IIT Madras

ONLINE DEGREE

Statistics for Data Science – 1
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Lecture No. 5.4
Permutations and Combinations – Permutations: Objects not distinct and Circular permutations

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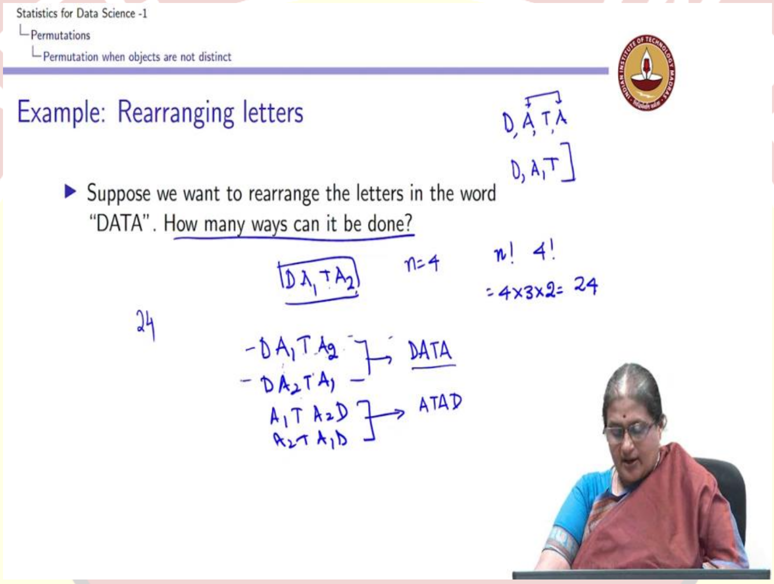

Now, let us look at the case when the objects are not distinct. So, when we looked at the case when the objects were distinct, we looked at different ways of arranging A, B, C. Now, these A, B, C could be people, it could be objects, but they are distinct. But now suppose I have a case where they are not distinct. So, let us look at an example where I am looking at how do I rearrange the letters in the word "DATA". Now, if you look at the word DATA and just look at this as alphabets, I see that D, A and T are the 3 distinct objects or distinct alphabets, because A appears twice.

Now, I am asking how many ways can it be done. To understand this, let us do the following, I will write the DATA, I will write as A_1 and A_2. Basically, A_1 and A_2 are the same alphabet, but I know how to find out for this, I already know that if I have DA_1TA_2, my $n = 4$, I am taking all at a time so this is going to be $n!$, this is what we have already seen because it is taking all at a time is $n!$. So, I can have $n!$ which is 4! which is again $4 \times 3 \times 2$ which is 24 ways of arranging DA_1TA_2

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Let us look at these two arrangements DA_1TA_2 and DA_2TA_1

. Now these two arrangements are different when I consider A_1 and A_2as distinct objects. But we know that A_1 or $A_2$are essentially the same object. So, these two together will just give me one arrangement which is DATA, whether the first A comes before the second A or the second A comes before this first A is irrelevant. So, DA_1TA_2 and DA_2TA_1 give me the same arrangement which is DATA. Similarly, A_1TA_2D and A_2DA_1D these two together will again give me the same arrangement which is ATAD.

So, for every two arrangements which I had in my original, original which is these 24 arrangements, I can see that every 2 arrangements will give me 1 arrangement that is, whenever I have A_1 and A_2, A_2 and A_1, that which can I have 2! ways because when I have 2 elements, I have 2! ways. So, I can erase that for every 2 I have 1 arrangement distinct because these two are the same. In other words, out of these 24 arrangements, I can eliminate half of them so I get 12 arrangements when I have one alphabet which is appearing twice.

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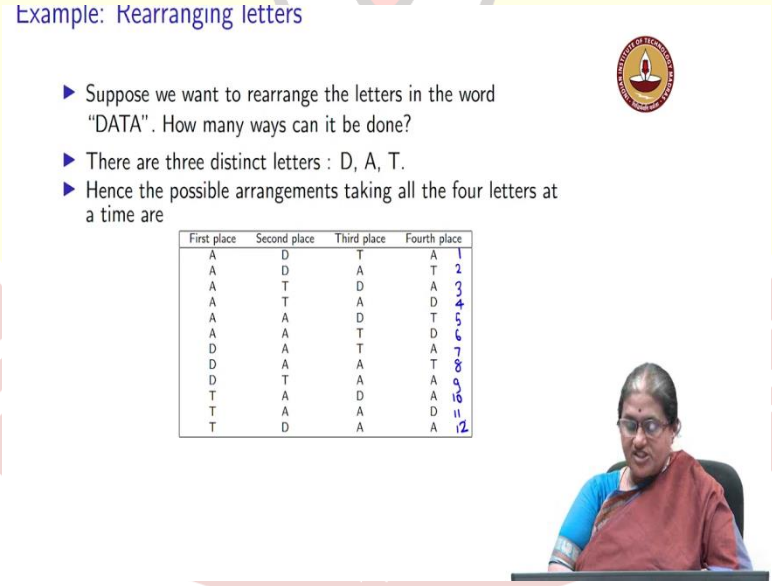

In other words, if I look at the 3 distinct alphabets, I have the possible arrangements that are possible are $1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11$ and 12 and how we have arrived at these 12 arrangements is what we had just discussed.

