Mathematics for Data Science-1 Term-2

Functions-1
Week - 7

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Functions

A function is like a machine which relates some inputs to its outputs. There are mainly three parts for defining a function.

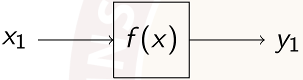
- Input (x)
- Relationships$(f(x))$
- Output (y)

$$x \longrightarrow \boxed{f(x)} \longrightarrow y$$

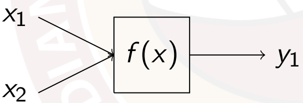
$$y = f(x)$$

Functions

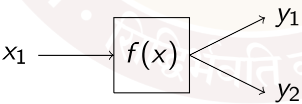
A function should always pass the the vertical line test i.e., there can not be more than one output for one input.



Function (Invertible or One to one)



Function (Non-invertible)



Not a Function

Sample question

Which one of the following is not a function?

a. $f(x) = x$

b. $f(x) = x^2$

c. $f(x) = x^3$

d. $f(x) = \pm\sqrt{x}$

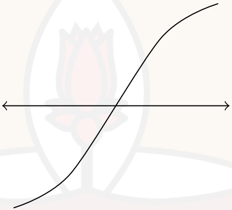
One to one functions

A function is one to one if it passes horizontal line test or any of the following conditions is true.

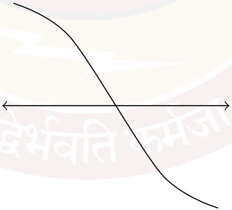
- Every output is related with only one input i.e., if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$ for all x_1 and x_2.
- Strictly increasing function i.e., if $x_1 > x_2$, then $f(x_1) > f(x_2)$ for all x_1 and x_2.
- Strictly decreasing function i.e., if $x_1 < x_2$, then $f(x_1) > f(x_2)$ for all x_1 and x_2.

One to one functions

- Strictly increasing function



- Strictly Decreasing function



Question 1

Selvi deposits ₹P in a bank A which provides an interest rate of 10% per year. After 10 years, she withdraws the whole amount from bank A and deposits it in another bank B for n years which provides an interest rate of 12.5% per year. $M_A(x)$ represents the amount in Selvi's account after x years of depositing in bank A. $M_B(y)$ represents the amount in Selvi's account after y years of depositing in bank B and the interests are compounded yearly for both the banks.

Question 1

Choose the correct option.

a. M_A is one to one function but M_B is not.

b. M_B is one to one function but M_A is not.

c. M_A and M_B both are one to one functions.

d. Both are not one to one functions.

Solution

- When the principal amount P is compounded annually, the amount M after q years is given by

$$M = P \times \left(1 + \frac{\text{Interest rate}}{100}\right)^q$$

- Amount $M_A(x)$ after x years in bank A with 10% interest rate will be

$$M_A(x) = P \times \left(1 + \frac{10}{100}\right)^x$$

- So after 10 years the amount $M_A(10)$ will be

$$M_A(10) = P \times (1.1)^{10}$$

Solution

- As Selvi has withdrawn all the amounts from bank A after 10 years so the new principal amount $P \times (1.1)^{10}$ is deposited in another bank B.
- After y years with 12.5% interest rate, the amount will be $M_B(y)$ which can be written as

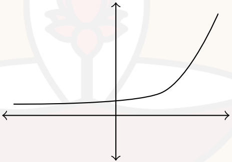
$$M_B(y) = P \times (1.1)^{10} \times \left(1 + \frac{12.5}{100}\right)^y$$

- So for n years

$$M_B(n) = P \times (1.1)^{10} \times (1.125)^n$$

Solution

- M_A and M_B both are the exponential functions.



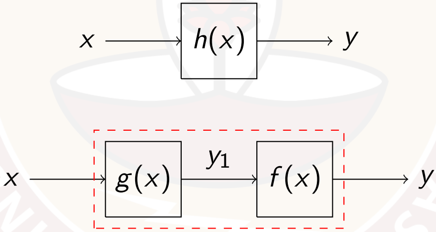
- $f(x) = a^{g(x)}$ is one to one function, if $g(x)$ is one to one.
- Therefore, both the functions are one to one functions.

