Logarithmic Functions

Monday, 14 September 2020 9:56 AM

Recall. $f(x) = a^x$ $(a > 0, a \neq 1)$

is one-to-one, it has its inverse

Def^n. The logarithmic function (to the base a) in standard form is

$$y = \log_a(x)$$

and is defined to be the inverse of

$f(x) = a^x$

$\log_a x = y$ $\boxed{7\text{-rule}}$ $a^y = x$

$$y = \log_a x \iff x = a^y$$

$$a^{\log_a x} = x \quad \& \quad \log_a a^x = x$$

$$f(f^{-1}(x)) = x \qquad f^{-1}(f(x)) = x$$

$\mathrm{Dom}(a^x) = \mathbb{R}$ $\qquad$ $\mathrm{Range}(a^x) = (0,\infty)$

$\parallel$ $\qquad$ $\parallel$

$\mathrm{Range}(\log_a)$ $\qquad$ $\mathrm{Dom}(\log_a)$

☆ $\mathrm{Dom}(\log_a) = \mathrm{Range}(a^x) = (0,\infty)$

☆ $\mathrm{Dom}(a^x) = \mathrm{Range}(\log_a) = \mathbb{R}$

Example. $\boxed{f(x) = \log_4(1-x)}$ Find the domain of f

$\boxed{\mathrm{Dom}(\log_4) = (0,\infty)}$ $\mathrm{Dom}(f) = (-\infty, 1)$

$1-x > 0 \iff 1 > x$ $x < 0$

$1-x > 0$

Example. $g(x) = \log_3\left(\frac{1+x}{1-x}\right)$, $x \neq 1$

$\mathrm{Dom}(g) = (-1, 1)$ ✓

$\mathrm{Dom}(\log_3) = (0,\infty)$

$\frac{1+x}{1-x} > 0$

Example $y = \log_3 x$

$$3^y = 3^{\log_3 x} = x$$

$$3^y = x$$

Example. $(1.3)^2 = m$

$$\log_{1.3} (1.3)^2 = \log_{1.3} m$$

$$a^{\log_a x} = x$$

$$2 = \log_{1.3} m$$

Observe $a^u = a^v \; (a > 0, a \neq 1)$

$$\Rightarrow u = v$$

Find $\log_3\left(\overline{9}\right)$

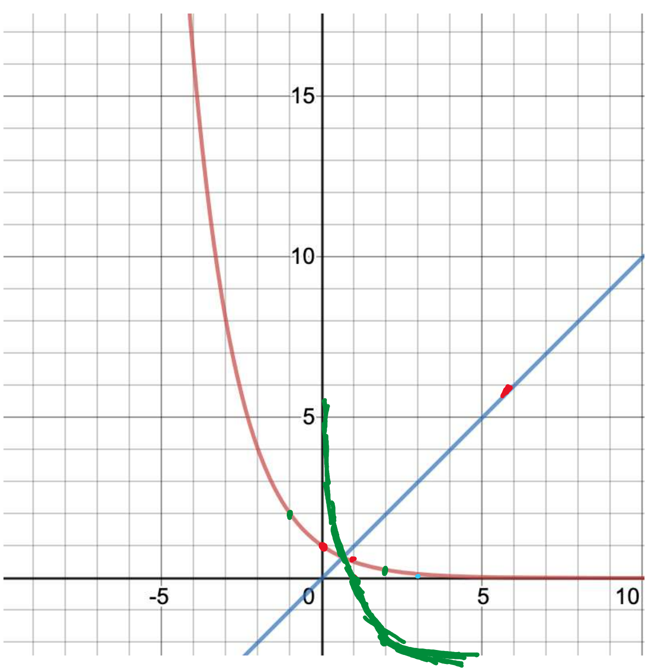
$$\log_3\left(\tfrac{1}{9}\right) = \log_3\left(3^{-2}\right) = \boxed{-2}$$

