IIT Madras

ONLINE DEGREE

Mathematics for Data Science 2
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Lecture No. 06
Limits and Continuity

Hello and welcome to the maths 2 component of the online degree programme on data science and programming. In this video, we are going to talk about limits and continuity.

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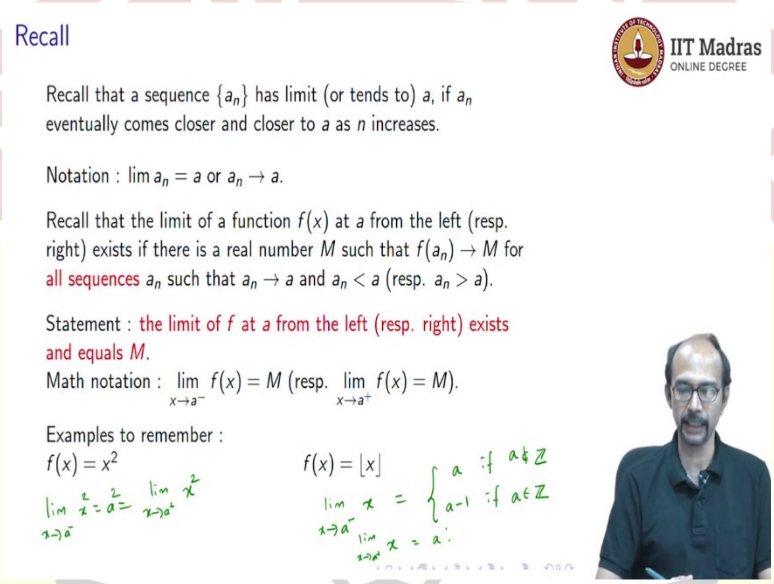

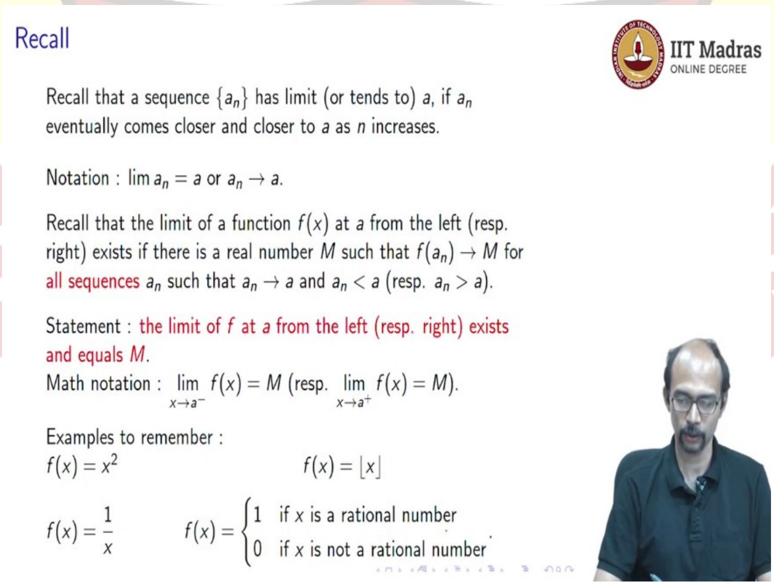

So, in the previous videos, we have been talking about the notion of limits let us recall what we have done. So, recall that a sequence a_n has limit or we say tends to a if a_n eventually comes closer and closer to a, as n increases. So, I have not made this very rigorous in a mathematical sense but I hope you have a feeling for what that means. So, the notation is lim $a_n = a$ or $a_n \rightarrow a$, so that arrows you should say tends to a.

So, recall that the limit of a function f(x) at a from the left respectively from the right exists if there is a real number M such that $f(a_n) \rightarrow M$ for all sequences a_n. So, the important part here is that it should tend for all sequences and not only for some sequences such that $a_n \rightarrow a$ and $a_n < a$ that when we say tends from the left and for the right, we have the same condition except that $a_n > a$.

