IIT Madras

ONLINE DEGREE

Mathematics for Data Science 2
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Lecture No. 07
Differentiability and the Derivative

Hello and welcome to the Maths 2 component of the online BSC program on Data Science and Programming. In this video, we are going to talk about differentiability and the derivative.

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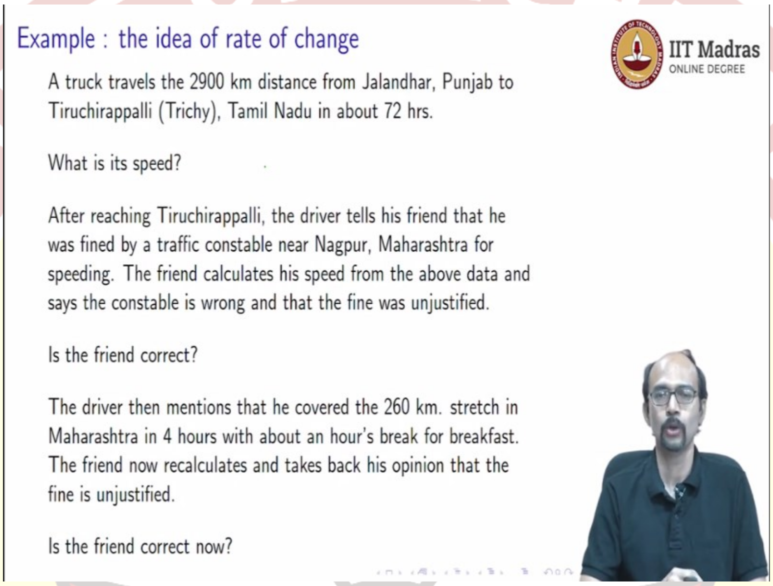

So, let us start by first recalling the idea of the rate of change. Let us do this example of a truck which is travelling from Jalandhar in Punjab to (Trichy) Tiruchirappalli in Tamil Naidu. So, the distance is about 2,900 kilometers and it takes the truck about 72 hours to get there. What is its speed? So, I will give you a second to think about that let me repeat the question. What is the speed of this truck in kilometers per hour?

So, if you have thought about this I think most of you would have done the following, you would have taken the distance covered which is 2,900 kilometers and divided by the time taken which is 72 hours. So, you would say that the speed is 2,900 kilometers by 72 kilometers per hour which is I guess about, how much would that be, about $\frac{1}{2.5}$ which is 0.4. So, 0.4×100 so that means this would be about 40 kilometers per hour. So, I think if you have done your computation this would be about 40 kilometers per hour.

So, after reaching Trichy the driver tells his friend that he was fined by a traffic constable near Nagpur Maharashtra. So, if you look at Google maps the route will go past various important and interesting places like Agra and Jhansi and so on. So, it will pass via Nagpur and Hyderabad and so on. So, near Nagpur this traffic constable stopped the driver and he was given a fine for speeding. By the way do you know what the speed limit on Indian highways is, probably not you should go and check that. Fine.

So, the friend calculates his speed from the above data and says that the constable is wrong and the fine was unjustified. So, what does the friend do? They do exactly what you probably did, they do $\frac{2900}{72}$ and compute that the speed must have been 40 kilometers per hour and let us say how can that be speedy. So, indeed the high speed is above 40 kilometers per hour allowed.

Is the friend correct? So, from this information can we really conclude that the speed of the truck, whenever the traffic constable stops the driver, was 40 kilometers per hour? I think you probably will agree that the answer is no. The driver then mentions that he covered the 260 kilometers stretch in Maharashtra. So, when they pass through Maharashtra it is about 260 kilometers they enter from Madhya Pradesh and they exit into Telangana and then close to Hyderabad or after sometime they hit Hyderabad.

So, the driver mentions that he covered the 260 kilometer stretch in Maharashtra in 4 hours with about an hour break for breakfast. So, in 4 hours the truck has travelled 260 kilometers, but out of those 4 hours one hour was spent on breakfast. So, that means in 3 hours the truck covered 260 kilometers. So, now we have a different data as you know Nagpur is in Maharashtra and now that means within Maharashtra the truck was travelling at a much higher speed than the rest of the journey.

So, what was the speed there if you calculate you will probably get in $\frac{260}{3}$ kilometers per hour which would be, what it would be, something beyond 85 kilometers per hour maybe something like 86 kilometers per hour which is well above the speed limit. The friend now recalculates and takes back his opinion that the fine is unjustified. So, the friend does exactly what we did that the truck is probably travelling at 86 kilometers per hour. Well, a constable is justified and fining the truck driver.

