Riemann sums and the integral

Sarang S. Sane

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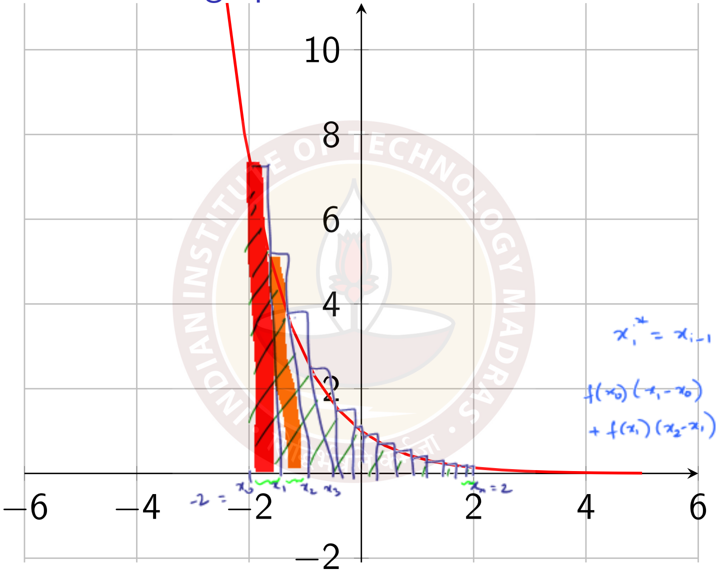
4. As the sizes decrease, the area of the shape is better and better approximated.
5. In the limit , you will get the area of the shape.

The area under the graph of a function



Graph of $y = e^{-x}$

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The Riemann sum of f w.r.t. the above data is defined as

$$S(P) = \sum_{i=1}^{n} f(x_i^*)\Delta x_i.$$

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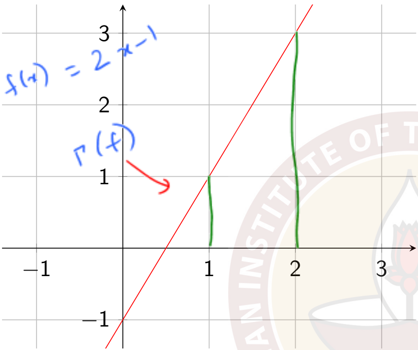
Let $f \geq 0$ (resp. $f \leq 0$) be piecewise continuous on this interval. Then the area under the graph of the function $\Gamma(f)$ above the interval $[a, b]$ is measured by $\displaystyle\int_a^b f(x)dx$ (resp. $\displaystyle -\int_a^b f(x)dx$).

Example



$$P_n = \{ x_0 = 1,\ x_1 = 1 + \tfrac{1}{n},\ x_2 = 1 + \tfrac{2}{n},$$

$$x_i = 1 + i/n.$$

$$1 = x_0 < x_1 < x_2 < \dots < x_n = 2.$$

$$\Delta x_i = \frac{1}{n}. \qquad \|P_n\| = \frac{1}{n}.$$

$$x_i^* = x_i = 1 + i/n$$

$$S(P_n) = \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i$$

$$= \sum_{i=1}^{n} \left\{ 2\left(1 + \frac{i}{n}\right) - 1 \right\} \frac{1}{n}$$

$$= \sum_{i=1}^{n} \left(1 + \frac{2i}{n}\right) \frac{1}{n}$$

$$= \sum_{i=1}^{n} \frac{1}{n} + \sum_{i=1}^{n} \frac{2i}{n^2}$$

$$= 1 + \frac{2}{n^2} \sum_{i=1}^{n} i$$

$$= 1 + \frac{2}{n^2} \cdot \frac{n(n+1)}{2}$$

$$= 1 + \frac{n+1}{n} = 2 + \frac{1}{n}.$$

$$\lim_{\|P_n\| \to 0} S(P_n) = \lim_{n \to \infty} 2 + \frac{1}{n} = 2.$$

Thank you