IIT Madras

ONLINE DEGREE

Mathematics for Data Science 2
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Examples: Computing Maxima and Minima

Hello and welcome to the maths 2 component of the online BSc program on Data Science and Programming. In this video we are going to continue from our previous video where we discussed critical points, computing maxima and minima and we will study examples of computing maxima and minima.

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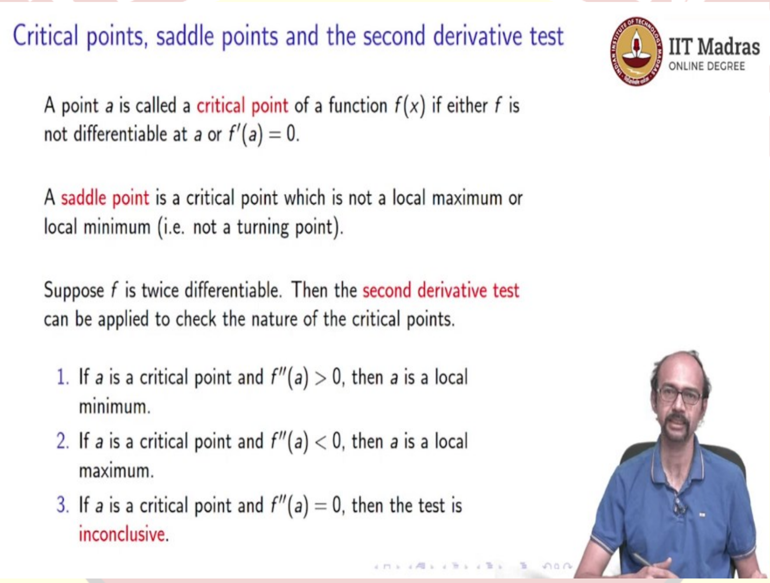

So, let us quickly recall what we studied in previous video. So, we have studied critical point, saddle points and the second derivative test. So, point a is called a critical point of a function f of x if either f is not differentiable at a or $f'(a)$ is 0. A saddle point is a critical point which is not a local maximum or local minimum which means it is not a turning point. So, remember turning points are local maxima which means a local maximum or local minimum and a saddle point is one where the derivative is 0, but it is not a turning point or it may not be differential.

So, suppose f is twice differentiable, then we had something called the second derivative test which we can apply to check the nature of the critical points. So, if a is a critical point and $f''(a)$ is positive, then a is a local minimum. If is a critical point and $f''(a)$ is negative, then a is a local maximum and if a is acritical point and $f''(a)$ is 0, then the test is inconclusive.

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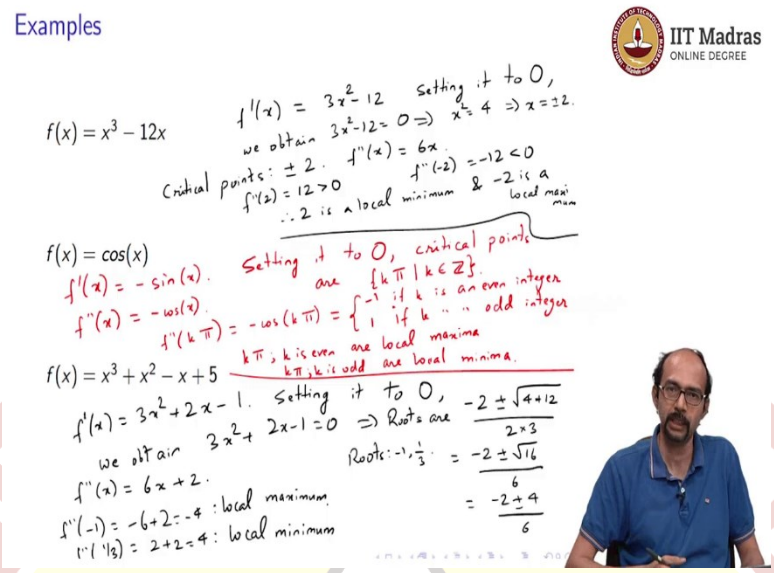