So, the statement that we will say for the sentence in mathematical terms is the limit of f at a from the left exists and equals M and similarly, for the right we will say the limit of f at a from the right exists and equals M and the math notation for the left hand side is $\lim\limits_{x \to a-.} f(x)$ and for the right is $\lim\limits_{x \to a+.} f(x)$.

So, the examples that we saw in the last videos where f(x) $=x^2$ and f (x)= $\lfloor x \rfloor$, the floor function of x which is a step function. So for the case f(x) $=x^2$, we saw that if you take any $a_n \rightarrow a$, then f(a_n) indeed tends to f(a), so f(a_n)$=a_n^2$, f(a)$=a^2$, so $a_n^2 \rightarrow a^2$, this was a property of convergence of sequences.

On the other hand, for the step function the floor function of x we saw that for all non integer values, indeed we have conversions from both the left and the right to the point the floor of x but, when x is an integer then, the limit from the left is one less than, the limit from the right which is indeed the floor function of x which in that case is x itself because it is an integer. So, just to be clear $\lim\limits_{x \to a-.} x^2 = a^2 = \lim\limits_{x \to a+.} x^2$.

Whereas $\lim\limits_{x \to a-.} \lfloor x \rfloor = a - 1$if a is an integer and $\lim\limits_{x \to a+.} \lfloor x \rfloor = a$if a is an integer.

So we have some points where the left and right limits do not match, namely the integers and then some other examples where f(x) =1/x where if you remember at 0 the positive coming from the positive side from the right hand side makes it shoot of to ∞ and from the left hand side makes it shoot of to - ∞.

And then there was this really crazy example f(x)=1 if x is rational

=0 if x is not rational

and here for all the limit was undefined from either the left or the right. So we had all these possible situations occurring.

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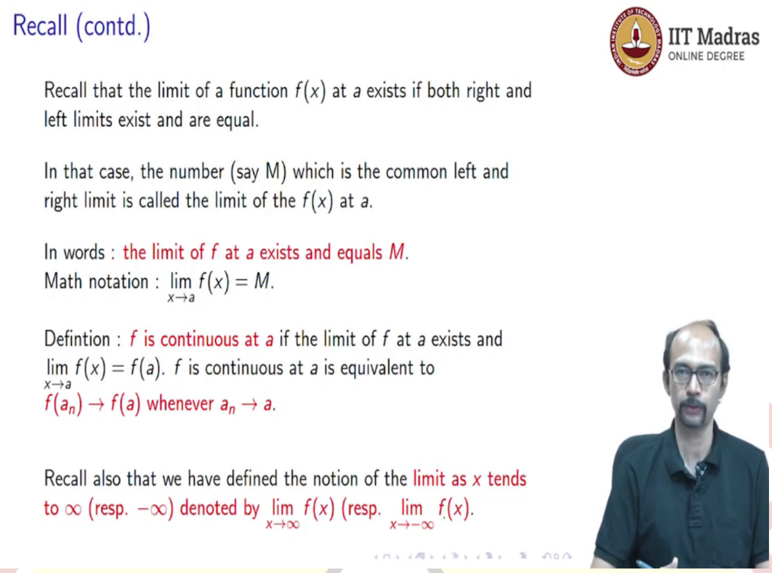

Recall it the limit of a function f(x) at a exists if both the right and the left limits exist and are equal and in that case, the number which is the common left hand or right limit, say, that is M that is called the limit of f at a so in words we will say the limit of f at a exists n equals M and the math notation for this is $\lim_{x \to a} f(x) = M$and then we had this definition of continuity, which was f is continuous at a if the limit of f at a exists and $\lim_{x \to a} f(x) = f(a)$.

So, not only do we want that these left and right limit exists and match, but that matching number must be equal to f (a), then we say that f is continuous at a. So, f is continuous at a is equal to saying that, if you take any sequence $a_n \to a$, then if you take $f(a_n)$, then that converges to f (a), so then we have also seen what we have defined this notion of the limit as x tends to ∞ or -∞.

So I will suggest you go back and check what that is or if x becomes very large and f (x) tends to some value then we will say that $\lim_{x \to \infty} f(x)$ exists and equals that number and similarly for $\lim_{x \to -\infty} f(x)$.

