Week - 8
Prcactice Assignment
Mathematics for Data Science - 1

1. If $b > 0$ and $\boxed{4\log_x b + 9\log_{b^5 x} b = 1,}$ then the possible value(s) of x is(are)
   - (a) $\boldsymbol{b^{10}}$
   - (b) b^9
   - (c) $\boldsymbol{b^{-2}}$
   - (d) b^5
   - (e) b^4

Soln:

$$4\log_x b + 9\log_{b^5 x} b = 1$$

$$\Rightarrow \frac{4}{\log_b x} + \frac{9}{\log_b(b^5 x)} = 1$$

$$\Rightarrow \frac{4}{\log_b x} + \frac{9}{\log_b b^5 + \log_b x} = 1$$

$$\Rightarrow \frac{4}{\log_b x} + \frac{9}{5\cancel{\log_b b}^{1} + \log_b x} = 1$$

Let $p = \log_b x$

$$\Rightarrow \frac{4}{p} + \frac{9}{5+p} = 1$$

$$\Rightarrow \frac{4(5+p) + 9p}{p(5+p)} = 1 \implies 20 + 4p + 9p = 5p + p^2$$

$$\Rightarrow \boxed{p^2 - 8p - 20 = 0}$$

$\Rightarrow p^2 - 8p - 20 = 0$

$\Rightarrow p^2 - 10p + 2p - 20 = 0$

$\Rightarrow p(p-10) + 2(p-10) = 0$

$\Rightarrow (p+2)(p-10) = 0$

$\boxed{p = -2, 10}$

We know that $\boxed{p = \log_b x}$

If $p = -2$

$-2 = \log_b x$

$\boxed{x = b^{-2}}$

If $p = 10$

$10 = \log_b x$

$\boxed{x = b^{10}}$

Note:-

$$\boxed{\log_{a^b} c = \frac{1}{b} \log_a c}$$

✓ * Can be used later

Proof:- LHS $= \log_{a^b} c$

$$= \frac{1}{\log_c a^b} = \frac{1}{b \log_c a} = \frac{1}{b} \times \frac{1}{\log_c a} = \frac{1}{b} \log_a c$$

$= \text{RHS}$

$\boxed{\text{LHS} = \text{RHS}}$

2. George deposits ₹5L in a bank that compounded quarterly at the rate of 20% per year. How long will it take to increase his money to 16 times the principal amount(in year)?

~~(a)~~ $\frac{\ln 16}{4}$

(b) $\mathbf{\frac{\ln 16}{4 \ln \frac{21}{20}}}$

(c) $\mathbf{\frac{\ln 2}{\ln \frac{21}{20}}}$

(d) $\mathbf{\log_{\frac{21}{20}} 2}$

~~(e)~~ $\frac{\ln 2^{\frac{1}{4}}}{\ln \frac{21}{20}}$

Soln Formula for compound interest

$$\boxed{A = P\left(1 + \frac{R}{100}\right)^{t}}$$

where

t = time period (years)

R = Interest rate per year

P = Principal or initial deposits

A = Amount after t years

n = No. of times it compounded in a year.

$$\Rightarrow \boxed{A = P\left(1 + \frac{R}{n \times 100}\right)^{nt}}$$

$n = 4$

$$A = P\left(1 + \frac{20}{400}\right)^{4t}$$

$$\boxed{A = 16P}$$

$$\Rightarrow 16\cancel{P} = \cancel{P}\left(1 + \frac{1}{20}\right)^{4t}$$

$$\Rightarrow 16 = \left(\frac{21}{20}\right)^{4t}$$

formula : $\ln a^b = b \ln a$

$$\Rightarrow \ln 16 = 4t \ln\left(\frac{21}{20}\right)$$

$$\boxed{t = \frac{1}{4}\left(\frac{\ln 16}{\ln\left(\frac{21}{20}\right)}\right)}$$

$$t = \frac{\ln(16)^{1/4}}{\ln\left(\frac{21}{20}\right)} = \frac{\ln(2^4)^{1/4}}{\ln\left(\frac{21}{20}\right)} = \boxed{\frac{\ln 2}{\ln\left(\frac{21}{20}\right)}}$$

