IIT Madras

ONLINE DEGREE

Statistics for Data Science-1
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Lecture No. 8.4
Discrete Random Variables - Probability Mass Function Properties

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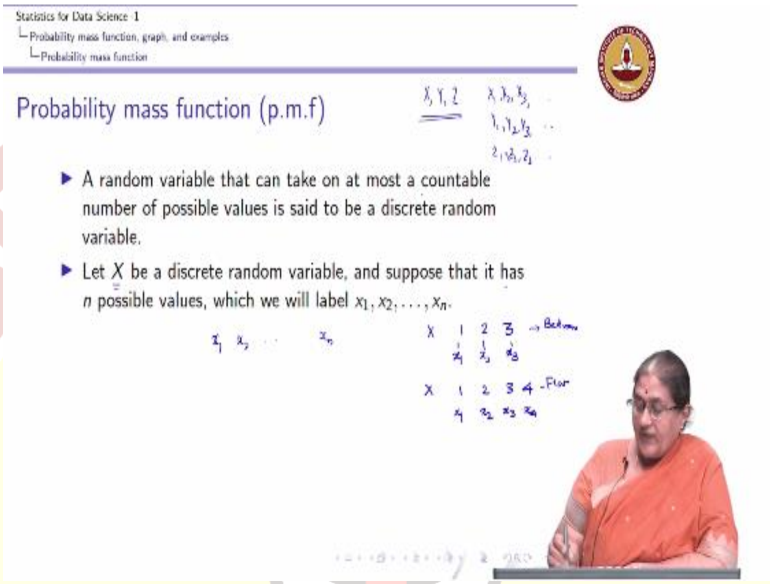

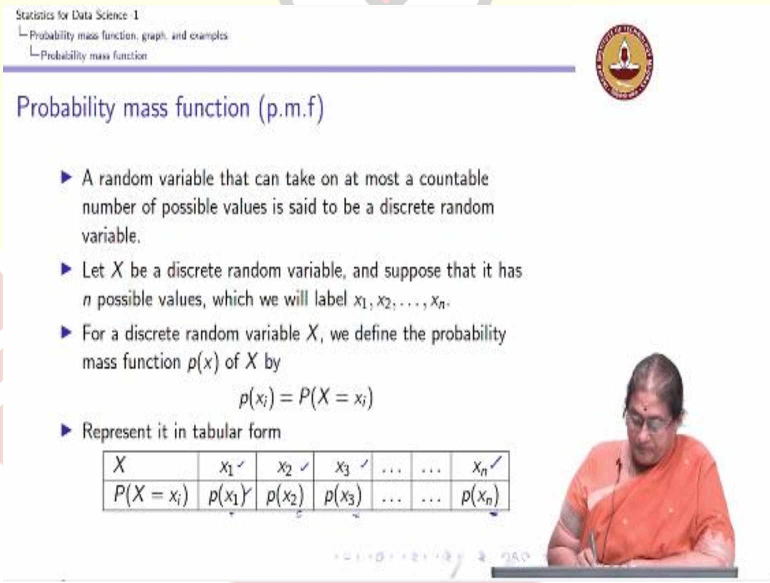

So, we continue our discussion on discrete random variables. Again recall a discrete random variable is a random variable that can take on at most a countable number of possible values. We refer to it as a discrete random variable we are now going to focus only on discrete random variables. Again when we look at discrete random variables let I denote a random variable with the upper case $X, Y, Z, X_1, X_2, X_3, Y_1, Y_2, Y_3$ and so forth or Z_1, Z_2, Z_3.

This is typically we represent a random variable through upper case alphabets capitals X, Y, Z. So, now let X be a random variable let it take a finite number of values. Now what do I mean by it takes a finite number of values. It takes n possible values. Let me represent that by $x_1, x_2, \dots x_n$. For example, if X takes the values 1, 2, 3 my x_1 would have been 1, x_2 would have been 2, x_3 would have been 3.

If X takes the value 1, 2, 3, 4 as in the case of the number of floors. This was the number of bedrooms, this is the floor, then x_1 is $1, x_2$ is $2, x_3$ is taking the value 3 and x_4 is 4. So, in general I can talk of this random variable X taking values $x_1, x_2, \dots x_n$ that is what I mean by X is taking n finite values $x_1, x_2, \dots x_n$. Once I know x_1, X takes these values we define the probability mass function. How do I define it?

This is the function for all the values what are the values X is taking X is taking values $x_1, x_2, \dots x_n$. So, associated with every value X takes I know there is a probability associated with it. So, I have what is a probability of X taking the value x_1 I know what is the probability of X taking the value x_2 and X taking the value x_n. This function the probability of X equal to x_i for each of these values. This function is referred to as the probability mass function of the random variable.

A nice way to represent it is in tabular form. So, X takes the values x_1, x_2, x_3 up to x_n the probability is with probability X equals to x_1 $is\ P(x_1),\ P(x_2),\ P(x_3)$ up to $P(x_n)$.

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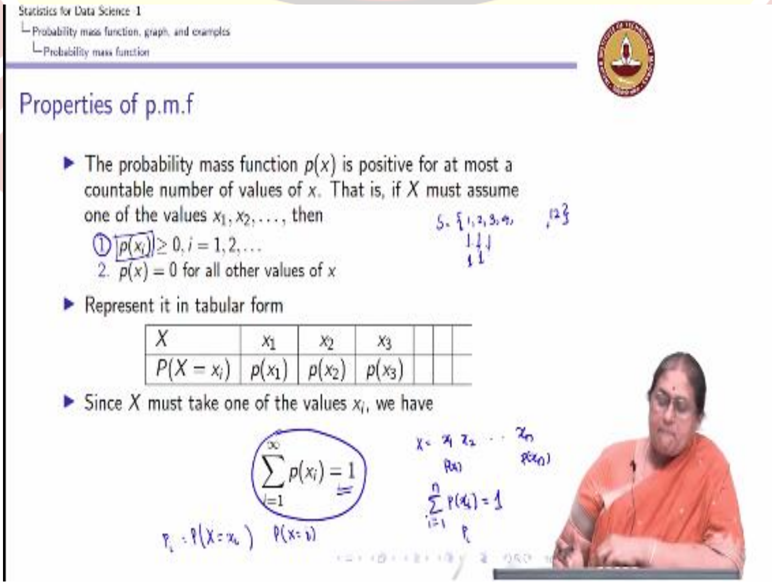

