Mathematics for Data Science - 1
Graded Assignment(Week 3) Solution
Jan 2022 term
Max marks: 20

1 Multiple Choice Questions (MSQ)

1. Which of the following functions may represent the graph given in the figure M1G3T4-1? (Answer: (a),(d))(1 mark)

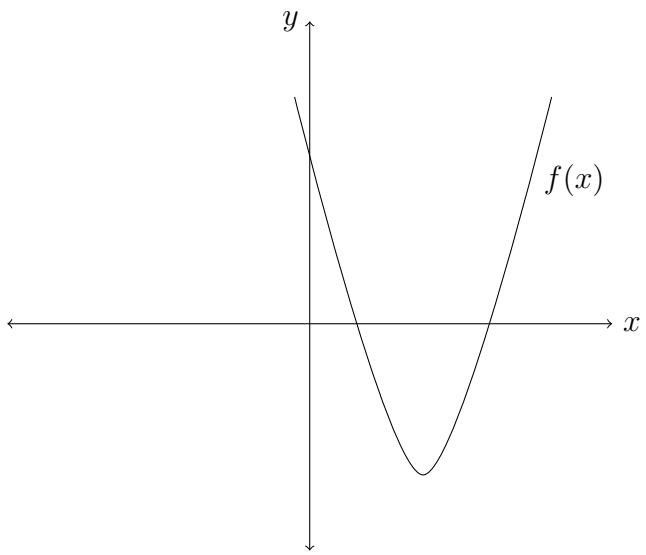

Figure: M1G3T4-1

- ◯ $f(x) = x^2 - 8x + 12$
- ◯ $f(x) = -x^2 + 10x - 21$
- ◯ $f(x) = x^2 - 7x + 15$
- ◯ $f(x) = x^2 - 6x + 4$
- ◯ $f(x) = x^2 - 6x - 12$

Solution:
Let function, $f(x) = ax^2 + bx + c$, represents the graph given in the figure. Also assume that α and β are the two roots of $f(x)$. Then the following statements can be said about function $f(x)$:

- $a > 0$, since it is an upward parabola.
- Discriminant, $D > 0$, for real and distinct roots.
- $\alpha > 0$ and $\beta > 0$, curve intersect on positive $x-$axis. Therefore $(\alpha + \beta) > 0$ and $\alpha\beta > 0$.
- At $x = 0$, $f(x) > 0 \implies c > 0$.

Now we check each and every options:
Option (a) $f(x) = x^2 - 8x + 12$: Roots are 2 and 6.
Here $a = 1 > 0$, $D = 16 > 0$, roots 2 and 6 which are greater than 0, and at $x = 0$ $f(x) = 12 > 0$.
Therefore the given function $f(x) = x^2 - 8x + 12$ fulfills all the criteria to represent the graph shown in figure. Hence option (a) is correct.

Option (b) $f(x) = -x^2 + 10x - 21$: Roots are 3 and 7.
Here, $a = -1 < 0$. Here no need to check further criteria. Therefore the given function $f(x) = -x^2 + 10x - 21$ does not fulfill the criteria, $a > 0$, to represent the graph shown in figure. Hence option (b) is incorrect.

Option (c) $f(x) = x^2 - 7x + 15$: Roots are not real.
Here $a = 1 > 0$ but $D = -11 < 0$.
Hence option (c) is incorrect.

Option (d) $f(x) = x^2 - 6x + 4$: Roots are $(3 - \sqrt{5})$ and $(3 + \sqrt{5})$.
Here $a = 1 > 0$, $D = 20 > 0$, roots $(3 - \sqrt{5})$ and $(3 + \sqrt{5})$ which are greater than 0, and at $x = 0$ $f(x) = 4 > 0$.
Therefore the given function $f(x) = x^2 - 6x + 4$ fulfills all the criteria to represent the graph shown in figure. Hence option (d) is correct.

Option (e) $f(x) = x^2 - 6x - 12$: Roots are $(-\sqrt{21} + 3)$ and $(\sqrt{5} + 3)$.
Here $a = 1 > 0$, $D = 84 > 0$, roots $(-\sqrt{21} + 3)$ and $(\sqrt{21} + 3)$ in which roots $(-\sqrt{21} + 3)$ is less than 0.
Therefore the given function $f(x) = x^2 - 6x - 12$ does not fulfill all the criteria to represent the graph shown in figure. Hence option (e) is incorrect.

