INDIAN INSTITUTE OF TECHNOLOGY MADRAS

सिद्धिर्भवति कर्मजा

IIT Madras

ONLINE DEGREE

Mathematics for Data Science 1
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Lecture 47
Natural Exponential Function

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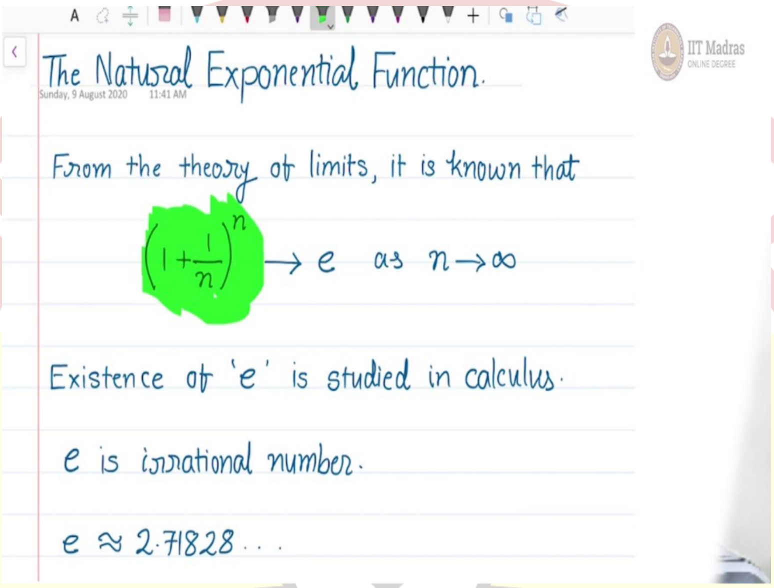

Hello friends, in this video we are going to talk about yet another important function among all exponential functions is natural exponential function. So, basically the theory of natural exponential function is derived from calculus. So, in order to understand how relation natural functional, natural exponential function arises, we need to study the theory of limits. In particular, this natural exponential function is dependent on something raised to the exponent that something is in irrational number that is called e.

And I will make sure by the end of this video you will understand why this number e is very important. In particular, when we talk about number e or ratio or a limit of some quantity is important which is shown here. So, from the theory of limits it is known that whenever you are talking about $(1+\frac{1}{n})^n$ this particular limit it actually converges to e.

So, now unless you understand the concepts of limits, you may not be able to have complete understanding of this concept, but still I will give you some intuition behind this number e. So, though, as I mentioned earlier, the existence of e is actually studied in the field of calculus. For that you may have to do the course which is maths 2, Math for Data Science 2 and you have to agree with me on certain facts without knowing them or you have to trust me that e is an irrational number and e is approximately equal to 2.71828 and so on. It is an irrational number. So, it will go to never ending decimal representation, it will continue on the right.

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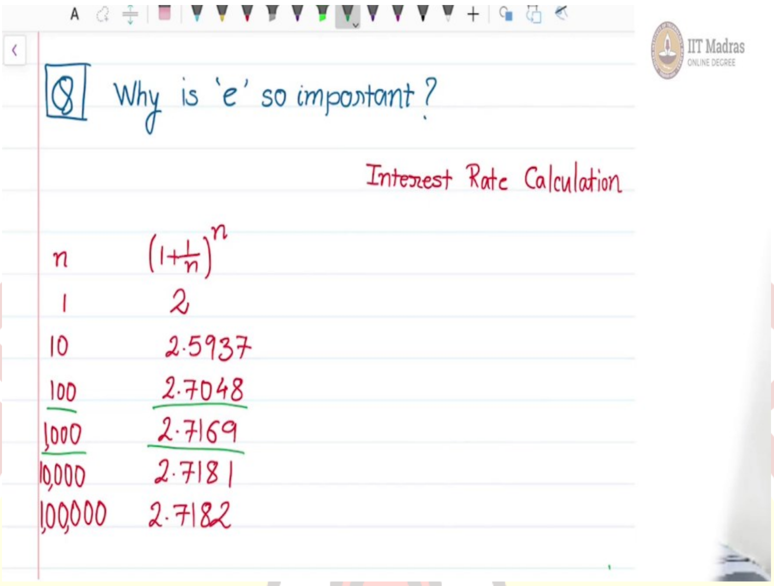

So, these are the facts about e. Now the question that we asked at the beginning of the video is why is 'e' so important? So, to answer that question, let us first look at the behaviour of this particular number as a limit. So, when I say that n goes without bounds, the number the this particular function $f(n) = (1+\frac{1}{n})^n$ converges to e. What do I mean? Let me put it in a proper formal way.