So, let us look at a very simple example. So, here we assume that this X takes finite number of values so X takes $x_1, x_2, \dots x_n$. possible values, but I could also have the case that X assumes value x_1, x_2, x_3 countable, but infinite. Again associated with each of these exercise I will a $P(x_1)$, I will have a $P(x_2)$, I will have a $P(x_3)$ so forth. So, whenever I talk about the discrete random variable, I could either have countably finite or countably infinite number of values.

But nevertheless whatever it is there are two key properties of the probability mass function. I repeat there are two key properties of the probability mass function. They are namely $P(x_i) \geq 0$. In other words $P(X = x_i)$ is always non negative and the second property is that some probability of X is equal to 0 for all values of x. So, if I represent it in tabular form I have $P(x_i) \geq 0 \; \forall i$ and the next property is since x I know that every point in my sample space.

For example, if my sample space in the apartment complex was 1, 2 up to 12. Every point was mapped to a random variable we also know from the axioms of probability the $P(S) = 1$. Since every point is mapped on to a random variable, it makes sense for us to say that the summation over all possible values X can take the summation of the probabilities should add up to 1. So, the two key properties are probability of x_i is now negative and the summation of over all possible values of x should be equal to 1.

Now if x takes only finitely many values with probability $P(x_1)$, $P(x_n)$, then I know $\sum_{i=1}^{n} P(x_i)$ should be equal to 1. Some books refer $P(x_i)$ with just P_i, but we need to understand that whenever you see a P_i this could be probability X takes the value x_i or probability X takes a value i. You need to understand how the probability mass function is defined, but nevertheless however you are defining the probability mass function the probability of x taking a particular values is always now negative and the sum of probabilities over all possible value should be equal to 1. These are the key properties of the probability mass function.

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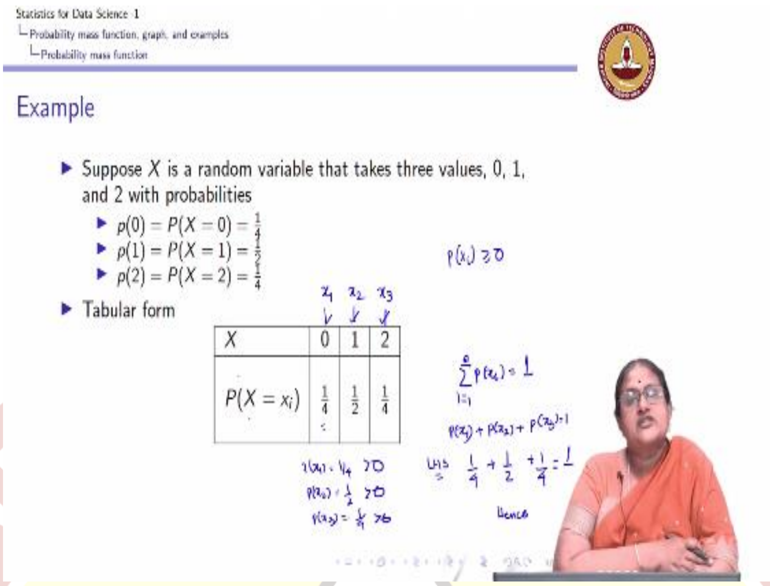

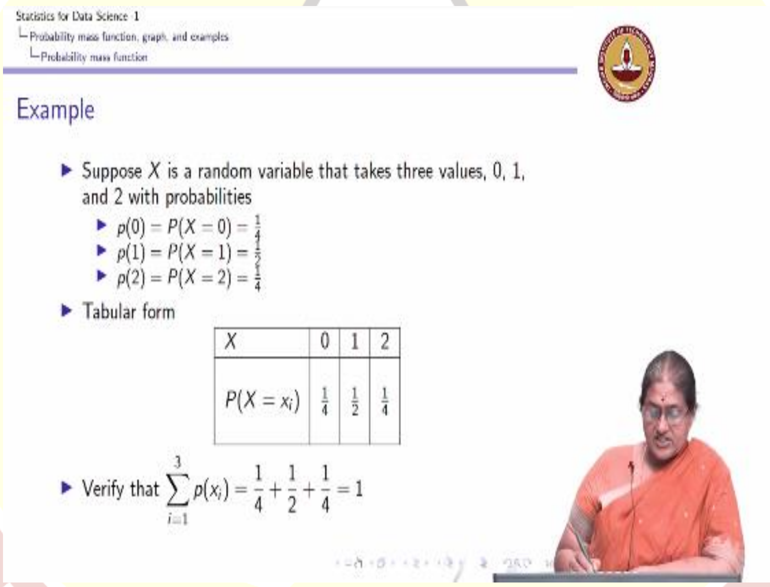

Now let us look at an example. Suppose X is a random variable that takes 3 values. So again I have a random variable which takes only 3 values it is finite 0, 1, 2 and what is the probability? $P(X = 0)$ is $\frac{1}{4}$, X is equal to 1 is half and X equal to 2 is $\frac{1}{4}$. Tabularly I can represent it as x takes the value 0, 1 and 2 so my x_1, x_2, x_3 x_1 takes 0, 1, 2 with $P(X = x_1)$ which is a 0 is $\frac{1}{4}$. X equal to x_2 which is a probability x equal to 1 is $\frac{1}{2}$ and X equal to x_3 which is X equal to 2 is $\frac{1}{4}$

So, what is the first property I need to see. $P(x_i) \geq 0$. I know $P(x_1)$ is $\frac{1}{4}$ this is greater than 0 $P(x_2)$ is $\frac{1}{2}$ which is greater than 0, $P(x_3)$ is $\frac{1}{4}$ which is also greater than 0. The second property is $\sum_{i=1}^{n} P(x_i)$ should be equal to 1. In other words I need to verify whether $P(x_1) + P(x_2) + P(x_3) = 1$. This is what I need to verify. I can see that $P(x_1)$ is $\frac{1}{4}$, $P(x_2)$ is $\frac{1}{4}$, $P(3)$ is $\frac{1}{4}$ I can verify that this is equal to 1. Hence, what we have here is a probability mass function of the random variable.

