IIT Madras

ONLINE DEGREE

Mathematics for Data Science 2
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Lecture No. 09
Derivatives, Tangents and Linear Approximation

Hello, and welcome to the Maths 2 component of the online BSc program on Data Science and Programming. This video is about Derivatives, Tangents, and Linear Approximation. So, in the previous videos, we have seen the notions of Tangents. And then we saw that it is difficult to we wanted a way to get a handle on when tangents exist and when they do not, when they do, how to compute them, and so on.

And then we started, I made this comment that we need to develop some theory in order to talk about tangents in a more fundamental way. And indeed, that is why we introduced the notion of limits, we talked about continuity, we talked about differentiability. Last video, we saw how to compute some derivatives. And now we are familiar with limits, computing limits, derivatives, continuity, and computing derivatives.

So, in this video, we are going to finally make the connection between Derivatives and Tangents. And on the way, we will obtain this idea of, we will obtain this extra information about a linear approximation to a function.

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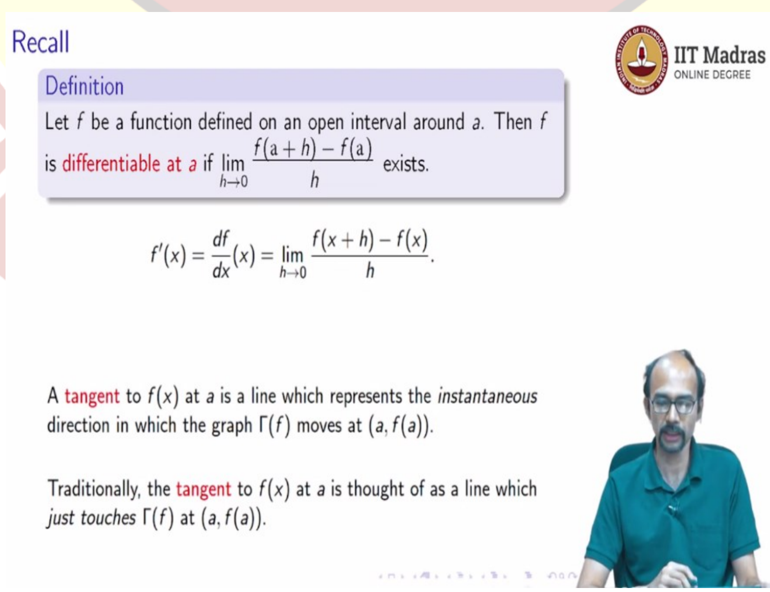

So, let us recall first, what is differentiability. So, let f be a function defined on an open interval around a, then f is differentiable at a, if $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ exists, and when it exists, then this number, whatever that limit is, is called the derivative of, of $f(a)$. And for each x, if we do this, we will get a function that is called a derivative function. So, it is denoted either byf', or $\frac{df}{dx}$.

Each notation has its advantage, by the way, and it is classic fact that one of these is due to Newton and the other is due to Leibniz, and one made life easier as compared to the other. So, let us know recall what is a Tangent. So, if you have a function f, and you consider a point a, which is in the domain of that function. So, tangent to $f(x)$ at a is a line which represents the instantaneous direction in which the graph $\Gamma(f)$ moves at the point $(a, f(a))$, which is a point on that graph.

So, you look at this graph as a curve and then on that point, we have on that curve, we have this point $(a, f(a))$ and then we ask at that point a, $f(a)$, what is the instantaneous direction in which the curve is moving. So, traditionally, the tangent is also thought of as a line, which just touches this graph, $\Gamma(f)$at the point a,f of a, so we have seen examples of this I, if you feel unsure of what that means, please go back and look at the examples that we did there and pictures that we drew.

So now, let us recall that when we started doing differentiability, we had this example of the instantaneous speed, that was what was the motivation for differentiability. So this, if f denotes the trajectory of a particle, then this $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ was supposed to denote the instantaneous speed. So here, meaning in the tangent, we want to talk about instantaneous direction, the derivative is talking about instantaneous speed not so surprising, they have a connection with each other.

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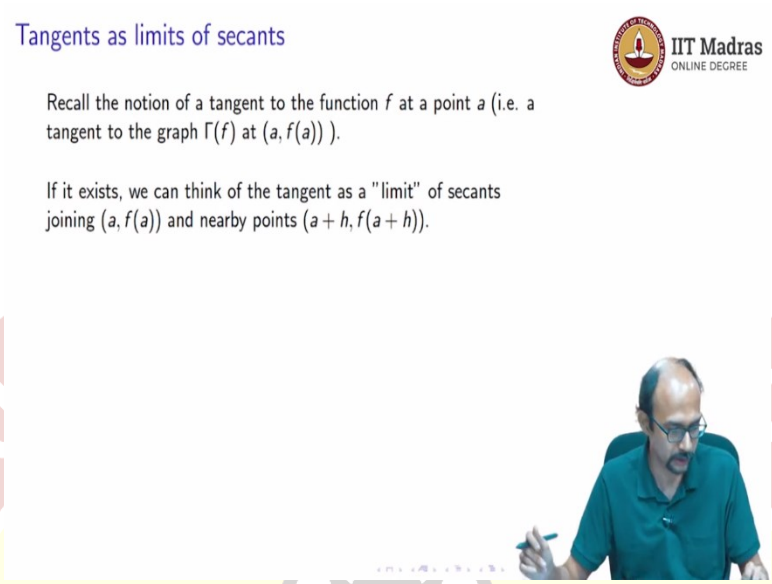

So, let us try to exploit what that this notion of a tangent as just touching. So, we will express tangents as limits of secants. So, recall the notion of a tangent to the function f at a point a, that is a tangent to the graph $\Gamma(f)$ at $(a, f(a))$. If it exists, and this is the key point, if it exists, we can think of the tangent as a limit of secants joining $(a, f(a))$ and nearby points($a+h, f(a+h)$).

Now, you may wonder what is a secant. If you have got have been away from geometry for a while. So, a secant is a line which is obtained by joining two points of a curve. So, in this case, you, you have your graph, you take two points on the graph, and you join them. And that is a Secant. So here, the suggestion is that you take the point $(a, f(a))$, and nearby point $a+h, f(a+h)$, and you join these two points and the line that you get is a Secant. And then you allow your h to come closer and closer to 0. So, your secant will slowly it is hoped move to the tangent. So, if this is not apparent, let us see an example of this.