We have

$$\boxed{t = \frac{\ln 2}{\ln \frac{21}{20}}}$$

$$\boxed{\text{Formula : } \log_b a = \frac{\log a}{\log b}}$$

Using change of base formula, we get,

$$\boxed{t = \log_{\frac{21}{20}} 2}$$

3. Choose the set of correct options.

(a) ~~$\log_5 2$ is a rational number~~

(b) If $0 < b < 1$ and $0 < x < 1$ then $\log_b x < 0$

(c) **If $\log_3(\log_5 x) = 1$ then $x = 125$**

(d) If $0 < b < 1, 0 < x < 1$ and $x > b$ then $\log_b x > 1$

(e) **If $0 < \mathrm{b} < 1$ and $0 < x < y$ then $\log_{\mathrm{b}} x > \log_{\mathrm{b}} y$**

Soln

(a) Let $\log_5 2$ be rational number, thus it can be written in $\frac{p}{q}$ form,

$\log_5 2 = p/q$

$\Rightarrow 2 = 5^{\frac{p}{q}}$

$\boxed{2^q = 5^p}$

2 & 5 are co-primes & 2 cannot divide 5

thus assumption is wrong.

So it should irrational

(b) Given:- $0 < b < 1$ & $0 < x < 1$ then $\boxed{\log_b x < 0}$

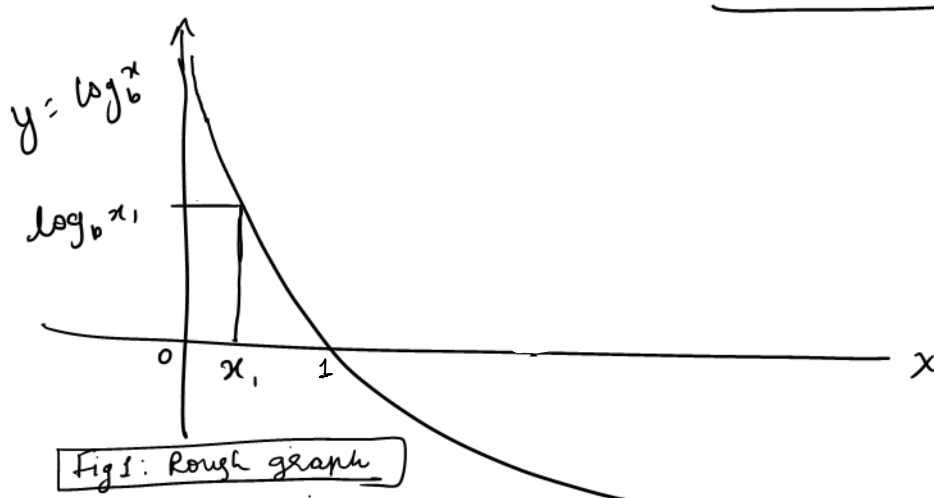

Fig 1: Rough graph

Let x_1 be in

$\boxed{0 < x_1 < 1}$; then from above graph (Fig 1)

$\boxed{\log_b x_1 > 0}$ $\therefore$ statement of option(b) is wrong.

✓ option (c): Given that,

$\boxed{\log_3 \log_5 (x) = 1}$ then $x = ?$

$\log_5 x = 3^1 = 3$

$x = 5^3 = 125$

$\boxed{x = 125}$

Formula:

