INDIAN INSTITUTE OF TECHNOLOGY MADRAS

सिद्धिर्भवति कर्मजा

IIT Madras

ONLINE DEGREE

Mathematics for Data Science 1
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Lecture No. 51
Inverse Functions

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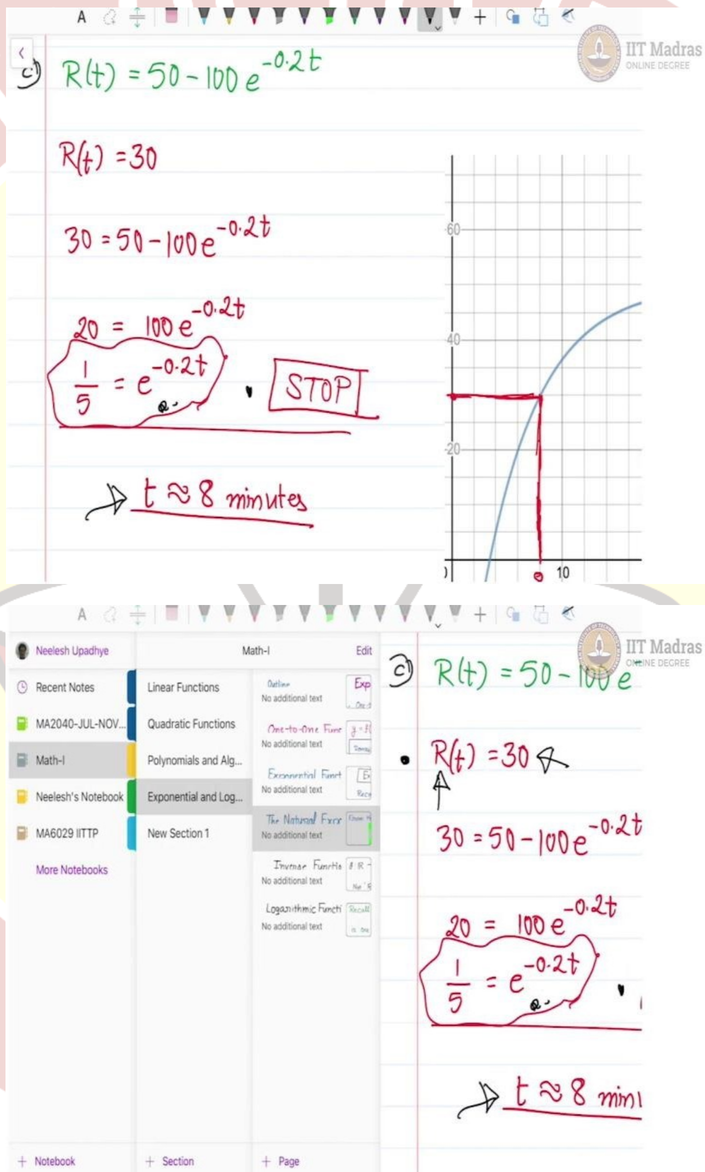

Hello Students, in the last video we have stumbled upon one concept, where we could not proceed. Then we came to…Let us go to the last videos last slide. So, here if you look at this particular concept we actually stopped while computing. And why we stop while computing is because we did not have enough information on, how to write t is equal to something given this equation.

So when, what we did we found escape out by plotting the lines across x and y $axis$, horizontal and vertical lines and figured out that the answer is 8 minutes. And that is how we concluded this is 8 minutes. Now when we start such a thing analytically that is R_t is given to be 30. What is the value of t? We want to answer such questions then we need to look at the function R and we need to understand whether this function is reversible or not.

Which is the case, in this case, in this particular function because we were able to map it uniquely. So, what are the important trades of this function R_t? R_t was a one to one function and it was increasing function. Hence, it was one to one. Therefore, we were able to find a reversal of the value 30 to the value of t which is 8.

