IIT Madras

ONLINE DEGREE

Relations

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Mathematics for Data Science 1
Week 1

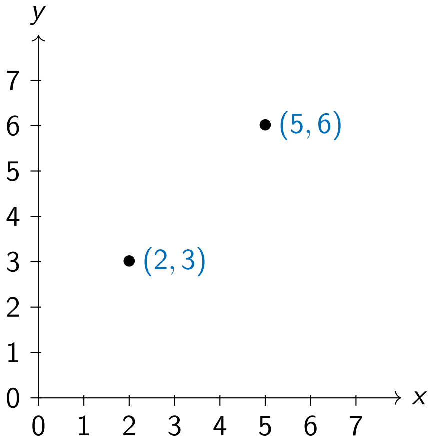
New sets from old

- A set is a collection of items
- We can combine sets to form new ones
  - $X \cup Y$, $X \cap Y$, $X \setminus Y$
  - $\overline{X}$ with respect to Y
- Define subsets using set comprehension
  - Odd integers
    $\{z \mid z \in \mathbb{Z}, z \bmod 2 = 1\}$
  - Rationals not in reduced form
    $\{p/q \mid p, q \in \mathbb{Z}, gcd(p, q) > 1\}$
  - Reals in $[3, 17)$
    $\{r \mid r \in \mathbb{R}, 3 \leq r < 17\}$

"New lamps for old"
Aladdin's Picture Book
Walter Crane (1876)

Cartesian product

- $A \times B = \{(a,b) \mid a \in A, b \in B\}$
  - Pair up elements from A and B
  - $A = \{0,1\}$, $B = \{2,3\}$
  - $A \times B = \{(0,2),(0,3),(1,2),(1,3)\}$
- In a pair, the order is important
  - $(0,1) \neq (1,0)$
- For sets of numbers, visualize product as two dimensional space
  - $\mathbb{N} \times \mathbb{N}$
  - $\mathbb{R} \times \mathbb{R}$



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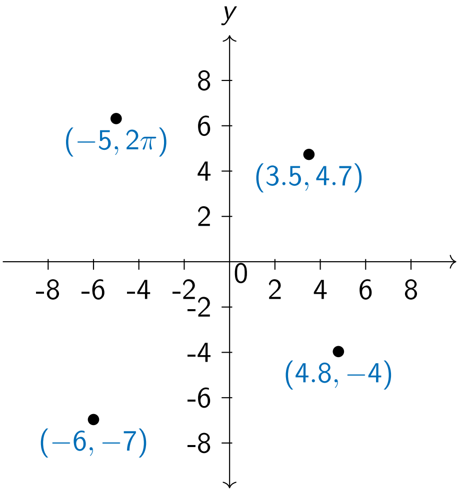
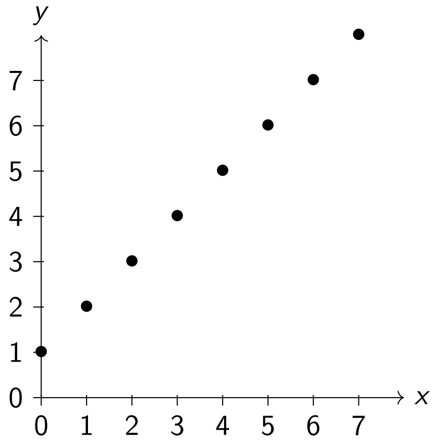


Binary relations

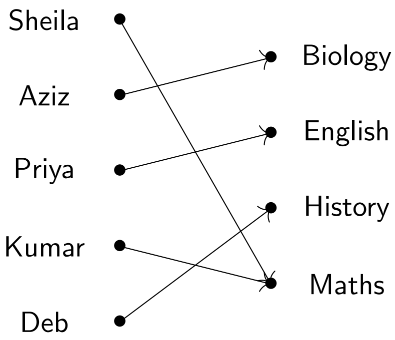
- Select some pairs from the Cartesian product
- Combine Cartesian product with set comprehension
- $\{(m,n) \mid (m,n) \in \mathbb{N} \times \mathbb{N}, n = m+1\}$
  - $\{(0,1),(1,2),(2,3),\ldots,(17,18),\ldots\}$
- Pairs (d,n) where d is a factor of n
  - $\{(d,n) \mid (d,n) \in \mathbb{N} \times \mathbb{N}, d|n\}$
  - $\{(1,1),\ldots,(2,82),\ldots,(14,56),\ldots\}$
- Binary relation $R \subseteq A \times B$
- Notation: $(a,b) \in R$, $a\ R\ b$