$\log_a x = b$

$\boxed{x = a^b}$

option (d) Given that:-

$0 < b < 1$, $0 < x < 1$, & $\boxed{x > b}$ then $\boxed{\log_b x > 1}$

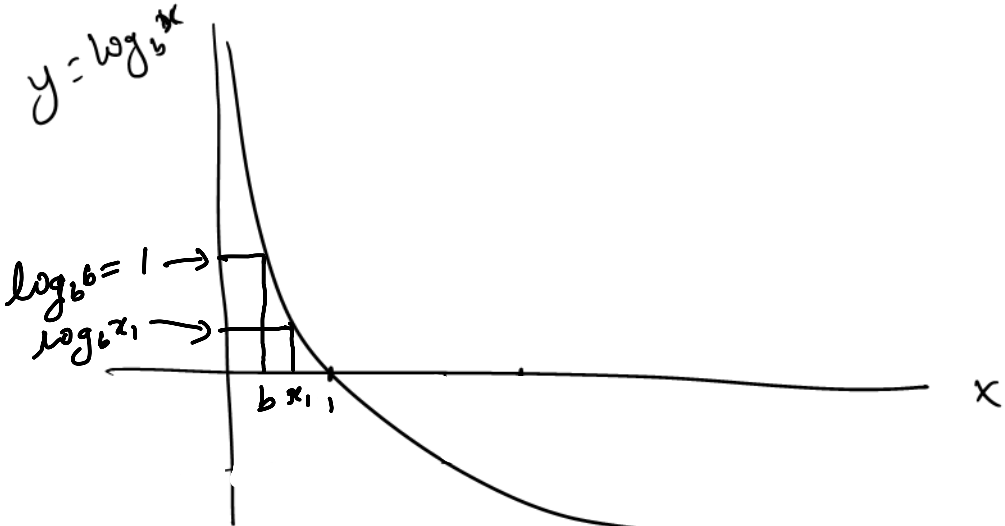

Fig 2: Rough graph

Let $b < x_1 < 1$

then, $\log_b x_1 < 1$ (see Fig 2; notice: $\log_b b = 1$)

Thus the statement of option (d) is wrong.

Option (e) Given that:

$0 < b < 1$ & $0 < x < y$ then $\boxed{\log_b x > \log_b y}$

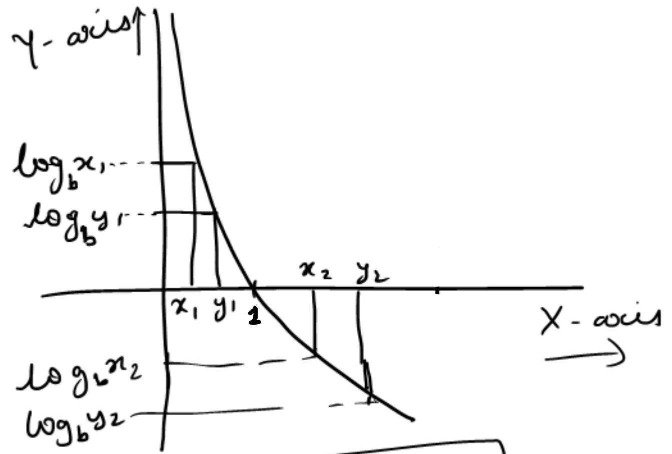

Fig 3: Rough graph

Case I: Let

$$x_1 < y_1 < 1$$

From graph (Fig 3)

$\log_b x_1 > \log_b y_1$

Thus in case I, the given statement is right

Case II: $b < x_2 < 1$ & $b < y_2 < 1$

also $1 < x_2 < y_2$

From graph (Fig 3)

$\log_b x_2 > \log_b y_2$

Thus in case II, the given statement is right again

Thus overall the given statement is right, thus the option (e) is correct.

4. Suppose that two types of insects are found in a pond. Their growth in number after t seconds is given by the equations $f(t) = 5^{3t-2}$ and $h(t) = 3^{2t+1} (t \neq 0)$. For what value of t will both insects be of same number in the pond?

(a) $\frac{\ln 3+2\ln 5}{3\ln 5-2\ln 3}$ ✓

(b) $\frac{\ln 75}{\ln \frac{125}{9}}$ ✓

(c) $\log_{\frac{125}{9}} 75$ ✓

(d) $\frac{\ln 5+2\ln 3}{3\ln 3-2\ln 5}$ ✗

Soln:-

Insects number will be same when

$$f(t) = h(t)$$

$$5^{3t-2} = 3^{2t+1}$$

$$\Rightarrow (3t-2)\ln 5 = (2t+1)\ln 3$$

$$\Rightarrow (3\ln 5)t - 2\ln 5 = (2\ln 3)t + \ln 3$$

$$\Rightarrow (3\ln 5)t - (2\ln 3)t = \ln 3 + 2\ln 5$$

$$t(3\ln 5 - 2\ln 3) = \ln 3 + 2\ln 5$$

$$\Rightarrow \boxed{t = \frac{\ln 3 + 2\ln 5}{3\ln 5 - 2\ln 3}}$$ ✓

formula:

