Single Source Shortest Paths with Negative Weights

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Mathematics for Data Science 1
Week 12

Dijkstra's algorithm

- Recall the burning pipeline analogy

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- Initially
  - No vertex is burnt
  - Expected burn time of source vertex is 0
  - Expected burn time of rest is ∞

Initialization (assume source vertex 0)

- $B(i) = \text{False}$, for $0 \leq i < n$
  - $UB = \{k \mid B(k) = \text{False}\}$
- $EBT(i) = \begin{cases} 0, & \text{if } i = 0 \\ \infty, & \text{otherwise} \end{cases}$

Dijkstra's algorithm

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  - The vertices that have been burnt
  - The expected burn time of vertices
- Initially
  - No vertex is burnt
  - Expected burn time of source vertex is 0
  - Expected burn time of rest is ∞
- While there are vertices yet to burn
  - Pick unburnt vertex with minimum expected burn time, mark it as burnt
  - Update the expected burn time of its neighbours

Initialization (assume source vertex 0)

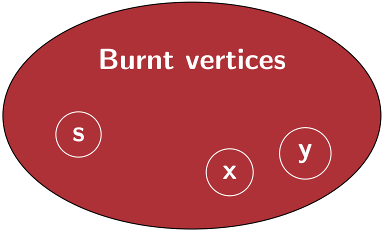
- $B(i) = \text{False}$, for $0 \leq i < n$
  - $UB = \{k \mid B(k) = \text{False}\}$
- $EBT(i) = \begin{cases} 0, & \text{if } i = 0 \\ \infty, & \text{otherwise} \end{cases}$

Update, if $UB \neq \emptyset$

- Let $j \in UB$ such that $EBT(j) \leq EBT(k)$ for all $k \in UB$
- Update $B(j) = \text{True}$, $UB = UB \setminus \{j\}$
- For each $(j, k) \in E$ such that $k \in UB$, $EBT(k) = \min(\ EBT(k),\ EBT(j) + W(j,k)\ )$

Correctness requires non-negative edge weights

- Each new shortest path we discover extends an earlier one
- By induction, assume we have found shortest paths to all vertices already burnt



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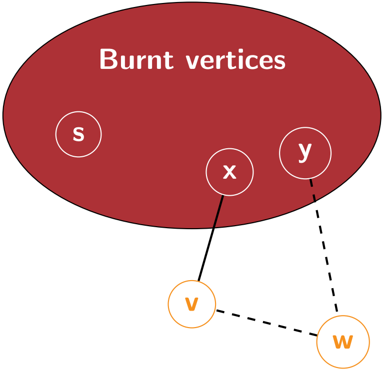
- Next vertex to burn is **v**, via **x**
- Cannot find a shorter path later from **y** to **v** via **w**
  - Burn time of **w** $\geq$ burn time of **v**
  - Edge from **w** to **v** has weight ≥ 0



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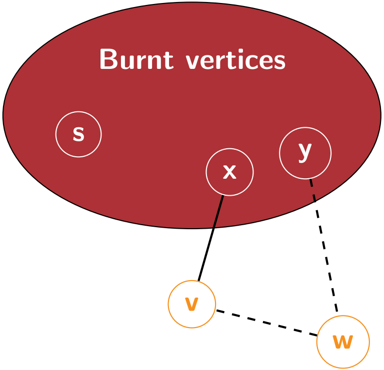
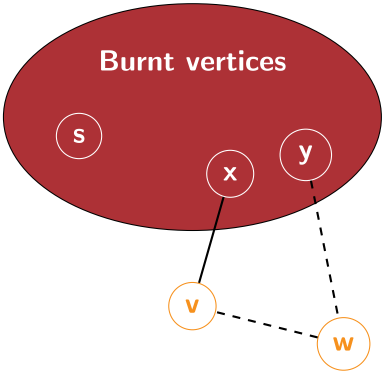
- This argument breaks down if edge (**w**,**v**) can have negative weight



Extending to negative edge weights

- The difficulty with negative edge weights is that we stop updating the burn time once a vertex is burnt



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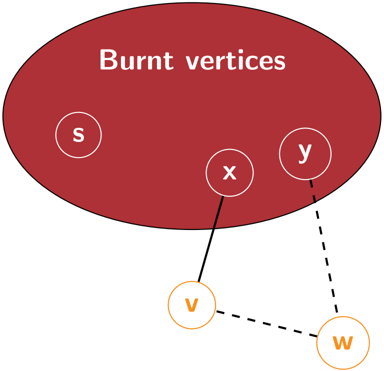
- What if we allow updates even after a vertex is burnt?