Hence options (a) and (d) are correct.

2. Surya has designed a new style of bicycle and want to set up a company that manufactures these new bicycles. Table M1G3T4-2 shows the different costs involved in setting up the company. Figure M1G3T4-3 shows his survey regarding the demand (number of bicycles manufactured) versus selling price of each bicycle(in ₹).

Cost type	Cost
Manufacturing set up	₹50,000
Raw material	₹250 per bicycle
Miscellaneous	₹150 per bicycle

Table: M1G3T4-2

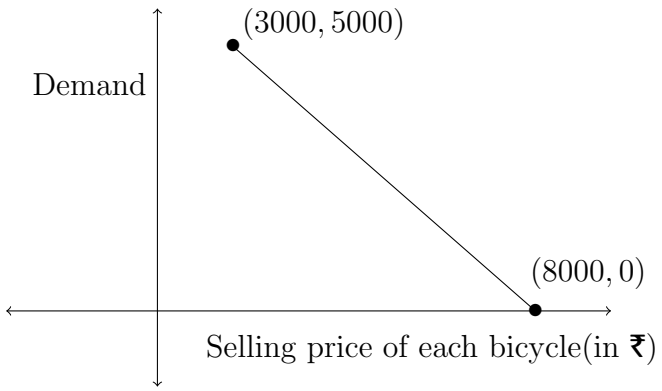

Figure: M1G3T4-3

Which of the following statements are correct? (Answer: (b))(3 marks)

- ◯ The maximum profit is independent on the selling price of each bicycle.
- ◯ To make maximum profit, the selling price of each bicycle should be ₹4200.
- ◯ The total selling price obtained by Surya by selling x bicycles is $-x^2 + 8400x$.
- ◯ The total profit obtained by Surya by selling x bicycles is $-x^2 + 8400x - 3250000$.

Solution:
Assume r and n be the selling price of each bicycles and demand respectively.
Then the equation of line shown in the plot will be $r + n = 8000$
The profit, p = Selling price - Cost price
$p = r * n - (250 + 150)n - 50000 = r(8000 - r) - 400(8000 - r) - 50000$
or, $p = -r^2 + 8400r - 3250000$

Now let us check every options,

Option (a): As we can see from the above expression, $p = -r^2 + 8400r - 3250000$, that profit, p, depends on the selling price of each bicycle, r. Hence, option (a) is incorrect.

Option (b):
We have, $p = -r^2 + 8400r - 3250000$
p will be maximum when $r = -8400/2(-1) = 4200$
Hint: If $f(x) = ax^2 + bx + c$ and $a < 0$, then $f(x)$ will be maximum at $x = -b/2a$
Therefore, the selling price of each bicycle should be ₹4200 for making maximum profit.
Hence, Option (b) is correct.
Option (c):
When we assume x be the number of bicycles then,
Selling price = No of bicycles $\times$ Selling price of each bicycle,
or, selling price $= x(8000 - x) = -x^2 + 8000$
Therefore, the given total selling price in the option (c) is incorrect.

Option (d):
When we assume x be the number of bicycles then,
Total profit

$$\begin{aligned} p &= \text{Total selling price} - \text{Total cost price} \\ p &= x(8000 - x) - (250x - 150x - 50000) \\ &= -x^2 + 7600x - 50000 \end{aligned}$$

Therefore, option (d) is incorrect.

3. A ball is thrown off from the top of a building. The height of the ball(in feet) from the ground after t seconds is given by $H(t) = -t^2 + 6t + 40$. Which of the following statements are correct? (Answer: (b),(c),(e))(3 marks)

NOTE: Consider the foot of the building is at the origin.

- ◯ The ball reaches its maximum height after 2.5 seconds.
- ◯ The height of the building is 40 feet.
- ◯ The ball travels in the air for 10 seconds before it reaches the ground.
- ◯ The maximum height reached by the ball is 48.75 feet.
- ◯ The maximum height reached by the ball is 49 feet.

Solution:
The given expression is $H(t) = -t^2 + 6t + 40$. We will get two roots -4 and 10 for $H(t) = 0$. Roots tell us that the time at which ball is at ground level. But one of the root, -4, is negative which is not possible because time can not be negative. The other root which is 10 tells us that after $t = 10$ seconds ball reaches the ground.

