INDIAN INSTITUTE OF TECHNOLOGY MADRAS

सिद्धिर्भवति कर्मजा

IIT Madras

ONLINE DEGREE

Mathematics for Data Science 1
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Lecture – 8.1
One-to-One Function: Definition & Tests

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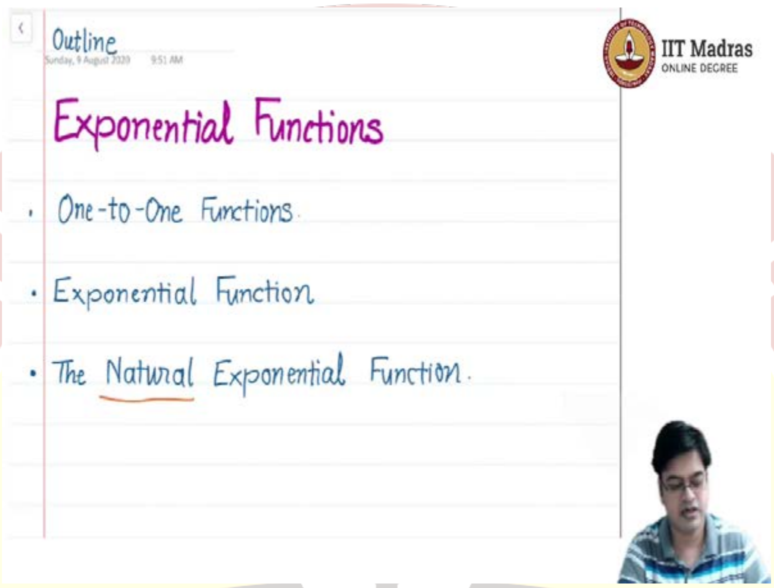

Hello students. Today, we are going to start a new unit called exponential functions and logarithmic functions. In that our goal today is to understand exponential functions. For studying exponential functions, a concept of a One-to-One Function is very much relevant.

So, we will first study that concept. Then, we will go to a definition of exponential function and properties of exponential functions as we studied for polynomial and straight line case. Then, we will define one interesting function which is called the natural exponential function, and I will justify why it is called natural and what are the properties that it shares with the mathematics world.

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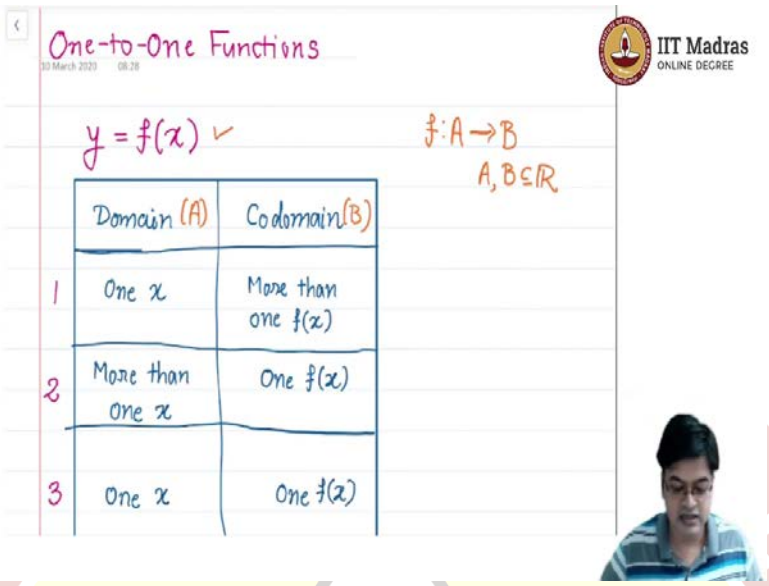

So, let us start and I think it is time to start and let us start. So, first let us go to a concept of one-to-one function. In order to define the concept of one-to-one function, let us quickly recall, what is a concept of a function? So, whenever I talk about function, I will talk about $y = f(x)$. This f when I say I need some domain let us say A and I need some co-domain B.