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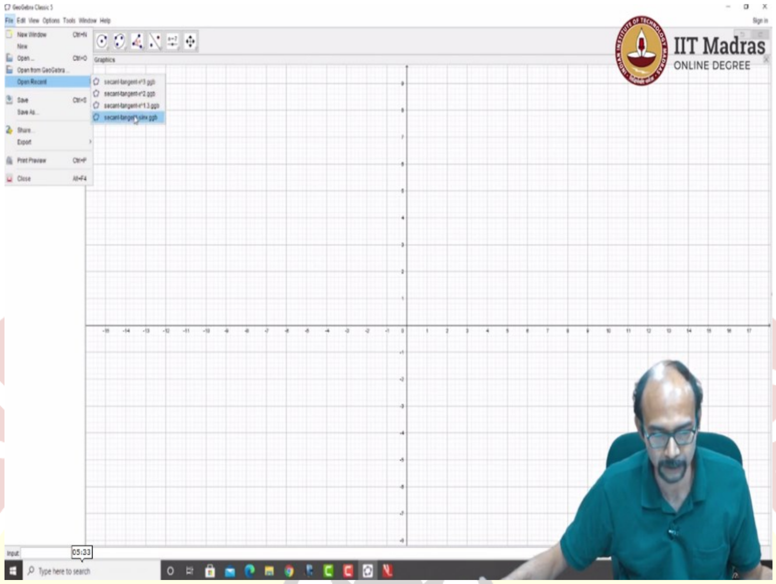

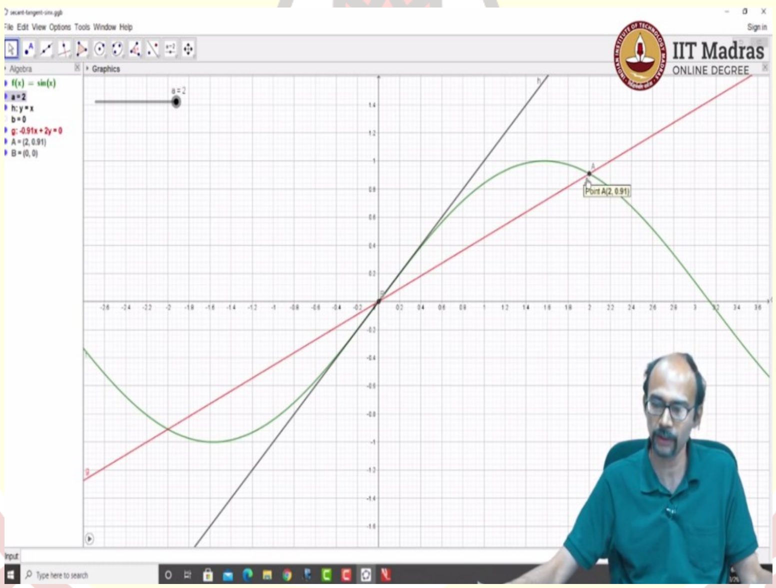

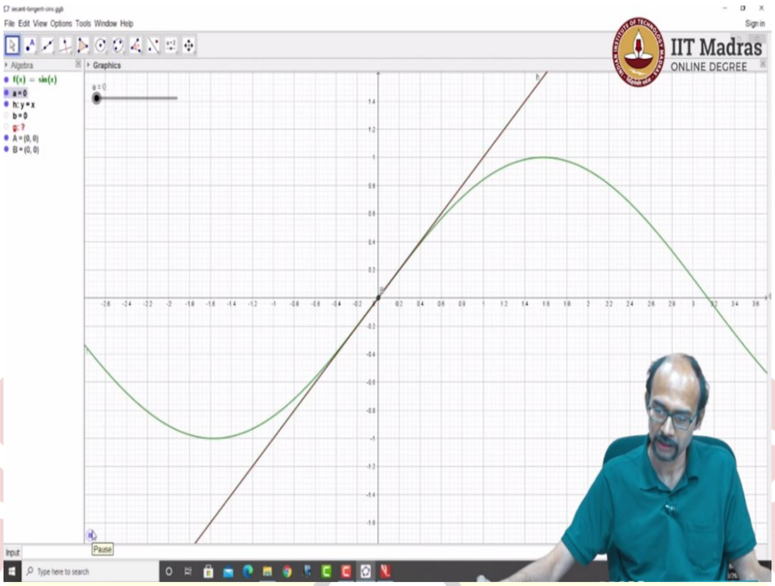

So, let us maybe look at, look at this, this animation that that I have here. So, what does this animation doing? So, let us stop this for a second. Begin from here, so here is your point B, which is the point (0,0). So, here is point (0,0). So, here are your $a = 0$, and $f(0)$ is 0. That is because we are plotting the sin function. So, $f(x)$ is $sin\ x$, so sin 0 is 0. And here is a point close by. So, I started this from the point (2,0.91).

So, the point capital A is your point, $(a + h, f(a + h))$. So, $a + h$ here is 2. And sin of 2 is whatever it is 0.91. And now we are going to vary h, and we have to bring it closer and closer to 0. So, this point will come along the function, that is a graph of the function it will come closer and closer to this point B. And this red line is the secant that is a line, line joining B and A and let us look at what happens to the secant.

And this black line, which I have called unfortunately labeled as h is the tangent to f at the point 0. That is what this is. So, let us play the animation now and see what happens. As you can see, it is coming closer and closer and closer and closer. And as the point comes further close, it is very close now. And as it approaches B, it is, it becomes the tangent. So, this is the idea of that the tangent is a limit of secants. So, I am this is not a formal statement, that one can make it formal, but we will utilize the statement. Let us just for I mean, just to get an idea.

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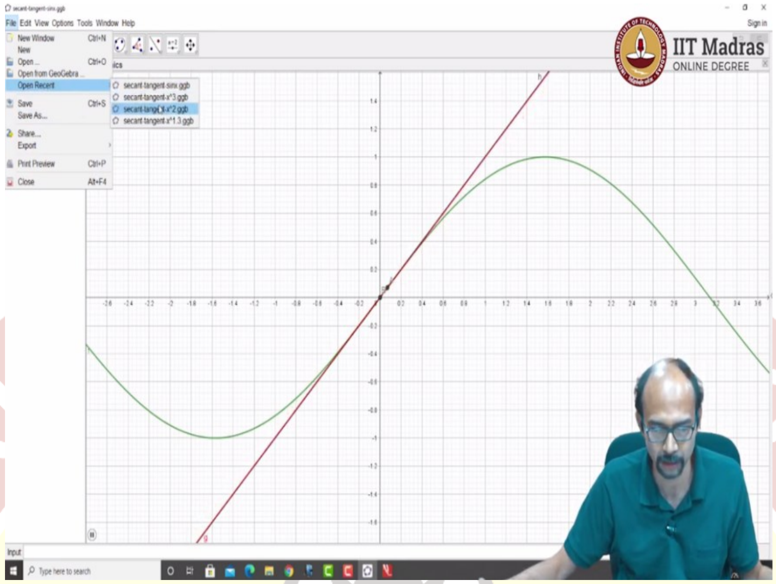