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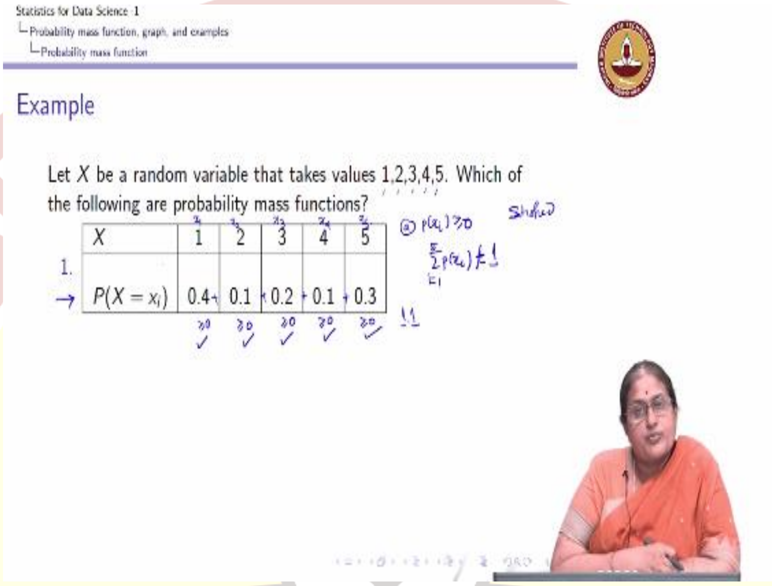

Let us look at certain more examples to understand the properties of the probability mass function. Now suppose I have X is a random variable that takes values 1, 2, 3, 4, 5. So I have this is my x_1, x_2, x_3, x_4, x_5. Again it takes finite number of values with the given probabilities. So, is this a probability mass function? So, the first condition is I need to check $P(x_i) \geq 0$. Yes, for this done this is greater or equal to 0 this greater or equal to 0, this greater or equal to 0 this is also greater or equal to 0. The next condition so the first condition is satisfied.

Now the second condition is I need to check whether $\sum_{i=1}^{5} P(x_i)$ is equal to 1. So, I have a 0.4 + 0.1 which is a 0.5. 0.5 + 02 which is a 0.7, 0.7 + 0.1 which is a 0.8, 0.8 + 0.3 which is 1.1 so I get the $\sum_{i=1}^{5} P(x_i)$ is not equal to 1. Hence, this does not satisfy the probability properties of a probability mass function. Hence, the first table is not a probability mass function.

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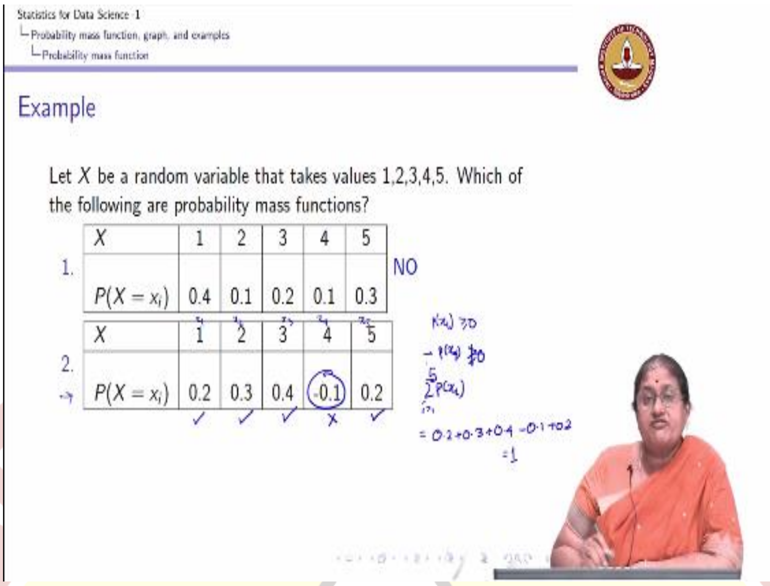

Now, let us look at the second example again X is taking the value 1, 2, 3, 4 and 5. So, the first property $P(x_i) \geq 0$ here it is yes, yes, yes. This is a no, this is a yes. So, you can see that $P(x_4)$ is not greater or equal to 0. However, if you notice $\sum_{i=1}^{5} P(x_i)$ which is equal to 0.2 + 0.3 + 0.4 - 0.1 + 0.2. You can see that this is 0.9 which is equal to 1. So, I have a situation where the function satisfies the second property, but not the first property. Hence, because of this it is not a probability mass function.

