IIT Madras

ONLINE DEGREE

Mathematics for Data Science 2
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Lecture 10
Critical points: local maxima and minima

Hello and welcome to the maths 2 component of the online BSc program on Data Science and Programming. In this video we are going to talk about Critical points: local maxima and minima.

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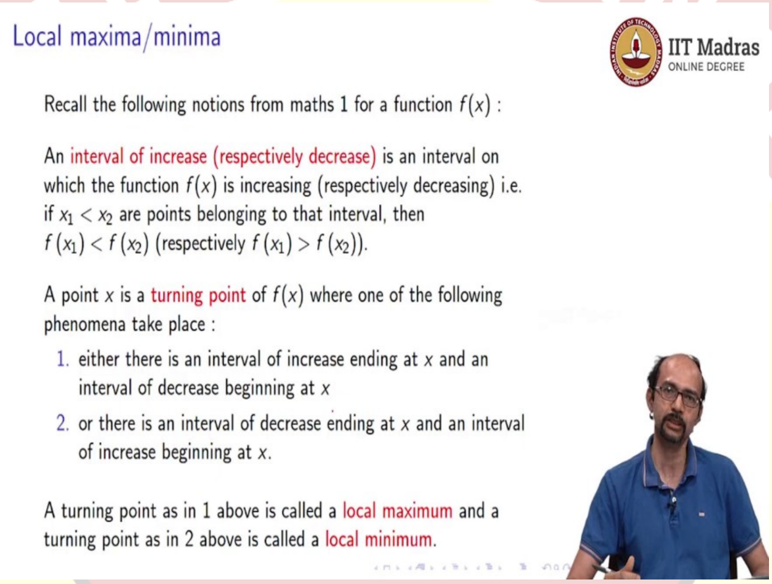

So, before we start discussing these terms, let us recall some things that we have done in maths 1. So, in maths 1, for a function f(x), we have studied something called the interval of increase or the interval of decrease. So, what is an interval of increase? It is an interval on which the function f(x) is increasing. And what is an interval of decrease? It is a function on which the interval is decreasing.

So, to put that into mathematical terms, what we are saying is that if you have two points x_1 and x_2 which belong to this interval and $x_1 < x_2$, then $f(x_1) < f(x_2)$. That is an interval of increase. So, $x_1 < x_2$ means $f(x_1) < f(x_2)$. And an interval of decrease is exactly the opposite which is to say that if x_1 and x_2 belong to that interval and $x_1 < x_2$, then the function decreases from x_1 to x_2 So, $f(x_1) > f(x_2)$.

So, these are notions that you have seen I believe in maths 1. We also saw in maths 1, what is a turning point for a function. So, point x is a turning point of f(x) where one of the following phenomena takes place; either there is an interval of increase ending at x and an interval of decrease beginning at x. So, that means the function increases till x and then decreases from x.

There is some interval, so it need not increase throughout till x, but there is some small interval, could be small, could be large which ends at x and then from x there is again another interval which begins and in that it decreases. Or there is an interval of decrease ending at x and an interval of increase beginning at x. So, we will draw pictures of these in the minute, but I am going to give you some new names to call them by. So, this is the definition which was in the title.

A turning point as in 1 above, so 1 is this phenomenon 1 which is to say the function increases till x and then decreases after x. So, such a point is called a local maximum that is because there is some, it means there is some interval you can build in which this, on that interval this point is where the maximum value of that function is attained. And a local minimum is where you have phenomenon 2 occurring which means that first you have an interval of decrease and then you have an interval of increase. So, the function decreases to x and then increases from x onwards.

