IIT Madras

ONLINE DEGREE

Why is $\sqrt{2}$ irrational?

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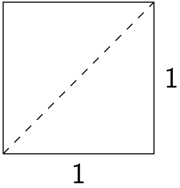
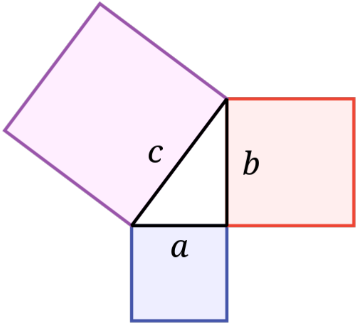
Mathematics for Data Science 1
Week 1

Irrational numbers

- The discovery of irrational numbers is attributed to the ancient Greeks
- Since Pythagoras, it was known that the diagonal of a unit square has length $\sqrt{2}$
- His followers spent many years trying to prove it was rational
- Hippasus is attributed with proving that $\sqrt{2}$ is irrational, around 500 BCE
- The followers of Pythagoras were shocked by the discovery
- Allegedly, they drowned Hippasus at sea to suppress this fact from the public





The proof of Hippasus that $\sqrt{2}$ is not a rational number

- If $\sqrt{2}$ is rational, it can be written as a reduced fraction p/q, where $gcd(p,q) = 1$
- From $\sqrt{2} = p/q$, squaring both sides, $2 = p^2/q^2$
- Cross multiplying, $p^2 = 2q^2$, so $p^2 = p \cdot p$ is even
- The product of two odd numbers is odd and the product of two even numbers is even, so p is even, say $p = 2a$
- So $p^2 = (2a)^2 = 4a^2 = 2q^2$
- Therefore $q^2 = 2a^2$, so q^2 is also even
- By the same reasoning, q is even, say $q = 2b$.
- So $p = 2a$ and $q = 2b$, which means $gcd(p,q) \geq 2$, which contradicts our assumption that p/q was in reduced form.

Hippasus
Engraving by Girolamo Olgiati, 1580

Summary

- The proof of Hippasus follows a pattern commonly used in mathematical reasoning
- To show that a fact P holds, assume $not(P)$ and derive a contradiction
- Using a similar strategy, can show that for any natural number n that is not a perfect square, $\sqrt{n}$ is irrational

Hippasus
Engraving by
Girolamo Olgiati, 1580