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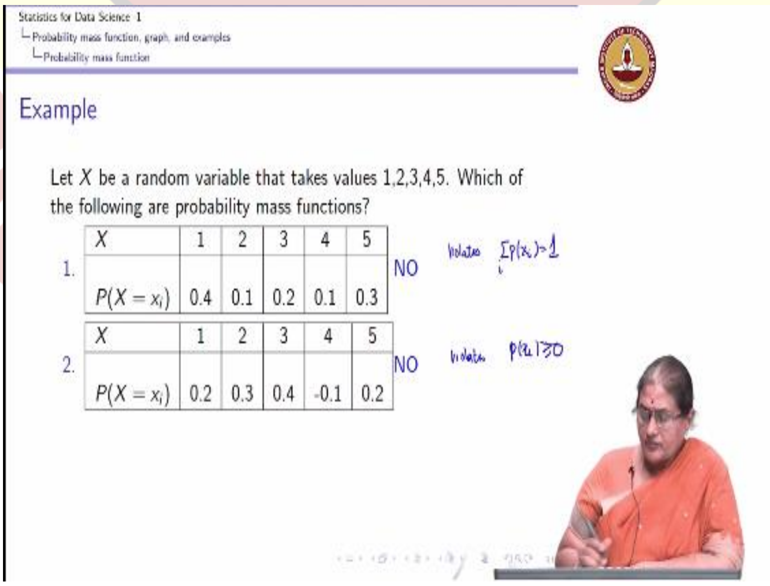

So, here it violates $\sum_{i=1}^{n} P(x_i)$ =1 over all values of i. Here it violates the $P(x_i) \geq 0$.

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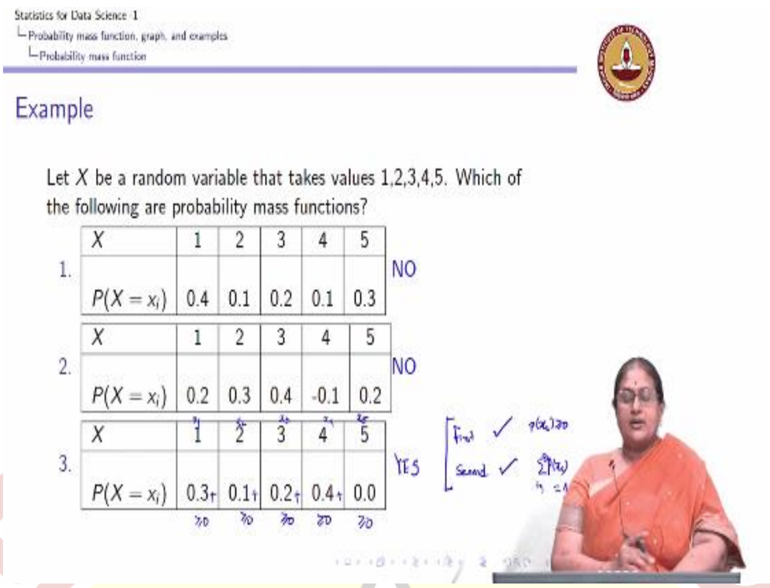

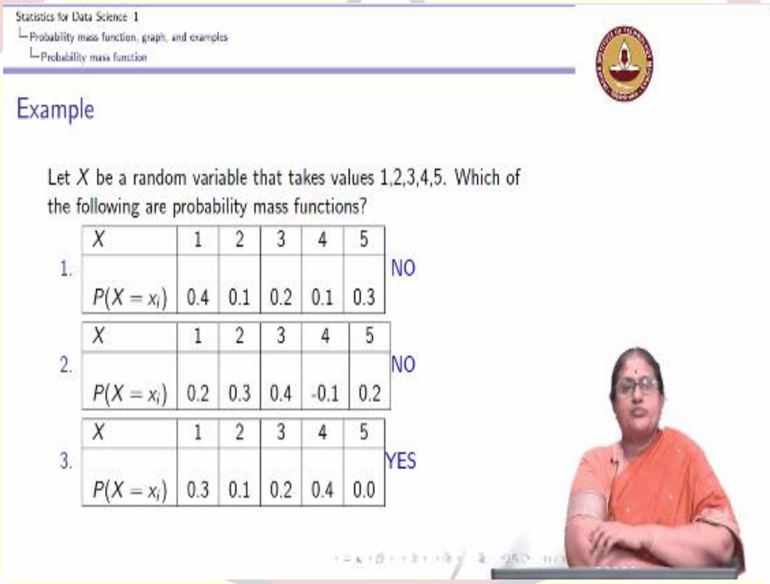

Now, let us look at a third example. In this example again X takes the value $x_1, x_2, x_3, x_4,$ and x_5 with these probabilities. Again, let me check the first condition greater or equal to 0, greater or equal to 0, greater than 0, greater or equal to 0, greater or equal to 0. First condition is satisfied first condition is $P(x_i) \geq 0$ satisfied. Now let us check the second condition which is $\sum_{i=1}^{5} P(x_i)$ 0.3 + 0.1 0.4, 0.4 + 0.2 0.6, 0.6 + 0.4 1, 1 + 0 is 1. Yes it is equal to 1. So, both the conditions are satisfied hence this is a probability mass function.

So, the first thing which we need to understand is given a random variable again I specify we are looking only at discrete random variables. So, given a random variable X that takes the values $x_1, x_2,$ up to x_n. For the first case I considered countably finite number of values if I am talking about a probability mass function I need the two properties. And what are the two properties we are looking at when we are talking about a probability mass function?

The first is $P(x_i) \geq 0$ and the second is summation probability of x_i over all possible values of x should be equal to 1. Now, let us look at another example where x does not take finite number of values. It takes countable number of values, but not finite.

