IIT Madras

ONLINE DEGREE

Mathematics for Data Science 1
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Lecture – 8.3
Exponential Functions: Definitions

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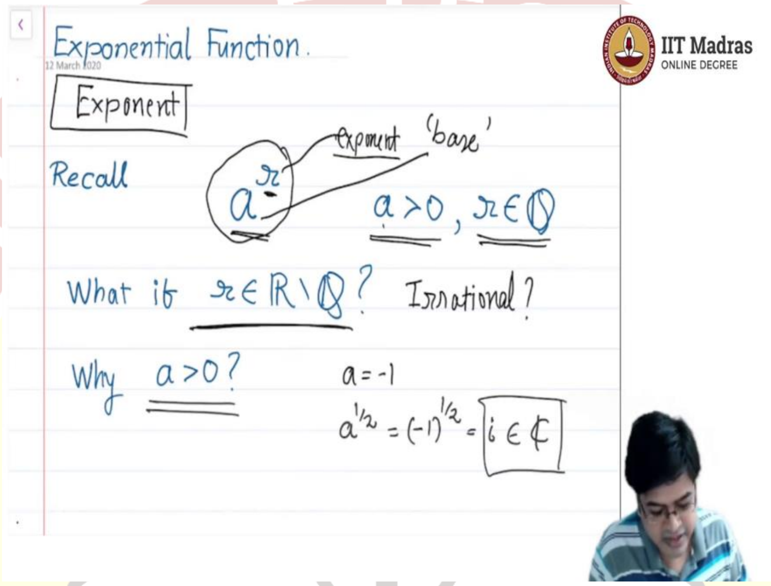

Welcome back. Next topic is Exponential Function. So, in this topic, what we will see is first we will identify with some known terminology that is exponent. We have already know seen this exponent. Where we allowed integer powers and then while defining the exponents, we allowed rationals also. So, when we defined exponents, they were of the form a^r and we always assume $a > 0$ and $r \in \mathbb{Q}$.

Now, I want to define an exponential function. So, as the name suggest, exponential has to do something with the exponent. So, what we are doing here when I am considering a function of this form, I am raising something, some number to the power of a where a will be popularly called as base and r is the exponent. So, this is base and this is exponent.

Now, if I want to define exponential functions on real line, then it is mandatory for me to define this a^r for $r \in \mathbb{R}\setminus\mathbb{Q}$. This is real line minus set of rational numbers that is I am talking about set of irrational numbers. So, I do not know as of now what is a definition of exponent form of set of irrational numbers ok.

The next question that we have seen is why is $a > 0$ that for which you know the answer. Let us say $a = -1$ now$a^{1/2} = (-1)^{1/2} = i$ which belongs to complex set of complex numbers, but I do not want to deal with complex number so, I am avoiding a to be greater than 0. In general, you can define a to be a negative number and then deal with complex numbers. We do not want to indulge into that conversation. So, I do not; I do not want to define $a < 0$.

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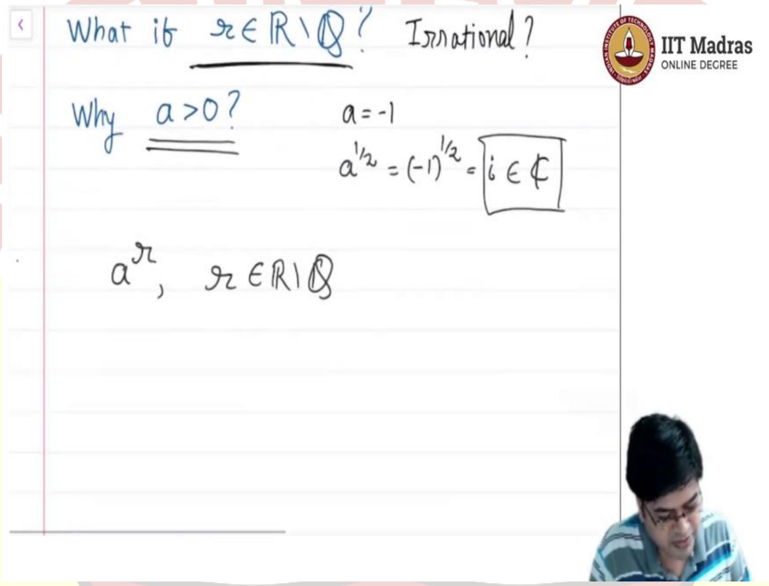

So, $a < 0$ is eliminated now the question is $a^r$and r belongs to irrational where $r \in \mathbb{R} \setminus \mathbb{Q}$ what will happen in this case? Or how will I define rational number?

