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IIT Madras

ONLINE DEGREE

Mathematics for Data Science 1
Prof. Neelesh S Upadhye
Department of Mathematics
Indian Institute of Technology, Madras

Lecture - 33
Algebra of polynomials: Multiplication

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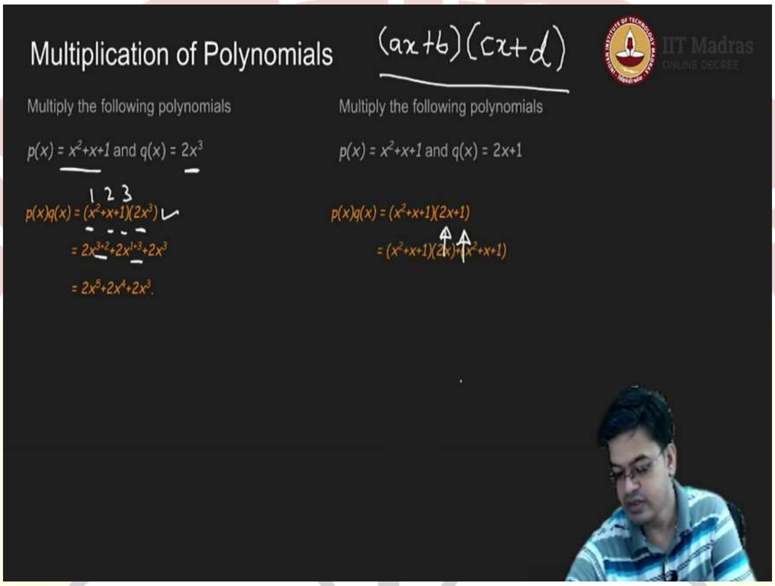

In this video, we will learn how to multiply two polynomials. Let us start with basics of multiplication of polynomials. We already know how to multiply two binomials. For example, if you have been given two binomials of the form $ax + b + cx + d$, then you know how to multiply these two binomials that is we will use the foil method. However, in this context, we want to generalize the settings for multiplication of polynomials of arbitrary degree.

So, let us see, let us start with some simple monomials with through examples. So, here is a polynomial given to you $p(x) = x^2 + x + 1$ and $q\ x = 2x^3$. The question is do I know how to multiply these two polynomials? Remember this one is called monomial, it has only one term. So, a standard rule of multiplication will mean we have seen this in our quadratic functions that I will consider the product in this manner.

Once I consider the product in this manner, what we will do is we will try to multiply each term of this $2x^3$ with each term of this polynomial. So, there are three terms. And for each term this $2x^3$ will be multiplied. So, if I do that the law of exponents will apply.

For example, $x^3 \times x^2$ will mean x^{2+3}. So, once I apply we apply the law of exponents and add the exponents, obviously, 2 was a constant coefficient of x^3 which will be multiplied throughout the expression. And therefore, the resultant is this which we can simplify as $2\,x^5 + 2\,x^4 + 2\,x^3$. This is how we will multiply a monomial.

Now, as you can see the this polynomial has three terms 1, 2 and 3. So, it is not a binomial; it is a trinomial. So, my foil method will not work here. So, foil method will work only for these kind of expressions which are binomials. So, let us go ahead and try to consider a similar expression that is a quadratic expression and another binomial, and try to see how can I extend the basis of foil method right.

So, here is a binomial $2\,x + 1$. And here is a general polynomial quadratic polynomial which is $x^2 + x + 1$ same. Now, what will you do? So, naturally you will consider $p\,(x) \times q\,(x)$ which will be written in this form. Now, if I want to extend the basis whatever I did for monomial, that means, I need to convert this into two monomials.

So, what are those two monomials? One monomial is $2x$; another monomial is 1. So, if I treat them separately that is if I write them in this manner, let me erase this, that is I have written them in this manner.

Then what can I do about it, that means, now this turned out to be a same expression instead of x^3, here it is x that is all is the difference right. So, whatever I did here, I can do it here. And the last term is actually multiplied with 1 which it suppressed because multiplication with 1 will not change anything. So, I do not have to worry about the last term.

