

# Derivatives, tangents and linear approximation

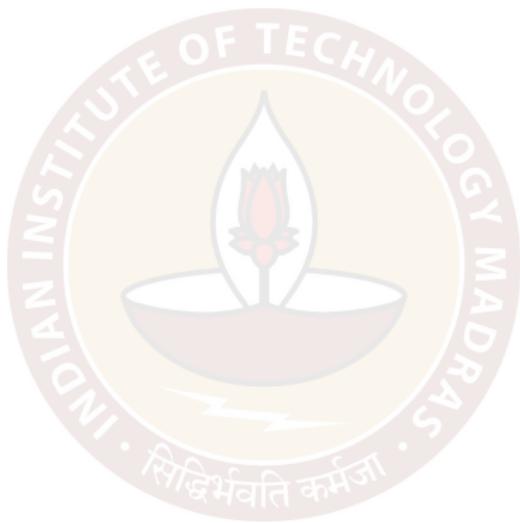


# Recall



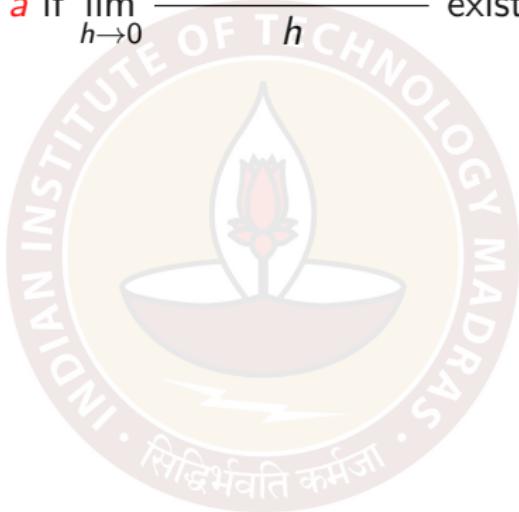
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Traditionally, the **tangent** to  $f(x)$  at  $a$  is thought of as a line which *just touches*  $\Gamma(f)$  at  $(a, f(a))$ .

# Tangents as limits of secants



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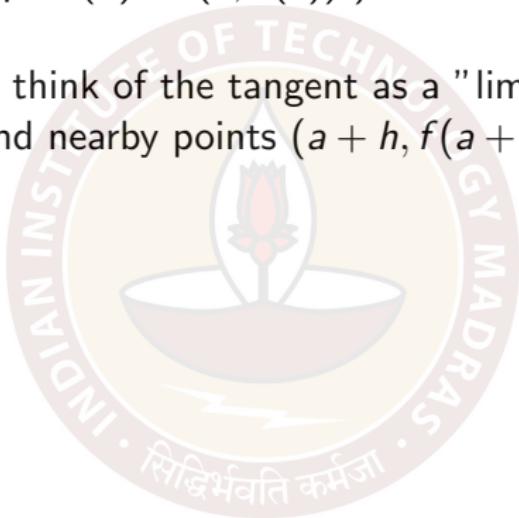
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# Tangents as limits of secants

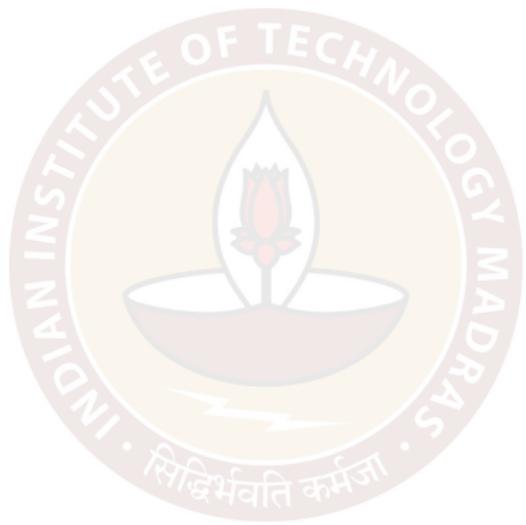
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$$y - f(a) = \frac{f(a+h) - f(a)}{a+h - a} (x - a) = \frac{f(a+h) - f(a)}{h} (x - a)$$