Composite functions

- The combination of more than one function is called the composite function (if it is defined).
- The combination of $f(x)$ and $g(x)$ could be either $f(g(x))$ or $g(f(x))$.
- let $h(x)$ be the composite function of functions $f(x)$ and $g(x)$ then we can write $h(x) = f(g(x)) = fog(x)$.

Composite functions

Let $y = h(x) = f(g(x))$ then,



Question 2

There are two offers in a shop. In the first offer, the discount in total payable amount is $M(n)\%$ if the number of products bought at a time is n. The second offer involves a discount of ₹1000 on the total payable amount. T_{12} is the final payable amount when the second offer is applied after the first offer and T_{21} is the final payable amount when the first offer is applied after the second offer. Geeta shops of ₹15,000 and $M(n) = -n^2 + 18n - 72$, where $n \in \{6, 7, 8, 9\}$, then answer the following questions.

Question 2(a)

Choose the correct option.

a. $f(n) = (100 - M(n)) \times 15000$

b. $g(n) = (100 - M(n)) \times 15000 - 1000$

c. $f(n) = (100 - M(n)) \times 150.$

d. $g(n) = 14000.$

Question 2(B)

Choose the correct option.

a. $T_{21}(n) = (100 - M(n)) \times 15000$

b. $T_{21}(n) = (100 - M(n)) \times 140$

c. $T_{12}(n) = (100 - M(n)) \times 15000 - 1000$

d. $T_{21}(n) = (100 - M(n)) \times 14000$

Question 2(C)

Choose a function which gives the minimum final payable amount.

a. $T_{21}(n)$

b. $f(n)$

c. $g(n)$

d. $T_{12}(n)$

Solution

- It is given that total payable amount without any offer is ₹15,000. Then, total payable amount after first offer is

$$f(n) = \frac{(100 - M(n))}{100} \times 15,000 = (100 - M(n)) \times 150$$

- The total payable amount if second offer is applied will be

$$g(n) = 15,000 - 1000 = ₹14,000.$$

- The amount payable when we apply the first offer after the second

$$T_{21}(n) = g(n)\frac{100 - M(n)}{100}$$

$$T_{21}(n) = \frac{(100 - M(n))}{100} \times 14000 = (100 - M(n)) \times 140$$

Solution

- The amount payable after applying the first offer would be

$$f(n) = \frac{(100 - M(n))}{100} \times 15000$$

- The amount payable when we apply the second offer after the first

$$T_{12}(n) = f(n) - 1000$$

$$T_{12}(n) = \frac{(100 - M(n))}{100} \times 15000 - 1000 = (100 - M(n)) \times 150 - 1000$$

Solution

$$T_{12}(n) = \frac{(100 - M(n))}{100} \times 15000 - 1000$$

$$T_{12}(n) = (100 - M(n)) \times 150 - 1000$$

$$T_{12}(n) = (100 - (-n^2 + 18n - 72)) \times 150 - 1000$$

$$T_{12}(n) = 150n^2 - 2700n + 24800$$

For minimum payable amount

$$n = \frac{-b}{2a} = \frac{-(-2700)}{2 \times 150} = 9$$

$$\boldsymbol{T_{12}(9) = 12,650}$$

Similarly

$$T_{21} = 12,740$$

Inverse Functions

- Let $f(x)$ and $g(x)$ are two functions such that,

$$g(f(x)) = f(g(x)) = x$$

  then $g(x)$ is called the inverse function of $f(x)$ or $f(x)$ is called the inverse function of $g(x)$.
- We can write $g(x)$ as $f^{-1}(x)$.

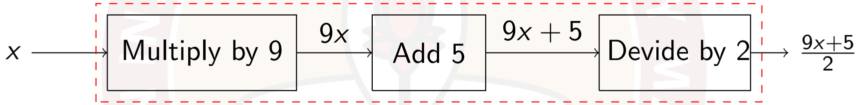
Inverse Functions

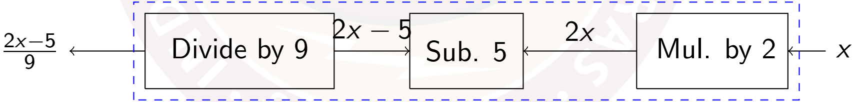
$$x \longrightarrow \boxed{f(x)} \longrightarrow y$$

$$x \longrightarrow \boxed{f(x)} \xrightarrow{y} \boxed{f^{-1}(x)} \longrightarrow x$$

