IIT Madras

ONLINE DEGREE

Mathematics for Data Science 1
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Week - 01
Lecture - 02
Rational Number

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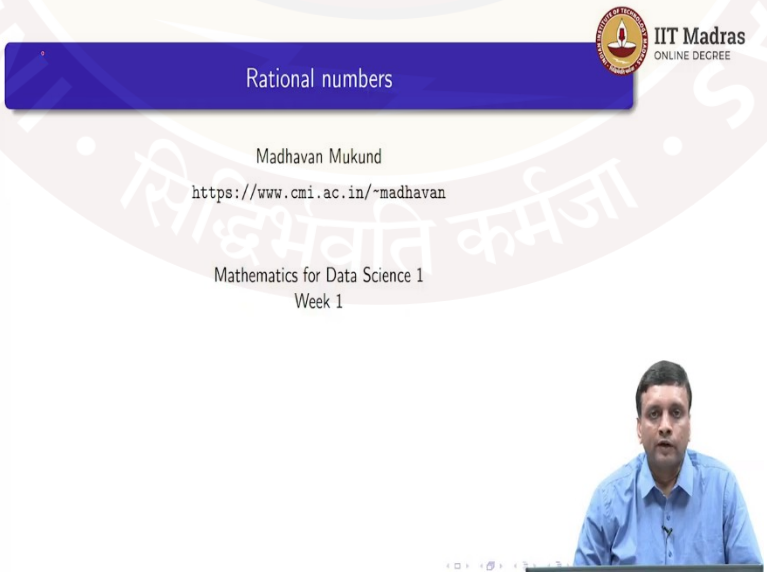

So, now first lecture on Numbers; we looked at natural numbers and integers. So, now, let see what happens when we try to divide. So, let us look at the rational numbers.

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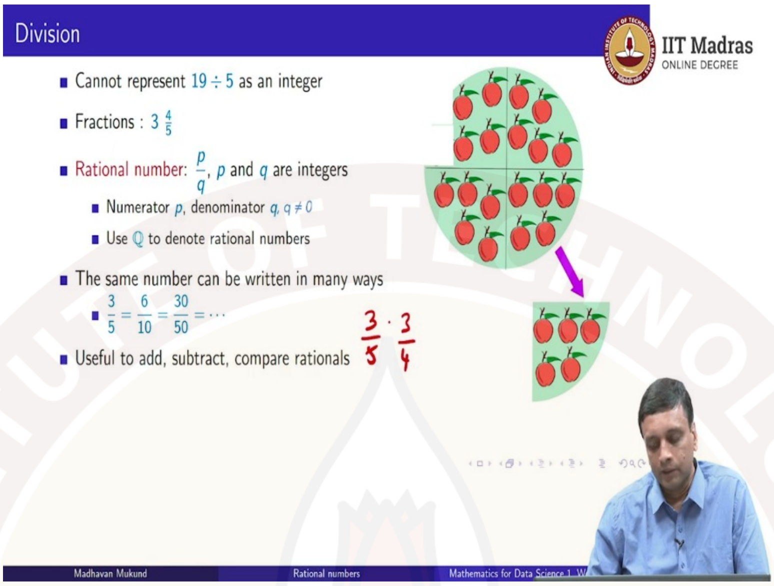

So, we said that we cannot represent 19 / 5 as an integer because we cannot find a number k such that 5 × k is 19. So, as we know the way we deal with this is to represent this quantity as a fraction. So, we say that 19 /5 is $3\frac{4}{5}$. So, this number is an example of a rational number. So, rational number what we usually called fractions in school, a rational number is something that can be written as ,$\frac{p}{q}$; where, p and q are both integers. So, as you probably remember from school, the number on the top is called the numerator. So for,$\frac{p}{q}$; p is called the numerator and q is called the denominator.

So, just like we had the symbols N and Z to represent the natural numbers and the integers, we have a special symbol which is somewhat unusual which is Q. So, Q stands for the rational numbers and again, to just say it is a special Q, we write these double lines on sides. So, this Q with these fat boundaries denotes the rational numbers. So, one thing about the rational numbers is that the same number can be written in many different ways. Now, this is not true of integers. Of course, we are not talking about changing base from binary to decimal or something.