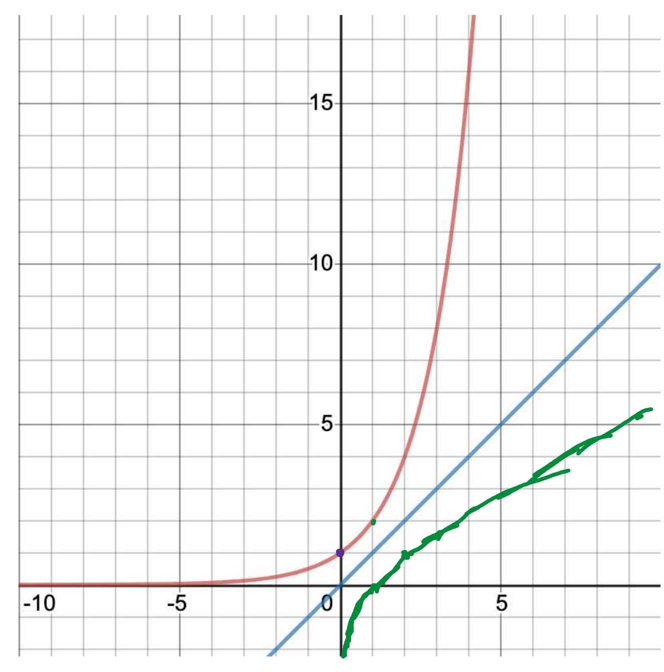
$$3^2 = 9 \Rightarrow 3^{-2} = \frac{1}{9}$$

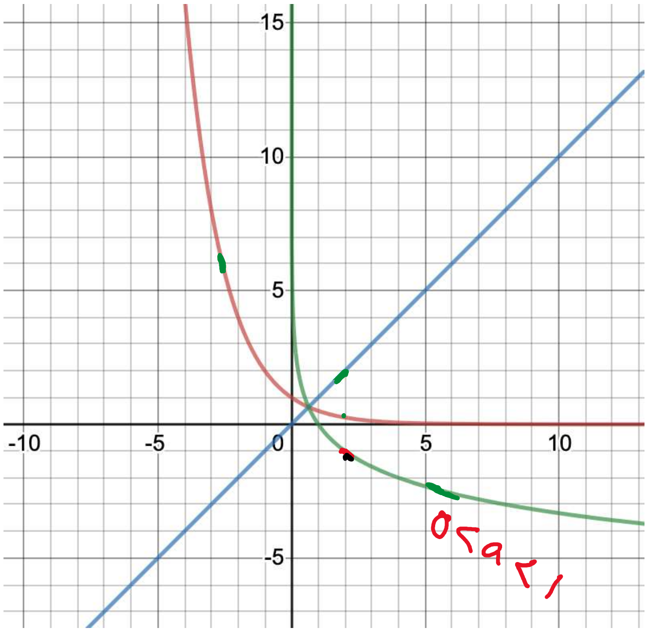
Graph $\underline{f(x) = \log_a x}$

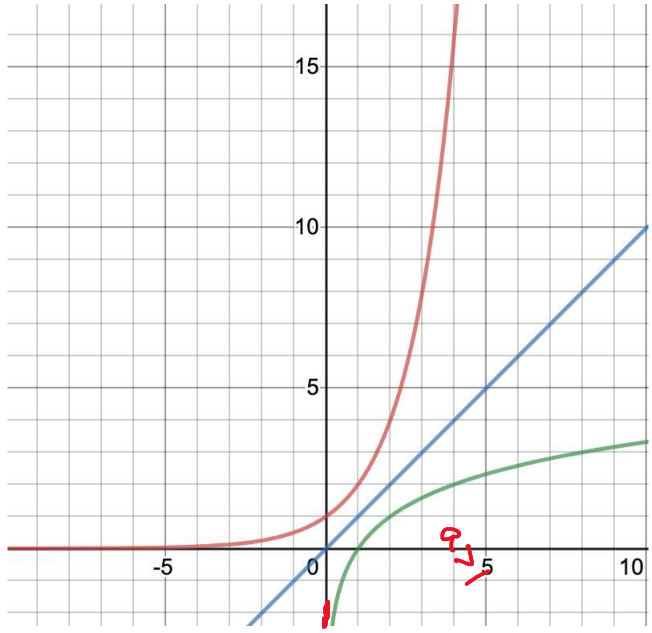
$a \quad 1 \quad a>1$

$* \; 0 < a < 1$ $\qquad\qquad$ $a > 1$









Properties for $f(x) = \log_a(x)$

$\text{Dom}(f) = (0, \infty)$ $\quad$ $\text{Range}(f) = \mathbb{R}$

x-intercept : $(1, 0)$

y-intercept : Nil

Vertical asymptote at $x = 0$ (y-axis)

