# Minimum Cost Spanning Trees: Prim's Algorithm

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Mathematics for Data Science 1
Week 12

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  $G = (V, E), W : E \to \mathbb{R}$
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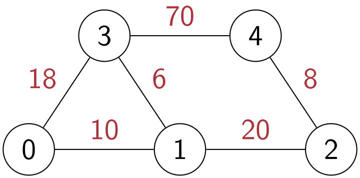

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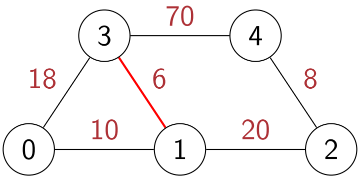


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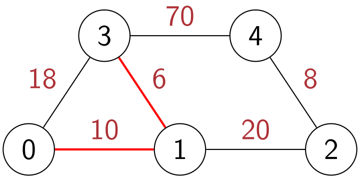


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- Extend the tree with $(1, 0)$

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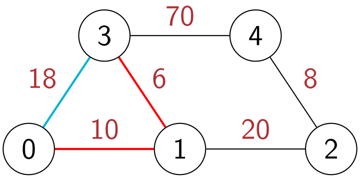


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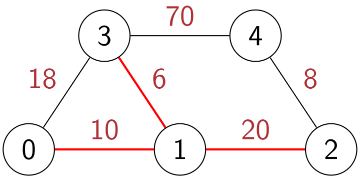


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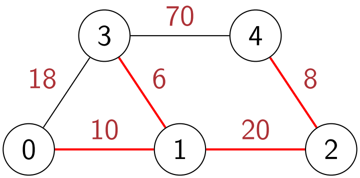


- Start with smallest edge, $(1, 3)$
- Extend the tree with $(1, 0)$
- Can't add $(0, 3)$, forms a cycle
- Instead, extend the tree with $(1, 2)$
- Extend the tree with $(2, 4)$

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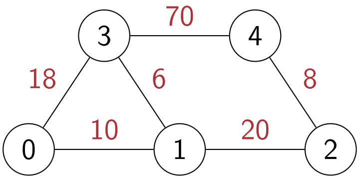


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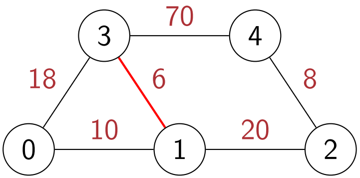


$TV = \{1, 3\}$

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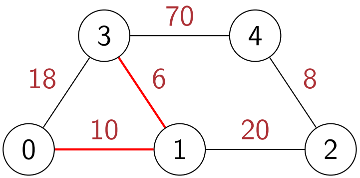


$TV = \{1, 3, 0\}$

$TE = \{(1, 3), (1, 0)\}$

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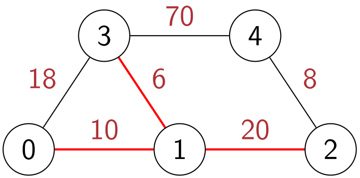


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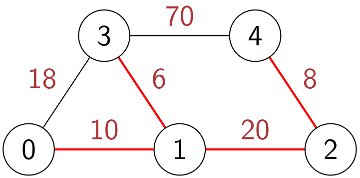


$$TV = \{1, 3, 0, 2, 4\}$$

$$TE = \{(1,3), (1,0), (1,2), (2,4)\}$$

# Correctness of Prim's algorithm

## Minimum Separator Lemma

- Let $V$ be partitioned into two non-empty sets $U$ and $W = V \setminus U$
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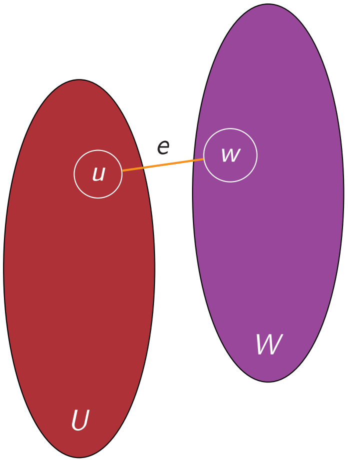


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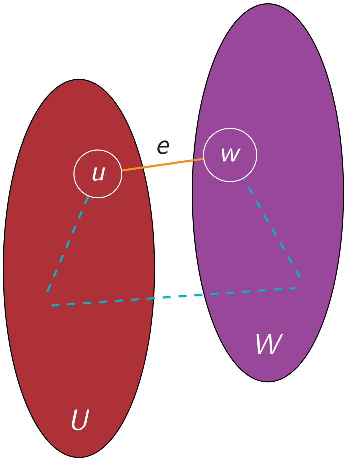