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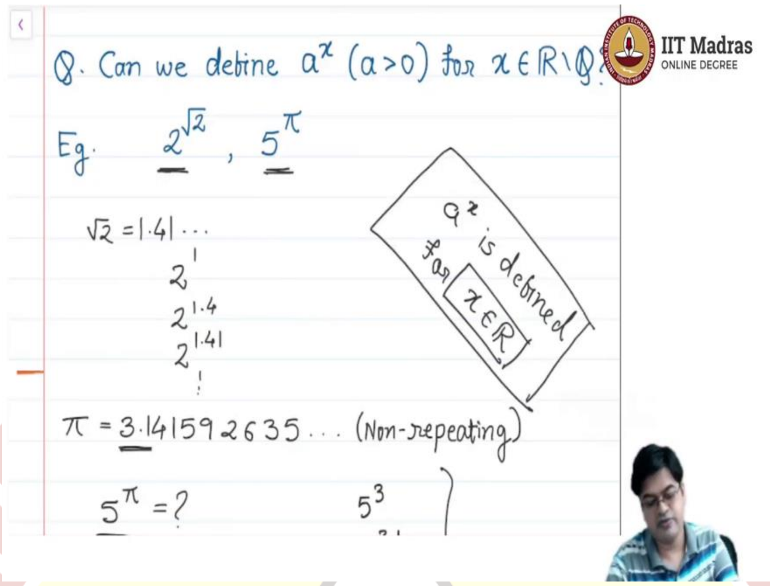

To be precise, let us ask a question that is can I define $2^{\sqrt{2}}$ or 5^{π} ok. So, in this case, there is no direct way to answer this question, but I will definitely have a strategy which is a calculus-based strategy to answer this question.

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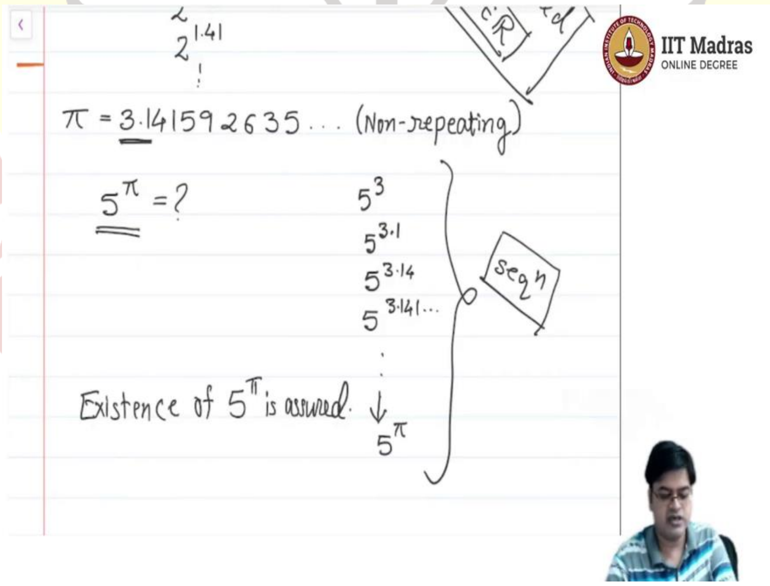

Let us consider this π and the value the numerical approximation of π is actually we all know π is an irrational number and 3.14592635… and this thing is non repeating it will continue till infinity right. So, now, what I need to understand is from what I know, can I define the number 5^{π}? So, anyway I cannot define it accurately right now.

So, based on my understanding, I am asking you a question that is 5^3? Right if so, then next question is $5^{3.1}$ defined? So, what I am doing here is this if yes, can I define $5^{3.14}$? Now, you remember all these a approximations are actually a rational approximations 3 is a rational number, 3.1 is 31 by 10 which is again a rational number, 3.14 is 314 by 100 which is again a rational number and I can go on like this there is 3.141 and so on.

