IIT ROORKEE

NPTEL ONLINE CERTIFICATION COURSE

QUANTITATIVE INVESTMENT MANAGEMENT

LECTURE 9

Bond Pricing with Binomial Trees

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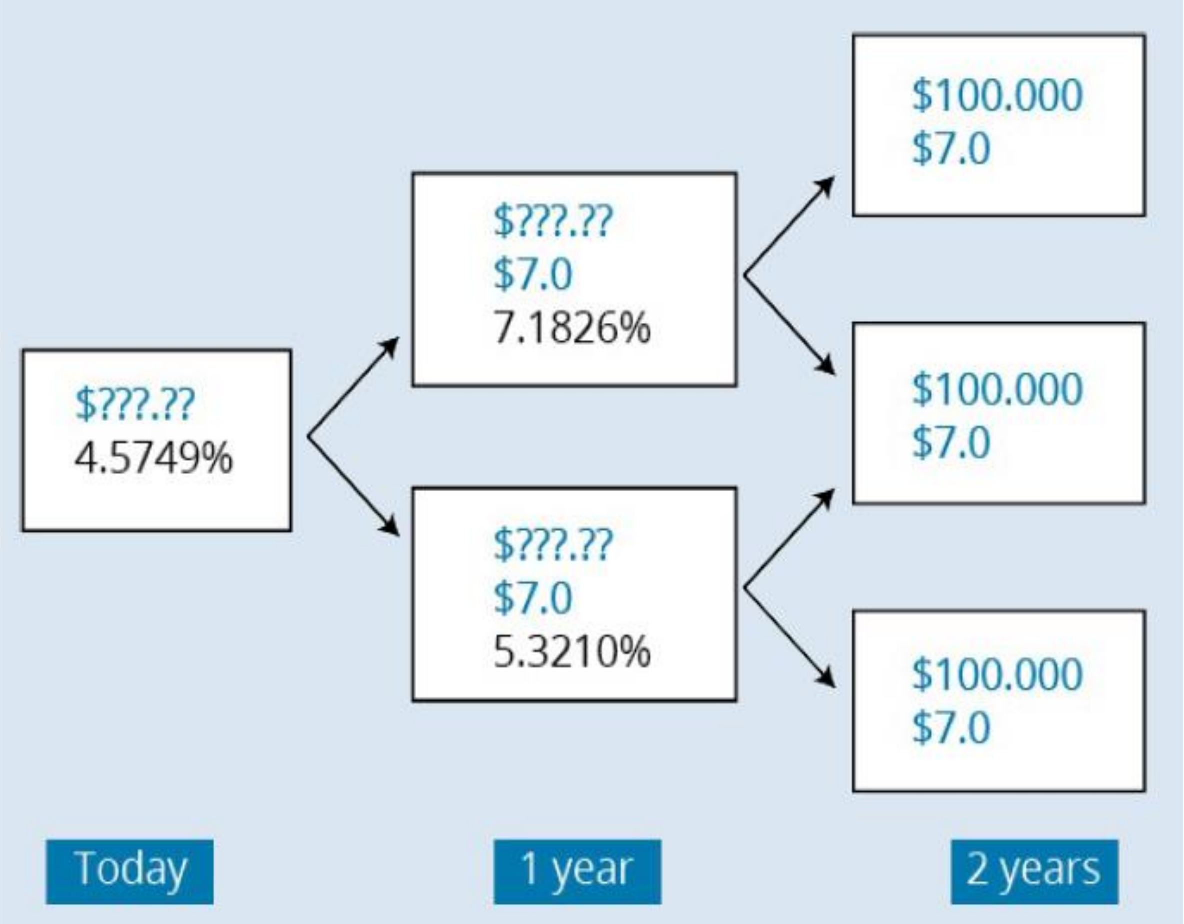
VALUING AN OPTION-FREE BOND WITH THE BINOMIAL MODEL

A 7% annual coupon bond has two years to maturity. The interest rate tree is shown in the adjacent figure. Fill in the tree and calculate the value of the bond today.

