IIT Madras

ONLINE DEGREE

Mathematics for Data Science 1
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Week - 01
Lecture - 03
Real and Complex Number

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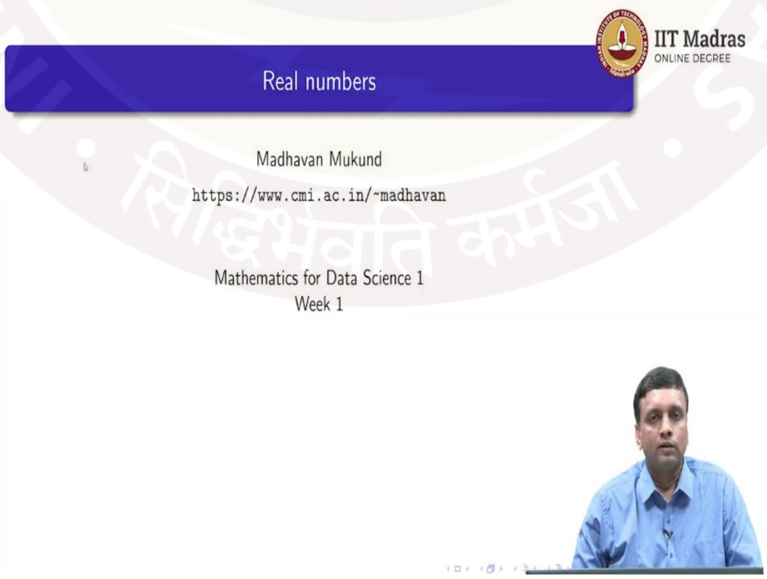

So, we started with the natural numbers and the integers and then, we moved on to the rational numbers which are defined as $\frac{p}{q}$; where p and q are both integers.

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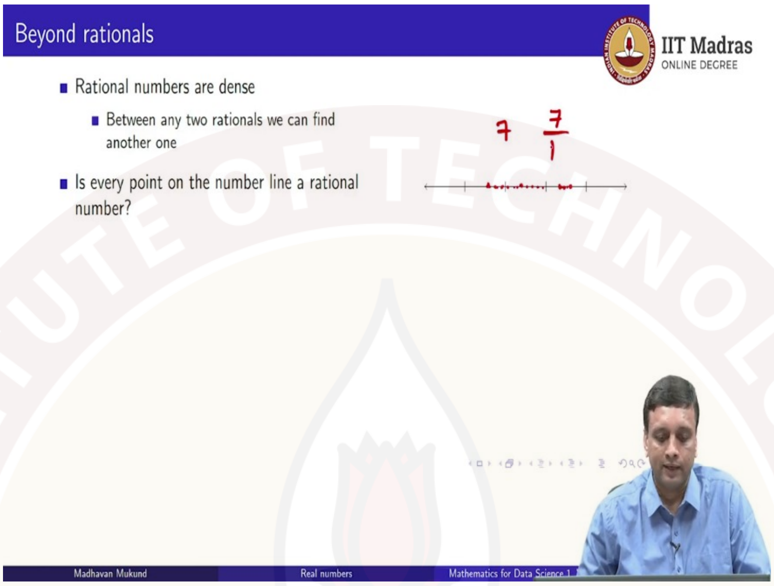

So, we decided that the rational numbers are dense right and that means that on this number line between any two rationals, you can find a rational. So, if I want to now talk about this number line, then I know that if I take any two positions, then I will find a rational between them and I will find a rational between them and so on. So, it makes sense to ask this question which is that if I take any two points and the rational between them any two points, then is this entire number line composed only of rational numbers. Of course, some of those rational numbers are integers.

So, an integer is a rational number because I can write 7; for instance, as $\frac{7}{1}$right. So, this is of the form $\frac{p}{q}$. So, any rational number which in reduced form as denominator 1 is an integer; so, an integer is a special case of a rational number. So, do all the rational numbers fill up this number line? That is the question.