In general function can be defined on any two sets, but for us let us consider to simplify our understanding. So, that we will understand in terms of coordinate plane A and B to be subsets of real line, ok. So, now my domain according to this definition is A, and co-domain is B. So, now, what is the function? Function is a relation between one set to the other set. So, it is a mapping that assigns values from one set to the other set.

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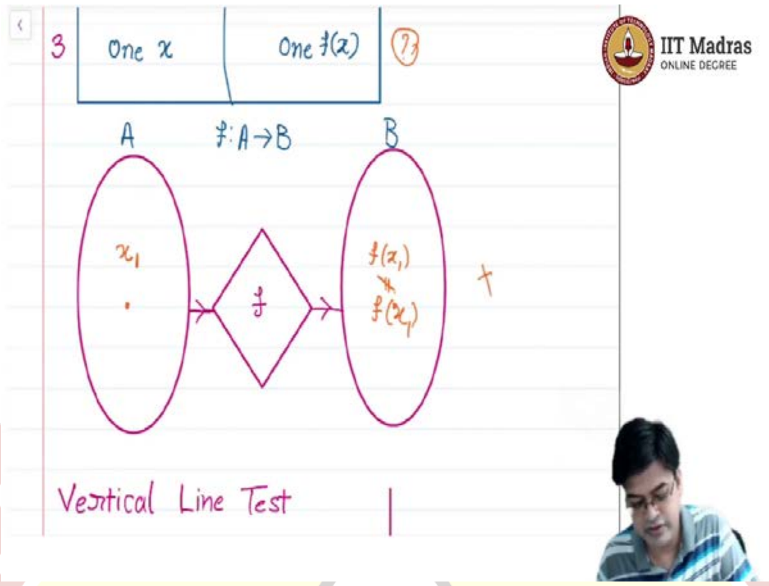

Let us describe this function as something like this, ok. So, let us see this is the set A which we will call as domain, and this is the; this is the set B that we will call as co-domain. This is A this is B. What does the function do? If we feed some value of x from this set, it will process the value and spit out or give us $f(x)$.

It is like a popcorn machine; you are feeding in the corn and getting out the popcorn. So, this is how the function works. Now, let us see, what are the cases that can happen? Suppose, for same value we are getting different outputs that, is the case, when you consider one x more than one $f(x)$, ok.

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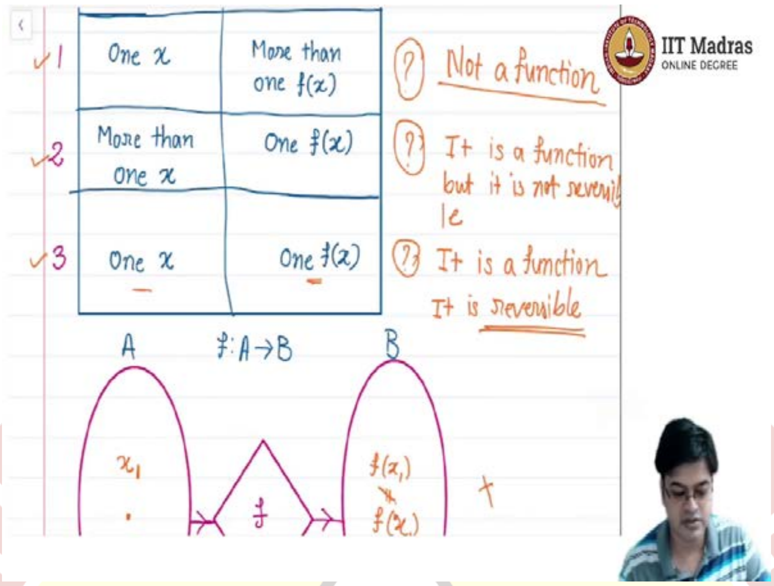

So, now, is this a function is a question mark. We will soon answer the question. Then, it can so happen that you have fed two different values and you are getting the same output, ok. Is this a function? We will answer the question. And, for every single output that you produce, there is a unique output that is produced by f, that is a one $f(x)$ only.

