IIT Madras

ONLINE DEGREE

Relations: Examples

Madhavan Mukund

https://www.cmi.ac.in/~madhavan

Mathematics for Data Science 1
Week 1

Relations

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  - $S = \{(a, b) \mid (a, b) \in A \times B, b = a^2\}$

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  - If there is a direct flight from Bangalore to Delhi, is there always a direct flight back from Delhi to Bangalore
  - For bigger cities, yes
  - For smaller cities, may have a triangular route
    Chennai $\rightarrow$ Madurai $\rightarrow$ Salem $\rightarrow$ Chennai

Tables as relations

- Flying distances between cities

Source	**Destination**	**Distance (km)**
Bangalore	Chennai	290
Chennai	Delhi	1752
Delhi	Bangalore	1735
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- Save space by representing only one direction in the table

Tables as relations . . .

Roll no	**Name**	**Date of birth)**
A71396	Abhay Shah	03-07-2001
B82976	Payal Ghosh	18-06-1999
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  - (key,value) pairs

Operations on relations

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- $\{(r, n, s, g) \mid (r, n, d) \in \text{Students}, (r', s, g) \in \text{Grades}, r = r'\}$

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Summary

- A relation describes special tuples in a Cartesian product
- Data tables are essentially relations
- Combining information on tables can be described in terms of operations on relations