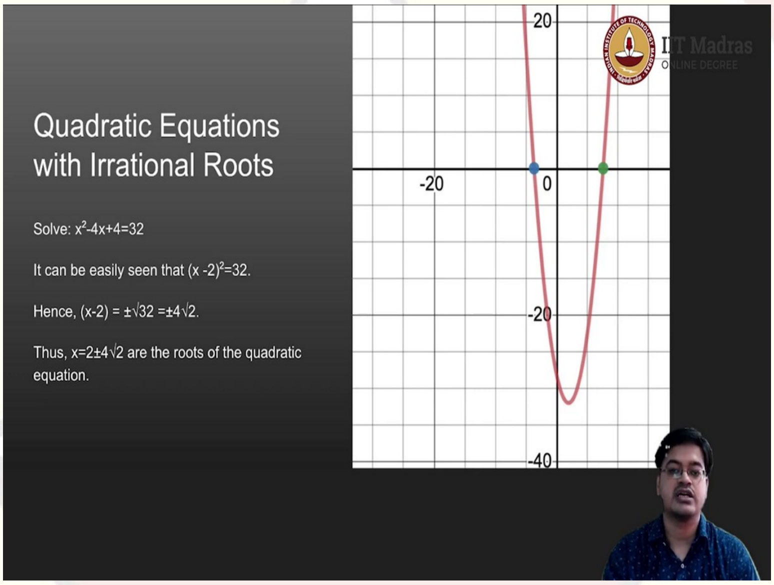
IIT Madras

ONLINE DEGREE

Mathematics for Data Science 1
Prof. Neelesh S Upadhye
Department of Mathematics
Indian Institute of Technology, Madras

Lecture – 29
Quadratic formula

(Refer Slide Time: 00:15)



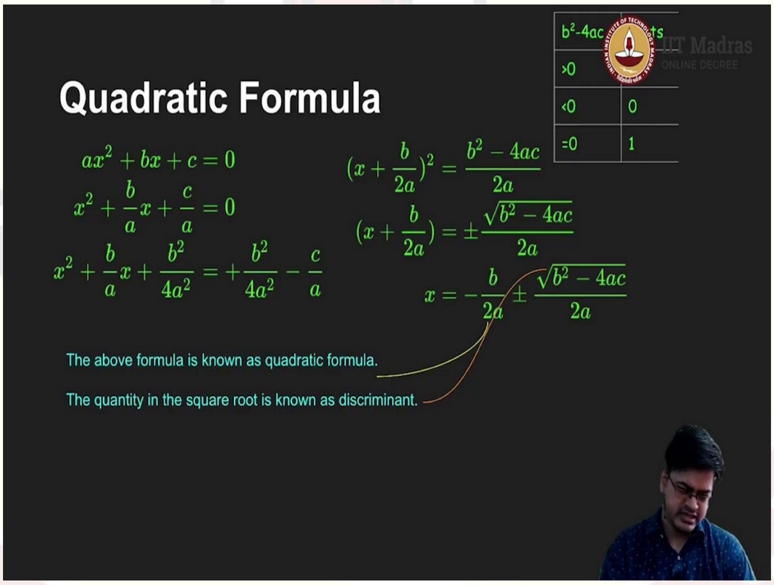
So, let us now go when the right-hand side is not a perfect square. In this case the right-hand side was a perfect square. So, when right hand side is not a perfect square, let us take this expression where; $x^2 - 4x + 4 = 32$. So, I have already done the first step, I have added it. So, you can see the left-hand side is nothing, but $(x-2)^2$ which is equal to 32 and 32 is not the perfect square.

So, in such case what will happen? So, you will equate you will go in a similar manner $(x-2)^2 = 32$, you will take a positive square root of the left-hand side and $\pm\sqrt{32}$. $\sqrt{32}$ can be decomposed into 16×2. So, $\sqrt{16}$ is 4. So, it is $\pm 4\sqrt{2}$.

