IIT Madras

ONLINE DEGREE

Mathematics for Data Science 1
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Week - 01
Lecture – 09
Why is a number irrational?

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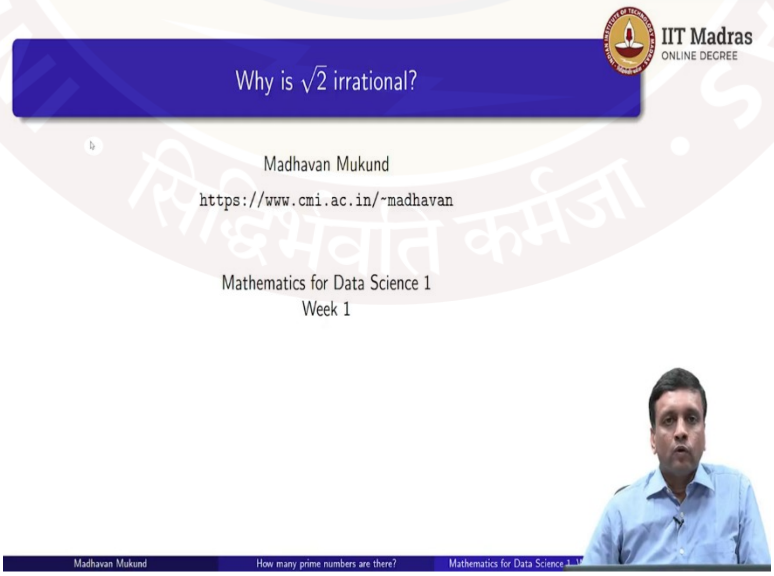

When we looked at the different types of numbers, we started with the natural numbers, move to the integers, then to the rationals which are expressed as $\frac{p}{q}$. And then we argued that the rationals do not exhaust all the numbers that we need; and in particular, we claim that the √❑ cannot be expressed as a rational numbers, so it is what is called an irrational number. So, let us try and ask why √❑is an irrational number.

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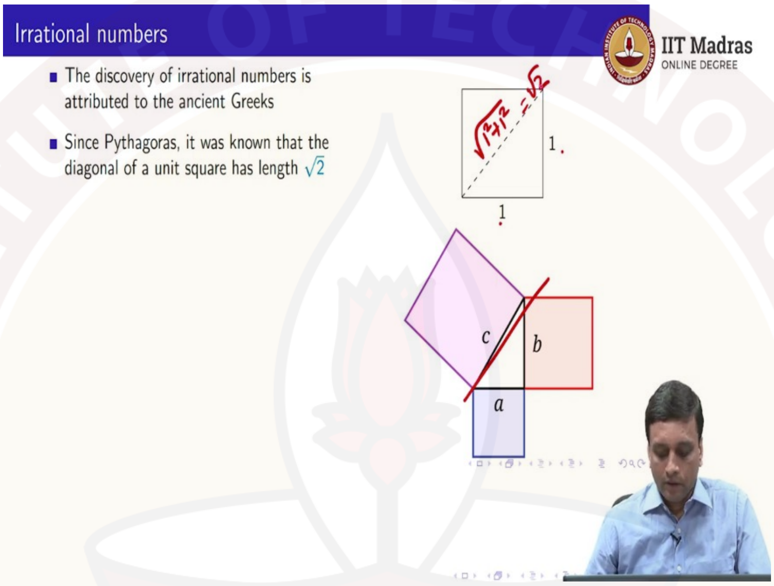

So, the discovery of irrational numbers actually is attributed to the ancient Greeks; and in particular, it comes from Pythagoras. So, remember that in Pythagoras's theorem which you must have studied in school. If you have a right angled triangle, then the square on the hypotenuse that is the square on the long diagonal side – this one, has an area which is the sum of the squares on the other side. So, in other words, if you have a right angled triangle and you measure the three sides, you get $a^2+b^2=c^2$. So, from this, knowing a and b, you can compute c.

So, in particular, if you draw a square which has one and one as its two sides, then this must be the √❑which is the √❑. So, you can actually physically draw if you assume that you can measure out a unit length using some kind of a measure, then by drawing a square, you can actually construct a length √❑.