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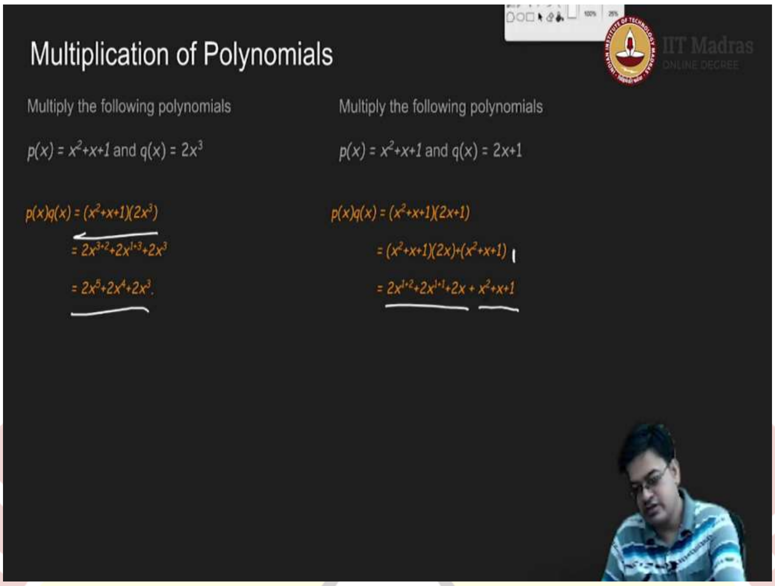

Now, I will multiply this $2x$ with all the terms in for of $p(x) = x^2 + x + 1$ which is similar to this particular thing. So, I will get $2\,x^{1+2} + 2x^{1+1} + 2x + x^2 + x + 1$. Now, the job is very simple.

You can treat this as one polynomial, and this one as a second polynomial, and then we have to add. How we add polynomials? We will add polynomials by matching the exponents, matching the exponents of x. So, if I want to add these two polynomials, what will I do, I will simply match the exponents and I will add them which is given here.

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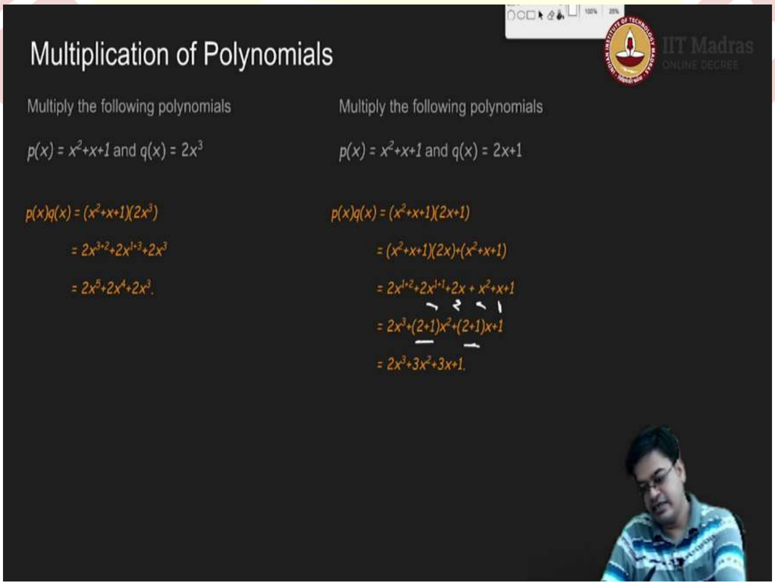

So, in this case 2 x^3, there is no competing term for x^3. So, it remains 2; x^2 comes here and here, therefore, I added the two which gives me 2+1, in a similar manner the terms containing x are these two. So, I have added these two, so 2+ 1 $x + 1$ which is similar to what we have seen in the last video of addition of polynomials. And therefore, we get the answer to be equal to $2x^3 + 3x^2 + 3x + 1$.

So, effectively what we have done is we know how to multiply the terms term by term. And finally, if at all I want to seek an extension of a foil method, it will be a term by term multiplication of polynomials, that means, you take the polynomial of least degree and multiply it with the polynomial of highest degree term by term, add those term match the powers and then write your answer. So, this is one prototype that we can follow for finding multiplication of polynomials or result of the multiplication of polynomials.