So, let us look some examples. So, let us apply to these examples to see how to use the test so looks at x cube-12x. So, of course, this is a nice function, so you compute $f'(x)$. So, $f''(x)$ is 3x squared-12. Set it to 0. So, setting it to 0, we obtain 3x square-12 is 0 which means x square is 4 which means x is+or-2. So, these are your critical points. So, critical points are+or-2. And now let us check what happens to the double derivative, so what is $f''(x)$?

So, $f''(x)$ is 6x. So, $f''(2)$ is 12 which is greater than 0 and $f''(-2)$ is-12 which is strictly less than 0. So, what does that mean? That means so therefore, 2 is a local minimum and-2 is a local maximum. So, that is what happens for this function $x^3 - 12x$.

Let us look at the function cosine of x. So, for cosine of x, well, we repeat the same thing. So, $f(x)$ is-sin of x what or the 0s. So, of course, sin of x we know what are the 0s, so they are multiples of π. So, equating to, setting it to 0, the critical points are all points of the form k times$\pi$where k is in the integers. So, now let us check out of these which ones are local maxima, which ones are local minima and which ones are saddle points.

So, for that let us compute what is $f''(x)$. So, $f''(x)$ is$-\cos x$and let us check this for k times pi. So, $f''(-\cos k\pi)$ and now this depends on the parity of k. So, if k is, so this is, so cosine of 0, $\cos 2\pi$, these are all 1. So,-of that is going to be-1. So, if is an even integer or the other way of saying this is if k is congruent to 0 mod 2 and it is 1 if k is an odd integer and what does that say? That says that the local, so k times$\pi$where k is even are local maxima and k times$\pi$where k is odd are local minima.

Let us do this last example. So, here, well, let us differentiate, so you get $f'(x)$ is $3x^2 + 2x - 1$. So, this looks somewhat complicated, but it is a quadratic; so if I set it to 0, I know what the roots are. Setting it to 0, we obtain so $3x^2 + 2x - 1$ is 0 that means roots are, so-b so-2+or-root b square, so 4-4 times a times c, so 4 times 3 times-1, so that is-12. So, 4+12 and then divide it by 2 times a, so 2 times 3, that is-2+minus root 16 divided by 6 so that is-2+minus 4 divided by 6.

So, the roots are, so $-2 - 4$ by 6 is one root, that is-1 and-2+4 that is 2 by 6, so one third. So, these are the two roots. Let us check what happens with the function at these roots. So, these are your critical points;-1 and one third. So, to see if they are local maxima or local minima, let us set this to, let us compute what is $f''(x)$.

So, that $6x + 2$, so evaluating this at-1, we get$-6 + 2\ is - 4$, so this is a local maximum, local maximum and if you evaluate it at one third, we get 6 times one third is 2+2 which is 4 so this is the local minimum. So, I hope the method is clear. So, this way we can check what are the local minima and maxima. So, it should be pointed out that sometimes the double derivative test does not work that is what we saw. So, if it is inconclusive, you have to really work harder to find out which ones are local maxima and minima. So, before we go to closed intervals and so on, let us ask why does it work?

