IIT Madras

ONLINE DEGREE

Degrees of infinity

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Mathematics for Data Science 1
Week 1

Are there degrees of infinity?

- Cardinality of a set is the number of elements
- For finite sets, count the elements
- What about infinite sets?
  - Is $\mathbb{N}$ smaller than $\mathbb{Z}$?
  - Is $\mathbb{Z}$ smaller than $\mathbb{Q}$?
  - Is $\mathbb{Q}$ smaller than $\mathbb{R}$?
- First systematically studied by Georg Cantor
- To compare cardinalities of infinite sets, use bijections
  - One-to-one and onto function
  - Pairs elements from the sets so that none are left out

Georg Cantor

Countable sets

- Starting point of infinite sets is $\mathbb{N}$
- Suppose we have a bijection f between $\mathbb{N}$ and a set X
  - Enumerate X as $\{f(0), f(1), \ldots, \}$
  - X can be "counted" via f
  - Such a set is called countable

Georg Cantor

$\mathbb{Z}$ is countable

- $\mathbb{Z}$ extends $\mathbb{N}$ with negative integers
- Intuitively, $\mathbb{Z}$ is twice as large as $\mathbb{N}$
- Can we set up a bijection between $\mathbb{N}$ and $\mathbb{Z}$?

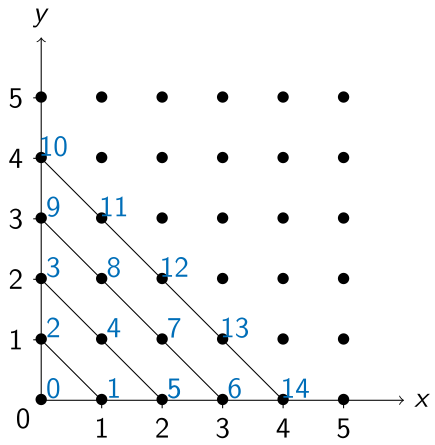
$$
\begin{array}{ccccccccccc}
\cdots, & -4, & -3, & -2, & -1, & 0, & 1, & 2, & 3, & 4, & \cdots \\
 & & & & 2, & 0, & 1, & & & & \\
 & & & 4, & 2, & 0, & 1, & 3, & & & \\
\cdots, & 8, & 6, & 4, & 2, & 0, & 1, & 3, & 5, & 7, & \cdots
\end{array}
$$

- The enumeration is effective
  - $f(0) = 0$
  - For i odd, $f(i) = (i+1)/2$
  - For i even, $f(i) = -(i/2)$
- $\mathbb{Z}$ is countable

Georg Cantor

Is $\mathbb{Q}$ countable?

- $\mathbb{Q}$ is dense, $\mathbb{Z}$ is discrete
- Are there more rationals than integers?
- There is an obvious bijection between $\mathbb{Z} \times \mathbb{Z}$ and $\mathbb{Q}$
  - $(p, q) \mapsto \frac{p}{q}$
- Sufficient to check cardinality of $\mathbb{Z} \times \mathbb{Z}$
  - For simplicity, we restrict to $\mathbb{N} \times \mathbb{N}$
- Enumerate $\mathbb{N} \times \mathbb{N}$ diagonally



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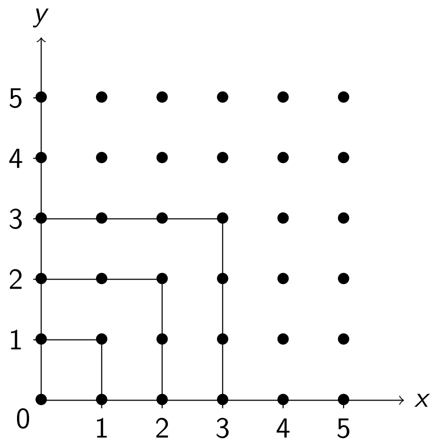
  - For simplicity, we restrict to $\mathbb{N} \times \mathbb{N}$
- Enumerate $\mathbb{N} \times \mathbb{N}$ diagonally
- Other enumeration strategies are also possible
- Can easily extend these to $\mathbb{Z} \times \mathbb{Z}$
- Hence $\mathbb{Q}$ is countable



Is $\mathbb{R}$ countable?

- $\mathbb{R}$ extends $\mathbb{Q}$ by irrational numbers
- Cantor showed that $\mathbb{R}$ is not countable
- First, a different set
  - Infinite sequences over $\{0, 1\}$
    $0\ 1\ 0\ 1\ 1\ 0\ \cdots$
- Suppose there is some enumeration

	b_0	b_1	b_2	b_3	b_4	$\cdots$
s_0	0	1	1	1	0	$\cdots$
s_1	1	0	1	0	0	$\cdots$
s_2	1	1	1	1	1	$\cdots$
s_3	0	1	1	0	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

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  - Infinite sequences over $\{0, 1\}$
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- Suppose there is some enumeration
- Flip b_i in s_i
- Read off the diagonal sequence
- Diagonal sequence differs from each s_i at b_i
- New sequence that it not part of the enumeration

	b_0	b_1	b_2	b_3	b_4	$\cdots$
s_0	1	1	1	1	0	$\cdots$
s_1	1	1	1	0	0	$\cdots$
s_2	1	1	0	1	1	$\cdots$
s_3	0	1	1	1	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Is $\mathbb{R}$ countable?

- Infinite sequences over $\{0,1\}$ cannot be enumerated
- Each sequence can be read as a decimal fraction

  0.011101110011
- Injective function from $\{0,1\}$ sequences to interval $[0,1) \subseteq \mathbb{R}$
- Hence $[0,1) \subseteq \mathbb{R}$ cannot be enumerated
- So $\mathbb{R}$ is not countable

	b_0	b_1	b_2	b_3	b_4	$\cdots$
s_0	1	1	1	1	0	$\cdots$
s_1	1	1	1	0	0	$\cdots$
s_2	1	1	0	1	1	$\cdots$
s_3	0	1	1	1	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Summary

- Any set that has a bijection from $\mathbb{N}$ is countable
- $\mathbb{Z}$ and $\mathbb{Q}$ are countable
- $\mathbb{R}$ is not countable — diagonalization
- Is there a set whose size is between $\mathbb{N}$ and $\mathbb{R}$?
- Continuum Hypothesis — one of the major questions in set theory
- Paul Cohen showed that you can neither prove nor disprove this hypothesis within set theory

Georg Cantor