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Mathematics for Data Science 2
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Lecture No. 14
Integrals as Anti-Derivatives

Hello and welcome to the math's two component of the online B.Sc program on data science and programming. In this video we are going to talk about integrals as anti-derivatives. So, in our previous video we have defined what are integrals via Riemann sums.

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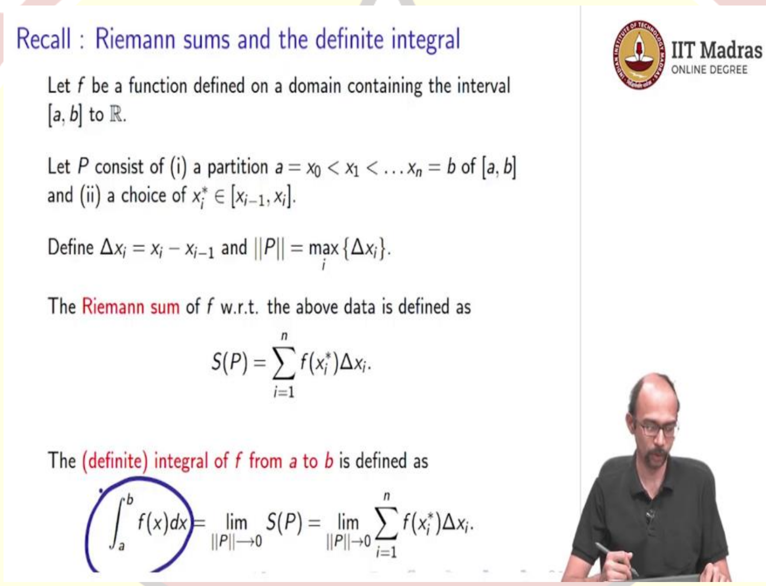

So, let us recall first what that was. So, suppose f is a function defined on the domain containing the interval $[a, b]$ to $\mathbb{R}$. We looked at partitions consisting of points x_1 through $x_n - 1$ where x_0 was a and x_n was b and x_1 through x_2 through x_n were increasing from a to b. And then we choose some particular points within each sub interval. So, $x_i{}^*$ as the point chosen from $[x_{i-1}$, $x_i]$, that interval. And based on this data we define something called the Riemann integral. So, $\Delta x_i = x_i - x_{i-1}$.

So, the length of each interval each sub interval and the norm of P was the maximum over the lengths of all the intervals in that partition. Then we define something called the Riemann $S(P) = \sum_{i=1}^{n} f(\, x_i{}^*\,)\Delta x_i$. And the idea here was that we were looking at rectangles which had the base height the base length was Δx_i and the height of the rectangle was $f(\, x_i{}^*\,)$.

And these rectangles were supposed to approximate the area above the interval x_{i-1}, x_i which is below the graph of *f*. And then as we summed it up this would the idea was that this would be some approximation of the area under the graph of *f*. And of course this is with *f* positive if f was negative then this would be the same thing. But on the other side it would be the area above the graph of *f* except that in that case we would be computing negative of that area.

So, this was supposed to approximate that. And as we made our partitions finer and finer the idea was that it will indeed approximate the area. And that was the definition of

$$\int_a^b f(x)dx = \lim_{\|P\|\to 0} S(P) = \lim_{\|P\|\to 0} \sum_{i=1}^{n} f(\,x_i{}^*\,)\Delta x_i$$

And the integral was denoted by this symbol. The leftmost sign whereas called the integral.

The smaller number is written on the bottom of that integral *a* the larger number is written the top that is *b*. And then the function that you want to integrate is written next. And then this we have this thing called *dx* which indicates many things in particular it indicates the end of the integral that we want to compute. So, this was a notation of the definite integral of *f* from *a* to *b*. Now in the last video I did not make clear why I call it the definite integral.

And that is because in this video we are going to talk about something called the indefinite integral. And it will help us in computing this number. So, in the previous video we have seen how to integrate $2x - 1$ from 1 to 2 we saw it was a fairly non trivial computation. We had to do some amount of work in order to get that. So, if we have more complicated functions presumably we would have to do much more work in order to compute the integral.

And is that defuses the purpose of defining the integral which is that the integral is supposed to quickly calculate these areas. So, in this video we have what we are going to do is we are going to link this integral with something that we have studied even before namely the derivative. And then using that we will compute these integrals and you will see that it becomes a very routine process. Once we understand how to link it up with the derivative or rather the anti-derivative.

