INDIAN INSTITUTE OF TECHNOLOGY MADRAS

सिद्धिर्भवति कर्मजा

# IIT Madras

ONLINE DEGREE

**Mathematics for Data Science 1**
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**Lecture 4.1 A**
**Quadratic functions**

Welcome students, today we are going to start the new topic in our syllabus that is Quadratic functions. Before starting this new topic, let us revise what we have studied so far. We started with some simple geometric objects like points and lines, after studying points and lines geometrically we plotted them on coordinate plane and seen how to derive the algebraic equation of a geometric line.

When we have seen the algebraic equation of a geometric line, we got a form of the form $y = mx + c$, it is also known as linear function that we have seen in last few lectures. And if you recall recollect it from the first week where you have studied functions, this is $f(x) = mx + c$is a linear function.

Now, we want to enhance our knowledge further and add 1 more intrication or 1 more complexity in this particular function and that is why we are studying quadratic function. Here we will take an approach where we will first state the algebraic form and then derive its geometric properties as opposed to what we did in straight lines. So, let us start with quadratic functions.

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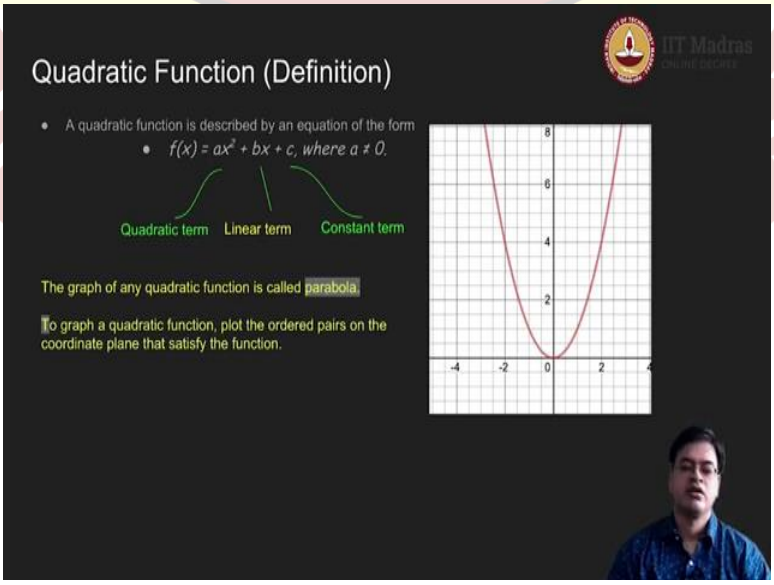

The first question is, how will I define quadratic functions? The answer to this question is given in this slide. So, a quadratic function is described by an equation of the form $f(x) = ax^2 + bx + c$, where $a \neq 0$ is a crucial condition. Why? If $a = 0$ it simply reduces to a linear function. Let us talk about the name quadratic function. The name quadratic function is derived from 1 foreign language where the quadra term, actual word quadratic term means square and quadratic means related to square.

So, a quadratic function is a function that is related to square of the variable as can be seen from the definition, it has a term containing a $x^2$. So, if $a = 0$ then it does not have a term containing square so it no longer remains a quadratic function and it is a linear function which is equivalent to a straight line as a geometric object.

So, we will put a condition that $a \neq 0$ that means we are studying a quadratic equation. The next question is how to plot a graph of this function. So, this equation is actually composed of 3 terms, let us describe them 1 by 1 that is a $x^2$, this term is a quadratic term.

As I mentioned earlier, when $a = 0$, the term $bx + c$ survives and that term $bx$ is a linear term. And finally, if you put $x = 0$, only term that survives is c so that is nothing but a constant term. So, a quadratic equation can be split into 3 parts. If $ax^2$ is not there, then I know how to handle this term on a coordinate plane, it just simply represents an equation of a non-vertical line.

So, I know how to handle these terms. So, what if the a $x^2$term remains that is $a \neq 0$? We can graph this particular function and graph of any quadratic function will be called as parabola. Graph of any quadratic function will be known as parabola. So, what are the important features of parabola? In order to do that we first need to plot the parabola.

