IIT Madras

ONLINE DEGREE

Functions

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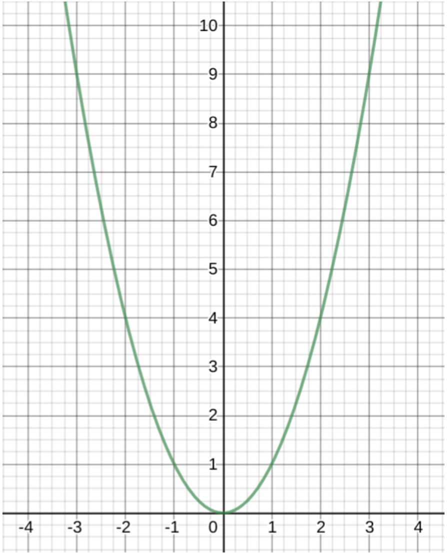
Mathematics for Data Science 1
Week 1

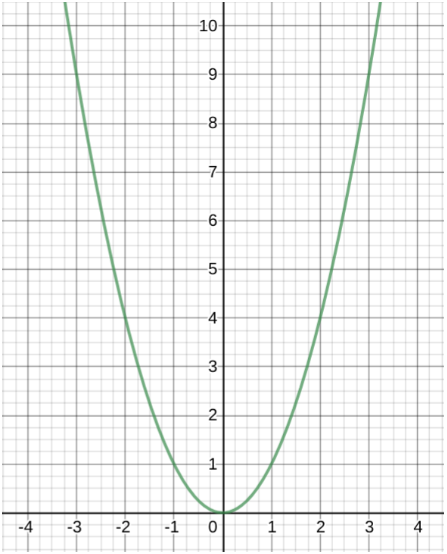
Functions

- A rule to map inputs to outputs
- Convert x to x^2
  - The rule: $x \mapsto x^2$
  - Give it a name: $sq(x) = x^2$
    - Input is a parameter
- Need to specify the input and output sets
  - Domain: Input set
    - $domain(sq) = \mathbb{R}$
  - Codomain: Output set of possible values
    - $codomain(sq) = \mathbb{R}$
  - Range: Actual values that the output can take
    - $range(sq) = \mathbb{R}_{\geq 0} = \{r \mid r \in \mathbb{R}, r \geq 0\}$
- $f : X \to Y$, domain of f is X, codomain is Y



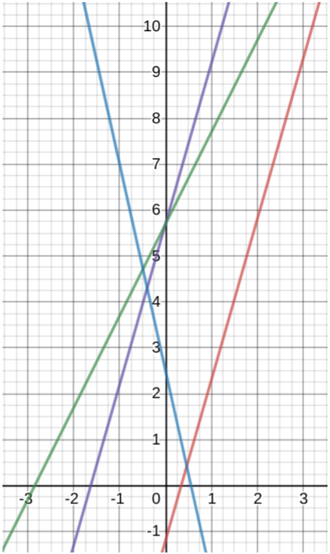
Functions and relations

- Associate a relation R_f with each function f
- $R_{sq} = \{(x, y) \mid x, y \in \mathbb{R}, y = x^2\}$
  - Additional notation: $y = x^2$
- $R_f \subset domain(f) \times range(f)$
- Properties of R_f
  - Defined on the entire domain
    - For each $x \in domain(f)$, there is a pair $(x, y) \in R_f$
  - Single-valued
    - For each $x \in domain(f)$, there is exactly one $y \in codomain(f)$ such that $(x, y) \in R_f$
- Drawing f as a graph is plotting R_f



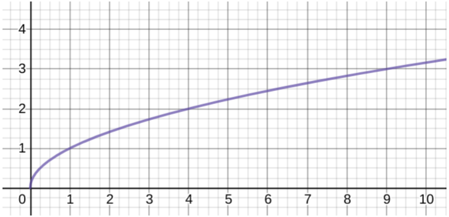
Lines

- $f(x) = 3.5x + 5.7$
  - 3.5 is the slope
  - 5.7 is intercept where the line crosses the y=axis, whe $x = 0$
- Changing the slope and intercept produce different lines
  - $f(x) = 3.5x - 1.2$
  - $f(x) = 2x + 5.7$
  - $f(x) = -4.5x + 2.5$
- In all these cases
  - Domain $= \mathbb{R}$
  - Codomain $=$ Range $= \mathbb{R}$



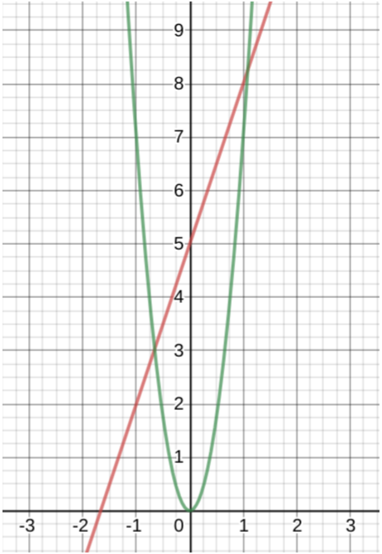
More functions

- $x \mapsto \sqrt{x}$
- Is this a function?
  - $5^2 = (-5)^2 = 25$
  - $\sqrt{25}$ gives two options
  - By convention, take positive square root
- What is the domain?
  - Depends on codomain
  - Negative numbers do not have real square roots
  - If codomain is $\mathbb{R}$, domain is $\mathbb{R}_{\geq 0}$
  - If codomain is the set $\mathbb{C}$ of complex numbers, domain is $\mathbb{R}$