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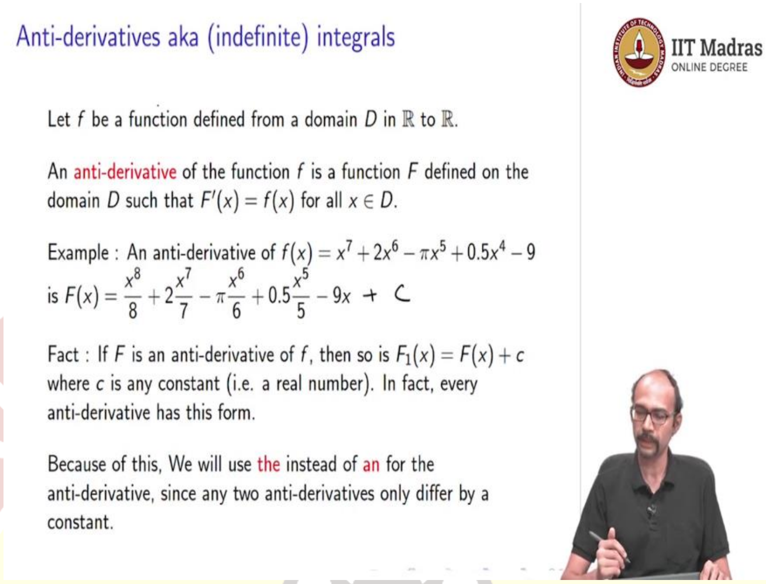

So, let us define what is an anti-derivative. So, let f be a function defined from a domain D in $\mathbb{R}$ to $\mathbb{R}$ an anti-derivative of the function f is a function capital F defined on the domain D such that

$F'(x) = f(x)$ for all $x \in D$. So, in particular of course this assumes that the function F is differentiable. And then when we compute its derivative that is little $f(x)$.

So, the $F(x)$ is the anti-derivative of little $f(x)$ right? Because little $f(x)$ is the derivative of $F(x)$. So, the anti-derivative is just reversing the operation of differentiation we are saying you differentiate and you get little $f(x)$. So, is there some function capital $F(x)$ such that when I differentiate that I get little $f(x)$ so that that is what we will call the anti-derivative.

So, just as an example if suppose little $f(x) = x^7 + 2x^6 - \pi x^5 + 0.5x^4 - 9$. Then its anti-derivative is $\frac{x^8}{8} + 2\frac{x^7}{7} - \pi\frac{x^6}{6} + 0.5\frac{x^5}{5} - 9x$. If you are wondering how this came take this capital $F(x)$ differentiate it and see that you get little $f(x)$.

So, what we did is we know how to differentiate. So, we try to go the other way and ask how do I sort of undifferentiated or anti differentiate? And that is the anti-derivative. So, we will see in a few slides how to get the anti-derivatives. Here is a fact if capital F is an anti-derivative of little f then if you add a constant so $F_1(x)$ is capital $F(x) + c$. Where c is some real number any real number in fact.

Then capital $F_1(x)$ is also an anti-derivative of little f. So, how do I get that? You differentiate capital $F_1(x)$ but we know by linearity that is the same as the derivative of $F(x) + c$. And we know that the differential derivative of constant is 0. So, $F_1{}'(x) = F'(x)$ which is little $f(x)$. So, $F_1(x)$ is an anti-derivative of little $f(x)$.

So, this fact is quite obvious we I just gave a proof. What is more remarkable is that every anti-derivative actually has this form. So, once you identify one anti-derivative of a function you know all possible anti-derivatives. So, if you know one antiderivative then if you add a constant that is that will be another anti-derivative. And if you allow that constant to vary over the real numbers that will give you all possible anti-derivative. So, there is no other anti-derivative that that is possible.

So, just as an example this example that we have here little $f(x)$ as we have your $F(x)$ is as we have here. And if I added let us say 5 to the to capital $F(x)$ that will also be an anti-derivative or if I add -3000 that will also be an anti-derivative. And in fact what we are saying is if you have some other antiderivative then it is going to be capital $F(x)$ along with some added to some constants. That your only possible anti-derivative for this function little $f(x)$.