So, what is the best way to do it? We have already seen to graph any function what we need to do is, we need to take the value of $x$, put it in the formula $f(x)$ and evaluate it and get the values of $y$. So, consider all ordered pairs and plot them on the coordinate plane so that they satisfy this function, is the best way to handle it. For example, let us take this let us take for example, when $b = 0$ and $c = 0$ and $a = 1$, let us take that particular function that is $y = x^2$.

In that case what I will do is, I will put $x = 0$, I will get back 0. So, I will plot a point (0,0), I will take $x = 1$ I will get 1, $x = -1$ I will get 1, and then $x = 2$ then I will get 4 and if I take $x = -2$ again I will get 4. So, $y = x^2$can be easily plotted by joining these points smoothly this is the curve $y = x^2$, this is how we plot our quadratic function.

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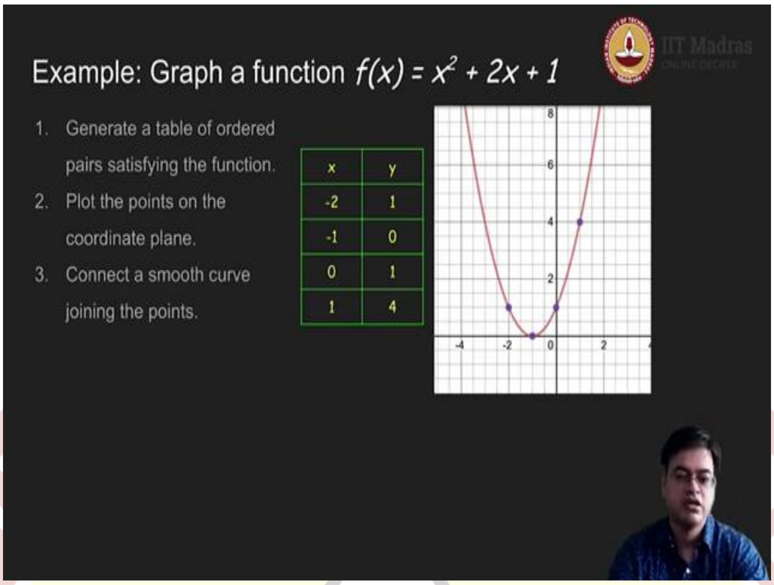


Let us take 1 example. Let us say I want to graph a function $f(x) = x^2 + 2x + 1$. How will I graph this function? In 3 steps. First, I will generate a table of ordered pairs satisfying the given function. Second, I will plot those points on the coordinate plane. Once I plot those points on the coordinate plane, I will connect a smooth curve joining the 2 points, this is the recipe for drawing a function. Let us draw it here, for that I have computed some points you can verify by yourself, if you put the value of $x = -2$, you will get $(-2)^2 + 2(-2) + 1$ and on solving you will get 1.

You take the value $x = -1$ you will get 1, for x square you will get $-2$ and 1 in the constant term. So, together they will cancel and you will get 0. Similarly, you can compute for $x = 0$ it is 1 and for $x = 1$ it is 4. Now, our job is to consider a coordinate plane and plot these points so I have plotted these points.

So, these points are plotted and now I need to draw a graph, which is connecting all these points. Now, here you remember I have plotted these 3 points, how will I know the shape of this graph in this zone? That is a major question that you can ask, but this parabola is somewhat symmetric in a sense, suppose I take this point, what is the point here, the point is (1, 4).

Now, if I consider this point which is $-1$ where it takes the value 0 and consider the point 1 it is 2 units apart. So, somewhere in this where $-3$ will come, which is 2 units apart from -1, the value of the parabola will be again 4. I will keep the cursor here, see. So, there is some kind of symmetry underlying this particular function, we need to understand that symmetry in a better way.

So, what essentially is happening is, if I consider this point which is the bottom of the curve, and if I draw a straight line, which is the line $x = -1$, then if you look at all these points for every point there is a similar point on $y - axis$ at the same distance from this particular point. This also can be called as a symmetry of a parabola. We will study this later in the next slide. So, right now, our job is to graph a function which we have plotted and let us explore further properties of this parabola like this symmetry, what is the meaning of the symmetry and all those things.