So, I have generated a table over here for our convenience and let us understand. So, when I substitute this n, the value of $n = 1$, this number $(1+\frac{1}{n})^n$ is simply 2. When I substitute $n = 10$, the number becomes 2.5937. So, does that mean this function will go without bounds? The answer is no that is why we get the convergence. And such type of questions are studied in Calculus.

So, when you substitute the further values of n that is, $n=100$ you have substituted, you got 2.7048. When you substitute $n = 1000$, you will get 2.7169. Now, you can see that you approaching closer to the ideal value of e. And when you put $n = 10000$, you get 7181 still because we are writing up to 4 decimal places, we are not really very close to it, but we will be, we are very close to it, but we are not at that point.

But when I put n is equal to 1 lakh, then I get a value of e which is 2.7182 and that is actually exact representation of this number e up to 4 decimal places, correct up to 4 decimal places.

Now, if you go on further and put higher and higher values e like you can put it to be a 1 million, 10 lakhs and then you will see you will get further improvement, but because we are focusing only on 4 digits after decimal, we this requirement is enough for us, this much calculation is enough for us. Another thing, another aspect in which this e becomes very crucial in accounts this interested calculations.

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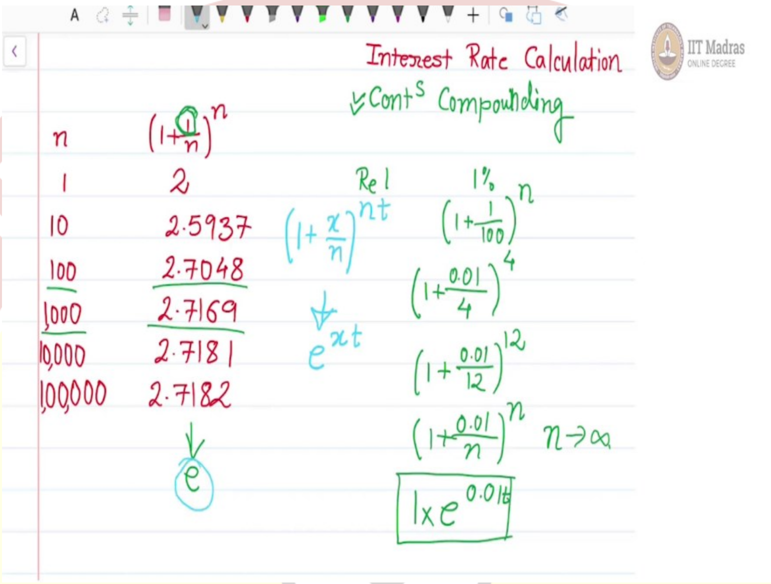

You must have heard the term by now of continuous compounding and continuous compounding actually means the taking ratios with respect to e or taking exponents with respect to e. So, let me demonstrate to you in this manner. Let us say you have invested rupee 1 in a bank and bank is offering 1 % interest rate and you have invested it for 1 year. So, in that case what is the answer? 1 plus 1 upon 100 raised to 1, this is the answer, 1.01 1 % you will get if you have invested rupee 1, you will get 1 paisa of interest.

So, now if you go on like this you will have something like $(1+\frac{1}{100})^1$ raised to so whatever number of years you have invested in raised to n. Now, when you actually look at the procedure of the bank, banks do not give you the interest which are given annually, but they credit the interest quarterly. So, in that case what you need to understand is the interest rate is actually given in a quarter, so it is computed on quarter and whatever interest you have accumulated, that interest will be taken into account for the next quarter.

So in that case, basically what bank is doing is bank is taking this interest rates which is 0.01 and it is actually dividing it into 4 quarters, 4 parts because they are giving you a quarterly interest and then you are actually getting this multiplied in this fashion. So here, if you look at the interest rate, instead of 1, I have 0.01 as the number and for 0.01 I got this number. So, if the bank decides, so I will revise the bank decides that I am revising, bank is revising the interest rate every month, then what will happen?

Then the same logic for single year, for single year remember $(1+\frac{0.01}{12})^{12}$. So, if the bank starts revising the interest rate infinitely often, then we are actually talking about something like $(1+\frac{0.01}{n})^{n}$ and this $n \to \infty$. In this case, according to our judgement, according to this, this number was 1 here, now it is 0.01 and this number converge to e. So, based on this understanding, if you apply the same logic and try to calculate this thing, then it will converge to e raised to 0.1.