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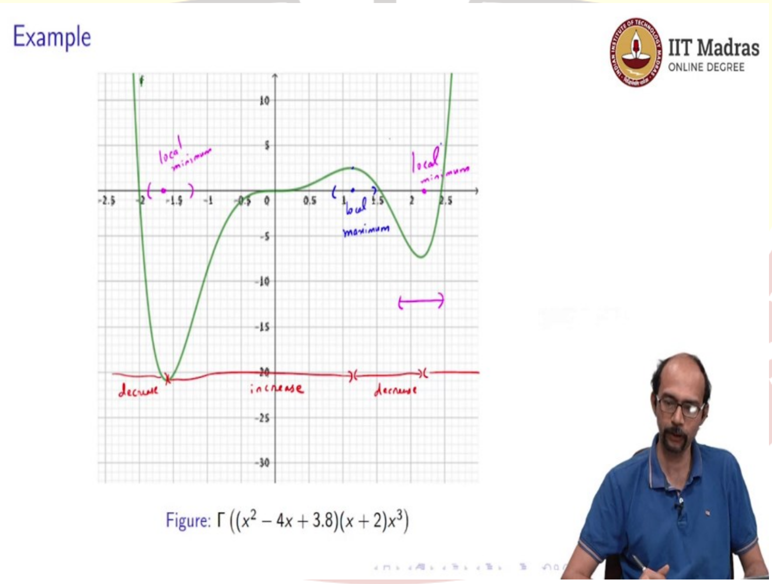

So, let us look at this example. So, in this example we have plotted this function $(x^2-4x+3.8)(x+2)x^3$. What are the intervals of increase and decrease? So, until this point, this entire part whatever is above here, this is an interval of decrease; so decrease. Then after this until you get approximately here, this is an interval of increase.

Although in the middle there is some strange thing happening close to 0 and then after this again there is a interval of decrease till about here and then after this again we have increase and probably increases all the way beyond this. So, what are these points? So, in terms of what we have studied, what are these points? So, this point here, this is I look somewhere here. This is the local minimum for the function and then this point here is another local minimum. So, why is this a local minimum? You can see there is a valley here or a trough, right.

So, if you chose some small interval something like this, on this interval, the value taken by this point is the smallest, it is the minimum. So, it is a local minimum. It is not as you can see it is not a global minimum, but this one actually looks like it is a global minimum. So, here, again you can choose some small interval something like this and it is a local minimum meaning the smallest value on that interval. And in this case it so happens that it, it is actually a global minimum meaning amongst the all possible choices of f(x), this is the smallest choice.

And then what about this point here? So, that point is a local maximum. So, this shows you how these behave and what is the interval here? Maybe you can take something like this or you could have a larger interval and its increasing, but you cannot say take this interval, -2.5 to 2.5 right over there, we may have values which are larger, in fact there are values which are larger. So, I hope this picture is illustrative of the local minima and local maxima.

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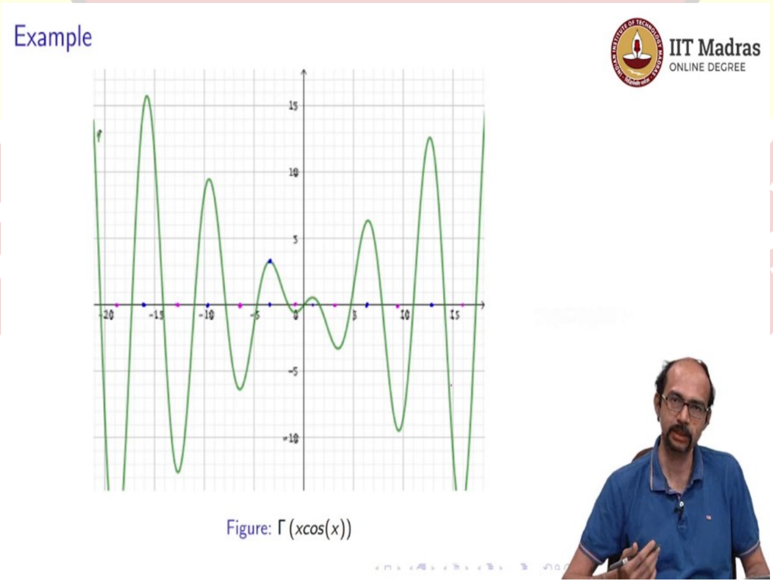

Let us do the same thing for this picture here. So, this is the graph of the function xcos(x) and let us look at the local maxima. So, here is one local maxima. So, this point is the local maxima. This point here is a local maxima. This point here is a local maxima, maximum, local maximum, local maximum, local maximum, and so on. So, this since it is a nice wave like function with increasing amplitude, meaning with increasing amplitude, but wave like, so you will have lots of local maxima, there will be infinitely many as you can see.

