More on Graphs

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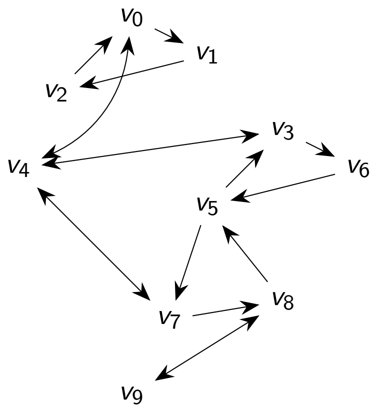
Mathematics for Data Science 1
Week 10

Graphs

- Graph $G = (V, E)$
  - V — set of vertices
  - $E \subseteq V \times V$ — set of edges

Airline routes



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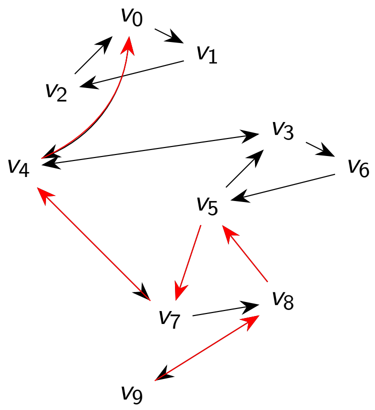

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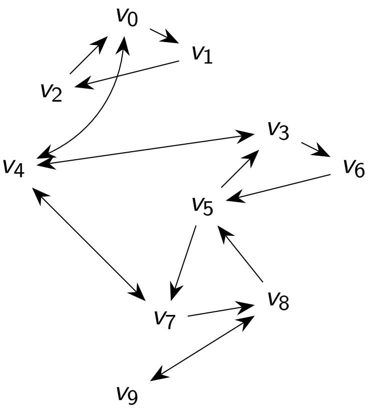

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- What more can we do with graphs?

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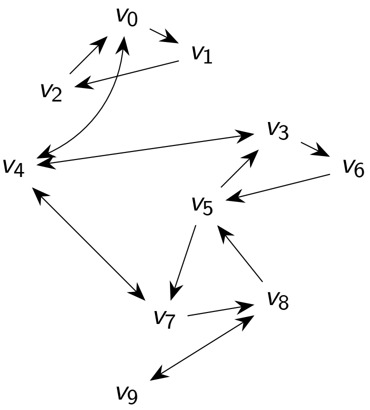

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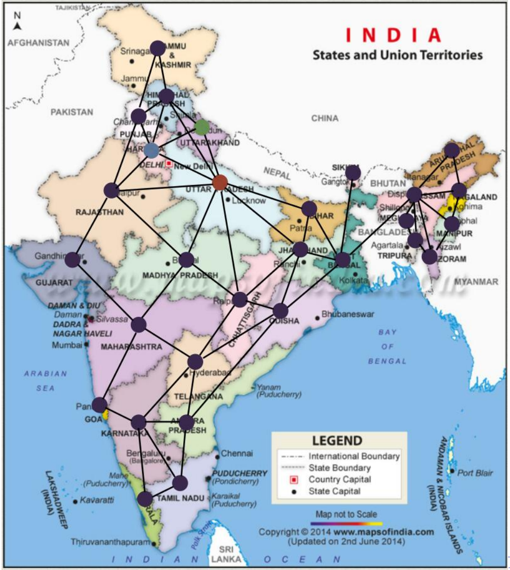

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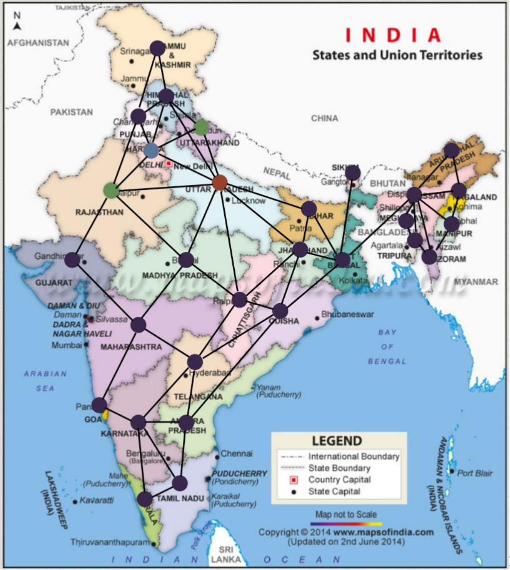