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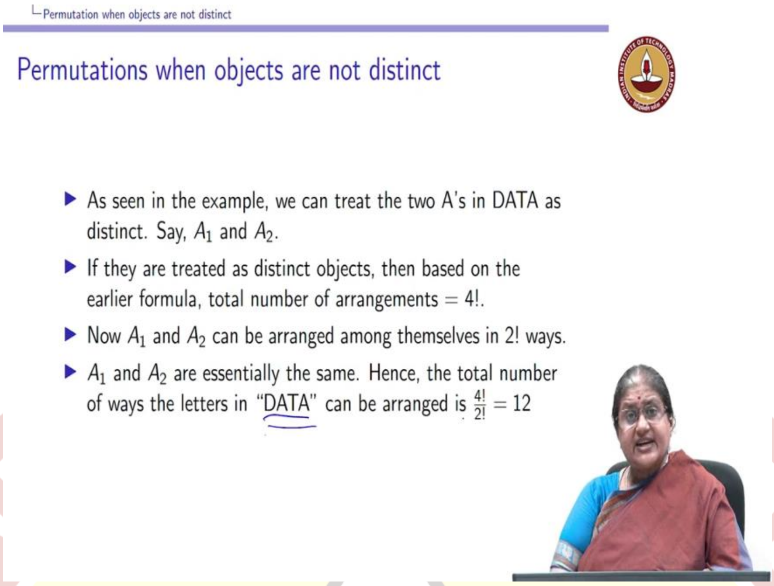

And the logic is I treat both the A's as A_1 and A_2. I looked them as distinct objects in which case I know I have 4! ways to do it, but I know effectively A_1 and A_2 are the same object hence I divide the total number of arrangements which is 4! by 2! and I get that the total number of ways the letters in DATA can be arranged is $\frac{4!}{2!}$ which is 12 ways.

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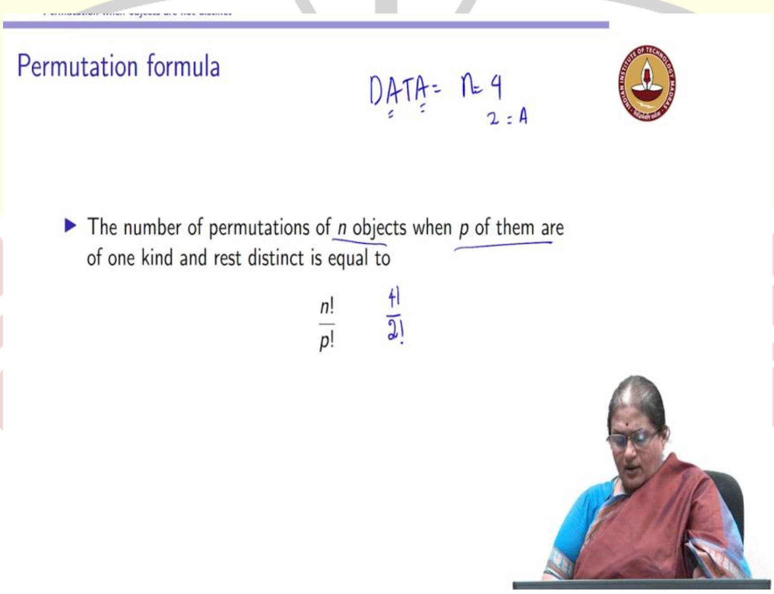

I can generalise this formula to say that the number of permutations of n objects when p of them are of one kind, so in DATA, my n was 4 so I can generalise this into what I refer to as the permutation formula. The number of permutations of n objects when p of them are of one kind and rest distinct. So, what are the p of them here? $n = 4$, because I have 4 objects but I have p of them, I have 2 of them A which are of one kind and other at distinct equal to $\frac{4!}{2!}$ and I get the answer to be 12.

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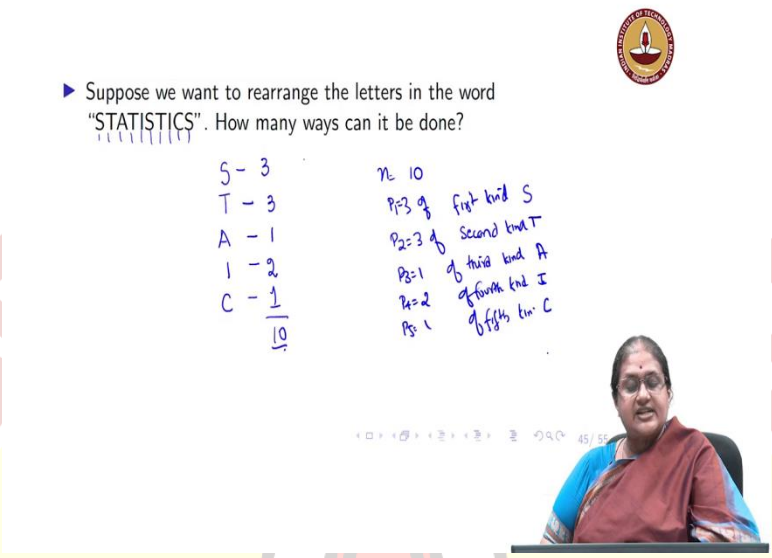

Now, let us extend this idea. Let us look at the same idea and say that now suppose I want to rearrange the letters in the word STATISTICS. How many ways can it be done? Now, if you look at this STATISTICS, I look at now the distinct alphabets in this word STATISTICS. Now if you look at the distinct alphabets, I have a S, I have a T, I have a A, I have an I and I have a C. The total number of alphabets in this word is $1, 2, 3, 4, 5, 6, 7, 8, 9, 10$. Now, of $n = 10$, I have 3 S, I have 3 T's. I have a 1 A, I have a 2 I and I have a 1 C. So, this adds up to 10.

In other words, of this $n = 10$ alphabets, I have say 3, I have 3 S, so I have p_1 of one kind, the first kind which is S. So, $p_1 = 3$ of the first kind, $p_2 = 3$ of second kind. What is my first kind? My first kind is S, my second kind if T. I have $p_3 = 1$ of third kind, my third kind is a 1, $p_4 = 2$ of fourth kind which is an I, this is A, this is I and $p_5 = 1$ of fifth kind which is C. So, how many possible arrangements can be done in this case?