So, I hope the statement is clear. So, because of this we will use the instead of an for the anti-derivative. So, what when we say anti-derivative in principle it is not at least we did not know before this that it is unique. And even if it is not unique the only possible difference between two anti-derivatives is that of a constant. And we will see that constants do not contribute for whatever we want this anti-derivative for. And so we are going to say the anti-derivative instead of an anti-derivative.

So, when we write down the integral we will always have this $+c$. So, the correct way of writing down the anti-derivative of little $f(x)$ here is $F(x)$ is this plus so $+c$. Where c is a real number some real number that is how any anti derivative is going to look like. So, we will use the instead of an when we talk about anti-derivatives. So, that is fine these are anti derivatives. But what is the connection with integrals? So, it says on top anti-derivatives *a k a* indefinite integrals.

So, we have talked about definite integrals which are about we are given 2 points a and b. And then we want to integrate we have defined via Riemann sums. What is $\int_a^b f(x)dx$.

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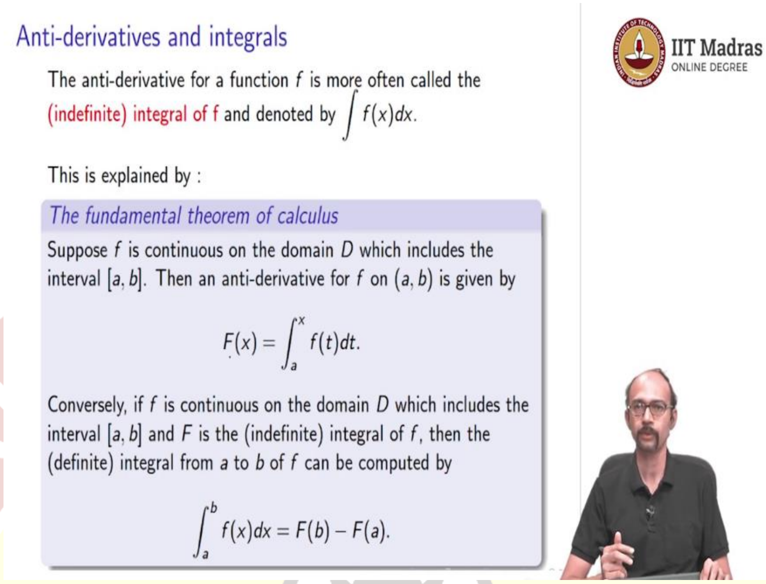

So, now we want to talk about the indefinite integral which I am claiming is the anti-derivative. So, the anti-derivative for a function f is more often called the indefinite integral of f. And it is denoted by $\int f(x)dx$. So, note here that when we have written this integral there is no a or b here. There are no limits for this integration and the point here is that when we do the definite integral that is a number. So, their definite integrals is a number which depends on the limits in question.

So, it depends on a and b of course if you change a and b you get a different number. But once you fix a and b it is some number. Whereas the indefinite integral is just the anti-derivative. So, it is not a number it is a function. So, as an example that is what we did in the previous slide. Capital $F(x)$ is the integral of little $f(x)$ in terms of what we have written here. So, this is called the indefinite integral or just the integral.

When we say integral of f we mean this function which is the antiderivative and again as I said in the previous slide I will be using the instead of an. So, let us pause for a second now and ask the remaining questions that that we should ask some in fact remaining from the previous video and some from this video.

First is let us talk about definite integrals. So, let us go back for a second and look at definite integrals. It is this limit as we know limits may not always exist? So, do we know the difference integral always exists? The answer is no. The definite integral may not exist. There are plenty of examples that we can talk about where the definite integral does not exist. However, if f is f is a continuous function from a to b then the different integral will exist.

So, that is fact that we have to remember. So, if f is a continuous function from a to b then the definite integral will exist. The other point to make here is what about anti-derivative? Do they always exist? And again the answer is that anti-derivatives may not always exist or integrals may not always exist. But once again if your function is nice in particular if it is continuous then indeed they will exist and we will shortly see how to go about doing that.

So, far we have seen the following we have seen the notion of the definite integral from a to b we have seen the notion of an anti-derivative and now we have given it a different name and call it the integral of f. The definite integral from a to b is a number it computes the idea is that it computes the area below or above the curve the graph of f and between the axes. So, you have to of course keep track of the fact that the area comes to the minus sign if it is the function is below. And it comes with a positive sign if the function is above.