① $\ln a^b = b\ln a$

② $\ln(ab) = \ln a + \ln b$

③ $\ln \frac{a}{b} = \ln a - \ln b$

$$\Rightarrow t = \frac{\ln 3 + \ln 5^2}{\ln 5^3 - \ln 3^2}$$

$$\Rightarrow = \frac{\ln 3 \times 25}{\ln\left(\frac{125}{9}\right)}$$

$$\Rightarrow \boxed{t = \frac{\ln 75}{\ln\left(\frac{125}{9}\right)}}$$

Formula:

$$\log_b a = \frac{\log a}{\log b}$$

Using change of base formula

$$\boxed{t = \log_{\frac{125}{9}} 75}$$

7. If $\log_{\sqrt{2}}(x+4) - \log_2\left(\frac{1}{2}x+2\right) = 1$ then x is

(a) $\mathbf{-3}$

(b) 1

(c) -4

(d) 5

Soln:-

Given that:

$$\log_{2^{1/2}}(x+4) - \log_2\left(\tfrac{1}{2}x+2\right) = 1$$

$$\Rightarrow \frac{1}{2}\log_2(x+4) - \log_2\left(\frac{x}{2}+2\right) = 1$$

$$\Rightarrow \log_2(x+4)^{1/2} - \log_2\left(\frac{x+4}{2}\right) = 1$$

$$\Rightarrow \log_2\left(\frac{(x+4)^{1/2}}{\left(\frac{x+4}{2}\right)}\right) = 1$$

$$\Rightarrow \frac{(x+4)^{1/2}}{\left(\frac{x+4}{2}\right)} = 2$$

$$\Rightarrow (x+4)^{1/2} = \cancel{2}\left(\frac{x+4}{\cancel{2}}\right)$$

Using derived formula

$$\boxed{\log_{a^b} c = \frac{1}{b}\log_a c}$$

formula:

$$\log_b x = a$$
$$x = b^a$$

Squaring on b.s

$$(x+4) = (x+4)^2$$

$$(x+4)^2 - (x+4) = 0$$

$$(x+4)\left((x+4)-1\right) = 0$$

we get

$x+4=0$ or $x+4-1=0$

$\boxed{x=-4}$ $\boxed{x=-3}$

From Question we have

$\log_{\sqrt{2}}(x+4) - \log_2\left(\frac{x}{2}+2\right) = 1$

$\boxed{\text{when } x=-4}$

$\log_{\sqrt{2}}(-4+4) - \log_2\left(\frac{-4}{2}+2\right) = 1$

Notice: $-4+4=0$

Thus -4 is out of domain of log function.

$\boxed{\text{Now when } x=-3}$

$\cancel{\log(-3+4)}^{0} - \log_2\left(\frac{-3}{2}+2\right) = 1$

$-\log_2\left(\frac{1}{2}\right) = 1$

$-\left[\cancel{\log_2 1}^{0} - \cancel{\log_2 2}^{1}\right] = 1$

$1 = 1$

Thus $x=-3$ is the right option.

8. Seismologists use the Richter scale to measure and report the magnitude of earthquake as given by the equation $\boxed{R = \ln I - \ln I_0,}$ where I is the intensity of an earthquake with respect to a minimal or reference intensity I_0(i.e $I = cI_0$, where c is a constant). The reference intensity is the smallest earth movement that can be recorded on a seismograph. If an earthquake in city A recorded of magnitude 8.0 in Richter scale and intensity of the earthquake in city B is the reference intensity, then what is the ratio of intensity of earthquake in city A with respect to city B?

(a) $e^0 : 1$

(b) $e^1 : 2$

(c) $\boldsymbol{e^8 : 1}$

(d) $e^5 : 1$

(e) $e^8 : 2$

Soln Using the given equation

To find: $\frac{I}{I_0} = ?$

$$\boxed{R = \ln I - \ln I_0}$$

$$\Rightarrow R = \ln \frac{I}{I_0}$$

$$\Rightarrow 8 = \ln \frac{I}{I_0}$$

$$\Rightarrow e^8 = \frac{I}{I_0}$$

$$\Rightarrow \frac{I}{I_0} = \frac{e^8}{1} \Rightarrow e^8 : 1$$

$\therefore$ The ratio of intensity of earthquake in city A w.r.t city B is $e^8 : 1$

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9. Suppose that the number of bacteria present in a loaf of rotten bread after t minutes is given by the equation $G(t) = G_0 3^{kt}$, where G_0 represents the number of bacteria at $t = 0$, k is a constant (Given $\ln 730 = 6.59$ and $\ln 3 = 1.09$). If the initial number of bacteria is 1000 and it takes 1 min to increase to 9000 then how long(in minutes) would it take for the bacteria count to grow to 730000(integer value of t)?

