Statistics for Data Science -1

Lecture 7.7: Conditional Probability: Bayes' Rule

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2. Distinguish between independent and dependent events.
3. Solve applications of probability.

Law of total probability

Bayes' rule

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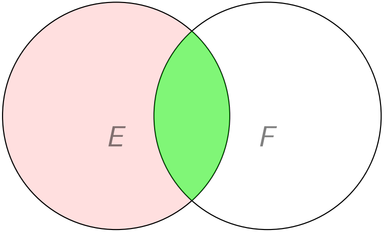

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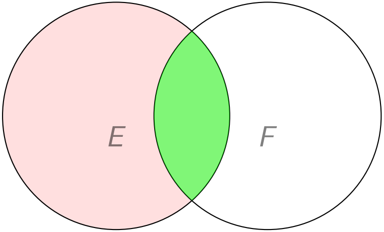

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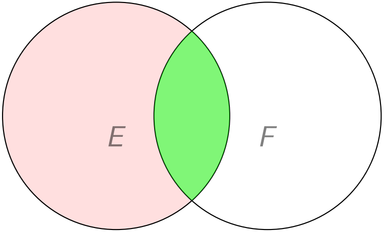

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- In other words, for, in order for an outcome to be in E, it must either be in both E and F or be in E but not in F.

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- Each conditional probability is weighted by the probability of the event on which it is conditioned.

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$$P(E) = \sum_{i=1}^{k} P(E|F_i)P(F_i)$$

Example 1: Application- Insurance policy

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- An insurance company[1] believes that people can be divided into two classes—those who are prone to have accidents and those who are not. The data indicate that an accident-prone person will have an accident in a 1-year period with probability 0.1; the probability for all others is 0.05. Suppose that the probability is 0.2 that a new policyholder is accident-prone. What is the probability that a new policyholder will have an accident in the first year?

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There is a 6 percent chance that a new policyholder will have an accident in the first year.

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Therefore, given that a new policyholder has an accident in the first year, the conditional probability that the policyholder is prone to accidents is $1/3$.

Section summary

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2. Bayes' rule