Lesson 3: Arbitrage Pricing Theory (APT)

Introduction

- Introduction to arbitrage pricing theory (APT)
- A simple proof of APT
- Testing the APT
- APT with CAPM
- Applications of asset pricing models
- Summary and concluding remarks

Arbitrage Pricing Theory (APT)

Arbitrage Pricing Theory (APT)

CAPM had its genesis in the mean-variance analysis

- Investors choose the optimum diversified portfolio on an efficient frontier based on the expected return and variance analysis
- The arbitrage pricing theory (APT) of Ross (1966, 1977) employs a multifactor (alternatively called multi-index) approach to explain the pricing of assets
- It relies on the single/multi-index approach to provide the return-generating process

Arbitrage Pricing Theory (APT)

Using the return-generating process, APT derives the definition of expected returns in equilibrium with certain assumptions

- At the heart of this approach is the arbitrage argument (and thus the name), similar to that employed in the CAPM
- Two items with the same cash flows cannot sell at different prices
- APT is more generic than CAPM in the sense that it does not assume that only expected return and risk affect the security prices

Arbitrage Pricing Theory (APT)

The assumption of homogenous expectations remains

- Instead of a mean-variance framework, we make assumptions about the return-generating process
- APT argues that returns on any stocks are linearly related to a set of indices

Arbitrage Pricing Theory (APT)

APT argues that returns on any stocks are linearly related to a set of indices

- $R_i = a_i + b_{i1}I_1 + b_{i2}I_2 + \ \ldots\ldots + b_{ij}I_j + e_i$, where
- a_i is the expected level of return on the stock "*i* " if all indices have a value of zero.
- I_j is the value of the *j* th index that affects the return on stock *i*.
- b_{ij} is the sensitivity of stock *i*'s return to the *j* th index.
- e_i is a random error term with a mean of zero and variance equal to σ_{ei}^2
- Essentially, the above-mentioned equation describes the process that generates security returns

Arbitrage Pricing Theory (APT)

APT argues that returns on any stocks are linearly related to a set of indices

- $R_i = a_i + b_{i1}I_1 + b_{i2}I_2 + \ldots\ldots + b_{ij}I_j + e_i$
- For the above model to be more accurate, the following assumptions are made
- $E\left(e_i e_j\right) = 0$; for all *i* and *j* where *i* ≠ *j*
- $E\left[e_i\left(I_j - \bar{I}_j\right)\right] = 0$ for all the stocks and indices
- It is an extension of a multi-index family of models

A Simple Proof of APT: Part I

A Simple Proof of APT

Suppose the following two-index model describes the returns

- $R_i = a_i + b_{i1}I_1 + b_{i2}I_2 + e_i$; also consider that $E\left(e_i e_j\right) = 0$
- Here, each index represents a certain systematic risk
- Now, if the investor holds a well-diversified portfolio, only the systematic risk – represented by the indices I_1 and I_2 – will matter
- The residual risk captured by σ_{ei}^2 will be close to zero
- The sensitivity of the portfolio to these two components of the systematic risk is represented by b_{i1} and b_{i2}

A Simple Proof of APT

Consider the three well-diversified portfolios shown below

Portfolio	Expected Return (%)	b_{i1}	b_{i2}
A	15	1.0	0.6
B	14	0.5	1.0
C	10	0.3	0.2

- The returns are provided at equilibrium: No arbitrage
- Remember our discussion of CAPM with sensitivity towards a single index (market portfolio) where all the securities in equilibrium were lying on a straight line (two axes: R and b_1)
- Here, since we have two sensitivities (two betas with respect to each axis), we can safely assume that these three portfolios will lie on a plane (three axes: $R, b_{i1},$ and b_{i2})

A Simple Proof of APT

Consider the three well-diversified portfolios shown below

Portfolio	Expected Return (%)	b_{i1}	b_{i2}
A	15	1.0	0.6
B	14	0.5	1.0
C	10	0.3	0.2

- The generic equation for a plane is as follows: $\bar{R}_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2}$
- Can we solve for the values of $\lambda_0, \lambda_1, \text{and } \lambda_2$ using the values provided in the table?

A Simple Proof of APT

We get the following equation: $\bar{R}_i = 7.75 + 5b_{i1} + 3.75b_{i2}$

Portfolio	Expected Return (%)	b_{i1}	b_{i2}
A	15	1.0	0.6
B	14	0.5	1.0
C	10	0.3	0.2

- Now, consider a third portfolio E with expected returns of 15%, b_{i1}= 0.6 and b_{i2}= 0.6
- Compare E with another portfolio D that places one-third in A, B, and C
- What is my expected return and sensitivities of D, and are there arbitrage opportunities?