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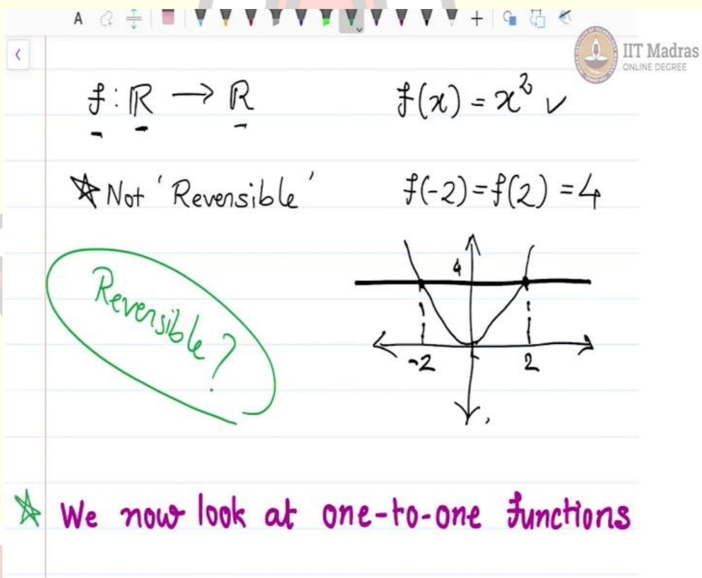

So, in order to find such reversible functions, we need to understand the theory which we will discuss now is the theory of inverse functions. So, when I talk about inverse functions, I am talking about functions from domain which is real line co domain which is real line. So, a function is define from real line to real line, then the immediate question that comes to our mind, are all functions reversible? And the immediate answer is a very well-known function that we have seen is, $f(x) = x^2$.

Now this function is not reversible because it feels to pass the horizontal line test, if you remember. So, $y = x^2$, if I try to plot, it will be something like this. Very close to something like this. And when I pass a horizontal line through this it passes through 2 points. And let these points be 2 and -2 . And that essentially means this, when I feed in the value 2, it will give you four. And when I feed in the value - 2, it will give you the answer to be equal to 4.

Now if this function is reversible, when I feed the value to 4 it can spit out the two values 2 and -2. So, it is not uniquely spitting out the value. Therefore, this function is termed as not reversible function. Such functions we cannot study the inverse properties or the properties of inverse functions. However, if you restrict the domain of this function instead of real line to only positive half of the real line, then you will get one to one correspondence between the values on x a$x$is and y a$x$is and then you can talk about inverse of these functions, when it is defined from 0 to ∞.

Now let us look, then the question is, this function is not reversible then which functions are reversible? That is a question that we can ask now, in order to answer this question, we need to study some class of functions. So, in last few videos we have already seen that one to one functions are nice functions. Any function that is either increasing or decreasing is one to one and therefore we can look at one to one functions for the class of reversible functions. So, here is our answer that we will start looking at the class of one to one functions.

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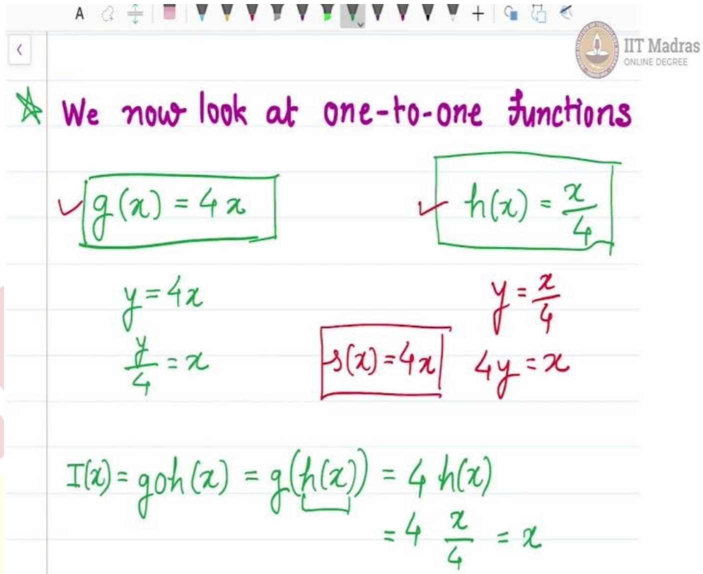

