Single Source Shortest Paths

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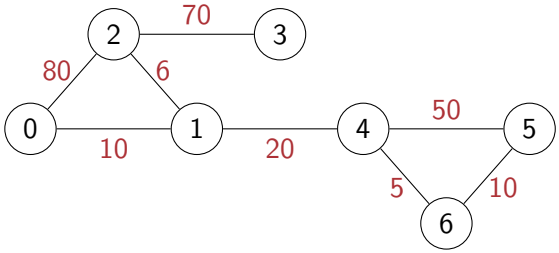
Mathematics for Data Science 1
Week 12

Single source shortest paths

- Weighted graph:
  - $G = (V, E)$
  - $W : E \to \mathbb{R}$



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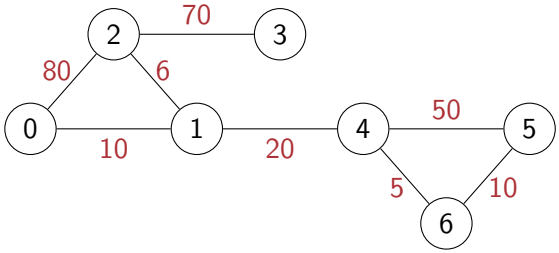

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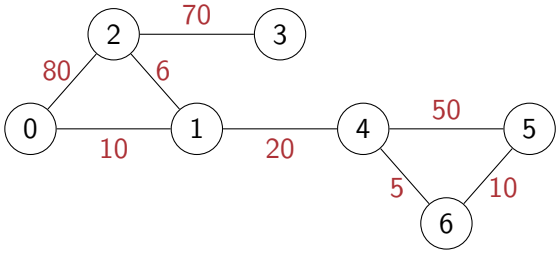
- Assume, for now, that edge weights are all non-negative



Single source shortest paths

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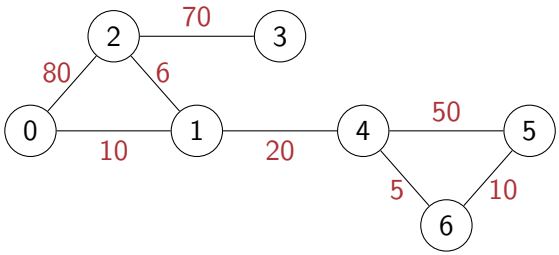

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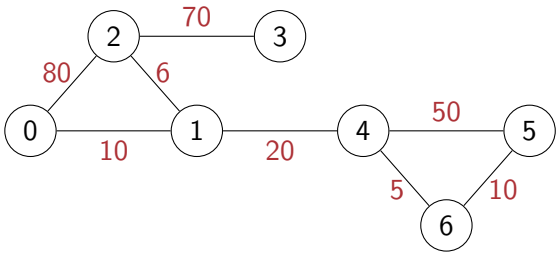
- Imagine vertices are oil depots, edges are pipelines



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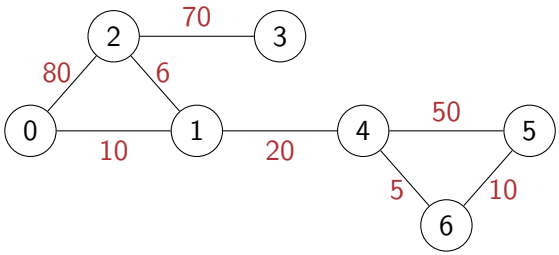

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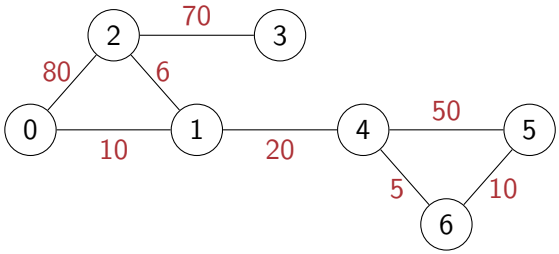

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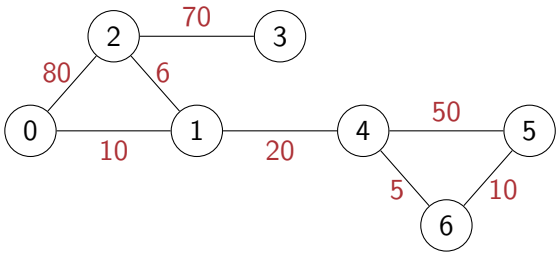