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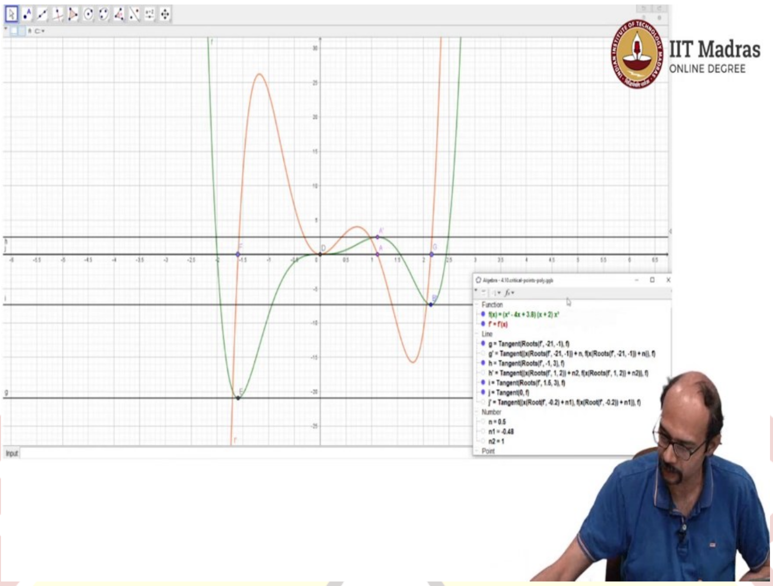