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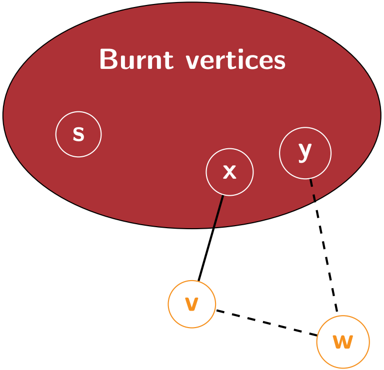
- Recall, negative edge weights are allowed, but no negative cycles



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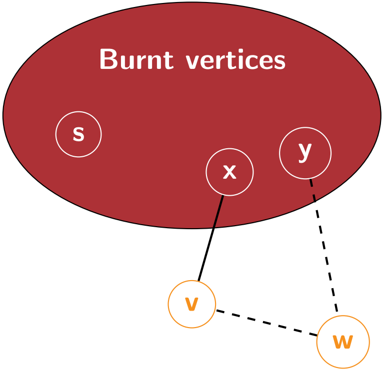

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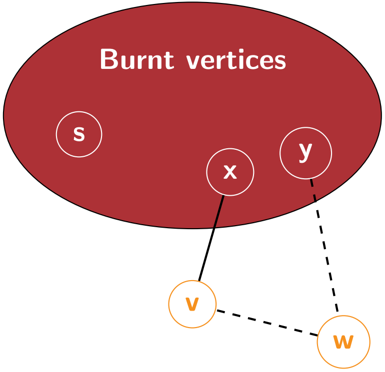
- Recall, negative edge weights are allowed, but no negative cycles
- Going around a cycle can only add to the length
- Shortest route to every vertex is a path, no loops



Extending to negative edge weights

- Suppose minimum weight path from 0 to k is

$$0 \xrightarrow{w_1} j_1 \xrightarrow{w_2} j_2 \xrightarrow{w_3} \cdots \xrightarrow{w_{\ell-1}} j_{\ell-1} \xrightarrow{w_\ell} k$$

  - Need not be minimum in terms of number of edges

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- Every prefix of this path must itself be a minimum weight path
  - $0 \xrightarrow{w_1} j_1$
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  - Update cannot push this distance below actual shortest distance

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- Repeatedly update shortest distance to each vertex based on shortest distance to its neighbours
  - Update cannot push this distance below actual shortest distance
- After ℓ updates, all shortest paths using $\leq \ell$ edges have stabilized
  - Minimum weight path to any node has at most $n-1$ edges
  - After $n-1$ updates, all shortest paths have stabilized

Bellman-Ford Algorithm

Initialization (source vertex 0)

- $D(j)$: minimum distance known so far to vertex j
- $D(j) = \begin{cases} 0, & \text{if } j = 0 \\ \infty, & \text{otherwise} \end{cases}$

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Repeat $n-1$ times

- For each vertex $j \in \{0, 1, \ldots, n-1\}$, for each edge $(j, k) \in E$,
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Works for directed and undirected graphs

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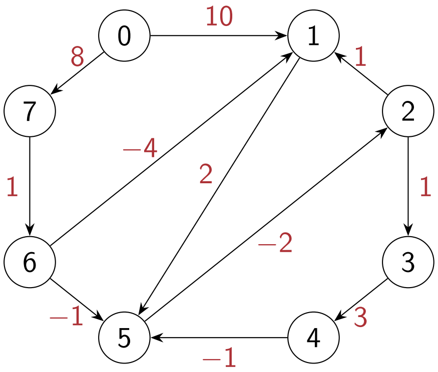
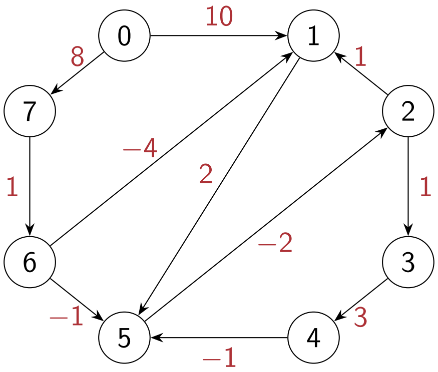


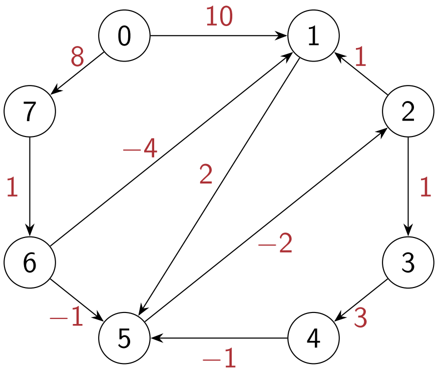
Bellman-Ford Algorithm

v	$D(v)$							
0								
1								
2								
3								
4								
5								
6								
7								