A Simple Proof of APT

Solving for D, we get the following values

- $b_{p1} = \frac{1}{3} * (1.0) + \frac{1}{3}(0.5) + \frac{1}{3}(0.3)$ = 0.6
- $b_{p2} = \frac{1}{3} * (0.6) + \frac{1}{3}(1.0) + \frac{1}{3}(0.2)$=0.6
- $\bar{R}_D = \frac{1}{3}(15) + \frac{1}{3}(14) + \frac{1}{3}(10)$=13
- D has an identical risk profile offered by a lower return
- We could also have computed the expected return on $\bar{R}_D$ using the equation of the plane
- $\bar{R}_D = 7.75 + 5b_{D1} + 3.75b_{D2}$=$7.75 + 5 * 0.6 + 3.75 * 0.6 = 13$

A Simple Proof of APT

- By the arbitrage argument (or law of one price), two portfolios with the same risk cannot sell at different prices (or have different expected returns)
- Arbitrageurs (or investors in general) would buy E and sell D short
- This would guarantee riskless profit (2%)
- This will continue until E falls back on the plane defined by A, B, and C

A Simple Proof of APT

- The plane that we draw on expected return is a b_{i1} and b_{i2} space
- If any security (like E) is undervalued/overvalued, it will be above or below this plane
- This would lead to an arbitrage opportunity, and such securities will converge back to this plane

A Simple Proof of APT: Part II

A Simple Proof of APT

The general equation of the plane in return, i.e., b_{i1} and b_{i2} space, is shown below

- $\bar{R}_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2}$
- This is the equilibrium model provided by APT when the returns are generated by a two-index model
- Here, λ_1 and λ_2 are the increases in returns for one unit increase in b_{i1} and b_{i2}
- Essentially, λ_1 and λ_2 reflect the returns for bearing the risks associated with the indices I_1 and I_2

A Simple Proof of APT

Consider a zero b_{ij} portfolio with no sensitivity to either index

- If it has no risk, then it should offer a risk-free return $\lambda_0 = R_F$
- In case the riskless rates are not available, then instead of R_F, we denote it by $\bar{R}_Z$, i.e., the return on the zero-beta portfolio (what is a zero-beta portfolio?)
- Imagine a portfolio that mimics index 1 and, therefore, has b_{i1}=1
- Also, it is not sensitive to I_2 and, therefore, has b_{i2}= 0

A Simple Proof of APT

For this portfolio, the equation $[\bar{R}_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2}]$ becomes

$\bar{R}_1 = R_F + \lambda_1$ and $\lambda_1 = \bar{R}_1 - R_F$

Similarly, $\lambda_2 = \bar{R}_2 - R_F$

The above analysis can be generalized to a *j* index case shown below

$$\bar{R}_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2} + \cdots + \lambda_j b_{ij}$$

$\lambda_0 = R_F$ and $\lambda_j = \bar{R}_j - R_F$ where the return-generating process can be described as

$$R_i = a_i + b_{i1} I_1 + b_{i2} I_2 + \ \ldots \ldots + b_{ij} I_j + e_i$$

A Simple Proof of APT

The above analysis can be generalized to a *j* index case shown below

- $\bar{R}_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2} + \cdots + \lambda_j b_{ij}$
- $\lambda_0 = R_F \text{ and } \lambda_j = \bar{R}_j - R_F$
- The derivation assumes here that both the indices are orthogonal
- In practical situations, there are always correlations between the risk factors represented by two indices
- Researchers orthogonalize both indices to remove any common component. In that case, the new indices may not be well-defined

A Simple Proof of APT

The straight line in the CAPM had two axes: return and beta axes

- Thus, two coordinates corresponding to each point denotes a portfolio
- In the context of a two-index APT model, we have one return and two beta axes (for each index)
- Thus, three coordinates that define the plane also define the efficient frontiers
- If a point is above (or below) this plane, this means that the security is under (or over) priced with respect to one or both of these indices
- Thus, it violates the law of one price

A Simple Proof of APT

If the law of one price is violated, then arbitrageurs may conduct risk-less arbitrage by selling (or buying) the under (or over) priced portfolio and taking a counter position in the portfolios that are fairly priced

- This will drive the prices of the inefficient portfolio towards this plane, that is, efficient frontier or efficient plane
- The implication of this riskless arbitrage is that all portfolios in the equilibrium would lie on this place, that is, an efficient frontier
- That is, in the space defined by three coordinates: expected return, b_{i1}, and b_{i2}

A Few Important Points About APT

A Few Important Points About APT

In the context of CAPM, it was needed to identify the “market portfolio,” and, therefore, all the risky assets

- While testing CAPM, one can always question whether all the securities are truly captured in the risky assets
- Therefore, have we achieved the true market portfolio?
- However, in the context of APT, arbitrage conditions can be applied to any security or portfolio
- Thus, it is not necessary to identify all the risky securities and market portfolio

A Few Important Points About APT

APT can very well be tested for a small number of stocks, for example, all the 50 stocks making up the “Nifty-50” index

- Given this advantage with APT, many studies argue that the tests designed for CAPM are actually the tests of single-factor APT
- Therefore, they utilized a limited number of securities, which arguably may not capture the entire market

A Few Important Points About APT

- The only caution needed here is that the systematic influences (or indices/factors) affecting these sets of stocks that are tested for APT should be adequately described
- This can be an issue when we have a large set of securities. Then, finding an adequate number of indices (or systematic influences) may become a challenge

