INDIAN INSTITUTE OF TECHNOLOGY MADRAS

सिद्धिर्भवति कर्मजा

IIT Madras

ONLINE DEGREE

Mathematics for Data Science 1
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Lecture - 32
Algebra of polynomials: Addition & Subtraction

In this video, we will start with polynomials and we will try to do some Algebra with Polynomials. Or in other words you can say we will try to understand some operations on polynomials like Addition and Subtraction. So, let us move on.

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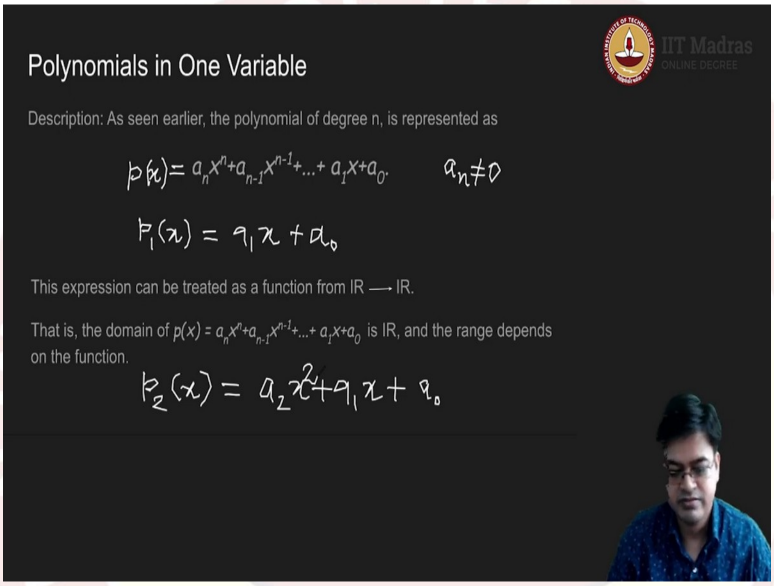

In order to simplify our calculations, we will only focus on polynomials in one variable; whereas all the operations that we are discussing can be done on polynomials with multiple variables. In order to pinpoint the thing, we will recollect how polynomials in one variable look like.

So, a polynomial of degree n in one variable can be represented in this form; $a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$. So, you can actually correlate this with, let us say this is a monomial with 0 degree, this is a monomial with 1 degree and so on, if you go on this way, this is the monomial with n th degree. In order that this polynomial to qualify as a polynomial with n th degree, we need something, we need one condition; that condition is actually I need this to be a polynomial of the term of degree n to be non-zero.

So, that forces me to write $a_n \neq 0$, this is a condition that require that is required for writing a polynomial of nth degree. Remember here the argument is only one that is the variable is only one x. So, I can also assign this as something called $p(x)$, and now you can as well treat this $p(x)$ as a function of one variable which is interesting.

So, if you assign this as a function of one variable, the next question is; how is this function, what is the domain and co-domain and range of this function? So, the function runs from $\mathbb{R}$ to $\mathbb{R}$. So, it is a function from real line to real line; whereas the range typically depends on function.

For example, if I take a function like let us say $p_1(x)$ is one function, which is $a_1 x + a_0$ and if I take this function, then it is a linear function; we have already seen this function, this is an equation of a real line, equation of a line. And if a 1 is not equal to 0, then this function actually represents a real line. If $a_1 \neq 0$, it also represents a real line, but it is some constant.

So, it is a horizontally real line. So, now, the range of this function for $a_1 \neq 0$ is entire real line. Whereas if you look at some other function, let us say $p_2(x) = a_2 x^2 + a_1 x + a_0$. Now, this particular function represents a parabola, which we have seen in our topic on quadratic functions.

And you know depending on the sign of a_2 , the parabola can open upward or downward; if it opens upwards, the range is the minimum value and any point beyond that; if it opens downwards, the range is the maximum value and any point below it. So, depending on the choice of the function, the ranges may differ. We will deal with polynomials as function when we will study the graphing of polynomials.

Right now, we are interested in algebraic properties of this polynomial. So, we will focus ourselves on the algebra of the polynomials that is addition, subtraction, multiplication, division.