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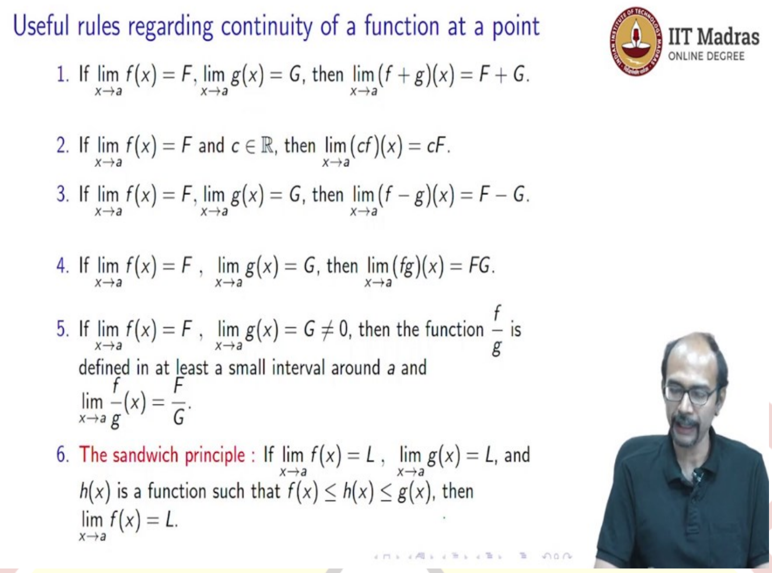

So, all this is fine, but how do we actually compute these limits and how do we know continuity of a function at a point if the function is fairly complicated. We have seen easy examples, like f (x) = x, f(x)=x^2 and so on. So here are some rules, we have seen similar rules for convergence of sequences and now we are going to see the same kind of rules for continuity of a function at points.

So, if $\lim_{x\to a} f(x) = F$and $\lim_{x\to a} g(x) = G$, then $\lim_{x\to a}(f+g)(x) = F+G$, so the limit just adds up. If $\lim_{x\to a} f(x) = F$ and you have a constant real number c, then $\lim_{x\to a}(cf)(x) = cF$. So, if $\lim_{x\to a} f(x) = F$and $\lim_{x\to a} g(x) = G$, then $\lim_{x\to a}(f-g)(x) = F-G$.

So, you can derive this third one from the first two by first noting that if you take limit of -g which is to say you take c to be -1 and multiplied to g, then you will get that the limit of that function -g is -G and then, you add f and -g, which is exactly f- g and then you will get F- G.

So, if you take the product fg, so since these are functions from real numbers to real numbers the the product makes sense and if you take if $\lim_{x\to a} f(x) = F$and $\lim_{x\to a} g(x) = G$, then$\lim_{x\to a} fg(x) = FG$. If you take f/g, first of all, it has to be defined around a which it is because, if $\lim_{x\to a} f(x) = F$and$\lim_{x\to a} g(x) = G$, and G is not 0, then that means , there is some small part, small interval around a where g(x) is not 0, something you have to prove and so the function f/g makes sense, at least in that small interval and hence, we can talk about the limit $\lim_{x\to a} \frac{f}{g}(x)$ a and that is indeed $\frac{F}{G}$.

So, the key point here is that, this G must be non zero. Finally, we have something called the sandwich principle, so this is very important, we have seen this sandwich principle in action in one of the examples before, so what is the sandwich principle?

If $\lim_{x \to a} f(x) = L$and $\lim_{x \to a} g(x) = L$, both are the same (remember) and you have h(x), which is a function such that f(x) ≤ h(x) ≤ g(x), then $\lim_{x \to a} h(x) = L$. Now, actually you do not need these inequalities for the entire domain, if these inequalities are true for some small intervals around your point a, then that is still good enough for the theorem to work and indeed that is a form we will use it.