But if you write a 7, there is only one way to write 7 fix, if you are fix the notation that you are using for writing numbers. With rational numbers, this is not true because there are many ways of writing $\frac{p}{q}$ such that $\frac{p}{q}$ is actually a same number. So, for instance if we take the number $\frac{3}{5}$, then we all know that $\frac{3}{5}$ is the same as $\frac{6}{10}$ and this is the same as $\frac{30}{50}$. So, when we take a rational number and multiply it by something the same quantity on the top and the bottom so, $\frac{3}{5}$, 3×2 and 5×2, we get the same number; $\frac{6}{10}$ or 3×10 and 5×10, we get the same number $\frac{30}{50}$. So, this is sometimes a nuisance, but it is also sometimes useful.

Now, there is no reasonable way to compare two numbers like say $\frac{3}{5}$ and $\frac{3}{4}$ or $\frac{2}{5}$ and $\frac{3}{4}$. If we have two fractions which have different denominators, there is no way to directly compare them. So, the only way to compare them is to somehow convert them into equivalent fractions such that they have the same denominator. So, the usual way is just to find a number such that both the denominators multiply into that number rather factors of that number. Now, you can find the smallest such number which is called the least common multiple; but you can find any number of this form.

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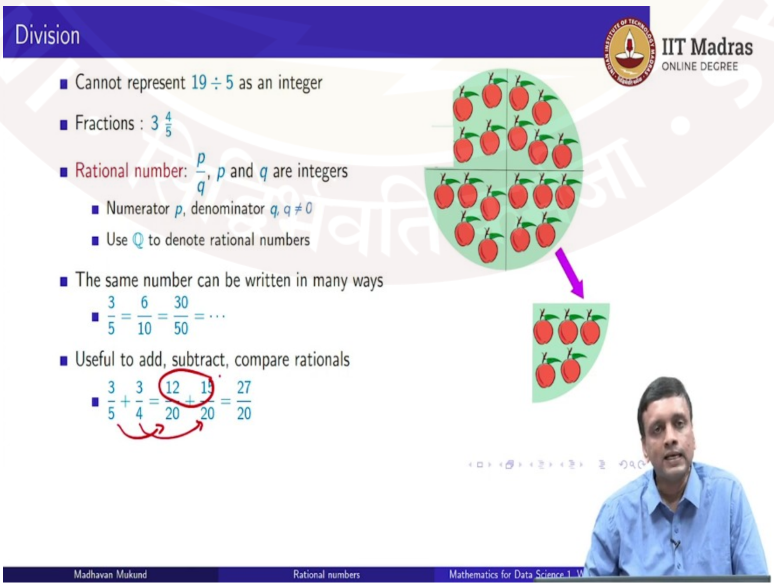

So, for instance, if you want to add $\frac{3}{5}$ and $\frac{3}{4}$, now you cannot do that directly; but you know that 20 is a number which divides which is a multiple of both 5 and 4. So, you can represent $\frac{3}{5}$ as equivalently as $\frac{12}{20}$; you can represent $\frac{3}{4}$ equivalent. So, this is equivalent and this is equivalent. So, you have converted these numbers into a different fraction of the same number; but this new representation has the same denominator.

And now once, the two denominator that the same, you can add the numerators and you can get (12 + 15)/20 is $\frac{27}{20}$. So, this kind of manipulation requires the denominators to be the same and therefore, it is actually extremely useful that we can write the same rational number in many different ways. The same is to we want to compare two numbers.

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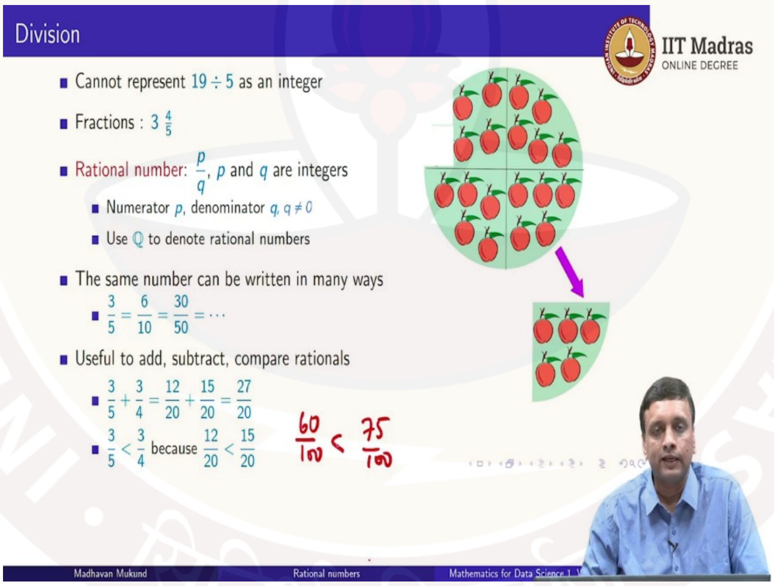