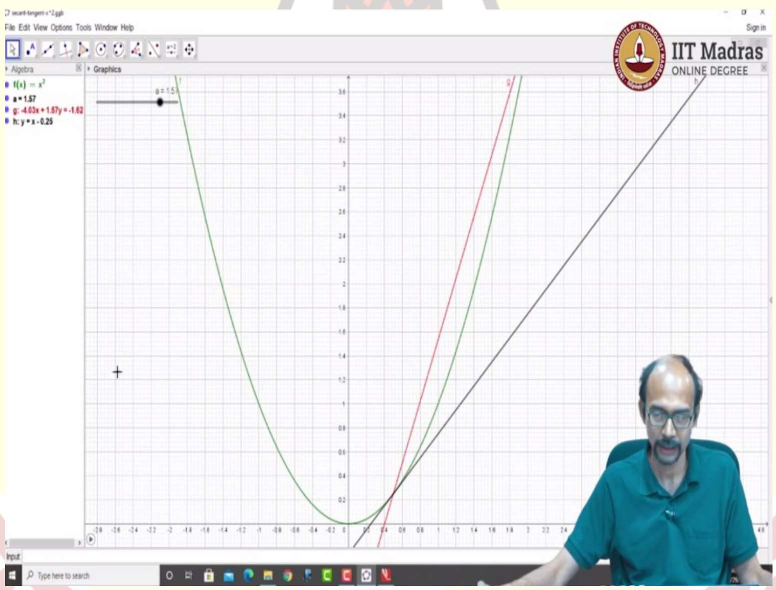

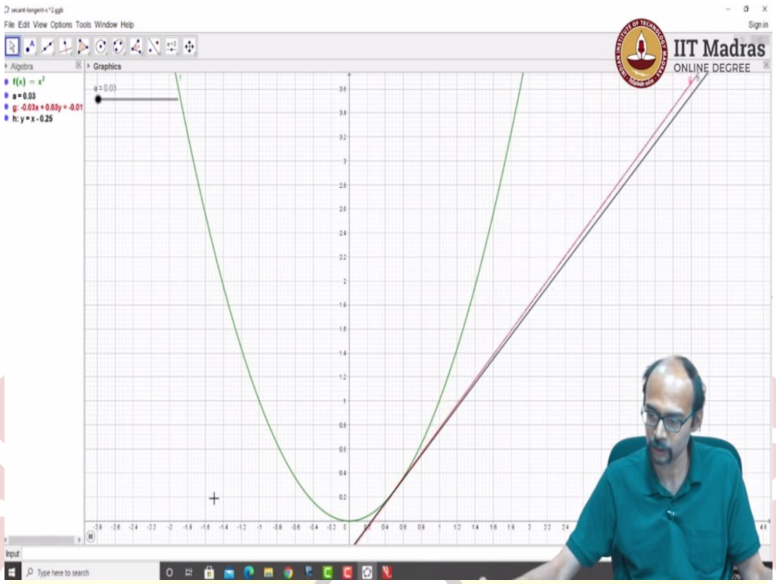

Let us look at another example. Maybe the example of x squared. So, here is the graph of x squared. So once again, we have, we have these points, and this time, I am not plotted the points. Let me stop this. So, this is at 0.5. So, we are looking at the tangent at 0.5. So, if the green curve is $f(x) = x^2$, at the point 0.5, f of that, which is 0.25, because it isx^2 so the function is x^2. So, 0.5 $x^2$is x^2.

So, this is the point 0.5,0.25 and we are joining this point with neighboring points, we are drawing the secant which is the red line, and the black line is attached. So, if we hover here, it says tangent to $f(x) = 0.5$,so x is 0.5. And let us play it now. So, if we play it now, you can clearly see that as, as your, as this point comes closer and closer and closer, the red line, which is a secant comes closer and closer to the tangent. So, just, just to show that it can come in a slightly strange way.

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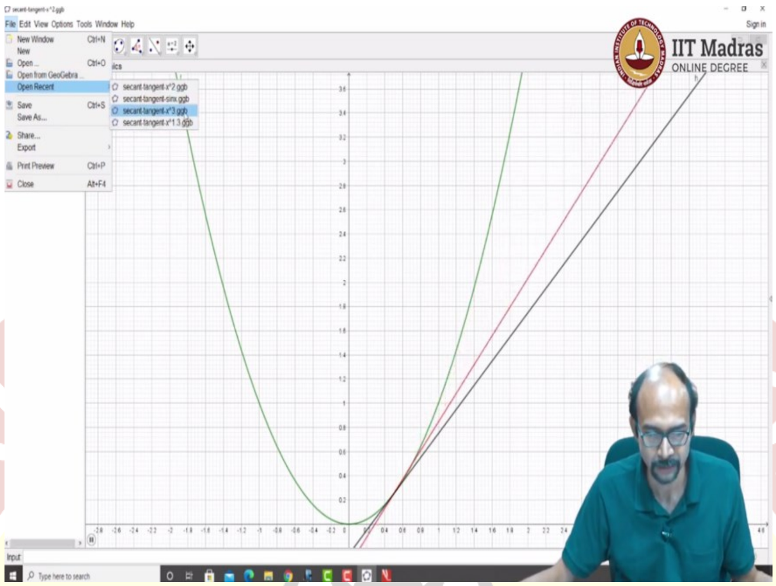