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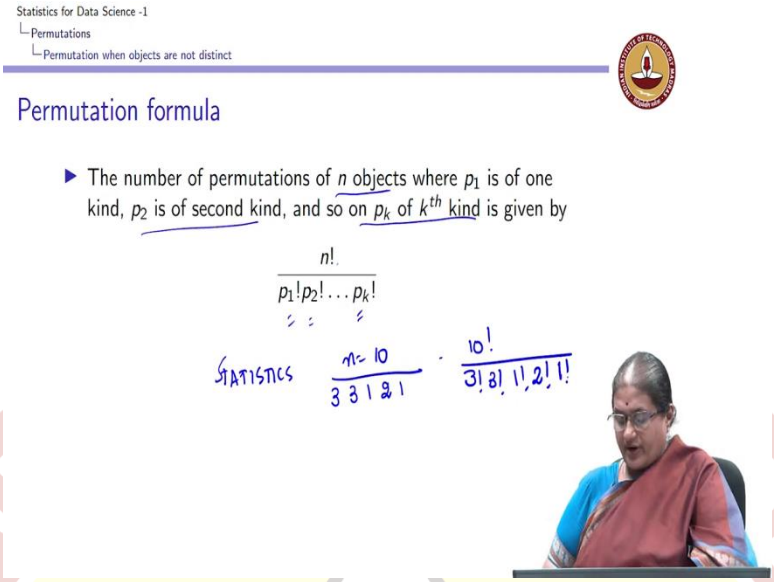

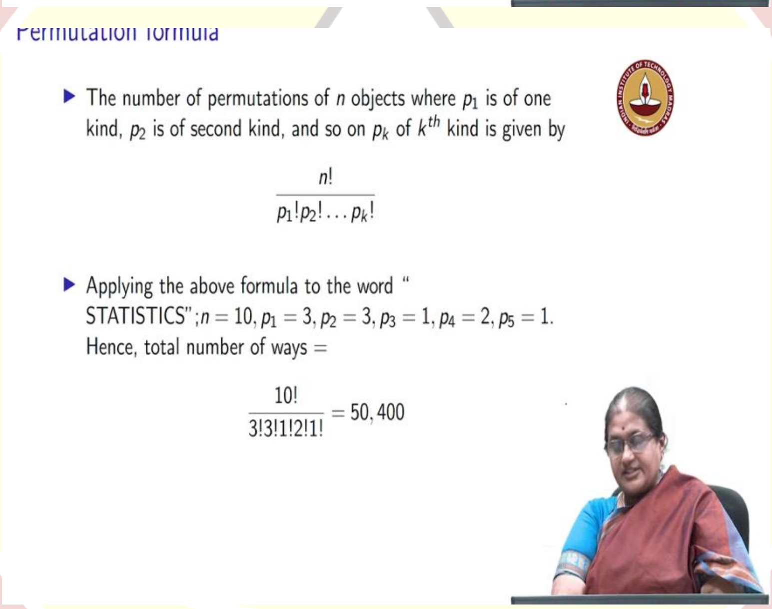

The formula that governs this of a thing which is just an extension of what we have seen before is that the number of permutations when of n objects, when p_1 is of one kind, p_2 of second kind and p_k of third kth kind is $\frac{n!}{p_1!p_2!\dots p_3!}$

In my STATISTICS example, we have already seen my $n = 10, p_1 = 3$, so I have, $p_2 = 3$, $p_3 = 1$, then I had a 2, $p_5 = 1$, so my total number of permutations is $3!,\ 3!,\ 1!,\ 2!$and $1!$ is the total number of things in this STATISTICS thing and you can see that, that is equal to, that value is equal to 50400.

So, if you apply the formula, the total number of arrangements that are possible out of this word STATISTICS is 50400. So, we have applied this when objects are not distinct formula to the STATISTICS example.

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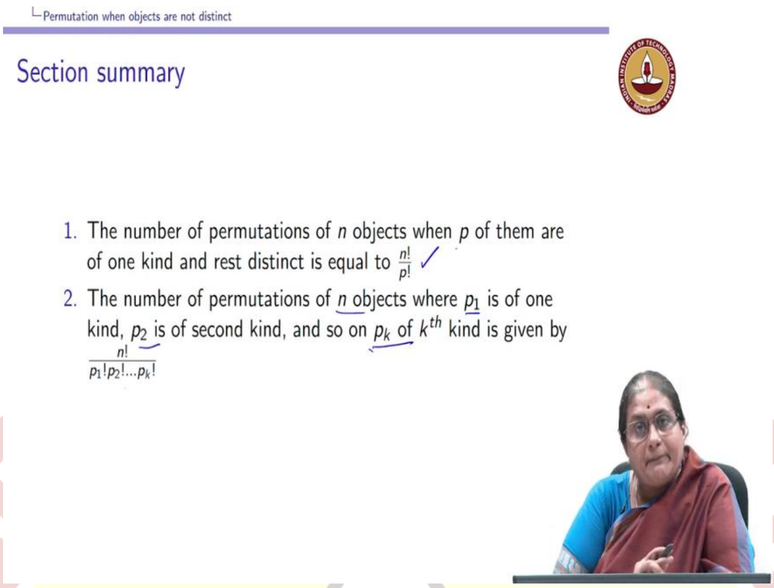

So, what we have learned so far is when I am looking at linear ordering that is by linear ordering, I am arranging objects or things or people in a row, then the number of permutations of n objects when p of them are of one kind and the rest distinct = $\frac{n!}{p!}$ The number of objects when p_1 is of one kind, p_2 of second kind, p_k of kth kind is $\frac{n!}{p_1!p_2!\ldots p_k!}$