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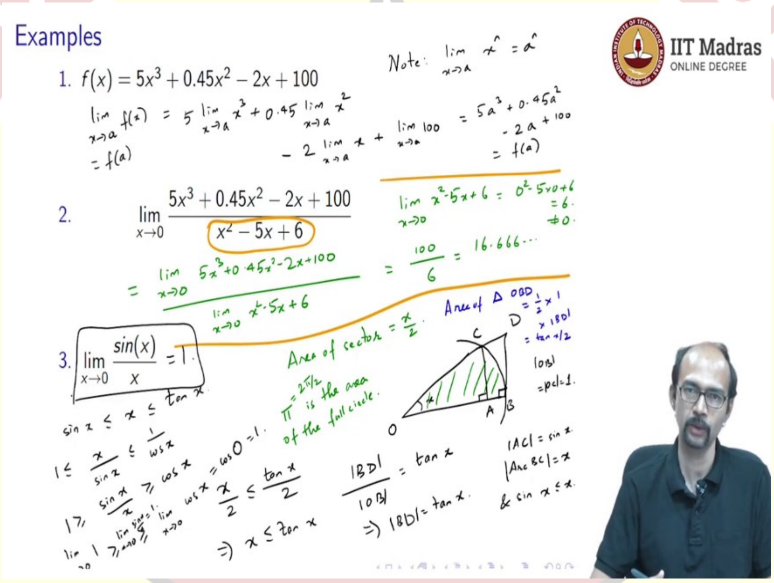

So, here is a couple of examples. Let us go through them, so if you take f(x)= $5x^3 + 0.45x^2 - 2x + 100$, how do I compute the limit. So, we will see quickly how to do that let us look at another example, suppose you take the same function as above and you divide it by $x^2 - 5x + 6$ and you want to compute the limit as x → 0 and then suppose I have $\frac{\sin x}{x}$. So we have seen this example before and we will talk about this in more detail in right now so, then what is the limit as x → 0?

So, let us first look at the first, the first example so, if you want this limit x → a, what is $\lim_{x \to a} f(x)$? Note, $\lim_{x \to a} x^n = a^n$, is something that we saw in the previous video and I will also remind you that you can do it from the rules that we saw in the in the previous slide, because x^n is like multiplying x, n times. And we know that, if the limit exists, then the limit of the product is the product of the limit, so from there this will follow.

So, from here, what we can do then is, that each of these limits individually exists and so we can apply the previous the result in the previous slide and we can say this is $5\lim_{x\to a} x^3 + 0.45\lim_{x\to a} x^2 - 2\lim_{x\to a} x + \lim_{x\to a} 100$so , you think of 100 as the function g (x)= 100.

So, now let us write down the limits. So the first limit is a^3 , the second limit is a^2, the third limit is a, and the fourth limit is 100. If you have a constant function, then the limit as x tends to a is again the same, it has the same value, whatever that constant is, so this is $5a^3 + 0.45 a^2 -2a +100$, so this is exactly f(a).

So, $\lim_{x\to a} f(x) = f(a)$, so in other words, this function is continuous at a and note that it does not matter what a is, a can be any real number, and note, how we got this result, so if you we can do this for any polynomial, but the main take home from this is the same proof will say that, if you take any polynomial function, then it is continuous at every point. In this case $\lim_{x\to a} f(x)$, where f is a polynomial is indeed f(a).

Let us do the second example, so in the second example I have, what is called a rational function, meaning a polynomial divided by a polynomial and here we want the limit as x tends to 0, so note that in the denominator here, you have x^2-5x+6. So of course, when you have denominators in general, you have to be careful. So, if you want to compute the limit, as x tends to, let us say 2 or 3 which are roots of this polynomial, so if you put x is equal to 3 you get 9 -15 + 6 which is 0 or if you put 2, then you have 4 - 10 + 6 which is 0.

So, if you, if you want to compute the limit of $x \to 2$ then there is trouble, but if you want to compute the limit as $x \to 0$, then you do not have any trouble, because the denominator in that case, the limit is non zero, why is that, what is the limit of the denominator? So, notice that the denominator is a polynomial so for the denominator, you get

x^2-5x+6 , you just evaluate, so this is $0^2-5(0) +6 =6$, which is non zero.

