INDIAN INSTITUTE OF TECHNOLOGY MADRAS

सिद्धिर्भवति कर्मजा

IIT Madras

ONLINE DEGREE

Mathematics for Data Science 1
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Lecture No. 58
Logarithmic Equations

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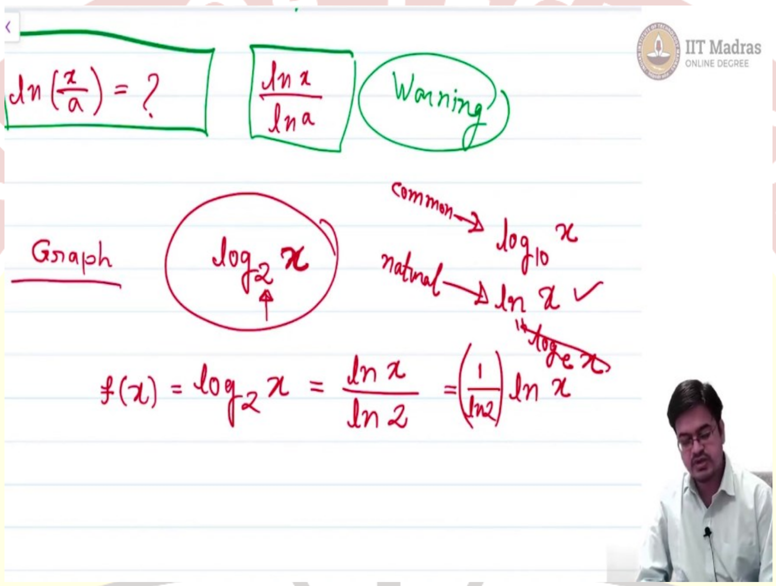

Hello students, in this video what we are going to do is, we are trying to look at the logarithmic function and how to solve equations using logarithmic functions. So, this is the goal of this particular lecture, so let us first get a simple πcture, we have already seen how to plot a logarithmic function. Suppose now you have been asked to plot a graph of a function which is $\log_2 x$.

If you have a graphing calculator, sometimes the old version of graphing calculators do not allow $\log_2 x$ to be taken and in that case what you have is either $\log_{10} x$or $\log_e x$which we have denoted by $\ln x$, this is not e, $\ln x$. So, only these two things are available to you and you can plot these two things, then can you plot the function log to the base 2 of x? This is what the first exercise that we will do when we try to solve the problems, so f(x) is log to the base 2 of x.

Now using the change of base formula which we derived in the last class, you can easily convert this function × a function with log to the base 10 or log to the base e. For convenience I will choose log to the base e. So in this case, I can simply convert using my change of base formula, this is $\frac{\ln x}{\ln 2}$ 2. That simply means, what I am doing is, I actually need to plot $\ln x$ and scale it appropriately with a constant which is $\frac{1}{\ln 2}$. We can easily evaluate the value of $\frac{1}{\ln 2}$ using any calculator and this is just multiplied with $\ln x$.

So, the graph will more or less have similar features of $\ln x$, only thing is it is scaled appropriately. You can try your hand in plotting this graph. This is a clear cut demonstration of the usability of the change of base formula. So, now you need not have to bother about, to what base the function is given. You can simply convert the function × log to the base e or log to the base 10.

This is I have roughly, I have already told this is called common logarithm and this is called natural logarithm. That is why the name ln and once again I reiterate ln means this is $\log_e x$. I will differ from this notation and I will use this notation for convenience and it is a standard convention in mathematics to write this as ln. So, this is a simplest application of graphing any log to the any base of a function.

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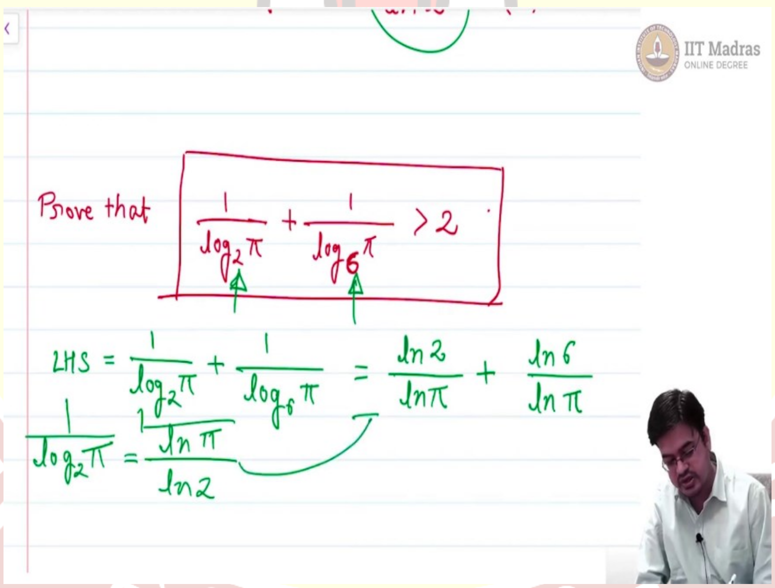

Let us now come to somewhat tricky application of this, that you have to prove that log to the base 2 of π + 1 by log to the base 6 of π is > 2. How will you prove this? Now the key tool while solving all the logarithmic equations or inequalities is exponentiation. So, we will use that tool over here. So, let us start and try to solve this problem. First of all, if you notice in this particular problem, one base is 2, another base is 6, I do not want this. I want everything to the same base.