So, that is what the definite integral does. And the indefinite integral is nothing but an anti-derivative. So, it is that function such that when you differentiate it gives you little $f(x)$. So, why did we call it the integral? So, of course we are aware that there is some connection between these two and that is explained by something called the fundamental theorem of calculus.

So, the fundamental theorem of calculus is that typically attributed to Newton although it was sort of computationally known before Newton. But Newton formulated it as a theorem possibly for the first time and that is why Newton is often credited with being one of the inventors of calculus. And when you say something is fundamental really it means that it gives you some very deep insight into the subject. And indeed that is what this theorem does.

So, what is the theorem? Suppose f is continuous in the domain D which includes the interval $[a, b]$. Then an anti-derivative for f on (a, b) is given by this function capital $F(x) = \int_a^x f(t)dt$. So, now just as a remark since I am using capital $F(x)$ I want to express my variable here as x. And the x appears in my integral as the upper limit. I cannot use my x inside my integral as well. So I have to integrate with respect to some other variable. So, I am calling that variable t. But do not get confused.

This is just the same function little f only thing is I am calling the variable a different name instead of x I am calling it t. And that is because I want to use x as a variable in capital F. So, let us understand what this is saying so suppose f is continuous on the domain D which includes the interval $[a, b]$. Then the first thing it is saying is that the anti-derivative exists or the integral exists. And the integral is in fact given by considering the definite integral from $\int_a^x f(t)dt$.

Now this is of course a theoretical statement right? Because as we know the integral the definite integral was defined via limit. And it is kind of hard to compute. So, this is a theoretical statement saying that the anti-derivative exists on (a, b). What is really useful and practically something that we are going to use soon is that if f is continuous on the domain D which includes the interval $[a, b]$. So, this is the second part of the theorem the converse in some sense. And capital F is the indefinite integral of little f that means if capital F is the has the property that when you differentiate capital F you get little f. Then the definite integral $\int_a^b f(x)dx = F(b) - F(a)$.

So, now this is giving us something interesting. It is saying that to compute the integral the definite integral from a to b find an anti-derivative or find the indefinite integral. And then evaluate that at b subtract out that same thing evaluated at a. This is what it is telling us. So, the second form is of immense use. And this is what we are going to use in order to compute integrals.

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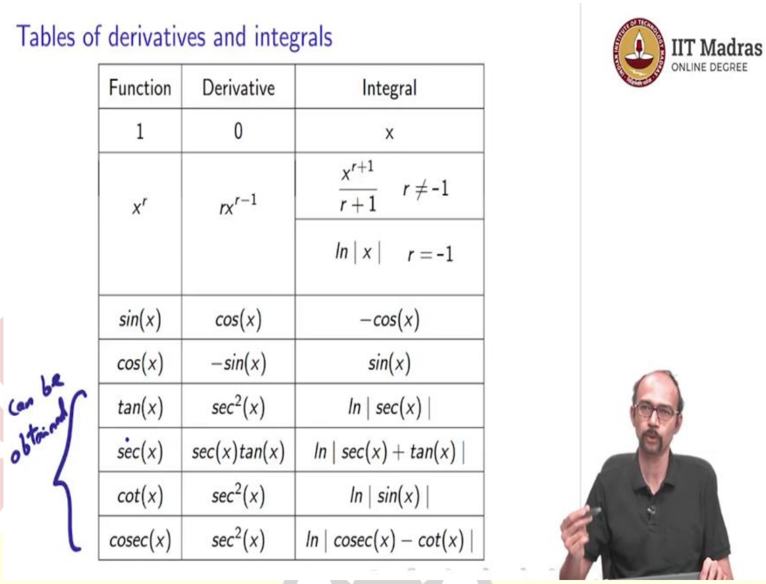

So, we will shortly do examples about this. Before we do that here is a table of derivatives and integrals. Why do I have a table of derivatives and integrals? This is what we are going to use. So, what the fundamental theorem is telling us is that if you want to find this definite integral, you better know the indefinite integral. And then once you know the indefinite integral evaluated at it the limits b and a and then subtract the values out. So, we better know these integrals.