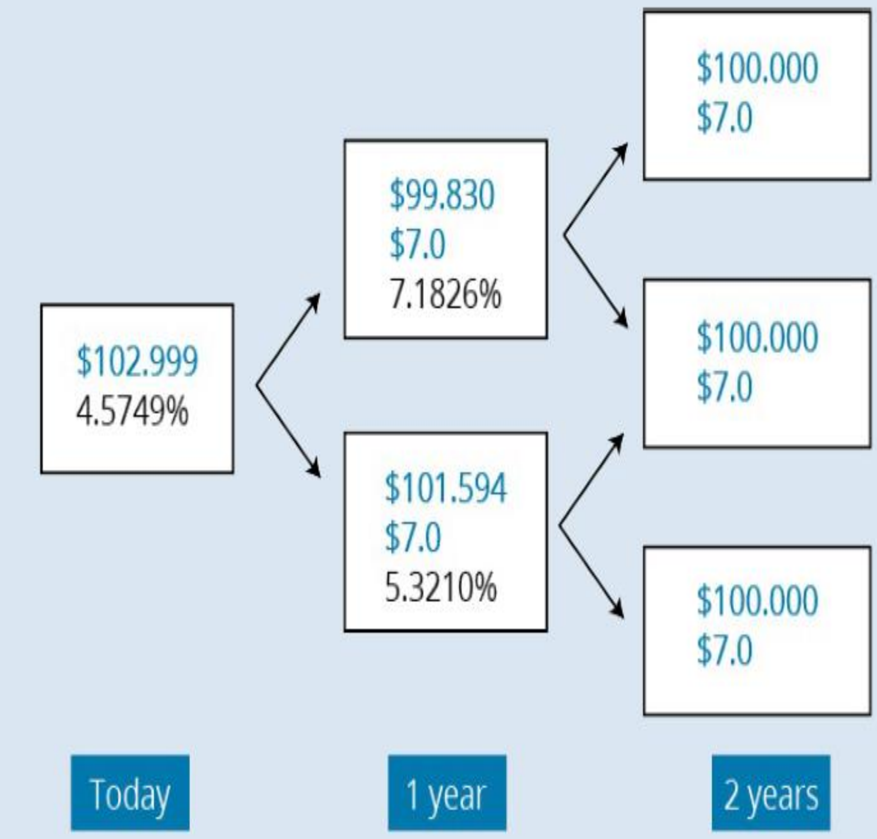


SOLUTION

Consider the value of the bond at the upper node for Period 1 $(V_{1,U})$: $V_{1,U} = \frac{1}{2}\left[\frac{\$100+\$7}{1.071826} + \frac{\$100+\$7}{1.071826}\right] = \$99.830$

Similarly, the value of the bond at the lower node for Period 1 $(V_{1,L})$is: $V_{1,L} = \frac{1}{2}\left[\frac{\$100+\$7}{1.053210} + \frac{\$100+\$7}{1.053210}\right] = \$101.594$

Valuing a 2-Year, 7.0% Coupon, Option-Free Bond



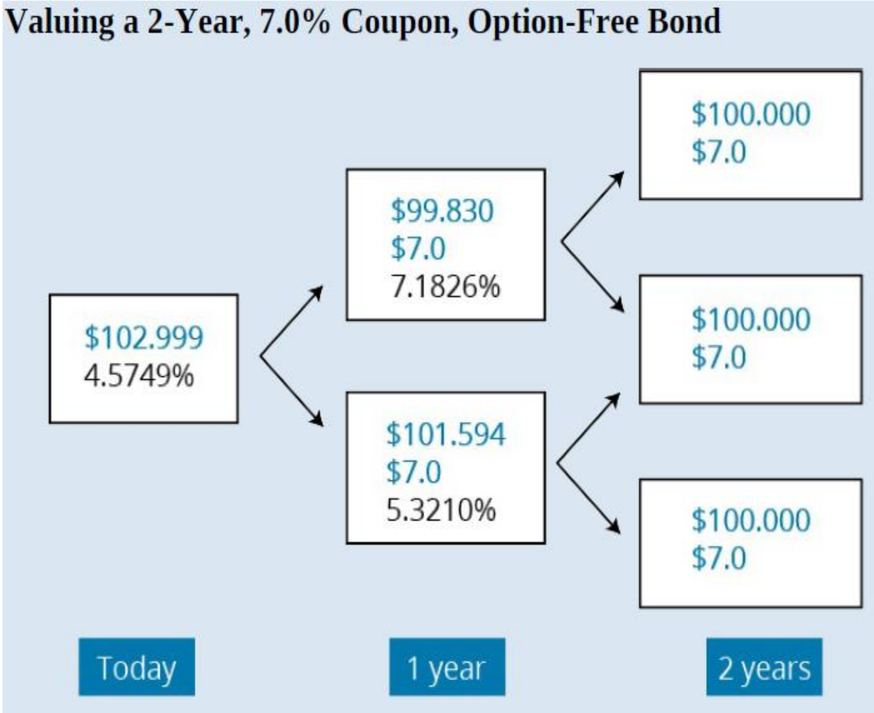
- **Now calculate V_0, the current value of the bond at Node 0.**
- $V_0 = \frac{1}{2} \times \left[\frac{\$99.830+\$7}{1.045749} + \frac{\$101.594+\$7}{1.045749}\right] = 102.999$