So, I will use the same principle that I have used here, that is change of base formula. So, let us try to take the left hand side, LHS, which is nothing but $\frac{1}{\log_2 \pi} + \frac{1}{\log_6 \pi}$, apply the change of base formula. So, $\log_2 \pi$ can naturally become $\frac{\log_e \pi}{\log_e 2}$log to the natural base e of π upon log to the natural base e of 2, which essentially gets converted $\times \frac{\ln 2}{\ln \pi}$. This is again simple application of change of base formula.

In a similar manner, I can write this as $\frac{\ln 6}{\ln \pi}$l, to be very precise what I did is, I substituted $\log_2 \pi = \frac{\ln \pi}{\ln 2}$ and because this thing was in the denominator because the original fraction was 1 over this, so this is 1 over that and therefore, it will give rise to this particular number. Fine. So, this is done, you do not have to worry about this.

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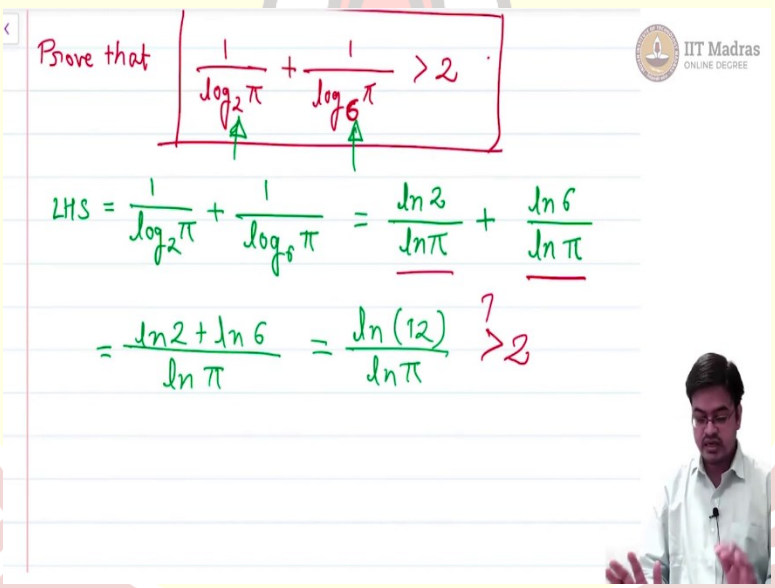

So, now you can see something amazing has happened. The denominator now is actually $\ln \pi$, denominator is same. So, I can rewrite this expression as, let us rightly write this expression as $\frac{\ln 2 + \ln 6}{\ln \pi}$. Wonderful. Now do I know something about the laws of logarithm, this is $log\ a + log\ b$, we have already solved this. This is $log\ ab$, so I know this is $\ln(2 \times 6) = \frac{\ln 12}{\ln \pi}$.

Now this is about LHS. So, LHS actually simplify it to $\frac{\ln 12}{\ln \pi}$. Now the question is whether this thing is > 2, I do not know still. Let us say, assume this thing holds true, then how will you proceed? So, let us try to do it in this fashion.

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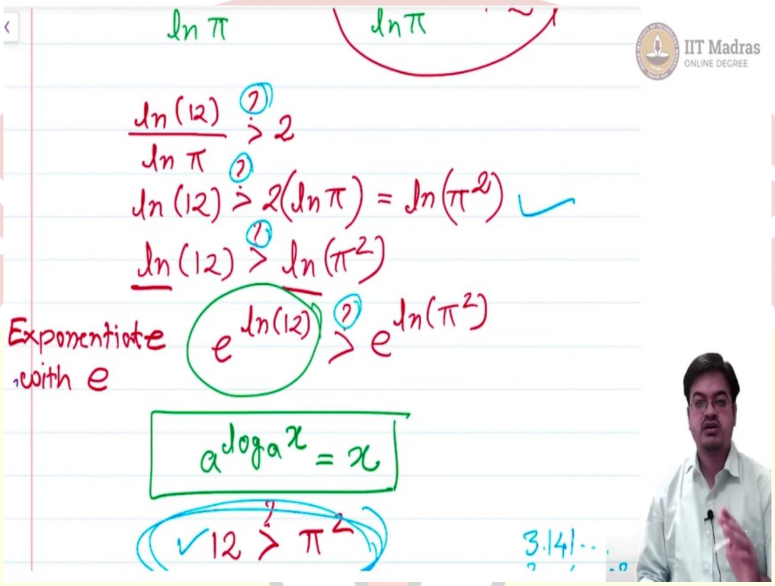

I will again consider this particular inequality and try to prove it, try to see whether this is true or not, l$\frac{\ln 12}{\ln \pi} > 2$ I do not know whether this is true or not. But I know $\ln \pi$ is a non-zero number. So, I can simply take $\ln \pi$, is $\ln\ \pi$ positive or negative? It is positive, so I can simply multiply throughout by $\ln \pi$, so $\ln 12 > 2 \times \ln \pi$.