(a) 2

(b) 1

(c) **3**

(d) 6

Soln :-

Given :- $G_0 = 1000$

At, $t = 1$ min $\quad G(t) = 9000$

To find :-

At what time (t), $G(t) = 7,30,000$

solve :-

$G(t) = G_0 3^{kt}$ —— ①

$\Rightarrow \dfrac{G(t)}{G_0} = 3^{kt}$

At $t = 1$ min

$\Rightarrow \dfrac{9\cancel{000}}{1\cancel{000}} = 3^{kt}$

$\Rightarrow \ln 3^2 = kt \ln 3$

$\Rightarrow 2 \cancel{\ln 3} = kt \cancel{\ln 3}$

$\Rightarrow \boxed{k = 2}$

On substituting the values of K & G_0, equation ① becomes

$G(t) = 1000\, 3^{2t}$; when $G(t) = 7,30,000$, then

$$\Rightarrow \frac{730\cancel{000}}{1\cancel{000}} = 3^{2t}$$

$$\Rightarrow 730 = 3^{2t}$$

$$\Rightarrow \ln 730 = 2t \ln 3$$

$$\Rightarrow t = \frac{\ln 730}{2 \ln 3} = \frac{6.59}{2 \times 1.09}$$

$$\boxed{t = 3 \text{ min}}$$

Thus at $t = 3$ min (integer value) bacteria count would be 7,30,000.

Let c_A and c_B be the luminosity(luminous efficacy) of the bulbs A and B respectively. The bulb A is $f(x)$ times brighter than the B, if $f(x) = 3^{x^2+1}$(i.e $c_A = f(x) \times c_B$), where x is the difference of the magnitude of supply voltage between the bulb A and the bulb B. Answer the questions ~~8 and 9~~ based on above information.

10. If the bulb A is 10 times brighter than the bulb B, then the difference of the magnitude of supply voltage between the two bulbs is

(a) $\sqrt{\log_3 5 - 1}$

(b) $\sqrt{\log_3 10}$

(c) $\sqrt{\frac{\ln 10}{\ln 3}}$

(d) $\sqrt{\log_3 \frac{10}{3}}$

Soln:-

given: $C_A = 10\, C_B$ — (1)

Luminosity is the measure of brightness

$C_A = C_B \times f(x)$

$C_A = C_B \times 3^{x^2+1}$ — (2)

Solve

$10 \cancel{C_B} = \cancel{C_B}\, 3^{x^2+1}$

$\Rightarrow \log_3 10 = (x^2+1)\cancel{\log_3 3}$

$\Rightarrow \log_3 10 = x^2+1$

$\Rightarrow x^2 = \log_3 10 - 1$

$\Rightarrow x = \sqrt{\log_3 10 - 1}$

$\Rightarrow x = \sqrt{\frac{\log 10}{\log 3} - 1}$

$$\Rightarrow x = \sqrt{\frac{\log 10 - \log 3}{\log 3}}$$

$$\Rightarrow x = \sqrt{\frac{\log \frac{10}{3}}{\log 3}}$$

$$\boxed{\text{formula: } \log_b a = \frac{\log a}{\log b}}$$

$$\boxed{x = \sqrt{\log_3 \frac{10}{3}}}$$

The difference b/w the magnitude of 2 bulbs is $\sqrt{\log_3 \frac{10}{3}}$

11. If 4 voltage and 3 voltage are the supply voltages for the bulbs A and B respectively then how many times the bulb A is brighter than the bulb B?
Ans : 9

Soln:- Since x is the difference b/w the supply voltage of A & B, thus

$x = 4 - 3 = 1$

We know that

$$C_A = C_B \times \underline{\underline{f(x)}}$$

We have to find $f(x)$

$$f(x) = 3^{x^2+1}$$

$$= 3^{1+1} = 3^2 = 9$$

$$f(x) = 9$$

$$\boxed{C_A = 9\, C_B}$$

$\therefore$ 9 times brighter

12. Find the number of values of x satisfying the equation $(5x)^{\log_{(5x)^{\frac{1}{5}}}(6x^3-36x^2+66x-35)^{\frac{1}{5}}} = 1.$
Ans: 3

