Statistics for Data Science -1

Lecture 9.4 Variance of a Random Variable

Usha Mohan

Indian Institute of Technology Madras

Learning objectives

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2. Types of random variables: discrete and continuous.
3. Probability mass function, graph, and examples.
4. Cumulative distribution function, graphs, and examples.
5. Expectation and variance of a random variable.

Introduction

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- Need for a measure of spread.

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- In other words, the Variance of a random variable X measures the square of the difference of the random variable from its mean, μ, on the average.

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- $(X - \mu)^2 = X^2 - 2X\mu + \mu^2$
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- Let's compute the variance of the random variables discussed earlier.

Rolling a dice once

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- Sample space: $S = \{1, 2, 3, 4, 5, 6\}$
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X	1	2	3	4	5	6
X^2	1	4	9	16	25	36
$P(X = x_i)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

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- $E(X) = 1\frac{1}{6} + 2\frac{1}{6} + 3\frac{1}{6} + 4\frac{1}{6} + 5\frac{1}{6} + 6\frac{1}{6} = 3.5.$

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- $E(X^2) = 1\frac{1}{6} + 4\frac{1}{6} + 9\frac{1}{6} + 16\frac{1}{6} + 25\frac{1}{6} + 36\frac{1}{6} = 15.167$

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- $Var(X) = 15.167 - 3.5^2 = 2.917$

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- Probability mass function

X	2	3	4	5	6	7	8	9	10	11	12
X^2	4	9	16	25	36	49	64	81	100	121	144
$P(X = x_i)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

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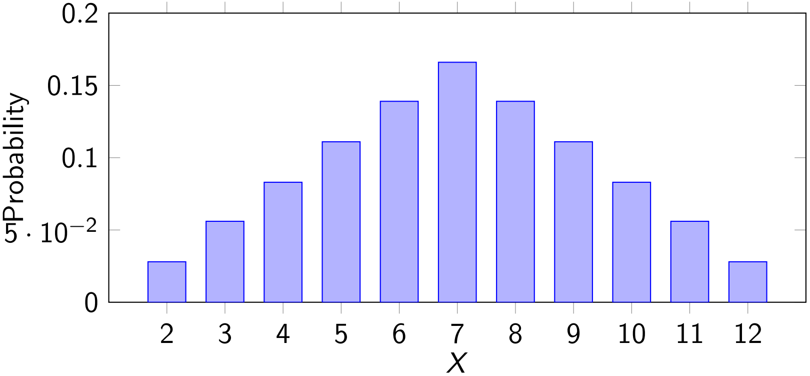
- $Var(X) = 54.833 - 49 = 5.833$


0.2
0.15
0.1
$5 \cdot 10^{-2}$
0
Probability
2
3
4
5
6
7
8
9
10
11
12
X

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    - $E(X) = \sum_{i=0}^{3} x_i p(x_i) = \dfrac{(0 \times 1) + (1 \times 3) + (2 \times 3) + (3 \times 1)}{8} = \dfrac{3}{2}$

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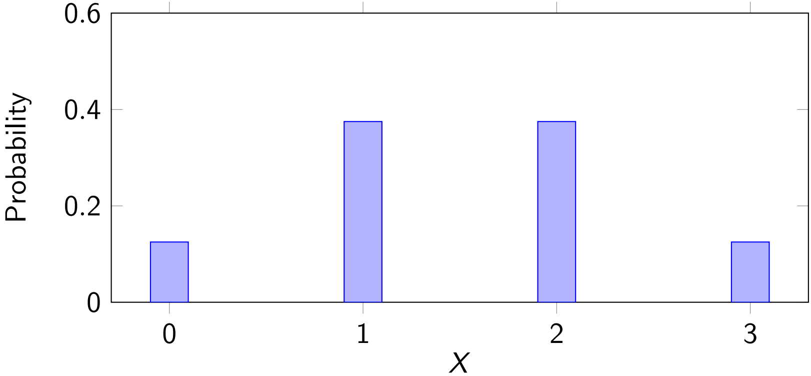
  - $Var(X) = 3 - 2.25 = 0.75$


0.6
0.4
0.2
0
Probability
0
1
2
3
X

Bernoulli random variable

- A random variable that takes on either the value 1 or 0 is called a Bernoulli random variable.
- Let X be a Bernoulli random variable that takes on the value 1 with probability p.
- The probability distribution of the random variable is

X	0	1

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  $E(X) = 0 \times (1 - p) + 1 \times p = p$

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- Expected value of a Bernoulli random varaible:
$E(X) = 0 \times (1-p) + 1 \times p = p$
- Variance of a Bernoulli random variable:
$Var(X) = p - p^2 = p(1-p)$

Discrete uniform random variable

- Let X be a random variable that is equally likely to takes any of the values $1, 2, \dots, n$
- Probability mass function

X	1	2	...	n

Discrete uniform random variable

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- Probability mass function

X	1	2	$\ldots$	n
X^2	1	4	$\ldots$	n^2
$P(X = x_i)$	$\frac{1}{n}$	$\frac{1}{n}$	$\ldots$	$\frac{1}{n}$

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- $E(X^2) = \frac{(n+1)(2n+1)}{6}$

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- $E(X) = \frac{(n+1)}{2}$
- $E(X^2) = \frac{(n+1)(2n+1)}{6}$
- $Var(X) = \frac{n^2-1}{12}$

Section summary

- Definition of variance
- Computational formula of variance of a random variable.