Let us look at a simple function a linear function g(x) is equal to $4x$. Is this function reversible or not? So in order to answer this question, let us look at $g(x) = 4x$. So, you can put $y = 4x$. If you look at $y = \ 4x$ from our basic understanding of linear equations or rather than linear equations an equation of a straight line. This is a straight line passing through origin having slope 4. So, if I want to find a point x on this a$x$is then I will simply transform this as $\frac{y}{4} = x$ and this transformation is unique. Therefore, I can write some function let us say *r(x)* as $\frac{x}{4}$. And this function will actually be giving be the inverse of this.

So, let us take this function, if this function $h(x) = \frac{x}{4}$. So, I do not need to write *r(x)*= $\frac{y}{4}$. $h(x) = \frac{x}{4}$. Now if I start with this function and I want to get value of x, what should I do? I will write, so I will write $y = \frac{x}{4}$ and in that case I will get $4y = x$. And therefore, I will get another function which is say $4x = s(x)$. So, essentially what we have seen is this g(x) and h(x) have something in common. So, let us recollect the notion of composition of two functions, and try to answer this question.

For e$x$ample, if I consider the function goh(x). Now this function is again a function and it will simply operate like g of h(x). And once you start with g of h(x), what you will do is, you will treat h(x) as an argument of g and put the values of h(x) inside. So, let us try to understand this, so it is like g(*h(x))* is actually, what is *g(x)?* 4 of x. So, it will be 4 $\times$ *h(x).* Now what is $h(x)$? $h(x)$ is nothing but $\frac{x}{4}$ So, 4 times x by 4 which will give me x. So, what this function is, this function actually gives me identity function. $goh(x) = x$ and a similar manner

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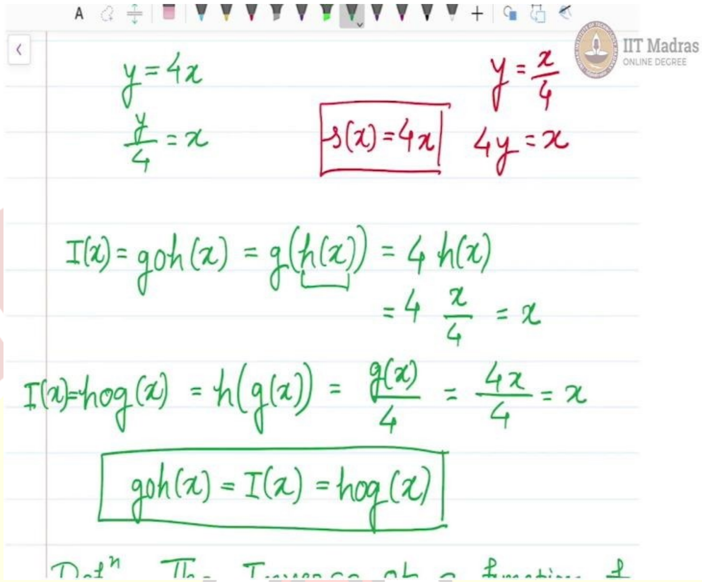

I can start thinking about *hog(x)*. now in this case, if you recollect the notion of composition of functions studied in week one, *h(g(x))*. So, if h of g(x) I will simply see what is *h(x)*. $\frac{g(x)}{4}$ and therefore what is g(x)? It is $4x$ therefore I will get $4x$/ 4 which is actually equal to x. And therefore this is also equal to identity function of x. So, now to summarize what I got is *G o h(x)*= $I(x) = hog(x)$. Now this becomes our definition of inverse function. And let us define it formally as the definition of inverse function.