Soln:- Given that:

$$(5x)^{\log_{(5x)^{1/5}}(6x^3-36x^2+66x-35)^{1/5}} = 1$$

Using derived formula

$$\log_{a^b} c = \frac{1}{b}\log_a c$$

$$(5x)^{5\log_{5x}(6x^3-36x^2+66x-35)^{1/5}} = 1$$

$$\Rightarrow (5x)^{\log_{5x}(6x^3-36x^2+66x-35)^{\cancel{5}/\cancel{5}}} = 1$$

Formula:

$$a^{\log_a x} = x$$

$$\Rightarrow 6x^3 - 36x^2 + 66x - 35 = 1$$

$$\Rightarrow 6x^3 - 36x^2 + 66x - 36 = 0$$

$$\Rightarrow 6(x^3 - 6x^2 + 11x - 6) = 0$$

$$\Rightarrow x^3 - 6x^2 + 11x - 6 = 0$$

$$\Rightarrow$$

Hit & trial method

when $x = 1$

$$1 - 6 + 11 - 6 = 0$$

$$0 = 0$$

Synthetic division to find other roots

$$
\begin{array}{r|l}
 & x^2 - 5x + 6 \\
x-1 & x^3 - 6x^2 + 11x - 6 \\
 & \cancel{x^3} - x^2 \\
 & (-) \quad (+) \\
\hline
 & -5x^2 + 11x - 6 \\
 & \cancel{-5x^2} + 5x \\
 & (+) \quad (-) \\
\hline
 & 6x - 6 \\
 & 6x - 6 \\
 & (-) \quad (+) \\
\hline
 & 0
\end{array}
$$

Factorizing to get other roots

$\Rightarrow x^2 - 5x + 6 = 0$

$\Rightarrow x^2 - 2x - 3x + 6 = 0$

$\Rightarrow x(x-2) - 3(x-2)$

$\Rightarrow (x-3)(x-2) = 0$

$x - 3 = 0$ or $x - 2 = 0$

$x = 2, 3, 1$

We got 3 values all together.

13. Which of the following is/are true?
(MSQ), (Ans:(a), (c))

- ◯ If m and n are positive real numbers, then $m^{\log(n)} = n^{\log(m)}$.
- ◯ $\log_5 12345678999999999999999$ is an rational number.
- ◯ The function $f(x) = \log_{10}(x^2 + x + 1)$ is one-one on the interval $(-0.5, \infty)$.
- ◯ None of the above.

Answer

(a) Given m & n are positive real number then

$$m^{\log(n)} = n^{\log(m)}$$

Taking log on both side

$$\cancel{\log n \cdot \log} m = \cancel{\log m} \cdot \log n$$

$$1 = 1$$

$$LHS = RHS$$

(b) Consider $a = 123456789\ldots9$

If $\log_5 a$ is a rational number then it can be represented in p/q form

$$\log_5 a = \frac{p}{q}$$

$$a = 5^{p/q}$$

$$a^q = 5^p$$

But 5 cannot divide a, thus $a^q = 5^p$ is never going to be satisfied. Thus it must be irrational.

(c) Given: $f(x) = \log_{10}(x^2+x+1)$ is one-one in interval $(-0.5, \infty)$.

Soln:- $f(x) = g(q(x))$; $q(x) = x^2+x+1$; $g(x) = \log_{10} x$

<u>Plotting $q(x)$</u>

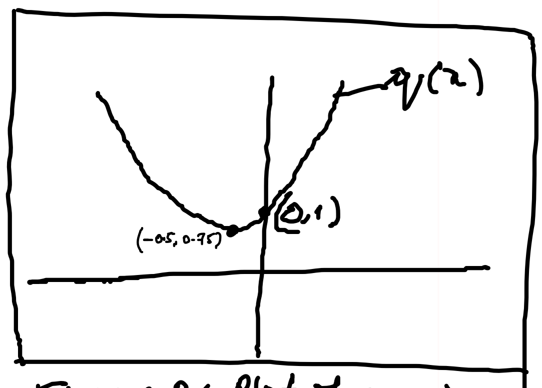

Figure 3: Plot of $q(x)$

- Y-intercept $q(0) = 1$
- No x-intercept as discriminant < 0
  $$\left[\begin{array}{c} b^2 - 4ac < 0 \\ -3 < 0 \end{array}\right]$$
- coefficient of $x^2 > 0$ thus parabola opens upward.
- Vertex $(-b/2a, q(-b/2a))$
  $(-0.5, 0.75)$
- A rough diagram is shown in Figure 3.