SHORT-CUT CALCULATIONS IN THIS EXAMPLE

$$V_{1,U} = \frac{\$100 + \$7}{1.071826} = \$99.830$$

$$V_{1,L} = \frac{\$100 + \$7}{1.053210} = \$101.594$$

$$\boldsymbol{V_0 = \left[\left(\frac{\$99.830 + \$101.594}{2}\right) + \$7\right] \times \left[\frac{1}{1.045749}\right] = 102.999}$$



CALIBRATING A BINOMIAL TREE

CALIBRATING A BINOMIAL TREE

- In practice, the interest rate tree is usually generated using specialized computer software. The underlying process conforms to three rules:
- The interest rate tree should generate arbitrage-free values for the benchmark security.
- This means that the value of benchmark bonds produced by the interest rate tree must be equal to their market price, which excludes arbitrage opportunities.

- This requirement is very important because without it, the model will not properly price more complex callable and puttable securities, which is the intended purpose of the model.
- As stated earlier, adjacent forward rates (for the same period) are $e^{2\sigma}$ apart.
- Hence, knowing one of the forward rates for a particular nodal period and the interest rate volatility allows us to compute the other forward rates for that period in the tree.

- **The middle forward rate (or mid-point in case of even number of rates) in a period is set approximately equal to the implied (from the benchmark spot rate curve) one-period forward rate for that period.**

EXAMPLE

X has collected the following information on the par rate curve. It is required to:

(i) calculate the arbitrage free implied forward rates.

(ii) generate a binomial interest rate tree consistent with this data and an assumed volatility of 20% given that $i_{1,U} = 5.7883\%$.

Maturity	Par Rate
1	3%
2	4%
3	5%

SOLUTION

- To generate the tree, we need to compute the forward rate for various initiation points and 1-year maturities.
- For this purpose, we need the various spot rates.

- For S_{01}, we have: $100 = \frac{103}{1+S_{01}}$ or $S_{01} = 3\%$
- For S_{02}, we have: $100 = \frac{4}{1+S_{01}} + \frac{104}{(1+S_{02})^2}$
- $= \frac{4}{1+0.03} + \frac{104}{(1+S_{02})^2}$ or $S_{02} = 4.02\%$
- Similarly, $S_{03} = 5.069\%$

- **Implied arbitrage free one-year forward rates:**

$$f_{01} = S_{01} = 3\%$$

$$f_{12} = \frac{(1 + S_{02})^2}{(1 + S_{01})} - 1 = \frac{(1 + 0.0402)^2}{(1 + 0.03)} - 1$$

$$= 5.05\%$$

$$f_{23} = \frac{(1 + S_{03})^3}{(1 + S_{02})^2} - 1 = \frac{(1 + 0.05069)^3}{(1 + 0.0402)^2} - 1$$