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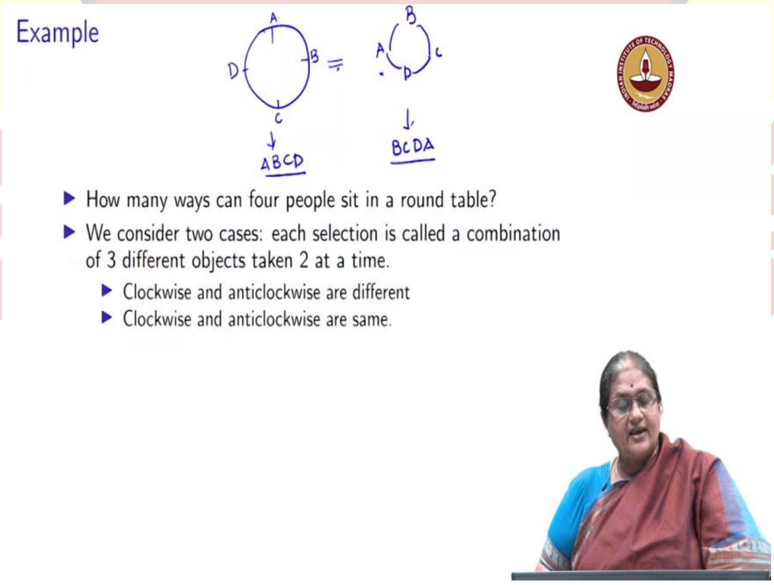

So, so far what we have seen is how do we count the number of arrangements when I am arranging n distinct or n objects, it could be distinct or not distinct objects in a linear row or in a linear fashion. But sometimes we might we might be interested in knowing about the number of ways we can arrange people in a nonlinear and the most common thing is if I have to arrange people in a circle. Then I am interested in knowing how many ways can I do it?

For example, suppose I have a round table, I am interested in knowing how can I seat 4 people in a round table? So, I have a round table, I want to know how can I seat 4 people. Again, let me assume the 4 people are A, B, C, and D, they are distinct. I want to know how can they sit in this table. Now suppose A occupies this position, I can have one arrangement where B occupies this position, and D occupies this position.

Now, if you look at this arrangement, you can see that B is next to A and next to C. So, B is between A and C. Now, this is same as if B is next to C and A and D, so this arrangement is same as this arrangement. The linear equivalence of this arrangement is A, B, C, D, whereas the linear equivalence of this is B, C, D, A.

A, B, C, D is different from B, C, D, A when we are talking about linear arrangements. But when you are looking at a circular arrangement, this arrangement is the same as this arrangement. So, you can see that it is circular arrangements need to be counted in a different way and that is what we are going to learn now. So, let us look at formulising this notion.

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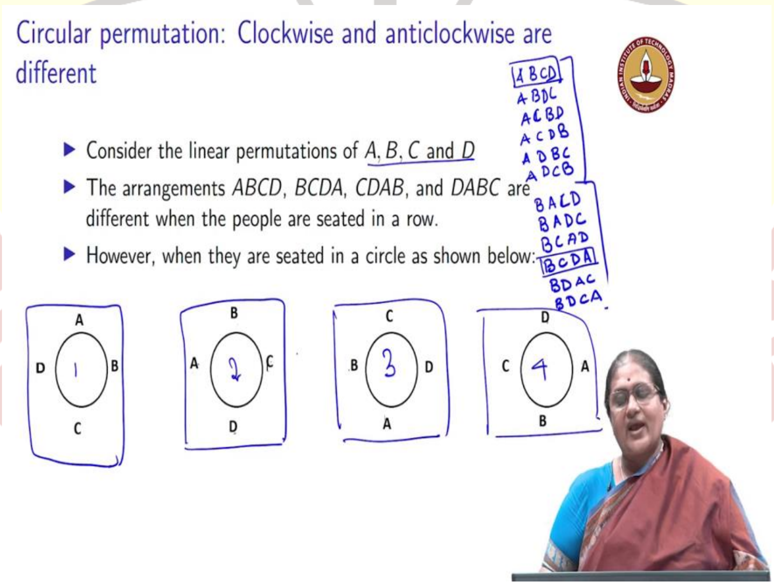

So, I have, first I will consider two cases again when clockwise and anticlockwise are different and when they are the same. So, I said that when you are looking at A, B, C, D, I know the possible suppose I fix A, I know it is a B, C, D. I know A, B, D, C; A, C, B, D; A, C, D, B; A, D, B, C and A, D, C, B. This is already something which we have seen. This has 6 possible linear arrangements if I fix A in one slot.

Now, if I fix B in the first slot, let me look at what are the possible arrangements here if I fix B, I can have a A, C, D again I am not allowing repetitions. B, I can have a B, A, D, C with B, C, A, D; B, C, D, A; B, D, A, C and a B, D, C, A. Now each one of these arrangements were distinct when I looked at a linear ordering. Now, let us look at these two A, B, C, D and let me look at this following B, C, D, A. Now, A, B, D, C corresponds to this circular arrangement. B, C, D, A corresponds to this arrangement.

Similarly, C, D, A, B corresponds to this arrangement and D, A, B, C correspond to this arrangement and we can actually see that these 4 circular arrangements 1, 2, 3 and 4 are the same arrangements.

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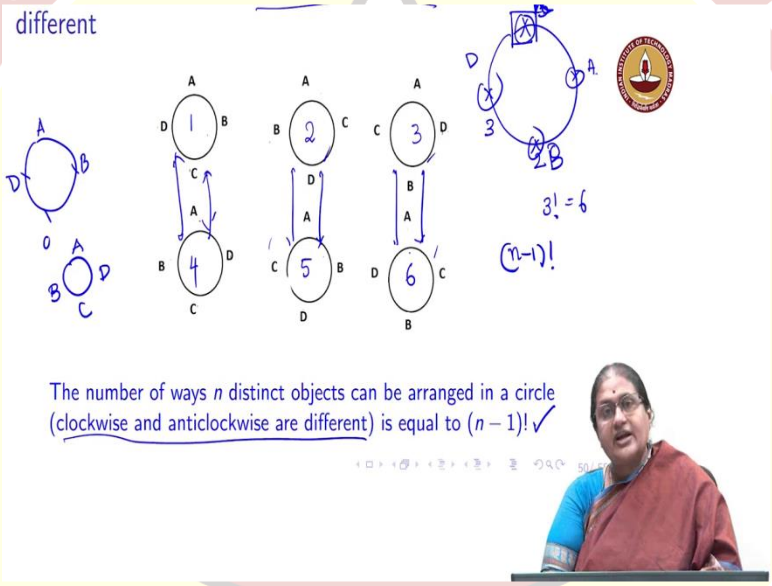