<u>Plotting $f(x)$</u>

$$f(x) = \log_{10}(x^2+x+1)$$

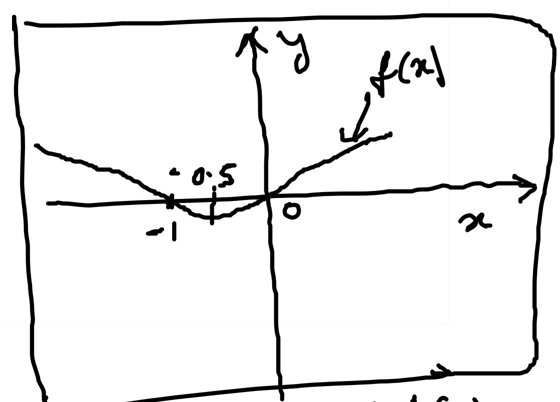

Figure 4: Plot of $f(x)$

- Minimum value $f(x)$ will be at minimum value of $q(x)$
  $\rightarrow f(-0.5) = \log_{10}(.75) = -0.125$
- <u>x-intercept</u> $(f(x)=0)$
  This will happen when
  $x^2+x+1 = 1 \Rightarrow x(x+1) = 0$
  $x=0$, $x=-1$

- Notice $f(x) > 0$ for all values of x except values between 0 and 1

- Rough plot of $f(x)$ is shown in figure 4

- Clearly, $f(x)$ is strictly increasing function in the domain $(-0.5, \infty)$ and thus it is one - one fun in domain $(-0.5\ \infty)$.

14. Which of the following is/are true?
(MSQ), (Ans:(a), (b), (c), (d))

- ◯ Suppose D is an arbitrary subset of $\mathbb{R}$ and f is one-one function on D. $\log f(x)$ whenever defined is also an one-one function on D.
- ◯ $(14!)^{\frac{1}{14}} < (15!)^{\frac{1}{15}}$, where ! denotes the factorial function, and for a non negative integer n, the value of $n!$ is $n \times (n-1) \times ... \times 2 \times 1$.
- ◯ The function $f(x) = 2^x + 3^x + ... + 100^x$ is one-one function on $\mathbb{R}$.
- ◯ There exists a function $f(x)$ on $\mathbb{R}$, such that $\log(f(x)) \geq 100$ for all $x \in \mathbb{R}$.

(a) Given: f is one-one function on D.
Also, $\log f(x)$ is defined on D (given)
We, know that log function is one-one function and therefore log of an one-one function (strictly increasing or decreasing) will give one-one function.

(b) Given: $(14!)^{\frac{1}{14}} < (15!)^{\frac{1}{15}}$
Taking $\log_{14!}$ on both side, we have

only if

$$\frac{1}{14}\log_{14!} 14! < \frac{1}{15}\log_{14!} 15!$$

$$\frac{1}{14} < \frac{1}{15}\log_{14!}(15 \times 14!)$$

$$\frac{1}{14} < \frac{1}{15}\left[\log_{14!} 15 + \log_{14!} 14!\right]$$

$$\frac{15}{14} < \log_{14!} 15 + 1 \Rightarrow \frac{15}{14} - 1 < \log_{14!} 15$$

$$\frac{1}{14} < \log_{14!} 15 \Rightarrow 1 < 14 \log_{14!} 15$$

$$\Rightarrow 1 < \log_{14!}(15)^{14} \Rightarrow \boxed{14! < 15^{14}}$$

which is true.

(c) We know that, exponential function of a natural number $\geq$ (except 0 & 1) is strictly increasing function & thus one-one function. Algbraic sum (linear combination) of such exponential function is also one-one function.

d) by properl of log fn.
over any range of a defined f.
log f exists

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