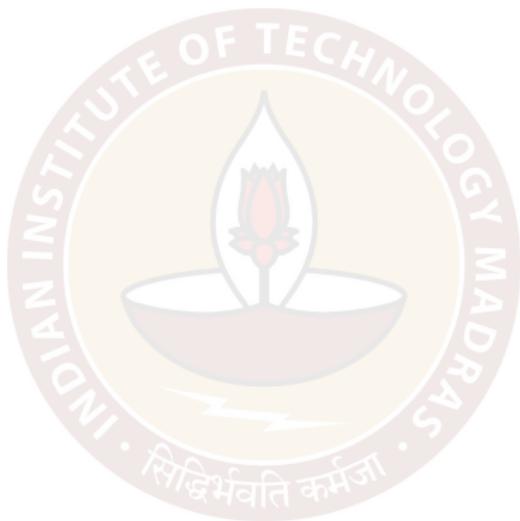
is the equation of the secant.  
what happens in the limit to this  
equation?

# Tangents and derivatives



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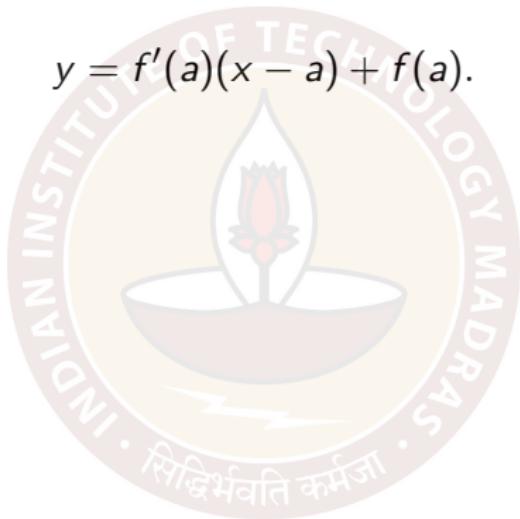
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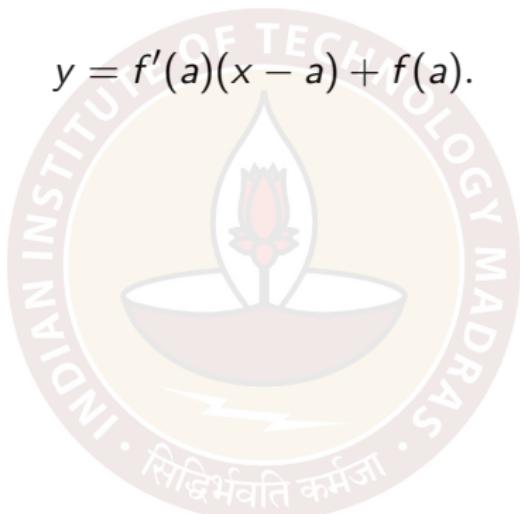
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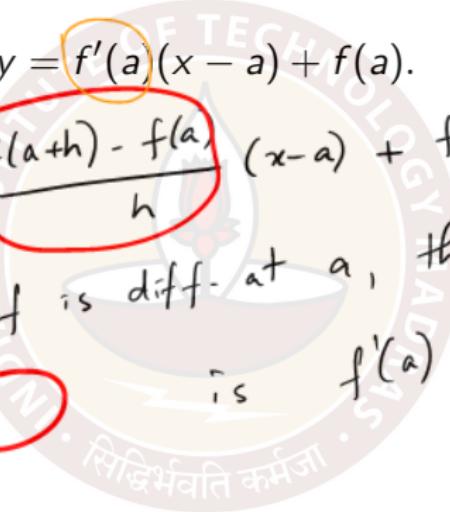
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$y = \frac{f(a+h) - f(a)}{h} (x-a) + f(a)$

If  $f$  is diff. at  $a$ , then the limit of  $\frac{f(a+h) - f(a)}{h}$  is  $f'(a)$ .



Suppose the tangent to  $f$  at  $a$  exists and is not vertical (i.e. is not the line  $x = a$ ). Then  $f$  is differentiable at  $a$  and the equation of the tangent is

$$y = f'(a)(x - a) + f(a).$$

# Examples



## Examples

$$f(x) = 5x^3 - 17x^2 + \pi x - 0.5 ; a = 0.$$

$$f'(x) = 15x^2 - 34x + \pi.$$

Eqn. of tangent to  $f$  at  $0$  is  
 $y = \pi(x - 0) + f(0) = \pi x - 0.5.$

$$f(x) = \cos(x) ; a = \frac{\pi}{3}$$

$$f'(x) = -\sin(x)$$
$$f'(\frac{\pi}{3}) = -\sin(\frac{\pi}{3}) = -\frac{\sqrt{3}}{2}$$
$$y = -\frac{\sqrt{3}}{2}(x - \frac{\pi}{3}) + \cos(\frac{\pi}{3})$$
$$= -\frac{\sqrt{3}}{2}(x - \frac{\pi}{3}) + \frac{1}{2}$$