Is the friend correct now? So what happened here? We initially knew that the truck travels the entire distance from Jalandhar to Trichy which is 2,900 kilometers in 72 hours. We computed that the speed is 40 kilometers an hour than we have this new set of information that within Maharashtra the distance was 260 kilometers which was travelled in about 3 hours, 4 hours, but with an hour's break for breakfast so which was about 86 kilometers per hour I mean more than that. The friend says that okay then maybe it is correct that you are fine. Is that correct really?

So, if you think carefully the same argument that applied for the first case meaning if you felt that the friend was incorrect in the first place, then you would agree that the friend was also incorrect in the second place and why is the friend incorrect? Because you cannot really say what speed the truck was going at exactly when the constable fined the truck driver based on the average speed.

So, the speed can increase or decrease as you go along so that is why we have the speedometer and as we go along the speedometer increases and decreases it tells you how much the speed is and instantaneous speed which is what the speedometer shows you cannot be saying that oh I travelled this much distance in this much time.

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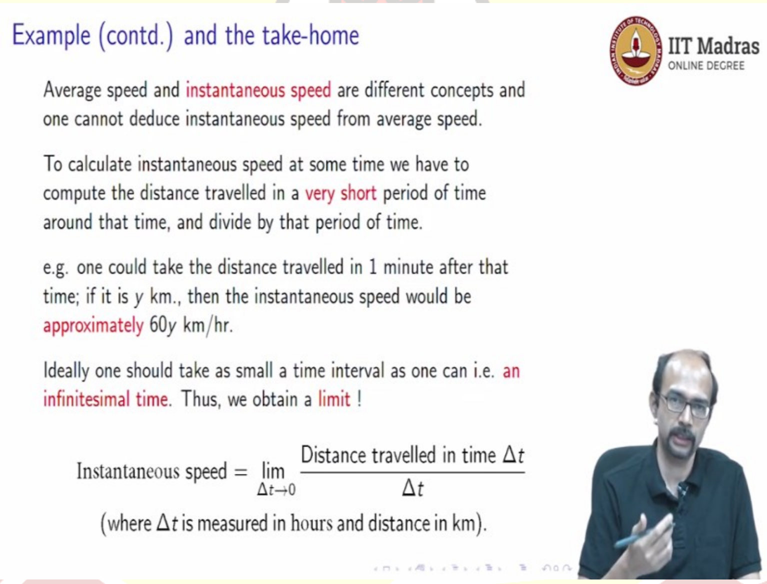

So, the take home from this example is that the average speed and instantaneous speed are different concepts and one cannot deduce the instantaneous speed from the average speed, you may be able to give some heuristic, but remember it is only a heuristic the average speed and the instantaneous speed are totally different concepts. So, for the instantaneous speed how do we get that?

Well, we really have no way of getting that from the information given. For that if there was a camera on the dashboard or behind the truck driver capturing the journey specifically the needle etcetera for the speedometer, then from there we have been able to get it. But unless you really kept track of the entire journey there is no way of getting it from the data given. The average speed different ballgame we could compute that based on the data given.

So, the constable may have had a speed gun of course this may not be so common in India especially on National highways, but they could have had a speed gun and typically what happens is if they have a speed gun and they have pointed it towards the vehicle they know instantaneously what speed the vehicle is going at. So, that computes instantaneous speed. So, to calculate instantaneous speed at some time what do you need?

We have to compute the distance travelled in a very short period of time around the time that we want to calculate it at and divide by that period of time. For example you could take the distance travelled in 1 minute after that time so if you have travelled y kilometers in 1 minute than the instantaneous speed would be approximately and again the underlying word here approximately 60 y kilometers per hour.

Why approximately? Well, even this is not instantaneous, within 1ne minute, the speed can vary so much maybe within 1 minute there was a series of speed breakers or a mountain side started or there was an accident and driver slowed down or they stopped for chai anything. So, even 1 minute could be too long. So, this is just an approximation only we may feel intuitively that it is a better approximation than the speed computation based on the entire journey duration.

So, ideally what would you do? You would take as small a time interval as you possibly can and mathematically we would call that an infinitesimal time. So, when you keep taking smaller and smaller times, this is really what you are saying is well, let me time shrink the time interval I take that shrink to 0 and that is exactly what we know is underlying a limit. So, what exactly is instantaneous speed?