And what are the local minima? So, let us also plot that whatever local minima. Here is a local minima, local minimum, here is another local minimum, local minimum, while somewhere here there is a local minimum, here is one, here is one, and somewhere here we cannot see the function value but it is a local minimum and the same thing happens as for the maxima, there are infinitely many local minima ok.

So, a local extremum is either a local minimum or a local maximum because it is an extreme value for a small interval around that point. And often what we want to do is to ask how do we find these local extrema? So, these values are, they often have, they are turning points and they often carry information about the function with them. So, often we want to ask, we want to study how to find them.

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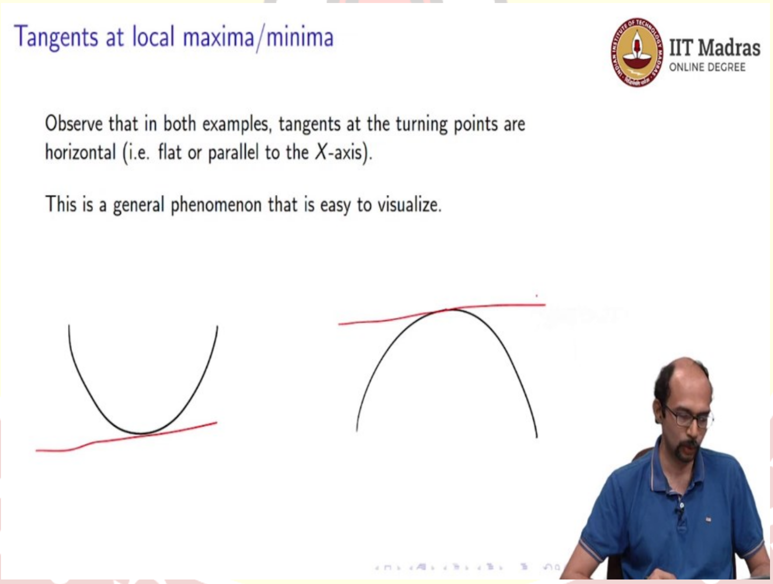