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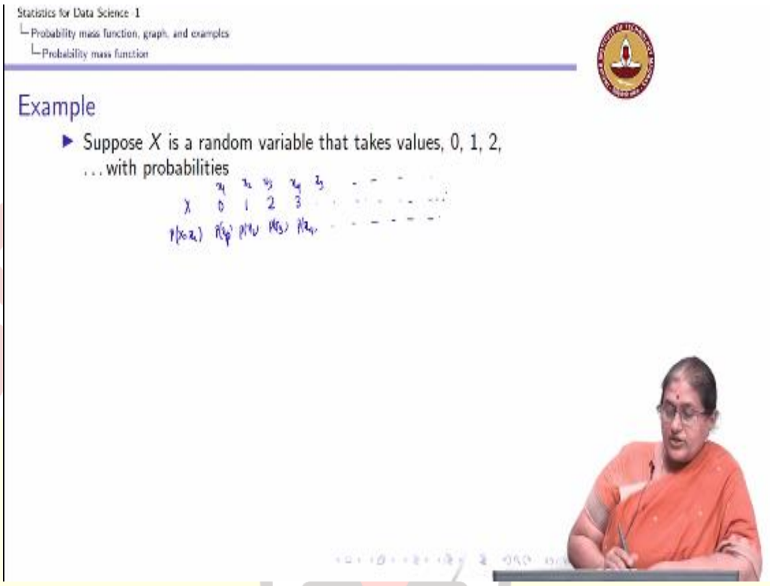

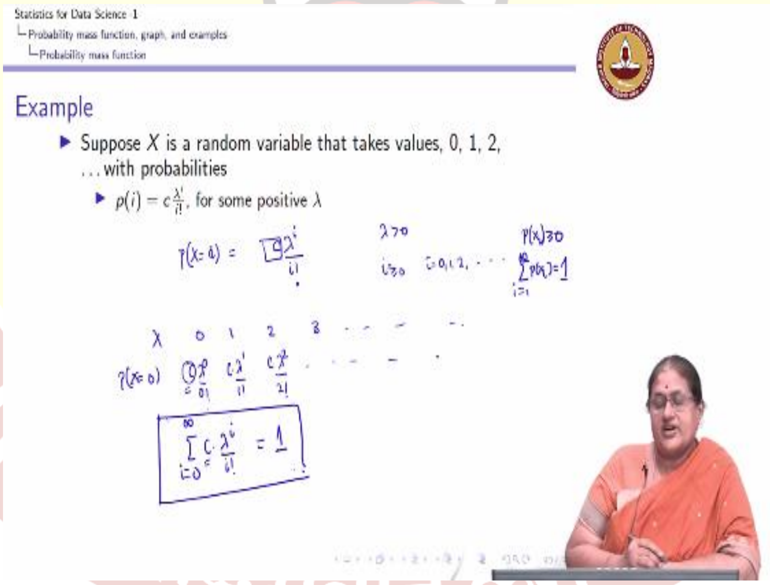

So, let us look at this example. So, what are the values this random variable is taking? X is taking the value 0, 1, 2, 3 it keeps taking these values. Let me write $P(X = x_i)$, probability of x_0 or x_1, x_2 probability of x_3, probability of x_4 so this is my x_1, x_2, x_3, x_4 so X is taking these values I am not having finite number of values. Now what is it the $P(X = i)$ is given by $\frac{c\lambda^i}{i!}$.

Now $\lambda$is positive $i \geq 0$ because i takes the value 0, 1, 2 all of it. So, I want to know for what value of c will this be a probability mass function? Again what are the values $X$is taking Xtakes the value 0, 1, 2, 3 so forth $P(X = 0)$ would be $\frac{c\lambda^0}{0!}$. This would be $\frac{c\lambda^1}{1!}$, this should be $\frac{c\lambda^2}{2!}$and so forth.

So, what value of c would make this a probability mass function? So, the first thing is I know λ is positive and i is positive. So, c has to be greater or equal to 0 because again what are the conditions? The conditions of $P(x_i)$ should be greater or equal to 0 and $\sum_{i=1}^{\infty} P(x_i) = 1$.

For the first condition I need c to be non negative because everything else is going to be now negative. For the second condition I need to check $\sum_{i=1}^{\infty} \frac{c\lambda^i}{i!} = 1$. So, what value of c would give this, this is what we need to check because for this to be a probability mass function I need this condition to be satisfied.

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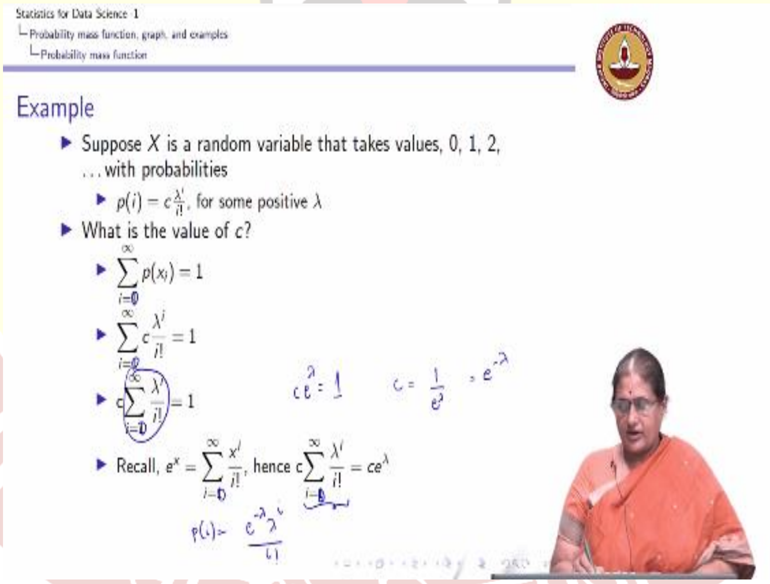