So, now we can apply our previous result of the previous slide, which said that if you have a quotient $\frac{f}{g}$and the limit of both numerator and denominator exists, which is indeed because the numerator is also a polynomial, we just computed what its limit is for any a and the denominator, the limit is non zero then indeed, this limit exists and this limit is actually equal to numerator limit divided by the denominator limit (you take the individual limits). So, we know what these two are the one on top you evaluated 0 that means you get just the constant term 100 and the denominator we have done over here, that $\frac{100}{6}$, I guess that is something like 16.666.

So, anyway value is not the main point. So, you can compute this limit by using the rule that we have described on the previous slide. So, you can see that many limits can be just done by using the rules in the previous slide and that is the point. So you get some limits that you know and then you use the rules that we have established and use them to compute other limits.

Now, here is $\lim_{x\to 0}\frac{\sin x}{x}$. I gave some brief indication of how to do this in the previous video. So now, let me discuss this at length over here. So the trick here is to really draw the picture. So you take your unit circle and suppose this angle is x in radians. So, the radians is important. You draw your perpendicular down here. You will extend this line until you can draw this. So this is how it should look like and maybe let me give it name so that, we can explicitly write down what is what, so now we have 2 right angle triangles, this here, this thing here is the unit circle, by unit circle, I mean, circle of radius one. So the length of OB is the length of OC is 1, so what is length of AC?

So, the length of AC is sin x and it is clear from here that since this is x radians, the arc BC has length x and sin x ≤ x. So this much we know. Now let us try to see how to get an inequality for the x part. So here, what we have to do is, we have to consider the length of BD, so what is the length of BD? So, $\frac{Length of BD}{Length of OB}$. So again, this is a right angle triangle remember, so this is $\frac{Opposite side}{Adjacent side}$. So, this is tan x , but length OB , remember , is 1 that means, the length of BD is tan x, that is something to keep in mind.

And now let us compute the areas the two areas that we will want one area is the area of the sector, so area of sector, let me put a line here, so the area of the sector, how do I get this? So, this is a general fact from high school geometry that if you have angle x, then the area of the sector is going to be $\frac{x}{2}$. So, in case you are wondering why that is the case remember, that if you have 2π radians, which is the entire circle, then you the area of that sector is πr^2 , which is in this case, just π, which I meant to write as $\frac{2\pi}{2}$, so the entire angle has is 2π radians and the area you get this π.

So, if you take half of it, you have π radians and the area you are going to get is $\frac{\pi}{2}$. one fourth, you have $\frac{\pi}{2}$ radians and you will get the area is $\frac{\pi}{4}$. So, in general, this the radians to area, the way it works is the area is half of the angle in terms of radians.

So, here we have x radians, the angle is x radians, so the area of the sector is $\frac{x}{2}$, so that is something we know and the other thing we can say is, what is the area of the triangle OBD, so area of the triangle OBD, this is our usual area rule for our triangles, particularly right angled triangle, it is particularly easy, so this is $\frac{1}{2} \times Length of OB \times Length of BD$, but length of BD is tan x, so this is $\frac{2\pi}{2}$ and it is clear that the area of the sector is less than or equal to the area of the triangle OBD, so what is the net result?

So, the net result of all this is that $\frac{x}{2} \leq \frac{\tan x}{2}$, which means x ≤ tan x. So now let us put together these inequalities. So, we have sin x ≤ x ≤ tan x, and now, we can divide everything by sin x, so we get $1 \leq \frac{x}{\sin x} \leq \frac{1}{\cos x}$. Let us flip it over so if you do that, you get $1 \geq \frac{\sin x}{x} \geq$ cos x , and now we can apply the Sandwich rule.

So, if you apply the Sandwich rule, you get $\lim_{x \to 0} 1$, $\lim_{x \to 0} \cos x$, but what happens to cos x as x tends to 0? as x tends to 0, cos x just goes to cos 0, you can evaluate and cos 0 = 1. So, by the Sandwich rule, $\lim_{x \to 0} \frac{\sin x}{x}$ is indeed 1. So, this is what I have stated this in the previous video and this is what we have now given a demonstration of.

