Derivatives, tangents and linear approximation

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A tangent to $f(x)$ at a is a line which represents the *instantaneous* direction in which the graph $\Gamma(f)$ moves at $(a, f(a))$.

Traditionally, the tangent to $f(x)$ at a is thought of as a line which *just touches* $\Gamma(f)$ at $(a, f(a))$.

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$$y - f(a) = \frac{f(a+h) - f(a)}{a+h-a}(x-a) = \frac{f(a+h)-f(a)}{h}(x-a)$$

is the equation of the secant.

What happens in the limit to this equation?

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$$y = \frac{f(a+h) - f(a)}{h}(x-a) + f(a)$$

If f is diff. at a, then the limit of is $f'(a)$

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Examples

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$f(x) = 5x^3 - 17x^2 + \pi x - 0.5 \quad ; \quad a = 0.$

$f'(x) = 15x^2 - 34x + \pi$.

$f'(0) = \pi$.

$\therefore$ Eqn. of tangent to f at 0 is

$y = \pi(x-0) + f(0) = \pi x - 0.5$.

$f(x) = cos(x) \quad ; \quad a = \frac{\pi}{3}$

$f'(x) = -\sin(x)$

$f'(\pi/3) = -\sin(\pi/3) = -\sqrt{3}/2$

$\therefore y = -\frac{\sqrt{3}}{2}(x - \pi/3) + \cos(\pi/3)$

$= -\frac{\sqrt{3}}{2}(x - \pi/3) + \frac{1}{2}$.

$f(x) = xtan(x) \quad ; \quad a = \frac{\pi}{4}$

$f'(x) = 1 \times \tan(x) + x \times \sec^2(x)$

$= \tan x + x \sec^2(x)$

$f'(\pi/4) = \tan(\pi/4) + \frac{\pi}{4}\sec^2(\pi/4)$

$= 1 + \frac{\pi}{4} \times 2 = 1 + \frac{\pi}{2}$.

$\therefore y = (1 + \frac{\pi}{2})(x - \pi/4) + \frac{\pi}{4}$

Example : $f(x) = x^{\frac{1}{3}}$

$f(x) = x^a$

$f'(x) = a x^{a-1}.$

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{h^{1/3} - 0}{h}$$

$$= \lim_{h \to 0} \frac{h^{1/3}}{h} = \lim_{h \to 0} \frac{1}{h^{2/3}}$$

diverges to ∞.

$\therefore$ This limit DNE.

If $x \neq 0$

$$f'(x) = \frac{1}{3} x^{1/3 - 1}$$

$$= \frac{1}{3} x^{-2/3}$$

$$= \frac{1}{3 x^{2/3}}$$

$y = mx + c$ works only for lines which are not vertical.

$x = 0.$

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$$\text{i.e. } f(x) \approx L(x) \quad \forall x \text{ close to } a.$$

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$f(x) = x^3$; $a = 1$

$f'(1) = 3 \times 1^2 = 3.$

$L(x) = 3(x-1) + 1 = 3x - 2.$

$f(x) = sec(x)$; $a = 0$

$f'(0) = \tan(0)\sec(0) = 0.$

$L(x) = 0(x-0) + \sec(0)$

$= 1.$

Thank you