In other words, when we want to compute the circular arrangements, one way to look at it is A, B, D, C, these are the same so the way I can look at it is I fix, so I have a circle. In, within this circle if I fix the first or one position in the circle, the other 3 positions that are available if I fix this position in the circle, the other 3 positions that are available are position 1, 2 and 3 and this can be filled with the 3 available alphabets which are B, C, D and that can be done in 3! ways which is equal to 6.

So, if you look at these arrangements A, B, C, D is different from A, C, D, B is different from A, D, B, C which is different from A, D, C, B, which is different from A, B, D, C which is different from A, C, D, B. So, I have 6 distinct arrangements of people around a table. So, I fixed A. Now, if I fix B, I can do the same thing. So, B, A, C, D is one of the arrangements. But if I look at B, A, C, D, that is the same as A, C, D, B. It is the same arrangement.

So, if I fix C, I have C, A, B, D as one of the arrangements. If I fix C, and look at C, A, B, D, that is the same as A, B, D, C. Hence, whatever I am fixing here is irrelevant but these are the possible distinct permutations. But I am allowing clockwise and anticlockwise are different, so my A, B, C, D is different from A, B, C, D. These are different. So, I have a A, B, D, C which is not same as A, B, D, C, the clockwise. So, the mirror images are different. If you look at it, this is a mirror image of this and this is a mirror image of this. So, they are different.

So, the number of possible circular permutations of n objects if the clockwise and anticlockwise are different is $(n-1)!$ and the reason behind this is I fix one of the objects, the other $n-1$ objects can be arranged among themselves in $n-1$ ways. And this is the same for any of these objects. So, there are n $(n-1)!$ ways of arranging my circular permutations. Now, this is when clockwise and anticlockwise are different.

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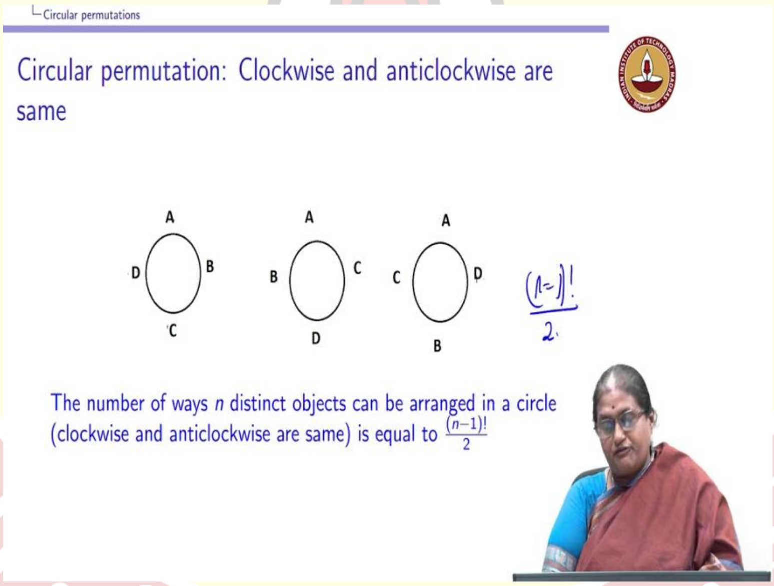

Now, if they are the same, these two are the same because the clockwise and anticlockwise are the same, these two are again the same and these two are again the same which gives me the total number of circular permutations when the clockwise and anticlockwise are same to be nothing but $\frac{(n-1)!}{2}$. So, this is how we compute the number of permutations when the order or when the objects are arranged in a circular fashion.

So, now we move forward. So, we had looked so far at linear transformations, when linear transformations, I mean, so we have looked so far at linear permutations, by linear permutations I mean that I am arranging n objects in a row, I am also looking at choosing r objects from these n objects and arranging them in a linear fashion or in a row.

First we looked at what would happen when the objects were distinct, where repetitions were allowed, repetitions are not allowed, we also looked when the objects were not distinct and then we looked at when I am placing the n objects, distinct objects in a circle, here whether clockwise and anticlockwise are same or different is what is considered.

Very often we would want to actually do some simplifications using the expressions we have gathered so far or we have described so far. Very often we would be looking at using the expressions that we have described so far. What are the expressions?

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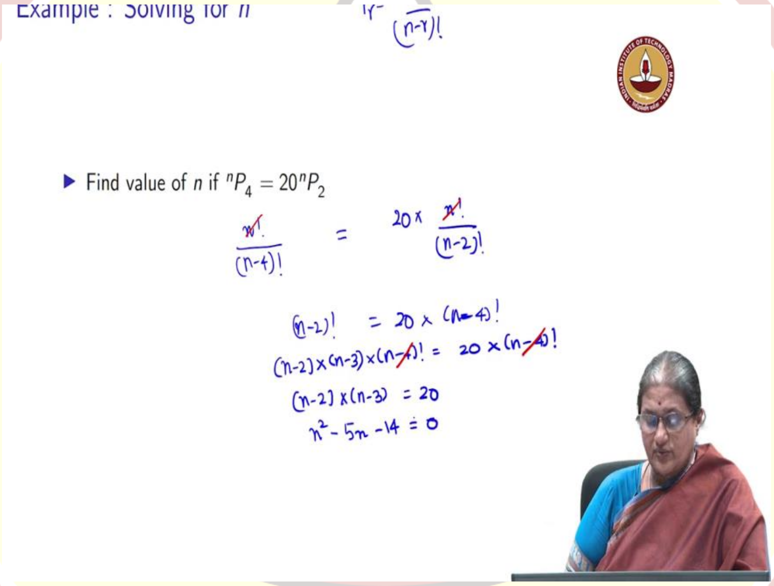

We looked at ^nP_r. ^nP_r was $\frac{n!}{(n-r)!}$. So, when we are simplifying our notation we might have to solve for the n or r. Let us look at a few examples where we can apply the formula to obtain n or r given either of them that is, obtain n when we are given a expression concerning r or obtain r when we are given a expression concerning n. Let us look at a few examples.