This is an interesting revelation. That means, if you invest rupee 1, you just take that rupee $1 * e^{0.01}$ that will be the interest accumulated along with the original capital in are bank if the bank follows continuous compounding. This is how whenever you study finance, you calculate the interest rate. So, this is for the period of 1 year. Now if you add the period in terms of time, then it will be $e^{0.01t}$. So, this is how e becomes important. Let us now replace this 1 % by a generic number which is x.

So, what I am talking about now is $(1+\frac{x}{n})^{n}$ and now from the discussion that we have done this will converge to e^{x} and when I add the time that is it is more than 1 year, then I have something like n_t and that is where I will get x_t. So, these are simple understanding why the number e is very important. e typically comes when you are considering a continuous compounding.

I hope I have made the relevance of the number e, irrational number e very clear and it is an irrational number and its exact value is given by this particular expression. It is not exact but it is approximate which is suitable for our purposes.

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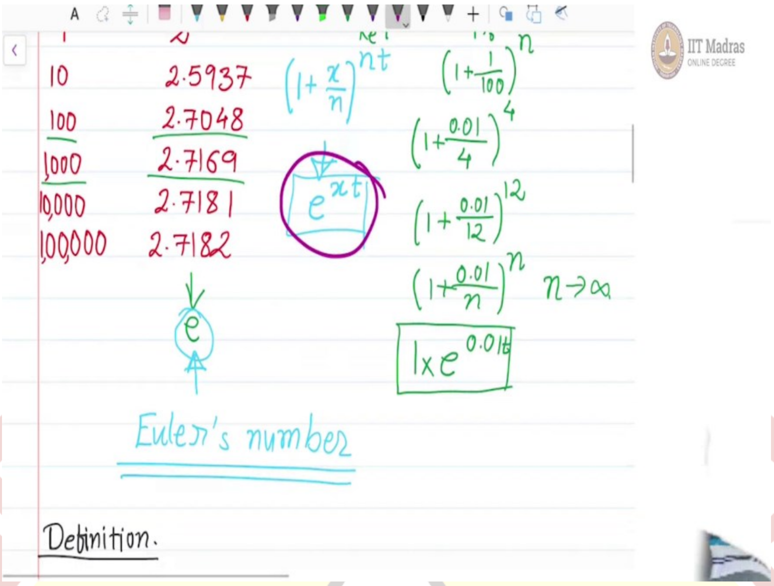

Now let us go further and understand what is a function that we have defined here and it is why it is called natural exponential function? So, this number *e* as I mentioned now naturally comes when you are considering continuous compounding. It also comes very naturally in the field of Differential Equations which is also relaying on our calculus. So, this number *e* has a special name when you consider differential equations as a area which is called Euler's Number.

So, you can Google and you can search the meaning of Euler's number and why it is relevant. So, that is how this *e* is called a natural exponential. So, now let us formally define the function that we have just now seen which is *e* raised to x_t as a natural exponential function.

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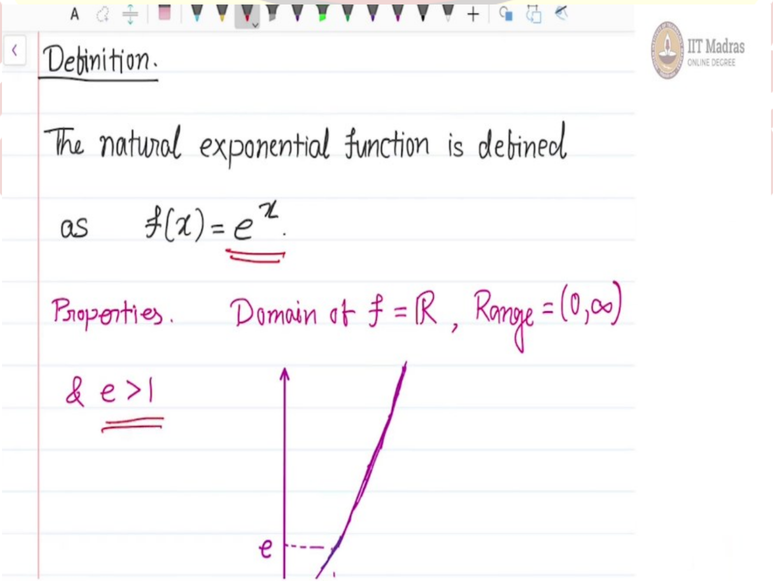

So, a natural exponential function is defined as $f(x) = e^x$. Then, you may ask a question, what are some interesting properties of this natural exponential function? Now, the properties will be very similar to the exponential function that we have studied, but it is special in some sense. We will see its specialness in a when we will study some special properties of this natural exponential function e^x. Let us list all the properties.