So, I hope you are convinced by the fact that we do not have to compute all the limits explicitly by hand by you keep taking sequences and then see what happens to the function when you apply to the sequence and so on. The rules in your previous slide, really are the ones that you have to use, only you have to know some limits beforehand. So, the easy ones you do by hand or by slightly harder ones by geometry like we did just know and then you keep applying these to newer and newer functions.

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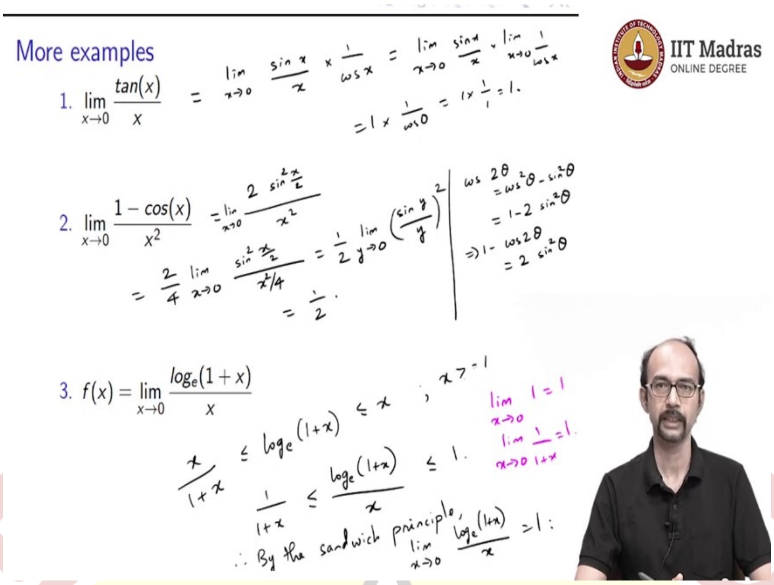

So, here are some more examples, what is $\lim_{x\to 0}\frac{\tan x}{x}$, what is $\lim_{x\to 0}\frac{1-\cos x}{x^2}$, and then finally, what is the $\lim_{x\to 0}\frac{\sin x}{x}$? So, I will maybe do one or two of these, and then I will postpone the others, or rather I leave the rest.

So, for example, if you take $\lim_{x\to 0}\frac{\tan x}{x}$, I can write this as $\lim_{x\to 0}\frac{\sin x}{x}\times\frac{1}{\cos x}$. Now, we can apply our previous slide, we have two functions, so we can call this first function, $\frac{\sin x}{x}$ as f(x), you can call the second one, $\frac{1}{\cos x}$ as g(x) and $\lim_{x\to 0}\frac{1}{\cos x}$ exists and equals $\frac{1}{\cos 0}$, which is 1, and the left hand side, we just computed to be 1. $\lim_{x\to 0}\frac{\sin x}{x} = 1$ So this limit is just the product of those two limits, 1×1. So this limit is 1.

Similarly, we can try and do this, so how do I do this? Well, so here you can use some trigonometric identities, so if you recall your trigonometry $\cos 2\theta = \cos^2\theta - \sin^2\theta$, which you can also write as $1-2\sin^2\theta$, so if you turn this over, that gives you $1 - \cos 2\theta = 2\sin^2\theta$. Now let us apply this here, so $1 - \cos x$ is $2\sin^2\frac{x}{2}$. So $\lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{x^2}$ That is what we get.

So, now what can I do, I can make the denominator of $\frac{x^2}{4}$. So I get $\frac{2}{4}\lim_{x\to 0}\frac{\sin^2\frac{x}{2}}{\frac{x^2}{4}}$.

Now, there are different ways of doing this, either you can just observe that x tends to 0 is the same thing as saying that $\frac{x}{2}$ tends to 0 or you can so called change variables.

And you can put $\frac{x}{2}$ to be y then, you can rewrite this as $\frac{1}{2}\lim_{2y\to 0}\frac{\sin^2 y}{y^2}$.