So, if I continue this way, I will reach somewhere; I will reach somewhere and that somewhere I will call as 5^{π}. So, in principle, I can actually define a raised to irrational number. This you will study when you will study a topic of sequences which is outside the scope of this syllabus. So, we will assume that you have to trust me on this that 5^{π} is well defined.

In a similar manner, you can do an exercise for $2^{\sqrt{2}}$. So, $\sqrt{2} = 1.41$ ... and something. So, again you will go with 2^1 is defined $2^{1.4}$ is defined, $2^{1.41}$ is defined and so on and you will reach somewhere that is $2^{\sqrt{2}}$.

So, this way we are very clear that a^x is defined for $x \in \mathbb{R}$. This sets up the platform for defining an exponential function, this is very important a raised to x is well defined for $x \in r$.

This answer is given by convergence of sequences which is outside the scope of the syllabus, but we know that it exist for sure. So, I am guaranteeing the existence of 5^{π}; existence of 5^{π} is assured. In case you are interested, you can take a basic course in analysis or; in analysis or calculus where you will study these things ok.

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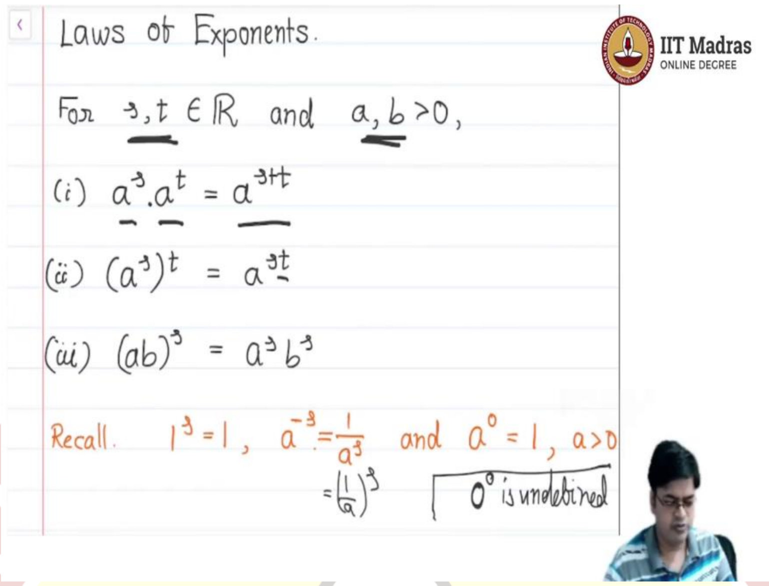

So, now let us go to let us recall all of these laws you already know simple laws of exponents. Earlier, we have we knew the laws of exponents for only rational numbers. Now, we are talking about the real numbers. So, $s, t \in \mathbb{R}$, $a, b > 0$, s and t will play a role of exponents, a and b will play a role of bases ok. So, then it is very easy to prove you might have proved. $a^s a^t = a^{s+t}$.

Remember here, product here is becoming addition here these are crucial points $(a^s)^t = a^{st}$. So, a raised to operation is becoming a product here. $(ab)^s = a^s b^s$ and then; obviously, you need to know that $1^s = 1$ for every $s \in \mathbb{R}$, $a^{-s} = \frac{1}{a^s} = \left(\frac{1}{a}\right)^s$, $a^0 = 1$. Remember where your $a > 0$ because 0^0 is undefined ok.

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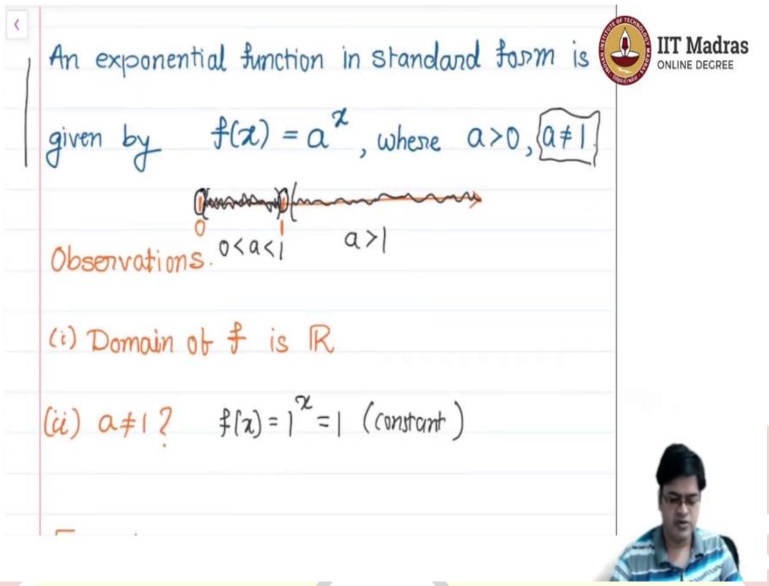