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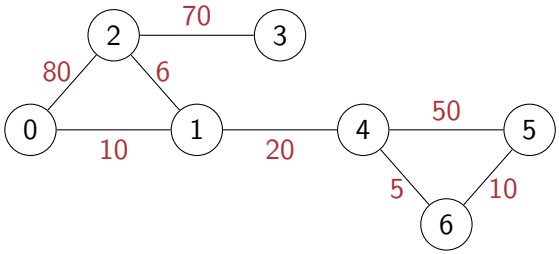

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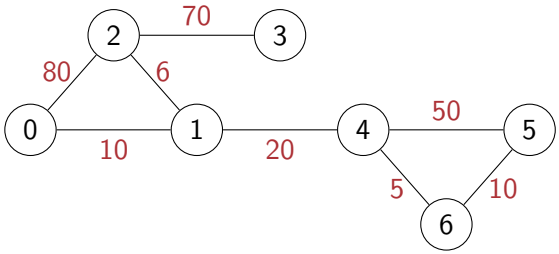

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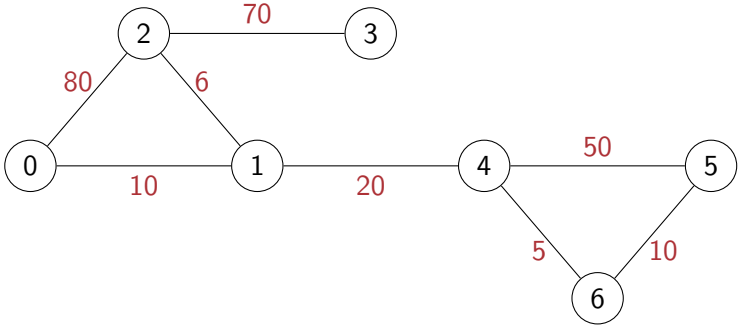

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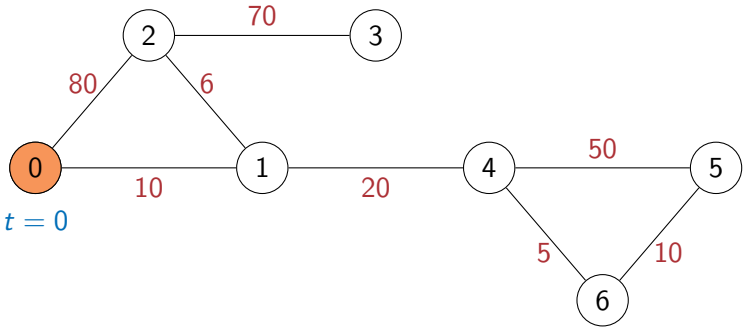

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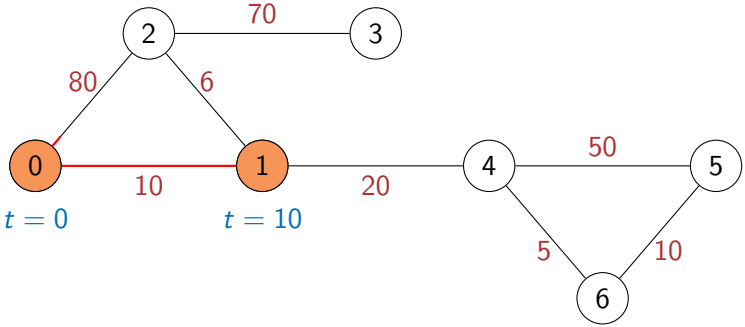

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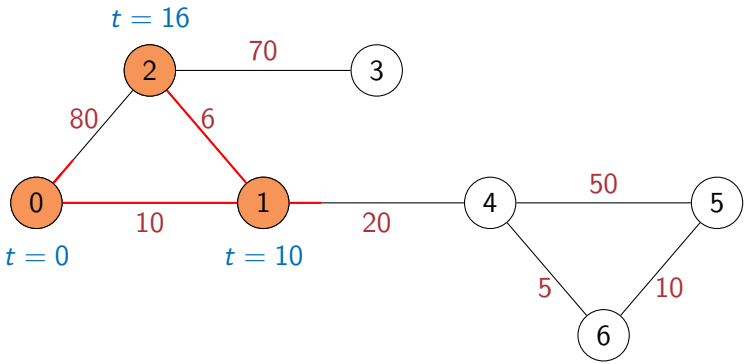