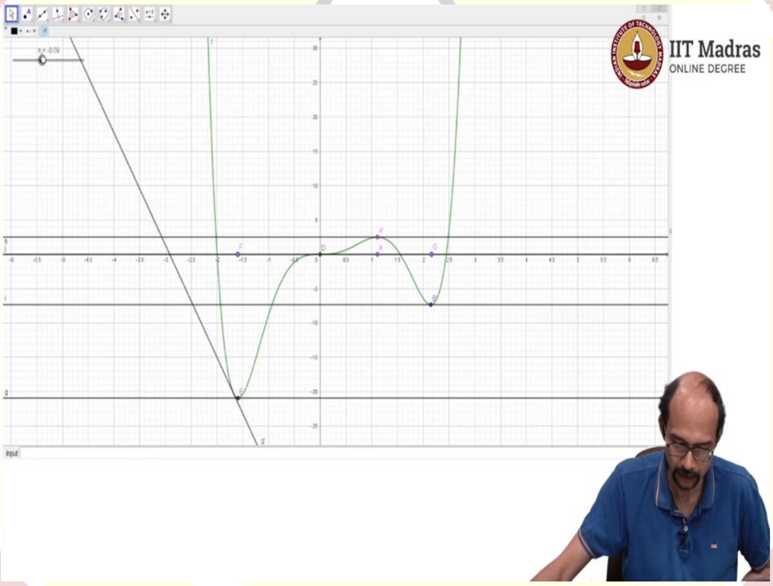

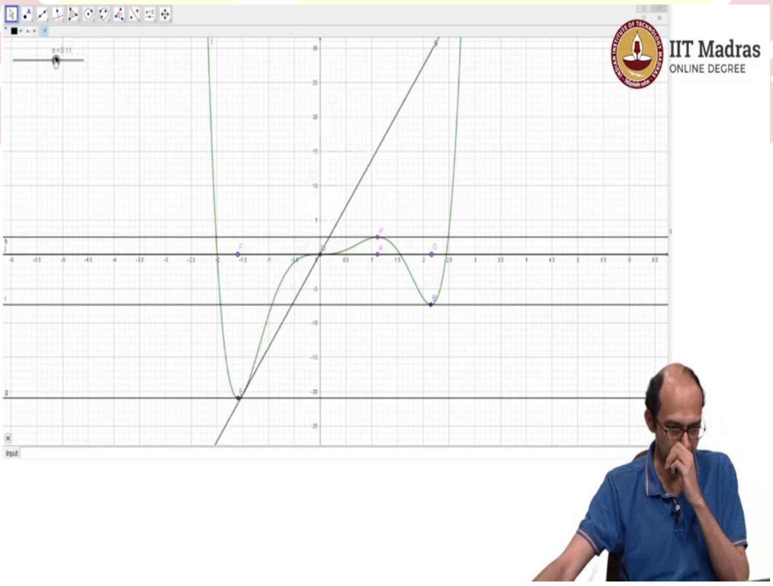

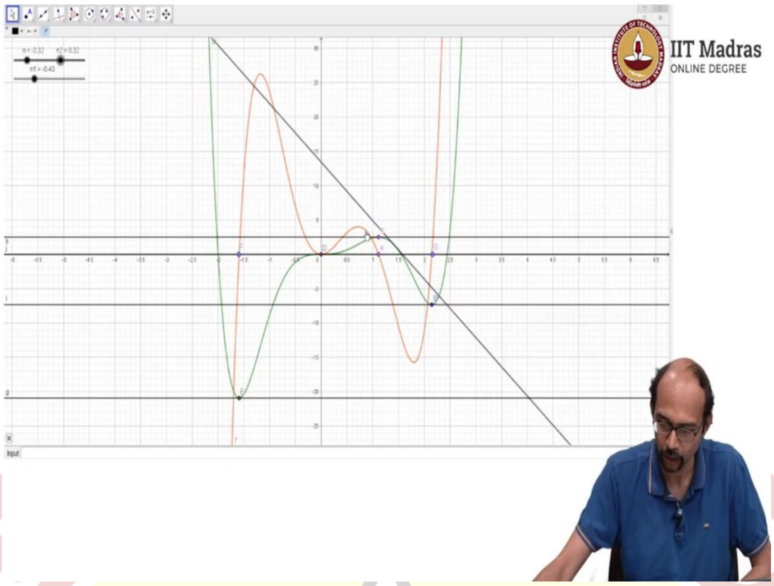

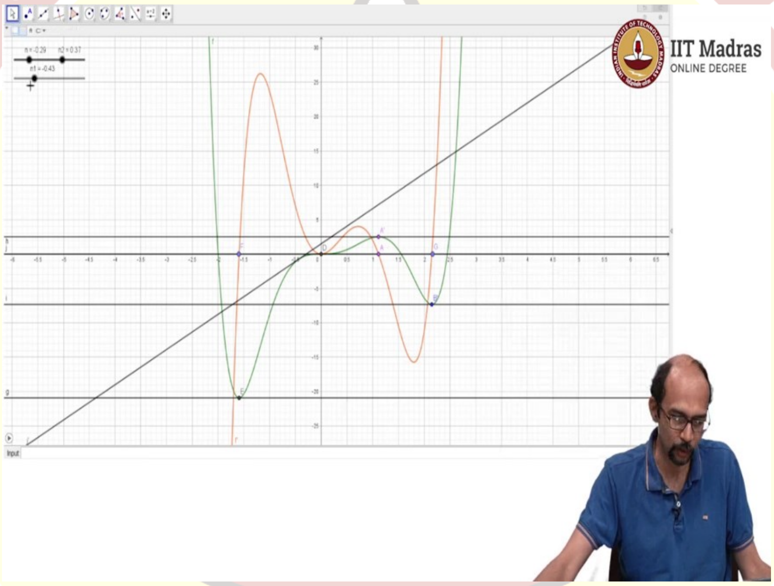

So, why does this test work? So, let us see what is happening to the tangents as we come close to this point here. So, let us look at this and so, as we vary this, the tangent going to vary in the small neighbourhood of that point. So, let us see how this goes. So, you can see the tangent is slowly varying and as it comes here, it become 0 meaning horizontal. f'' become 0 and then again it changes. So, what happened in this entire part? If we start again, initially this derivative value was negative, then it became 0, then it became positive, this is what happened.

So, when you have the double derivative, the double derivative will be positive because this is an increment because f' is increasing and we have plotted f'as well. So, let us stop this, we have plotted f'as well and let us check that phenomenon is occurring here. Indeed, as you can see $f'$is increasing over here, all the way until you get to some intermediate point.

So, and you can see what that intermediate point is that is exactly where, here it is concave upwards and at some point is changes and became concave downwards. So, that point is where the double derivative become 0. So, here is n2, here is slighter n2 and as you can see this is a small, we chose a small neighbourhood around this point, then the tangent. So, at this point the tangent is not we have a problem. So, you have to choose a small neighbourhood. So, if you choose a neighbourhood which is-0.61 from that point to something ahead, we will be fine.

