Minimum Cost Spanning Trees

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Mathematics for Data Science 1
Week 12

Examples

Roads

- District hit by cyclone, roads are damaged
- Government sets to work to restore roads
- Priority is to ensure that all parts of the district can be reached
- What set of roads should be restored first?

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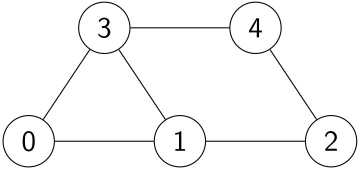
Fibre optic cables

- Internet service provider has a network of fibre optic cables
- Wants to ensure redundancy against cable faults
- Lay secondary cables in parallel to first
- What is the minimum number of cables to be doubled up so that entire network is connected via redundant links?

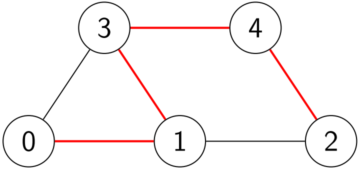
Spanning trees

- Retain a minimal set of edges so that graph remains connected



Spanning trees

- Retain a minimal set of edges so that graph remains connected
- Recall that a minimally connected graph is a tree
  - Adding an edge to a tree creates a loop
  - Removing an edge disconnects the graph



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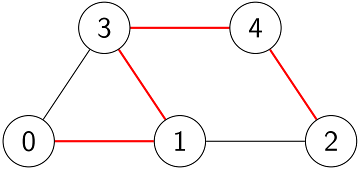

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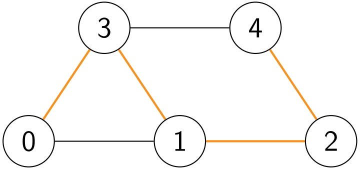
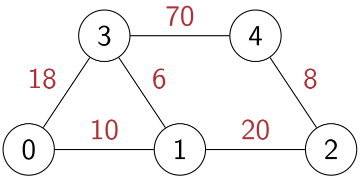
- More than one spanning tree, in general



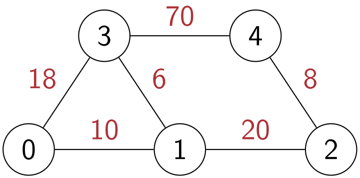
Spanning trees with costs

- Restoring a road or laying a fibre optic cable has a cost



Spanning trees with costs

- Restoring a road or laying a fibre optic cable has a cost
- Minimum cost spanning tree
  - Add the cost of all the edges in the tree
  - Among the different spanning trees, choose one with minimum cost



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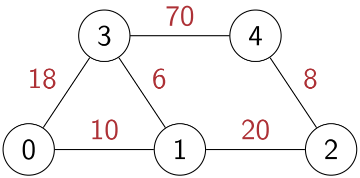

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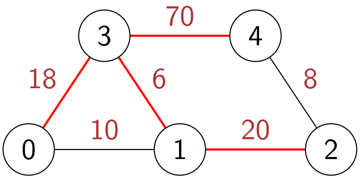

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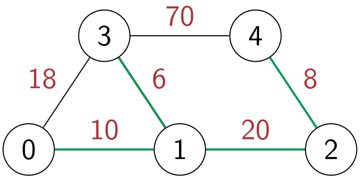
  - Anonther spanning tree, Cost is 44 — minimum cost spanning tree



Some facts about trees

Definition A tree is a connected acylic graph.

Fact 1

A tree on n vertices has exactly $n - 1$ edges

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Fact 2

Adding an edge to a tree must create a cycle.

- Suppose we add an edge (i,j)
- Tree is connected, so there is already a path from i to j
- The new edge (i,j) combined with this path from i to j forms a cycle

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In a tree, every pair of vertices is connected by a unique path.

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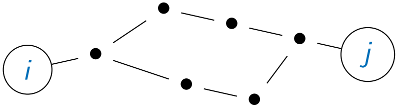
- If there are two paths from i to j, there must be a cycle



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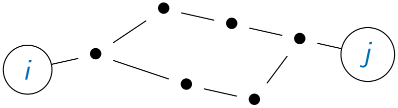

Observation

Any two of the following facts about a graph G implies the third

- G is connected
- G is acyclic
- G has $n-1$ edges

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- We will use these facts about trees to build minimum cost spanning trees
- Two natural strategies
- Start with the smallest edge and “grow” a tree
  - Prim’s algorithm
- Scan the edges in ascending order of weight to connect components without forming cycles
  - Kruskal’s algorithm