You take the distance travelled in time Δt so suppose you want to take, you want to know the instantaneous speed at a particular time t_0. So you take $t_0 + \Delta t$, take the distance travelled within that time between time t_0 and time $t_0 + \Delta t$ and divide by Δt and you let limit delta Δt tend to 0. Now, of course you have to measure this correctly because if you want in kilometers per hour than delta t has to be measured in hours and distance has to be measured in kilometers.

So, even if you take Δt it to be 1 second and distance is 1 meter you should think of that as 10^{-3} kilometers and 1/3,600 hours and then that is how we should calculate that is how you will get instantaneous speed as a limit and this is really one of the main reasons why we are interested in limits in the first place because limits are really at the heart of things like instantaneous stuff, for example, instantaneous speed. So, now we can extrapolate this and we can talk about the rate of change after all what is speed? Speed is the rate of change of so how much distance you covered in time so the rate of change.

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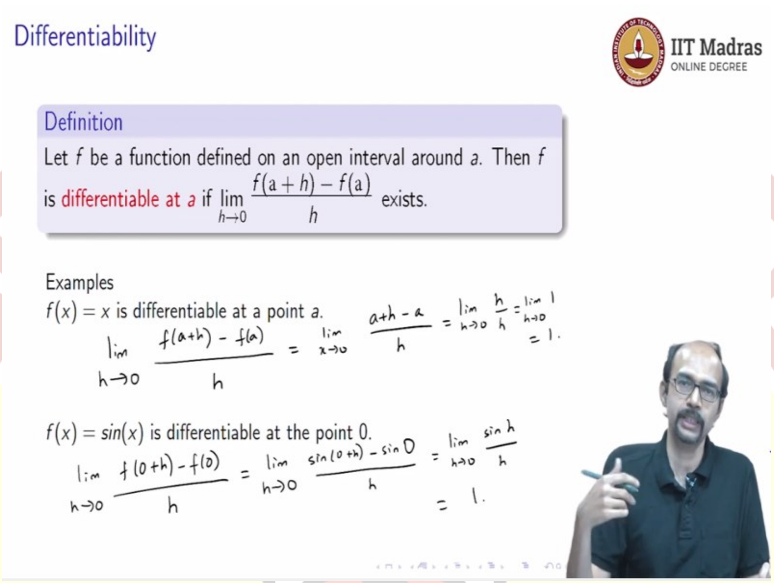

So, that concept is called differentiability. So, let f be a function defined in an open interval around a so a is a point then f is differentiable at a if $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ exists. This is exactly the kind of idea we had before. In the previous slide your f(x+ h) – f(x) so f represented the distance covered so x was the time so x was a function of time.

So, if you wanted the instantaneous time at t_0, we would have written this as $\frac{f(t_0+\Delta t)-f(t_0)}{\Delta t}$. So, the Δt here is replaced by h and the t_0 is replaced by a. So, this is this limit we say that the function is differentiable at a and what is this limit doing if indeed it is differentiable, then this limit is computing for you the instantaneous rate of change of the function.

So, the rate of change of distance with respect to time is exactly the instantaneous speed. So, the rate of change of any function f with respect to this variable x that is what this limit is computing. Let us look at these two examples so f(x) is differentiable at a point a why is that? Let us do this computation? So, if you do limit let me do this explicitly so limit h→ a f of a plus h so what is a here? a is any point this is what you want and what is my function?

My function is just f(x)=x. So, we have limit term to be 1 as shown. So, what are we saying? If you take f(x) is x, the rate of change of this function is 1.

This is not a very difficult fact to appreciate, it is changing if you take 1 unit then it is changing by 1 so this is more or less what it says. Let us look at f(x) is sin x at the point 0. So, this is $\lim_{h\to 0}\frac{\sin(0+h)-\sin(0)}{\mathrm{h}}$, but what is this? This looks like something familiar. So, the first term is sin (h.) The second term meaning the second sine value is 0 because sin 0 is 0. So, sin(h) - 0 so I will just write sin (h) then divided by h.

So, now this is the limit $\lim_{h\to 0}\frac{\sin(h)}{\mathrm{h}}$ and in the previous video we have given a very detailed argument to show that this limit is 1. So, sin x is indeed differentiable at the point 0 and the value of the limit is 1 and we are going to say more about this as we go ahead. So, I hope these two examples tell you how to compute. So, when you take this limit h tends to 0 we have converted this function f to a function of the variable h. So, remember that when you compute this limit we have to think of this entire expression as a function of h.