So, let us play this animation and now it changes. So, this is why you have to get a local, this is local. So, you have to get, choose a small interval around this point. Then, it will be strictly decreasing and the graph at this point of f' bares this, this graph is decreasing at the point a. So, that is why the f'' is negative and the same phenomenon happens at this point. So, let us stop that and look at this third one which is what happens that 0 so, let us.

So, for 0 we will see something strange is happening. So, now let us play that animation again. Here it is decreasing, decreasing, decreasing up to 0 and as soon as it hits 0 it starts increasing again. So, it is not strictly increasing or strictly decreasing around this neighbourhood of 0. And that is why the f''over there is 0.

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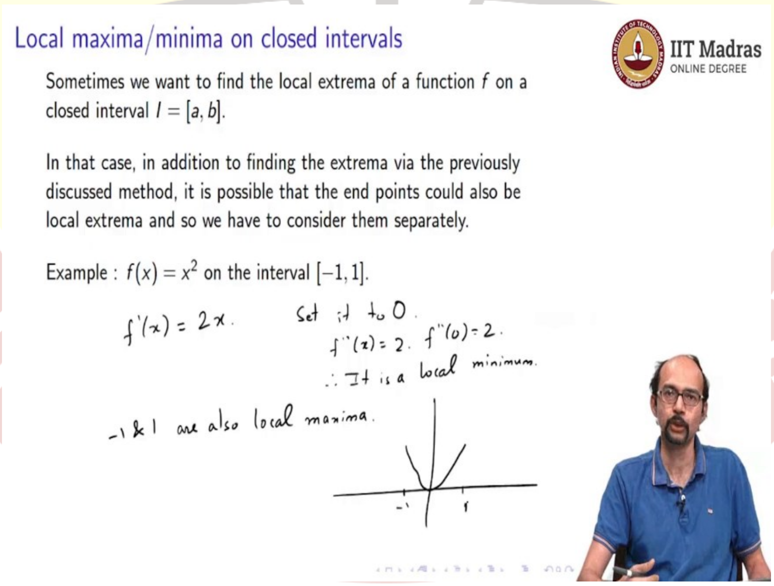

So, I hope this pictures have been illustrative of what we had in mind. So, let us go on and study local maximum or minimum on closed interval. So, sometimes you want to find the local extrema of a function on a closed interval a comma b. So, in that case in addition to finding the extrema via the previously described method which means to find $f'(a)$ set it to 0, $f'(x)$ set to

0 and so on, it is possible at the end points could also be local extrema and we have consider them separately.

So, let us consider this example of f x is x squared on the interval-1 , 1. So, x^2 is-1, 1, if we compute the first derivative this is $2x$ set it to 0. So, I think it is clear of course that what is happening at 0, it is a global minimum in fact. So, $f''(x)$ is 2 so $f''(0)$ is 2, therefore, it is a local minimum. In fact, it is global minimum because we know how the graph of x^2 is, but if you want to consider only on this interval, then-1 and 1 are also local maxima.

Why is that? Because if you look that the graph of the function x squared it is something like this and if we want to look at it only in the part between-1 and 1. So, so on that part this is it and so this point here which is-1 and this point here which is 1 are actually local minimum and local maxima. Because in a small neighbourhood around those points remember now neighbourhood means only, you are considering only points which are within minis 1 comma 1. So, they will be local maxima. So, for closed intervals you have to also consider the end point separately.