Bellman-Ford Algorithm

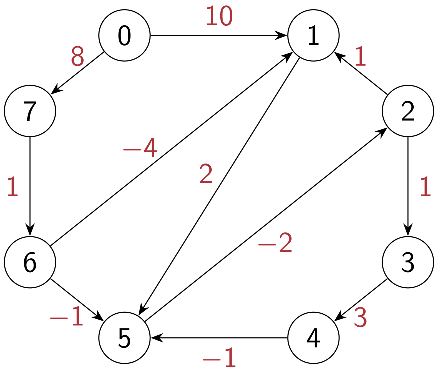
v	$D(v)$							
0	0							
1	∞							
2	∞							
3	∞							
4	∞							
5	∞							
6	∞							
7	∞							



- Initialize $D(0) = 0$

Bellman-Ford Algorithm

v	$D(v)$							
0	0	0						
1	∞	10						
2	∞	∞						
3	∞	∞						
4	∞	∞						
5	∞	∞						
6	∞	∞						
7	∞	8						



- Initialize $D(0) = 0$
- For each $(j,k) \in E$, update
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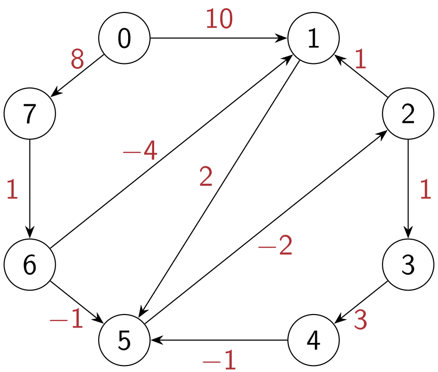
Bellman-Ford Algorithm

v	$D(v)$							
0	0	0	0					
1	∞	10	10					
2	∞	∞	∞					
3	∞	∞	∞					
4	∞	∞	∞					
5	∞	∞	12					
6	∞	∞	9					
7	∞	8	8					



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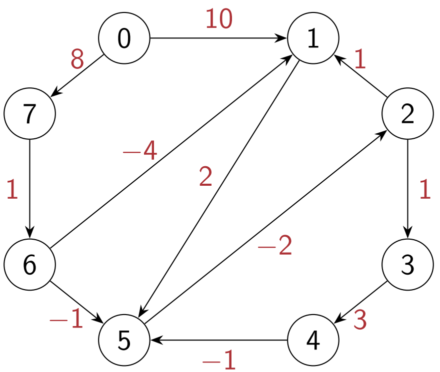
Bellman-Ford Algorithm

v	$D(v)$							
0	0	0	0	0				
1	∞	10	10	5				
2	∞	∞	∞	10				
3	∞	∞	∞	∞				
4	∞	∞	∞	∞				
5	∞	∞	12	8				
6	∞	∞	9	9				
7	∞	8	8	8				



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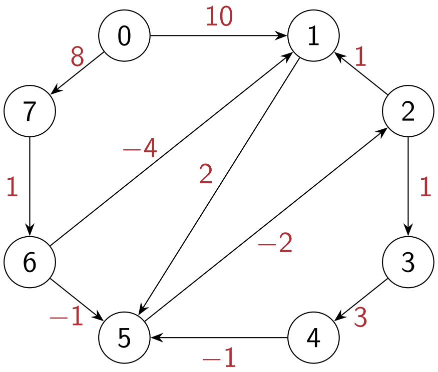
v	$D(v)$							
0	0	0	0	0	0			
1	∞	10	10	5	5			
2	∞	∞	∞	10	6			
3	∞	∞	∞	∞	11			
4	∞	∞	∞	∞	∞			
5	∞	∞	12	8	7			
6	∞	∞	9	9	9			
7	∞	8	8	8	8			



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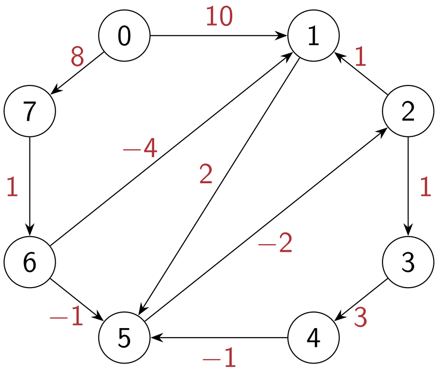
v	$D(v)$							
0	0	0	0	0	0	0		
1	∞	10	10	5	5	5		
2	∞	∞	∞	10	6	5		
3	∞	∞	∞	∞	11	7		
4	∞	∞	∞	∞	∞	14		
5	∞	∞	12	8	7	7		
6	∞	∞	9	9	9	9		
7	∞	8	8	8	8	8		