Now, the next question is can I generalize this method or can I answer it programmatically, that means, can I give a simple formula for what the coefficient of one part x^m will be? For example, in this case can I give a general formula what will be the coefficient of $3x^2$ provided I know polynomials $p(x)$ and $q(x)$. So, to answer that, let us go ahead and try to find a general formulation of this form of this formula.

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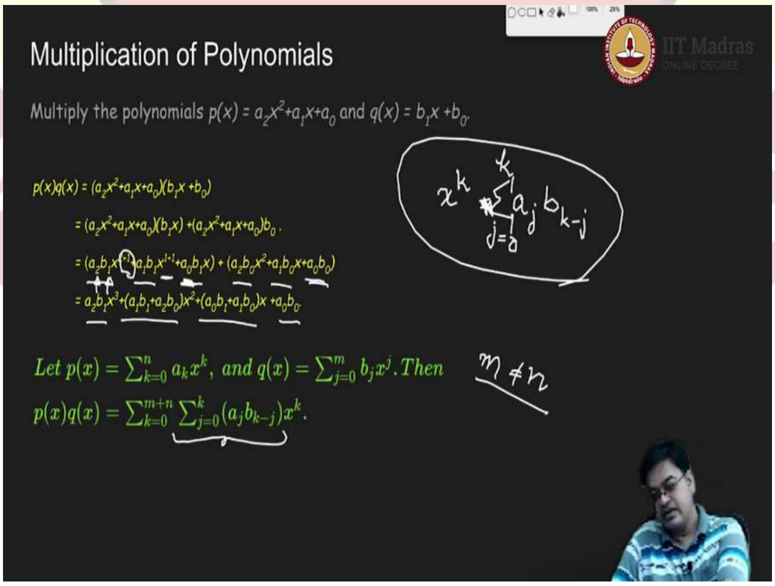

Let us go ahead. And if you are asked given one quadratic polynomial and one linear polynomial, you are asked to compute $p(x) \times q(x)$, how will you go about this? This is what our task is now. So naturally I will write $p(x) \times q(x)$, and then I will convert each of them into monomials that is one monomial will be b_1x, and second monomial will be b_0.

In this case, what will happen is we will simply multiply them as a separate term by term multiplication. So, in earlier case our b_0 was 1 when we studied one example. But here we are considering a general expression, and none of the expressions are 0 that is what we are assuming none of the coefficients at a_2, a_1, a_0, $b_1$and b_0 none of them are 0.

For example, if you consider $b_0 = 0$, then this term itself will vanish the second term itself will vanish; you will not have the second term. So, we are assuming that all terms remain in the loop ok. So, now it simple, the job is multiplying these two polynomials, and you will get some answers that is ok, but now our main worry is to find a pattern in these answers ok.

So, now, when I multiplied this, if you look at this particular expression that is $(a_2b_1x^{2+1} + a_1b_1x^{1+1} + a_0b_1x^1) + (a_2b_0x^2 + a_1b_0x^1 + a_0b_0)$. Here you take a pause and examine the terms. For example, this term contains the coefficient of x^3, this is $2 + 1$.

So, x^3. So, in that case, what is happening here is if you look at the suffixes of the coefficients this is a_2, this is b_1, so together they will sum to 3. In a similar manner, you look at this term which contains x^2. And you look at the suffixes of the coefficients that is a_1b_1, together they will sum to the exponent that is a 1+1=2. So, this should be a coefficient of x^2.

Then if this logic is correct, what should be the coefficient of a constant? The coefficient of the constant that is x^0. So, the coefficient of the constant must be a_0b_0. In a similar manner you can ask the question what is a coefficient of x? If you asked that question, you will naturally get the answer you collect all the in all the coefficients such that their suffixes will sum to 1 that is $a_1b_0 + b_1a_0$. So, is there anything called b_1a_0? Yes, it is here.