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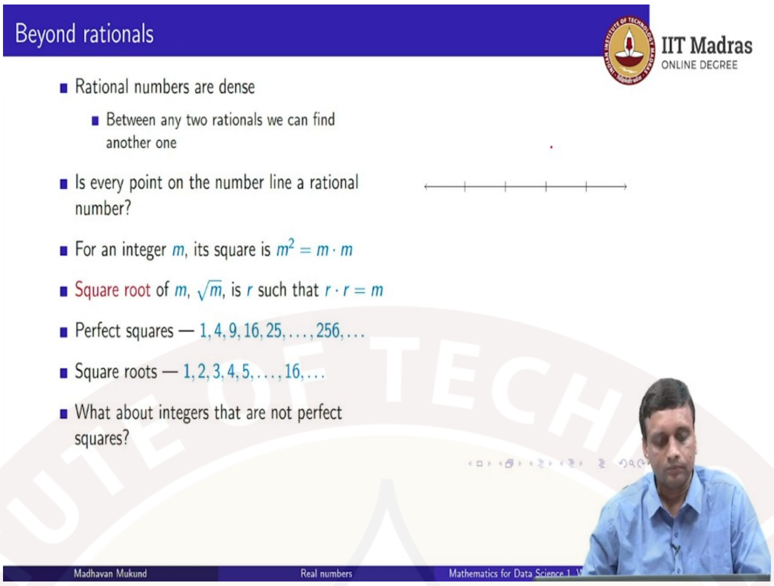

So, it turns out this is not the case. So, remember that a square of a number is the number multiplied by itself. So, if I take a number m and multiply it by itself, I get m^2 which is $m \times m$ and if I take this operation and turn it around, then the square root of a number is that number r such that $r \times r$ is equal to m right. So, I want to find out which number, I have to square in order to get m and that is called the square root.

So, if we take the so called perfect squares, like 1, 4, 9, 16, 25 and so on their square roots are integers. So, 1^2 is 1. So, the $\sqrt{1}$ is 1; $2^2$is 4, so the $\sqrt{4}$ is 2; $5^2$is 25, so $\sqrt{25}$ is 5; $16^2$is 256, so $\sqrt{256}$ is 16 and so on. So, some integers are clearly squares of other integers and so, you can get the square root and find an integer. Now, what happens if something is not a square right? So, supposing I take a number which is not a square like 10 and I take its square root, I know that the square root is not an integer, its somewhere between 3 and 4 because $3^2$is 9 and $4^2$is 16. Question is, is it a rational number or not?

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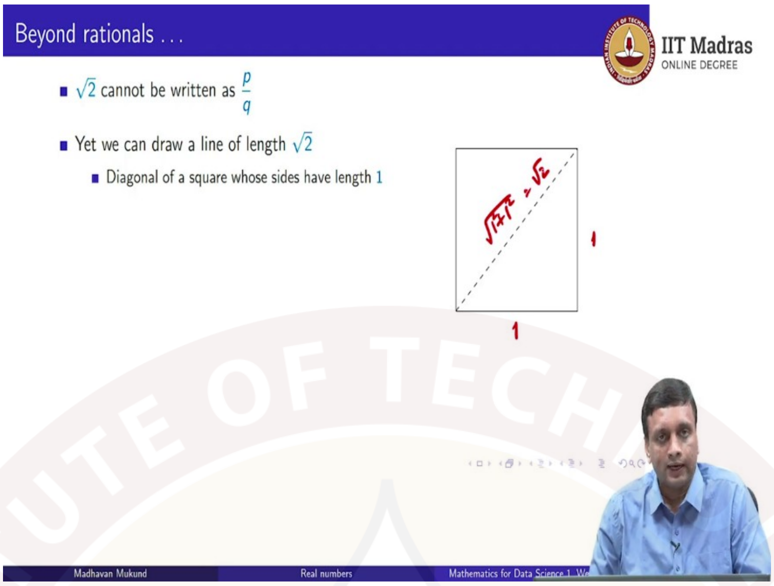

So, what happens to the square roots of integers that are not perfect squares? So, the smallest such number which is not a perfect square because 1 remember is a perfect square, 1 × 1 is 1. The smallest such number that is not a perfect square is actually 2 and it is one of the very old results that the √2 cannot be written as $\frac{p}{q}$. This was certainly known to the ancient Greeks, in fact, to Pythagoras and one way to do this is to see that you can actually draw a line of; so, this is not an unreal number in that sense right.

So, you can actually draw a line of this length because if you take a square, whose sides are 1 right. So, this is 1, then if you remember your Pythagoras theorem; then, the hypotenuse of this triangle is going to be $\sqrt{1^2+1^2}$,technically which is √2. So, I can actually physically draw a line whose length is √2. So, this is a very real quantity.