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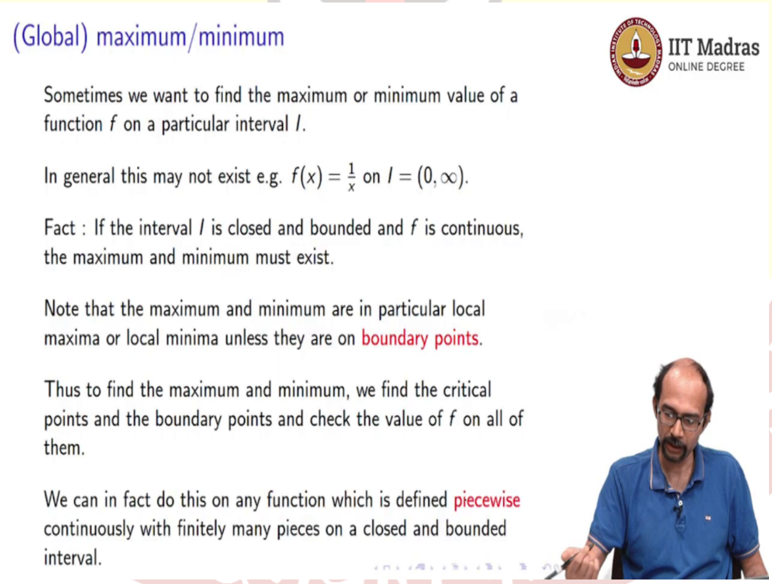

How about global maxima or minima? So, sometimes you want to find the maximum or minimum value of a function on a particular interval I and here by maximum or minimum we mean the global maximum or minimum. Now this may not exist in general so sometimes it may and sometimes it may not.

So, for example if you take the function 1/ x on the interval 0 comma infinity, it has neither a global maximum nor or a global minimum. Why is that? Because this function approaches infinity as you come close to 0 and it approaches 0 as you come close to infinity but it never attains the value 0 nor it ever attain the value infinity obviously because it is not a real number and so it can never attain a maximum or minimum value.

So, in general maxima or minima meaning global maxima or global minima may not exist. So, in this case actually there is no local maximum or local minimum either. But here is a fact if the interval I is closed and bounded so closed means you have you include the boundary points, and bounded means you do not go to infinity. You have to be within some a comma b and the function f is continuous so this is important, then the maximum and minimum must exist.

So, note that the maximum and minimum are in particular local maxima or local minima unless they are on boundary points, if they are on boundary points. If they are on boundary points, then they need not be a local maximum or a local minimum in the sense that we have studies before. So, to find the maximum and minimum we find their critical points and the boundary points and check the value of f on all of them. So, we can in fact do this on any function which is defined piecewise continuously with finitely many pieces on a closed and bounded interval.

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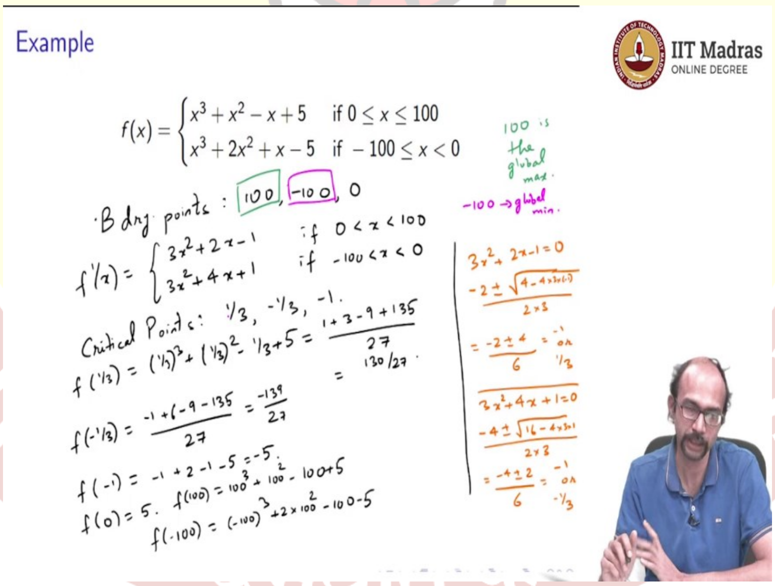

So, let us do an example of this just to get a clear picture of what were we saying. So, in this example $f(x) = x^3 + x^2 - x + 5$ between 0 and 100 and its x cube+2 x squared+x-5 between-100 and 0 and we want to find let say the global maximum or the global minimum, so let us see how to do this. So, first of all we ask what are the boundary points, so here the boundary points are 100 and-100 and we have to also include 0 as a boundary point because that is where the definition of the function changes. So, here we include 0 as a boundary point.