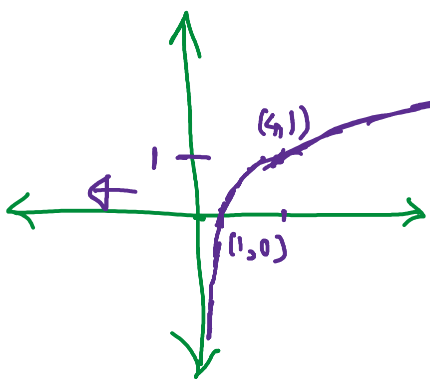
- f is one-to-one & passes through $(1,0)$ & $(a,1)$
- $0 < a < 1$, f is decreasing
- $a > 1$, f is increasing

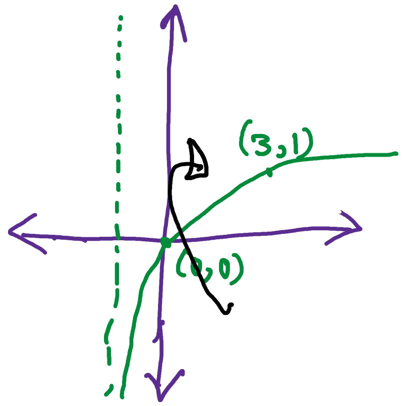
Example. Draw graphs of the functions

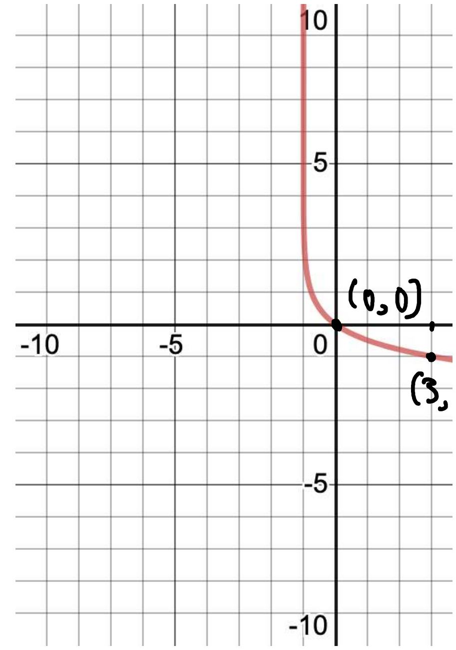
$f(x) = -\log_{a}(x+1)$

★ $g(x) = \log_{1/4}(-x) + 1$

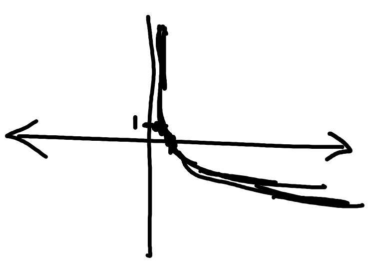
$\log_4(x)$ $\quad\quad$ $-\log_4(x+1)$

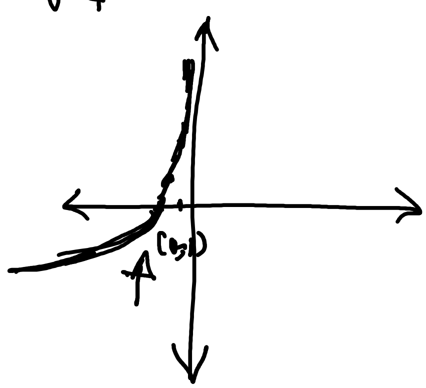


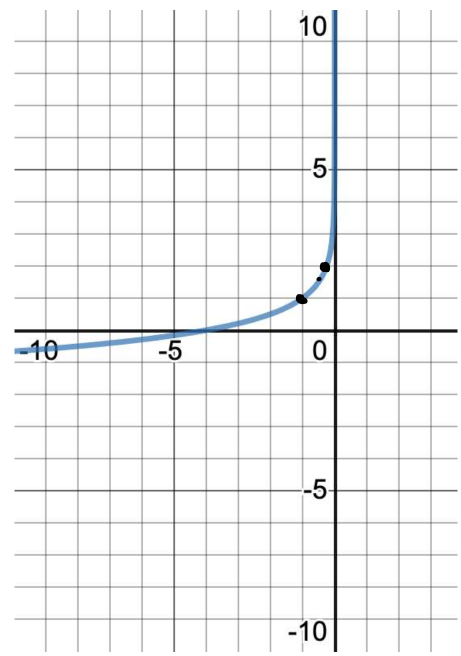




$\log_{1/4}(x)$ $\quad\quad$ $\log_{1/4}(-x)+1$







Properties of Logarithmic Function

★ $a \in (0,1)$ or $a > 1$

Recall. $\boxed{\log_a 1 = 0}$ ⟵ $(1,0)$ $\quad$ $(0,1)$

$$\boxed{\log_a a = 1}$$

$\cdot (a,1)$ $(1,a)$

$$* \; a^{\log_a x} = x = \log_a(a^x) = x$$

$$3^{\log_3(\pi/2)} = \boxed{\pi/2} \qquad 4^{\log_4(1)} = 1$$

Laws of Logarithm

Let $x \in \mathbb{R}$, $\underbrace{0<a<1}$ or $\underbrace{a>1}$; $\underbrace{M, N > 0}$.

Then

① $\log_a(MN) = \log_a M + \log_a N$.

② $\log_a(M/N) = \log_a M - \log_a N$.

③ $\log_a(1/N) = -\log_a N$

④ $\log_a(M^x) = x \log_a M$

Proof of ①. Put $A = \log_a M$ & $B = \log_a N$

$$A+B = \log_a M + \log_a N$$

$$a^{A+B} = a^{\log_a M + \log_a N} = a^{\log_a M} a^{\log_a N}$$

$$= MN$$

$A-B$

M/N

$$a^{A+B} = MN$$

$$\log_a(a^{A+B}) = \log_a(MN)$$

$$\log_a(a^{A-B}) = \log_a(M/N)$$

$A+B = \log_a(MN)$

① $\log_a(M) + \log_a(N) = \log_a(MN)$ //