So, let us analyze these three again is this a function this. So, let us analyze all these three things together. So, let us start with the first case that is, one x more than one $f(x)$. The, what do I mean by one x more than one $f(x)$. Suppose, I have put x_1. If I put x_1 as a value if ones gave me $f(x_1)$, other time it gave me $f(x_1)$ which are not equal.

Is this a function? Is this a well-behaved function? It is not. So, I will say no, no this is not a function. Therefore, the first thing the first case I will say is not a function. Because, we are dealing with coordinate plane, let us understand this function with the coordinate plane.

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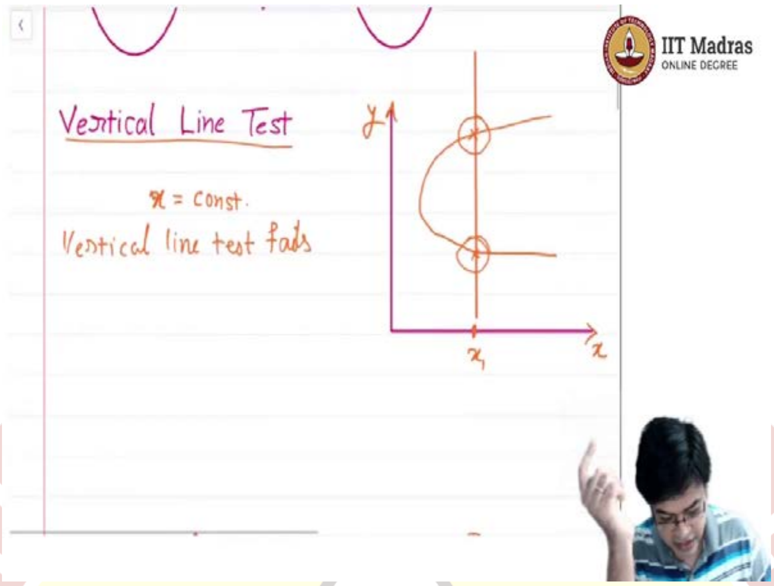

So, what happens here is suppose, I have been I am taking the value one value x_1 on x −axis. Let us say this is x −axis, let us say this is y −axis. And now, I am taking one value of x_1. So, one I once I fed in I got something which is $f(x_1)$ here. And, other time I fed in I got something which is $f(x_1)$ here, ok. Now, what is this? This is actually while if at all some curve I have to draw, it can be like this.

And, what happens here is for same value of x you are getting two different values. Then, your function is not well behaved, because I do not know which value will come when I fed in x_1. So, in that case how will you identify whether something is a function or not? Sometimes, if you have a graph of a function the way it is given; it is very easy to see; what is not a function.

For example, if you take a line which is $x =$ constant. If you take your line which is $x =$ constant and if I draw that line vertically. For example, let us say here this is the line $x =$ constant; $x = x_1$ in fact I have drawn. So, if I draw this line vertically, then I can see that there are two points at which, this line intersects the graph of a function.

When such a thing happens, we say a vertical line test that is, this is the vertical line right it is a parallel to y −axis. Therefore, this line passes through two points; that means, there is something fishy about the function and we will use this as a vertical line test. And say that, because vertical line test fails, this particular function is not a function sorry, vertical line test fails and hence, this cannot be a function.

Can you imagine such a function where can you imagine such a relation where this is not a function. For example, ok. Let us not imagine right now. Let us come to the next case and generalize it to the other set up. So, now let us see here. So, first thing is not a function that is very clear. Now, let us take the second case which is more than one x and one $f(x)$, ok.