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So, there are a few important observations, if you consider equation of $y = ax^2 + bx + c$, these are all parabolas, I have shown you two parabolas $y = x^2$, $y = x^2 + 2x + 1$both parabolas have axis of symmetry. Inevitably all parabolas will have an axis of symmetry that is, what is axis of symmetry. Let us go to the previous slide and see.

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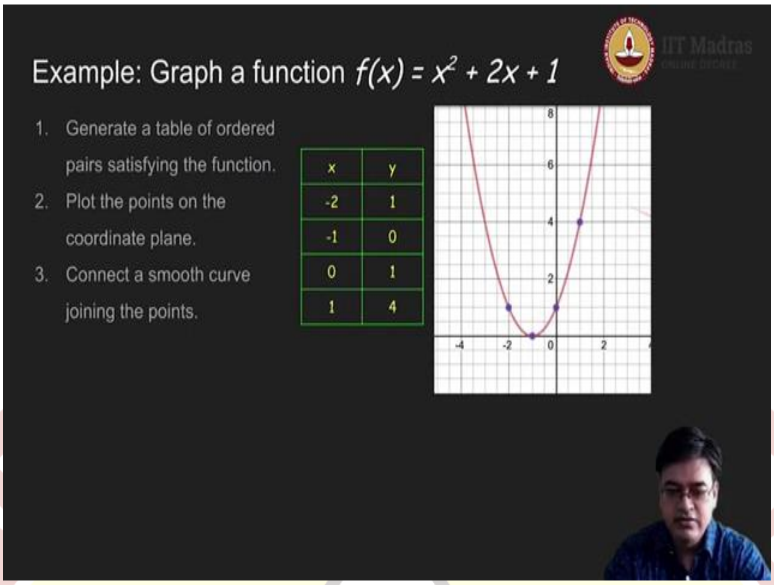


The axis of symmetry over here, as I mentioned was $x = -1$. If I take this graph paper and fold along $x = -1$, then the curves that we have plotted here must exactly match each other that gives us a recipe to draw a parabola.

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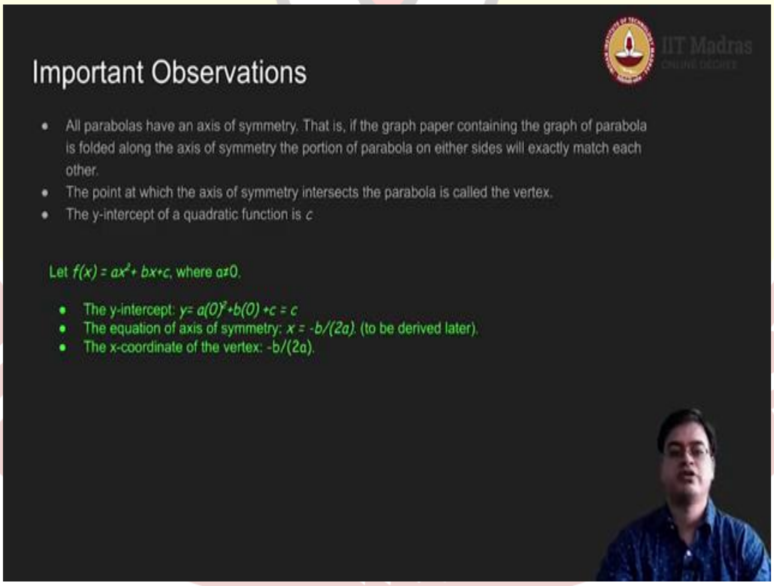


So, all parabolas will have axis of symmetries that is if you take a graph paper containing the parabola, and if you fold it along the axis of symmetry, the portions of the parabola on both sides will exactly match with each other, this is the beauty of a parabola. So, now if I know how the parabola appears on one side, I know how the parabola appears on the other side of the axis of symmetry. It is a pure reflection of whatever is happening on one side.

Then, the point this axis of symmetry as we have seen in the previous graph, the point at which this axis of symmetry meets parabola, we will call that point as a vertex of the parabola. This is again a nomenclature we will call that point as a vertex of the parabola and the point at which $x$, if you put $x = 0$, then the point at which the $y$ coordinate is taken is called the value c or you can simply refer to the equation $ax^2 + bx + c$, put $x = 0$, that will be the y intercept which will be given by $c$.