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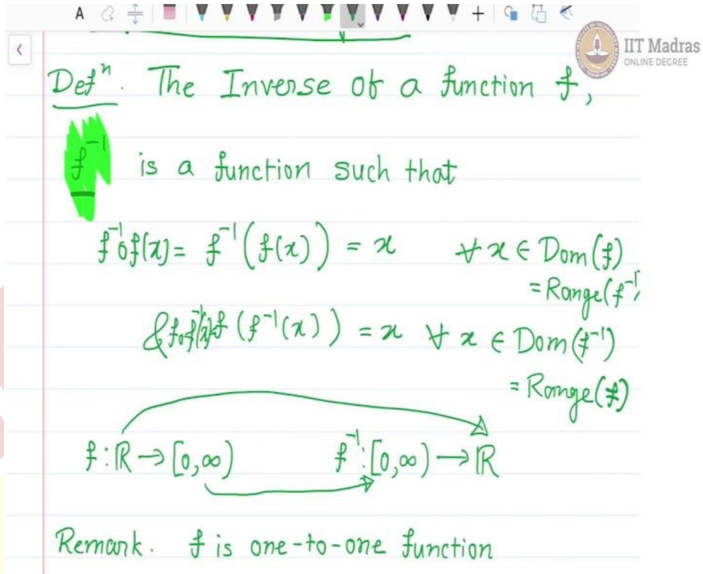

So, here is a definition of inverse function. The inverse function inverse of a function f, we denote it by .$f^{-1}$is actually a function this is our notation f^{-1} is actually a function such that.$f^{-1}f(x)$ or I can rewrite this as $f^{-1}f(x) = x$. Now here is a typical thing that comes for all x belonging to domain of f which is equal to range of f^{-1}. And $f(f^{-1}(x))$ or you can write this as $f(f^{-1}(x))$ being equal to $f(f^{-1}(x)) = x$ for all x belonging to domain of f^{-1} and range of f^{-1}

So, right now when I did this particular calculation I have assume that everything goes from real line to real line there was no such event. Because this function is define from real line to real line. And this function is also define from real line to real line. So, there was no consideration for domain and ranges. But sometimes it may so happen that your original function maybe define, let us say f is define from ℝ to[0,∞). If such a definition is there, then you need to worry about the domain of a function and the range of a function. Because here the domain of f is ℝ and range of f is 0 to ∞.

So, if I talk about f^{-1} of this, then naturally I cannot go over entire real line. I have to go over 0 ∞ and then I have to come to ℝ. So, this is how it will be define and therefore, the domain of f will become the range of f will become the domain of f and the domain of f will become the range of f^{-1}. This is the typical factor that you need to always remember. Now let us go ahead and improve our understanding about one to one functions.

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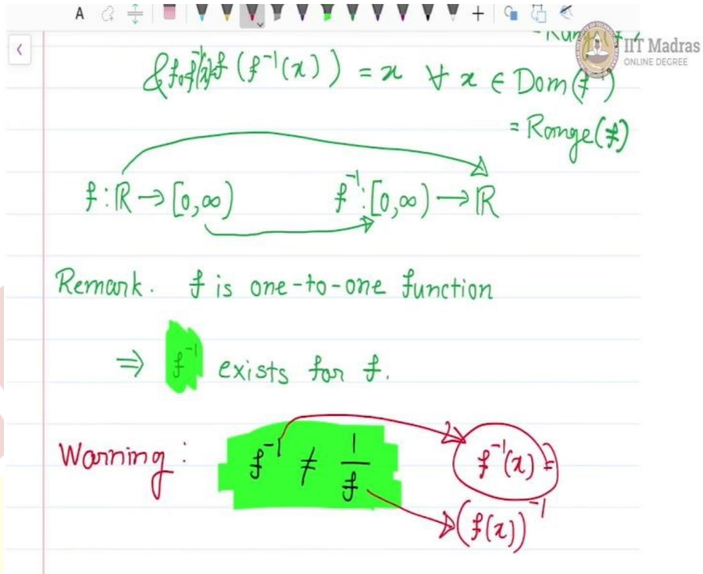

So, if the given function is one to one function then f^{-1}always exist for *f*. Now the notion may confuse you. So, let me give you one precise warning that the notion f^{-1}does not mean $\frac{1}{f}$. This is very important. Because you may quite often confuse f^{-1} with $\frac{1}{f}$. So, whenever we want to discuss in this course or in Mathematics, whenever we talk about $f^{-1}(x)$ it is simply means it an inverse function.