So, how do I know these integrals? I am just back calculating, I know how derivatives work. And from there I am back calculating how integrals work. So, what you have to do here is you have to check out these functions the functions in the rightmost column which are integrals differentiate them and see that you get the function which is in the leftmost column. And the derivative I am just I have just written it.

So, to jog your memory about the fact that this is something that we have actually done before. So, for the function 1, the derivative is 0 and the integral is x. How do I know that? Because if I differentiate x then I get 1. So, the next is the function x^r. We have already seen this rx^{r-1}. How about its integral? That depends on the value of r.

If $r \neq -1$ then it is $\frac{x^{r+1}}{r+1}$. And as you can see if $r = -1$ then there will be a problem because it denominator is becoming 0. And indeed when $r = -1$ you do not have a polynomial instead you have $\ln|x|$. So, the in other words if $r = -1$ then this function is $\frac{1}{x}$ and the integral of $\int \frac{1}{\mathrm{x}}$ is $\log(x)$. So, this is something that we have seen when we did our derivatives part.

The function $\sin x$ the derivative is $\cos x$ and its integral is $-\cos x$. For $\cos x$ the derivative is $-\sin x$ and its integral is $\sin x$. For $\tan x$ the derivative is $\sec^2 x$. This is something we computed and it is integral is the $\ln|\sec x|$. Similarly, for $\sec x$ its derivative is $\sec x\ \tan x$. And it is integrals is $\ln|\sec x + \tan x|$.

And similarly, we have values we have the derivatives or integrals of the $\cot x$ and the $\csc x$ as given here. Now some of these formulae in specifically these last four can be worked out from these two. These can be obtained from here can be obtained. And in the next video we will do an example of this. Of course I will not do all of them. But I will try to do at least one example. But what you can do is you can ask if I differentiate this function on the right do I get the function on the left that you can do by hand.

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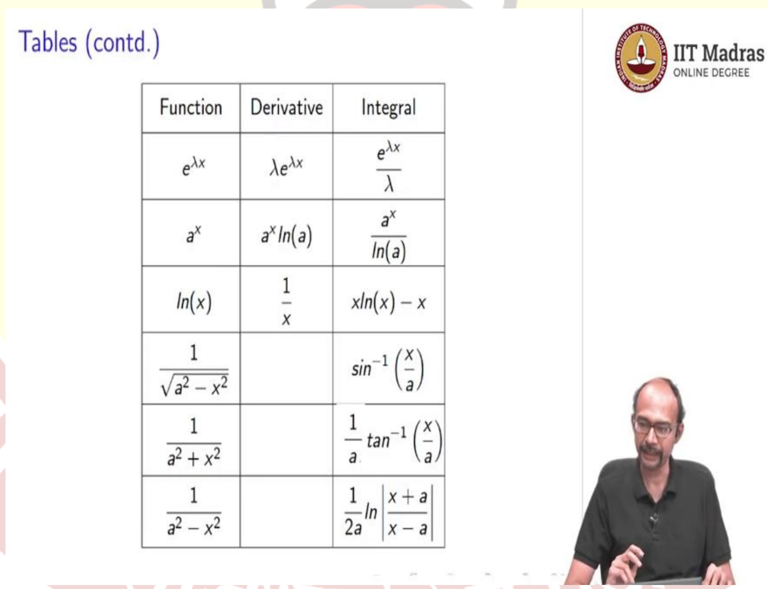

Function	Derivative	Integral
$e^{\lambda x}$	$\lambda e^{\lambda x}$	$\frac{e^{\lambda x}}{\lambda}$
a^x	$a^x \ln(a)$	$\frac{a^x}{\ln(a)}$
$\ln(x)$	$\frac{1}{x}$	$x\ln(x) - x$
$\frac{1}{\sqrt{a^2 - x^2}}$		$\sin^{-1}\left(\frac{x}{a}\right)$
$\frac{1}{a^2 + x^2}$		$\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$
$\frac{1}{a^2 - x^2}$		$\frac{1}{2a}\ln\left\lvert\frac{x+a}{x-a}\right\rvert$

So, I will encourage you to do that to continue. Let us look at the exponentials and the logarithms. If you have the function $e^{\lambda x}$ very important function. Then its derivative is $\lambda e^{\lambda x}$. And its integral is it is $\frac{e^{\lambda x}}{\lambda}$. How did I get that? We will differentiate it and see what you get. So, some of these are usually some of these anti derivatives are being just obtained by observation of what the derivative function is.