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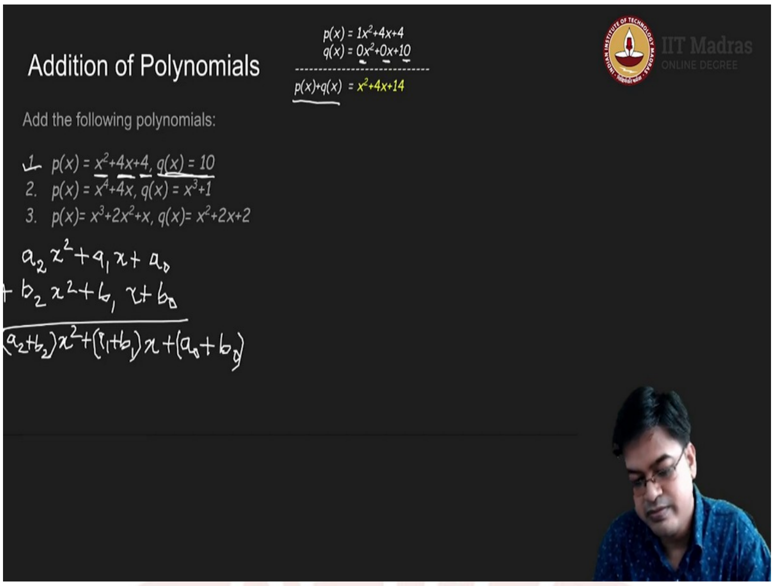

So, let us move ahead and try to understand addition of polynomials. We have done this in some sense, for example, currently the polynomial that we have written that $a_n x^n + a_{n-1} x^{n-1} + \dots$, is also addition of some kind; but it is addition of monomials. So, let us try to see, if I have been given to two polynomials, how will I add them?

To help us in understanding and developing general theory for addition of polynomials, we will consider these three examples. Remember, the first example is actually one polynomial added with another monomial; second one both are two polynomials, but there are no clashing terms, like they the exponents are different for both the polynomials, you can check and the third one has few clashing terms.

So, we will demonstrate the addition of polynomials through these three things and we will formalize this into a theory. So, let us start with the first expression, $p(x) = x^2 + 4x + 4$. So, we are starting with this, this particular expression. So, $p(x)$ is x^2 . So, if I am writing x^2 , then it essentially means I am multiplying this with 1, the coefficient is 1; if I am starting with $4x$, then I do not have to do anything and this is 4.

So, in this case in our standard form for a quadratic polynomial, what is a standard form for a quadratic polynomial? $a_2 x^2 + a_1 x + a_0$ this is our standard form. So, in this particular thing, you can identify $a_2 = 1, a_1 = 4$ and $a_0 = 4$.

In a similar manner, I will look at this particular expression which is $q(x)$. Now, you notice the fact that $q(x)$ is just a constant polynomial, $q(x)$ do not have any terms which are related to square or related to a linear term. And, I want to add this polynomial to a given expression.

So, how will I add? So, let us bring in the terms related to square term and related to linear term; if I bring in those terms, the associated coefficients will be 0 right, the associated coefficients will be 0. So, I can write this term as $0x^2+0x+10$.

Now, because of this, let us write it in a generalized setting; $b_2x^2+b_1x+b_0$, right. So, now, I am trying to add these two polynomials. So, what is our recipe? We will consider the terms with like powers that is like exponents, ok. So, let me try to add the things.

So, if I consider this, this particular expression that is given here. So, I have $1x^2$, 1 minute. So, let me bring in my mouse pointer here. So, I have $1x^2+0x^2$. So, 1 + 0, I will get again singleton x^2; then 4 + 0 that will give me $4x$, 4 +10 will give me 14. So, essentially I can see that this expression should have a formulation which is of the form $x^2+4x+14$.

So, if I now try to do it in a more general settings, then how will I compare with this general setting. Let us see it here. So, I want to add these two. So, just add. So, in a similar manner, if I add these two; what I am getting is $(a_2+b_2)x^2+(a_1+b_1)x+a_0+b_0$, just to remember this format. So, what I am writing here is essentially.

Another point that to note with this example is; the first one was a polynomial of degree 2, the second expression $q(x)$ was a polynomial of degree 0. Now, the resultant expression that is $p(x)+q(x)$, what is the degree of this polynomial? It is a polynomial of degree 2. So, it is the maximum of degree of the first polynomial and degree of the second polynomial. So, we have roughly understood the settings that we need maximum of 1 and 2, let me write the findings in a different way.

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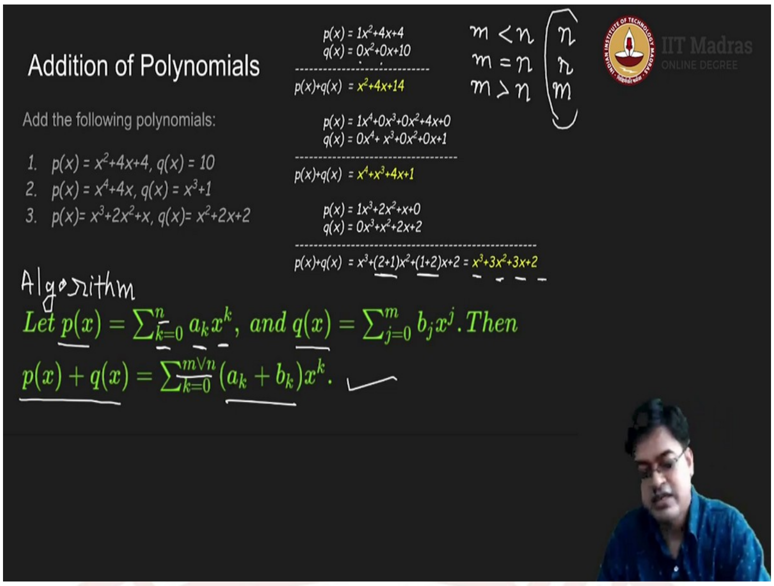