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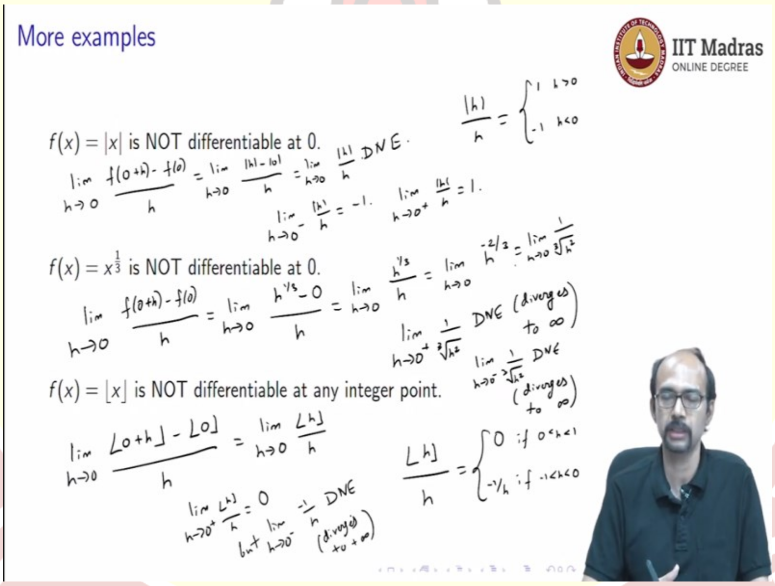

So, let us do a couple of other examples so f(x) =|x| is not differentiable at 0. So, indeed this was one of the places we were stuck at vis-à-vis tangents. f(x) = $x^{1/3}$ is not differentiable at 0 and f(x) is slower function is not differentiable at any integer point. So, let us do the first one. If you take f(x) = |x| what happens? So, at 0 let us take the limit as h tends to 0. So, this is what we want to compute? So, limit is $\lim_{h\to 0}\frac{|h|}{h}$.

So, $\frac{|h|}{h}$ what is this? So, if h is strictly positive then this function is just 1 and if h is negative, then, it is -1. So, this is what it is and of course it is not define a priori when h is 0, but anyway we want to only compute the limit as h tends to 0.

So, now we have to compute this limit as the left hand and the right hand limit. So, this limit exists first of all we have to ask what happens to this function from the left. So, from the left you can see this is -1 from the right you can see this is 1 and hence this limit does not exist and once this limit does not exist that means the function is not differentiable at 0. What happens to f(x) = $x^{1/3}$ why is this not differentiable at 0? So, let us write down what this limit is.

So, let us again do from first principle so this is $\lim_{h\to 0}\frac{(0+h)^{1/3}-0^{1/3}}{\mathrm{h}} = \lim_{h\to 0}\frac{h^{1/3}}{h}$ So, now what happens to this function? So, for this function as h tends to 0 from the right so again we have to ask what is this function look like? So, maybe I should write this more explicitly as $\frac{1}{\sqrt[3]{h^2}}$.

So, you can see as h is positive and h is small meaning it tends to 0. When you divide, it is going to become very, very, very large. So, this limit is going to diverge to infinity or does not exist whichever you prefer. So, I will say does not exist as it diverges to infinity. This is exactly the behavior that we encountered for the limit $\frac{1}{h}$ so this is similar.

And similarly, if you take from the left the same thing happens it does not exist which is to say it diverges to infinity again the same behavior and because of that this limit does not exist so the function is not differentiable at 0. What about f(x) =$\lfloor x \rfloor$ this is I think is very apparent that there will be a problem, but let us do this computation anyway let me do it for 0, but the same argument holds for any integer.

So, now we have to ask for the limit from the left and the right. So, what is this function? So, this function is going to be 0 if h is between 0 and 1 and it is going to be $\frac{-1}{h}$ if h is between -1 and 0. So, I hope this is clear and what happens. So, now if you take the limit from the right so this limit is 0, but the limit from the left is undefined. So, this actually diverges to plus infinity, why plus infinity? Because we have a $\frac{-1}{h}$. So, if you have $\frac{1}{h}$ it would have gone to minus infinity because you have a $\frac{-1}{h}$ it goes to plus infinity so in any case this does not exist. So, all the results that we have proved in the previous or seen in the previous videos have now come to help us in checking whether a function is differentiable or not for these elementary functions.