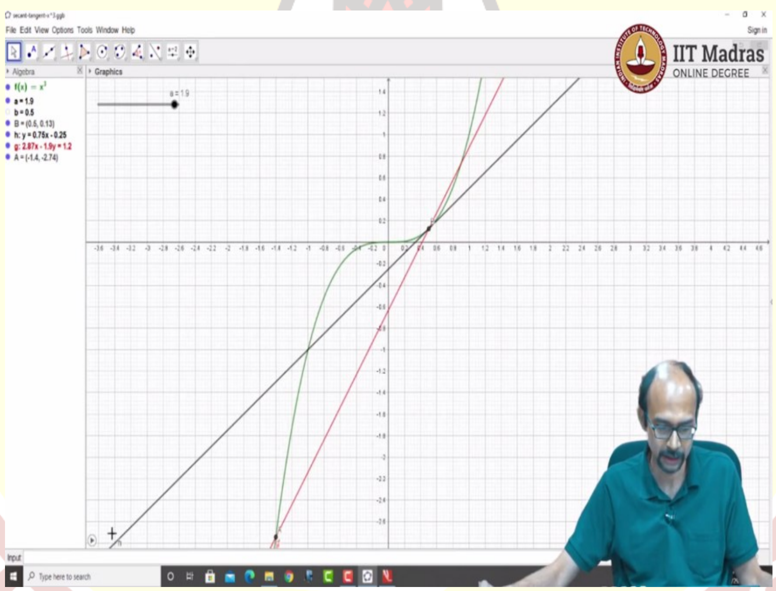

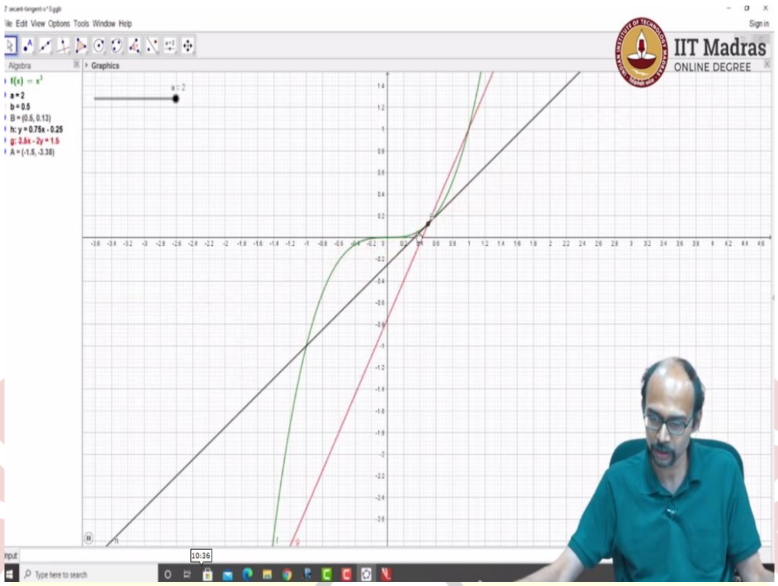

Let us look at another example, which is the example of x cubed. So, let us look at x cubed. So, once again, the green curve is $f(x)$ is x^3. The red line is the secant and the black line is a tangent and so this is the line joining this point $b, f(b)$ and $b-a, f(b-a)$, which is this point A over here. So, keep an eye on this red line. Let us play this and see what happens. So, as you play it, these red lines, come comes closer, it actually goes over.

And then it starts moving back. And now again, it is moving back. And now as A is coming to B, it becomes the tangent. So, you could have this behavior depends on how close you are, so in this case, it is crossed over. But then it is again, coming close by. So, really, if we had chosen something very close to this point B, it would have come closer and closer and closer automatically, but your points are far away, it may first go away, and then come back.

But that is allowed when you do limits, because when you do limit, the idea is that eventually it comes closer and closer. So, that is what we are seeing here. So, I hope you get a you have gotten the idea of that, that was expressed in, in this slide.

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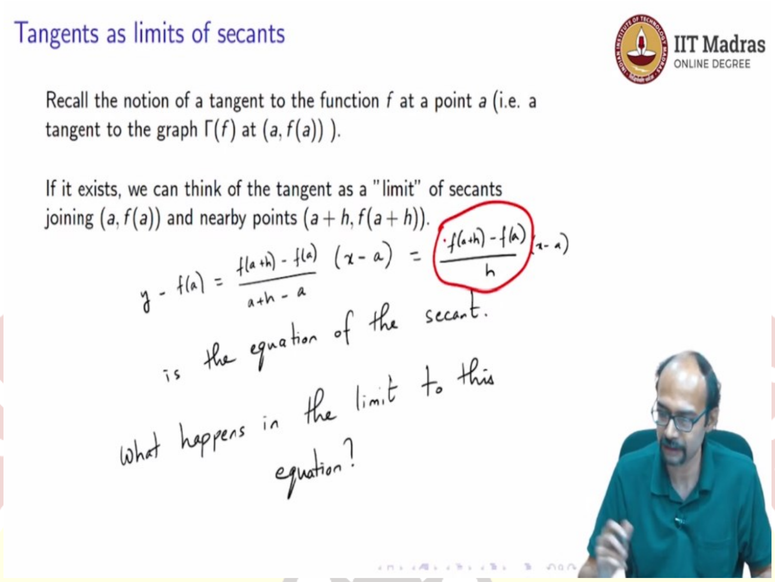

Namely, that you can think of a tangent as the limit of secants. So, if it exists, we can think of the tangent as a limit of secants joining $a, f(a)$ and nearby points $a+h, f(a+h)$. So, let us write down what is the equation of, of the secant which joins these two points. So, we have done the, in Maths 1, you must have studied how to write down the equation of a line which passes through two points.

So, so the equation of this line is going to be y minus, so first, what is the slope? So, so $y - f(a)$ is equal to slope times, so what is the slope? So, in this case, the slope is $\frac{f(a+h)-f(a)}{(a+h)-a} \times (x-a)$,. So, this is the equation of the secant. Let us just cross check that this is indeed the correct equation. So, if you, if you put $y = x = a$ and $y = f(a)$, then indeed, both sides are 0. So, this passes through $a, f(a)$.

And if you put $x = a+h$, and $y = f(a+h)$, then the right hand side, well, I should have written the right hand side, one more step here, $\frac{f(a+h)-f(a)}{(h)(x-a)}$, the denominator becomes h. So, this expression is familiar $f(a+h) - f(a)$. Anyway, if you substitute x $is\ a+h$ and y is $f\ (a+h)$, indeed, the dissatisfied so it passes through both points.

So, in case you have forgotten the equation of a line passing through two points, I have anyway, check that it passes through these two points. So, this is the equation of the secant that is given about. And now what we are going to try and do is ask, what is the limit of this? So, what is what happens in the limit to this equation?

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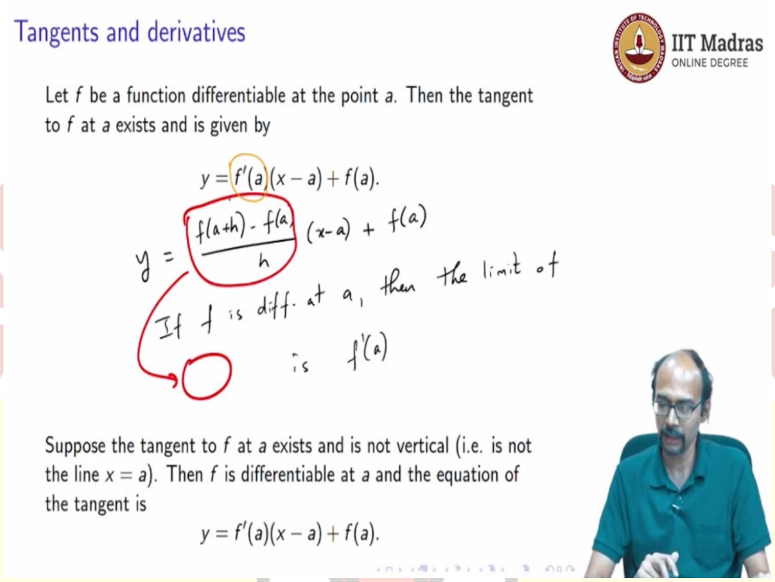

So, let us make that precise. So, let f be a function differentiable at the point a. Then the tangent to f at a exists, and is given by y is $f'(a)(x-a)+f(a)$, that that expression in the previous slide, let us just look at that again. This is the expression $y - f(a)$ is $\frac{f(a+h)-(a)}{h}$. So, this, this expression here, the limit becomes a $f'(a)$ and then the rest is exactly what we had.

So, y is $f'(a)(x-a)+f(a)$, and so this is saying that if the function is differentiable, which means that in the limit that expression that we had that has a limit and that x, that limit is we have called it f prime a, then the tangent exists and is given by this equation. And on the other side if the tangent to $f(a)$ exists, and it does not vertical. So, that is it is not the line $x = a$, then f is differentiable at a and the equation of the tangent is the same as above.

Once it is differentiable at a its force that exists. How did we get this? Well, let us write down the equation that we had expression that we had in the previous slide. So, I am if I take the f of a on the other side, we had y is $f'(a)$ times excuse me, y is $\frac{f(a+h)-(a)}{h} \times (x-a)+f(a)$, this was the expression. Now in the limit, meaning as limit as h tends to 0 if this limit exists, which is exactly saying that the function is differentiable, then of course, in the limit, you will get, you will get the expression above.