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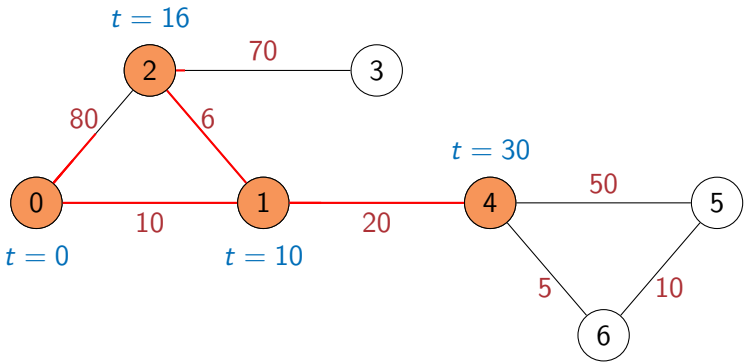

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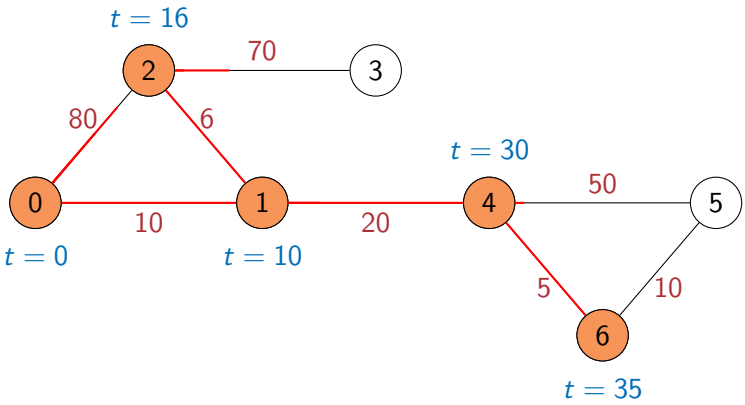

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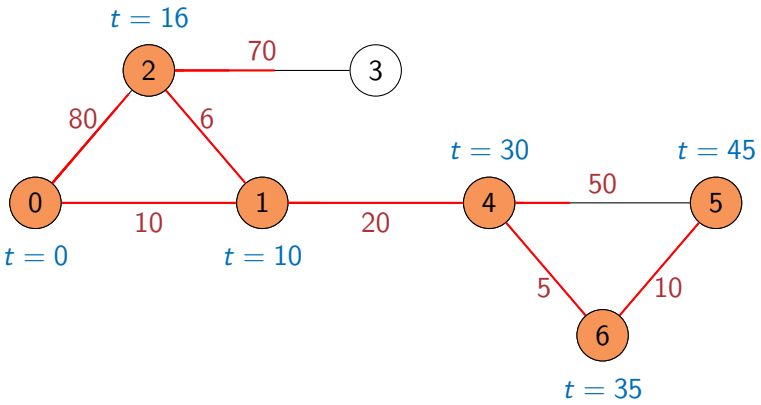

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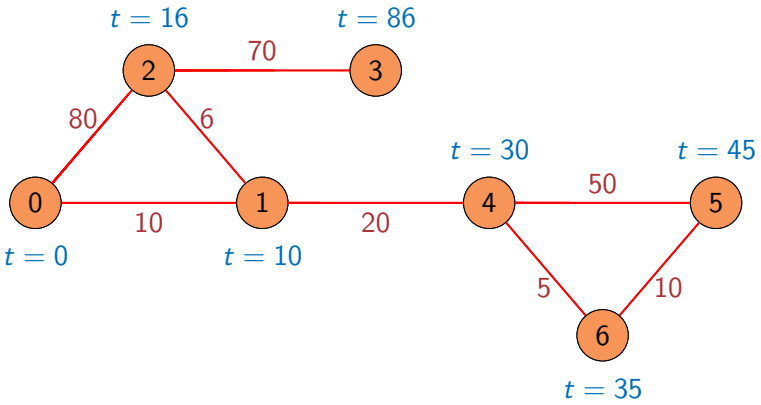