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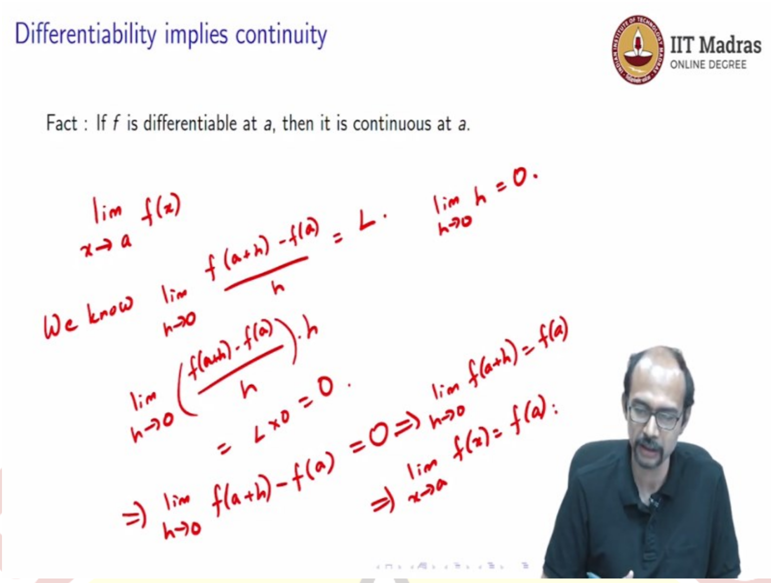

Let us put that last example in more general context differentiability implies continuity. So, if f is differentiable at a, then it is continuous at a. So, why is that? Let me sketch out the argument. So, we want to say that $\lim_{x \to a} f(x)$ we want to conclude that this is equal to f(a). What do we know? We know limit $\lim_{x \to a} \frac{f(a+h)-f(a)}{h}$. So, we know that this limit exists and so let us call it L.

So, now let us compute limit maybe I can do it from this above one better. Let us multiply this by h. So, we have two things this is one and the other is that this is 0. This is the function f(x) is x instead I am taking that function f(h) is h and this limit is 0. We if know from our rules for limits that if you multiply the two functions whose limits at a point exists, then the limit is the, limit of the product is product of the limits.

So, if you take these two functions and multiply this is exactly the product so that is L times 0 is 0, but what does that tell us that means the limit.

So, if you do that you will get that $\lim_{h \to 0} f(a+h) - f(a)$ is equal to f(a), but now instead of writing this as a+ h you could write it as x and just write this as x tends to a because x equal to a + h, then h tends to 0 means x tends to a; f(x) is f(a+h). So, this is a proof that f is continuous at a. So, differentiability is a much more powerful concept than continuity. So, if it is differentiable it is definitely at a point it is definitely continuous at a point.

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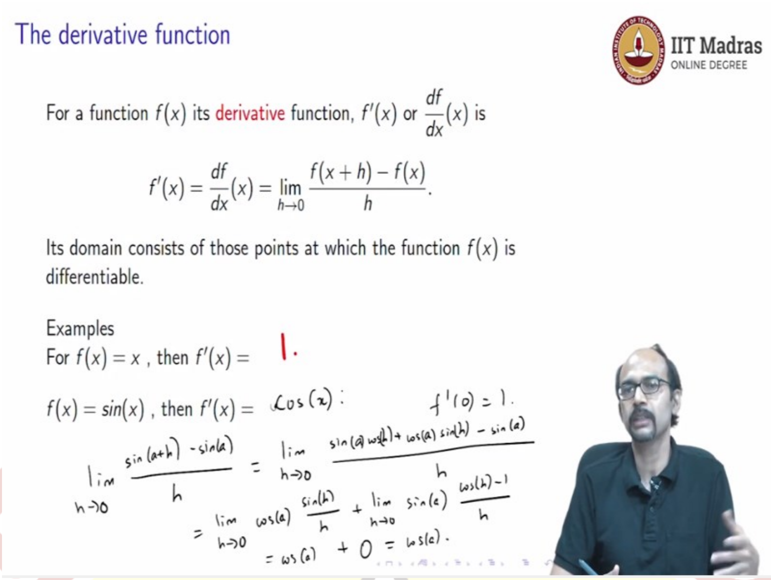

So, we can now talk about the derivative function for a function f(x) its derivative function which is denoted by f'(x) or df/dx is the limit $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ the value of that limit at the point x that is what we mean by the derivative. So, if you do it at different points you get different values and that is exactly what the derivative function is.