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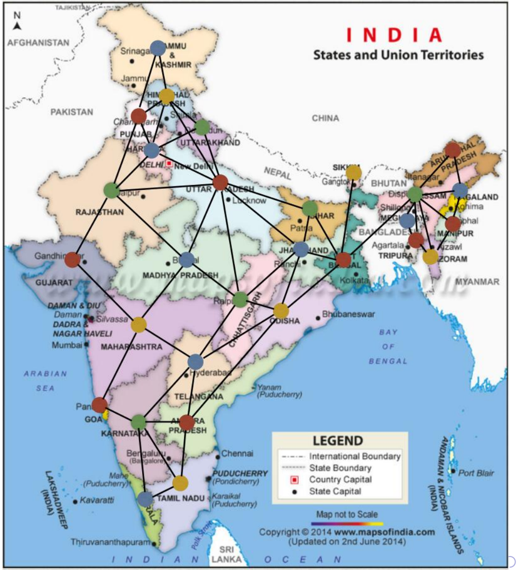

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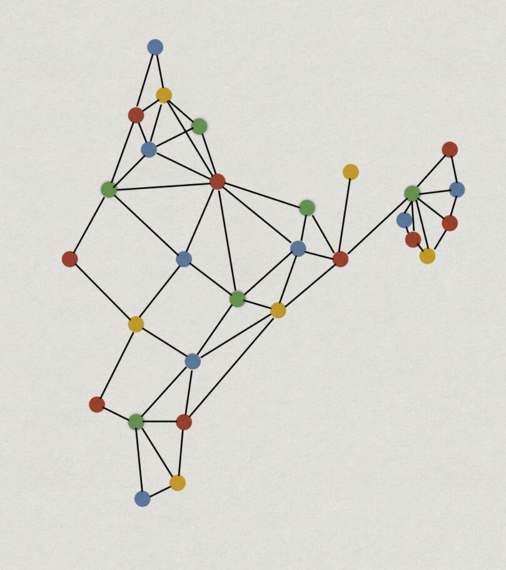

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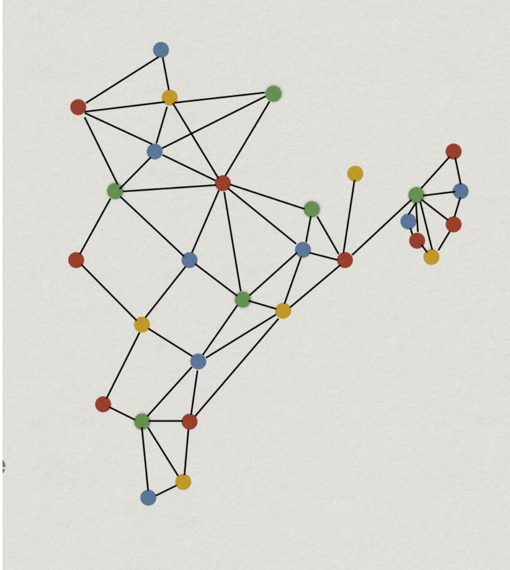
- Abstraction: if we distort the graph, problem is unchanged



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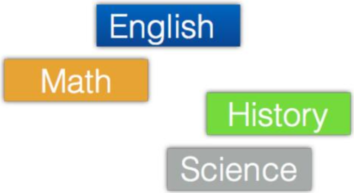