More relations

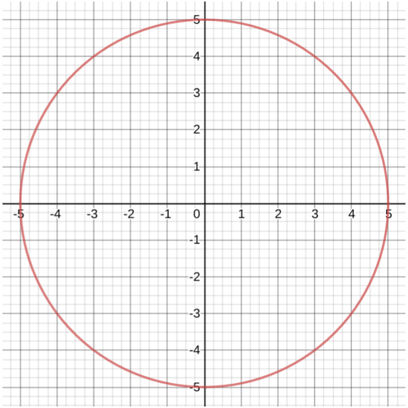
- Teachers and courses
  - T, set of teachers in a college
    C, set of courses being offered
  - $A \subseteq T \times C$ describes the allocation of teachers to courses
  - $A = \{(t, c) \mid (t, c) \in T \times C, t \text{ teaches } c\}$
- Mother and child
  - P, set of people in a country
  - $M \subseteq P \times P$ relates mothers to children
  - $M = \{(m, c) \mid \ (m, c) \in P \times P,$
    $m \text{ is the mother of } c\}$

A relation as a graph



More relations

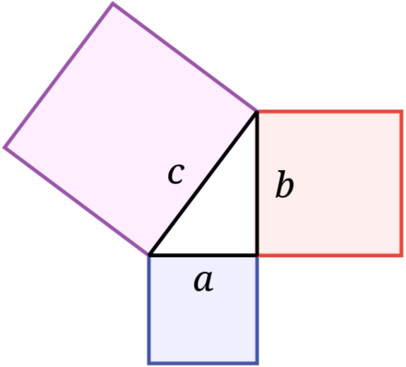
- Points at distance 5 from $(0,0)$
  - Distance from $(0,0)$ to (a,b) is $\sqrt{a^2+b^2}$
  - $\{(a,b) \mid (a,b) \in \mathbb{R} \times \mathbb{R}, \sqrt{a^2+b^2} = 5\}$
  - $\{(0,5),(5,0),(3,4),(-3,-4),\ldots\}$
  - A circle with centre at $(0,0)$



- Rationals in reduced form
  - A subset of Q
    - $\{p/q \mid (p,q) \in \mathbb{Z} \times \mathbb{Z}, gcd(p,q) = 1\}$
  - ... but also a relation on $\mathbb{Z} \times \mathbb{Z}$
    - $\{(p,q) \mid (p,q) \in \mathbb{Z} \times \mathbb{Z}, gcd(p,q) = 1\}$

Beyond binary relations

- Cartesian products of more than two sets
- Pythagorean triples
  - Square on the hypotenuse is the sum of the squares on the opposite sides
  - $\{(a, b, c) \mid (a, b, c) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N},$
    $a, b, c > 0, a^2 + b^2 = c^2\}$



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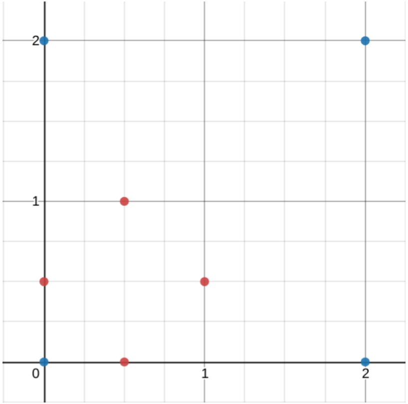
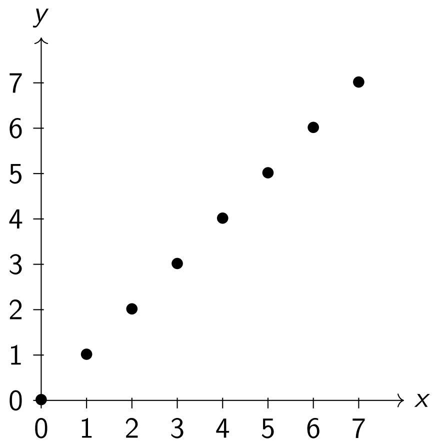
- Corners of squares
  - A corner is a point $(x,y) \in \mathbb{R} \times \mathbb{R}$
  - $((x_1,y_1),(x_2,y_2),(x_3,y_3),(x_4,y_4))$ are related if they are four corners of a square
  - For instance:
    - $((0,0),(0,2),(2,2),(2,0))$
    - $((0,5,0),(0,0.5),(0.5,1),(1,0.5))$
  - $Sq \subset \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$