And of course, it may happen that for some points the derivative does not exist for example we saw the examples in the previous slides for we took f(x) let us say the floor function of x and we say that at integers it does not exist, but for other point certainly the derivative exist and so you can talk about the derivative function of the floor function at all points other than the integers.

So, in general you can talk about the derivative at all points the derivative functions defined at all points where the function is differentiable so that is the domain of the derivative function. So, let us do some examples if f(x)= x, excuse me, then f'(x) is what? If f(x) is sin x, what is f'(x?) So, if f(x)= x we have computed what happens for any point we saw that this limit that we have gave us the value 1 so this is 1. So, f'(x) is 1.

Let us see what happens for sin x. We have to compute what this limit is. So, we did for 0. We know that f'(0) is 1 we did this, but let us do it for any arbitrary point. So, what is $\lim_{h \to 0} \frac{sin(a+h)-sin(a)}{h}$that is what we have to compute limit as h tends to 0. So, let us see what this is now we can use the expansion of sin(a +h).

So, what is this? So, we will break this into two parts $\lim_{h \to 0} \frac{\cos(a)\sin(h)}{h} + \lim_{h \to 0} \frac{\cos(h)-1}{h} \sin(a)$. So, the first limit this is constant cos(a) because limit is over h.

And limit of $\frac{\sin(h)}{h}$ is 1 and the second limit now I think you can do this from our previous computations. Remember, we saw that $\frac{1-\cos(h)}{h^2}$ that limit exists and equals half. So, here you can pull the sin (a) out and then for the inside limit you can multiply and divide by h and then you can compute what it is so you will get that this is 0 and so the answer is cos (a).

So, what does that mean? That means if you take the function f(x) = sin x and you compute its derivative you get the function cos (x). So, the derivative of sin (x) is cos (x) and we have demonstrated that with a proof. So, we will do more properties of the derivatives as in the next video on similar lines as we did for continuity, because we cannot show.

So, in this video we have computed everything from what we call first principle. So, we took this limit and then we evaluated and so on, but as we saw for continuity as well we do some limits and after that we keep using properties, we use the rules for continuity. So, the same kind of idea we will first describe we will describe the rules for derivatives and then we will use those to compute more limits, more derivatives.

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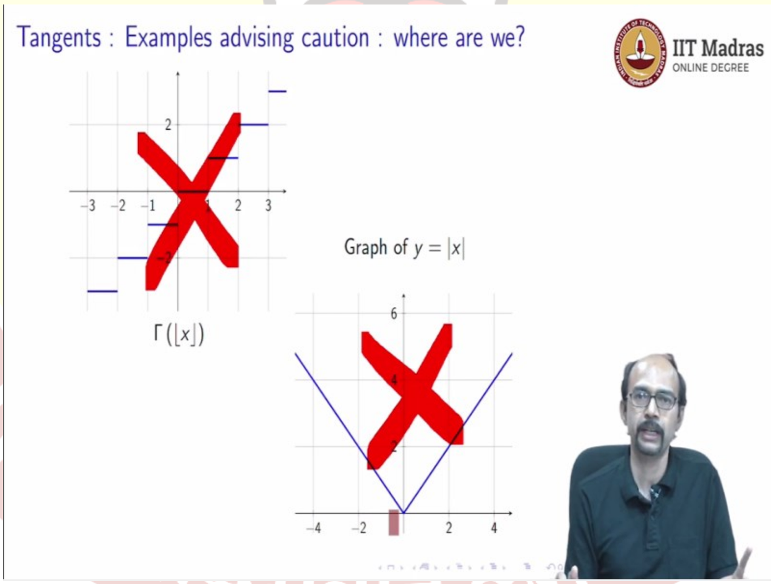

So, before we end let us look again at these examples. So, why did we do all this? We were studying tangents and we came across these two examples which gave us some pause for thought and we saw in the previous videos that if we demand that our function is continuous before we start talking about tangents, then we can get rid of this example, but this function graph of y = |x|. Here still continuity was not good enough to throw out this example.

So, now let us ask what happens for differentiability and for differentiability we saw that at 0 this function is not differentiable. So, we can get rid of this function also by making the following hypothesis. We will talk about tangents only at points of functions where those functions are differentiable. So, we will gladly talk about the tangent to this function when the point is not 0 even for |x| when the point is not 0 we will talk about tangents.

And as we saw if you remember our video about tangents, we did not have a problem. The problem was at 0. So, we will talk only about tangents for points where the function is differentiable and that is something we will study also in our next video. Thank you.