If we want to check whether $\frac{3}{5}$ is bigger or smaller than $\frac{3}{4}$, there is no way to do it directly. What we have to do is again take the denominators and make them the same and then, say that $\frac{12}{20}$ is less than $\frac{15}{20}$ because you are dividing something 20 parts and you are taking 12 of them that is less than taking 15. Now, as I said there is no reason why this must be the smallest one. So, for instance you could take a bigger number like 100, right. So, 5 goes into 100 and 5 goes 4 also goes into 100.

So, we could also say that $\frac{3}{5}$ is the same as $\frac{60}{100}$ and, $\frac{3}{4}$ is the same as $\frac{75}{100}$ and therefore, since 60 is less than 75; $\frac{60}{100}$ is less than $\frac{75}{100}$ and therefore, $\frac{3}{5}$ is less than $\frac{3}{4}$. So, it is not really important that the denominator is the smallest common multiple of the two denominators; but it must be some common multiple so that you can bring it all to a common number that you can then compare.

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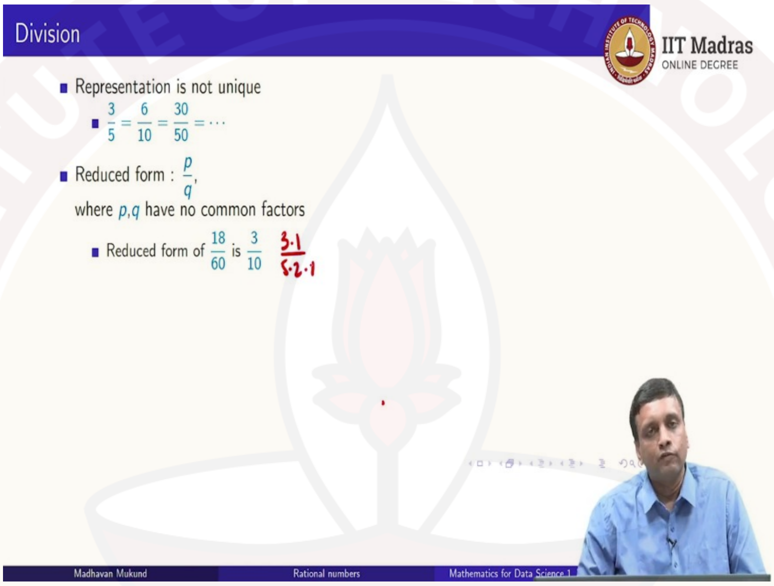

So, we saw that representation is not unique for rational numbers. So, how do we find actually the best way to represent a rational number? So, normally if you are not using it for some arithmetic operation of some comparison, we would prefer to have it in a reduced form. So, the reduced form of a rational number is one, where there are no common factors between the top and the bottom. So, $\frac{p}{q}$ is of the form, where we cannot find any factor f such that $f \mid p$ and $f \mid q$.

So, for instance, if we take $\frac{18}{60}$, then its reduced form will be $\frac{3}{10}$. Notice that 3 is of the form 3×1 and 10 is of the form $5 \times 2 \times 1$. So, therefore, there is no common factor between the top and the bottom and therefore, this is in reduced form.

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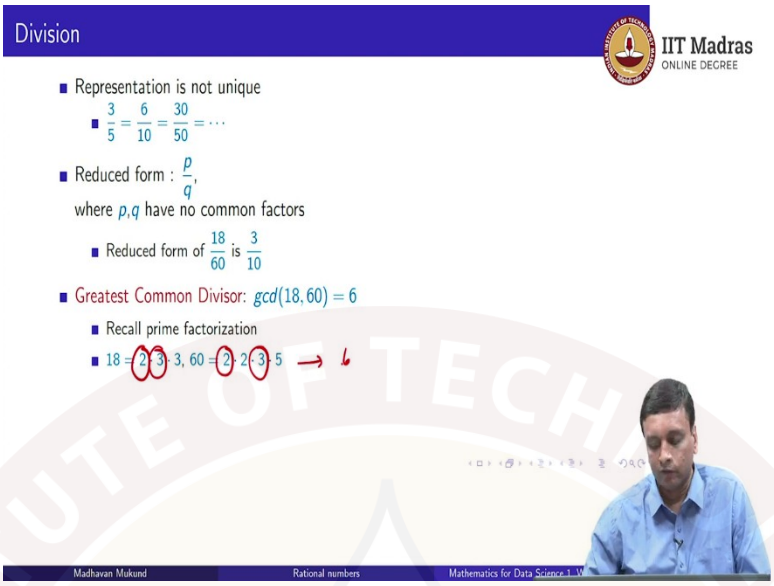