If this limit exists so, if f is differentiable at a, then this expression here, the limit is exactly what, what we have here, this is the limit, then the limit of this thing in the red, red box is f prime a, which is in the orange box, and that is how you get this expression. So, I have more or less given you a proof. Well, not exactly, but at least give you given you a fairly good holding point for, for the statement, why the statement holds.

So, I just repeat the statement, if the function is differentiable at the point a, then the tangent to f at a exists, and is given by this expression here y is $f'(a)(x-a)+f(a)$. And what is this saying. The other statement, it is saying more or less the converse. Suppose the tangent exists, and it is not vertical. So, let us, in a minute, we will study why we have this caveat.

So, vertical, meaning it is not like this, there is nothing preventing the tangent from being like this after all. So, so if that is not the case, then f is differentiable at a and the equation of the tangent is y is $f'(a)(x-a)+f(a)$. So, in other words, what we are saying is that differentiability is equivalent to the tangent existing and being given by this expression except in the case where you have a vertical tangent. That vertical tangent is not captured by the definition of derivative that we have made, we can actually modify that definition if we allow something called infinity, then, then we will also be able to deal with vertical lines. But for now, that is not what we are doing.

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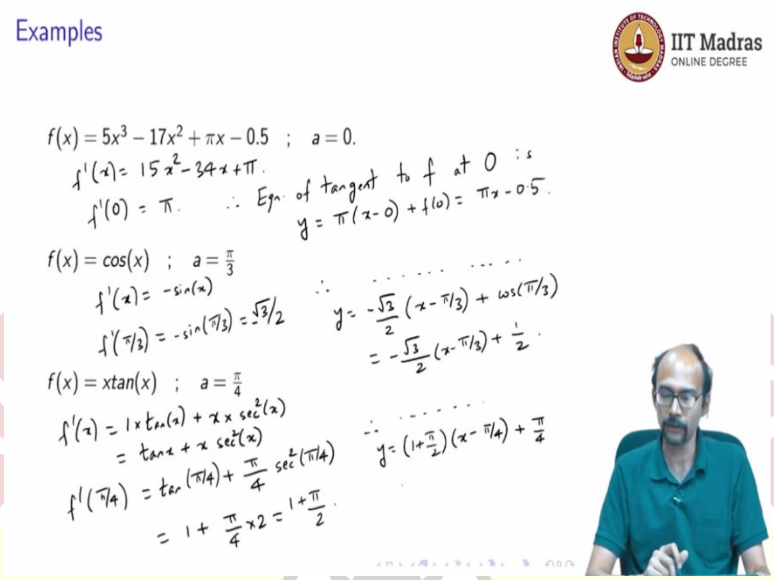

So, let us talk about this getting the tangents. So, let us compute the tangent the equation of the tangent line for these examples. So, we have seen these examples before. So, we know what are the derivatives. So, the tangent line here, we want to find it at the point a is 0. So, $f'(x)$ is $15x^2 - 34x + \pi$. So, that means $f'(a)$which is 0, so $f'(0)$is, well, π, so therefore, the equation of the tangent at a tangent to f at a, so a is 0, so add 0 a is y $=f'(0)$, which is π times x-0,+f of 0. And what is f of 0, f of 0 is-0.5.

So, this is π times x-0.5. This is the equation of the tangent line to f at 0. Let us do the second example. So, f prime x here is-sin of x. So, we want to evaluate this at π by 3, which is the point we are given. So, f prime of π by 3 is-sin of π by 3, so I suppose-$\frac{\pi}{3}$ is $\frac{3}{2} - \frac{\sqrt{3}}{2}$. And so therefore, the equation of the tangent to $f(a)$ is $y = \frac{-\sqrt{3}}{2}x - \frac{\pi}{3}$+$f(\frac{\pi}{3})$, what is $f(\frac{\pi}{3})$? So, cosine of $\frac{\pi}{3}$, so, cosine of 60, if I remember is $\frac{1}{2}$, so this is$\frac{-\sqrt{3}}{2}x - \frac{\pi}{3} + \frac{1}{2}$. That is the equation of your tangent line.

And let us do this final example. This will give us some. So here, let us find $f'(x)$. But what is f prime x here? This looks like a complicated function. But last time, we have studied the product rule for functions, so if it is a product of two functions for which we know how to find the derivative, at a point, we can do it for by the product rule. So, by the product rule, this is 1 times tangent of $x + x$ times derivative of tangent of x, which is secant squared x. So, this is tan $x + x$

secant squared of x. So now, I want to do the, evaluate this at the point π by 4. So, at π by 4 is at is 45 degrees.

So, that is $\tan\frac{\pi}{4}$, which, if I remember, is 1, and then the rest $sec^2\frac{\pi}{4}$, so I think so, $\cos\frac{\pi}{4}$ was $\frac{1}{\sqrt{2}}$, so secant of π 4 is root 2. So, secant squared is 2. So, this is going to be $1+\frac{\pi}{4}\times 2$. So, this is $1+\frac{\pi}{2}$. And now the so therefore, the equation of the tangent line is y =$1+\frac{\pi}{2}x-\frac{\pi}{4}$,+ $f(\frac{\pi}{4})$, so it is$\frac{\pi}{4}$ times tan, $\tan\frac{\pi}{4}$, which we just saw was 1. So, this is just $\frac{\pi}{4}$.

And I guess we can simplify this much more beyond this. So this is what it is. So, I so, this is the tangent line to this function, x times tan x at $\frac{\pi}{4}$. So, I hope you can see that computing the tangent is a very, the derivative is useful, because now I can actually get the equation of the tangent line, this is something we can plot in a graph. So, I hope you understand how to compute the equation of the tangent line, once we know how to compute derivatives.

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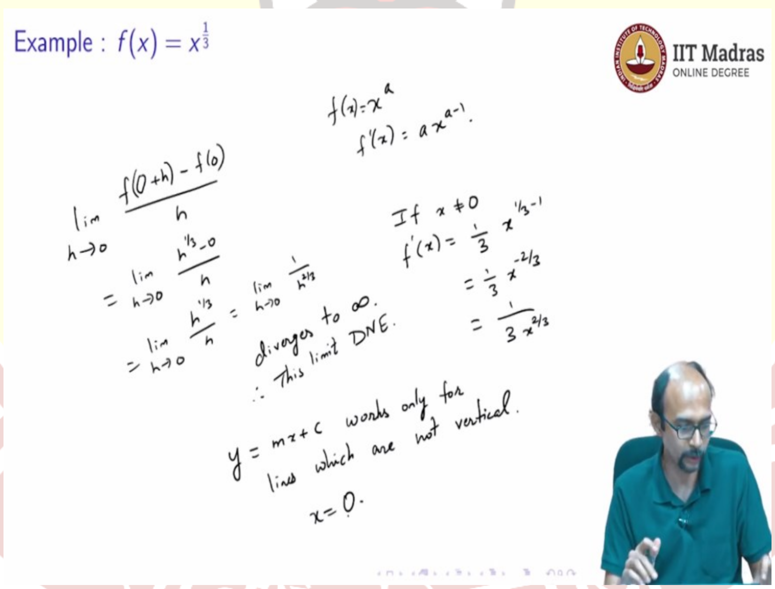