These 3 points play a crucial role in graphing the parabola. How? Let us do it one by one, Let us say, our quadratic function is $ax^2 + bx + c$ where $a \neq 0$, you can easily figure out that the $y$ intercept of this point by putting the value 0 in $c$.

Now, I want to know the axis of symmetry, this plays a crucial role. So, I will derive the expression for axis of symmetry later but right now you memorize this equation as $x = \frac{-b}{2a}$ as this needs some algebraic skills which we do not have right now. So, I will derive it later. But right now, you understand that $x = \frac{-b}{2a}$.

Remember, the equation of the quadratic function is given by$ax^2 + bx + c$, and $c$ will not play any role in this and $b$ and $a$ will play a role. So, it is $\frac{-b}{2a}$ is the axis of symmetry and where the graph meets this parabola, it is called vertex. So, the $x$ coordinate of the vertex is $\frac{-b}{2a}$ obviously, because the axis of symmetry has $x = \frac{-b}{2a}$. So, the $x$ coordinate of the vertex is $\frac{-b}{2a}$.

Let us see how this knowledge helps us in understanding how to draw a parabola. So, there are 3 steps in drawing the parabola, first you need to generate a table of values, but if you generate a table of values only on one side and you do not have a table of values on vertex, then you may not be able to draw the parabola appropriately, that is way the knowledge of these facts is important, so let us see how to draw a parabola by example.

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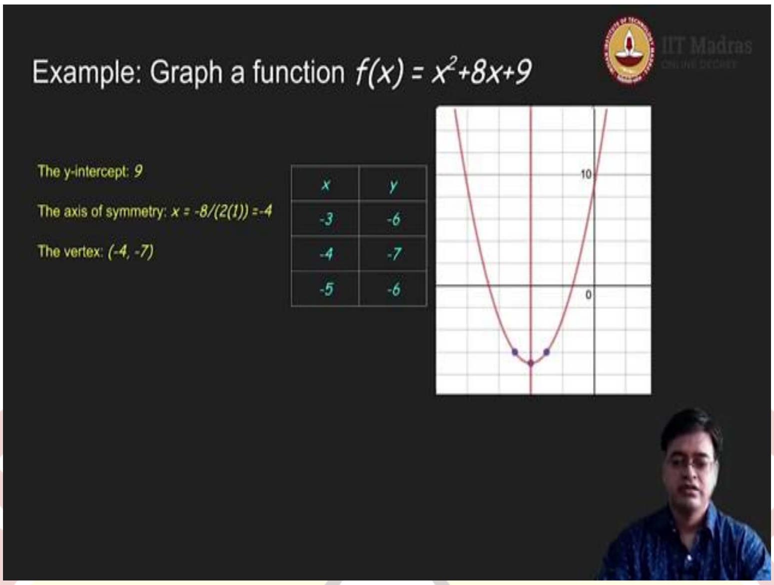


$f(x) = x^2 + 8x + 9$, I want to graph this function. So, I will reiterate on previous points. So, what is the $y$ intercept, $y$ intercept is 9, because if you put $x = 0$, the y intercept is 9. Next, I want to know the equation for axis of symmetry. So in this case, what is $b$, $b$ is 8, $a$ is 1, so $\frac{-b}{2a}$ is $\frac{-8}{2}$, which will give me axis of symmetry to be $x = -4$, $y$ intercept is 9, axis of symmetry is $x = -4$ so I can evaluate the coordinate that is $(-4, -7)$ will be the vertex. How this $-7$ comes, you just substitute $-4$ over here in this expression, you will get the value to be equal to $-7$.

So, now with these 3 terms, how will I draw the function? So, now I know that around vertex I need to find the points. So, based on this, I will draw a table. So, around $-4$ I have simply taken three points, fourth point is already with me, (0, 9) is the 4th point. So, around the around the point $-4$ I have taken the values so $-3$ which is the value of $-6$ when you substitute in the function $-4$, $-7$ already known and $-5$, $-6$. So, I have 3 points and the point (0, 9).