We can also know the height of building at which ball is thrown by putting $t = 0$ in $H(t)$. So, $H(t = 0) = 40$. Therefore height of building is 40 feet. The possible path of ball shown in the given below figure: To find the maximum height reached by the

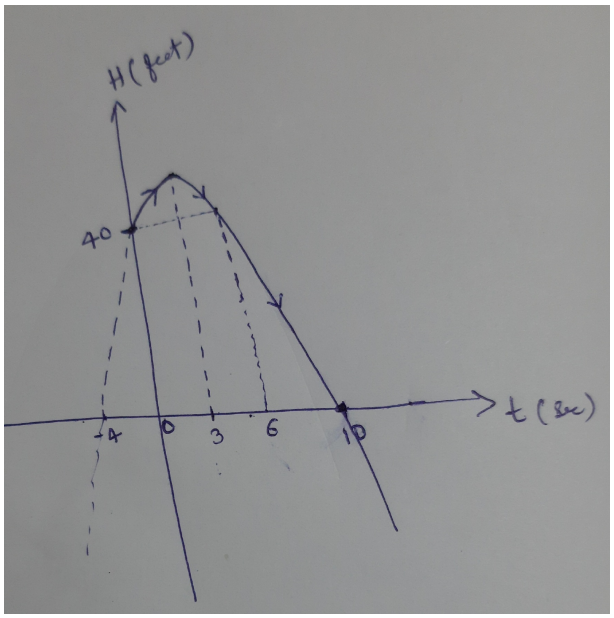

Fig. Path followed by the ball

ball, we need to find the $y-$coordinate of the vertex point and $x-$ coordinate gives us the time after which the ball reaches its maximum height.

So the ball reaches its maximum height after $t = \frac{-b}{2a} = \frac{-6}{2(-1)} = 3$. Therefore, The maximum height reached by the ball is $H(3) = 49$. Hence, the ball reaches its maximum height after 3 seconds and the maximum height it reaches is 49 feet.

Hence, when we look at the given options, option (b), (c) are (e) are correct.

2 Numerical Answer Type (NAT)

4. Suppose $f(x) = ax^2 + bx + c$, where $a, b, c \in \mathbb{Z} \setminus \{0\}$ is a quadratic function. If the sum of roots and product of roots of $f(x)$ are $\frac{-7}{4}$ and $\frac{-1}{2}$ respectively, then find the value of $2a + 3b + 4c$. (1 mark)

Solution: Let α and β are the roots of $f(x) = ax^2 + bx + c$. Then sum of roots will be $-b/a$ and product of roots will be c/a. It is given in the question
$\alpha + \beta = -b/a = -7/4$, and
$\alpha\beta = c/a = -1/2$

If α and β are the roots of a quadratic function then the quadratic equation can be can be written as,
$x^2 - (\alpha + \beta)x + \alpha\beta = 0$
or, $x^2 - (-7/4)x + (-1/2) = 0$
or, $4x^2 + 7x - 2 = 0$

On equating we get,
$a = 4, b = 7$, and $c = -2$

Hence, $2a + 3b + 4c = 21$

5. The product of two consecutive odd natural numbers is 143. Find the largest number among them. (2 marks)

Solution:
Let x be the first odd number then $(x + 2)$ will be the next consecutive odd number. Then from the question,
$x(x + 2) = 143$,
$x^2 + 2x - 143 = 0$,
$(x - 11)(x + 13) = 0$,
Therefore, $x = -13$ is not possible because -13 is not an odd natural number.
Hence $x = 11$ is the only possible value. So, the largest odd number among them will be $x + 2 = 13$.

6. The slope of a parabola $y = 3x^2 - 11x + 10$ at a point P is -11. Find the $y-$ coordinate of the point P. (3 marks)

Solution:
The slope of a parabola, $y = ax^2 + bx + c$, at a point (x_0, y_0) is $2ax_0 + b$.
So for the given parabola $y = 3x^2 - 11x + 10$, slope will be $6x_0 - 11$ at a point (x_0, y_0).
At point P the slope is given in the question which is -11.
So, the $x-$ coordinate of the point P can be found by equating the slopes, $6x_P - 11 = -11 \implies x_P = 0$
Hence $y-$ coordinate of the point P can be found by putting the value of x_P in equation of parabola.
$y_P = 3(0)^2 - 11(0) + 10 = 10$.