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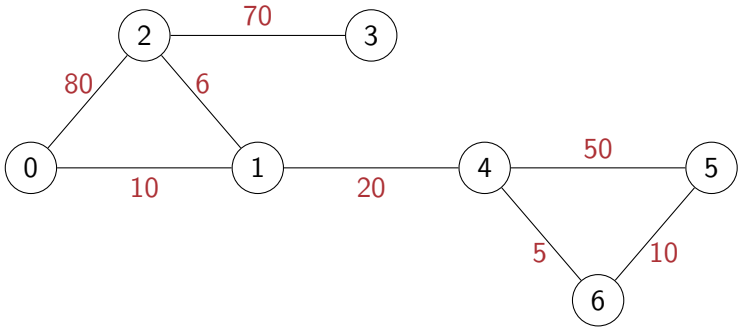
- Each time a new vertex burns, update the expected burn times of its neighbours



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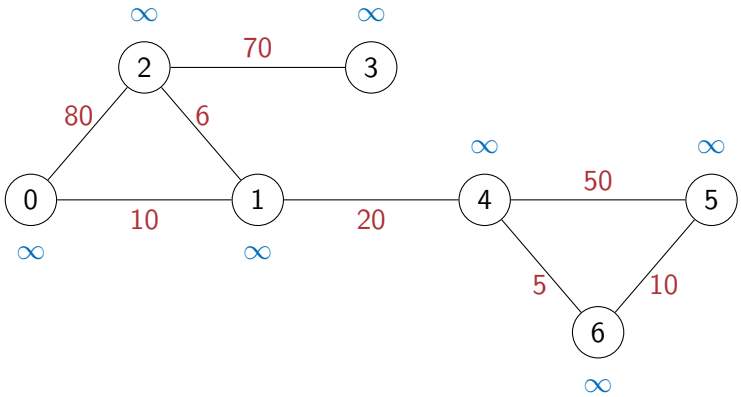

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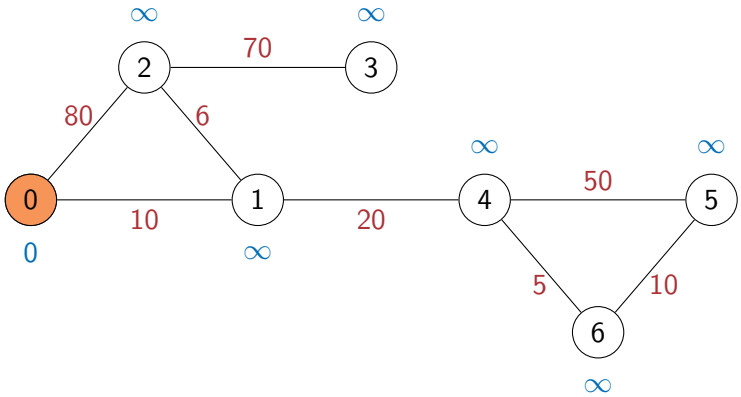

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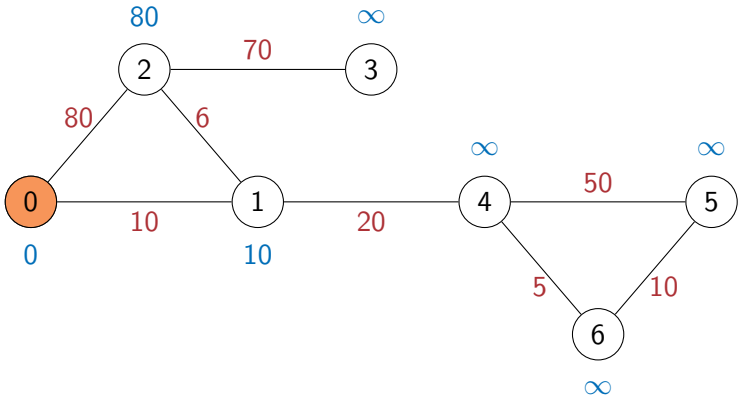