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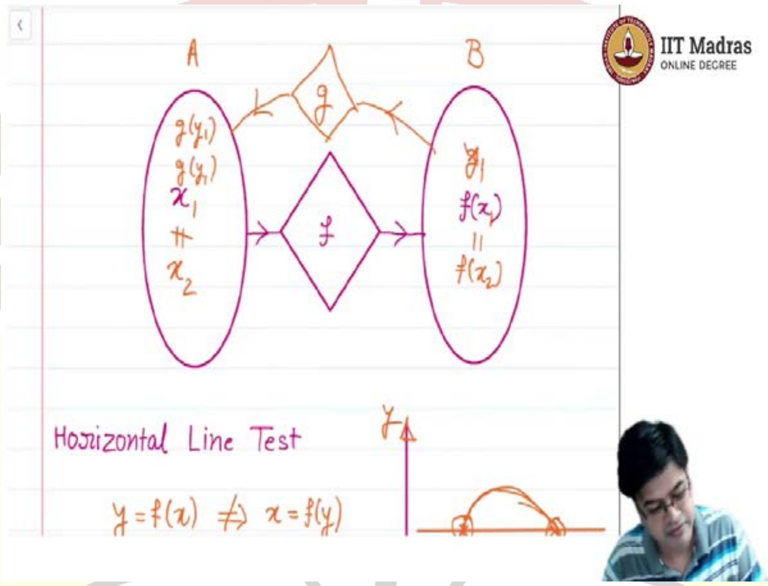

So, let us understand it on a paper. So, they are more than one x, x_1 and x_2, both are not equal, but somehow, they give the output which is equal, ok. In such case, what happens? So, as usual general this is domain set A, this is a co-domain B; f is processing unit and I have processed I have given I have fed in two different values of exercise, but the output produced is same.

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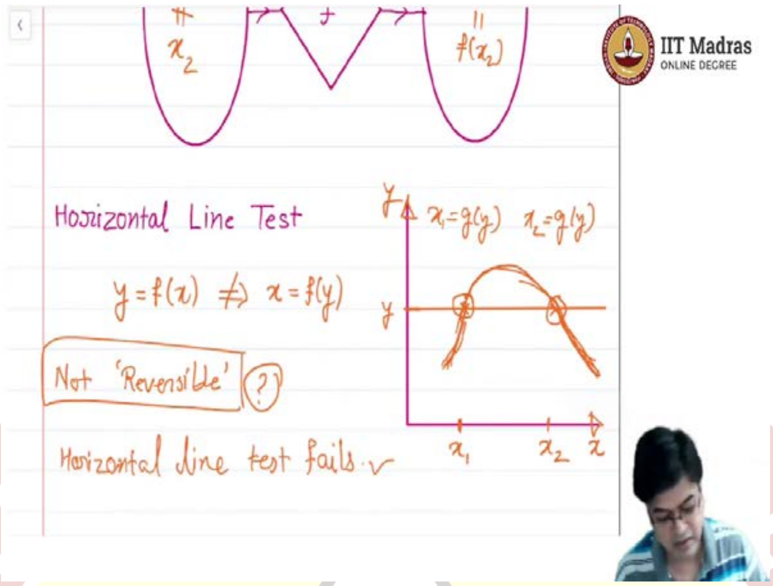

So, is this a function or not? The answer is this is a function. And, in this case let us see, how it happens? So, let us have some understanding about x_1 and x_2. So, naturally $x_1 \neq x_2$ and I got $f(x_1)$ which is this and I got $f(x_2)$which is this, ok. Both are same.

And, then how will the function look like? I can join a curve like this, where the function actually passes through these two points and I have a curve like this. So, if this is the curve then these two points are same. And, do you call this as a function? Yes, we call this as a function. Based on our understanding of the function lecture we call this as a function. In this case, something interesting has happened. Let me analyze it in a more thorough manner.

For example, when I considered the first case let me go to the first case. I have drawn a vertical line and I said that, because of this vertical line I can say this is not a function; I can say this is not a function. Now, the similar graph has appeared over here, but now if I draw a horizontal line I have a horizontal line, which passes through two points and I am saying it is a function, correct?

So, if I rotate this graph by say 90 degrees and flip it over then, what I am getting is a graph similar to this function. So, this actually helps me in understanding that, if I want to write y as a function of x, I am able to write it. But, if I want to write x as a function of y that is simply just flip this by rotate this by 90 degrees and flip the y axis. That will give you the exact understanding of the picture. And, from this to this I cannot go, ok. And therefore, the horizontal line if I draw it passes through more than one points.