Now I have one law again to my aid that is multiplication rule, so log of x^a is a×log of x, that rule I will use and this in fact will become=$\ln \pi^2$. So, now I am checking $\ln 12$ is > $\ln \pi^2$d or not. This is the question that we are asking again. So, all these are questions, we have not yet proved anything. Now both side logs are to the same base, so I can exponentiate this, I can exponentiate this with e, Euler's number, Euler's number.

So, if I exponentiate this, then because exponentiation, the operation of exponentiation is nothing but applying exponential function to a particular argument that is monotone, it is monotonically increasing. So, I will have inequalities intact, therefore $e^{\ln 12} \geq e^{(\ln \pi)^2}$.

So, now you can simply go ahead and try to solve this particular problem. What is $e^{\ln 12}$ this particular thing, $e^{\ln 12}$? $e^{\log_e 12}$, so I already have proved that $a^{\log_a x}$ is nothing but x, this we already know using our inverse function definition. So, I will apply this definition over here and therefore, my $e^{\ln 12}$ will simply convert to 12 which is $>\pi^2$ π or not? Is it true or not, we have to check.

So, now you look at the way we started, we started with some complicated inequality which is given here, the inequality, if you include this, some complicated term on the right hand side some number was there and now we made it tailored to our understanding which is, whether 12 is $>=\pi^2$ or not? Now how to prove this? Very easy, what is π basically? It is 3.141 something, something. This number is strictly > 3.15, strictly smaller than 3.15.

So, the π^2, the π is smaller than this, therefore, π^2 will also be smaller than 3.15^2. And 3.15^2 will not exceed 10, you can check for yourself, it will not exceed 10. Therefore, this inequality which is under question is certainly true because this π^2 will always be less than 10, which is less than 12 and therefore, naturally this inequality holds true. So, I do not need to put a question mark over here, I do not need to put a question mark over here, and all these inequalities are true.

And therefore, we have proved that this particular inequality in particular is true. This is the way we will solve a question which is using logarithms. Now you can simply identify what technique we have used, we have used 3 laws of logarithm; first law of logarithm that we used is change of base, second law is the multiplication rule, next the multiplication rule repeated again, here multiplication rule repeated again and then the third law, it is not a law, it is actually the definition of inverse function that we have used. Using these 3 together we were able to find our answer.

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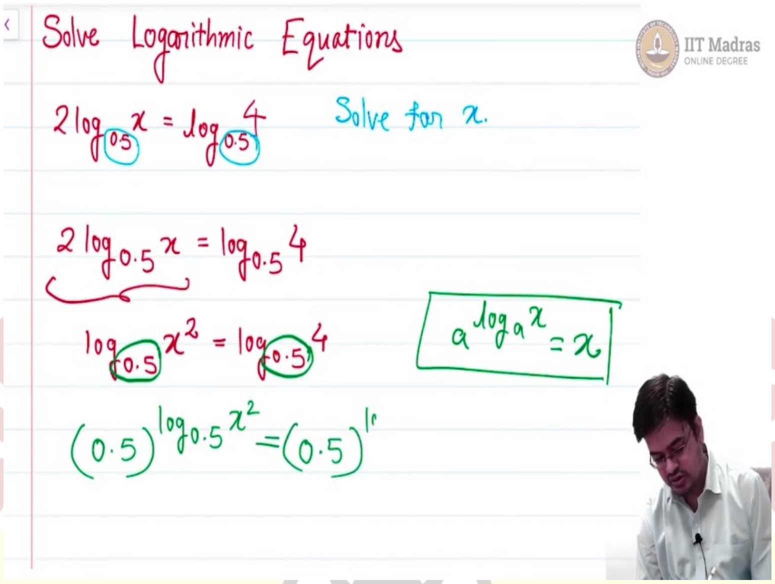

Let us try to solve some slightly complicated problem like this. Let us say you are asked to find, solve log to the base 0.5 of x = the base 0.5 of 4. So, what do I mean by solve, this is actually solve for x. So, you are interested in finding the feasible values of x which will satisfy this equation. Now at the beginning you may be worried about this 0.5 in the denominator because if you remember for 0 less than a, if a is the base, a less than 1, the behavior of log function was somewhat different.