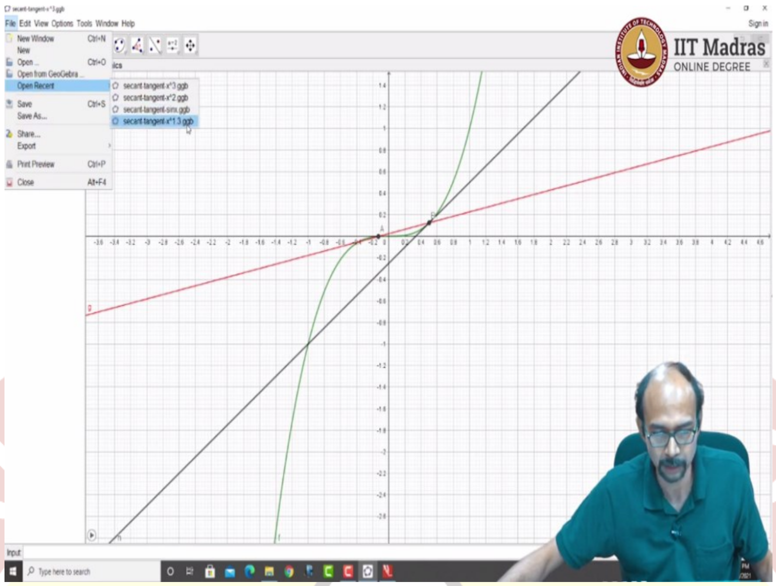
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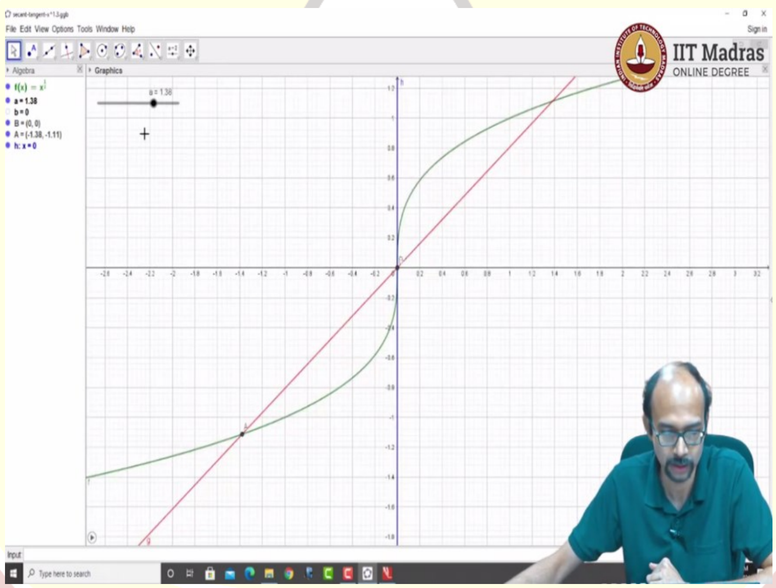
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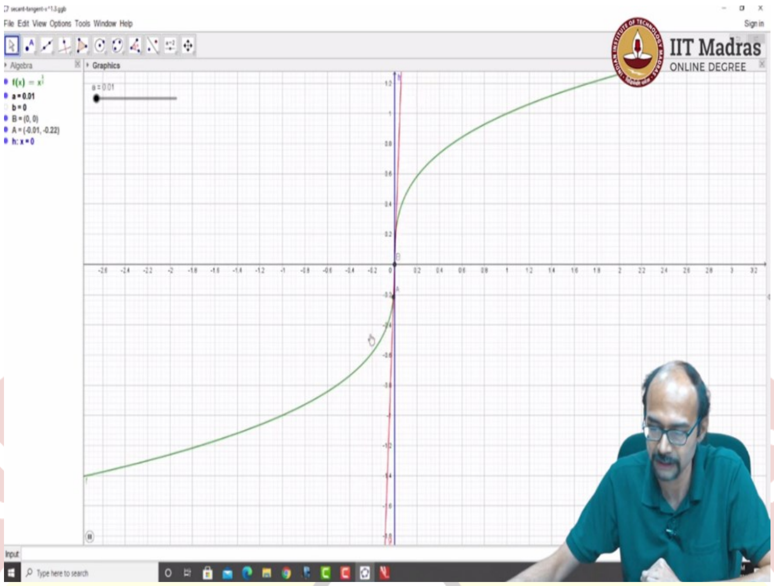

Fine. So, let us look at this example $f(x)$ is $x^{\frac{1}{3}}$. So for this example, let us go back to our animations. So here, here is $x^{\frac{1}{3}}$, we bring this into the middle. So here, the green curve here is the graph of the function. So, this function is $x^{\frac{1}{3}}$. The red line here is the secant. So, that is the line joining B, which is a point (0,0). And the point A, which is going to move towards B and where is the tangent? Well, here is the tangent.

The tangent is the y axis in this case, and this is this is a vertical tangent. And why is this the tangent? Well, you can see that as, as you come close, as the as the graph comes close. Indeed, this the instantaneous direction of this graph, at the point (0, 0), is indeed is indeed vertical. So this is the tangent. But what happens though, once you play this, let us look at what happens to the secant. So, the secant comes close, close, close and indeed, it approximates the tangent.

So, the secant approximating the tangent is not, not really a problem here. So, why is, why is this a big deal? Why are we making a big deal about this example? So let us try to compute what happens to the derivative, we have seen that if you have x^a, then the derivative of this function I claimed was ax^{a-1} and I made some small remarks about what happens if a or $a-1$ is negative, that case, we have a problem at 0 and really, that is exactly what happened here. So, for x to the power one-third, if you have a point for if x is not 0, $f'(x)$, where here f' means $f(x) = x^{\frac{1}{3}}$.

And $f'(x)$ is $\frac{1}{3}x^{\frac{1}{3}-1}$which is $\frac{1}{3}x^{\frac{2}{3}}$. In other words, $\frac{1}{3}x^{\frac{2}{3}}$, so you square x and take it cube root. That is what the function f' is. If x is not 0, but add 0, you can see there is a problem. So, you have x in the denominator and that is exactly what happens. So, if we try to compute this limit at 0, let us see what that gives us. So, $\lim_{h \to 0} f(0+h)$So, this is $f(h)$ to $h^{\frac{1}{3}-0}/h$, so, this l$\lim_{h \to 0} h^{\frac{2}{3}}/h$, which is and this limit does not exist or, so again, it depends on what how the convention is.

