Minimum Cost Spanning Trees: Kruskal's Algorithm

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Mathematics for Data Science 1
Week 12

Minimum cost spanning tree (MCST)

- Weighted undirected graph, $G = (V, E), W : E \to \mathbb{R}$
  - G assumed to be connected
- Find a minimum cost spanning tree
  - Tree connecting all vertices in V

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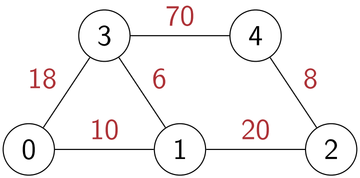

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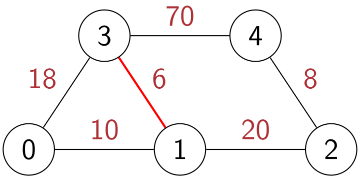

- Start with smallest edge, $(1, 3)$

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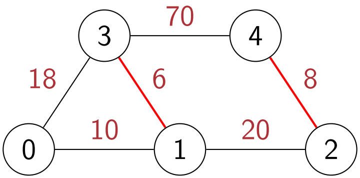

- Start with smallest edge, $(1, 3)$
- Add next smallest edge, $(2, 4)$

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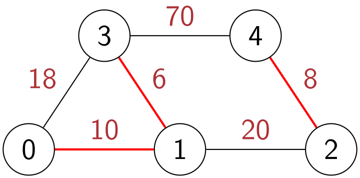

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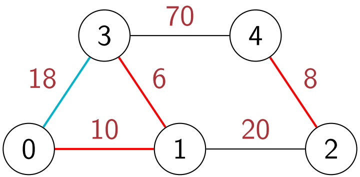

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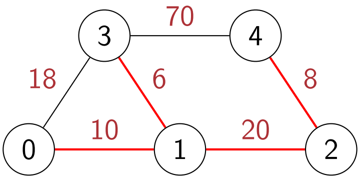

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Kruskal's algorithm

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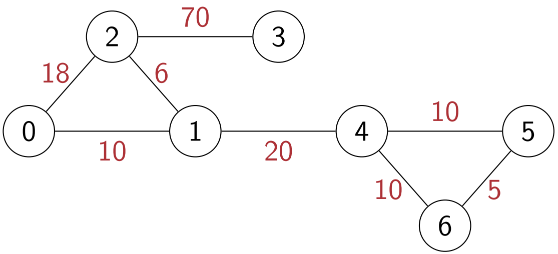

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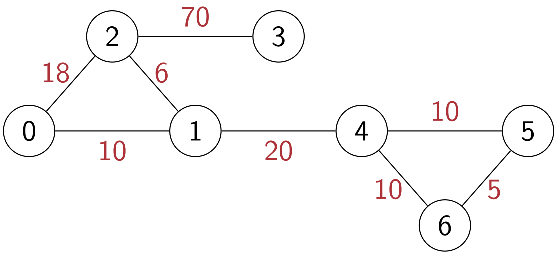

Sort E as
$\{(5,6), (1,2), (0,1), (4,5), (4,6), (0,2), (1,4), (2,3)\}$

Set $TE = \emptyset$

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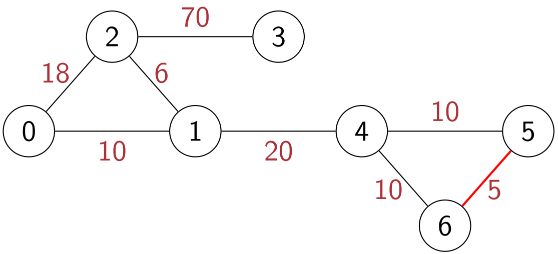

Sort E as
$\{(5,6),(1,2),(0,1),(4,5),(4,6),(0,2),(1,4),(2,3)\}$

Add $(5,6)$

Set $TE = \{(5,6)\}$

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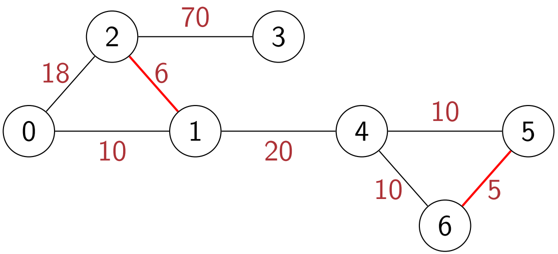

