Limits and continuity

Sarang S. Sane

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$f(x) = \lfloor x \rfloor$

$\lim_{x \to a^-} x = \begin{cases} a & \text{if } x \notin \mathbb{Z} \\ a-1 & \text{if } x \in \mathbb{Z} \end{cases}$

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Recall also that we have defined the notion of the limit as x tends to ∞ (resp. $-\infty$) denoted by $\lim_{x \to \infty} f(x)$ (resp. $\lim_{x \to -\infty} f(x)$.

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Examples

Note: $\lim_{x \to a} x^n = a^n$

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$\lim_{x\to a} f(x) = 5 \lim_{x \to a} x^3 + 0.45 \lim_{x\to a} x^2 - 2 \lim_{x\to a} x + \lim_{x \to a} 100 = 5a^3 + 0.45a^2 - 2a + 100 = f(a)$

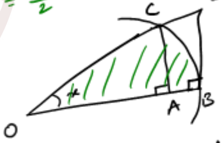
$= f(a)$

2. $f(x) = \lim_{x \to 0} \dfrac{5x^3 + 0.45x^2 - 2x + 100}{x^2 - 5x + 6}$

$\lim_{x\to 0} x^2 - 5x + 6 = 0^2 - 5\times 0 + 6 = 6 \neq 0.$

$= \dfrac{\lim_{x\to 0} 5x^3 + 0.45x^2 - 2x + 100}{\lim_{x\to 0} x^2 - 5x + 6} = \dfrac{100}{6} = 16.666\ldots$

3. $\boxed{\lim_{x \to 0} \dfrac{\sin(x)}{x} = 1.}$



Area of Δ OBD $= \frac{1}{2} \times 1 \times |BD| = \tan x / 2$

$|OB| = |OC| = 1.$

Area of sector $= \frac{x}{2}.$

$= \frac{2\pi}{2}$ π is the area of the full circle.

$\frac{x}{2} \leq \frac{\tan x}{2}$

$\Rightarrow x \leq \tan x$

$\frac{|BD|}{|OB|} = \tan x$

$\Rightarrow |BD| = \tan x.$

$|AC| = \sin x.$

$|\text{Arc } BC| = x$

& $\sin x \leq x.$

$\sin x \leq x \leq \tan x$

$1 \leq \frac{x}{\sin x} \leq \frac{1}{\cos x}$

$1 \geq \frac{\sin x}{x} \geq \cos x$

$\lim_{x \to 0} 1 \geq \lim_{x\to 0} \frac{\sin x}{x} \geq \lim_{x\to 0} \cos x = \cos 0 = 1.$

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   $= 1 \times \frac{1}{\cos 0} = 1 \times \frac{1}{1} = 1.$

2. $\lim_{x\to 0} \frac{1 - cos(x)}{x^2}$ $= \lim_{x\to 0} \frac{2\sin^2 \frac{x}{2}}{x^2}$

   $= \frac{2}{4} \lim_{x\to 0} \frac{\sin^2 \frac{x}{2}}{x^2/4} = \frac{1}{2} \lim_{y\to 0} \left(\frac{\sin y}{y}\right)^2$

   $= \frac{1}{2}.$

   $\cos 2\theta = \cos^2\theta - \sin^2\theta$
   $= 1 - 2\sin^2\theta$
   $\Rightarrow 1 - \cos 2\theta = 2\sin^2\theta$

3. $f(x) = \lim_{x\to 0} \frac{log_e(1 + x)}{x}$

   Try this.

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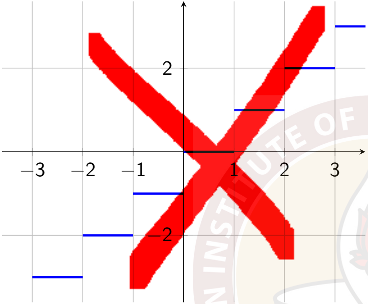
Examples : Polynomials, rational functions with non-zero denominators, e^x, $log(x); x > 0$, $sin(x)$, $cos(x)$.

Tangents : Examples advising caution : where are we?



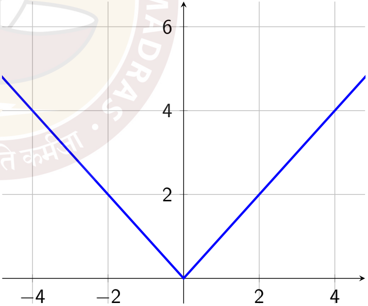
$\Gamma\left(\lfloor x \rfloor\right)$

Will not talk about tangents at points of discontinuity

$\lim_{x\to 0^-} |x| = 0 = \lim_{x\to 0^-} -x = 0.$

$\lim_{x\to 0^+} |x| = 0 = \lim_{x\to 0} x = |0|.$

Graph of $y = |x|$



Thank you