② $\log_a(M) - \log_a(N) = \log_a(M/N)$.

③ $\log_a(1/N) = \log_a 1 - \log_a N = -\log_a N$.

④ $\log_a(M^x) = x \log_a M$

$x \in \mathbb{N} := \{0, 1, 2, \dots\}$

Partially proved

$$\log_a(M^x) = \log_a(\underbrace{M \dots M}_{x \text{ times}}) = \log_a M + \dots + \log_a M$$
$$= x \log_a M$$

$\boxed{x \in \mathbb{Q} \quad x \in \mathbb{R}}$ $\boxed{\log_a(M^{\pi}) = \pi \log_a M}$

Applications of Laws of logarithm

Simplify using logs.

$$\log_a \left[\frac{x^3 \sqrt{x^2+1}}{(x+3)^4} \right]$$

$$= \log_a\left(x^3 (x^2+1)^{1/2}\right) - \log_a\left[(x+3)^4\right]$$

$$= \log_a(x^3) + \log_a\left[(x^2+1)^{1/2}\right] - 4\log_a(x+3)$$

$$= 3\log_a(x) + \frac{1}{2}\log_a(x^2+1) - 4\log_a(x+3)$$

Warning: $\log_a(M+N) \neq \log_a M + \log_a N = \log_a(MN)$ ✓

$\log_a(M-N) \neq \log_a M - \log_a N = \log_a(M/N)$

Combine using logs

$$\underbrace{2\log_a x + \log_a 9}\ + \underbrace{\log_a(x^2+1) - \log_a 5}$$

$$= \log_a(9x^2) + \log_a\left(\frac{x^2+1}{5}\right)$$

$$= \log_a\left(\frac{9x^2(x^2+1)}{5}\right)$$

$$\boxed{\begin{aligned} \log_a(M+N) &= ? \\ \log_a(M-N) &= ? \end{aligned}}$$ ~~X~~ X

XX

.

Thm. Let $0<a<1$ or $a>1$ and $M, N>0$.

$$M=N \iff \log_a M = \log_a N$$

Two important values of a are 'e' & '10'

ln log

natural $\boxed{\ln}$ $\boxed{\log}$ common

Natural Common

Change of base Rule

old base new base

Thm. If $0<a<1$ or $a>1$ & $0<b<1$ or $b>1$.