$$f(x) = x \tan(x) ; a = \frac{\pi}{4}$$

$$f'(x) = 1 \times \tan(x) + x \times \sec^2(x)$$
$$= \tan x + x \sec^2(x)$$
$$f'(\frac{\pi}{4}) = \tan(\frac{\pi}{4}) + \frac{\pi}{4} \sec^2(\frac{\pi}{4})$$
$$= 1 + \frac{\pi}{4} \times 2 = 1 + \frac{\pi}{2}$$
$$y = (1 + \frac{\pi}{2})(x - \frac{\pi}{4}) + \frac{\pi}{4}$$

Example :  $f(x) = x^{\frac{1}{3}}$

$$f(x) = x^a$$
$$f'(x) = a x^{a-1}.$$

$$\lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h}$$
$$= \lim_{h \rightarrow 0} \frac{h^{\frac{1}{3}} - 0}{h}$$
$$= \lim_{h \rightarrow 0} \frac{h^{\frac{1}{3}}}{h}$$

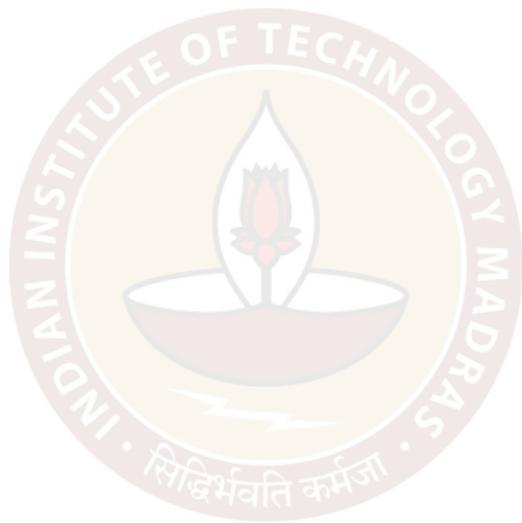
$\lim_{h \rightarrow 0} \frac{1}{h^{\frac{2}{3}}}$   
diverges to  $\infty$ .  
 $\therefore$  This limit DNE.

If  $x \neq 0$

$$f'(x) = \frac{1}{3} x^{-\frac{2}{3}}$$
$$= \frac{1}{3} x^{-\frac{2}{3}}$$
$$= \frac{1}{3} x^{\frac{-2}{3}}$$

$y = mx + c$  works only for  
lines which are not vertical.  
 $x = 0$ .

# Linear approximation



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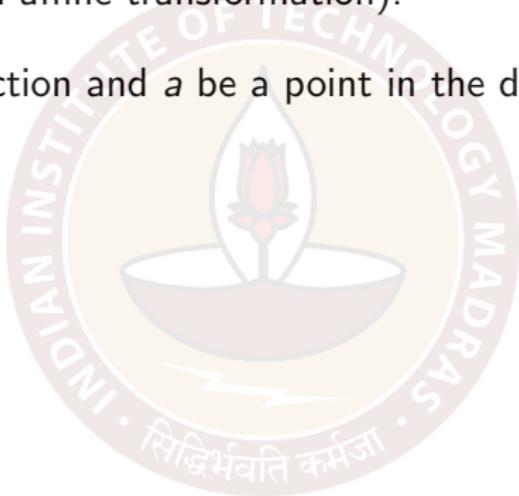
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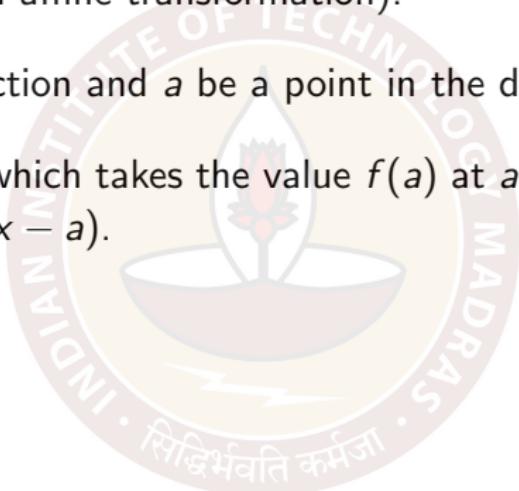