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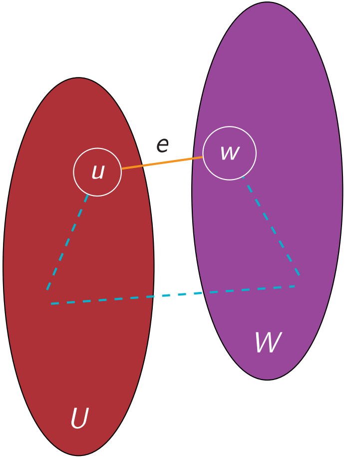

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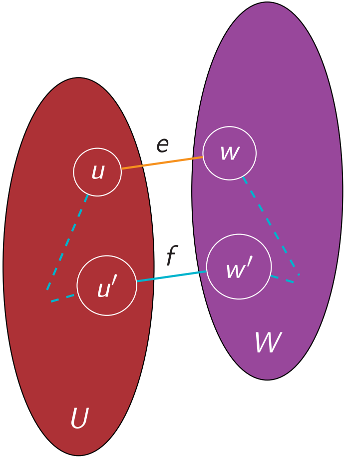


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- $T$ contains a path $p$ from $u$ to $w$
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  - Let $f = (u', w')$ be the first edge on $p$ crossing from $U$ to $W$

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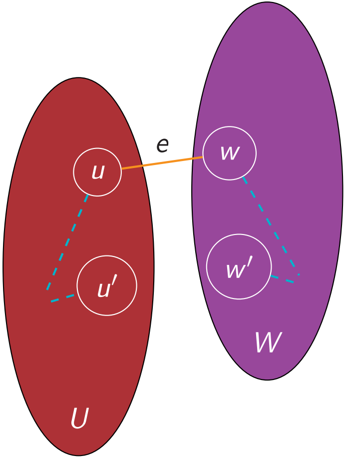


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  - Let $f = (u', w')$ be the first edge on $p$ crossing from $U$ to $W$
  - Drop $f$, add $e$ to get a cheaper spanning tree

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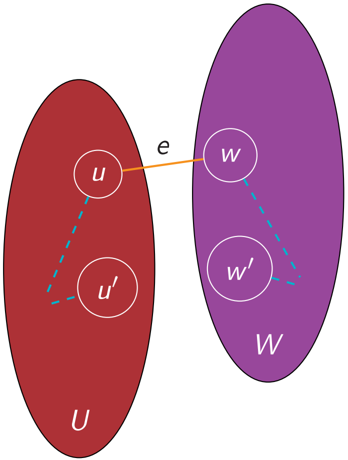


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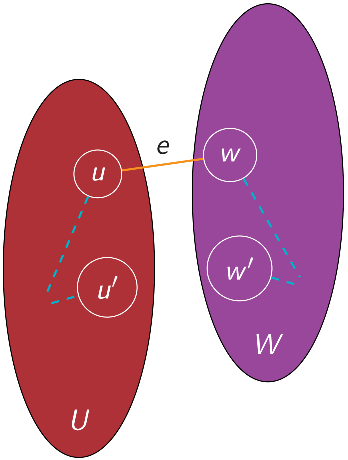
- What if two edges have the same weight?
- Assign each edge a unique index from $0$ to $m - 1$



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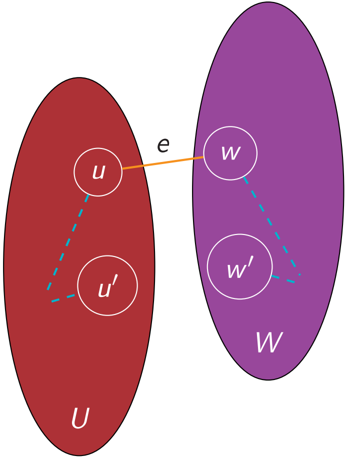


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- Define $(e, i) < (f, j)$ if $W(e) < W(j)$ or $W(e) = W(j)$ and $i < j$

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- Algorithm picks smallest edge connecting $TV$ and $W$, which must belong to every MCST

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- The smallest weight edge leaving any vertex must belong to every MCST

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- The smallest weight edge leaving any vertex must belong to every MCST
- We started with overall minimum cost edge
- Instead, can start at any vertex $v$, with $TV = \{v\}$ and $TE = \emptyset$
- First iteration will pick minimum cost edge from $v$

# Summary

- Prim's algorithm grows an MCST starting with any vertex
- At each step, connect one more vertex to the tree using minimum cost edge from inside the tree to outside the tree
- Correctness follows from Minimum Separator Lemma