Then, for $x>0$,

$$\boxed{\log_a x = \frac{\log_b x}{\log_b a}}$$

Proof. $M=\log_a x$ $N=\log_b x$ $R=\log_b a$

$\boxed{a^M = x}$ $\boxed{b^N = x}$ $\boxed{b^R = a}$

$(b^R)^M = x$

$b^{RM} = x$

$\log_b(b^{RM}) = \log_b x$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$RM = \log_b x \Leftrightarrow (\log_b a)(\log_a x) = \log_b x$$

Examples. $\log_5 89 = \frac{\ln 89}{\ln 5} \approx 2.78$ ✓

$$\log_{\sqrt{2}} \sqrt{5} = \frac{\ln\sqrt{5}}{\ln\sqrt{2}} \approx 2.32$$ ✓

$\boxed{\ln\left(\frac{x}{a}\right) = ?}$ $\boxed{\frac{\ln x}{\ln a}}$ Warning

Graph: $\log_2 x$

common → $\log_{10} x$

natural → $\ln x$ ✓ ~~"$\log_e x$"~~

$$f(x) = \log_2 x = \frac{\ln x}{\ln 2} = \left(\frac{1}{\ln 2}\right)\ln x$$

Prove that $\boxed{\frac{1}{\log_2 \pi} + \frac{1}{\log_6 \pi} > 2}$ ✓ ✓

$$LHS = \frac{1}{\log_2 \pi} + \frac{1}{\log_6 \pi} = \frac{\ln 2}{\ln \pi} + \frac{\ln 6}{\ln \pi}$$

$$= \frac{\ln 2 + \ln 6}{\ln \pi} = \frac{\ln(12)}{\ln \pi} \overset{?}{>} 2$$

$$\frac{\ln(12)}{\ln \pi} \overset{?}{>} 2$$

$\ln(12) \overset{?}{>} 2(\ln\pi) = \ln(\pi^2)$ ✓

$\ln(12) \overset{?}{>} \ln(\pi^2)$

Exponentiate with e $\quad e^{\ln(12)} \overset{?}{>} e^{\ln(\pi^2)}$

$$\boxed{a^{\log_a x} = x}$$

$\checkmark\ 12 \overset{?}{>} \pi^2$

$3.141\ldots$

$\pi^2 < (3.15)^2$

$\pi^2 < 10 < 12$

Solve Logarithmic Equations

$2\log_{0.5} x = \log_{0.5} 4$ $\quad$ Solve for x.

$x = 2$

$2\log_{0.5} x = \log_{0.5} 4$

$\log_{0.5} x^2 = \log_{0.5} 4$ $\qquad \boxed{a^{\log_a x} = x}$

$(0.5)^{\log_{0.5} x^2} = (0.5)^{\log_{0.5} 4}$

$x^2 = 4$

$x = \pm 2$ $\qquad \boxed{x = \cancel{-2}, 2}$

$\boxed{x = 2}$

$\cancel{-3+1=-2}$ $\quad$ $\cancel{-3-1=-4}$

$3+1=4$ $\quad$ $+3-1=2$

Solve for x: $\log_8(x+1) + \log_8(x-1) = 1$

$3+1 \qquad 3-1$

$$\underbrace{\log_8(x+1)+\log_8(x-1)}=1$$

$$\log_8[(x+1)(x-1)]=1$$

$$8^{\log_8[(x^2-1)]}=8^1$$

$$x^2-1=8 \iff x^2-9=0$$

$x=\pm 3$ possible sol^n

$$\boxed{x=3}$$

Solve for x.

$$\log_3 x+\log_4 x=4$$

$$\log_3 x+\log_4 x=4$$

$$\frac{\ln x}{\ln 3}+\frac{\ln x}{\ln 4}=4$$

$$\ln x\left[\frac{1}{\ln 3}+\frac{1}{\ln 4}\right]=4$$

$$\ln x=4\left[\frac{1}{\frac{1}{\ln 3}+\frac{1}{\ln 4}}\right]$$

$$=4\left[\frac{\ln 3\cdot\ln 4}{\ln 3+\ln 4}\right]$$

$$\ln x=4\frac{\ln 3\cdot\ln 4}{\ln(12)}$$

$$\boxed{x=e^{4\frac{\ln 3\cdot\ln 4}{\ln 12}}}$$

Example. Solve for x.

$$\ln(x^2) = \boxed{(\ln x)^2}$$

$$\ln(x^2) = (\ln x)^2$$

$$2(\ln x) = (\ln x)^2 \qquad \boxed{\ln x = t}$$

$$2t = t^2$$

$$2t - t^2 = 0 \qquad t(2-t) = 0$$

$$\boxed{t=0} \text{ or } \boxed{t=2}$$

$$\ln x = 0 \text{ or } \ln x = 2$$

$$e^{\ln x} = e^0 \text{ or } e^{\ln x} = e^2$$

$$x = 1 \text{ or } x = e^2$$

$$\bullet \boxed{x=1} \text{ or } \boxed{x=e^2} \bullet$$