So, I will plot these points on a graph paper, take a graph paper, plot the axis of symmetry because around this the curve should be symmetric, take these 3 points, these 3 points are here and I know (0, 9) is another point. So, it should be somewhere here (0, 9). Now, let us plot a graph. So, now we have plotted a graph with much ease because of the knowledge of axis of symmetry I know where the point where the minimum has occurred or the vertex point is that is the beauty of axis of symmetry. So, this is how you will be able to plot any function any quadratic function given to you, this is about the graphing of a function.

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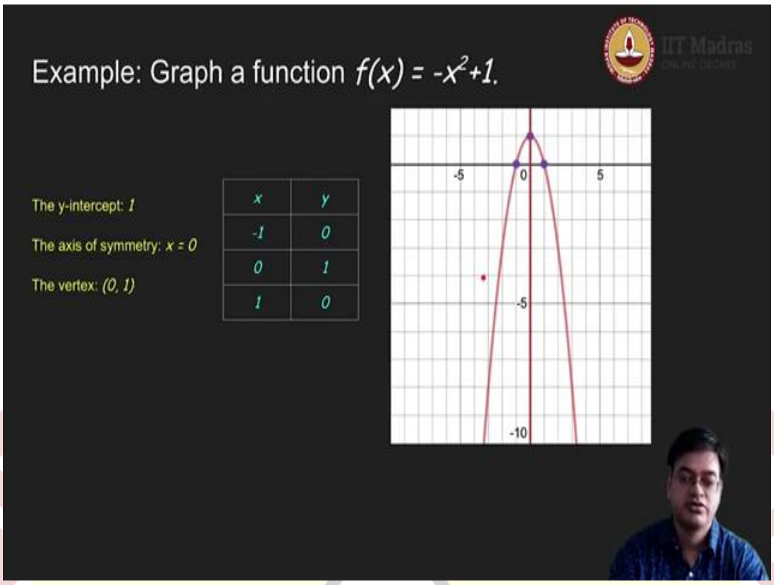


Let us try to see, is this the only shape that is possible that is the upward shape. Let us try to figure out whether is this a quadratic function first of all, $-x^2 + 1$, the answer is yes, $a = -1$, $b = 0$ and $c = 1$. Now, in this case, let us try to figure out the 3 summaries that is what will be the $y$ intercept for this? $y$ intercept will be 1, what will be the axis of symmetry for this because b is 0, it does not matter what is the value of $a$ it will be 0.

So, $x = 0$ is the axis of symmetry that is $y$ axis is the axis of symmetry for this particular function. And the vertex is (0, 1). In this case, we are not really getting much information because what this is saying is (0, 1) is the $y$ intercept that is (0, 1) is the coordinate, axis of symmetry is 0 that means (0, 1) is the vertex as well, right?

But still this information will suffice because I know I have to find the points around 0. So, let Figure out the points around 0; $-1$, 0 and 1, these are the 3 points, their y coordinates respectively are 0, 1, 0. Now you see there is a change, earlier we were only dealing with positive side of $y$ axis or the $y$ axis where the curve is opening up, here the curve is opening down. For example, if I plot an axis of symmetry over here, which is $y$ axis and if I plot these 3 points, these 3 points look like this that means the curve will go downward. So, the curve is opening down, why this has happened.

In earlier examples, if you look at it closely then this the form, general form of this expression $ax^2 + bx + c$, in all of them $a$ was equal to 1, and in this particular expression $a = -1$ therefore, the curve is actually opening down instead of opening up, this point needs to be noted.

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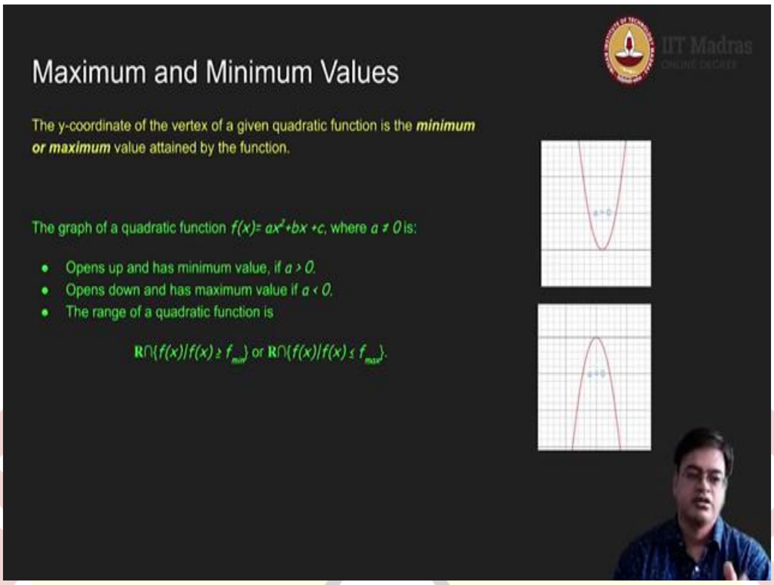