On the other hand, for reasons that we will not described here, but there will be a separate lecture explaining this for if you are interested. √2 cannot be written as a rational number $\frac{p}{q}$. So, here is a number which is a very measurable quantity, I can actually draw this quantity as a length. At the same time, it does not fit into this number line of rational numbers which seems to cover all the rational numbers, all the numbers because they are dense.

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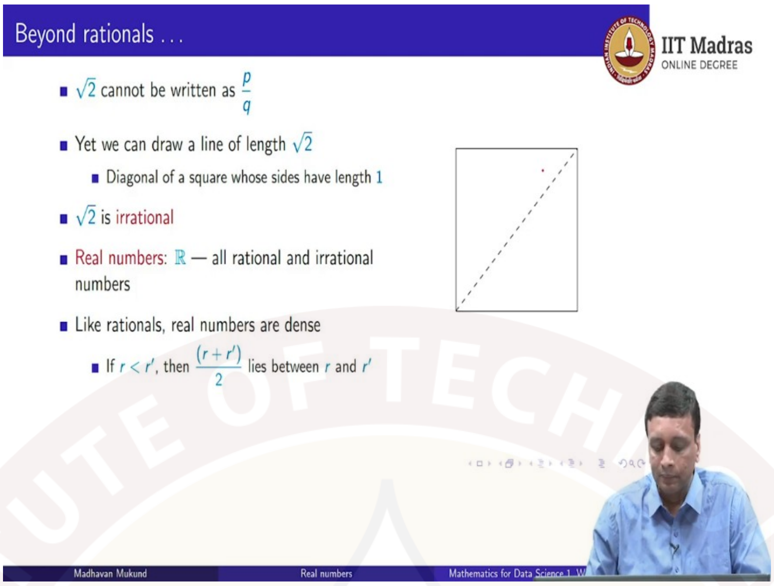

So, √2, since it is not a rational number right and yet it exists is called an irrational number and these numbers which constitute all the rational numbers and the real irrational numbers together are called the real numbers. So, the real numbers are denoted by this double line R. So, we had N for the natural numbers, Z for the integers, Q for the rational numbers and now, we have the real numbers R.

So, the real numbers extend the rational numbers by these so called irrational numbers which are very much on the number line, but which cannot be written on the form $\frac{p}{q}$. Now, it is not difficult to argue that like the rationals, the real numbers are dense for the very same reason. Because if you have two real numbers r and r' such that r is smaller than r prime, then you can just take their average $r + r'$ divided by 2. This must be a number πwhich is bigger than r and it is smaller than r' and therefore, it must lie between them. So, between any 2 real numbers, you will find another real number. So, the real numbers are also dense.

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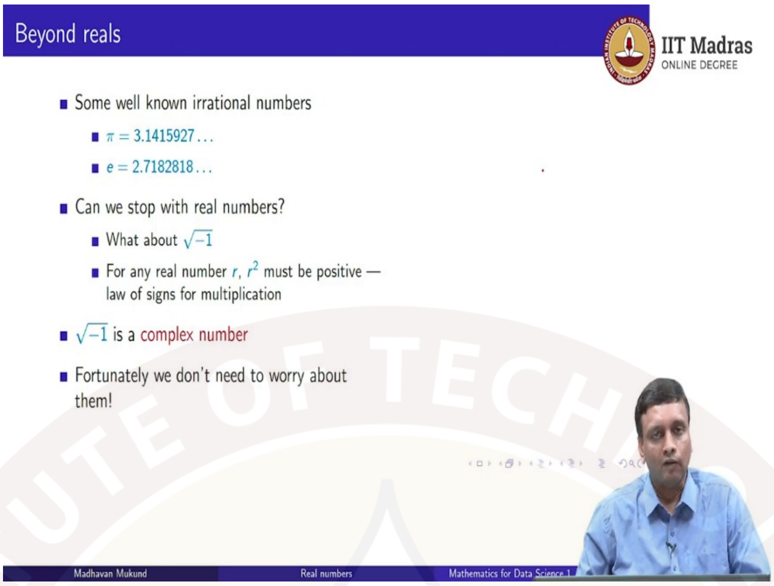

So, there are some irrational numbers which we use a lot in mathematics and which you have probably come across; one of them is this famous number π which comes when we are talking about circles. Because it is the ratio of the circumference to the diameter and this is an invariant. π is always; the circumference divided by diameter for any circle is π ok.