So, with this understanding, we have revised laws of exponents which will which we will use the left and right. So, you better remember all these laws and therefore, we are ready to set a framework of exponential function. So, here is our definition. An exponential function in the standard form is given by $f(x) = a^x$, where $a > 0, a \neq 1$. These are new condition that we have introduced.

We have seen why $a > 0$, but here they are saying $a \neq 1$. So, this needs further analysis, we will analyze it in due course. So, right now, if you look at the values of a, $a > 0$; that means, all these values are allowed and $a > 1$; that means, all these values are also allowed. Bearing the values 0 and 1 right.

So, the first from the definition, the first observation that you can figure out is because you have bared the value 0 and 1, the function $f(x) = a^x$ will have a domain which is entire real line. For every , $x \in \mathbb{R}$, we should be able to compute a^x ok.

Then, let us analyze this is then observation: why $a \neq 1$? Let us put $a = 1$. So, $f(x) = 1^x$, but from the laws of exponent what you know? $1^s = 1$.

Therefore, $1^x = 1$ in fact; it is nothing, but a constant function. I am not interested in handling a constant function right which nothing, but a horizontal line $y = 1$ is the graph of a function; I am not interested in this. So, let us not call this as exponential function that is what we are saying in the definition.

So, hence forth, we will never talk about $a = 1$, $a = 0$ or $a < 0$. So, if you have a real line, you will have an expression of this form where you are talking about this interval, open interval and this interval, which is an infinite interval. So, you have two characterizations which is $0 < a < 1$, $a > 1$ these are the two characterizations that you got over this thing.

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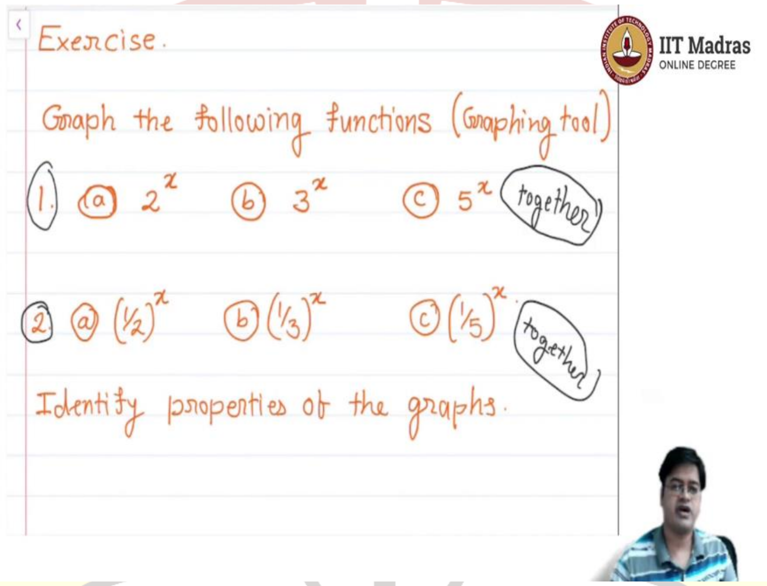

Now it will be interesting to use some graphical tool and see what are some functions of this kind look like. So, here is an exercise that I will give you. Use some graphing tool like Desmos and plot these functions together. For example, you plot the functions given in 1 using Desmos we just put $f(x)$ is equal to this, $f(x)$ is equal to this and this and plot all these three graphs together without any understanding about the behavior of the function you plot all three of them together.

Then, use the 2nd graph and put all these three things together. Identify the properties of the graph that is through which points they pass through is there any difference in the graphs of 1 and 2. So, identify all these properties like we did in polynomials and after doing that again return back to this video and we will see some of the functions that are given here by a graph and we will analyze those functions. So, right now you pause the video, come back in the next video.