So, this is an inverse function and whenever you want to talk about the $\frac{1}{f}$. Then you should talk about $f(x) - f^{-1}$. So, this $f(x) - f^{-1} = \frac{1}{x}$ and this f^{-1} is actually has a meaning $f^{-1}(x)$ with this you always remember. Now if f is one to one function f^{-1} always exist for *f*. This you have to trust me. I cannot prove it right now with the current tools, so f^{-1} always exists.

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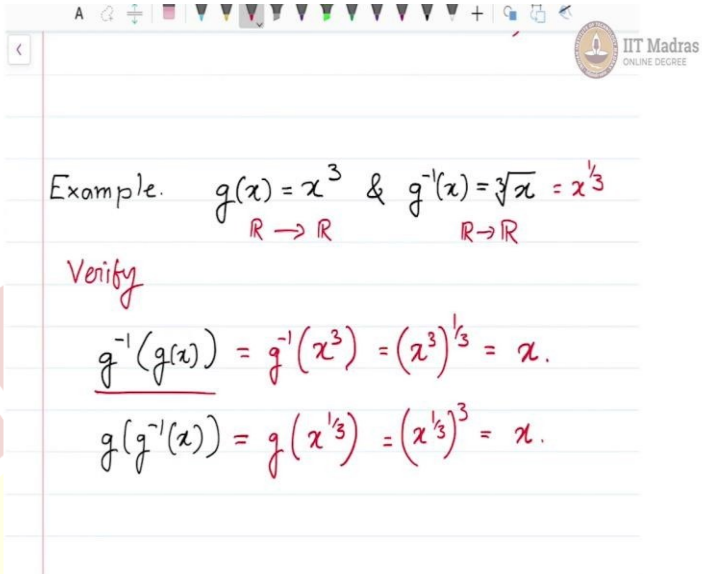

Let us take one example g(x) is equal to x cube and g^{-1} of x is $\sqrt[3]{x}$. This you can write as x raise to 1 by 3 as well. So that, this is simple to verify. So, now you want to verify that the given functions are actually inverses of each other. So, in this case let us first identify the domains, it is a real line to and range is real line. So, naturally for inverse also it's real line to real line. Now question about it. So, let us talk about g^{-1} $g(x)$.

Now if you recollect the definition of inverse function then naturally the inverse function is a function such that all this combinations, all this combinations should produce x $f^{-1}(f(x))$ or $f(f^{-1}(x))$ f. So, let us talk about $g^{-1}g(x)$). So, let us keep g^{-1}intact and put what is $g(x)$ which is x^3. Now this you substitute the function $g^{-1}$of x as $x^{1/3}$ then this becomes $(x^3)^{1/3}$. Then multiplication of indices a^{mn} applicable, so it will x. So, one way it is true.

Now the second way also you have to check. So, what you will do now, is you just write g within the box you write $g^{-1}(x)$ here, x raise to 1 by 3. And then simply put the function g so x raise to 1 by 3 the whole thing raise to 3 which again a raise to m n. So, this will also give you x domain and ranges we have already seen. So, whatever the conditions of that therefore g and $g^{-1}$are inverses of each other. So, $g^{-1}$is inverse of g .