Do I really need to worry about it, is the first question? Before worrying about anything, let us try to simplify this expression, so what is this equation saying, let us write this 2×log to the base 0.5 of x = log to the base 0.5 of 4. Now first thing, if you look at the left hand side, is there any rule that I can apply? Yes, I can apply that power law, power rule. So, I can simply convert this × $\log_{0.5} x^2$ and then let it be as it is, so it is $\log_{0.5} 4$.

Now if you look at this particular expression and if you look at the denominator which is 0.5, here also 0.5, not denominator, base. So, if you look at this base, you can easily say that the base is common. Therefore, the exponentiation trick which we have floated in the last class that is $a^{\log_a x}$ is x, so this trick will work. And therefore, I will exponentiate this with respect to 0.5. So, I will write $0.5^{\log_{0.5} x^2} = 0.5^{\log_{0.5} 4}$. What does this mean?

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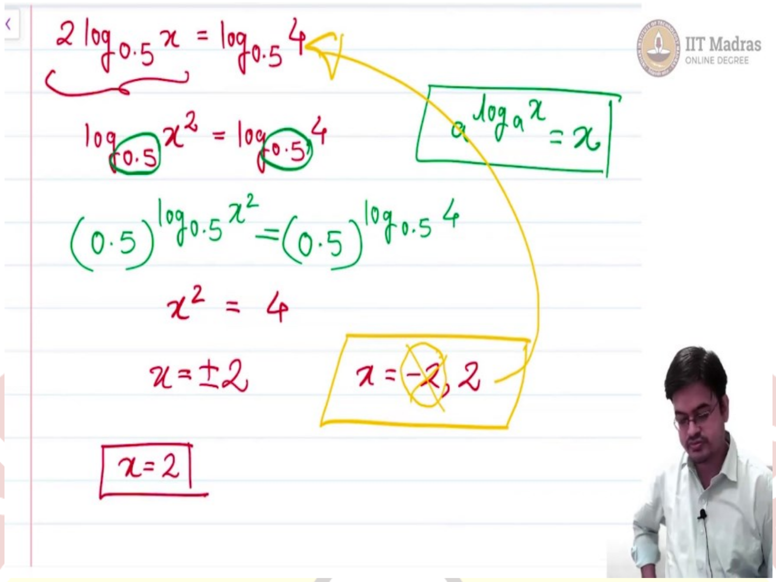

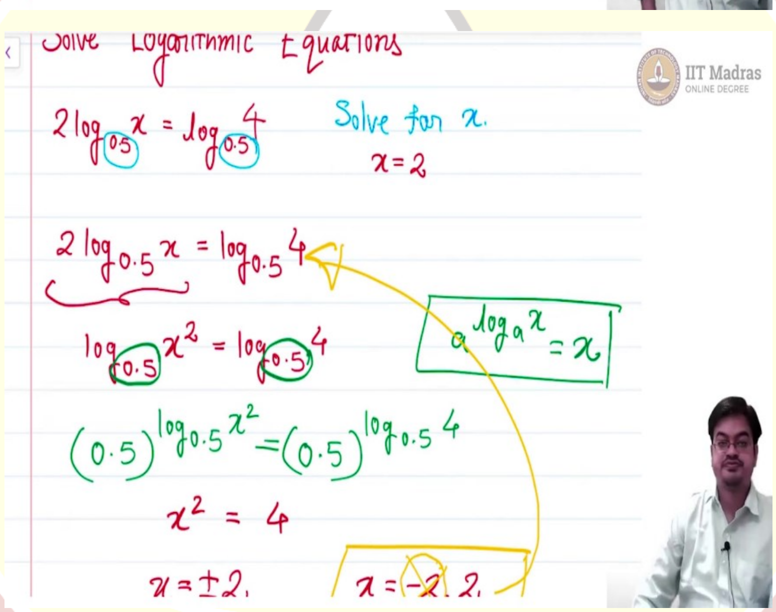

This simply means I will get here $x^2 = 4$, $x^2 = 4$ and now because $x = 4$, based on my knowledge about the quadratic function I know $x^2 - 4 = 0$ has two roots, that is $x = +$ or $- 2$, these are the two roots that are available. Now that means I have two solutions to this particular problem $x = - 2$ and $x = + 2$.

The next question that you should ask is are these both solutions feasible when I substitute them × this expression? So, what is log to the base 0.5 of - 2? When you look at the logarithmic function, it is defined only on the positive real line, it is not defined on negative real line. So, log to the base 0.5 of - 2 is indeterminate, you cannot determine the value, the function is not defined, it is outside the domain. So, this value you can easily chuck off.

And therefore, your solution, solution to your problem is x = 2. This is the solution for the logarithmic equation which we are finding, so solve for x, the answer will be x = 2. You can simply substitute it here and check, you put it to 2^2 is 4 and log to the base 0.5 of 4 = the base 0.5 of 4. What if you put - 2? This is not valid. Correct. So, this way you have to verify once you get the final answer.