For example, in the example we look at it, I want to find the value of n if $^nP_4$ is equal to $20 \times {}^nP_2$. Now the left hand side I have $^nP_4$ is $\frac{n!}{(n-4)!}$. This is from my definition. My right hand side I have $20 \times \frac{n!}{(n-2)!}$ and what is given to me is these two are equal and I have to solve for n. Now, if you are cross multiplying it, I can see that I can cancel off $n!$ here and this leaves me with $(n-2)! = 20 \times (n-4)!$.

Now, again applying what we learned from factorial, I know $(n-2)!$ is $(n-2) \times (n-3) \times (n-4)!$, again this comes from our discussion from factorials, this is equal to $20 \times (n-4)!$. I can again cancel out these two $(n-4)!$ and now I have what I remain is I have $(n-2) \times (n-3) = 20$. Now, this is what I have and I can quickly see that this is I can simplify this to get a quadratic which is $n^2 - 5n - 14$ which is equal to 0.

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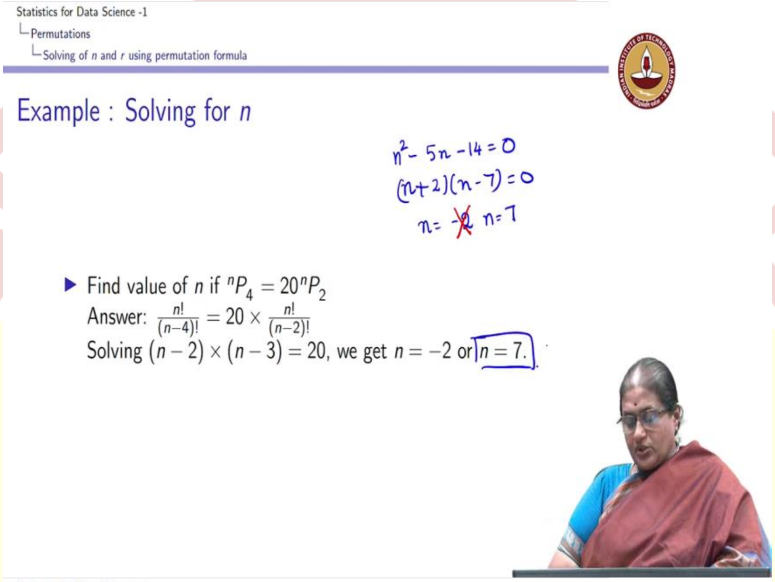

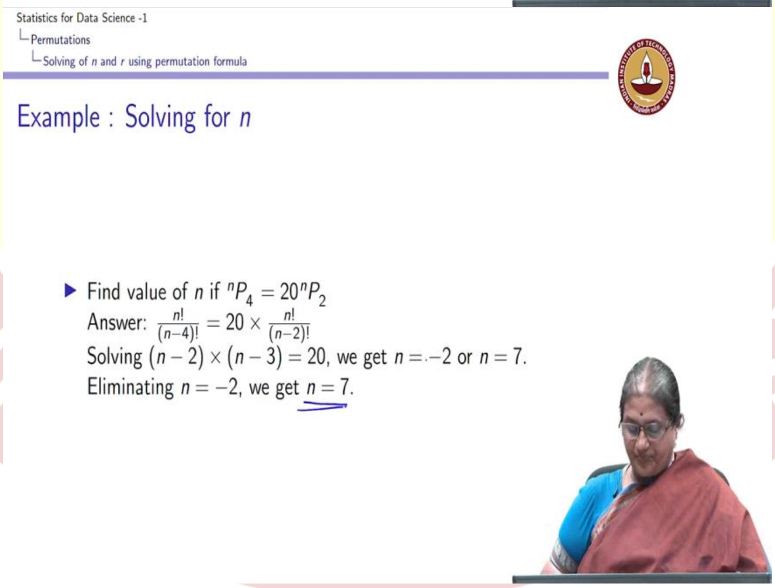

Now, if I solve this quadratic equation, I can so I have the following thing if I solve this quadratic equation, I get that n so I have $n^2 - 5n - 14 = 0$, n square which I can write as $n-$, $n+2$ *and* $n-7=0$ factoring which gives me a $n = -2$ or $n = 7$ and taking a negative value has to be cancelled out which leaves me with only the choice of $n = 7$. So, the solution to this problem is $n = 7$ because I am eliminating $n = -2$.