Domain of f, domain of this function will be set of real numbers and range of the function is positive real line that is 0 to ∞. As you have seen *e*, the value of *e* is 2.7182 so *e* is natural greater than 1.

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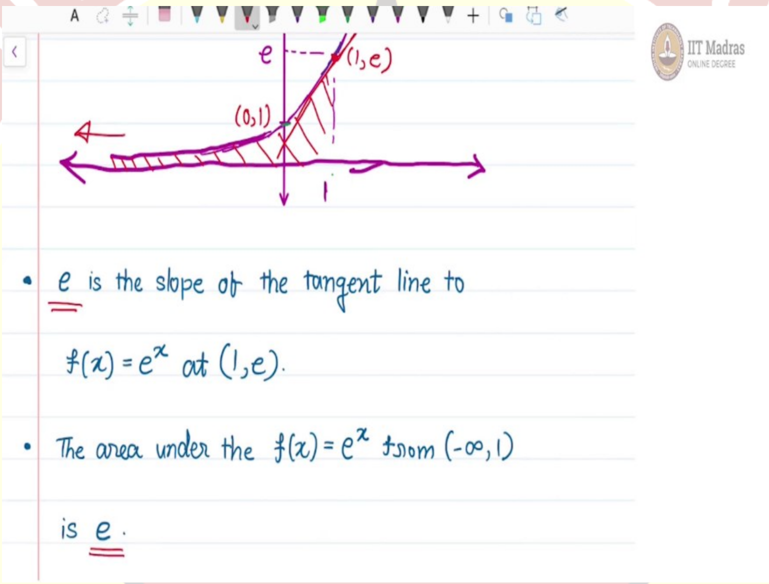

So, if you recollect whatever we studied for exponential functions, you will get the graph of exponential function in this manner where there are two typical points 0,1 is one typical point that it passes through and it will always pass through 1,e. As x tends to infinity, the function goes without bounds, as $x \to -\infty$, the function asymptotically goes to the *x* axis so *y = 0* is the horizontal asymptote for this function, we have already seen that. For general exponential function same properties hold true.

Now what makes *e* special? And what is something special that is not true with general exponential function. So, in this case, if you look at the point (1,*e*) and if you draw a tangent to a line tangent to the curve, that is a line passing through this particular point, the slope of this line will be *e*, that is very special. So, *e* is the slope of the line that is tangent to the curve y is equal to *e* raised to x at (1,*e*). So, that is one thing.

Then if you look at the area that is covered under this curve from $-\infty$ to 1, that area is actually *e,* the irrational number *e.* This you will learn when you will study calculus in maths 2. So, that is very important.

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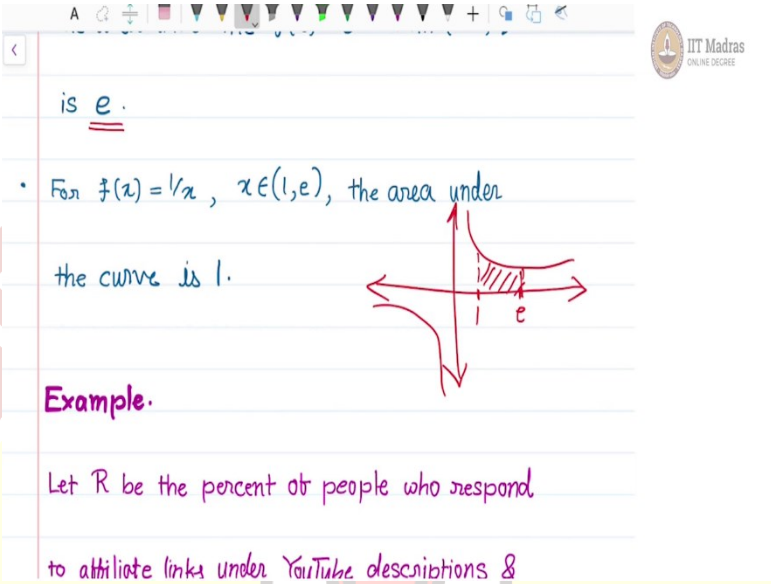

And the third thing that is very important which will not happen in general with other exponential function is if you draw a curve *f(x)* is *e*, if you draw a function $f(x) = \frac{1}{x}$. So, you may be familiar with that function, it will be something like this and something like this. And in this particular case, if you look at the area under the curve in the in the range 1 to *e*, this particular area, this area is a unit area for $f(x) = x$ and remember this *e* is an irrational number so still it will be a unit area.