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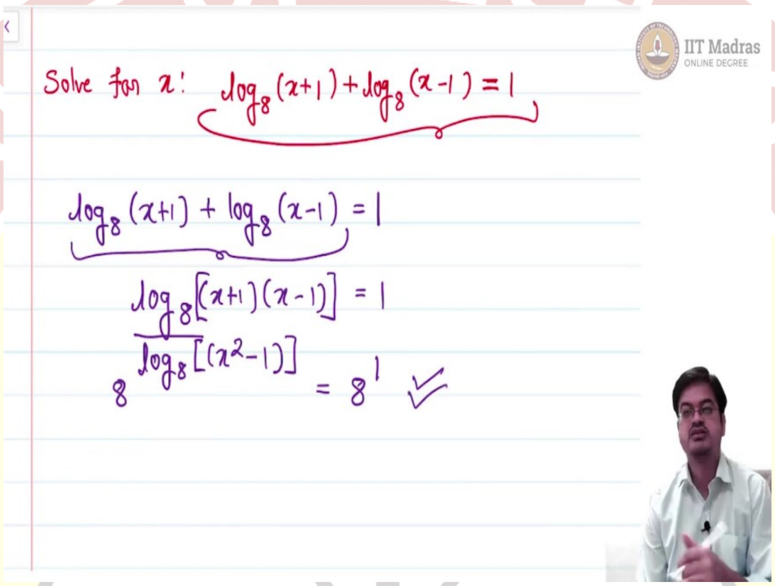

Let us handle somewhat more difficult problem which is again going in a similar line but exponentiation will again help you, but it will reveal some important traits over here. So, let us look at this particular problem where the LHS is log to the base 8 of x + 1 and log to the base 8 of x - 1. Let us try to simplify, let us start with LHS. We want to solve for x, so solving, taking LHS will not help. So, let us take the entire, entire thing that is log to the base 8 of x + 1 + log to the base 8 of x - 1 = 1.

It is important to write the equation as it is, because if you write the equation as it is, you will understand the nitigrities of this equation. So, it is good practice to write once, repeat once whatever is written there, therefore I am writing this. This, do I know any rule, any law of logarithm which will help me to simplify this? Yes, I know multiplication rule, log of m + log of n = log of mn. So, I will apply that rule and I will get log to the base 8 of x + 1 × x - 1 = 1.

Now what can I do? What is the way to simplify? Now I want to get rid of factor of law, so how will I get rid of this log factor? I will exponentiate. So, what I will do is, I will to the, what is the base 8, so $8^{\log_8}$ of this particular factor, I can rewrite this as x^2 - 1, which is easy for me to do and then this = 8^1. Remember if you do not write this step, you may miss out on this, you may write this to be=1. So, just write all these things.

Then 8 log to the base 8 will vanish and therefore you will get $x^2 - 1 = 8$. Do I know something like this? We can, I know that $x^2 - 9 = 0$. That means x = + or - 3 are the possible solutions. Now is any, so in the earlier case when x = + or - 3 were the solutions is there any, so for example here in this case - 2 was eliminated. So, here also you need to do a similar check, if you put x = + 3, if you put x = + 3, then what will happen?

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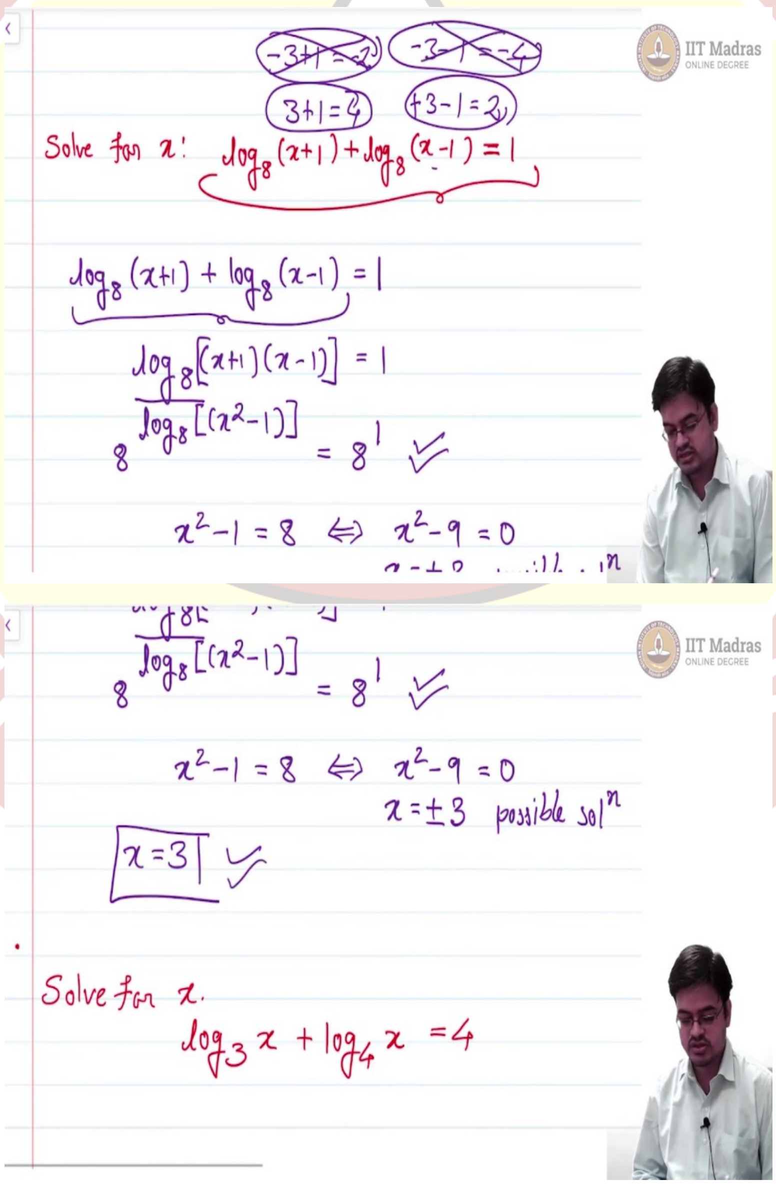