So, if I have a polynomial of degree m and let us say if I have a polynomial of degree m and degree n, these are the two polynomials; and if they satisfy a relation that m is less than n, then the resultant polynomial will have a degree n. If I have a polynomial where degree m is equal to n ok, then the resultant polynomial will still have a degree n.

And if I have a case where m is greater than n, then the resultant what will be the; so if we switch this $q(x)$ to $p(x)$ and $p(x)$ to $q(x)$, this case will happen and the resultant polynomial will have degree m. So, just remember this in mind; that means it is always maximum of m and n, if the resultant polynomial is having polynomials underlying polynomials with different degrees. So, with this understanding, let us attack the second problem.

So, the second problem has $p(x)$ which is x which is a polynomial of degree 4. So, I have written all other terms which were not there in the polynomial by multiplying with 0. In a similar manner, I have written the second polynomial $q(x)$ which is a polynomial of degree 3; but we want the maximum degree to survive right or is essentially in this expression the maximum degree will survive, therefore I have this kind of expression.

So, the resultant polynomial we know from our discussion will be a polynomial of degree 4, and therefore I need to consider all the terms that correspond to each of the degrees. So, what is that term corresponding to degree 4? In the first expression that is

$p(x)$ is 1, the coefficient is 1 and the term corresponding to degree 4 in the second polynomial that is $q(x)$ is degree 0.

So, I will get 1 + 0, which is 1; $1x^4$. In a similar manner you can see, for x^3 it is $1x^3$; x^2 there is no survivor both are 0, so $0x^2, 4+0x$ and 0 + 1 times 1. So, it is 1. So, the resultant that you are interested in is x^4+x^3+4x+1 . Again I will reiterate, this time it will be; if you consider a generalized polynomial, it will be $a_4 x^4+a_3 x^3+a_2 x^2+a_1 x+a_0$.

And in a similar manner $q(x)$ will $b_4 x^4+b_3 x^3+b_2 x^2+b_1 x+b_0$. And if you sum over them, what we have written in yellow is essentially sum of $a_4+b_4=1, a_3+b_3=1, a_1+b_1=4, a_0+b_0=1$; a_2 and b_2 will sum to 0, because it does not have any non-zero coefficient. So, this is how we will handle the third, this is how we have handled the second problem.

And the term containing the highest degree survive, therefore the degree of polynomial is 4, ok. So, let us go back to the third problem, let us come ahead and solve the third problem; $p(x)$ and $q(x)$, again similar setting highest degree is degree 3. So, the term corresponding to degree 3 will survive. So, the polynomial with lower degree, I will bring it to degree 3. So, essentially, I will multiply with coefficient 0 for a degree 3 term.

Again by same logic, I will add the two terms and therefore, I will get the corresponding answer. So, there were cross terms, like the term corresponding to x^2 was crossing; for example, both polynomials had terms corresponding to x^2 .

So, you can see the difference here, we are just adding 2 + 1, 1 + 2. So, all these things are happening and together we are writing the result x^3+3x^2+3x+2 . So, from this we can derive, if you are clear with these three examples then we can derive a general formula; otherwise pause and look at each of the terms, you will be able to understand the general formula in a bit better manner if you pause and review these additions.

So, let us come to the general formula, you must have paused and understood the additions. So, if I have a polynomials of the form $\sum_{k=0}^{n} a_k x^k$. And if you assume that $a_n \neq 0$, then this is a polynomial of degree n.

In a similar manner you have taken a second polynomial $q(x)=\sum_{j=0}^{m} b_j x^j$. Now it does not matter whether m is greater than n or m is less than n, this particular thing will give you the answer, ok. What is the resultant? So, if I want to add these two polynomial functions $p(x)$ and $q(x)$, then $p(x)+q(x)$ will essentially show this kind of representation.

So, what we are actually saying is, choose which one is the maximum m or n ok? Whichever is the maximum? Take that maximum. So, it is maximum of m and n ; sum will run from k is equal to 0 to n. Let us say for argument purposes this is the highest degree that is n is the highest degree, then match the degree of the highest degree for other coefficients, for example, for j equal to $m+1$ to n put all $b_j's$ to be equal to 0.