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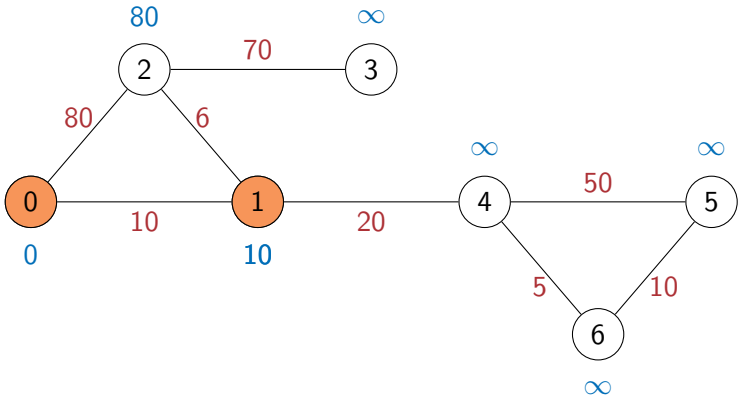

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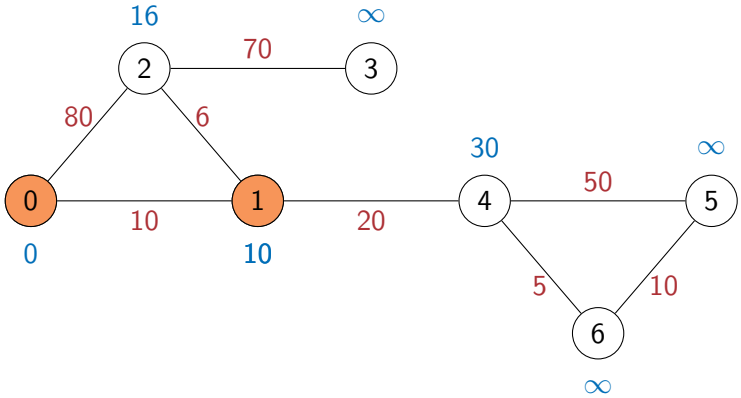

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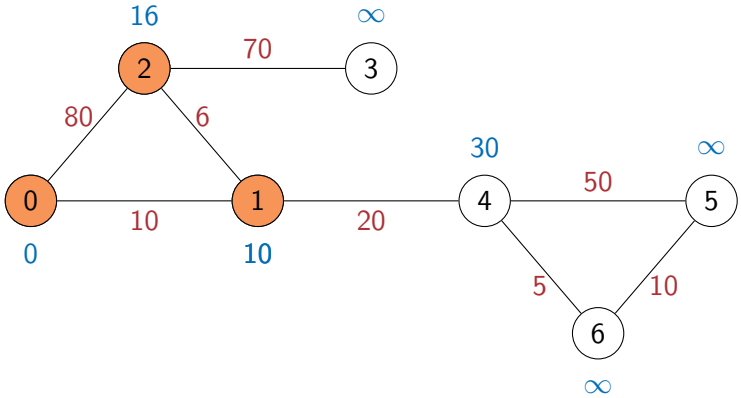

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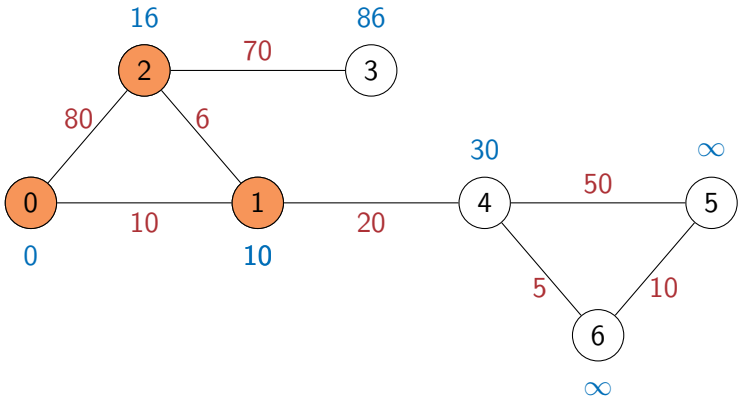