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v	$D(v)$							
0	0	0	0	0	0	0	0	
1	∞	10	10	5	5	5	5	
2	∞	∞	∞	10	6	5	5	
3	∞	∞	∞	∞	11	7	6	
4	∞	∞	∞	∞	∞	14	10	
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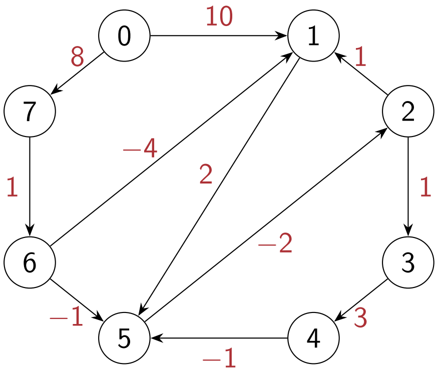

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v	$D(v)$							
0	0	0	0	0	0	0	0	0
1	∞	10	10	5	5	5	5	5
2	∞	∞	∞	10	6	5	5	5
3	∞	∞	∞	∞	11	7	6	6
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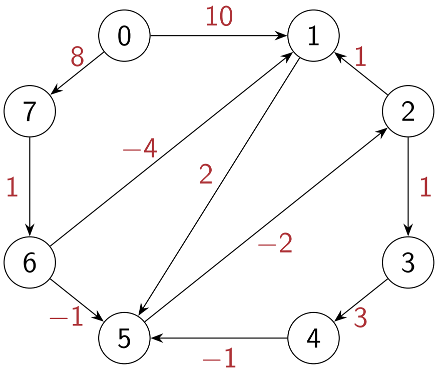

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0	0	0	0	0	0	0	0	0
1	∞	10	10	5	5	5	5	5
2	∞	∞	∞	10	6	5	5	5
3	∞	∞	∞	∞	11	7	6	6
4	∞	∞	∞	∞	∞	14	10	9
5	∞	∞	12	8	7	7	7	7
6	∞	∞	9	9	9	9	9	9
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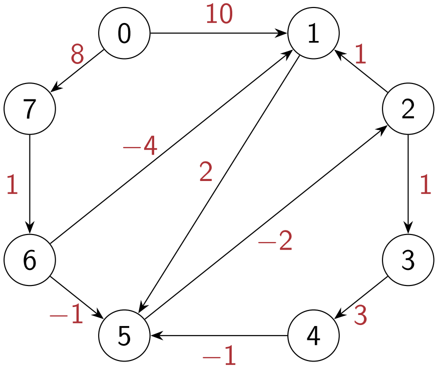

- What if there was a negative cycle?

Bellman-Ford Algorithm

v	$D(v)$							
0	0	0	0	0	0	0	0	0
1	∞	10	10	5	5	5	5	5
2	∞	∞	∞	10	6	5	5	5
3	∞	∞	∞	∞	11	7	6	6
4	∞	∞	∞	∞	∞	14	10	9
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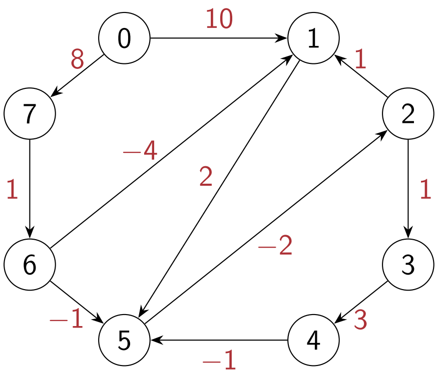

- What if there was a negative cycle?
- Distance would continue to decrease

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0	0	0	0	0	0	0	0	0
1	∞	10	10	5	5	5	5	5
2	∞	∞	∞	10	6	5	5	5
3	∞	∞	∞	∞	11	7	6	6
4	∞	∞	∞	∞	∞	14	10	9
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6	∞	∞	9	9	9	9	9	9
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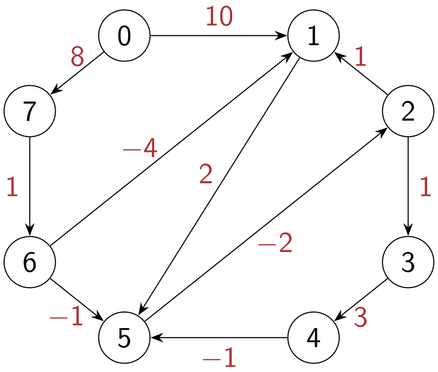

- What if there was a negative cycle?
- Distance would continue to decrease
- Check if update n reduces any $D(v)$

Summary

- Dijkstra's algorithm assumes non-negative edge weights
  - Final distance is frozen each time a vertex "burns"
  - Should not encounter a shorter route discovered later
- Without negative cycles, every shortest route is a path
- Every prefix of a shortest path is also a shortest path
- Iteratively find shortest paths of length $1, 2, \ldots, n-1$
- Update distance to each vertex with every iteration — Bellman-Ford algorithm
- If Bellman-Ford algorithm does not converge after $n-1$ iterations, there is a negative cycle