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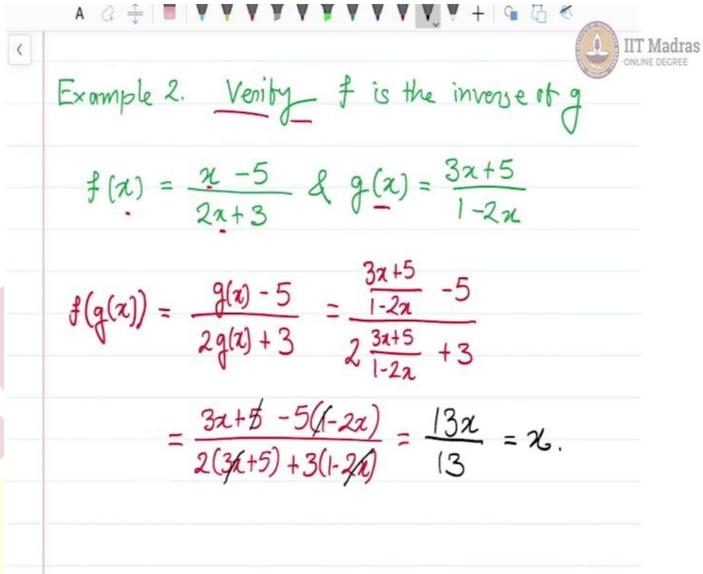

Let us take this example where we are suppose to verify, whether f and g are inverses of each other. So, let us try to verify, you can check the domain and co ranges of these functions. I will simply start with $f(g(x))$. So, if I start with $f\ (g(x))$ as per our notion what we will do? We will simply keep this $g\ x$ in place wherever f has an argument x. So, we will take this and we will put $g\ x$ wherever x is written there.

So, let us do that exercise that is, $\frac{g(x)-5}{2g(x)+3} = \frac{\frac{3x+5}{1-2x}-5}{2\times\frac{3x+5}{1-2x}+3} = \frac{3x+5-5(1-2x)}{2(3x+5)+3(1-2x)} = \frac{13x}{13} = x$. this is f o g(x). Now what is g(x)? g(x) is this so let us go ahead and substitute those values over, those functions in place of $g(x)$. So, it is $\frac{\frac{3x+5}{1-2x}-5}{2\times\frac{3x+5}{1-2x}+3}$. So, now it is a matter of your Algebra just simplify this. So, denominator both have $1-2x$ in common, so multiply the numerator by $1-2x$ and denominator by $1-2x$. So, that we will get rid of this. So, it will be $\frac{3x+5-5(1-2x)}{2(3x+5)+3(1-2x)}$.

So, what is a question, we have to verify that f is the inverse of g So, essentially I want to come up with a number with a function which is $x\ f$ o g$(x)= x$, this is my end goal just remember this. Now you can simply (multi) simplify this $3x$ + 5 - 5 will get rid of this + 5. Let me change the color over here. So, this will get this one will get rid of this then this is - $2x$ - $10x$ and - $10x$ will become + $10x$. Because of this - sign and then 3. So, I will simply get here $13x$.

Now you look at the denominator which is 2 into $3x$ that is, $6x$ then look at the corresponding term here - $6x$. So, this x, terms corresponding to x will vanish and 3 and $2 \times 5 = 10$. So, I will give get the denominator to be equal to 13 and that will give me x as my answer. So, f o g(x) is verified. Does this complete, will this complete our verification of whether f is the inverse of g? No, because I want to check whether g is also the inverse of f. Then only the verification will be complete.

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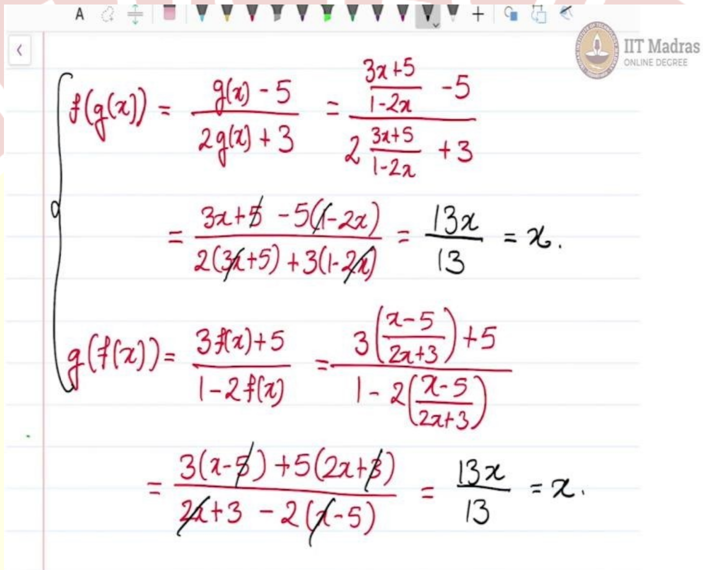

So, let us go ahead and do that, that is we will consider g f of x and that should give me x that is my end goal. So, now you look at what is a function g and put f x as it is everywhere. So, it is 3 times f x + 5 1 - 2 times f x. What is the next step take the functional form of f x and substitute it in the expression.