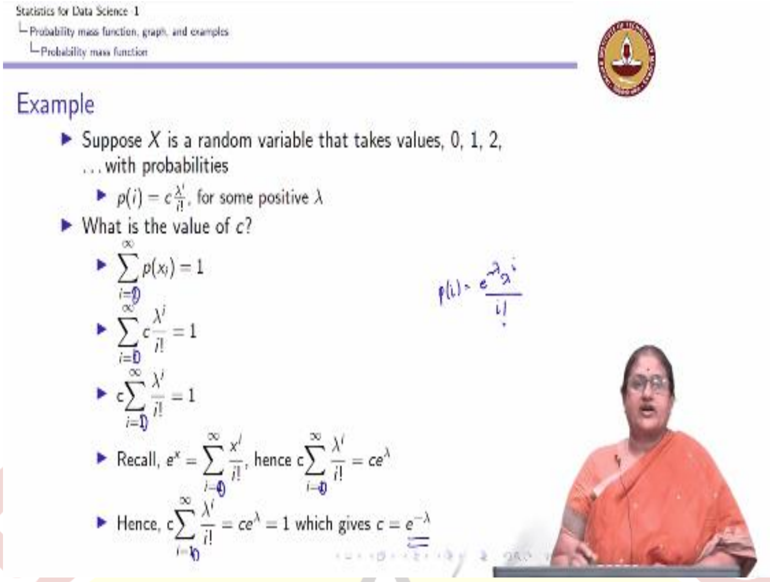

So, the question is what is the value of c for which this is a probability mass function. So, this is i goes from 0 to infinity. This is again i goes from 0 to infinity. So I know it is again 0 to infinity here so i goes from 0 to infinity $c\lambda^i$ because X takes the value 0, 1, 2. I need to look at i goes from 0 to infinity. Now we all know the following that e^x is so we all know the following that e^x is $\sum_{i=1}^{\infty} \frac{x^i}{i!}$.

Hence, I have summation $\sum_{i=1}^{\infty} \frac{c\lambda^i}{i!}$ is ce^λ because this would be e^λ. So, now I have from here ce. So, this term is going to be e^λ that is what this tells us that $\sum_{i=1}^{\infty} \frac{x^i}{i!}$ is e^x. So, this term would be e^λ. So, I get $ce^\lambda = 1$ which should give me $c = \frac{1}{e^\lambda}$ or $e^{-\lambda}$.

Hence, I get my $P(i)$ is $\frac{e^{-\lambda}\lambda^i}{i!}$. So, I can get my c to be $e^{-\lambda}$ which is going to give me the solution that $P(i)$ is $\frac{e^{-\lambda}\lambda^i}{i!}$.

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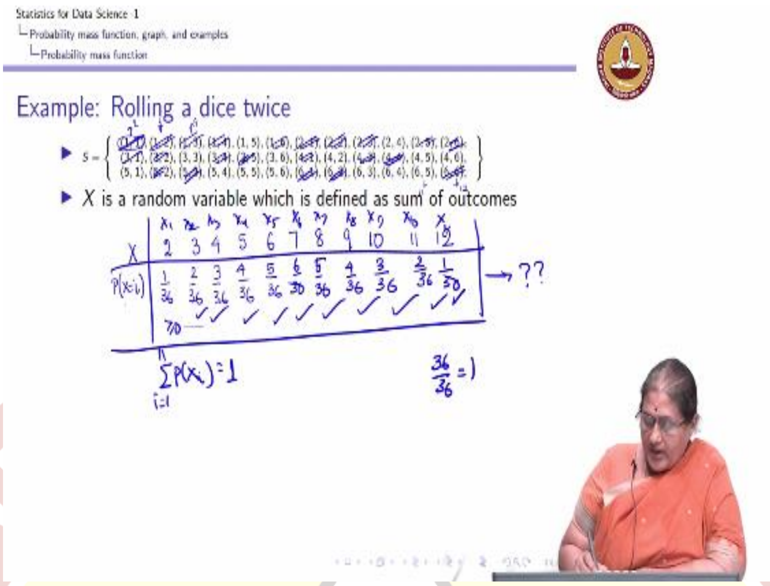

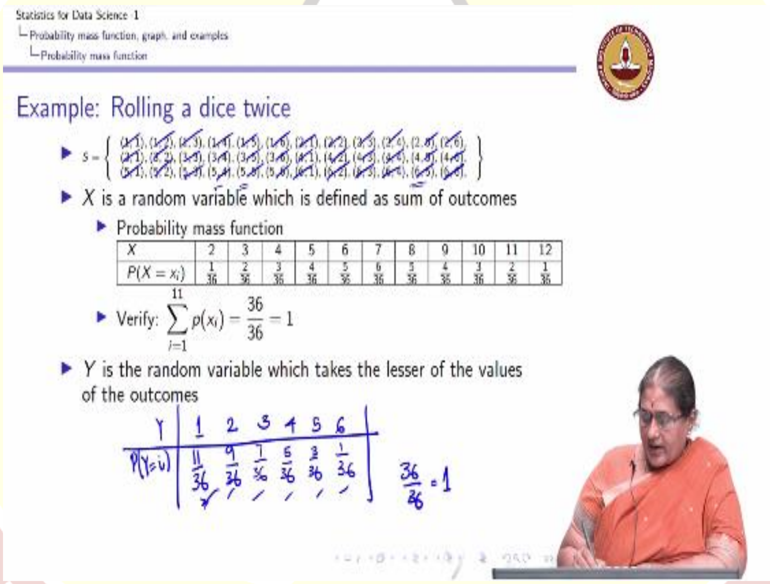

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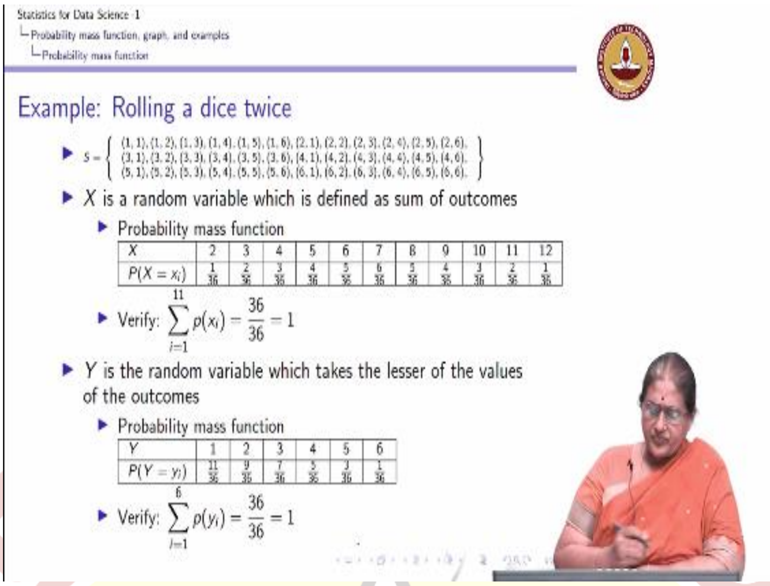

Now let us go back to the examples which we have discussed earlier. So, we looked at the first example we looked at was rolling a dice twice. We know that the sample space has 36 outcomes and I have listed down the 36 outcomes here which are (1, 1) to (6, 6). Now let us define the random variable which is defined as a sum of outcomes. Again, we have seen this in the earlier lecture that every outcome is mapped to a particular values.