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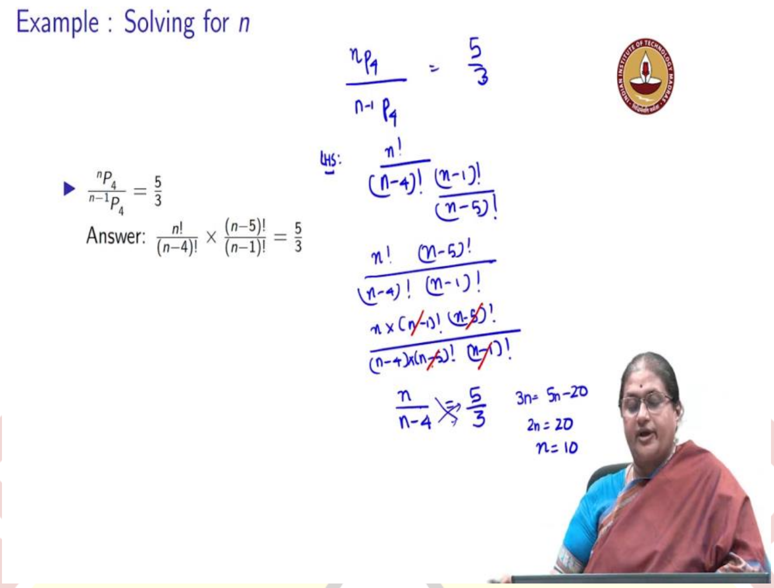

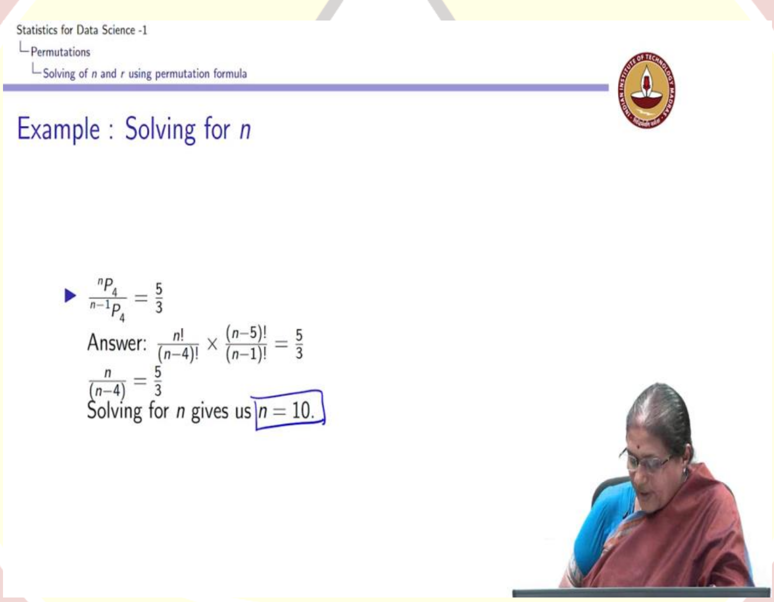

Now, let us look at another example. I have $n -, -= \frac{5}{3}$. Again I solve for n. Now, again my left hand side $^{n}P_4$, the numerator is $\frac{n!}{(n-4)!}$. The denominator is $\frac{(n-1)!}{(n-5)!}$, I can simplify this to write it as $\frac{n!}{(n-4)!} \times \frac{(n-1)!}{(n-5)!}$. Now, I know $n!$ is $n \times (n-1)!$, this is what we know.

Similarly, $(n-5)!$, $(n-4)!$ is $(n-4) \times (n-5)!$. I retain $(n-1)!$. Again, I can cancel out $(n-1)!$ and $(n-4)!$ and this leaves me with the following, this leaves me with in the numerator n, the denominator $(n-4)$ equal to $\frac{5}{3}$. Cross multiplying I get, $3n = 5n - 20$ which further simplifies to $2n = 20$ and I can solve for $n = 10$.

So, you can see that I just solved from the first principles that $n = 10$ by expanding or applying whatever is the formula we have derived. Now, this kind of simplifying expressions would be very useful again when we get into probability and counting when we are doing the probability module. So, in the earlier case we solved for n given a particular expression.

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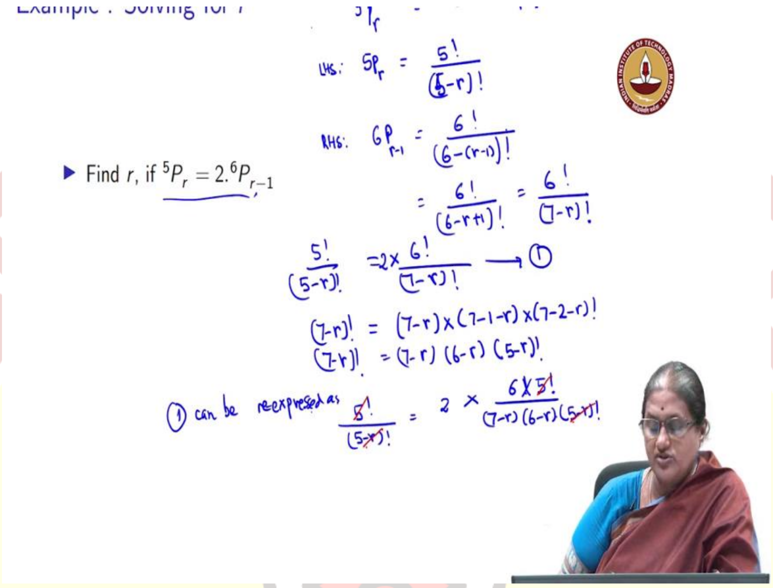

The next thing which we are interested in doing is how do we solve for r? So, in the earlier case I was solving for n, n was my unknown. Now, for example, I am given ${}^5P_r = 2 \times {}^6P_{r-1}$. Now, let us look at the left hand side. It is ${}^5P_r$ again applying the formula ${}^5P_r$ is $\frac{5!}{(5-r)!}$ this is what I have if I apply the formula. Let us look at the right hand side.

In the right hand side I have ${}^6P_{r-1}$. Again this is $\frac{6!}{(6-(r-1))!}$. We look at the denominator I can further simplify this denominator as $(6 - r + 1)!$. So, I have this as $\frac{6!}{(7-r)!}$. So, what is given to us is I $\frac{5!}{(5-r)!} = 2 \times \frac{6!}{(7-r)!}$ this that is what is given to us.