Types of functions

- Injective: Different inputs produces different outputs — one-to-one
  - If $x_1 \neq x_2$, $f(x_1) \neq f(x_2)$
  - $f(x) = 3x + 5$ is injective
  - $f(x) = 7x^2$ is not: for any a, $f(a) = f(-a)$



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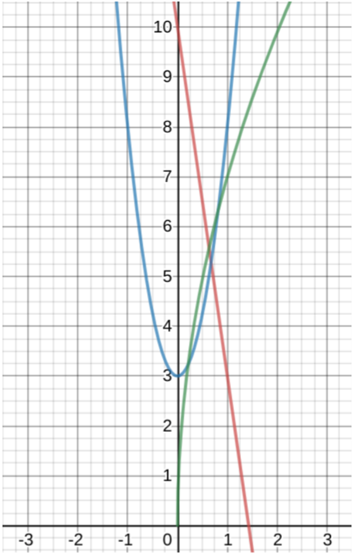
- Surjective: Range is the codomain — onto
  - For every $y \in codomain(f)$, there is an $x \in domain(f)$ such that $f(x) = y$
  - $f(x) = -7x + 10$ is surjective
  - $f(x) = 5x^2 + 3$ is not surjective for codomain $\mathbb{R}$
  - $f(x) = 7\sqrt{x}$ is not surjective for codomain $\mathbb{R}$



Properties of functions ...

- Bijective: $1-1$ correspondence between domain and codomain
  - Every $x \in domain(f)$ maps to a distinct $y \in codomain(f)$
  - Every $y \in codomain(f)$ has a unique pre-image $x \in domain(f)$ such that $y = f(x)$

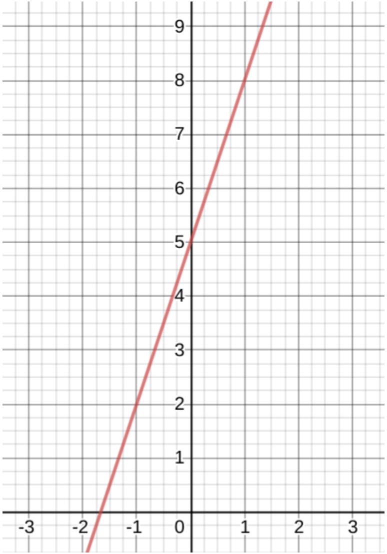
Theorem

A function is bijective if and only if it is injective and surjective

- From the definition, if a function is bijective it is injective and surjective
- Suppose a function f is injective and surjective
  - Injectivity guarantees that f satisfies the first condition of a bijection.
  - Surjectivity says every $y \in codomain(f)$ has a pre-image. Injectivity guarantees this pre-image is unique.

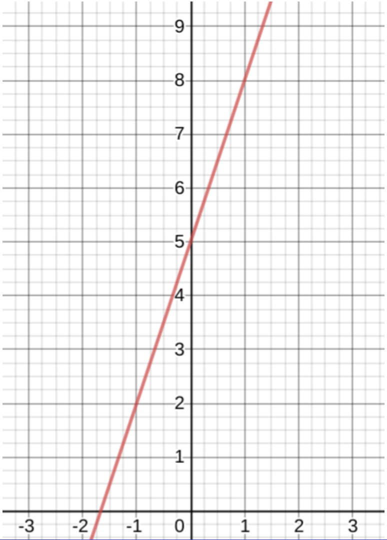
Bijections and cardinality

- For finite sets we can count the items
- What if we have two large sacks filled with marbles?
  - Do we need to count the marbles in each sack?
  - Pull out marbles in pairs, one from each sack
  - Do both sacks become empty simultaneously?
  - Bijection between the marbles in the sacks
- For infinite sets
  - Number of lines is the same as $\mathbb{R} \times \mathbb{R}$
  - Every line $y = mx + c$ is determined uniquely by (m, c) and vice versa



Bijections and cardinality ...

- For every pair of points (x_1, y_1) and $(x_2.y_2)$, there is a unique line passing through both points
- Number of lines is same as cardinality of $\mathbb{R} \times \mathbb{R}$
- Does this show that $(\mathbb{R} \times \mathbb{R}) \times (\mathbb{R} \times \mathbb{R}) = \mathbb{R}^2 \times \mathbb{R}^2$ has the same cardinality as $\mathbb{R} \times \mathbb{R}$?
- The correspondence is not a bijection — many pairs of points describe the same line
- Be careful to establish that a function is a bijection



Summary

- A function is given by a rule mapping inputs to outputs
- Define the domain, codomain and range
- Associate a relation R_f with each function f
- Properties of functions: injective (one-to-one), surjective (onto)
- Bijections: injective and surjective (one-to-one and onto)
- A bijection establishes that domain and codomain have same cardinality