So, this what we have actually done is we have figured out a pattern; that means, if I want to find the coefficient of x^k, then better the sum should be some a_jb_{k-j}, so that they both will sum, they both will sum to it is not equal to the this is I am saying x raise to coefficient of x^k will be equal to of the will be of the form a_j+b_{k-j}. So, with this understanding, let us go further and try to rewrite this sum ok.

So, once I have rewritten this sum, my analogy is further amplified. For example, if you look at the coefficient of x^2, yes, it was it is a_1b_1 and a_2b_0 which is the coefficient of x^2, so that also means this means if I can sum over this j from 0 to what point to a point where I want the sum the exponent is raised to k, then I will get all possible combinations where sum is actually k.

In a similar manner, you can pause this video and verify whether you are getting the same expression for x^1 and all others right. So, with this understanding, I am ready to generalize this demonstration or this theory for a polynomial of an arbitrary order.

Let us consider polynomials of degree n and m, and try to find the general answer for them, and that answer will be in this form. So, if you are given a polynomial of degree $n, p(x)$, and if you are given another polynomial of degree $m, q(x)$, let us say $m \neq n$.

Even if $m = n$ it does not matter, but for our purposes let us take $m \neq n$, then what will be the coefficient of each of the x^k's? The coefficient is actually given here, $\sum_{j=0}^{k} a_j b_{k-j}$ this is what we have figured out in this expression is the coefficient of x^k.

Then the question is how far the degree will go? The degree will go till $m + n$ $m \neq n$; if $m = n$ then the degree will go to $2\,n$ that is ok. So, $k = 0$ to $m + n$, and each of the coefficient of x^kwill be $\sum_{j=0}^{k} a_j b_{k-j}$. Now, let us demonstrate this idea with one example. Let us go ahead and see one example of this idea.

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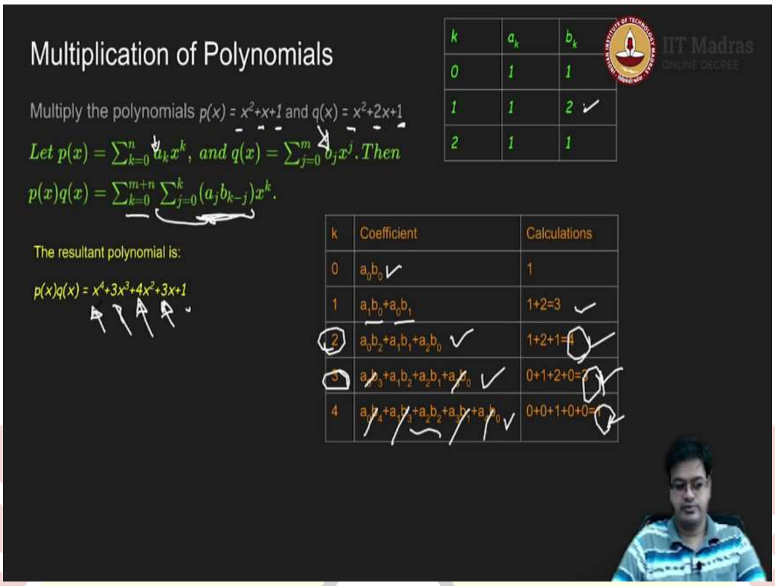

So, now, you have been given two polynomials two quadratic polynomials and you are asked to compute the multiplication of these two polynomials. One way is very simple you will go with term by term multiplication, and it simply means you have to multiply the terms of second polynomial with the first polynomial in a term by term fashion, or you can actually use the formula that I have given you in the previous slide. So, you can pause this video, and try to compute by yourself or you can go along with me.

So, let us recall that formula again that is $p(x)$ is equal to sum a, so my polynomial is a polynomial of degree n, and $q\ (x)$ is a polynomial of degree m. In this case, in this particular example, the polynomial the first polynomial is of degree 2 as well as the second polynomial is of degree 2.

So, in order to find the product of these two polynomials, what do we need to find is we simply need to find the coefficients of x^k. So, let us first identify what are a_k's and what are b_k's, j is a dummy index. So, it does not matter.