This 3 + 1, 4 and - 3 - 1, sorry + 3 - 1 which is 2 and therefore, this is 4, and therefore it is a valid expression. So, it is in the domain. If you put x = - 3 and - 3 + is - 2. This is not a log, what happens here - 3 - is - 4, which is also not a log for putting × log. So, - 3 cannot be in the domain.

So, only thing that is possible is x = 3 should be there in the domain, again when you have solved the equation, it is better to verify, so you write here $\log_8$ 3 + 1 which is 4 and 3 - 1 which is 2 = 1. This is valid, so you have good answer. So, x = 3 is the answer to this question. Let us move ahead and solve one question which has different bases.

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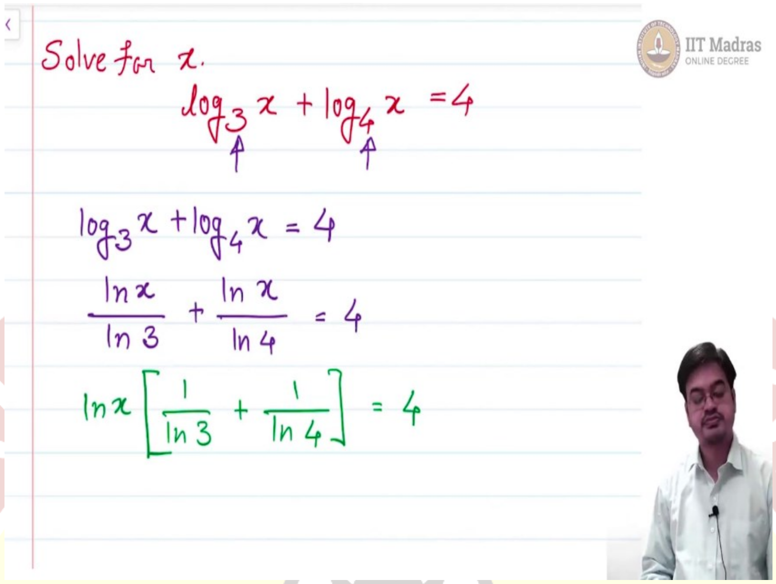

So, here is a question which is log to the base 3 of x + log to the base 4 of x = 4. Now what can you do? Because you cannot use your trick of exponentiation, because this bases are different. So, what should you do? The first thing is to make the bases equal, how will you make it? One formula is change of base formula, so you apply change of base formula, so as I mentioned it is better to write the expression once more, log to the base 3 of x + log to the base 4 of x = 4.

If I want to apply change of base formula, I will simply use this as log to the natural base, so $\frac{\ln x}{\ln 3}$+ $\frac{\ln x}{\ln 4}$ = 4. So, ln x is taken in common and therefore I have $\frac{1}{\ln 3}$+ $\frac{1}{\ln 4}$ = 4. Now you can notice that this particular within the brackets is a number which is non-zero.

