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{3}{2}$ and this is precisely the expectation of a random variable where I am counting the number of heads we have checked this also which is going to be 3/2.

Now, let us look at the $Var(X_1)$. If X_1 takes the value 0 and 1 I know X_1 square also takes the value 0 and 1 with probability 0 and 1/2 recall this is a Bernoulli random variable. We know the variance of a Bernoulli random variable is $p \times (1-p)$ in this case is 1/2. So, $Var(X_1) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$. Similarly, $Var(X_2) = \frac{1}{4}$ and $Var(X_3) = \frac{1}{4}$. Because X_1, X_2, X_3 are all Bernoulli random variables with parameter $p = \frac{1}{2}$.

They are independent also so I can apply my property that the variance of a sum is sum of variances which we have just seen, so, $Var(X_1 + X_2 + X_3) = Var(X_1) + Var(X_2) + Var(X_3) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}$, and you can see that matches with the variance which we computed using the computation formula which is 0.75 and 3/4 is the same this is what we have already seen.

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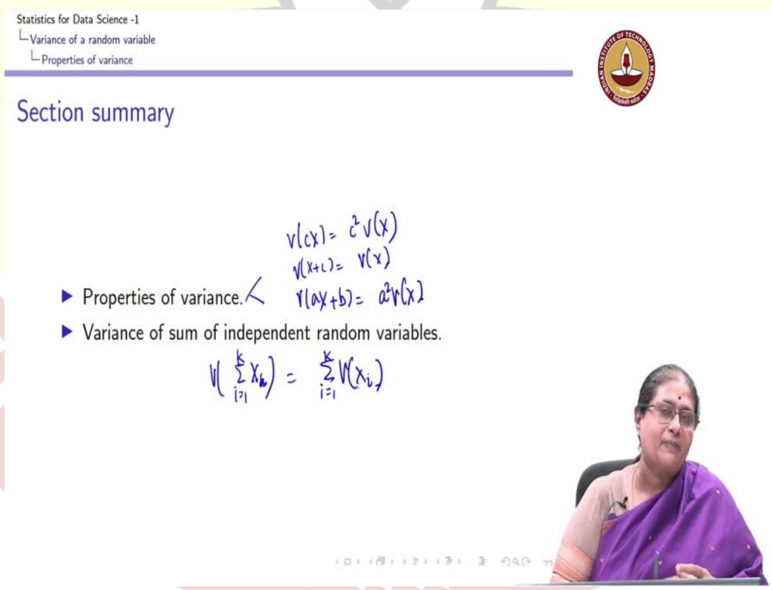

So, in summary what we have seen so far is the main properties of the variance namely $Var(cX) = c^2 Var(X)$, which can be generalized to $Var(aX + b) = a^2 Var(X)$ and $Var(\sum_{i=1}^{k} X_i) = \sum_{i=1}^{k} Var(X_i)$, where $X_1, X_2, \dots . X_k$ are independent random variables. So, these are the two important properties we have seen and we computed the earlier distributions applying this property.