If you have e^xthen the derivative is e^x. And its integral is e^x. So, again this is by observation see that the right hand side when differentiate it gives you the left hand side. And when your function is $\log x$ its derivative is $\frac{1}{x}$ gives in that few in the previous slide. And it is integral is $x\ln x - x$ again this is by observation.

If you differentiate the right hand side $x\ln x - x$ you indeed will get $\log x$. And here is a couple of other important interesting functions which you may run across. So, $\frac{1}{\sqrt{a^2-x^2}}$. And we can write down of course, what is the derivative? So, but I have not written it because it is only a tedious calculation you can do it on your own by using the U by V rule. And then successively using that if you have composition of functions how that behaves.

So I will encourage you to check what is the derivative of this? But what is its integral? It is integral something interesting. It is the $\sin^{-1}\left(\frac{x}{a}\right)$. So, $\sin^{-1}\left(\frac{x}{a}\right)$. And what is $\frac{1}{a^2+x^2}$? What is the integral? So, the integral is $\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$. And how about $\frac{1}{a^2-x^2}$ for that the integral is $\frac{1}{2a}\ln\left(\frac{x+a}{x-a}\right)$.

So, again these are all checkable I strongly urge you to check what is the derivative if you differentiate the integrals. Then you should get the function. So, this is just some table that is helpful. Many of these things you can derive or some of these things you can derive on your own but it is useful to have it in your mind. So, that you can quickly compute integrals and derivatives.

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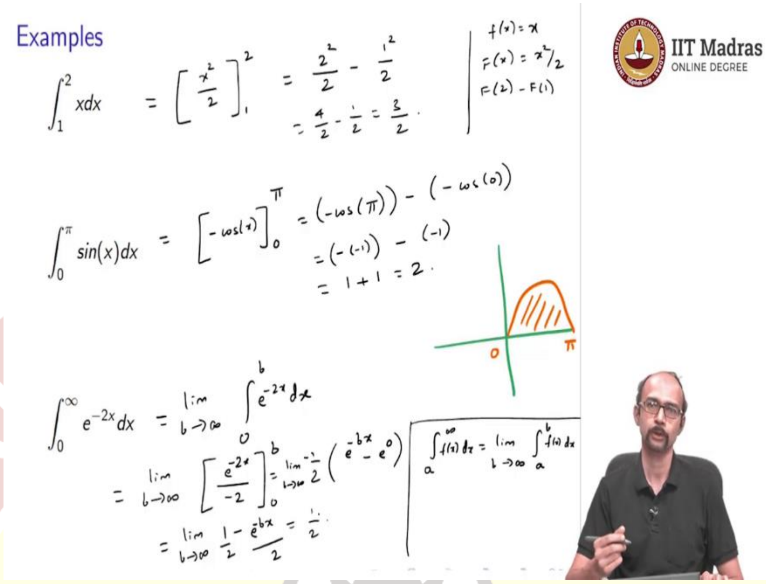

So, let us do a couple of examples of computing definite integrals to get to clear our ideas about how to how we have to use a fundamental theorem. So, here is integral $\int_1^2 x dx$. So let us apply the fundamental theorem of calculus. So, what I have to do is I have to find the definite the indefinite integral evaluate it at 2, then find the definite integral indefinite integral evaluate it at 1 subtract out. So, the way we do this is this is how it is written.

So, the indefinite integral or the anti-derivative of x is $\frac{x^2}{2}$. So, this you can do by observation or you can use the table that we just had. And we want to now do this from 1 to 2. So, I will write this with a with a bracket like this and write the 1 below and the 2 above this is just notation. If you want to follow what we are doing what we are saying is $f(x)$ is x capital $F(x) = \frac{x^2}{2}$ I should evaluate $f(2)$ and subtract out of 1 and that will be my answer.

So, instead of writing these separate steps this is how it is typically written down. So, this is $\frac{x^2}{2}$ you put square brackets and then put 1 at the bottom and 2 at the top on the right. And then we evaluate whatever is inside the bracket at 2 that is $\frac{2^2}{2}$ and subtract out the evaluation at 1 which is $\frac{1^2}{2}$. So, this gives us $\frac{4}{2} - \frac{1}{2} = \frac{3}{2}$. So, the next $\int_0^{\pi} \sin(x)\, dx$. What is the anti-derivative of $\sin(x)$?