Why it is so? this is a matter of calculus to explore, but these are the things, these are some of the things that makes the function $f(x) = e^x$ special function. Let us now understand this function better by considering an example which will deal with our real life problems.

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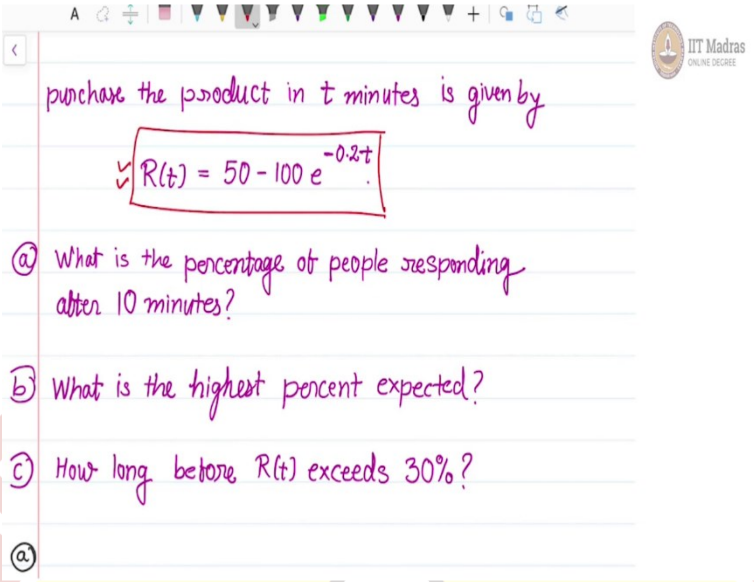

So, here is an example which says that let R be the percentage of people who respond to affiliate links under YouTube descriptions and the purchase and they purchase the product in t minutes and that particular purchasing thing is a function of time so it is given as the $R_t = 50 - 100e^{-0.2t}$. So, let me give you a brief understanding of the problem.

So, now when you watch some video on YouTube if you, the speaker in the YouTube says that there are some affiliate links below in the description. Now, if you click on that link and go to the affiliate site, then what you will do is, either you will purchase or you will not purchase. If you will purchase, the speaker or the channel owner will get some amount of commission.

Now, here the person who is actually giving the affiliate links is interested in finding the number of people who are responding in t minutes. So, he has devised a function which is available in YouTube statistics so based on the data available, he has derived a function, we are taking the function as it is. So, that function is $R_t = 50 - 100e^{-0.2t}$.

Now, he is interested in answering these questions. What percentage of people responding after 10 minutes? So, how many percentage of people responded after 10 minutes? Then, based on this function, what is the highest percentage expected? And the third question is how long before $R_t > 30$ %? The response rate being 30 % is also a good enough rate.

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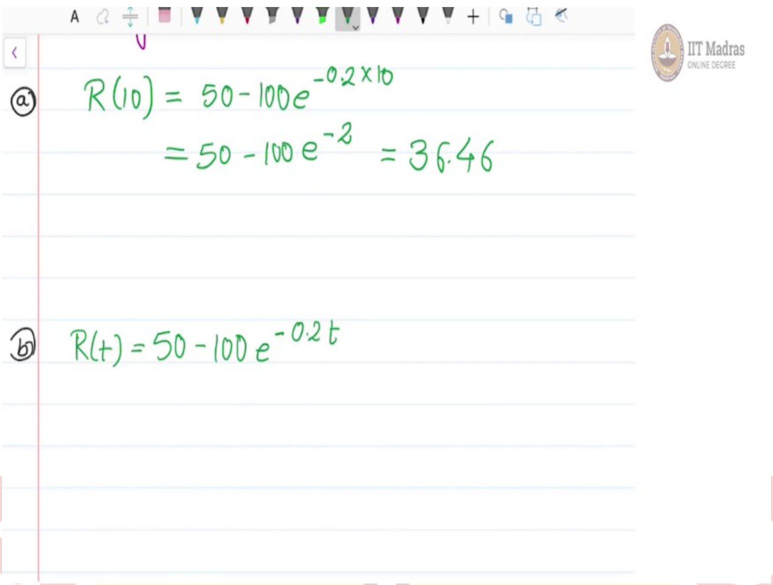