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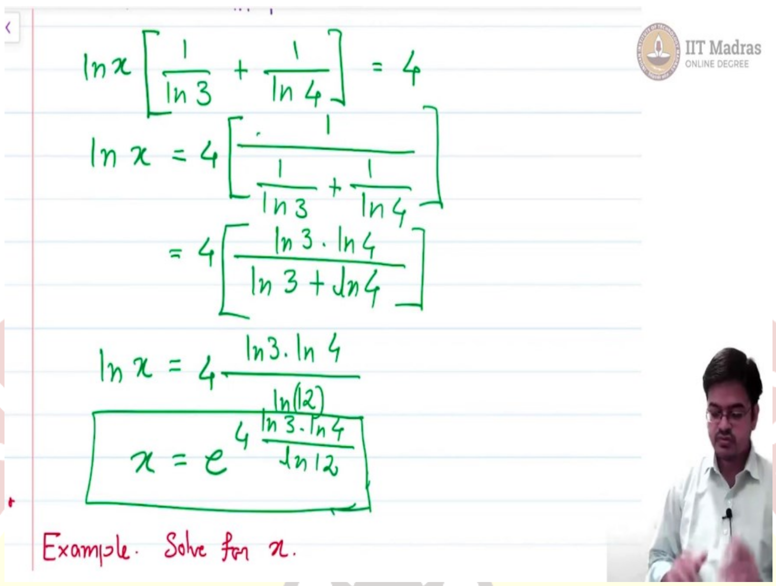

Therefore, I can actually take this number on the other side. So, let me simplify this, ln x = 4 times, when I take this on the other side it will be a reciprocal of that, so $\frac{1}{\ln 3}+\frac{1}{\ln 4}$, this is what it will be. So, can I simplify this? Yes, I can simplify this further, which will give me 4 times, so ln 3 + ln 4 will be in denominator and in the numerator it will give me ln 3×ln 4. Fine.

Then I need to do something which is let us say, I will do it in this fashion, ln 3 × ln 4 cannot be combined × anything, they will remain as it is. But what can be done is, this is ln 3 × ln 4 upon ln 12, 4 × 3, I have used the multiplication rule. So, let this, all these things remain as it is and because it is ln, I will exponentiate and I will get $x = e^4$ ×ln 3 ×$\frac{\ln 4}{\ln 12}$.

If you want you can merge these 4 with one of these ln's and write, with one of these ln's and write this as $\ln 3^4$ or $\ln 4^4$, whatever you are comfortable with, but I will leave this as it is. So, this is the solution and it is a perfectly valid solution. This is how you will solve a problem which has different bases. So, this is a perfectly valid solution. Let us go ahead and try to solve one more example, which is, which looks somewhat ambiguous.

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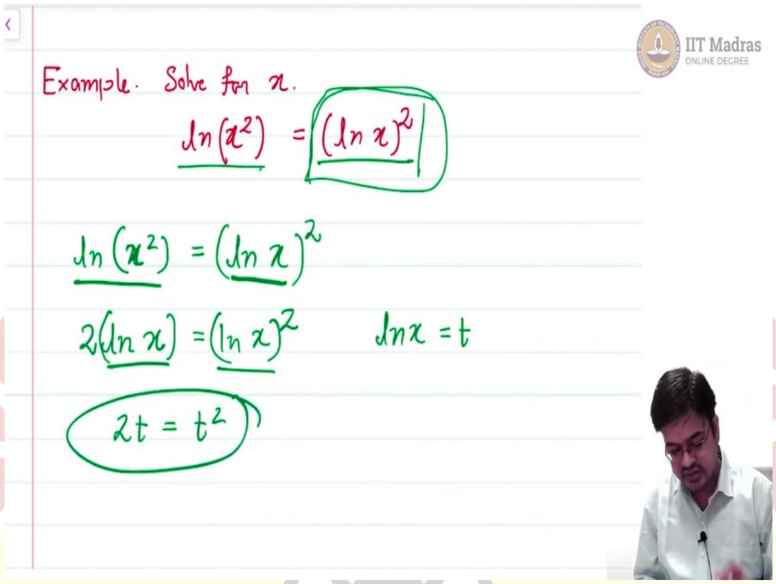

In a sense you have been given ln x^2 = (ln x) ^{2}. How will you solve this problem? That is a first question. So, we need to resolve to some methodology, let us look at this particular problem and try to simplify the things over here. So, here you typically, because this isd, you typically need some knowledge about quadratic functions in order to solve this problem. Let us try to understand this.

So, let me write ln x^2 = (ln x) ^{2}, in such a problem can you figure, the first question is, are the bases common? Yes, the bases are common, so there is no problem with this. Now what is happening is, here the term is ln x, here the term is ln x^2. Can I get the terms which both are in the form of ln x, then I can do something with it. So, I will simply ask that question and what comes to my aid is the multiplication rule or the power law, ln a^k is k×ln a. So, 2×ln x = (ln x) ^{2}.

Now comes the real power or the real strength. What is happening? It is 2×ln x, now the argument both are same, so you can treat this as composition of functions. So, what is happening is, if you write ln x = t, you are actually talking about 2t = t. So, what I am doing is, I am putting ln x = t, this I am defining. Fine. So, this particular thing is fine.