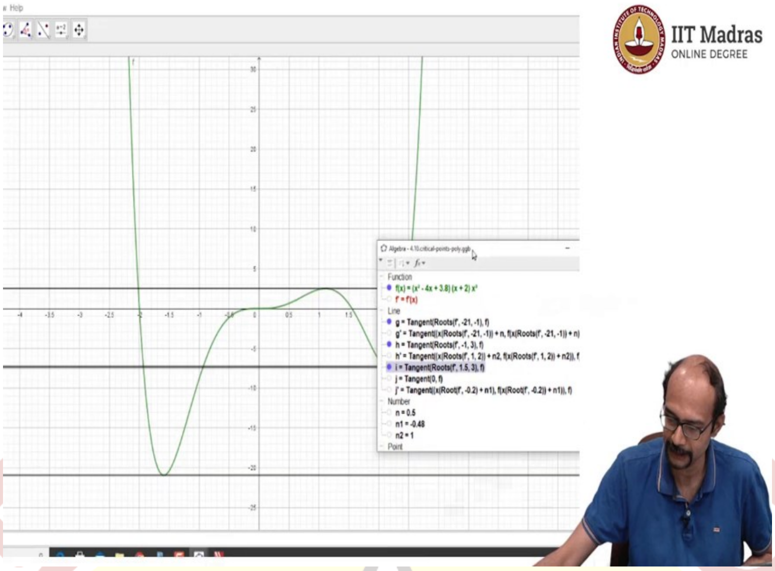

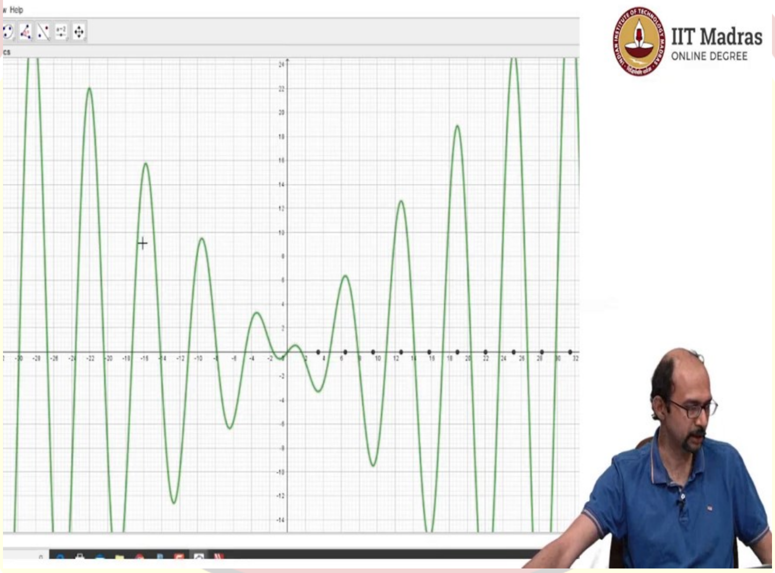

So, one way that we can think about to find them is to look at the tangents. So, we have been studying tangents for a while now and this is another application of tangents and well once we have tangents, you know now what is going to come, but let us hold on to that idea for a minute. So, observe that in both examples, tangents at the turning points are horizontal, so they are flat or parallel to the X-axis.

So, let us look at the first example. Let us look at what are the tangents at these various points. So, if you look at the tangents at the various points, let us look at this tangent. Well, you can see it is parallel to the X-axis or it is horizontal. Here is the tangent to this local maximum, it is again horizontal and let us look at the tangent to this local minimum here, that is also horizontal. So, these were the 3 points that we identified as local extrema in this case and all three, the tangents are horizontal. Let us look at the other picture.

So, it is clear that these points, if you draw the tangent over here, it will be a local, it will be horizontal. So, at this point or this point here, or this point here, so at all these points, they would be horizontal. So, I think at least the picture is very clear and unambiguous in telling us that at turning points the tangents are horizontal. So, this is a general phenomenon that is, it is easy to visualize this and let us me just draw this picture so that we have an easy way to visualize this.

So, if you have a local minimum, it is going to be something like this, if you have a local maximum, it will be something like this and now how is the tangent going to look like? So, if you draw the tangent, the tangent to this thing over here will look like this, the tangent to this thing over here will look like this. So, you can see it is horizontal, although it is not very well drawn. So, it is something that is easy to visualize. So, but how do we make this formal?

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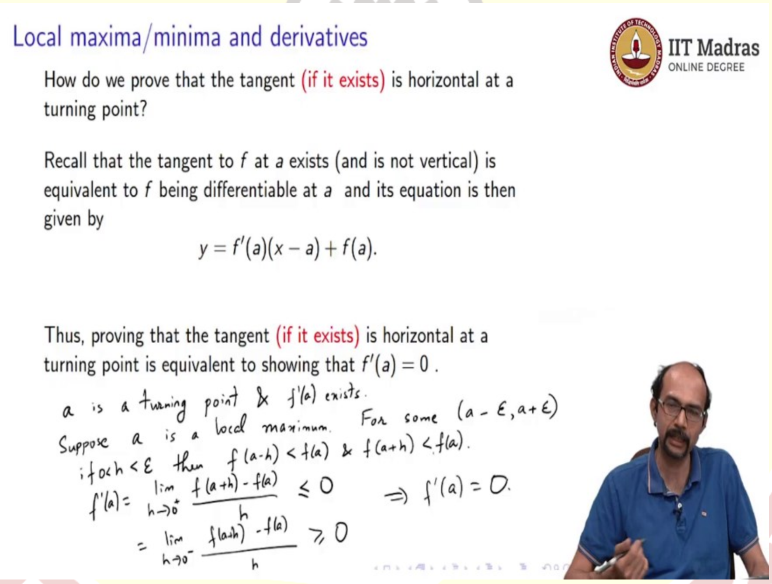