7. Suppose the functions $f(x) = -x^2 + 4x - p$ and $g(x) = x^2 - px + 9$ has no real roots for some $p \in \mathbb{Z}$. Find the value of p. (Answer: 5)(2 marks)

Solution:
As we know that for no real roots,
Discriminant, D should be less than zero i.e. $D = b^2 - 4ac < 0$
For $f(x) = -x^2 + 4x - p$, $D = 16 - 4p < 0 \implies p > 4$

And for $g(x) = x^2 - px + 9$, $D = p^2 - 36 < 0 \implies p < 6$

From the above two conditions, $p > 4$ and $p < 6 \implies 4 < p < 6$.
But in the question it is mentioned that $p \in \mathbb{Z}$, therefore value of p will be only 5.

3 Comprehension type question

Use the following information for questions 8-10:

A footpath of uniform width x meters is constructed around a rectangular garden of length 7 meters and breadth 5 meters. The total area of the garden including the footpath is 99 m^2. Figure M1G3T4-4 shows the rough diagram of the above data.

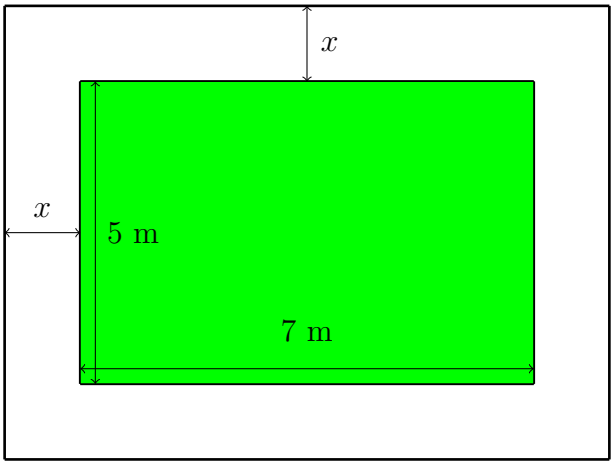

Figure: M1G3T4-4

Solution:
The length of garden including footpath from both sides (left and right) = $(7+2x)$, and the breadth of garden including footpath from both sides (top and bottom) = $(5+2x)$
Therefore, total area of the garden including the footpath will be $(7+2x)*(5+2x) = 4x^2+24x+35$.
Since total area of the garden including the footpath is given, then
we can write, $4x^2+24x+35 = 99$
On simplification, $x^2+6x-16 = 0$
or, $(x-2)(x+8) = 0$
So, $x = 2$, and -8. Since x can not be negative because x represents width of the footpath. Therefore x will be 2.

8. Which of the following statements are correct? (Answer: (b),(e))(1 mark)

   ◯ The total area of the garden including the footpath in terms of x will be equal to $x^2 + 12x + 35$.

   ◯ The total area of the garden including the footpath in terms of x will be equal to $4x^2 + 24x + 35$.

   ◯ By finding the roots of the quadratic equation $4x^2 + 24x + 35 = 0$, we get the value of x.

   ◯ By finding the roots of the quadratic equation $x^2 + 12x + 35 = 0$, we get the value of x.

   ◯ By finding the roots of the quadratic equation $4x^2 + 24x + 35 = 99$, we get the value of x.

   ◯ By finding the roots of the quadratic equation $x^2 + 12x + 35 = 99$, we get the value of x.

   Solution:
   From the above explanation we find from the given options that options (b) and (e) are correct.

9. Find the value of x. (Answer: 2)(2 marks)
   Solution:
   The value of x is 2.

10. If the footpath is covered with square tiles and the length of the side of each tile is 0.2 meters, then find the number of tiles used to cover the footpath. (Answer: 1600)(2 marks)
    Solution:
    The area of footpath can be found by subtracting area of garden from the total area of the garden including the footpath. So,
    Area of the footpath $= 99 - (7 * 5) = 64\ m^2$,
    Area of each tile $= 0.2 * 0.2 = 0.04\ m^2$

    So, number of tiles to cover the footpath $= 64/0.04 = 1600$