

# Shortest Paths in Weighted Graphs

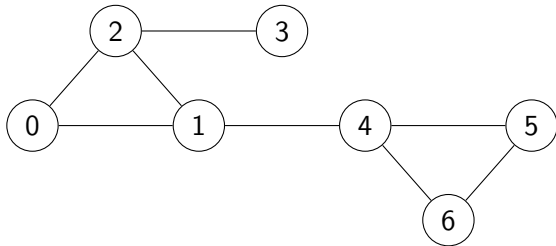
Madhavan Mukund

<https://www.cmi.ac.in/~madhavan>

Mathematics for Data Science 1  
Week 12

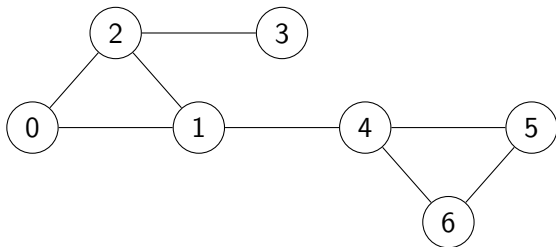
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- Recall that BFS explores a graph level by level



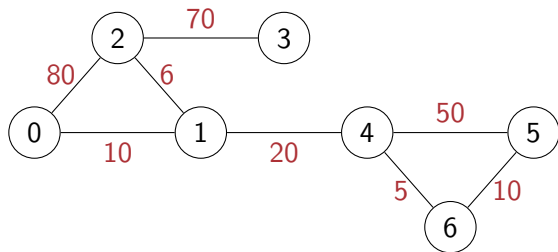
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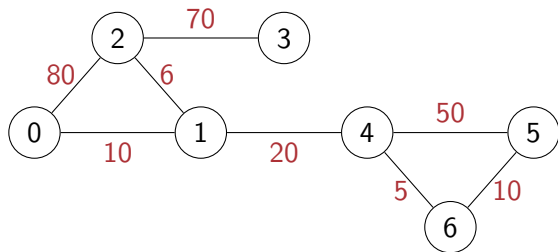
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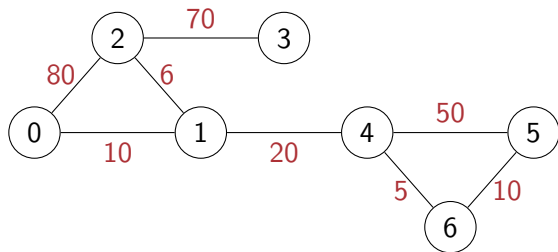
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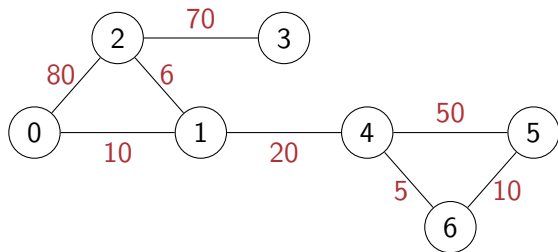
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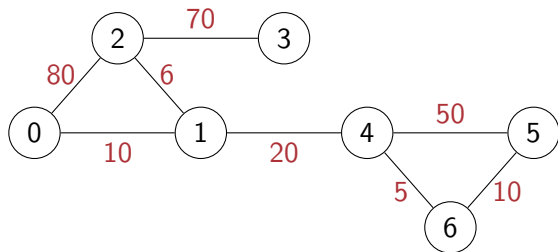
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	0	1	2	3	4	5	6
0	0	10	80	0	0	0	0
1	10	0	6	0	20	0	0
2	80	6	0	70	0	0	0
3	0	0	70	0	0	0	0
4	0	20	0	0	0	50	5
5	0	0	0	0	50	0	10
6	0	0	0	0	5	10	0

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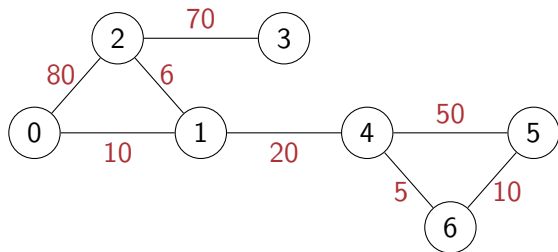
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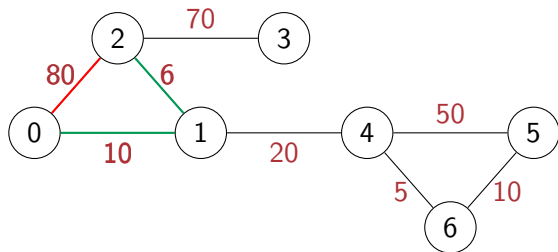
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- In a weighted graph, add up the weights along a path
- Weighted shortest path need not have minimum number of edges
  - Shortest path from 0 to 2 is via 1



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- Find shortest paths between every pair of vertices  $i$  and  $j$
- Optimal airline, railway, road routes between cities



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- Without negative cycles, we can compute shortest paths even if some weights are negative

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- Negative edge weights
  - Should not have negative cycles
  - Without negative cycles, shortest paths still well defined