So, the way to make it formal is to of course introduce derivatives as we have been doing for a while now. We have been capturing the idea of tangents in terms of derivatives and so derivatives give us an algebraic way of getting hold of equations and so on and that tells us in analysing how to first prove that the tangent will be horizontal and second; to utilize it for further study of local maxima and minima.

So, how do we prove that the tangent if it exists, note that tangents may not exists in general, so if it exists is horizontal at a turning point. So, recall that the tangent to f at a exists and is not vertical is equivalent to f being differentiable at a. This was something that we observed in our previous video. And in fact we know how its equation looks like. Its equation is then given by $y = f'(a)(x-a) + f(a)$.

So, proving that the tangent if it exists, is horizontal at a turning point is equivalent to showing that $f'(a)=0$. So, let us show that $f'(a)=0$ on turning points. So, if a is a turning point, so a is a turning point and f'(a) exists, so that of course, we cannot guarantee, but assuming that is the case. Let us show that $f'(a)=0$. So, suppose just for the sake of argument that we have a local maximum.

So, suppose a is a local maximum. So, what does that mean? That means that for h>0, there is some interval or maybe I should first say that is a, so for some open interval a comma or maybe not a, a minus some small number, let us call it ε. So, for some $(a-\varepsilon, a+\varepsilon)$ if h<ε, then $f(a-h)<f(a)$ and f(a+h)<f(a) right? That is, when you will say that that this is a local maximum.

So, now let us apply the definition of the derivative. So, $f'(a) = \lim_{(h \to 0)} \frac{f(a+h)-f(a)}{f(a)}$. But notice what happens to the numerator when you have h is between 0 and epsilon. So, f(a+h)<f(a). So, this numerator is negative, so this value is going to be less than or equal to 0. So, this is the derivative from the right. So, this is $\lim h \to 0+$. So, this is going to be negative.

On the other hand, we know that $f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{f(a)}$. So, here, of course, the h is between -ε and 0, so in that case, this is going to be greater than or equal to 0. Why is that? Because here the numerator is negative, but the denominator is also negative. So, negative by negative gives you a positive number.

So, now we know that f'(a) exists and it is greater than or equal to 0 and we also know it is less than or equal to 0 that tells us that f'(a) must be equal to 0. So, you can see the power of the left limit and the right limit coming in here. So, if it is a turning point, then f'(a) must be 0; f'(0) is exactly the same as saying that the tangent is horizontal or parallel to the X-axis.

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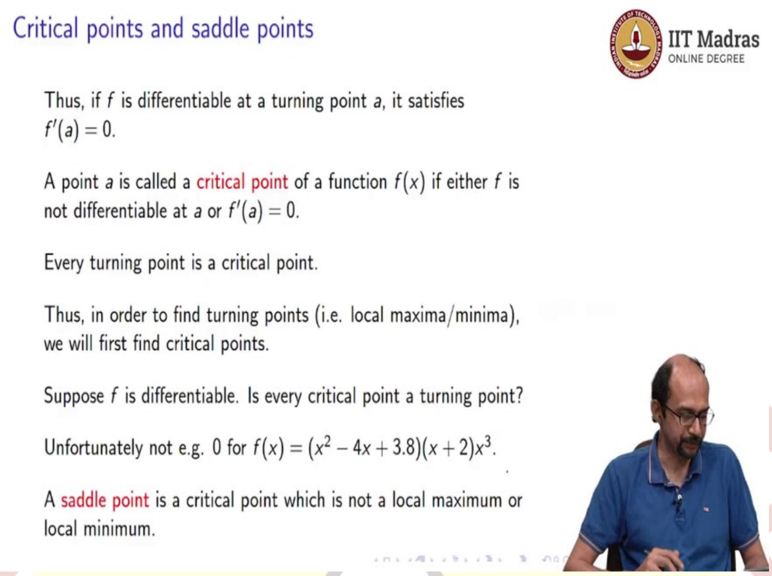