Then again 2 y goes to 0 is the same thing as saying that y goes to 0. So either which way, you can say this is $\frac{1}{2}\lim_{y\to 0}(\frac{\sin y}{y})^2$.

So this is again, you can think of it as a product, $\frac{\sin y}{y}\times\frac{\sin y}{y}$and both of them have limits as y tends to 0. It is 1. So this gives you that this is $\frac{1}{2}\times 1^2$, which is $\frac{1}{2}$. So this is how you can do this one.

Let us look at the third example, so we have seen this example in our previous video and we wrote down some inequalities for this example.

So, let us write down those inequalities again, $\frac{x}{1+x}\leqslant \log_e(1+x)\leqslant x$, this is for x>-1. So, now if we divide this by x, we get $\frac{1}{1+x}\leqslant\frac{\log_e(1+x)}{x}\leqslant 1$and now we can apply the sandwich principle.

And if you apply the Sandwich principle to the right hand function you get 1 because it is a constant function if you apply the limit as x tends to 0 to the left hand function, it is again an $\frac{f}{g}$type situation where g does not become 0 when x tends to 0, so you can use substitution, so if you substitute x = 0, you get the limit $\frac{1}{1}$, which is 1, so both limits tend to 1.

So, let us write that down, so $\lim_{x\to 0}1 = 1$, $\lim_{x\to 0}\frac{1}{1+x} = 1$, as I said is by substitution, which works because both numerator and denominator have limits as x tends to 0 and the denominator has limit which is non zero, so both have limit 1, so $\frac{1}{1}$ is 1. This is how you get 1 here. So therefore, by the Sandwich principle, we can calculate this limit to be 1.

So, this is not extremely intuitive and or at least rather than unintuitive let me say it depends on these inequalities for the logarithm function which or not tremendously hard to prove, but it is something which is beyond the scope of what we are doing in this course. However, we will see this limit in a different way in a subsequent lecture, when we will use something called L'Hopital's Rule.

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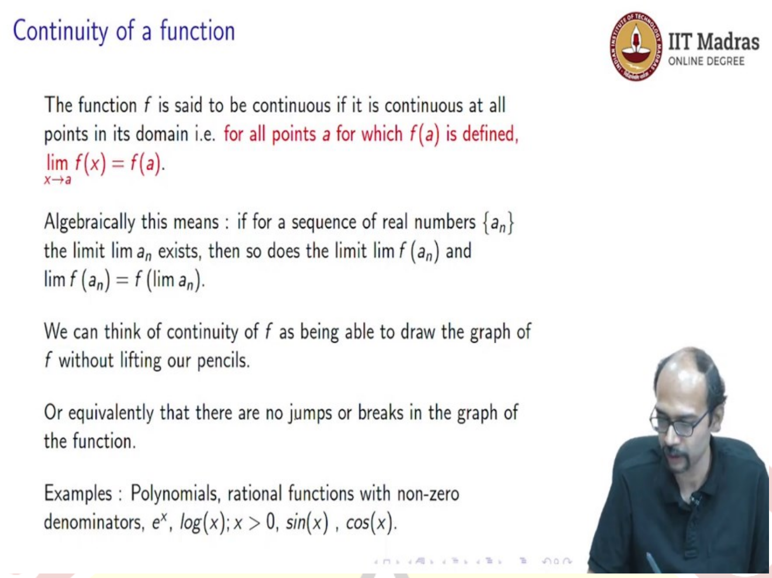

Finally, let us define the continuity of a function, so we have talked about when a function is continuous at a point, so now we want to talk about when is the function itself continuous. So, the function f is said to be continuous, if it is continuous at all points in its domain, that is for all points a for which f (a) is defined, when you take $\lim_{x \to a} f(x)$, then that is equal to f(a), this is our definition of continuity. So how do we think of this? So algebraically this means, that if you take a sequence of real numbers, such that lim a_n exists, then the lim f(a_n) also exists.