Sort E as
$\{(5,6),(1,2),(0,1),(4,5),(4,6),(0,2),(1,4),(2,3)\}$

Add $(1,2)$

Set $TE = \{(5,6),(1,2)\}$

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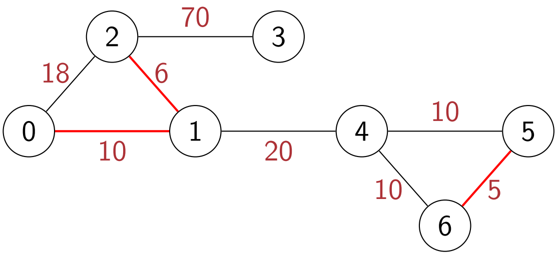

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Add $(0,1)$

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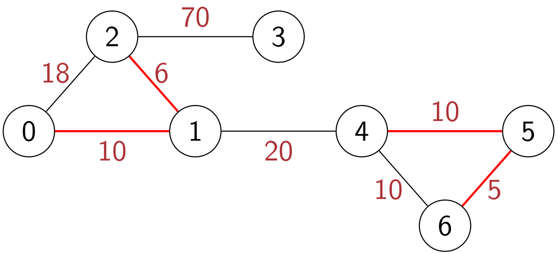

Sort E as

$\{(5,6),(1,2),(0,1),(4,5),(4,6),(0,2),(1,4),(2,3)\}$

Add $(4,5)$

Set $TE = \{(5,6),(1,2),(0,1),(4,5)\}$

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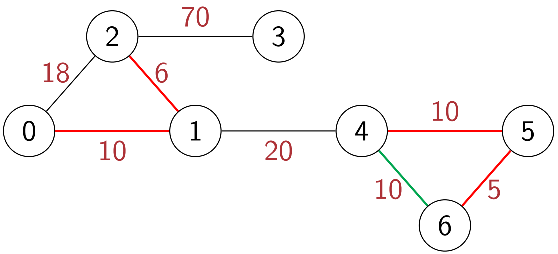

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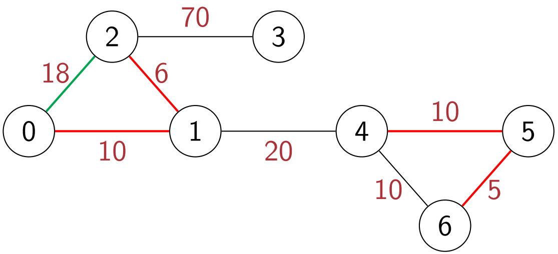

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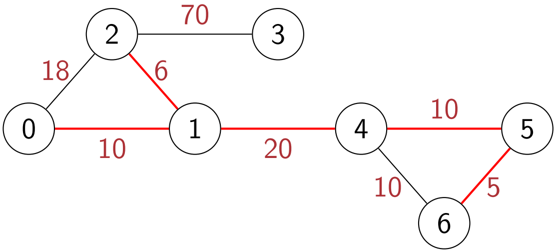

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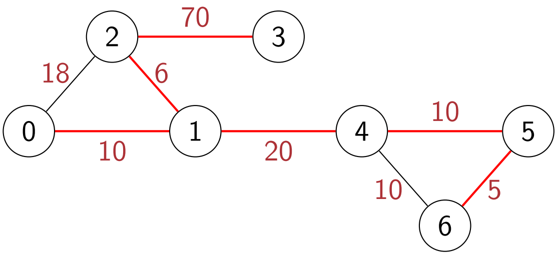

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Add $(2,3)$

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Correctness of Krusksal's algorithm

Minimum Separator Lemma

- Let V be partitioned into two non-empty sets U and $W = V \setminus U$
- Let $e = (u, w)$ be the minimum cost edge with $u \in U$, $w \in W$
- Every MCST must include e

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  - U, W form a partition of V with $u \in U$ and $w \in W$
  - Since we are scanning edges in ascending order of cost, e is minimum cost edge connecting U and W, so it must be part of any MCST

Summary

- Kruskal's algorithm builds an MCST bottom up
  - Start with n components, each an isolated vertex
  - Scan edges in ascending order of cost
  - Whenever an edge merges disjoint components, add it to the MCST
- Correctness follows from Minimum Separator Lemma

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- Different choices lead to different spanning trees
- In general, there may be a very large number of minimum cost spanning trees