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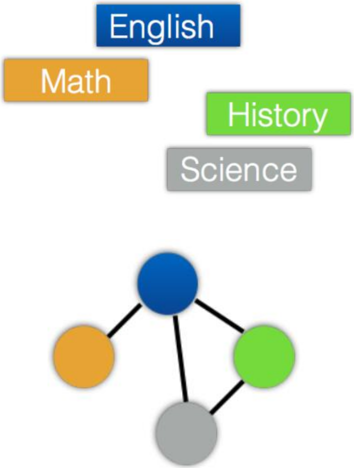

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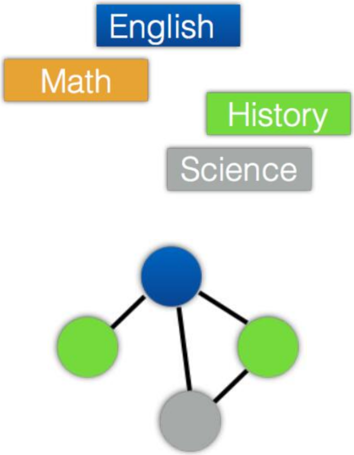

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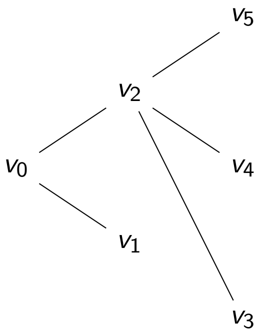

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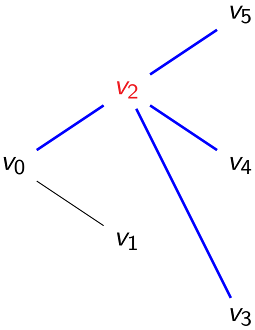

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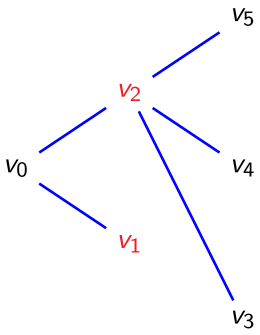

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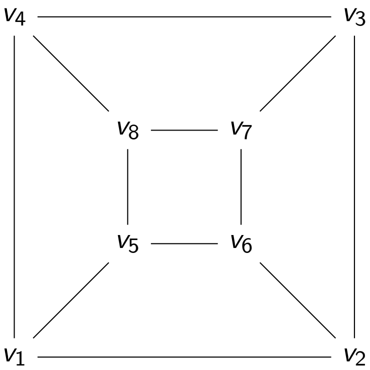

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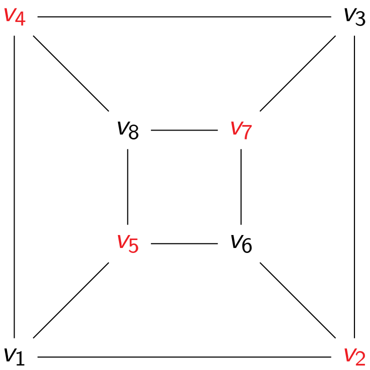

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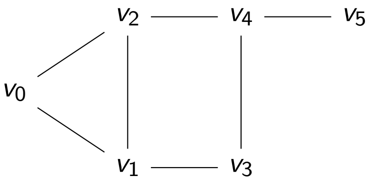

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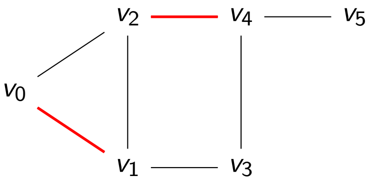

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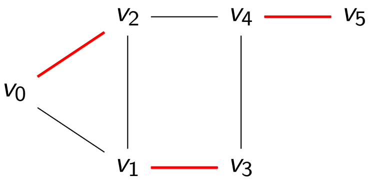
- Is there a perfect matching, covering all vertices?



Summary

- Graphs are useful abstract representations for a wide range of problems
- Reachability and connectedness are not the only interesting problems we can solve on graphs
  - Graph colouring
  - Vertex cover
  - Independent set
  - Matching
  - ...