So, we are actually proved that. So, let us now ask how we can harness this information to actually study turning points. If f is differentiable at a turning point a, it satisfies f'(a) is 0. So, point a is called a critical point of a function f if either f is not differentiable at a or f'(a) is 0. So, either it is a point at which the function is not differentiable or it is differentiable and when you take the derivative, the derivative is 0 at that point.

So, of course, every turning point is a critical point. We just saw that if you have a turning point, then f'(a) is 0 if it is differentiable and if it is not differentiable, it is a critical point anyway. So, in order to find turning points which is to say local maxima or local minima, we will first find critical points. So, suppose f is differentiable, so is every critical point a turning point?

So, what do we want to do? We want to find turning points. So, that is local maxima or local minima, by the way I should point out that, the plural of maximum is maxima and the plural of minimum is minima. So, we want to find local maxima or local minima and in order to do this what we are going to do is to find critical points. But then, there will be some of them which are local maxima, some will be local minima, but are there any others? That is the question. So, is every critical point a turning point?

So, if there are no others, we really have the entire set of local maxima and local minima just by finding the set of critical points. So, unfortunately that is not the case. So, if you look at that function that we had earlier, $(x^2-4x+3.8)(x+2)x^3$ and we look at the value 0, then we can see that it is going to be a critical point. So, I will just make this definition. So, a saddle point is a critical point which is not a local maximum or local minimum.

So, I want to make a small caveat here, somehow different sources use different names for this, so saddle point is common when we go to beyond one dimension which we will do in subsequent weeks. In one dimension, there are different words used. Some people seem to use the word inflection point, so be careful of, depending on the source that you study, they might use a different word for this. So, let us look at the picture of this function.

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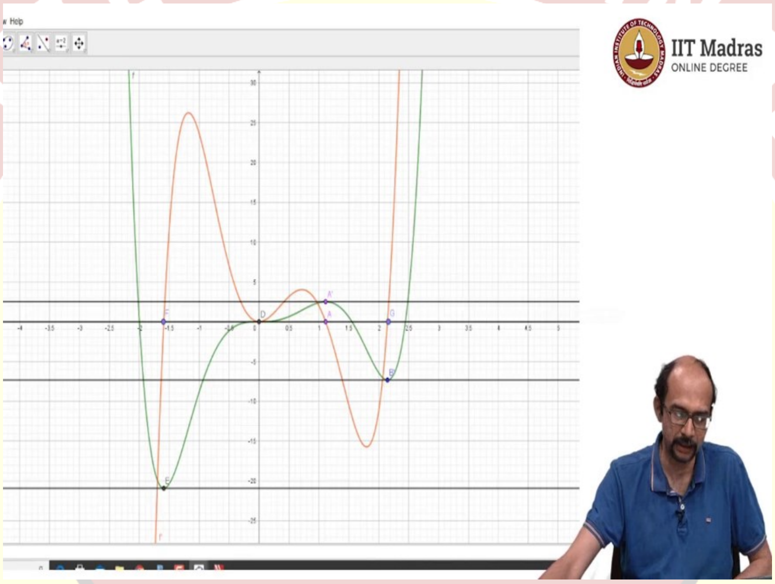