So, you will get 2 two roots; $2 \pm 4\sqrt{2}$ are the roots of the quadratic equation and they are irrational roots because, $\sqrt{2}$ is an irrational number ok. It is interesting to verify this result using a graph because, that will give us the clear cut understanding where does this $2+4\sqrt{2}$ are mapped. The two green dots over here represent the location of the roots ok. So, this is how you will solve a quadratic equation using the method of completing the squares.

Now let us explore the relation between quadratic equations method finding the roots using quadratic equations. Sorry; finding the roots of quadratic equations using completing the square method and its connection with quadratic formula.

(Refer Slide Time: 02:05)



So, for this let us take a general quadratic function equated to 0 that is; $ax^2+bx+c=0$. I n is the second step is because $a \neq 0$. I can easily divide a I can easily divide by a; so that will give us the second step.

Now, as far as we can understand what we have here is, $\frac{c}{a}$ is the constant term. So, as per if we go by our method of completing the square, we will push this $\frac{c}{a}$ on the other side. So, it will take a negative sign that is what you are seeing here $\frac{-c}{a}$.

And, now, $\frac{b}{a}$ was the term $\frac{b}{a}$ was the term, if it is a complete square $\frac{b}{2a}\times 2$ will have come. So, I will get a term which is our a in the earlier expression that will be $\frac{b}{2a}$. So, $(\frac{b}{2a})^2$ will be $\frac{b^2}{4a^2}$, which is the term that I will add on both sides. So, I have added on both sides $\frac{b^2}{4a^2}$.

Now, look at this expression carefully, what is this expression? This is $(x+\frac{b}{2a})^2$ and this is some constant. So, I will use this that is; $(x+\frac{b}{2a})^2$ is equal to we can rearrange this term, $4a^2$ is the LCM. So, you just multiply by 4a over here you will get $4a^2$ there and divide by 4a so you will get $\frac{b^2-4ac}{2a}$. This is what you will get if you rearrange these terms hm. Sorry, it is wrong.

It is $\frac{b^2-4ac}{4a^2}$ it is. This is wrong. It should be $4a^2$ and then once you take the square root of this then you will get $(x+\frac{b}{2a})=\frac{\pm\sqrt{b^2-4ac}}{2a}$.

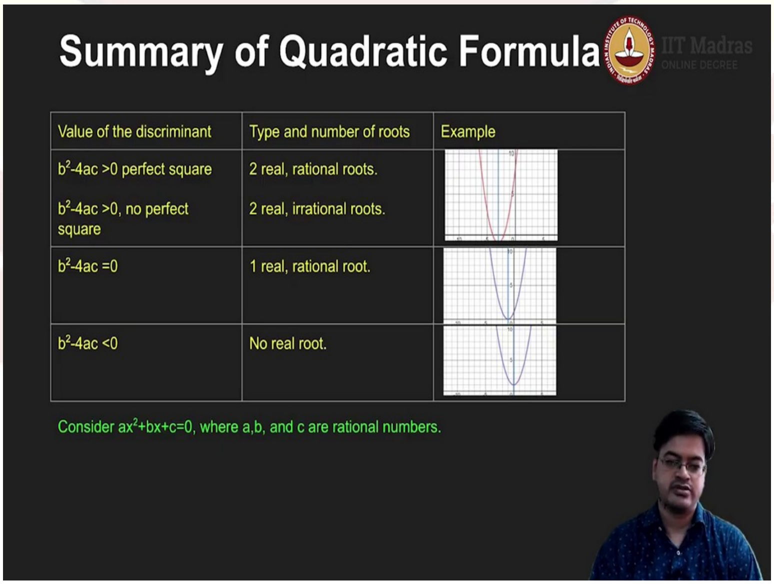
So, effectively using the same method of completing the square, the root of this equation will be $x=\frac{-b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$; as easy as this. So, this method is very powerful; this is what we have done using method of completing the squares and it gives us a general formula which is called quadratic formula. So, this formula is known as quadratic formula and the term over here in the square root is known as discriminant.

What is the quadratic formula? This complete expression on the right-hand side is a quadratic formula. And the term in the square root since b^2-4ac is called the discriminant. Why? Because it discriminates.

