Computing derivatives and L'Hôpital's rule

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Fact

If f is differentiable at a, then it is continuous at a.

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If $f(x)$ and $g(x)$ are differentiable at a, then so is $(fg)(x)$ and

$$(fg)'(a) = f'(a)g(a) + f(a)g'(a).$$

$$\lim_{h\to 0} \frac{f(a+h)g(a+h) - f(a)g(a)}{h} = \lim_{h\to 0} \frac{f(a+h)\left(g(a+h) - g(a)\right) + \left(f(a+h) - f(a)\right)g(a)}{h}$$

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Composition : the chain rule

If $f(x)$ and $g(x)$ are differentiable functions, then so is the function $f(g(x))$ and its derivative is :

$$(f(g))'(x) = f'(g(x))g'(x).$$

Examples

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Guess: $f(x) = x^n$
$f'(x) = nx^{n-1}$.

$f(x) = x^2 = x \times x$

$f'(x) = (x)' \times x + x \times (x)'$
$= 1 \times x + x \times 1$
$= x + x = 2x$.

$f(x) = x^n; n \in \mathbb{N}$
$= x^{n-1} \times x$

$f'(x) = (n-1)x^{n-2} \times x + x^{n-1} \times 1$
$= (n-1)x^{n-1} + x^{n-1} = nx^{n-1}$.

$f(x) = x^a$
$f'(x)$
$= ax^{a-1}$.

$f(x) = x^3 = x^2 \times x$
$f'(x) = 2x \times x + x^2 \times 1$
$= 2x^2 + x^2$
$= 3x^2$.

$f(x) = 5x^3 - 17x^2 + \pi x - 0.5$

$f'(x) = 5 \times 3x^2 - 17 \times 2x + \pi \times 1 - 0$
$= 15x^2 - 34x + \pi$

Gen. statement for a polynomial.

$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$
$\Rightarrow f'(x) = n a_n x^{n-1} + (n-1) a_{n-1} x^{n-2} + \cdots + a_1$

$f(x) = x^7 sin(x)$

$f'(x) = 7x^6 \times \sin(x) + x^7 \times \cos(x)$
$= 7x^6 \sin(x) + x^7 \cos(x)$.

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$f(x) = tan(x) = \frac{\sin(x)}{\cos(x)}$

$$f'(x) = \frac{\cos(x) \times \cos(x) - \sin(x) \times (-\sin(x))}{\cos(x)^2}$$

$$= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)}$$

$$= \sec^2(x).$$

$f(x) = \frac{1}{x^r} \quad ; r > 0$

$= x^{-r}$

$$f'(x) = \frac{\frac{d}{dx}(1) \times x^r - 1 \times \frac{d}{dx}(x^r)}{(x^r)^2}$$

$$= \frac{-r x^{r-1}}{x^{2r}} = -r\, x^{r-1-2r}$$

$$= -r\, x^{-r-1}$$

$f(x) = tan(2x) = \frac{2\tan(x)}{1-\tan^2(x)}$

$= \frac{\sin(2x)}{\cos(2x)}$

$g(x) = 2x. \quad h(x) = \tan(x)$

$f(x) = h(g(x)).$

$f'(x) = h'(g(x))\, g'(x)$

$= \sec^2(2x) \times 2$

$= 2\sec^2(2x)$

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$\lim_{x \to 0} \log_e(1+x) = \log_e(1+0) = \log_e(1) = 0.$

$\lim_{x \to 0} x = 0.$

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In this situation, we can try and use L'Hôpital's rule.

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e.g. $\lim_{x \to 0} \dfrac{log_e(1+x)}{x} = \lim_{x \to 0} \dfrac{1/_{1+x}}{1} = \lim_{x \to 0} \dfrac{1}{1+x} = 1.$

$\dfrac{d}{dx}\left(\log_e(1+x)\right) = \dfrac{1}{1+x}$, $\dfrac{d}{dx}(x) = 1.$

More examples

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$$\lim_{x\to 2}\frac{x^2-5x+6}{x-2} = \lim_{x\to 2}\frac{2x-5}{1} = \lim_{x\to 2}(2x-5) = 2\times 2-5 = -1.$$

$$\lim_{x\to 0}\frac{sin(x)}{x} = \lim_{x\to 0}\frac{\cos(x)}{1} = \lim_{x\to 0}\cos(x) = \cos(0) = 1.$$

$$\lim_{x\to\infty}\frac{a+be^x}{c+de^x} = \lim_{x\to\infty}\frac{be^x}{de^x} = \lim_{x\to\infty}\frac{b}{d} = \frac{b}{d}.$$

$$\lim_{x\to 0}\frac{1-cos(x)}{x^2} = \lim_{x\to 0}\frac{\sin(x)}{2x} = \frac{1}{2}\lim_{x\to 0}\frac{\sin(x)}{x} = \frac{1}{2}.$$

Thank you