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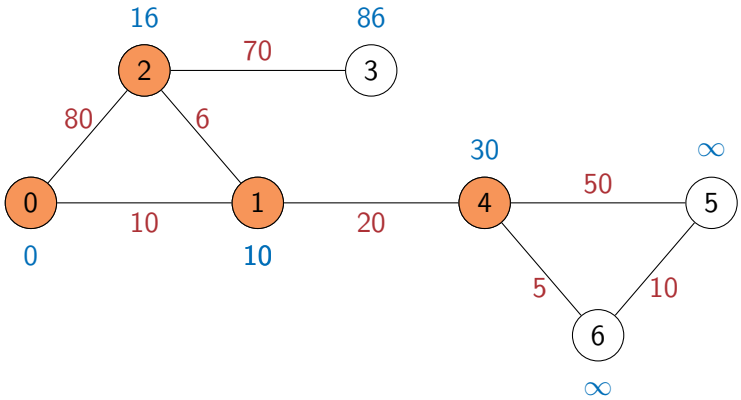

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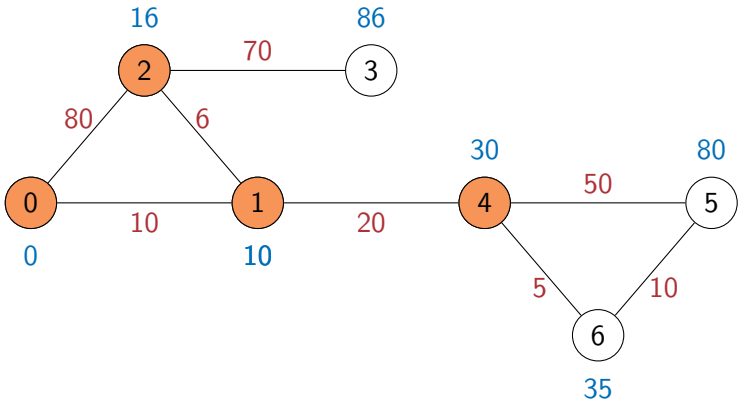

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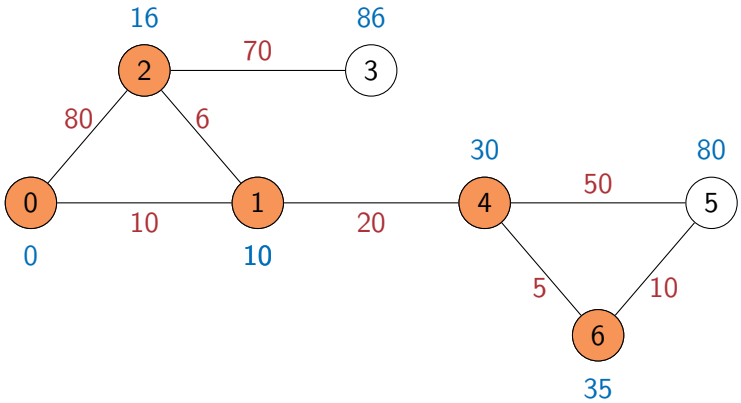

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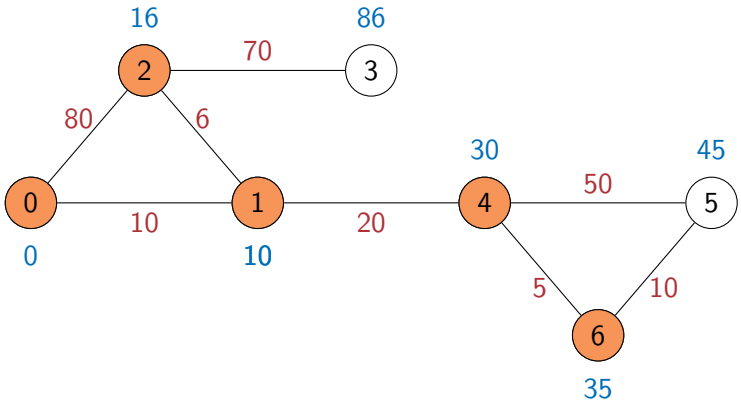