So, one way of thinking of this is that we have this number called infinity and if we allow infinity, then really we can salvage the situation. But since we are not doing that, we will say this diverges to infinity. And because that is the case we say this, so therefore it does not exist. So, therefore, this limit in our notation, we say it does not exist. So, this function is not differentiable at 0. And what is the issue, the issue is that we get y is equal to, or rather $x = 0$ as the, as the derivative sorry, as the tangent. And the problem is, if we write the form $y = mx + c$, this works only for works only for lines which are not vertical.

So, for the vertical line, you have to separately say x is 0. So, the problem is your m is only a real number. So, if you allow m to be infinity, you can sort of divide by m, and that will give you a $\frac{1}{m}$, so, 1 by ∞, which is 0, so, you will get $x = 0$. And if you allow that, then indeed, we can salvage the situation. So, that is what, what the problem is. So, that is why in the previous slide, we had to make this caveat about the tangent not being vertical.

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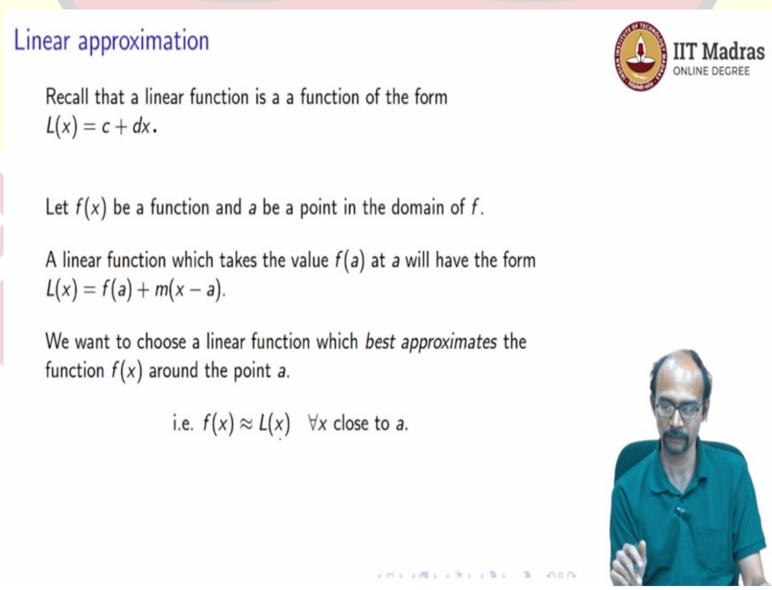

So, let us now consider the notion of Linear Approximation. So, what is Linear Approximation? So, we have a function f, and we want to approximate it by a linear function. So, what is a Linear

Function? Let us recall that a linear function is a function of the form $L(x)$ is $c + dx$. So, we have seen this kind of thing before in our Linear Algebra videos, if you have seen them before, $L(x)$ is $f(a) + m(x - a)$.

Again, with the caveat that if you allow vertical functions, vertical linear functions, then that function will just be the function $x = a$ point form for a line. So, if you have a $L(x)$ is f of a+m times x-a then m is the slope. And how do I know that? This, this, this is the linear function which takes a value f of a at a, because if you put x =a, then indeed on the right hand side, you get a f of a. So, $L(a) = f(a)$, so it will pass through the point $(a, f(a))$.

So, that means the graph of the function L will add the point a will be the same as the graph of the function f at the point a. So, we want to choose a linear function which best approximates the function f of x around the point a. So, that is f of x is approximately= $L(x)$ for all x close to a. And you can see that I have written this best approximates in italics. So, really the point is here, we want to, we want to get this get a hand on what it means for best approximate. Unfortunately, this is a slightly technical notion, somewhat difficult to explain.

But what, what really we are saying is the following, amongst all possible linear functions, which linear function approximates this best in the sense that if you have some other approximation, then the difference between fx and Lx will be smaller as compared to the difference between fx and that other approximation for all x, which are close to a, so, there is some small interval for which that the difference of fx and Lx must be smaller than the difference of fx and any other linear function. So, in this sense, it is best approximates. So, there are many issues here. I mean, first of all, why do we even know that such a thing exists?

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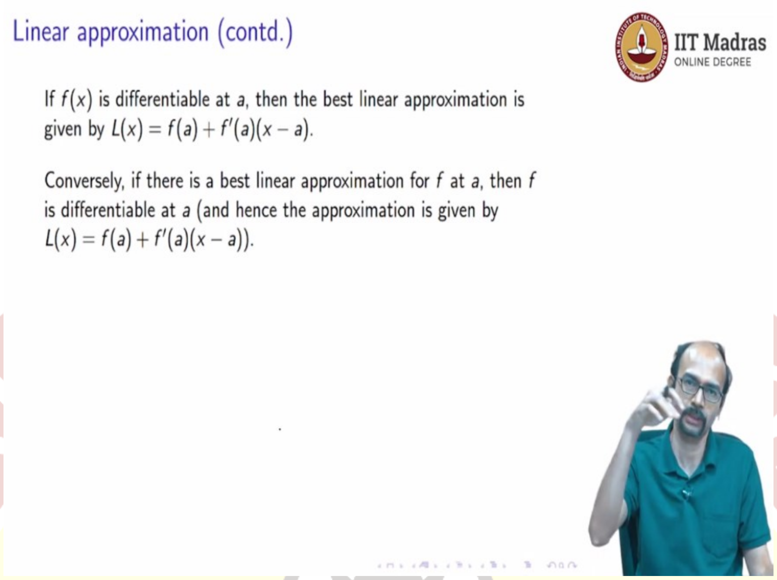

So, turns out that if f is differentiable at a, then best linear approximation is, in fact, given by the expression for the tangent. So, $L(x)$ is f of a+f prime a times x-a. Conversely, if there is a best linear approximation for f at a, then f is differentiable at a. And once we know it is differentiable at a, the best linear approximation must be of this form, f of a+f prime a times x-a. So, this is not very difficult to imagine, it is a very intuitive statement, because if you looked at the pictures that we had, indeed, as, as the function came closer to the point a as x came closer to the point a, the function value came, the graph of the function f was very closely approximated by the tangent.

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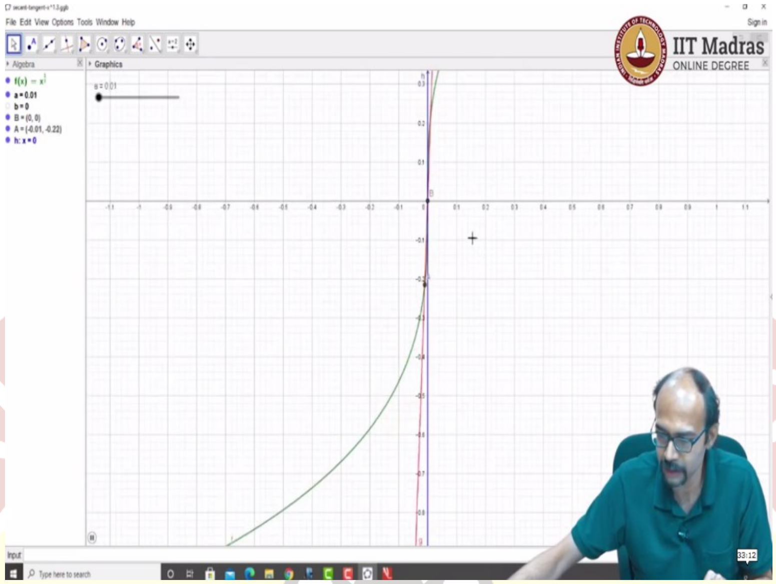