So, this is called the greatest common divisor problem. So, we want to find the largest number which divides both the top and the bottom; both the numerator and the denominator; divide them both by this and then come to something in the reduced form. So, in this case, what we are saying is that the gcd of 18 and 60 is actually 6 and we can do this using our prime factorization that we talked about before.

So, if we look at prime factorization for 18, then 18 is $2 \times 3 \times 3$ right; its 2×3 is 6 and 6×3 is 18 and the prime factorization of 60 is $2 \times 2 \times 3 \times 5$; its 4×3, 12 and 12×5. So, now, you can look at what are common. So, we have one 2 here and one 2 here. So, we can say that this is part of the same factor, we have one 3 here and another 3 there. The second 2 is not present in the first term.

So, we have a 2 and 3 and 18 which are factors. We have a 2 and 3 in 60 which are factors and this gives us the fact that 6 is a common factor. There is no bigger common factor because we want to assemble a bigger common factor, we have to pull out one more prime from each side; but there is no prime left which is present on both sides. 3 is there in 18; 2 and 5 are there on 60, but we do not have a matching one of the other side right.

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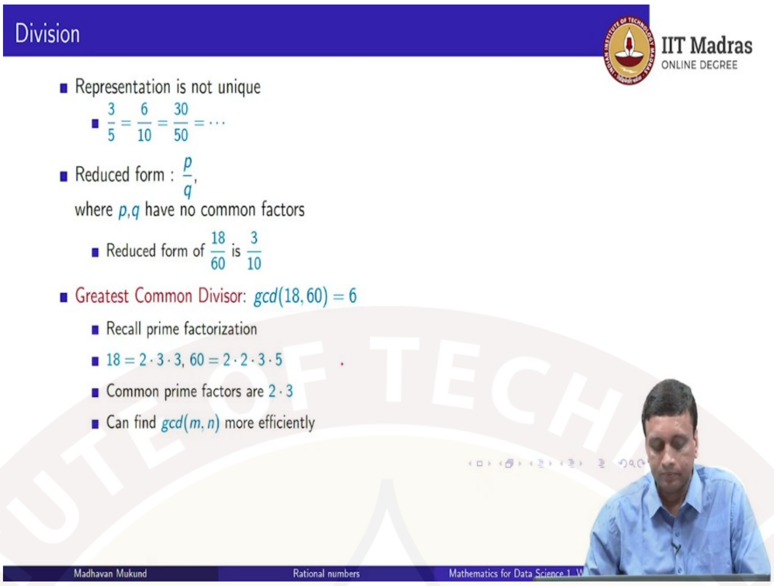

So, this way, the common prime factors are one 2 and one 3 and so, 2 × 3 equal to 6 is the gcd. Now, this is not the best way to find the gcd, there are more efficient ways to find the gcd. But this intuitively tells us what the gcd is. You take the prime factorization of both the numbers and you collect together all the primes that occur in both the numbers, the same number of times.

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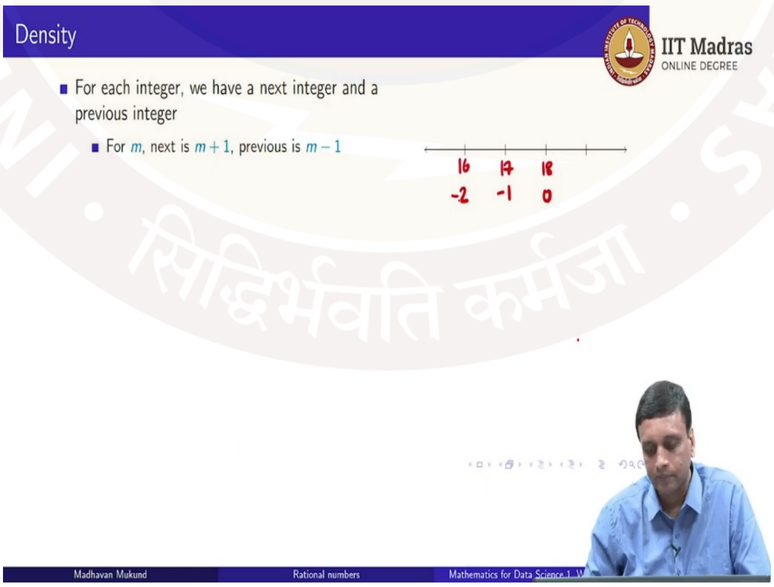