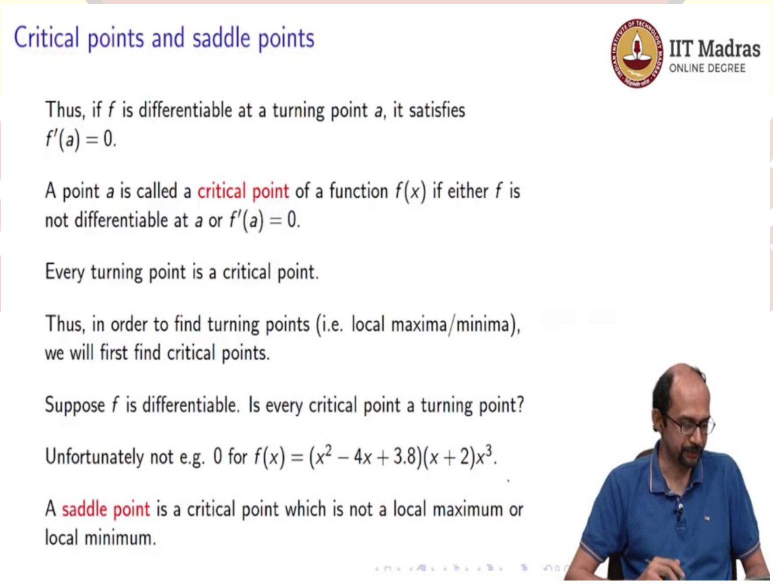

So, here is the function that we had, $f(x)=(x^2-4x+3.8)(x+2)x^3$ and let us draw all the points. So, here is a local maximum and maybe corresponding to that local maximum, we have

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the corresponding value. Here is your local minimum; the point v' and here is the point (0, 0) which is the saddle point. Why is that a saddle point?

As you can see as if you draw the tangent to this point, tangent to f at this point, let us draw that tangent, here it is. So, if you draw that tangent, it is actually the X-axis and you can check this by writing down the equation. So, you have an x^3 in the expression, so if you differentiate the function, we will obtain that f'(0)=0. So, it is a good exercise actually to do that.

So, there is also a point here which unfortunately has not gotten marked, this and that corresponding number somewhere here. So, this is a local minimum, the corresponding point here is a local minimum, the point A is a local maximum and D is a saddle point. So, I hope now we have drawn all the critical points of this picture. So, I hope now it is clear what a saddle point is.

So, just to again retrieve what is happening for a saddle point, you can see that for a saddle point there is something special happening. It is not the local minimum or maximum, but the function behaves in a slightly weird way. So, it is concave below before it hits that point and it is concave above after it hits that point. So, between E and D, it is concave below and between D and A, it is concave above. So, there is something happening, there is some change which occurs and that change is really what is reflected by this point and whenever such a thing occurs, such points are going to be saddle points.

So, let us also draw the f', so this is f'. So, if you look at f', these are exactly the zeros of f'. F is a zero, D is a zero, A is a zero and so this point G is zero. So, we have identified all the critical points correctly for this function. Fine. So, let us go back to our slides. So, I hope in this example it is clear what is a saddle point and, what are the local maxima and minima.

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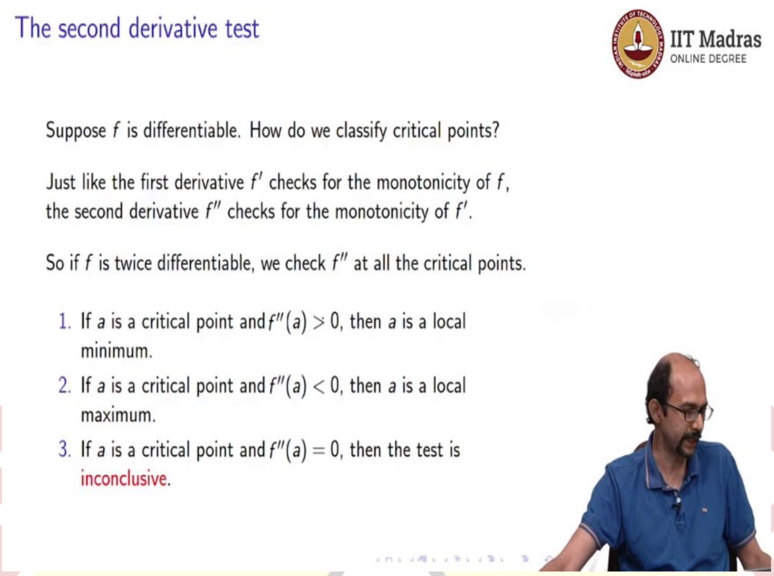