So, 3 into x - 5 upon $2x$ + 3 + 5 upon 1 - 2 times x - 5 upon $2x$ + 3 and then again the same logic applies multiply both sides by the $2x$ + 3 and then you will get 3 times x - 5 + 5 times $2x$ + 3 to be, upon 2 1 is there. So, $2x$ + 3 as it is - 2 into x - 5. Let us look at the simplified form let me change the color. So, $3x$ + $10x$ that will give me $13x$ here 3 into 5 - 15 + 5 into 3 + 15. So, this is taken care of vanished upon again the same logic applies $2x$ - $2x$ will vanish 2 into 5 will give me 10 and this 3 will give me 13.

Therefore, I got this domain and ranges you have already verified for yourself and therefore process is now complete because f of g(x) is x g o f of x is again x. So, we can verify that f is inverse of g as stated. So, this completes our discussion on inverse functions.

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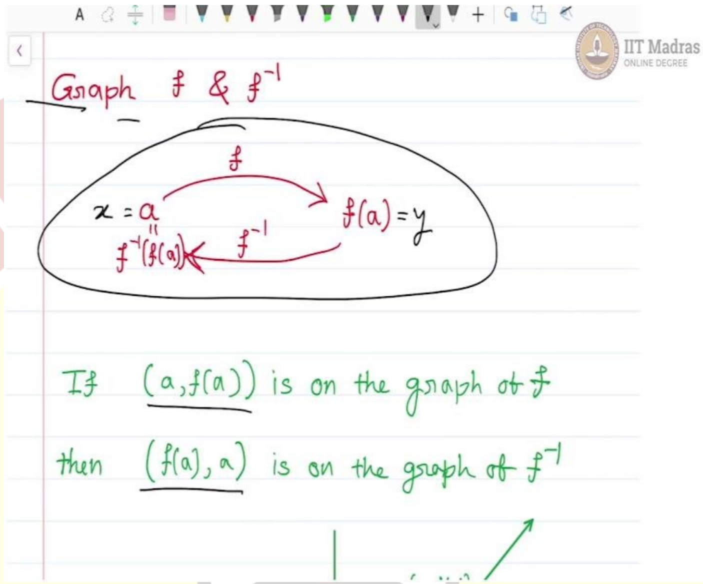

Now it is important to understand graphically what the inverse function is or how the graph of f and $f^{-1}$is changes. So, we already have a wage understanding of the graph of f and f^{-1}. Now let us look at it formally, so if I know something about f or the graph of f, then given a value of a. I am able to calculate f of a and a f of a is the payer which we call as graph of x, graph of f. You look at f^{-1}, what happens when you talk about f^{-1}, here f of a is actually on y a$x$is and a is on x a$x$is.

So, when you look at the inverse function the values on y a$x$is actually get convert into values of x a$x$is. And the values on x a$x$is will actually get converted into values on y a$x$is. So, this is the mapping that we have given. So, if you start with f of a which is y then you will talk about $f^{-1}(y)$of y and when you talk about $f^{-1}(y)$ you will actually get it to be equal to a. Because $y = f(a)$. and this is how the entire circle is complete. So, in particular if a and $f(a)$ is on the graph of f then f(a), a is the graph of f^{-1}. That is obvious.