So, let us know this point and figure out what happens when this $a$ is greater than 0 or is less than 0. So, that leads us to the next question that is maximum and minimum values. So, the $y$ coordinate of the vertex of a given quadratic function is minimum or maximum value attained by the function.

Do you all agree with this, we have seen 3 to 4 graphs of the functions, first we have seen $y = x^2$ where it goes to bottom and 0 is the minimum value, then we have seen $y = x^2 + 2x + 1$which again gave us 0 value, then finally we have seen Third graph that we have seen the last graph that is$-x^2 + 1$, because the value of a was negative, it was going downward, and that will give me the maximum value and all the values are below that value.

So, the $y$ coordinate of the vertex of a given quadratic function gives us the minimum or maximum value attained by the quadratic function. In particular, given any graph given any function $f(x) = ax^2 + bx + c$ where $a \neq 0$. The graph of this quadratic function if a is greater than 0 will open upwards and will have a minimum value.

If a is less than 0, the graph will open downwards and will have the maximum value and there will be either maximum or minimum values, not both, this is the beauty of the quadratic function. Another thing that you can see is the range of the quadratic function, if you relate to your weak 1 background, where you are discussing about the domain-codomain range, so the range of this quadratic function will be.

So, let us say a is greater than 0, then it attains the minimum value then it will be the minimum value and all of the real line that is above the minimum value. And if a is less than

0, then it will be the maximum value and an entire real line which is below that particular thing, I can denote this using the set theoretic notation as it is a set of real numbers R intersected with a set of all $f(x)$, these $y$ values such that $f(x)$ is greater than or equal to $f$ min when $a > 0$, or if $a < 0$ it is set of real numbers intersected with $f(x)$ such that $f(x)$ is less than or equal to the maximum value that $f$ has achieved.

So, let us try to visualize this. For example, if $a > 0$ your graph looks like this. So, in particular the range of the value, range of $y$ values is from this point to upward. So, this is the entire real line above this value. Similarly, if $a < 0$, the range of the values that is taken by this function is this, if you relate this to domain codomain terminology, what is the domain of this quadratic function, it is an entire real name and range is restricted to some subset of real life.

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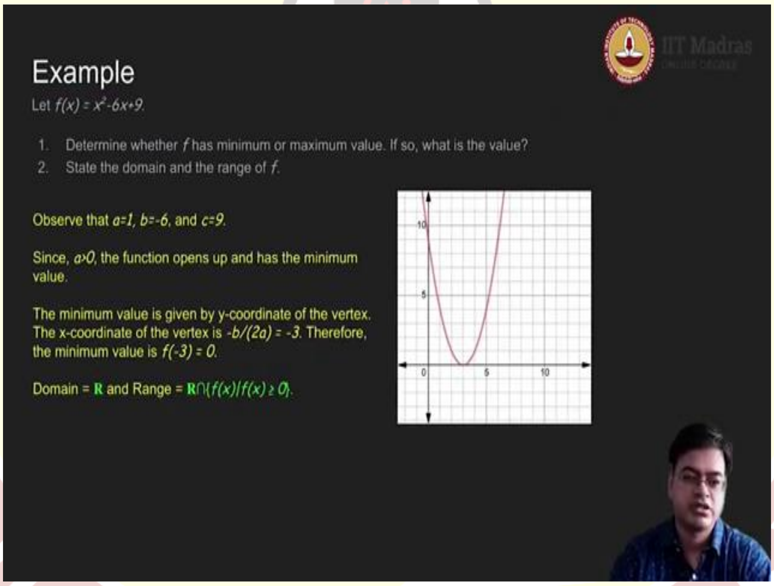


So, we will try to improve upon this concept using this example.