So, here is another interesting property about rational numbers. Now, for each integer, we know intuitively that there is something which is the next integer and the previous integer. If

I tell you 22 and ask you what is the next integer? Then, you will know it is 23. What is the previous one? It will be 21. So, for every integer m, the next one is m + 1 and the previous one is m - 1 and it does not matter, if this is positive or negative. So, for instance if I am at 17, then the next integer is 18, the previous one is 16; right. If I am at -1, then the next integer is 0 and the previous integer is -2. So, I can always take the integer that I am at, add 1 and get the next integer, subtract 1 you will get the previous integer.

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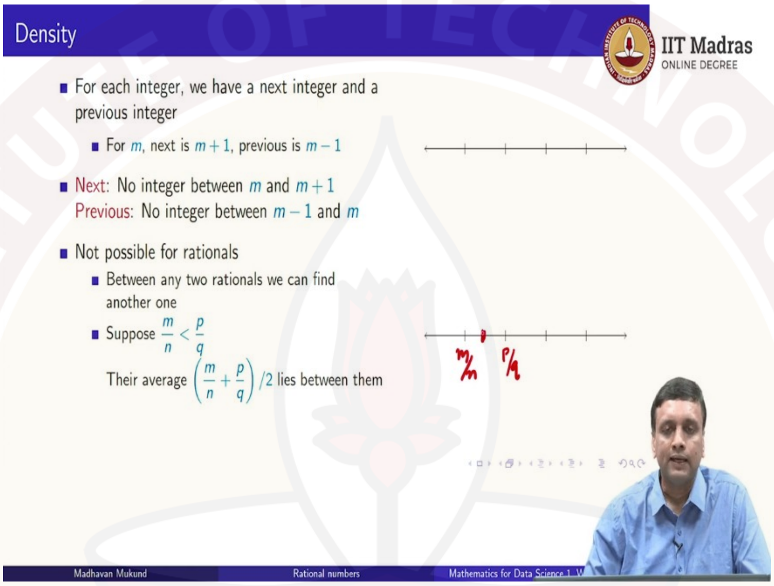

So, the property of this next and previous is that there is nothing in between right. So, there is no integer between m and m +1, there is no integer between m and m -1. So, that is what next means, it is not some bigger integer or some smaller integer. It is the immediate neighbor in the integer of the in this number line. Now, what about rationals? Is it possible to talk about the next and the previous rational number? Now, it turns out that this is not possible for a very simple reason.

So, between any two rationals, we can always find another one because we can always take the average of 2 numbers. So, remember that if you take the average of any 2 numbers, then it must be between those 2 numbers right because it is the sum of the numbers divided by 2. So, the average cannot be smaller than both or cannot bigger than both. So, if the 2 numbers are not the same, then it must lie strictly between them. If the numbers are the same, then the average is the same.

So, if somebody has 37 marks and 37 marks, then their average marks is 37. But if they have 37 marks and 52 marks, even without calculating the average, you know that their average is bigger than 37, but smaller than 52; right. So, in the same way, if I give you 2 fractions $\frac{m}{n}$ and $\frac{p}{q}$ and I tell you that $\frac{m}{n}$is smaller than $\frac{p}{q}$. Remember that in order to do this, we would have to normally get the denominators to be the same and so on.

But supposing I know that $\frac{m}{n}$ is smaller than $\frac{p}{q}$. So, I know that say $\frac{m}{n}$is here and I know that say $\frac{p}{q}$is here and supposing you claim that $\frac{m}{n}$ and $\frac{p}{q}$ are adjacent, that is $\frac{p}{q}$is the next rational after $\frac{m}{n}$. Well, I will say no; let me take these 2 numbers and find its average right. So, this average now is also a rational number because you can also represent it as $\frac{a}{b}$ right. If you just workout this $\frac{m}{n}$ plus $\frac{p}{q}$ divided by 2, you can simplify this whole expression and you will get a new number which is also of the form $\frac{a}{b}$.

So, this is also a rational number and this rational number as we argued must be between the 2 numbers and therefore, between any 2 rational numbers by just taking the average of the mean of the 2 numbers, I can find another one.