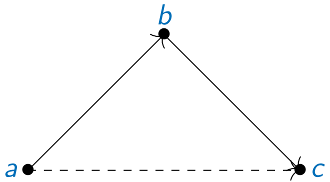
Back to binary relations

- Identity relation $I \subseteq A \times A$
  - $I = \{(a,b) \mid (a,b) \in A \times A, a = b\}$
  - $I = \{(a,a) \mid (a,a) \in A \times A\}$
  - $I = \{(a,a) \mid a \in A\}$
- Reflexive relations
  - $R \subseteq A \times A$, $I \subseteq R$
  - $\{(a,b) \mid (a,b) \in \mathbb{N} \times \mathbb{N}, a, b > 0, a|b\}$
    - $a|a$ for all $a > 0$
- Symmetric relations
  - $(a,b) \in R$ if and only if $(b,a) \in R$
  - $\{(a,b) \mid (a,b) \in \mathbb{N} \times \mathbb{N}, gcd(a,b) = 1\}$
  - $\{(a,b) \mid (a,b) \in \mathbb{N} \times \mathbb{N}, |a - b| = 2\}$



Back to binary relations ...

- Transitive relations
  - If $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$
  - $\{(a, b) \mid (a, b) \in \mathbb{N} \times \mathbb{N}, a|b\}$
    - If $a|b$ and $b|c$ then $a|c$
  - $\{(a, b) \mid (a, b) \in \mathbb{R} \times \mathbb{R}, a < b\}$
    - If $a < b$ and $b < c$ then $a < c$



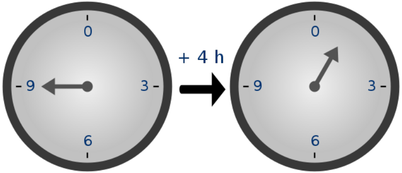
- Antisymmetric relations
  - If $(a, b) \in R$ and $a \neq b$, then $(b, a) \notin R$
  - $\{(a, b) \mid (a, b) \in \mathbb{R} \times \mathbb{R}, a < b\}$
    - If $a < b$ then $b \not< a$
  - $M \subseteq P \times P$ relates mothers to children
    - If $(p, c) \in M$ then $(c, p) \notin M$

Equivalence relations

- Reflexive, symmetric and transitive
- Same remainder modulo 5
  - $7 \bmod 5 = 2$, $22 \bmod 5 = 2$
  - If $a \bmod 5 = b \bmod 5$ then $(b - a)$ is a multiple of 5
  - $\mathbb{Z}Mod5 = \{(a, b) \mid a, b \in \mathbb{Z}, (b - a) \bmod 5 = 0\}$
  - Divides integers into 5 groups based on remainder when divided by 5
- An equivalence relation partitions a set
- Groups of equivalent elements are called equivalence classes

Measuring time

Clock displays hours modulo 12



2:00 am is equivalent to 2:00 pm

Summary

- Cartesian products generate n-tuples from n sets
  - $(x_1, x_2, \ldots, x_n) \in X_1 \times X_2 \times \cdots \times X_n$
- A relation picks out a subset of a Cartesian product
  - $\{(m, r) \mid (m, r) \in \mathbb{N} \times \mathbb{R}, r = \sqrt{m}\}$
- Properties of relations
  - Reflexive, symmetric, transitive, antisymmetric
- Equivalence relations partition a set