So, you can see that (1, 1) is mapped to 2; (1, 2) is mapped to 3; (1, 3) is mapped to 4; (6, 6) is matched to 12; (6, 5) is mapped is 11 the sum of outcomes. So, I can see that this random variable X takes the values 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 and 12. These are the values this random variable take X takes the value 2, (1, 1) is the outcomes which maps to it X takes the value 3 (1, 2) and (2, 1) are the outcomes that give the value 3. (4, 1), (3, 2), (2 , 3), 1 so each one of them you can see is mapped 12 is from the outcome (6, 6).

Now again associated with each one of these values $P(X = i)$. So, associated what is the $P(X = 2)$? I know only outcome gives this value so it is $\frac{1}{36}$ (1, 2) and (2, 1) here so this is $\frac{2}{36}$, X equal to 4 comes from (1, 3); (2, 2) and (3, 1) $\frac{3}{36}$, X equal to 5 I have (1, 4), I have (2, 3) I have (3, 2), I have (4, 1). So, it is $\frac{4}{36}$ this should we can check is $\frac{5}{36}$.

X equal to 7 is (1, 6); (2, 5); (3, 4); (4, 3); (5, 1) and (5, 2) and (6, 1) which is $\frac{6}{36}$. Similarly, this would again 8 I know 8 would come from a (6, 2); (5, 3); (4, 4); (3, 5) and a (2, 6). So, again this is $\frac{5}{36}$ you can verify for all other values this is going to be my probability mass function. So, I know X takes the values 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 with the probability this one.

Now is this a probability mass function? Again what are the properties of the probability mass function? This should be greater or equal to 0 I can see each one of them is greater or equal to 0. The second thing we need to verify is the sum of the all probabilities i going from all x_i so this is $x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}$ and x_{11} i going from 1 to 11 should be equal to 1. 3 + 3 6, 10, 15, 21, 26, 26 plus again you can see + 10 is 36 so you can see that this is $\frac{36}{36}$ which is equal to 1 and I can see that this indeed is a probability mass function.

Now let us go to the second random variable which we defined. Again we defined the second variable which takes the lesser of the values of the outcomes. So, again I have Y takes the value 1 for these outcomes. So for all these outcomes Y takes the values 1, Y takes the value 2 for these outcomes, it takes the value 3 for these outcomes, it takes the value 4 for these outcomes, it takes the value 5 for these outcomes and it takes the value 6 for this outcome.

So, this is $\frac{1}{36}$ 5, 1, 2, 3 so this is a $\frac{3}{36}$ 4 would be $\frac{5}{36}$ for 3 it was $\frac{7}{36}$, 2 it was $\frac{9}{36}$ and 1 11 out of these outcomes $P(Y = i)$. So, I know Y takes these values with the respective probabilities. Again is this a probability mass function. Let us verify I know all of them are greater than or equal to 0. So, the first property is satisfied the second property 11 + 9 20, 20 + 7 27, 32 35, 36. It is actually $\frac{36}{36}$ which is equal to 1. So, I have both the properties are satisfied and hence this also defines a probability mass function.

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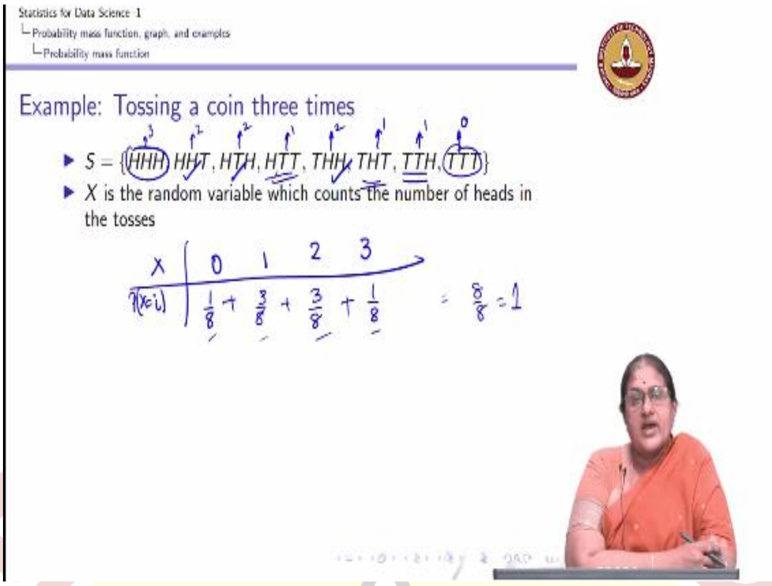

Now let us look at the next example of tossing a coin 3 times. Now when I toss a coin 3 times again this is my sample space. Again we define X to be the random variable that counts the number of heads in the toss. Again this outcome it is 3 here I have 2, here I have 2, here I have 1 again 2 I have 1 head, 1 head and no head. So, X takes the value 0, 1, 2 and 3 with what probabilities.

The probability with X takes no head it corresponds to this outcome it is $\frac{1}{8}$, 3 heads corresponds to this outcome which is again $\frac{1}{8}$. One head it corresponds to this, this and this so it is $\frac{3}{8}$, 2 corresponds to this, this and this outcome which is again $\frac{3}{8}$. Now is this a probability mass function? All of them are greater or equal to 0 and the sum of the probabilities which is equal to 1 + 3 + 3 + 1 is $\frac{8}{8}$ which is equal to 1. So, hence it is indeed a probability mass function.