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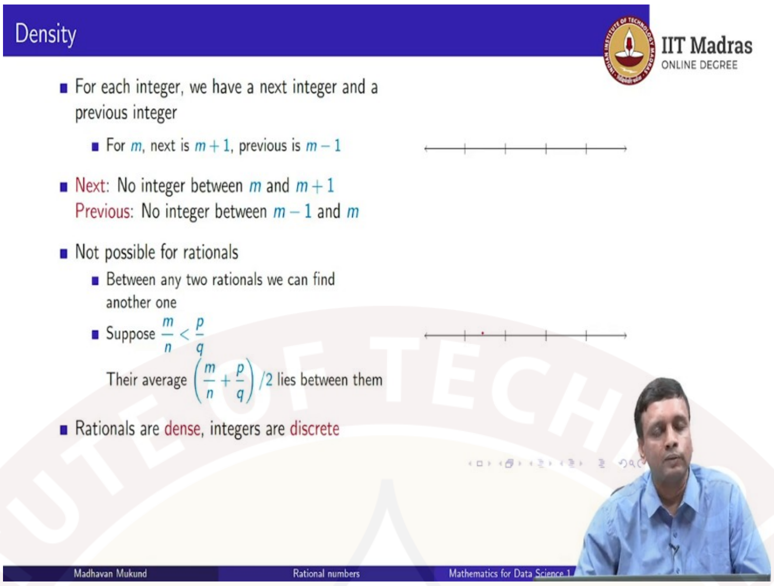

So, in other words, the rational numbers are dense right. So, dense in the usual sense, so dense just means that they are closely packed together. So, basically you cannot find any gaps in the rational numbers because any between any 2 rational numbers, you will find another rational number and this is not true of the integers because we saw that in the number line, there is a gap between m and m + 1, there is no integer there right. So, we say that the rational numbers are dense and conversely, we say that the integers and the natural numbers are discrete. So, a discrete set has this kind of next property and a dense set has no next property between any 2 numbers, will find another number right.

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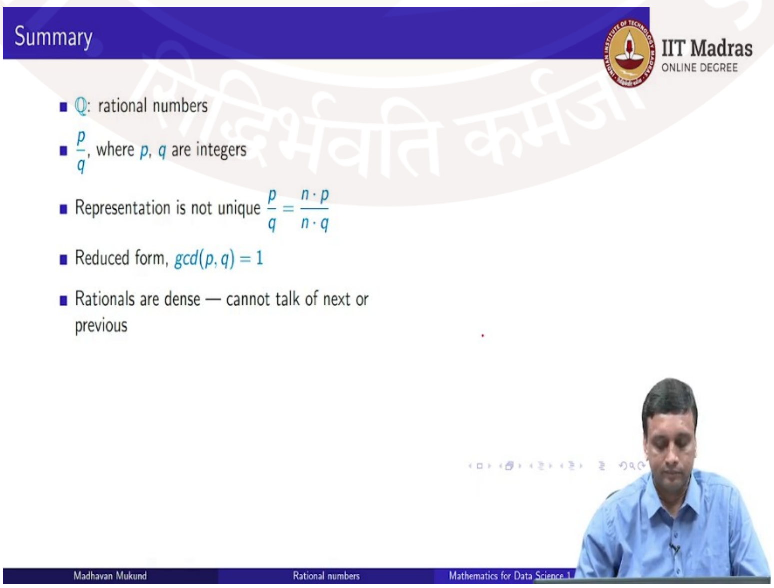

To summarize, we use this funny symbol Q to denote the rational numbers and a rational number is just the ratio. So, that is where it comes from actually; so, ratio. So, rational number comes from the word ratio and so, it is a ratio of 2 integers p divided by q. Now, there is no unique representation of a rational number because we can multiply both the numerator and the denominator by the same quantity and get a new rational number which is exactly the same in terms of the quantity that it represents.

And we use this fact for things like arithmetic and comparisons, but if we really want to talk about rational numbers in a canonical way, in a unique way; then, we get this reduced form, where we cancel out the common factors using prime factorization. So, that we get a number whose gcd of the numerator and the denominator is 1.

And finally, we saw that we cannot talk about the next or the previous rational number because between any 2 rational numbers, there is another rational number. In particular, if you take the average of the 2 numbers, you will find a number that is in between. So, unlike the integers and the natural numbers which are discrete for which next and previous makes sense; for the rational numbers, there is no such quantity.