Inverse Functions

Let $f(x) = \frac{9x+5}{2}$





$f^{-1}(x) = \frac{2x-5}{9}$

Inverse function: Steps

Let

$$f(x) = \frac{9x+5}{2}$$

- First check if function is one to one.
- Replace $f(x)$ by x and x by $f^{-1}(x)$

$$x = \frac{9f^{-1}(x)+5}{2}$$

- Solve for $f^{-1}(x)$

$$2x = 9f^{-1}(x) + 5$$

$$\frac{2x-5}{9} = f^{-1}(x)$$

Question 3

Given two real valued functions $f(x) = \frac{5x+9}{2x}$, $g(y) = \sqrt{y^2 - 9}$. If $h(x) = f(g(x))$, then answer the following questions.

Question 3(a)

Domain of $h(x)$ is.

a. $(-\infty, -3) \cup (3, \infty)$

b. $(-\infty, -3) \cap (3, \infty)$

c. $(-\infty, \infty)$

d. None

Solution

- Find the domain of $g(x)$ let us say g_d.

$$g(x) = \sqrt{x^2 - 9}$$

$$x^2 - 9 \geq 0$$

$$g_d = (-\infty, -3] \cup [3, \infty)$$

- Write the expression for finding the domain of $f(x)$.

$$f(x) = \frac{5x + 9}{2x}$$

$$2x \neq 0$$

Solution

- Replace x with $g(x)$ and then find the accepted values of x let us say g_r.

$$2x \neq 0$$

$$2g(x) \neq 0$$

$$\sqrt{x^2 - 9} \neq 0$$

$$g_r = (-\infty, -3) \cup (3, \infty)$$

- The domain of $h(x) = f(g(x))$ (let us say h_d) would be the intersection of g_d and g_r.

$$g_d \cap g_r = ((-\infty, -3] \cup [3, \infty)) \cap ((-\infty, -3) \cup (3, \infty))$$

$$h_d = (-\infty, -3) \cup (3, \infty)$$

Question 3(b)

Domain of $f^{-1}(x)$ is.

a. $(-\infty, -2.5) \cup (2.5, \infty)$

b. $(-\infty, 2.5) \cap (2.5, \infty)$

c. $(-\infty, \infty)$

d. None

Solution

Check if $f(x)$ is invertible or one to one function.

$$f(x) = \frac{5x+9}{2x}$$

$$f(x) = 2.5 + \frac{9}{2x}$$

$$f(x) = 2.5 + \frac{4.5}{x}$$

Solution

Let $f(x_1) = f(x_2)$, then

$$2.5 + \frac{4.5}{x_1} = 2.5 + \frac{4.5}{x_2}$$

$$\frac{4.5}{x_1} = \frac{4.5}{x_2}$$

$$\frac{1}{x_1} = \frac{1}{x_2}$$

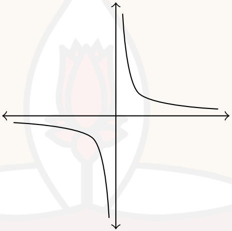
$$\implies x_1 = x_2$$

Which means

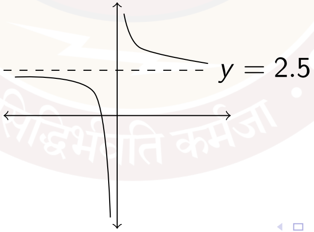
$$f(x_1) \neq f(x_2) \ if \ x_1 \neq x_2$$

Solution

- We know that the graph of $y = \frac{1}{x}$ is



- Similarly we can draw the graph of $y = 2.5 + \frac{4.5}{x}$ as



Solution

- As we can see that the function is one to one, inverse is possible.
- Range of $f(x)$ is $(-\infty, 2.5) \cup (2.5, \infty)$.
- Therefore, domain of $f^{-1}(x)$ would be the range of $f(x)$ which is $(-\infty, 2.5) \cup (2.5, \infty)$.

THANK YOU