Let us see, if this $b^2-4ac>0$; that simply means we have two real roots. If this $b^2-4ac=0$; that means, we have only one repeated root. And if $b^2-4ac<0$, then we are actually taking square root of a negative number which will go to the complex domain. So, it has no real roots.

So, let us summarize this method or the summary of this method into a table.

(Refer Slide Time: 06:03)



So, value of the discriminant suppose $b^2 - 4ac > 0$ and the discriminant $b^2 - 4ac$ is a perfect square; that means, I know the square root of it then we have two real rational roots. If $b^2 - 4ac > 0$, but it is not a perfect square; then I have two real irrational roots.

We have already seen in week 1 that real number line is divided into rational numbers and irrational numbers. So, this is the splitting which will help. So, if $b^2 - 4ac > 0$, but not a perfect square I will get irrational number. If $b^2 - 4ac$ is a perfect square, I will get a rational number. If $b^2 - 4ac = 0$, then I will get one real rational root. And if $b^2 - 4ac < 0$, I do not have any root.

Let us demonstrate it through some graphs which we have seen which we have already seen. So, here is the example; where $b^2 - 4ac > 0$. These are the two roots of this quadratic equation which are given here.

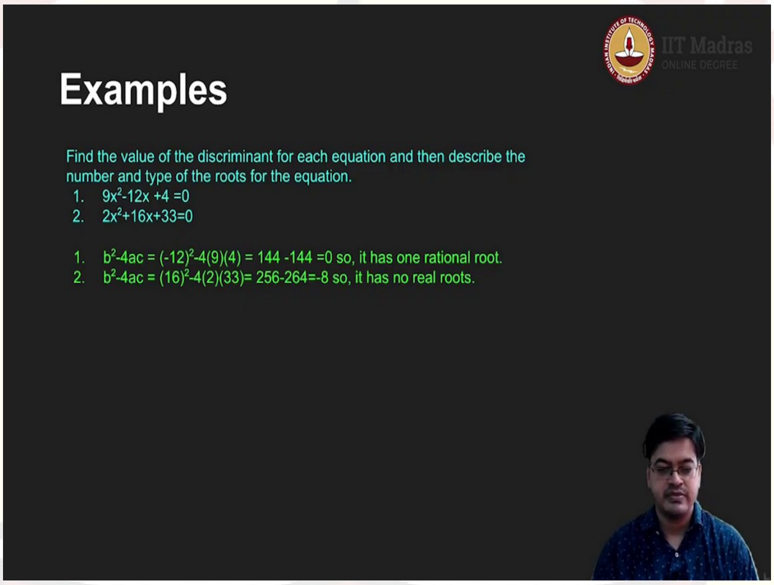
Let us say $b^2 - 4ac = 0$, this is the root only one root it has and it is repeated. And if $b^2 - 4ac < 0$, our example was $x^2 + 1$. So, in this case right it never touches the minimum; so you will get this particular expression ok. $b^2 - 4ac < 0$, it never really touches the x axis. That is the verification that we have something like this ok.

So, let us go to the next slide which is actually yeah of course, these are the conditions that are required where a, b, c are rational numbers because, I am telling that this is this is will be a rational root. So, if it is not rational then what you need to do is; you suppress this, you do not need to say anything about this.

If they are rational numbers then whatever I am saying over here is true and whatever I am saying over here is true. If they are not rational numbers still you will have two real roots, but I cannot say whether they will be rational or irrational. And you will you may have one real root, but it can be irrational also. For example, $(x - \pi)^2$ or $b^2 - 4ac = 0$ and then still you will get root as a π which is an irrational number.

So, that is so in order to distinguish between rational and irrational you need a condition that a, b, c are rational numbers. If you do not want to distinguish between rational and irrational numbers you do not need this condition. You can have a, b, c as real numbers; only condition that will prevail is $a \neq 0$.