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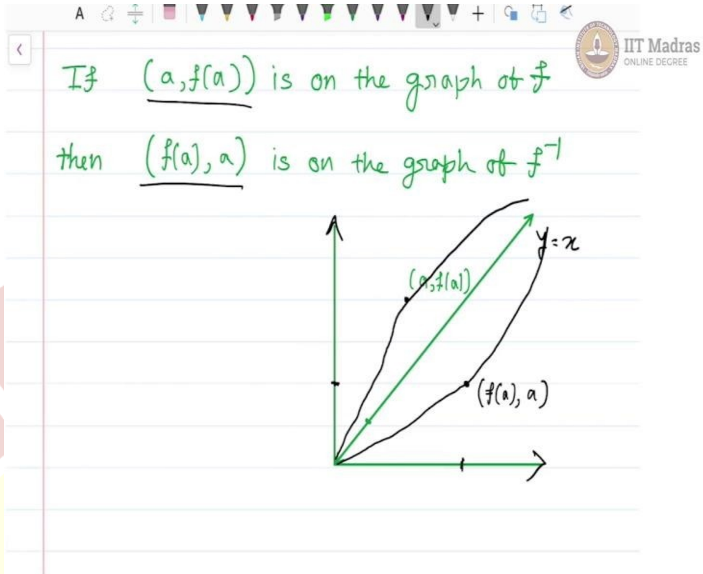

So, let us look at, let us imagine this is the graph. This is the graph of a straight line a, fa. So, you plot a line $y = x$ here. This is a line in $y = x$ and this is a point which is on the graph of $f(x)$. So, now you are saying that where will this point be, when I talk about f inverse. So, then we are also answering this question that wherever a was there it will be *f(a)* now. And wherever $f(a)$ was there now there will be a.

So, in this case you just take this distance and you plot it, you just take the distance on the y a$x$is and choose the distance on x a$x$is here and take the distance on x a$x$is for this point and put that distance over here. That means it will be somewhere here. And therefore, the point will be somewhere here and this point is actually $f(a)$. So, what we are actually doing when we are plotting the graph is actually we are reflecting our original function, in the original function is somewhat like this. Let us say, so it is somewhat like this.

Then what we are doing is we are actually reflecting it along y a$x$is and it will be very similar function. Which will look like this. So, this is how the graph of inverse function will look like it is actually a reflection along y is equal to x or reflection along a function y $f(x)$=x.

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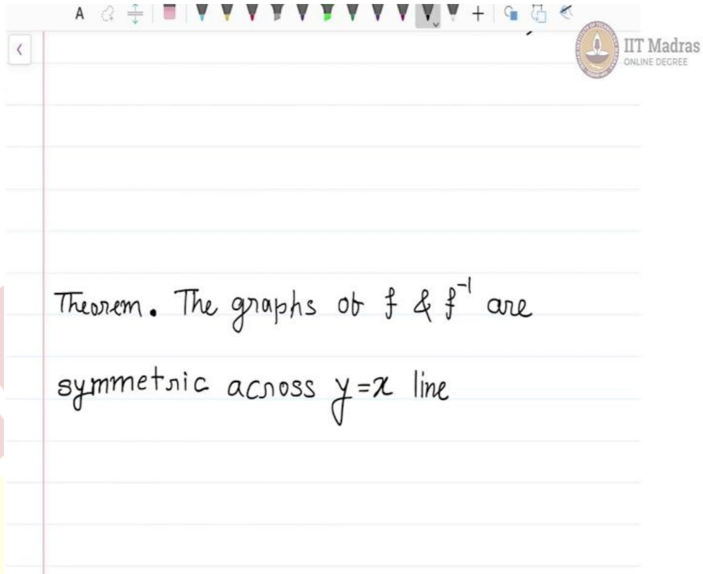

So, in particular the graph of f and f^{-1} are symmetric across the line y= x. This is what you have to remember all the time. If you want, you can prove the theorem but there is nothing it just a graphical prove that, if I want to compute this particular point and if I know that the inverse of this function e$x$ist, then you just take length on y and plot it across x and length of, length in x direction plot it across y direction. That is what I did a this is actually the prove of the theorem that the graph of f and f^{-1} are symmetric on y =x line. That completes our topic on inverse function. In the ne$x$t video we will deal with the inverse functions in a more restricted manner that is, inverse of this e$x$ponential functions.