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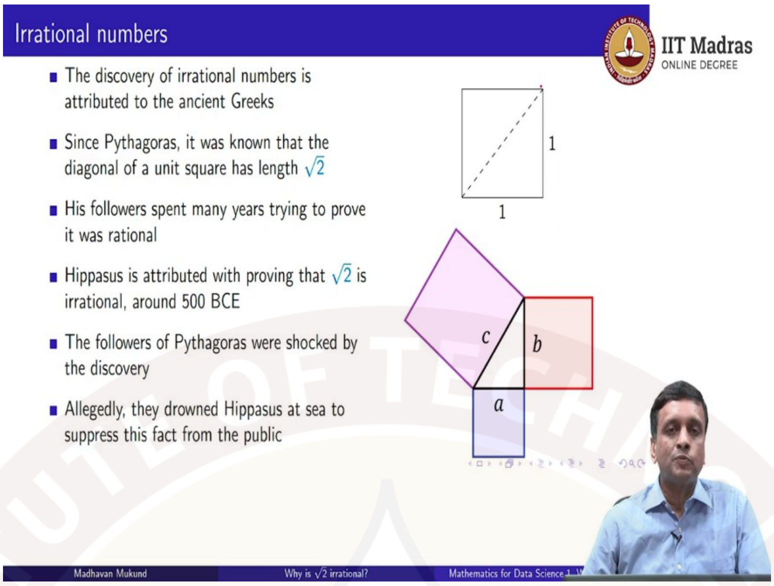

So, for Pythagoras it was very important to understand how to describe the √❑as a rational number, and he and his followers and many times many years trying to prove that in fact it could be expressed as a rational number. Much after Pythagoras, about 50-60 years after Pythagoras, one of his followers Hippasus is claimed to have proved that √❑is irrational this was around 500 BCE.

Now, the followers of Pythagoras had a very mystical idea about numbers, and they felt that numbers could solve everything. And in particular they were very keen that rational rational numbers should form the basis of all of what we could call it modern day time science and philosophy. So, the followers of Pythagoras were really shocked by this discovery of Hippasus, they found it to be a, I mean they could not argue with it; at the same time they felt that this discovery could not be revealed to the public because they felt it was very dangerous. So, in fact, it is said that they allegedly drowned him in the sea to prevent this from being made public. So, the √❑being irrational has a rather colorful history. And let us see now how Hippasus proved that this was actually the case.

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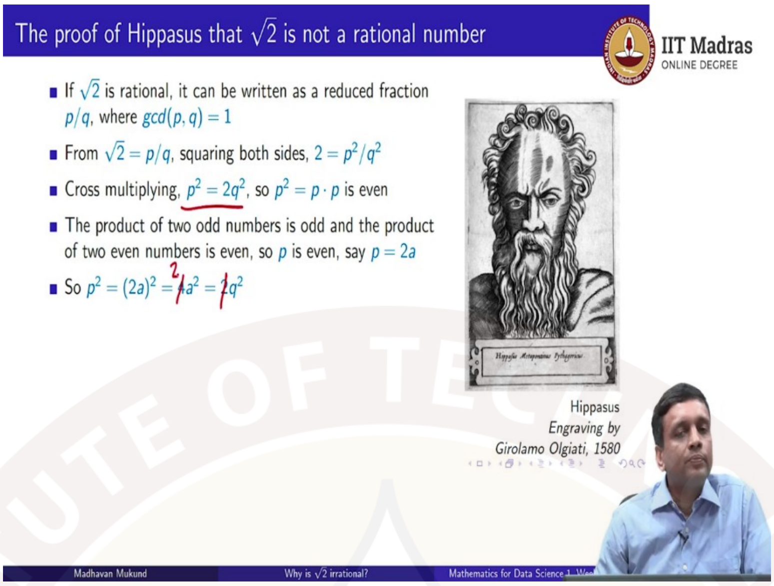

So, let us assume as in many of our arguments. Let us assume that √❑was rational. So, if it is rational, then we know that it can be written as a ratio or fraction of two integers p and q; and in particular we can assume that it is in reduced form. So, p and q have no common divisor, their gcd is 1. So, if we take √❑is equal to $\frac{p}{q}$, and we square both sides, then √❑times √❑is 2 on the left hand side, and $\frac{p}{q}$times $\frac{p}{q}$is $\frac{p^2}{q^2}$. So, we get 2 is equal to $\frac{p^2}{q^2}$. So, we can cross multiply as usual, take the $q^2$from the denominator on the right hand side to the left hand side numerator, and we get $2q^2$ is equal to p^2.

So, what is p^2? p^2 is p X p. And if it is of the form 2 times something, then it is an even number, because an even number is something which has 2 as a factor. So, p^2has 2 as a factor. So, $p^2$is an even number. Now, it is a basic fact about natural numbers that if you multiply two odd numbers, you get an odd number; and if you multiply two even numbers, you get an even number. So, if $p^2$is even, and p^2 is p X p, then both p and p – the two copies must both be even; so p must be an even number in other words.

So, if p is an even number, then we can write p as 2 times something because p is even p must be of the form two times something say 2a right. So, from this initial assumption, we have concluded that the numerator of this fraction which represents √❑is actually an even number of the form 2a.

So, now, let us substitute in this equation for p^2right. So, $p^2$is $(2a)^2$ is 4 times a^2. So, now $4a^2$ is equal to $2q^2$. So, now, we can cancel right. So, we can take this 2, and this 2, and cancel it.