So, let us first identify what are a_k's and b_k's. So, a_0 as you can see is 1, b_0 is 1, a 1 is 1 again, b_1 is 2, correct, this is correct, and then a_2 and b_2 both are 1. So, I have enlisted all the coefficients of this particular expression, $p\ (x)$ and expressions $p\ (x)$ and $q\ (x)$. Now, we need to use this formula, then this formula which gives me the sum. So, let us use this formula and figure out.

Remember, all the coefficients that are not listed here. For example, what will be a_4, if at all, I will write a_4, what will be a_4 in this expression? It will be 0. What will be a_3 in this expression? It will be 0. So, all the coefficients that are not listed here are 0s. Keep this in mind and try to answer the question.

So, now, computation of coefficient; it is very easy. So, let us start with 0th degree term that is constant term. So, here $k = 0$. So, the summation will actually go from $j = 0$ to 0, that means, it will have only one term which is $a_0 b_0$.

What is $a_0 b_0$? Look here 1 into 1, so it will give you 1 ok. Let us go for a degree 1 term. So, j is equal to 0 to 1, j is equal to 0 to 1, so it will have, $a_0 b_1 + a_1 b_0$ these two terms are there. So, let us compute them through this table a_1 is 1, b_0 is 1, so this will retain 1. a_0 is 1; b_1 is 2, so it will give you 2. So, together it is 1+ 2 = 3.

Let us go for a second order term that is the monomial with degree 2. So, in this case, j will run from 0 to 2. So, I will have $a_0 b_2$, $a_1 b_1$, $a_3 b_0$, $a_1 b_1$, $a_2 b_0$, $a_0 b_2$, $a_1 b_1$, this is correct. Just go ahead and compute these terms, a_0 is 1, b_2 is 1, so you will get 1, $a_1$is 1, b_1 is 2, so you will get 2. And $a_2 b_0$ that is a_2 is 1, b_0 is 1, so you will get another 1. So, you will get the sum to be 4.

Let us go for a third term x^3 term, and just simply substitute this. So, we need to find all possible combinations. So, if it is a degree 3 term and we start with a_0, it will be $a_0 b_3$, $a_1 b_2$, $a_2 b_1$, $a_3 b_0$, these are the terms. And then you simply compute them.

Remember here now we came up with b_3. What is b_3? b_3 is not listed here, that means, b b_3 must be 0. In a similar manner here a_3 must be 0 correct. So, these 2 terms are chopped off right away they are 0. So, let us focus on the other 2 terms the first term you can easily verify because b_2 is 1, and a_1 is 1. And $a_2 b_1$, b_1 is 2, a_2 is 1, so it will be 2. So, 1+ 2=3; this is correct.

Now, the final term is a degree 4 term, correct. If you do a term wise multiplication, what you will come up with is because the degree 4 will be contributed by the highest order terms.

So, you will simply multiply $x^2 \times x^2$, and you will get only 1 term. But in this formulation what we are doing here is we are taking all possible terms of degree 4. So, even though they are 0, we will first list them, and we will put them as 0s.

So, now, when we consider degree 4 term, I will get a_0b_4, a_1b_3, a_2b_2, a_2b_2, a_3b_1 and a_4b_0. So, all these terms are here. And most of the terms will obviously, be 0 only 1 term is a contributor.

For example, a_0b_4 is 0, a_4b_4 will be 0, a_1b_3 is 0, a_3b_1 is 0. Why? Because b_4, b_3, a_3, a_4 all are 0 only term that will contribute is a_2b_2 which will be 1×1, so 1. So, this gives us a clear cut answer, and this is a systematic way to multiply two polynomials.

Therefore, the resultant polynomial $p(x) \times q(x)$ simply write the terms from this table, so this is a coefficient of x^0 is 1, so the constant term 1 is here coefficient of x^1 is 3, so $3x$ is here. So, in a similar manner x^2 coefficient of x^2 is 4. So, you will get $4x^2$ here ok; so $3x^3$ correct.

So, this is also done. And then x^4 has only 1 term as 1, so x^4. Therefore, you got the resultant polynomial to be equal to this. Now, remember one side note the multiplication of two polynomials will always fetch you a polynomial again ok. Next operation is division which we will see in the next video, but the division of two polynomials will not always lead to a polynomial. We will see that in the next video.

Bye for now.

Thank you.