And in fact, you can switch lim f(a_n) with f (lim a_n), and to the sort of pictorial or graphical way of thinking of this is that we will be able to draw pictures and you draw the graph of this function, you can draw it without lifting your pen unlike the jump function the step function where you had this, then this so I have to lift my pen, so that does not happen for continuous functions, you can draw the graph continuously without lifting your pencil. Equally there are no jumps or breaks in the graph of the function that is not what this means.

And let us again recall how maybe before going ahead, what are examples of continuous functions we have actually seen plenty of examples, polynomials, we saw in the example slide that $\lim_{x \to a} f(x) = f(a)$ for polynomials, for rational function with non zero denominators, we saw the same thing. So, of course, if you have a 0 denominator you have to be careful, but if you are denominator is always non zero for example, suppose you take the function $\frac{x}{x^2+1}$, so the function $x^2 + 1$ is never 0 and so, for this function, the denominator will not cause problems of being 0 and so all these limits at every point will exist.

So, such rational functions are continuous other examples of continuous functions are e^x , the logarithm function when x is positive, trigonometric functions such as sin x, cos x, and so on.

tan x is not continuous, in general as we have seen, there are issues at $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, and for the multiples of those, but within $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ we are fine.

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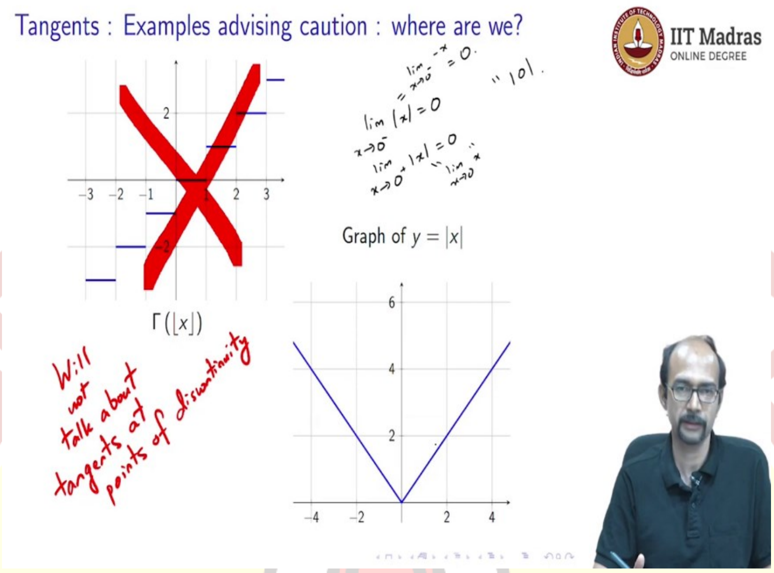

So, finally let us come back to why we started talking about limits and continuity in the first place. So remember that, this whole conversation started because, we wanted to talk about tangents and we recall that there were two examples that we said these are examples that advised caution, so now what have we done by introducing the notion of continuity, we are going to say the following, that, we will talk about tangents only for those functions, where the function is at least continuous.

So, this function here, which is the floor function, the graph of this function is not continuous, so for such functions we will not talk about tangents at points of discontinuities which are integers. So, we will not talk about tangents, at the points of discontinuity. So what do I mean by points of discontinuity, I mean points at which the function is not continuous, let us look at this example.

This example is continuous, you can draw it in a very nice way you can take your pen and draw it and let us just quickly check at other points. Of course, there is no issue what happens at 0, so at 0 if you take the left limit |x|, this is 0. What happens is, $\lim_{x\to0-.}|x| = \lim_{x\to0-.} -x =$ 0and $\lim_{x\to0-.}|x| = \lim_{x\to0-.} -x = 0$.

So, both of these are 0. So the limit exists and this limit is actually equal to |0|, which is the value of the function at the point 0 so this function is continuous, so we still have some more work to do, because remember, even for function, this function, where we had corners, we had an issue with what the tangent was.

So, we are going to now step into the world of what is called differentiability and we are going to we are going to impose further condition, so that we do not have functions like this, so that we can talk about tangents. Thank you.