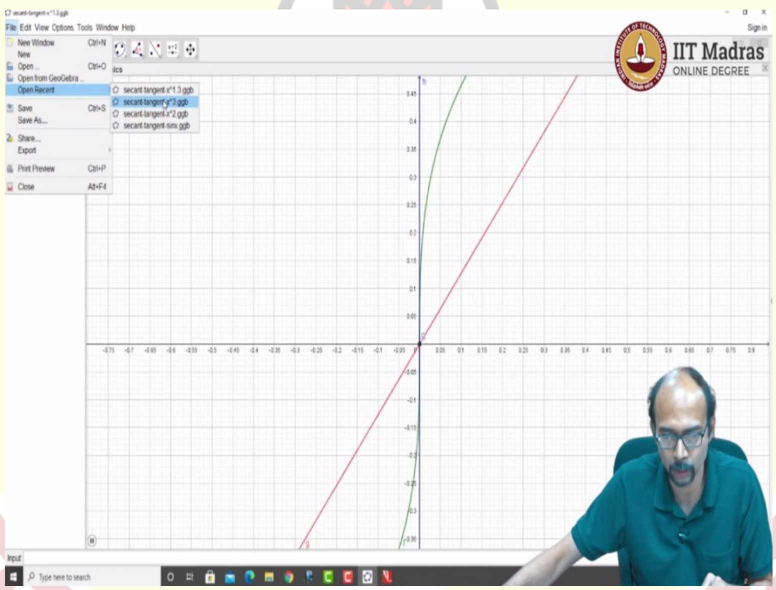

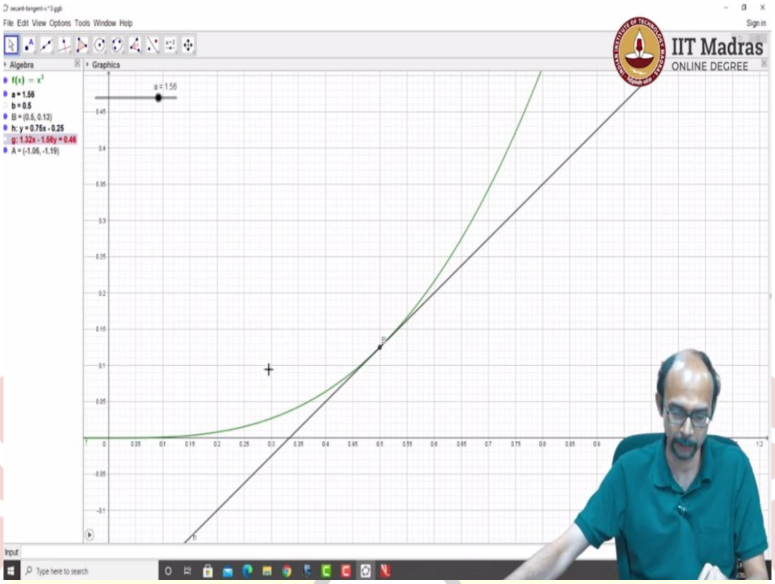

So, for example, in this, this video here, when we play this, as the function is coming closer, you can see the graph is, it is it is very, the tangent line is very close to the graph. So, this blue line is very close to the green line, the green curve, at the point (0,0). In fact, here, it is almost indistinguishable. Even in the other examples. Let us do one, let us look at one more example. Maybe the sample of x cube. You can see here, stop the animation, you can see here, at the point that the point 0.5, we do not need the red line.

Close to this point, the curve, meaning the graph of the function and the tangent, they are very, very, very close. So, that is, that is the intuition behind the best linear approximation.

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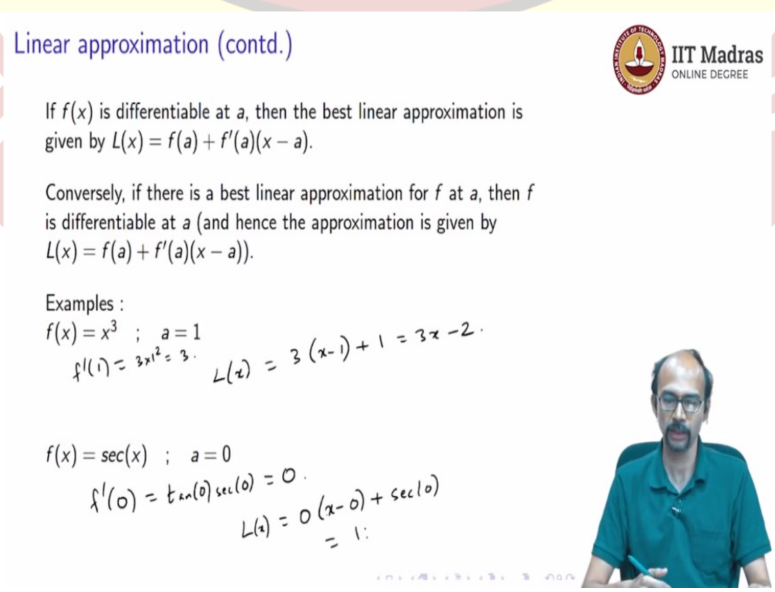

And this is the statement that indeed, it is the best Linear Approximation. Unfortunately, we cannot prove it in this series of lectures, but those of you who are more mathematically inclined can try give it a shot by using the definition of the tangent. So, let us do this, these examples $f(x)$ is x^3, a is 1, what is the best Linear Approximation? This is exactly the same as finding the tangent. So here, you will have to find f' at 1. So, $f'(1)$is 3×1^2.

I am doing it faster now than then I did earlier. So, the best linear approximation is $L(x)$ is $f'(1)$ which is $3(x-1)+f(1)$, which is 1. So, this is $3x-3+1$ so $3x-2$. So, if you have secant, you want f', so add 0. So, f' and 0 is, so, that Derivative of the secant of x is $tan\ x$ times secant of x. So, this is tan of 0 secant of 0, a tan of 0 is 0, so this is 0.

And that tells us that y is equal to, so L of not y, but $L(x)$, the best linear approximation is 0 times $x-0, +f\ (0)$. So, secant of 0, and what is the secant of 0? Secant of 0 is 1, so at 0. The best Linear Approximation is the constant function 1. So, this is a very interesting situation when, when the best linear approximation is a constant function, or a lower the tangent is flat.

So, we will be studying this phenomenon in the next video, and we will be picking up things called Turning Points. This was something you may have read, you can recall from Maths 1. And this phenomenon will be explored in more detail in the next video. So, let us just recall what we did in this video we have finally being able to link up the tangent with the derivative. Namely, the tangent has an equation which is given by f' of a times $f'(a)(x-a)+f(a)$. And it turns out that, that expression is also the best linear approximation to your function f. Thank you.