Statistics for Data Science -1

Lecture 9.3: Properties of expectation

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Learning objectives

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5. Expectation and variance of a random variable.

Properties of expectation

Expectation of a function of a random variable

Proposition

Let X be a discrete random variable which takes values x_i along with its probability mass function, $P(X = x_i)$. Let g be any real values function, The expected value of $g(X)$ is

$$E(g(X)) = \sum_i g(x_i)P(X = x_i)$$

Corollary

If a and b are constants, $E(aX + b) = aE(X) + b$

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- NOTE: $0.5 = E(X^2) \neq (E(X))^2 = 0.01$

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  - Let X denote Anita's bonus. Given $E(X) = 15,000$.
  - Let Sanjay's bonus be Y. Given $Y = 0.9X$
  - Hence $E(Y) = 0.9 \times E(X) =$₹13,500

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- If Sanjay's is set to equal ₹1000 more than Anitha's, find his expected bonus.
  - In this case, $Y = X + 1000$
  - Hence $E(Y) = E(X) + 1000 =$ ₹16,000

Expectation of sum of two random variables

- The expected value of the sum of random variables is equal to the sum of the individual expected values. i.e Let X and Y be two random variables. Then,

$$E(X+Y) = E(X) + E(Y)$$

Example: Rolling a dice

- Let X be the outcome of a fair dice. Let Y be the outcome of another fair dice.
- We know $E(X) = E(Y) = 3.5$
- $X + Y$ is the sum of outcomes of both the dice rolled together. Then,

$$E(X + Y) = E(X) + E(Y) = 3.5 + 3.5 = 7$$

This is the same expectation of the sum of outcomes of rolling a dice twice.

Hypergeometric random variable

- Suppose that a sample of size n is to be chosen randomly (without replacement) from a box containing N balls, of which m are red and $N - m$ are blue.
- Let X denote the number of red balls selected, then

$$P(X = i) = \frac{\binom{m}{i}\binom{N-m}{n-i}}{\binom{N}{n}}, i = 0, 1, 2, \ldots, n$$

- X is a said to be a hypergeometric variable for some values of n, m, and N
- $E(X) = \frac{nm}{N}$

Example

Two students are randomly chosen from a group of 20 boys and 10 girls. Let X denote the number of boys chosen, and let Y denote the number of girls chosen.

1. Find $E(X)$.
   - X is a Hypergeometric rv. $N = 30, m = 20, n = 2$. Hence $E(X) = \frac{2\times 20}{30} = \frac{4}{3}$
2. Find $E(Y)$.
   - Y is a Hypergeometric rv. $N = 30, m = 10, n = 2$. Hence $E(Y) = \frac{2\times 10}{30} = \frac{2}{3}$
3. Find $E(X+Y)$.
   - $E(X+Y) = E(X) + E(Y) = \frac{4}{3} + \frac{2}{3} = 2$

Expectation of sum of many random variables

- The result that the expected value of the sum of random variables is equal to the sum of the expected values holds for not only two but any number of random variables.
- Let $X_1, X_2, \ldots, X_k$ be k discrete random variables. Then,

$$E\left(\sum_{i=1}^{k} X_i\right) = \sum_{i=1}^{k} E(X_i)$$

Example: Tossing a coin three times

- Toss a coin i times.
- Let X_i be a random variable which equals 1 if the outcome is a head, 0 otherwise.
- $E(X_i) = 0.5$
- $X_1 + X_2 + \ldots + X_n$ is the total number of heads in n tosses of the coin.
- $E(X_1 + X_2 + \ldots + X_n) = \sum_{i=1}^{n} E(X_i) = 0.5 \times n$
- For $n = 3, X_1 + X_2 + X_3$ is equal to the number of heads in three tosses of a coin.

$$E(X_1 + X_2 + X_3) = 3 \times 0.5 = 1.5$$

This is the same expectation of number of heads in three tosses of a coin.

Section summary

- Notion of expected value as long-run average.
- How to compute expected value of a random variable.
- Expectation of a function of a random variable
- Expectation of sums of random variables. .