And then let us compute what are the critical points on each piece so remember that when we are computing critical points say, I want to compute what are the critical points for the first piece that is x cube for the function $x^3 + x^2 - x + 5$ between 0 and 100, this is within the open interval 0 bounded, so when we differentiate and put this function into 0, we may get several critical points which are not inside this open intervals, so we have to only take those which are inside the open interval 0 to 100 not even the boundary points.

So, let us write down what is $f'x$. So, $f'x$ is $3x^2 + 2x - 1$ hat is if 0 is strictly less than x is strictly less than 100 and it is 3x squared+4x+1 if-100 is strictly less than x, is strictly less than 0. So, now let us put this to 0 and find out what the critical points are so in the first piece the critical points that you get are, so let me do that computation here because remember that I do not have to, I do not want to take all the critical points but only those which lie in that interval.

So, this is a quadratic equation so to solve this equation. So we want to find when 3x squared+2x-1 is 0. So, we know how to solve quadratic equation. So, the solutions are-b+minus root b squared-4 ac by 2a if the equation is $ax^2 + bx + c$. So, that gives us-2+minus root 4-4 times 3 times-1 divided by 2 times 3. So, the two possibilities are where we have-2+minus root of 16, so that is 4 divided by 6. So, this is either-2-4 which is-6 divided by 6 so that is-1 or-2+4 which is 2 divided by 6 so one third.

But now as you can see-1 is not in the interval 0 to 100, so I should not take that as a critical point. So, one of the critical points is one third coming from the first part, for the second part let us do the similar computation and see what that yields. So, there I have to find the roots of $3x^2 + 2x - 1$, so that is-4+minus root of 16-4 times 3 times 1 by 2 times 3 and so the computation is similar, you get-4+minus root of 16-12 which is 4. So, root of 4 is 2 divided by 6.

So, again the answers here are-4-2, so-6 by 1 so-1 or-2 by 6. So, that is-one third. So, both of these are inside the open interval that we are considering. So, this is-one third and-1. Fine. So, now we have these 3 critical points and we have our 3 boundary points 100,-100 and 0 and so now we can evaluate f at all these points and see where the maxima, find out which of these are maxima and minima. When I mean maxima and minima, I mean global maxima and global minima.

So, what is f of one third? So, f of one third is one third cubed+one third squared-one third+5, so that is 1 by 27+1 by 9-one third+5. So, that is 1+3-9+5 times 27, that is 135 by 27. So, then that is 139-9, so 130 by 27. So, close to 5, slightly less than 5, between 4 and 5. And then we have f of-one third and you will have a similar computation, so in the end you are going to get a-1+3 minus, but here though I have to use the second equation, so this 3 is multiplied by 2, so-1+6,-9 and then-135 by 27.

And again that should give us something close to 5 or rather-5, so-135+5, so-130-139 by 27. So, it is between-5 and-6. And then I have f of-1. So, if I substitute that you, we get-1+2-1-5 so that is-5 and then we have f of 0, so f of 0 is 5 and then finally we have f of 100 and f of-100. So, $f(100) = 100^3 + 100^{20} - 100 + 5$f of 100.

So, I am not going to compute this value because it is obvious that this number is really, really large as compared to the other numbers which are all in the range of 5 or-5 or between-10 and 10. And this number is of the order 100 cubed. So, this is clearly a huge number. So, this certainly larger than all of these others and similarly if we have f of-100 and that is-100 cubed+2 times 100 squared-100-5 again of the order-100 cubed.