Further notice that $(7 - r)!$ can be re expressed as $(7 - r)(7 - 1 - r)(7 - 2 - r)!$ This is what we have already seen which is as $(7 - r)(6 - r)(5 - r)!$. So, I can re express as $(7 - r)!$ in this way. Now, once we do that, this expression which I have labelled as 1 can be re expressed as 1 can be re expressed as the following.

How can I re express 1? I can write it as 5!, but $(5 - r)!$ is $2 \times \frac{6!}{(7- r)(6- r)(5- r)!}$. Now, I can also write 6! as $6 \times 5!$, this is something which we have already seen. So, again we go back and we can see that I can cancel out $5!$ from that expression, $(5-r)!$ also from the expressions which gives us the following.

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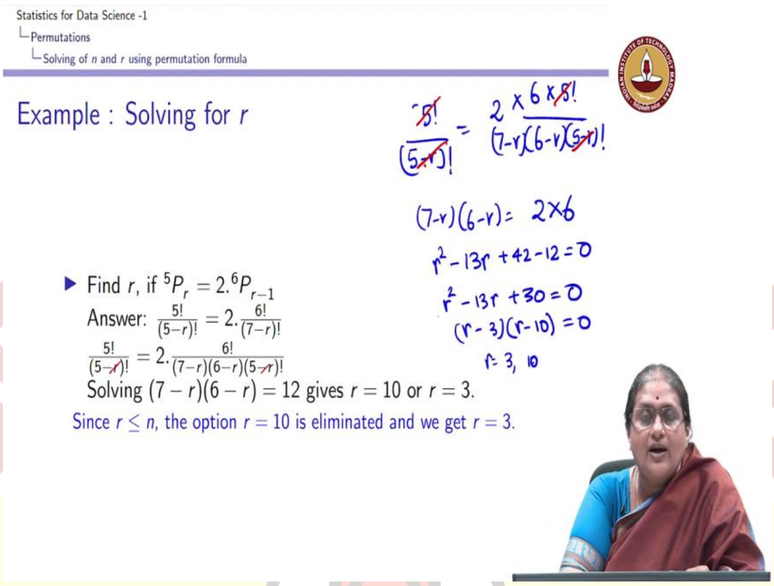

That when I cancel out both of them, it gives me this following. It gives me the expression that because I am cancelling out $(5-r)!$ and I can write $5, 6$. I have $7-$ so what I have earlier was I had $5!$ so earlier what I had earlier was $\frac{5!}{(5-r)!} = 2 \times \frac{6\times5!}{(7-r)(6-r)(5-r)!}$, I cancel out 5 factorial and $(5-r)!$i, and what I get is $(7-r)(6-r)$ is 2×6 which is 12.

I further I simplify this I get it as an r^2, this is an r^2, I get the I can write this as $r^2 - 13r + 42 - 12$ which is $r^2 - 13r + 30 = 0$ which I can see as $(r-3)(r-10) = 0$. I can factor this as $(r-3)(r-10) = 0$ which gives me r is either 3 or 10. So, this is how we can solve for r. Whenever you know what is an expression but you have an unknown you can simplify the and solve for either n or r by applying the formula.

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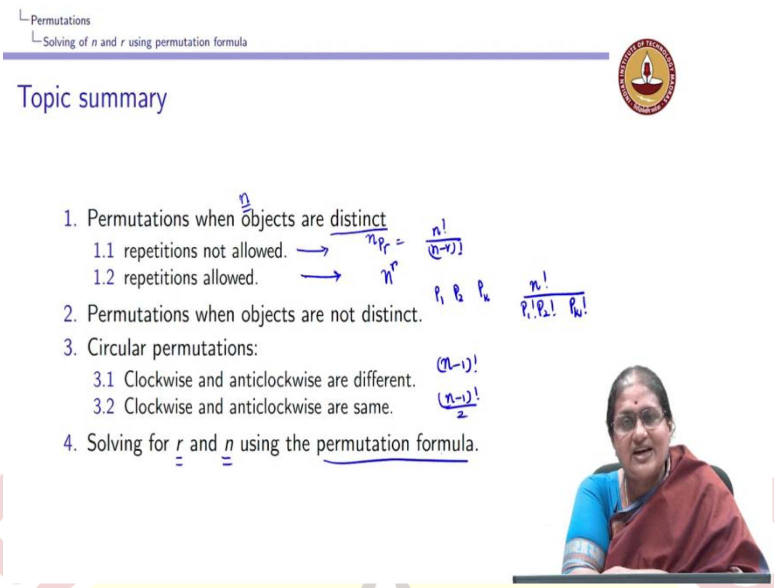

So, when we look at the permutations what a summary of what we have seen in the permutation segment, first we started with n objects, these n objects were considered to be distinct. We looked at how many ways we can choose r objects from these n objects. When repetitions are not allowed we saw it was ^nP_r which was $\frac{n!}{(n-r)!}$. When repetitions are allowed, we saw it was n^r.

When objects are not distinct that is I had p_1 of one kind, p_2 of another kind and p_k of a kth kind, it was $\frac{n!}{p_1!p_2!\ldots p_k!}$. Now, when we looked at circular permutation when anticlockwise and clockwise were different, we saw there are $(n-1)!$ ways to arrange these objects. When clockwise and anticlockwise were the same, it was $\frac{(n-1)!}{2}$. And finally, we saw how you can solve for r and n using the permutation formula when we are given an expression and we are asked to solve for r and n.

So, in permutation we actually looked that how do you arrange n objects or r objects out of n objects arrangement. A order was important. AB was different from BA. But many a time we might be just interested in how can we choose 2 objects out of 3 objects or how can we choose 3 people from a group of 10 people. In situations like this the order is not important. AB is same as BA. For example, I have 3 people and I just have to choose 2 people from 3 people, the order is insignificant.

Hence, we are going to look at what is called combinations, the number of ways you can choose 2 people or you can select 2 people or you can select r people in general from n

people gives us to what we referred to as combinations. That would be the next topic we are going to discuss.