So, because you are just putting some affiliate links. So, let us try to understand what percentage or people will respond after 10 minutes? That means, I want to essentially evaluate the function as R_{10}. So, if I substitute this, it will be $50 - 100e^{-0.2\times10}$. that is simply if you rewrite this as 50 -100 times, this is 2/ 10 which can be simplified to e^{-2}. And then you can actually calculate the function e to by value of e^{-2} and you can put the value, that value is 36.46. So, this you can do it using calculator.

Now, let us look at the second question. What is the highest percentage expected? Now you have to think about this function which is $50 - 100e^{-0.2t}$. Now, you look at the function which is $e^{-0.2t}$ or e^{x}.

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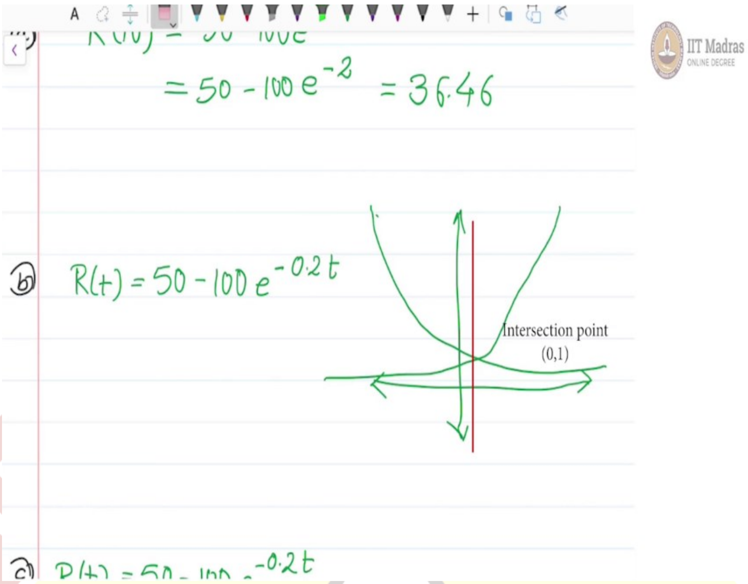

You already know the graph of the function which is e raised to x, $f(x) = e^x$. Now, how will the function look like when you are talking about $f(x) = e^{-x}$ has a graph of this form, roughly this form. Now, when you are talking about − of x, when you are talking about − of x, you are actually talking a reflection of this graph along this so that will give you some graph of this kind, it will never cross x axis but it will go this way.

So, now you have a understanding of how the graph of e^{-t} will look like. But here are some scaling versions, scaled versions like 100, this is 100 and this is 50. So, now this graph is actually multiplied with − 100, this graph is actually multiplied with -100, but multiplying with − 100 will again, what it will? It will actually keep the graph in a similar manner but it will actually because it is multiplied with − 100, it will shift in some sense like this.

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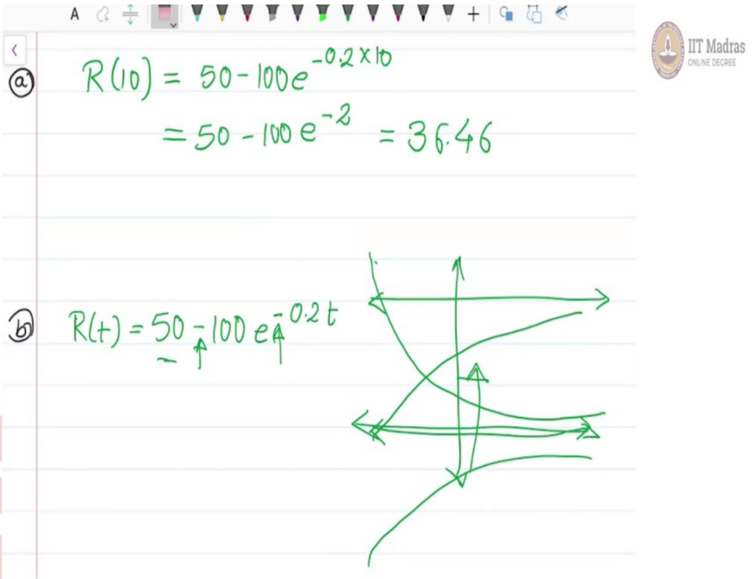

One minute let me chose and erase it. So, it will shift like, it will flip here and it will shift like this. And then, now when you are adding 50 to it, this graph will actually go up by 50 units. So, this way the changes will happen to the graph and finally graph will look something like this. You can actually check for yourself. So, basically first multiplying this − sign will have an effect of reflecting the graph along y axis, then multiplied with − 100 will reflect the graph along x axis and then adding 50 will shift the graph by 50 unit.