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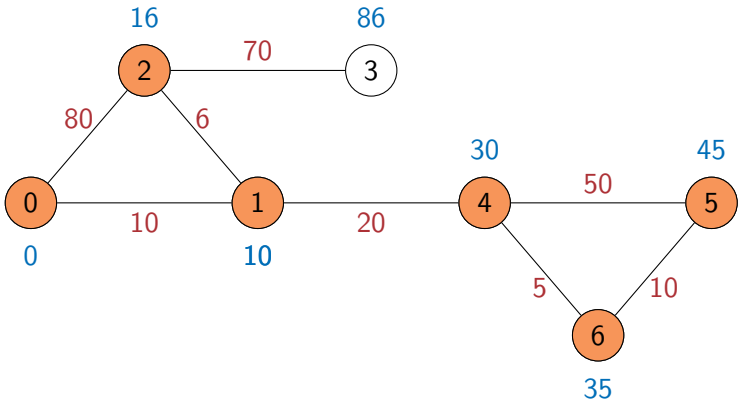

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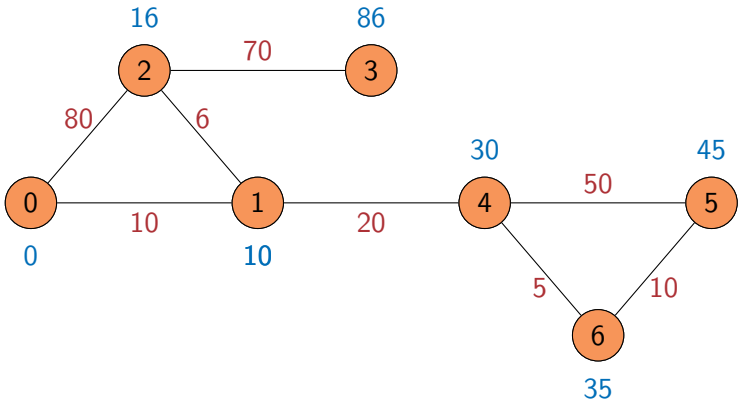
- Algorithm due to Edsger W Dikjstra



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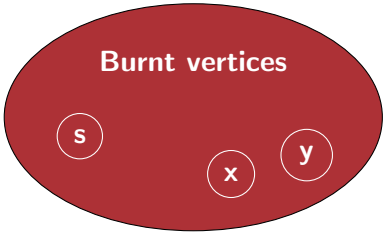

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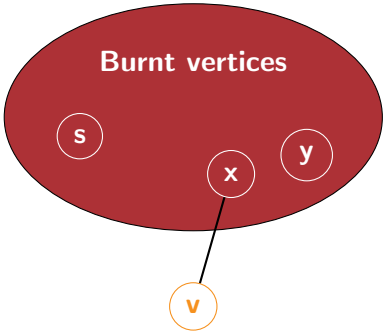

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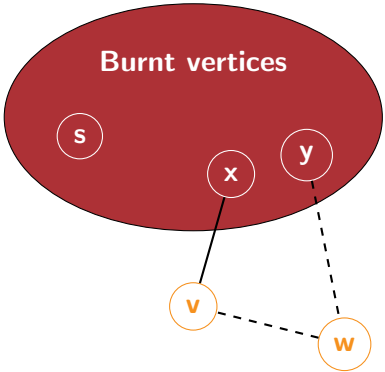
  - Edge from **w** to **v** has weight ≥ 0



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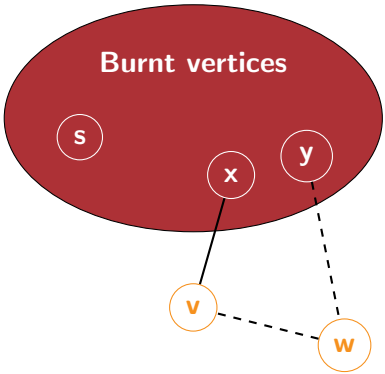
  - Edge from **w** to **v** has weight ≥ 0
- This argument breaks down if edge (**w**,**v**) can have negative weight
  - Can't use Dijkstra's algorithm with negative edge weights



Summary

- Dijkstra's algorithm computes single source shortest paths
- Use fire analogy
  - Keep track of expected burn times for each vertex
  - Update burn times of neighbours each time a vertex burns
- Correctness requires edge weights to be non-negative