So, what will happen is; suppose, if I take; if I take x_1 here, if I take a point in this domain y_1. And, if I try to setup another processing unit let us say g, and if I feed that $y_1$into g, what I get if I use this f, I will not get something which is similar that is I will get something called $g(y_1)$ once. And, if I feed in again y_1, I will get something else as $g(y_1)$. That is what is happening.

For example, here if you locate this point, this point it is say y. Then, once you feed y into g then, $g(y)$ you will get as x_1 and if you feed it other time $g(y)$, you will get as x_2. Something interesting happens. What I am trying to do is, I am trying to reverse the function and I an easily say that this function is not reversible.

If this is a interesting point which will help us in gaining more understanding of exponential functions. So, if this function is not reversible and a horizontal line test actually detects whether a function is reversible or not; so, this horizontal line test fails; horizontal line test actually fails in this case.

So, what happens here? Here, when you applied vertical line test that is case 1, it was not at all a function. Here it is a function, but our conclusion is it is not reversible. Let us look at this 3rd case, where only one x is there and one $f(x)$ is there. So, here is x, $f(x)$.

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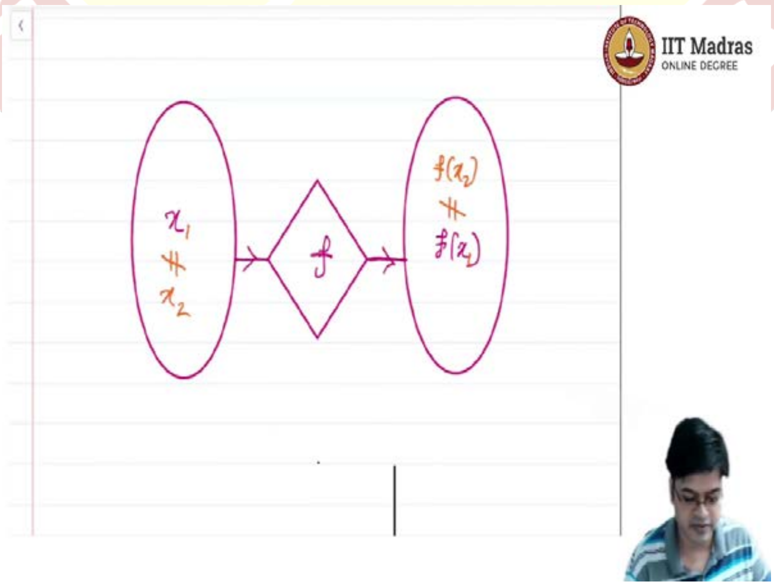

So, if I substitute x_1 and x_2 as two different values, then I will get through this processing unit, I will get $f(x_1)$and I will get $f(x_2)$ and both of them will not be equal to each other for $x_1 \neq x_2$. That is the only way it can happen right; one x, one x to one $f(x)$. So, if I take different $f(x)$ I will get different values, ok.

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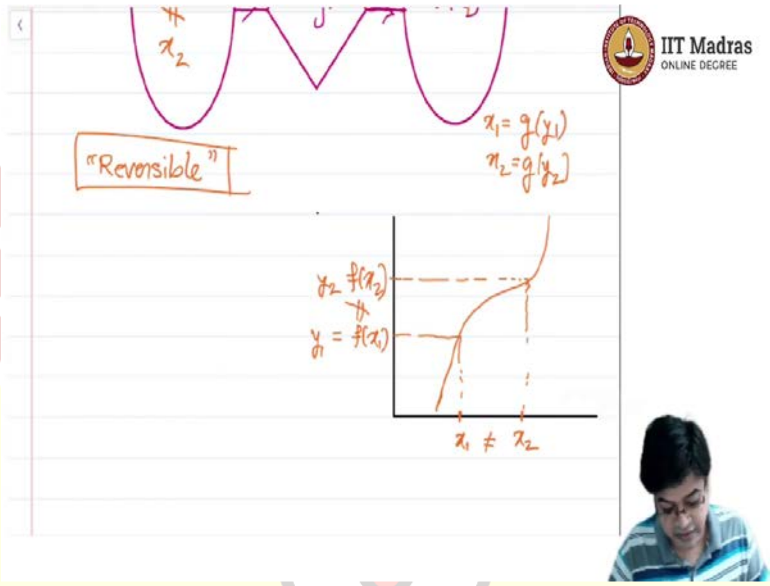