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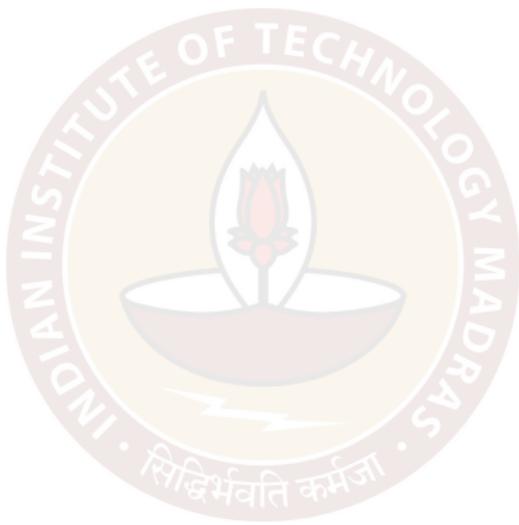
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i.e.  $f(x) \approx L(x) \quad \forall x$  close to  $a$ .

# Linear approximation (contd.)



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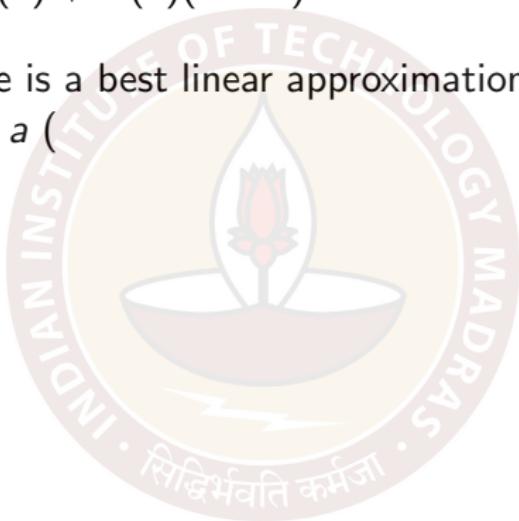
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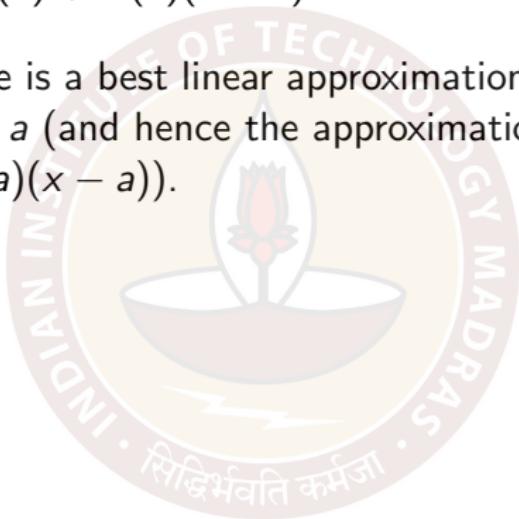
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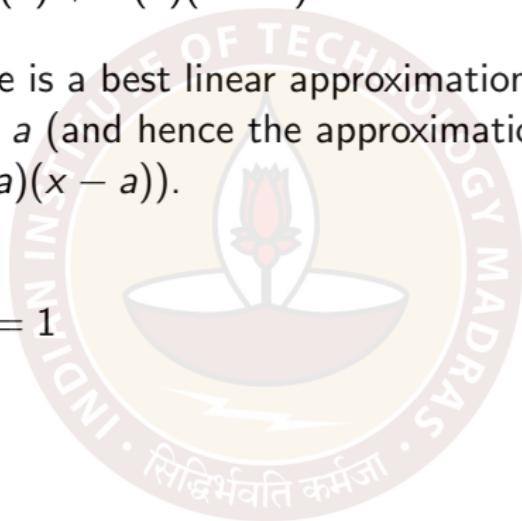
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Examples :

$$f(x) = x^3 \quad ; \quad a = 1$$
$$f'(1) = 3x^2 \Big|_{x=1} = 3.$$
$$L(x) = 3(x-1) + 1 = 3x - 2.$$

$$f(x) = \sec(x) \quad ; \quad a = 0$$

$$f'(0) = \tan(0) \sec(0) = 0.$$
$$L(x) = 0(x-0) + \sec(0)$$
$$= 1.$$

# Thank you