(Refer Slide Time: 09:21)



So, let us go ahead and see some examples where I will use the discriminant formula or quadratic formula to distinguish between the roots. So, the question itself says; find the value of discriminant for each equation and then describe the number and type of roots for the equation.

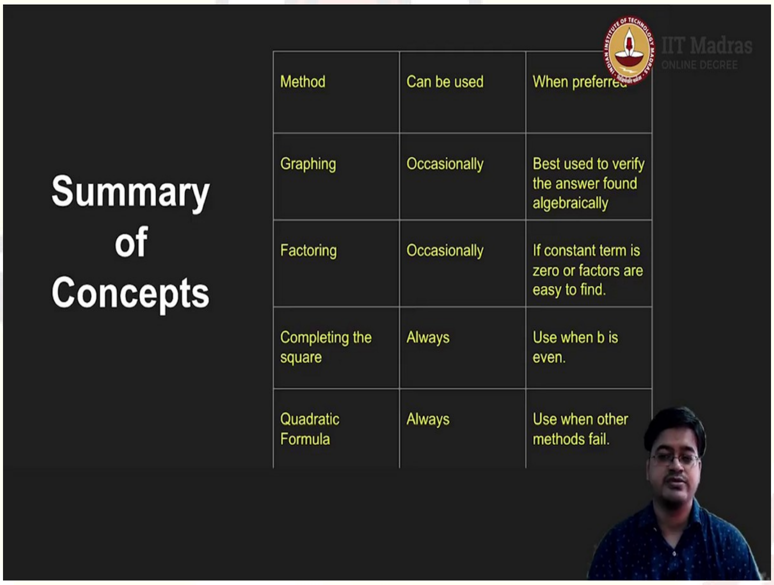
So, let us take the first example is $9x^2 - 12x + 4 = 0$. So, I want to evaluate $b^2 - 4ac$. So, it is b is -12, a is a is 9 and c is 4. So, $b^2 - 4ac$ I want to evaluate for this. In a similar manner let us take the next equation which $2x^2 + 16x + 33 = 0$; where b will be 16, a will be 2 and c will be 33.

Let us evaluate $b^2 - 4ac$ for the first equation that is; $(-12)^2 - 4 \times 9 \times 4$. So, 9 4s are 36, 36 4s are 144 and 12^2 is also 144. So, 144 - 144, if you refer to the previous table it has only one repeated rational root.

You go to the second example b^2-4ac ; b is 16, a is 2 and c is 33. So, if you look at it 16^2 is 256, 33×8 is 264 yes. So, I got 256 - 264; that means, I got -8. Therefore, the $b^2-4ac<0$, and hence it will have no real root ok. This is the summary of using the discriminant method.

The discriminant method or the quadratic formula actually gives you a number of ways to handle the problem. So, in short what we have seen today is, we can solve an equation given that I know the values of a, b and c using the quadratic formula. So, let us summarize what are all the methods that we have studied in this particular example ok.

(Refer Slide Time: 11:45)



So, the summary of concepts is; let us say I have a method which is called graphing method. This is the method which we started with. When do we use this method? The graphing method actually unless your solutions are integers will not give you a good result.

But it is a best method to verify your results or verify the results that you have actually found algebraically. If there is any calculation mistake or something it will be revealed very easily. So, the graphing method is very helpful when you want to verify the result, but you can also use it to find roots of the equation occasionally.

Similarly, factoring method also suffers from the disadvantage that; it may not be helpful if the factors are not easily visible. For example, you may get the constant term to be equal to 26.2 in the quadratic equation. The constant term is 26.2 and then, you may have tough time in visualizing the factors.