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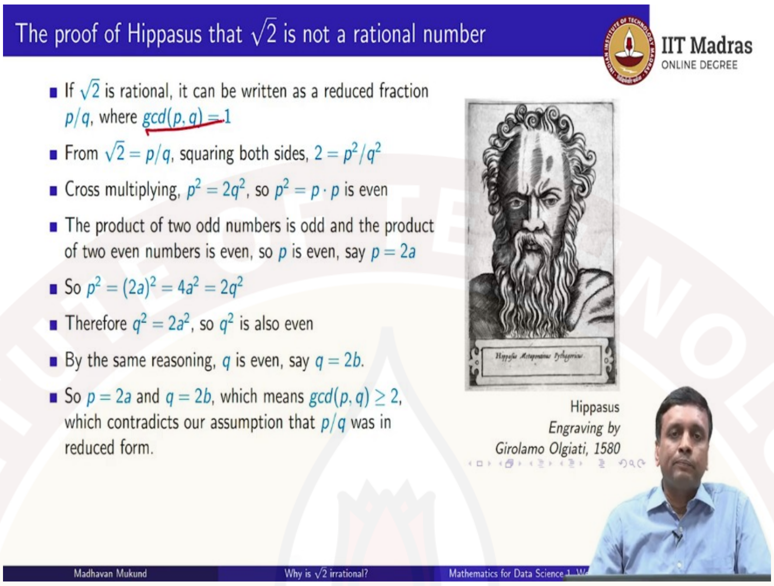

So, we have in other words that $q^2$is $2a^2$. And if $q^2$is $2a^2$, then by the same argument as before $q^2$is also even, and so q must be even. And therefore, q can be written as the form of 2 times some other number b. So, we have that p is of the form 2 times a and q was of the form 2 times b. But what this means is that the gcd of p and q must be at least 2, because both of them are even numbers. So, they are both multiples of 2. So, we claimed initially that the gcd of p and q is 1. We said that they were actually both in reduced form. So, there was no common factor other than 1. And now we have shown that if we assume that we in fact generate 2 as a common factor. So, this cannot be the case. So, the only contradiction that we can resolve with this is by assuming that $\frac{p}{q}$could not have been there. So, therefore, $\sqrt{\square}$ cannot be represented by any reduced fraction $\frac{p}{q}$.

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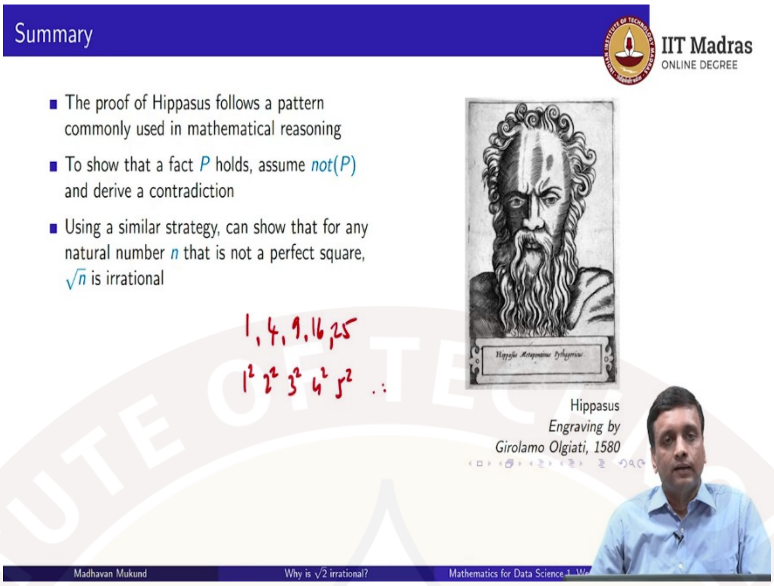

So, this argument of Hippasus is a common way of arguing things in mathematics right. To show that some fact capital P holds you first assume that not P holds, it is negation holds. So, we wanted to show that there is no way that $\sqrt{\square}$cannot be expressed as rational. So, we said let us assume the negation. Let us assume that $\sqrt{\square}$can in fact be express as a rational, and then you take that assumption and derive a contradiction. And since you cannot accept a contradiction, your assumption must be wrong and therefore, what you tried to prove originally was correct.

So, in fact, it is not just $\sqrt{\square}$that is irrational, $\sqrt{\square}$is also irrational. Now, 4 is a perfect square. So, we know that $\sqrt{\square}$ is 2. What about$\sqrt{\square}$; that is also irrational. So, among the integers among the natural numbers we have the perfect squares 1, 4, 9, 16, 25 and so on which consists of 1^2, 2^2, 3^2, 4^2, $5^2$and so on. So, a perfect square is one whose square root is also a natural number.

Now, it turns out that anything which is not a perfect square has an irrational square root, and the proof is not exactly the same because we have used a property of 2, and evenness in this proof, but with a very similar argument you can show this is the case. So, therefore, there are a lot of irrational numbers that you can generate just by taking square roots of non-perfect squares.