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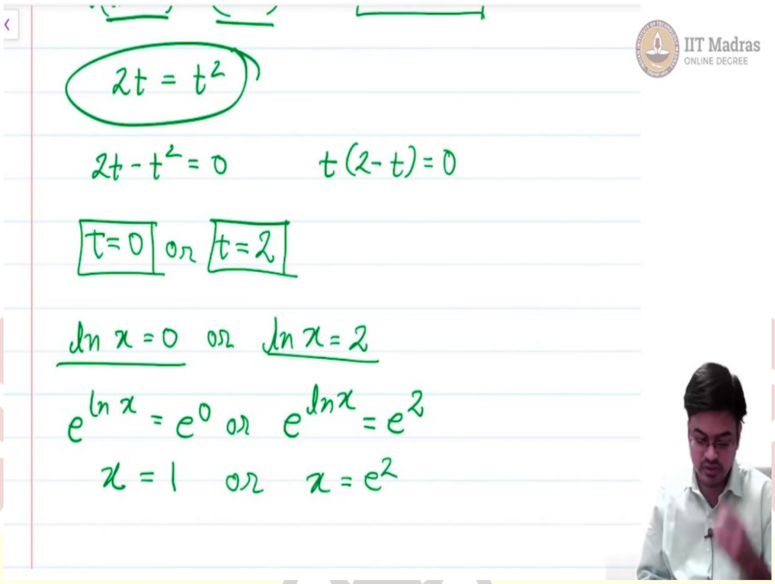

So, what I will do here, very easy then, it is 2t -, basically $2t - t^2 = 0$, take out one t common, so $t \times 2 - t = 0$ and therefore, either t will be=0 or t will be=2. These are the two possible solutions. So, now, but what is t? According to our substitution it is $\ln x$, so that means I am saying $\ln x = 0$, when is $\ln x$ 0? And $\ln x = 2$, so when is $\ln x$=2? So, these are the two questions that we have asked. So, let us say or.

And then in both cases you have a natural log, is not it? So, I can exponentiate this particular function, so $e^x = e^0$. You already know logarithmic function has a point where it passes through 0 and e^0 is, $e^{\ln x}$ will be x itself, e^0 will be 1, in this case exponentiating will give me $e^{\ln x} = e^2$ by default I wrote, so I will write it as e^2. So, that will give me $x = e^2$ or $x = 1$. This is what the answer is.

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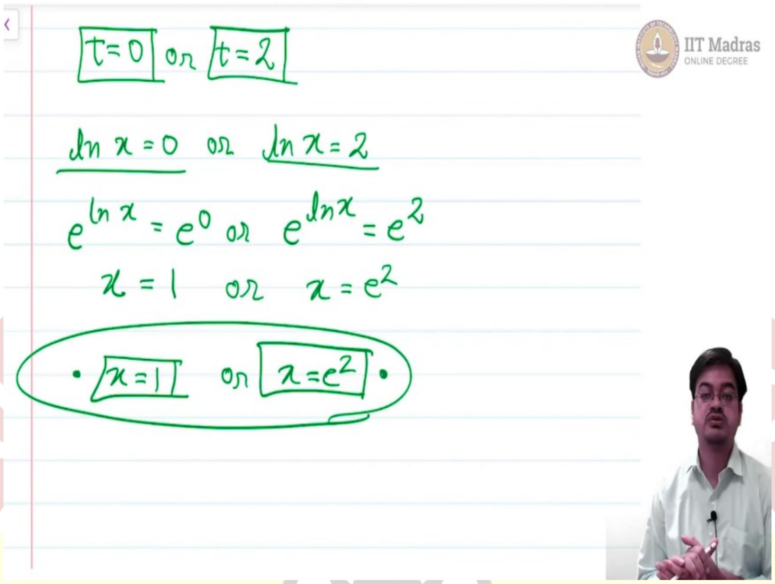

So, in this case $x = 1$ is one answer or $x = e^2$ is another answer. It is good to go to the original problem and check whether these conditions are satisfied in the original problem or not. So, first let us check $x = 1$, so if $x = 1$, $\ln 1 = \ln 1^2$. What is ln 1? You know the answer log of 1 is 0 and ln, so 0^2 is 0. And then if you put $x = e^2$, then $\ln e^2$, that is 4 and $\ln x$ that is $\ln e^2$ is 2 and 2^2 is also 4.

So, we have verified that both these are valid answers. So, what I have said just now is $\ln 1^2$ is 1, so it is 0. 1 is 1, so $\ln x^2$ is 0. Similarly, ln 1 is 0, so 0^2 is 0. So, the first $x = 1$ is the solution. In the similar manner, you substitute $x = e^2$ and you can plot.

Now it is good think if you can use a calculator, graphing calculator like Desmos and plot these two curves and see how, where they are intersecting. We have actually given the points of intersection of these two curves. These are the points of intersection of these two curves. You can verify for yourselves and I will see you in the next class. Thank you.