If you do so, this is what we have done here. So, in this case the first degree was 2 and the second degree of was 0. So, in this case we matched the degree and substituted all the coefficients here to be equal to 0. So, you do a similar thing over here in general and then just add the coefficients $(a_k+b_k)x^k$, and this should give you the final answer. So, this is in fact an algorithm for adding the polynomials, so this is algorithm for adding the polynomials.

What is the, what are the steps in the algorithm? First identify degrees of both polynomials, choose the polynomial with highest degree that will be the degree of the resultant polynomial. Take the polynomial of least degree that is step 2, take the polynomial of least degree, add all the coefficients which are of the degree higher than the polynomial and multiply them with coefficients 0.

Once you do that you are ready to do the addition, add the two using this formulation. So, this is how you can program, you can actually program into a computer for addition of polynomials. Now, let us try to understand this with subtraction. What is the difference between subtraction and addition? Both are essentially same, but in subtraction you are multiplying the second polynomial by -1.

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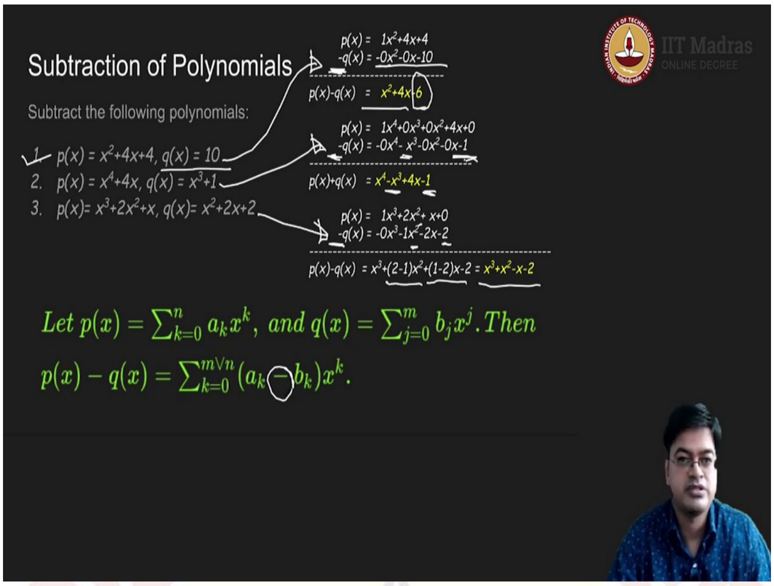

We have already seen how subtraction will happen. So, here is a quick overview of these examples. So, this is, these are the polynomials, same polynomials now we are subtracting. And what we are doing by subtracting? I mean we have multiplied with -1, just look at here all these terms.

The procedure is exactly the same, it is just that first we have to multiply by -1 and put the polynomial appropriately. So, let us start with first example, but it will be a quick run, because $p(x)$ is this. So, there is no change, but I want to subtract $p(x)$ from, I want to subtract $q(x)$ from $p(x)$. So, this polynomial will be multiplied with -1 that is what is done here, so $-q(x)$.

So, correspondingly all coefficients are negated, just look at these terms; all coefficients are negated and therefore. Because there were no cross terms, so you will not find any difference in the first two terms; but the significant difference is there in the third term which is actually -6. In a similar manner, take the second question and you are multiplying $q(x)$ with -1.

So, $-0x^4-x^3-0x^2-0x-1$, right. Again because there were no cross terms, there were no additions. So, this also will have a minimal effect where the second term, these terms will be with the negative sign, right. So, this -1 was there. So, this will have a negative sign.

So, there is a minimal impact. The third example that we have taken will face a major impact; for example, $p(x)-q(x)$. Now, you take this $-q(x)$ here, if you take that $-q(x)$ here, then $p(x)-q(x)$; the first term will be as it is x^3, because it is coming from this point here there was a clash of x^2. So, 2 in when we added, it was 2 + 1. So, everything became 3; here it is $(2-1)x^2$. In a similar manner $(1-2)x$ and then the final term was -2.

So, essentially you got the expression in this form, which is x^3+x^2-x-2; but the key principles remain the same, except for multiplying with -1. Multiplying with -1, because that was a polynomial of degree 0 will not change the degree of the polynomial. So, once it is not changing the degree of the polynomial, all the rules which were possible for addition remain intact.

For example, you have to choose the degree which is maximum of the polynomial; there is no change in the degree except for the multiplication of a minus sign. So, that multiplication of minus sign is absorbed here. So, now, in the new rule it will be $p(x)-q(x)$ will be k is equal to 0 to maximum of m and n there is a remained intact; and earlier when we were adding it was $a(k)+b(k)$, now it is $a(k)-b(k)$, ok.

So, I hope you have understood addition and subtraction of the polynomials, both are essentially same and the resultant what we are getting is again a polynomial. In the next video, we will take a closer look at multiplication of polynomials.

Thank you.