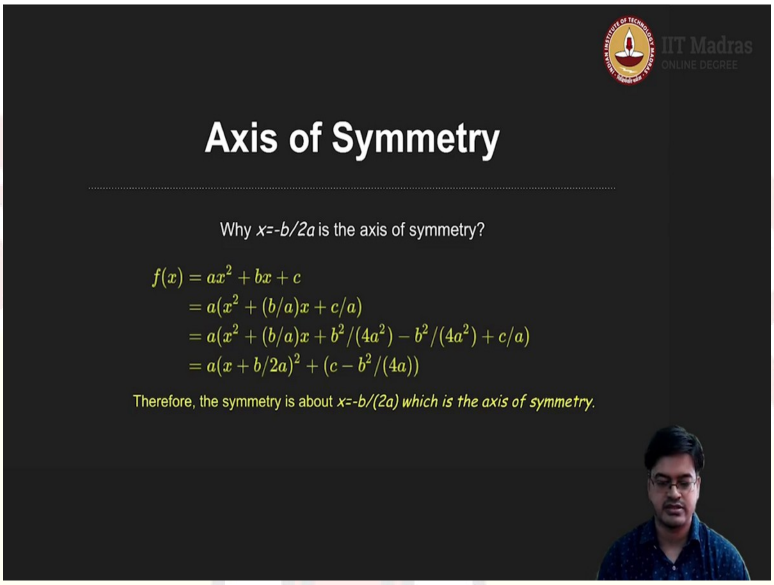
So, in such cases factoring method need not be used, but it is very helpful if the constant term is 0 or the factors are easy to find you can actually guess the factors if the nice numbers like 49, 24 all these nice numbers are there then you can very well go with factorization method.

The method of completing the squares all the time it works; it is very easy when b is even. Otherwise, you will have problems if the right-hand side in the method of completing the square is negative, that is you will go to complex domain and then you may have some problems which we are not dealing with in this course. So, for us it can be always used when b is even ok.

Now last method is quadratic formula, which is derived from which is derived from method of completing the square. And this gives you all the time this is this will give the answer all the time it is always helpful for. So, for our purposes when we are studying these methods these two methods; completing the square and quadratic formula will always give the answer irrelevant of whether the coefficients are rational numbers, irrational numbers, or some absurd numbers ok.

Now, let us go to one more concept which is called axis of symmetry. I have not given any derivation about axis of symmetry yet.

(Refer Slide Time: 14:33)



So, let us start with axis of symmetry. We already know while graphing the graphing the quadratic function it is very important to know the axis of symmetry. And we have boldly claimed at $x = \frac{-b}{2a}$ is the axis of symmetry. Now I will answer the question why $x = \frac{-b}{2a}$ is the axis of symmetry. This is an application of method of completing the square.

So, let us assume that I have been given a general quadratic function $f(x) = ax^2 + bx + c$. $a \neq 0$. So, I will pull out a common and therefore, my expression will become $a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$. Now, when I was completing the square I was throwing $\frac{c}{a}$ on the right hand side, but this time that provision is not there.

So, I will retain $\frac{c}{a}$ only thing is I will split the entire expression. So, when I split the entire expression, I will get a as it is and this expression as it is here $\frac{b}{2a}$ should come;

that means, $\frac{b^2}{4a^2}$ I will add and subtract $\frac{b^2}{4a^2}$ ok. And therefore, I will get this expression. Once I get this expression, I will recognize this term this entire term so these three terms together as $(x+\frac{b}{2a})^2+c-\frac{b^2}{4a}$ ok.

Now you look at this equation correctly. For example, this is a quadratic function written in a different form. What is this number? This is some constant ok. This is some constant and now you look at this number this is actually deciding the symmetry around x symmetry on x axis.

If you put $\frac{-b}{2a}$ as one vertical line; everything because $y=x^2$ it is symmetric about that y axis, everything will be symmetric about $x=\frac{-b}{2a}$. This expression defines the symmetry of the relation or the symmetry of the function because this is nothing, but just a constant on y axis.

Therefore, this x is equal x. So, basically you will write $(x+\frac{b}{2a})^2=0$. So, $x=\frac{-b}{2a}$ that vertical line is the axis of symmetry for this expression. So, the symmetry about $x=\frac{-b}{2a}$. Therefore, this is known as axis of symmetry that answers the quadratic equation axis of symmetry question. Why $x=\frac{-b}{2a}$ is the axis of symmetry?

This ends our topic on quadratic functions and quadratic equations.