So, π is an irrational number. We cannot write it in the form $\frac{p}{q}$ and it has this. If you write it in this decimal form, it has this infinite decimal expansion. Another number which is very popular as an irrational number is this number e which is used for natural logarithms. So, it is 2.7182818 and so on right. So, there are a lot of irrational numbers. So, $\sqrt{2}$ as we have seen as an irrational number. It will turn out that square root of anything, $\sqrt{3}$ is also an irrational number, $\sqrt{6}$ is also an irrational number.

Anything which is not a perfect square, its square root is actually an irrational number. But many of these numbers are not very useful to us, but π and e are certainly very useful irrational numbers. So, now, we have seen that we can find more numbers on the line than just the rationals and these are the real numbers. So, do we stop here? Well, let us look at the square root operation which we use in order to claim that there are irrational numbers. So, what happens if we now take the square root of a negative number like -1?

So, remember that we had a sign rule for multiplication. The sign rule for multiplication said that if I multiply any two numbers, then if the two signs are the same that is their two negative signs or two positive signs, I will get a positive sign in the answer. Only if the two signs are different, if I have one minus sign and one plus sign, will I get a negative answer. So, if I want to multiply two numbers and get a -1, one of them must be negative and one must not be negative. But by definition, a square root is a number which is multiplied by itself, the same number has to be multiplied by itself. So, it will have the same sign.

So, any square root which multiplies by itself must give me a positive number. So, if I take a negative number, there is no way to find a square root for it. So, if we want to find square roots for negative numbers, we have to create yet another class of numbers called complex numbers. So, complex numbers extend the real numbers, just like real numbers extend the rational numbers and rational numbers extend the integers and so on. But the good news for you is that we do not have to look at complex numbers for this course.

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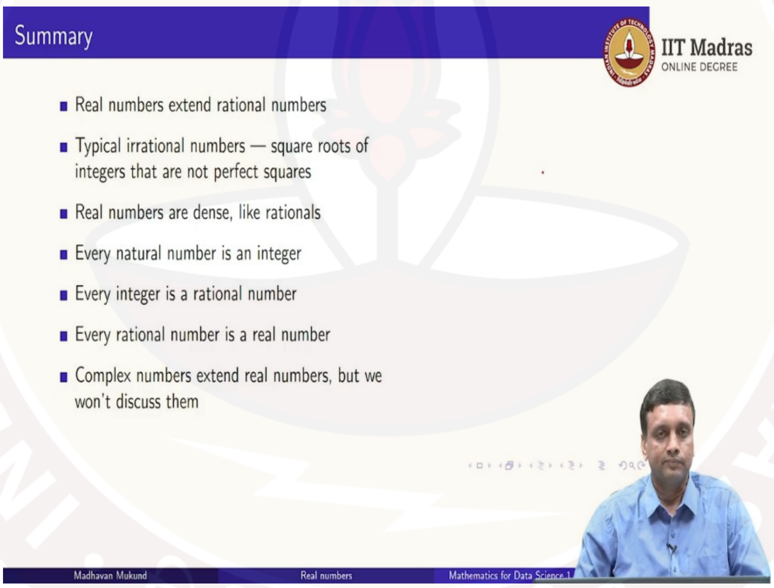

So, to summarize, a real numbers extend the rational numbers by adding the so called irrational numbers which cannot be represented of the form $\frac{p}{q}$ and a typical example of an irrational number is the square root of an integer that is not a perfect square. So, $\sqrt{2}$ for example is not a rational number and this is also of the case was $\sqrt{3}$, $\sqrt{5}$, $\sqrt{6}$ and so on. So, except for the perfect squares, none of the square roots are actually rational numbers. Now, just like we said that the rational numbers are dense because the average of any two rational numbers is a rational number. Similarly, the real numbers are dense because the average of any two real numbers is a real number.

So, we have a progression in terms of numbers. So, every natural number that we started with is also an integer because the integers extend the natural numbers with negative quantities. Now, every integer is also a rational number because we can think of every integer as a ratio $\frac{p}{q}$; where, the denominator is 1. And finally, every rational number is a real number because we said that the real numbers include all the rational numbers plus all the irrational numbers. And finally, we said that there are even things beyond rational numbers like complex numbers, but we will not discuss them.