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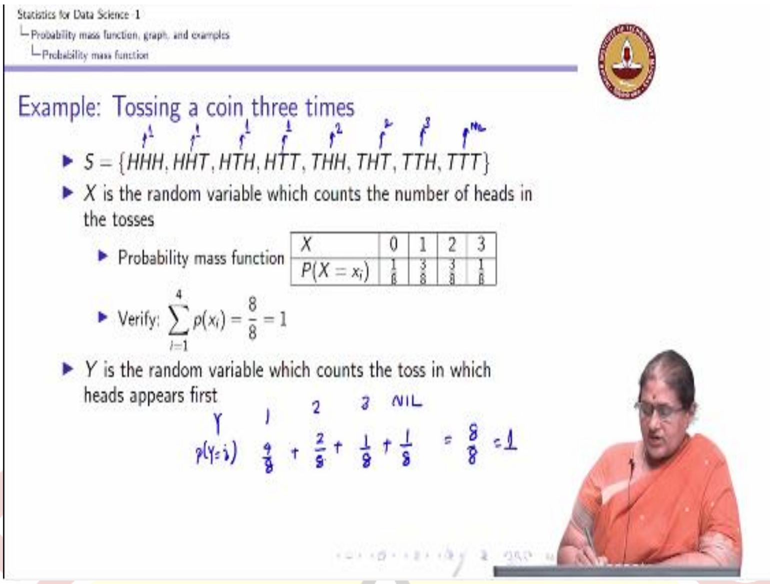

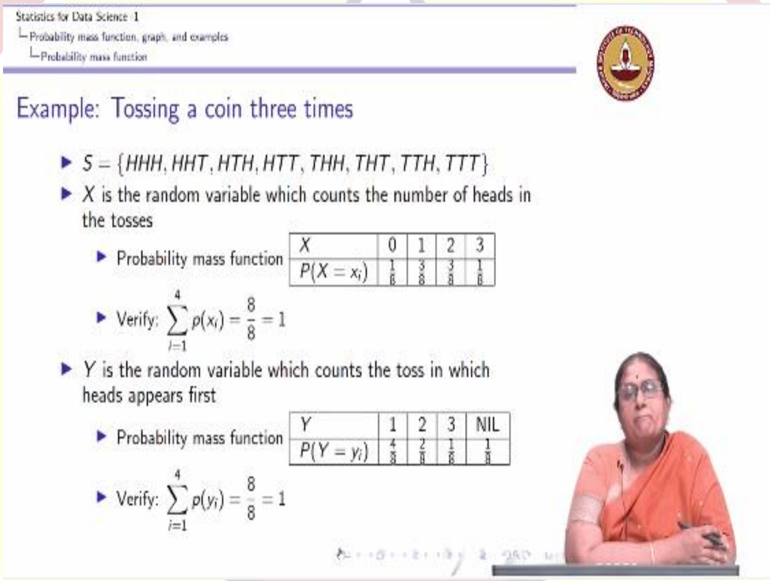

So, the next thing which we can look at is the random variables which counts the toss in which head appears first. Again, here head appears first in the first toss again here it appears in the first toss here it again appears first in the first toss, here it appears in the first toss, here it appears in the second toss this is again second toss, this is third toss and this is the nil toss. Remember, we defined it as nil we did not define it as 0.

So, the values Y takes are again 1, 2, 3 and nil and the probability with Y takes those values. It takes the value 1, 4 out of 8, 2 out of 8, 3, 1 out of 8 and nil 1 out of 8 all of them are greater or equal to 0. The second property I need to check whether they add up to 1 which is $\frac{8}{8}$ which is equal to 1 and I can see that this indeed is again a probability mass function.

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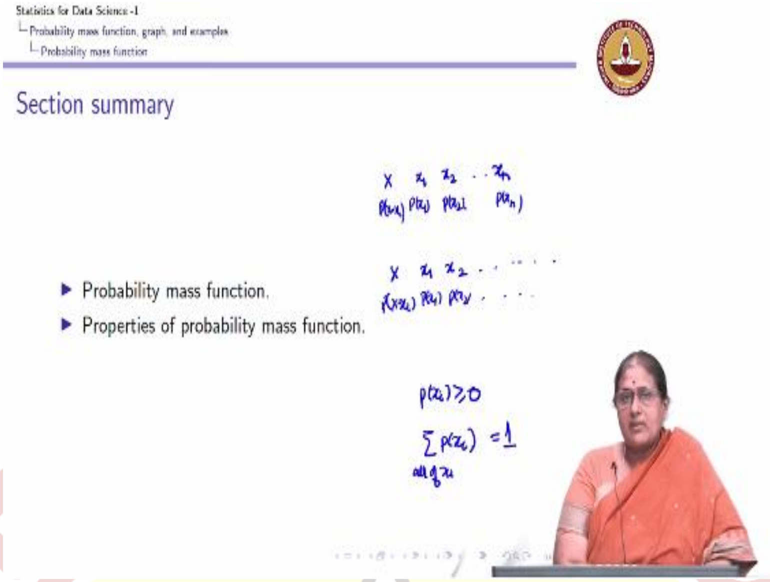

So, by this time what you have to understand is what is the probability mass function. Again remember we are talking only about a discrete random variable. So, what we are looking at is we have defined a discrete random variable which can take finite number of values or it can take countably infinite number of values with $P(X = x_i)$ which is $P(x_1),\ P(x_2), \dots\ P(x_n)$.

Again, same thing when it takes countably infinite number of values $P(x_1),\ P(x_2)$ say it is a probability mass function if each of the $P(x_i) \geq 0$ and summation over all possible values of x_i should be equal to 1, then it is a probability mass function. So, the next is can I graph this probability mass function. So, that is what we are going to look next.