So, you have a fairly good understanding of the graph. Now, you just apply your knowledge that what is the highest percentage expected? So, in this case, if you understand this, the horizontal asymptote over here is actually shifted to 50 units because you are transferring to 50 and the graph actually let me clear up the image.

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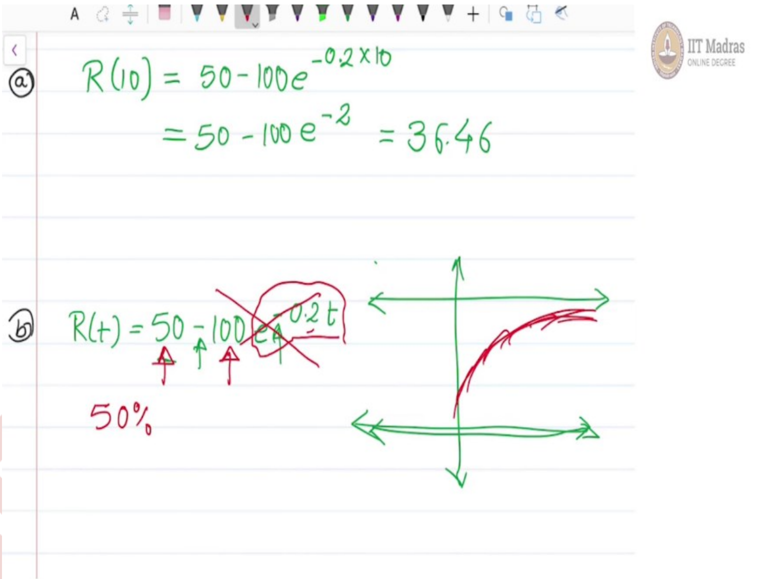

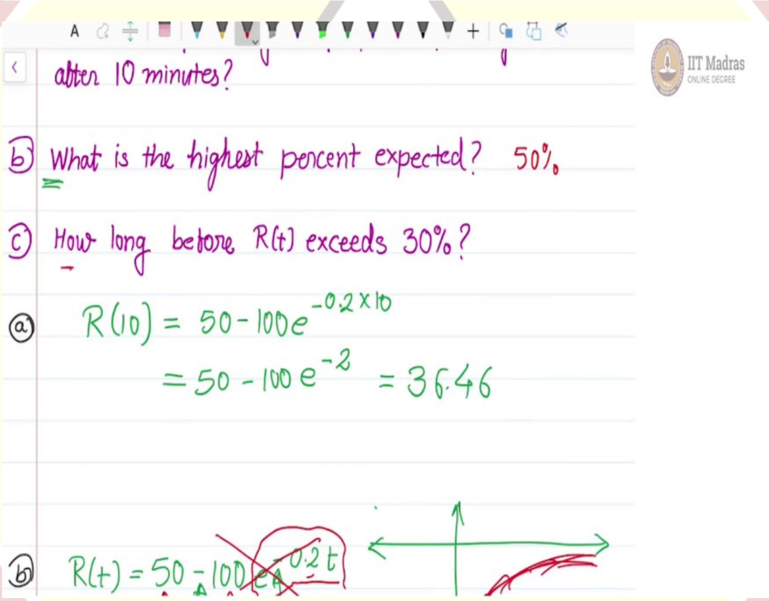

You can use your graphing tool also and verify that this is the graph. So the graph will look somewhat like this and it will asymptotically reach to 50 units. So, the highest percentage that is expected will be 50 %. It will not exceed 50 % based on the graphical analysis. Let us analyse this graph, instead of graphically analysing, let us look at this function e raised to − 0.2t, this function in itself will never exceed 1 and as x tends to infinity, this function will actually tend to 0.

That means, whether I am multiplying by 100 or I am multiplying by 10000, as $t \to \infty$, this function has to go to 0. So, this entire thing will go to 0 and therefore, 50 is the maximum that I can achieve. Therefore, my question b is answered as 50 %. Now let us look at this particular thing, how long it takes before R_t exceeds 30 %?