$$= 7.20\%$$

TREE CALIBRATION

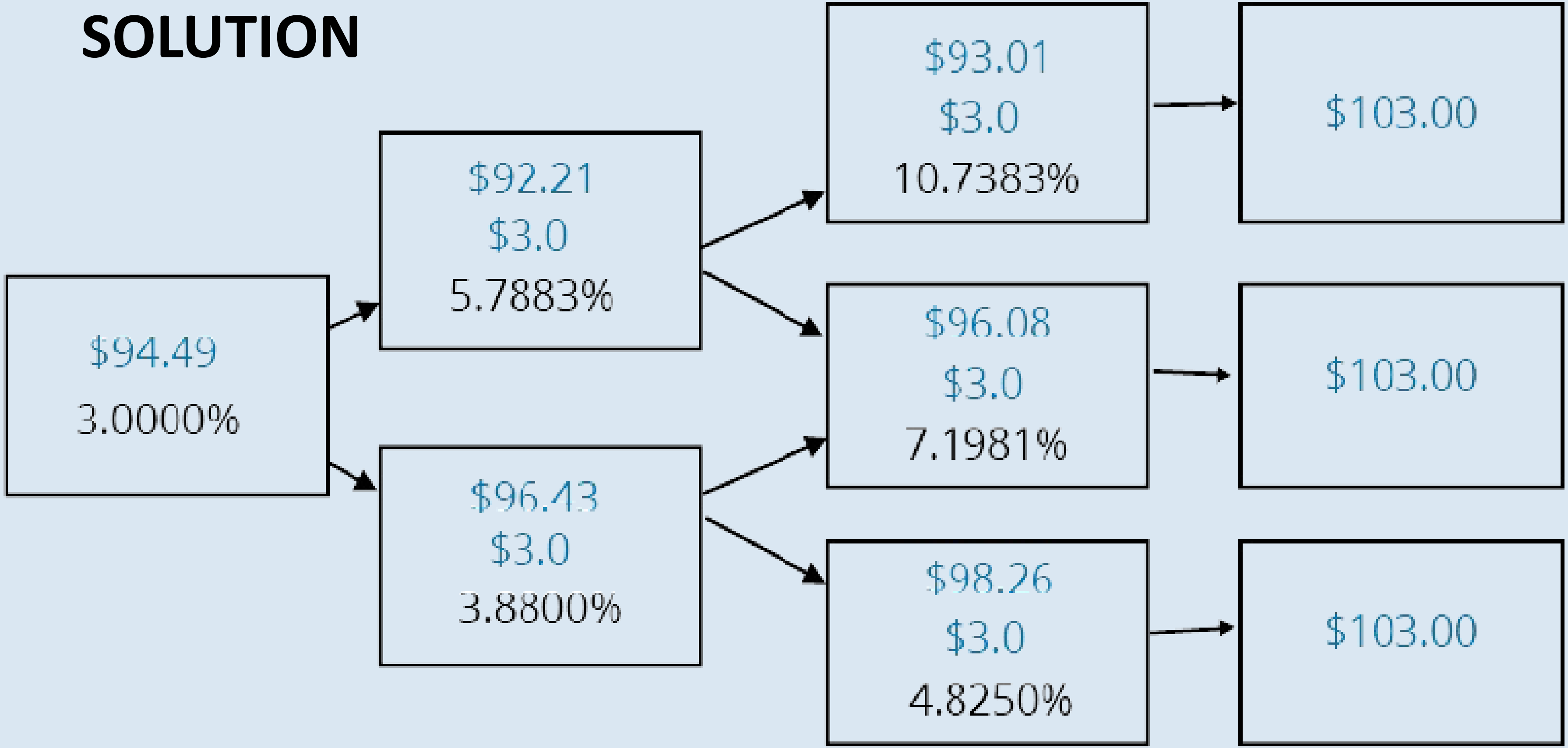
- Given that $i_{1,U} = 5.7883\%, \sigma = 20\%$ so that: $i_{1,L} = 5.7883\% \times e^{-0.40} = 3.8800\%$.
- For the rates $i_{2,UU}, i_{2,UL}$, $i_{2,LL}$ no information other than volatility is given.
- We make the assumptions that (i) the tree is recombinant and (ii) the implied forward rate $f_{23} = 7.20\%$ corresponds to the middle rate $i_{2,UL} = i_{2,LU} = 7.20\%$. Then,
- $i_{2,LL} = i_{2,UL}e^{-2\sigma} = (0.072)e^{-0.40} = 0.0483$ or 4.83%
- $i_{2,UU} = i_{2,UL}e^{+2\sigma} = (0.072)e^{+0.40} = 0.1074$ or 10.74%

EXAMPLE

- X is interested in valuing a three-year, 3% annual-pay Treasury bond using the adjacent binomial tree. Value the bond.

0	1	2
3%	5.7883%	10.7383%
	5.7883%	7.1981%
	3.8800%	7.1981%
	3.8800%	4.8250%

SOLUTION


$94.49
3.0000%
$92.21
$3.0
5.7883%
$96.43
$3.0
3.8800%
$93.01
$3.0
10.7383%
$96.08
$3.0
7.1981%
$98.26
$3.0
4.8250%
$103.00
$103.00
$103.00

- $V_{2,UU} = \frac{103}{(1.107383)} = \$93.01$
- $V_{2,UL} = V_{2,LU} = \frac{103}{(1.071981)} = \$96.08$
- $V_{2,LL} = \frac{103}{(1.048250)} = \$98.26$
- $V_{1,U} = \frac{1}{2} \times \left[\frac{93.01+3}{1.057883} + \frac{96.08+3}{1.057883}\right] = \$92.21$
- $V_{1,L} = \frac{1}{2} \times \left[\frac{93.08+3}{1.038800} + \frac{98.26+3}{1.038800}\right] = \$96.43$
- $V_0 = \frac{1}{2} \times \left[\frac{92.21+3}{1.03} + \frac{96.43+3}{1.03}\right] = \$94.485$