So, what will be a typical behavior of such functions? Let us try to figure out. Let us say this is the; this is one function. So, is this function one-to-one? The answer is yes, because if I take x_1 x_2 here this is $f(x_1)$, this is $f(x_2)$. So, for every one, if $x_1 \neq x_2$, $f(x_1)$ is not going to be equal to $f(x_2)$. Such function is called one-to-one function. And is it a function? It is a properly defined function, yes.

So, now, I can summarize using this as, this is a function. In earlier case, we actually characterized whether it is reversible or not. What can you say about this new function that you have defined? This function; so, now because for $x_1 \neq x_2$, $f(x_1) \neq f(x_2)$. So, if I start this as y_1 and if I start with this as y_2, I can easily retrace back x_1 x_2 that is g there will be some function g such that, $x_1 = g(y_1)$ and $x_2 = g(y_2)$.

So, this function in some sense is actually reversible. We will come to this point in more detail in the next section. Right now, remember that the function that is one-to-one is reversible and it is reversible. We need to be more precise and I will give you a word for reversible function in the later lectures. But right now, it is an important observation that there are three cases. These three cases actually deal with a function.

The first one is not a function; first one is not a function, why? Because we had dealing with the coordinate plane. The vertical line test fails when then you pass a vertical line that vertical line will pass through two points or more than two points, then it is not a function.

So, then we said the second case. The second case was more than one x and one $f(x)$. In this case, what happened is we were able to find a function properly, but we were not able to revert the function. So, we said the function is not reversible. And, on what basis we have said? We have said on the basis of horizontal test, ok. And, we have related this with our first case that is, if the function is not reversible then the other part that is g is not also is not a function as well.

Third case where we have one x and one $f(x)$ we showed that it is a function and it is reversible also. This brings us to a question that, one-to-one functions how easy are they to identify; how easy are they to identify? So, let us properly define a one-to-one function. One in the domain, one in the co-domain and there is a clear cut association that is given by two.

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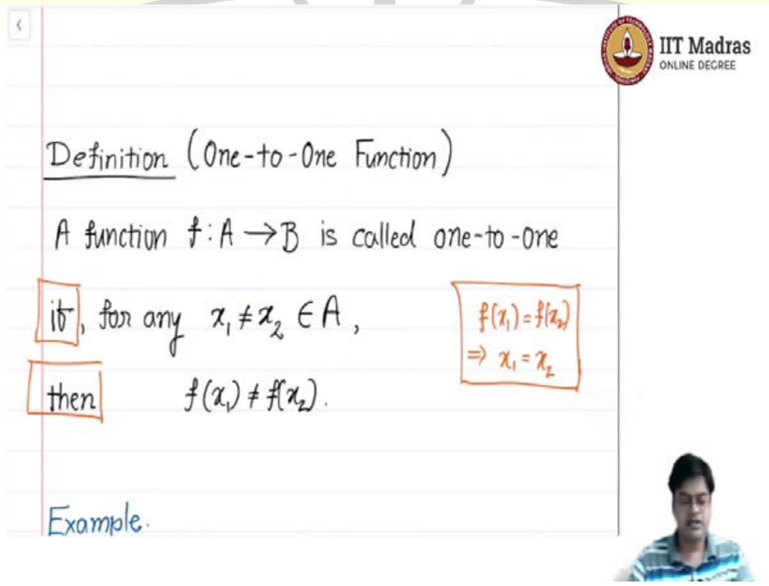

So, our definition; give a function f is said to be one-to-one if, for every $x_1 \neq x_2$ which belong to the domain of a function, $f(x_1) \neq f(x_2)$, ok. The other interpretation is, if $f(x_1) = f(x_2)$, this should imply $x_1 = x_2$ this is the other interpretation of definition, but we will use this as a definition. But, sometimes it may be difficult to prove this thing in that case, you can prove the one written in the orange.