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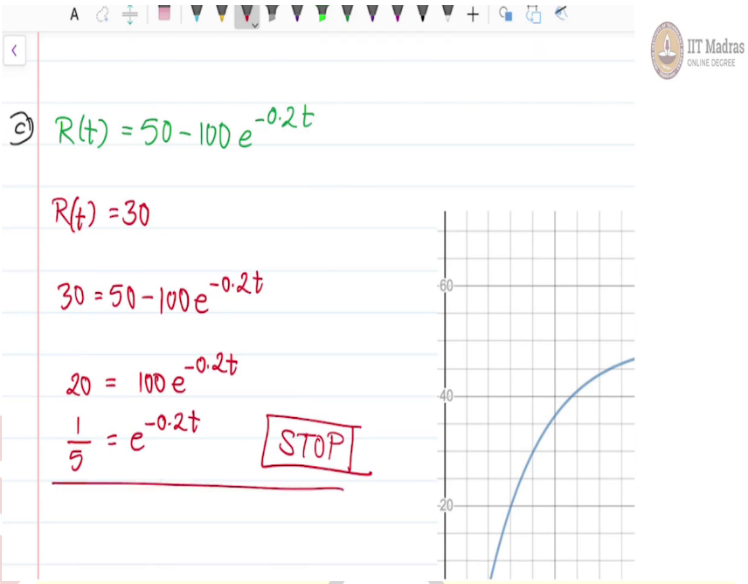

So, in this case you just look at the graph of the function, so as I have already described this will, this is how the graph of the function will look like. Now, there are two ways in which you can solve. Let us try to see whether we can go ahead formally and solve this. So, essentially I have been given that how long till R_t >30 %? That means R_t= 30, find are value of t such that $R_t = 30$?

So, 30 = 50 − 100$e^{-0.2t}$. So, 30 − 50 will give me something like − 20 = − 100$e^{-0.2t}$. Now, this is, these − signs will cancel themselves off so, this is this and then you simply rewrite this expression $\frac{20}{100}$is nothing but $\frac{1}{5} = e^{-0.2t}$. Now, I have to stop here because right now I do not have any ways to see what t will be when $\frac{1}{5} = e^{-0.2t}$. No analytical way is possible.

Then, what I will do is, so analytically I am stopping here and if I somehow I am able to figure out how to find t is equal to something, then I can answer this question. But let us now try to compute this graphically.

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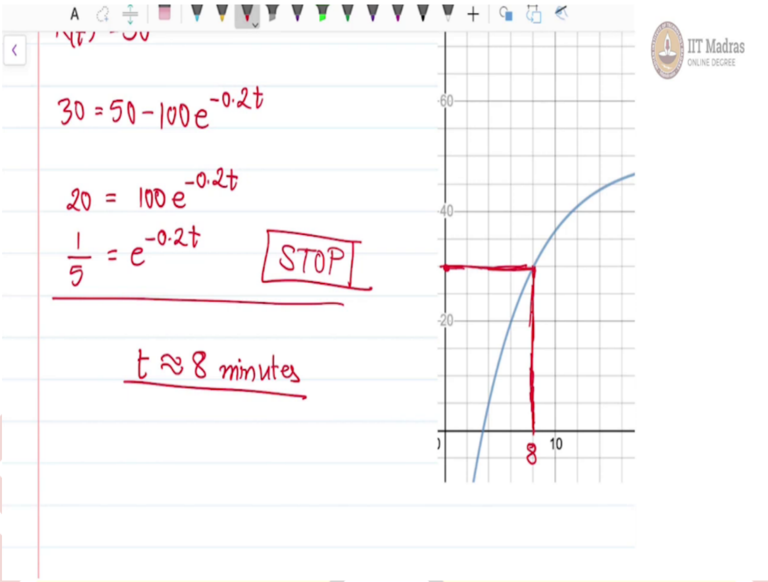

So, in this case, $R_t = 30$ is this point. So, now if you go along this line and then you map this onto x axis, remember x axis is nothing but the value of t. So, in this case, now you look at the mesh, this roughly turns out to be 8 that means, t will be approximately equal to 8 minutes. So, this is how we can without even solving the expression for this, graphically solve the expression for exponential functions. So, this is one live demo of that. That is all for now. So, we will meet in the next video where we will actually try to understand how to solve this particular problem. Thank you.