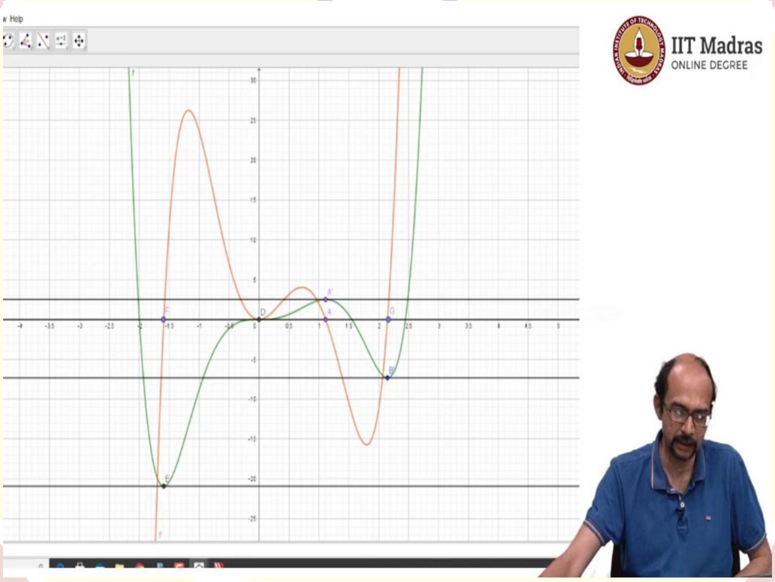

So, now how do we find, out of these critical points, how do we find which ones are local maxima and local minimum, those are the ones that we want to find. So, let us assume that the function f is differentiable, how do we classify critical points? So, we use the second derivative test. So, just like the first derivative f prime checks for the monotonicity of the function f, that is what we saw, the second derivative f'' checks for the monotonicity of f'. And based on this monotonicity, we can see how the behaviour is.

So, let us go back to our picture. So, here is our picture for f' and you can see that for the point F, it is a zero of this function f', but this function is very nicely increasing at F and similarly for the point A, the function is 0 at that point A, but the function is decreasing for that point A. And similarly for the point G, the function is increasing at that point G. But for D, this function is behaving totally different way. Namely, if you, this function is a, so D is actually a local minimum for the derivative.

So, when such a thing happens that is going to be a critical point, sorry, that is going to be a saddle point. So, let us see what is the second derivative test. So, the second derivative test says the following; so, if f is twice differentiable, we check f double prime at all the critical points. If a is a critical point, if $f''(a)>0$, then a is a local minimum. If a is a critical point and $f''(a)<0$, then a is a local maximum. If a is a critical point, if $f''(a)=0$, then the test is unfortunately inconclusive and that is a little sad that there, this is not an absolute test and we will see why that is the case later on.

So, let us quickly conclude by recalling what we have done in this video. So, we have checked how to obtain critical points, that is when you put the derivative to 0 or if the function is not differentiable at the point, assuming that the function is differentiable and in fact, twice differentiable. We saw the second derivative test, which tells us that if at a critical point the second derivative is negative, then it is a local maximum; if it is positive, it is a local minimum and if it is 0, then it is inconclusive. Thank you.