So, this is, in absolute value, this number is very large, so is a negative sign, it is the smallest number amongst all these and so that gives us that this is the first one which is 100 is a global maxima. So, let us write that down. So, this number here is a global maximum, 100 is the global maximum and then we have-100 which is the global minimum. So,-100 is the global minimum. So, amongst all possible values taken by the function, the one taken by-100 is the smallest and the one taken by 100 is the largest.

We can also go ahead and ask about these critical points, what kind of points they are, local maxima, local, local maxima, local minima or saddle points and to do that we have to compute f double prime. Again this has to be done only on the open intervals 0 to 100 and-100 to 0. So, if you do that you will get something like $6x + 2$ and $6x + 4$ for the respective intervals and then by plugging in the values, you can check whether they are positive or negative and hopefully they are non-zero and then you can check if they are local maxima or local minima using the second derivative test.

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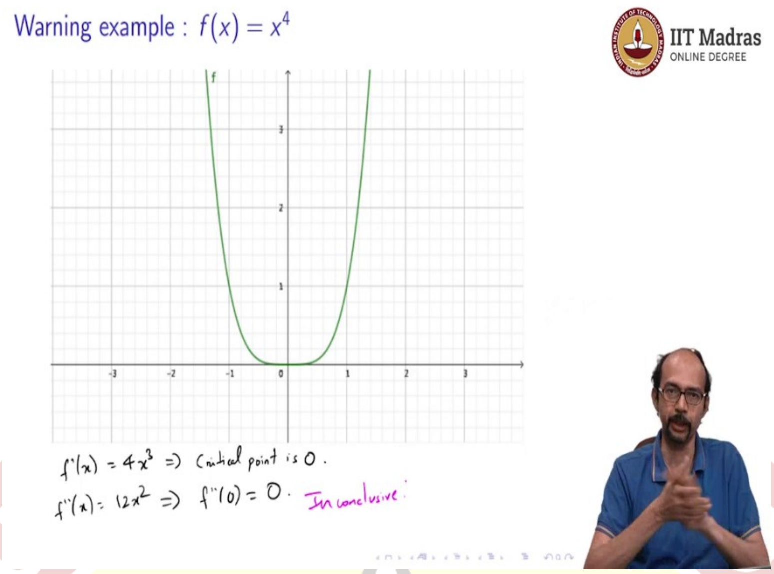

So, let us end with this warning example. So, if you have $f(x) = x^4$, clearly 0 is global minimum, but what happens, I mean what does our test give us? So, if you, so in particular it is a local minimum. So, we hope that it will show up in the test. So, we do $4x^3$, then $f'$of, then the critical point is 0, so now we do the double derivative test. So, that is $12x^2$ and when you apply it at 0, you get 0. So, in fact the double derivative test ends up being inconclusive, so inconclusive.

Although this function is so clearly and obviously, is a function which you can clearly and obviously analyze, meaning what are the global minima, the test is inconclusive. So, you want to be careful that when we end up in this inconclusive situation, you may still have a local, in fact a global minimum or maximum. So, that you have to analyse by different methods.

So, let us quickly conclude by recalling what we have done in this video. We saw the second derivative test which is, which tells us that is at the, at a critical point, the second derivative is negative, then it is a local maximum; if it is positive, it is a local minimum. And if it is 0, then it is inconclusive. We saw that the inconclusive situation can occur because of local minimum, local maximum or saddle points, any of them and to do that we have to really check what happens to the derivative function close to that point.

Further we saw that if you have a closed and bounded interval, or you have a piecewise continuous function, then you have to check the to get these phenomena of local maximum and local minimum, you have to also check the function values close to the, sorry, at the boundary points or at the break points where it is piecewise defined. And only from there can you conclude about local or global minimum or maximum. Thank you.

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