It is $-\cos x$. I put my brackets I put 0 to π. And then I evaluate this at the limits. Now here you have to be careful this is the minus sign floating around. So, best to use a lot of brackets so that you will not make any mistakes. So, this is what you get $-\cos(\pi) - (-\cos(0))$. So, what is $\cos(\pi)$? $\cos(\pi)$ if you remember your cosine is -1. So, this is $-(-1)$. And then what is cosine of 0? That is 1 so this is (-1).

So, what does that give us? The first thing gives us 1 the second thing gives us 1 with a plus sign because you have a minus minus so this gives us 2. So, what is this telling us? This is telling us that the if you take the function $\sin x$ and if you draw that curve. So, here is the axis and here is $\sin x$. So, 0 to π so if you take one one such thing and you ask what is the area here? We have just evaluated that that area is 2. And now you can see the power of the integral.

This area is not at all easy to compute otherwise we would have had no real way of guessing what this area is we would have had to maybe go through Riemann sums get rather difficult sum evaluate the limit and then get this. But the fundamental theorem of calculus tells us in one shot that the area here is 2 wonderful. So, I will do one final example. And there is a very important reason I am doing this example.

You will see examples like this in other courses for example statistics where this exponential decay will appear as what is called a density function. So, I also want to emphasize here are our notations for the function and its anti-derivative the function was little f and the anti-derivative was capital F and if you go back to your statistics course you might find a connection with things going on there.

So, for the exponential decay function here we have e^{-2x}, from 0 to ∞. So, what do I mean by 0 to ∞ we have only talked about definite integrals where we have two numbers a and b but we can also do this for 0 to ∞. So, by I should let me quickly emphasize what this means. So, if we want 0 to ∞ $f(x)dx$. We define this as limit of b tends to infinity integral sorry this is a to infinity a to b $f(x)dx$.

This is what we mean. Now of course within this definition there is many things involved. One is that the integral itself from a to b has to be defined then the limit b tends to infinity that limit of these numbers have to be defined. So, it is a fairly non trivial definition. The good part is that you compute it in the same way. So, I am going to tell you how to compute it rather than how to theoretically justify all the steps. So, I am what I will do is I will use this definition.

So, it is about $(-2x)dx$ we know an anti-derivative for $e^{\lambda x}$ that was $\frac{e^{\lambda x}}{\lambda}$. So, this is $\frac{e^{-2x}}{-2}$ 0 through b. Let us evaluate that so we get so we can pull out our $-\frac{1}{2}$. So, $-\frac{1}{2}(e^{-bx} - e^{0x})$. I forgotten my limit excuse me. So, there is a limit here.

So, this is $\lim_{b\to\infty} e^0 = 1$. So, I get 1 minus I am taking the minus sign inside. So, I get 1 minus or rather $\frac{1}{2}$ minus limit as b tends to ∞ $e^{-\frac{bx}{2}}$. And as x tends to ∞ we know that e^{-bx} not x tends to ∞ as b tends to ∞ we know that e^{-bx} is going to tend to 0. So, this is going to be just $\frac{1}{2}$. This is how we compute e^{-bx} the integral is 1 half.

So, if you if instead of e^{-2x} I had $-2e^{-2x}$. Then the integral would have been 1. And that may be should ring a bel from your statistics course. If it has come already or if not it will shortly come something called the exponential random variable. So, let us quickly recall what we have done in this video. We have defined what is the anti-derivative or the integral also called the indefinite integral of a function f when f is continuous this such a thing exists.

And how do we get what it is? You use you have the knowledge of derivatives you know how to differentiate functions. So, you can manipulate and you can get what the integrals are. And then the key point was we can use these indefinite integrals or these anti-derivatives to compute definite integrals. And this is the main thrust of the fundamental theorem of calculus. Which tells us that if you want to compute integral a to b little $f(x)dx$ find the indefinite integral capital F. And then evaluate it at b evaluate it at a subtract the difference. So, that is the summary of what we have done here.

In the next video we will see more examples of computing definite integrals and some rules to compute different integrals. Since this has a connection with differentiation it should not come as a surprise that we will be able to use the same kind of idea the same properties that we had for derivatives similar properties for and they apply also to integrals. So, we will study that and we will use that in the computation of more integrals. Thank you.