A Few Important Points About APT

APT is extremely general in nature

- It allows us to describe the equilibrium in terms of a single/multi-index model
- However, it does not define what would be the most appropriate multi-index model
- We do not know $\lambda's$ or I_js's
- They are generated from the data available (e.g., through factor analysis)
- For example, what risk factor a given I_j indicates (inflation risk, market risk, etc.) that is not provided by the model
- So, one does not have the direct specific economic rationale for a given factor

Testing the APT: Introduction

Testing the APT

The multifactor return-generating process is provided below

- $R_i = a_i + b_{i1}I_1 + b_{i2}I_2 + \ldots\ldots + b_{ij}I_j + e_i$
- The corresponding APT model is shown below
- $\bar{R}_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2} + \cdots + \lambda_j b_{ij}$
- In order to test the APT, one has to identify I_js, that is, risk factors
- Subsequently, one can define the sensitivity of a given security b_{ij} to this risk factor
- Unfortunately, APT does not offer a direct economic rationale or description of I_js
- What do we know about b_{ij}, I_j, and λ_j?

Testing the APT

Each firm has a unique sensitivity b_{ij} for each index I_j

- Thus, b_{ij} is a security-specific attribute (such as dividend yield) or security-specific sensitivity to an index
- The value of I_j is the same for all the securities
- These I_js are systematic influences affecting a large number of securities and, therefore, are the source of covariance between those securities
- λ_j is the extra-expected return required because of the sensitivity of a security to the j th attribute of the security

Testing the APT

For CAPM b_{ij} (sensitivity to the market, beta), I_j (market index), and $\lambda_j(R_m{-}R_f)$ were well-defined

- For APT, these are not defined in the model
- One has to test the model below with the observed returns
- $\bar{R}_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2} + \cdots + \lambda_j b_{ij}$
- This requires estimates of b_{ij} and λ_j

Testing the APT

Most of the APT tests use the following equation on a set of predefined indices to obtain b_{ij}

- $R_i = a_i + b_{i1}I_1 + b_{i2}I_2 + \ldots\ldots + b_{ij}I_j + e_i$
- Then, the following equation is used to obtain the estimates of $\lambda_j s$ and thus the APT model
- $\bar{R}_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2} + \cdots + \lambda_j b_{ij}$
- In this manner, one can keep identifying risk factors until a sizable portion of expected returns are identified
- Effectively, these are joint tests of APT as well as the factors/influences/portfolios considered in the model

Testing the APT

Since there is no generalizable theory that explains all the factors, the following methods are used to provide a broad set of factors in the APT model

1. Factor analysis approach

2. Specifying the attributes of the security

3. Specifying the influences (factors) affecting the return-generating process

4. Specifying a set of portfolios that capture the return-generating process

Testing the APT: Factor Analysis

Testing the APT: Factor Analysis

A slightly purer and advanced method calls for factor analysis of the security returns

- The analysis determines a specific set of I_js and b_{ij}s and also aims to reduce the covariance of the residual returns to as low as possible
- In the factor analysis terminology, I_js become the factors and b_{ij}s become the factor loadings
- One can keep adding factors to the model till the ability of the additional factors to explain the covariance matrix drops below a certain level

Testing the APT: Factor Analysis

Post factor analysis, the following equation is used to obtain the estimates of $\lambda_j s$ and thus the APT model

- $\boldsymbol{\bar{R}_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2} + \cdots + \lambda_j b_{ij}}$
- The challenges with the factor analysis are discussed as follows
  - Like any similar analysis, the estimates of I_js and b_{ij}s are subject to the error of the estimate
  - The factors produced in the analysis have no meanings
  - For example, the signs of factors and three betas (and therefore, the lambdas) can be reversed with no change in the resulting expected return

Testing the APT: Specifying the attributes of the Security

Testing the APT

Specifying the attributes of the security

- If we can establish, a priori, that a certain set of attributes of security that affect the return
- Then, the extra return required on account of these attributes can be measured through the following equation: $\bar{R}_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2} + \cdots + \lambda_j b_{ij}$
- Here, b_{ij}s would represent the level of an attribute (*j*) associated with the security "*i* " associated with each characteristic
- λ_j would represent the extra return because of the sensitivity to that characteristics

Testing the APT

Specifying the attributes of the security: $\bar{R}_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2} + \cdots + \lambda_j b_{ij}$

- “*n* % increase in dividend of the portfolio is associated with Δ% increase in the expected returns.”
- Once these b_{ij}s are directly obtained, risk premiums for these attributes are computed using the APT model
- These attributes directly affect the expected returns
- Once major firm attributes and the corresponding risk premiums (λs) are identified, the equation [$\bar{R}_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2} + \cdots + \lambda_j b_{ij}$] can be estimated to define the APT

Testing the APT: Specifying a Set of Systematic Influences or Portfolios

Testing the APT

Specifying the influences (factors) affecting the return-generating process

- Another alternative is to determine and pre-decide the set of risk factors (influences) that affect the return-generating process
- A set of economic variables that affect the cash flows associated with the security
- For example, inflation, term structure of interest rates, risk premia, and industrial production

Testing the APT

Specifying the influences (factors) affecting the return-generating process

- Another set of tests involve time-series regressions of the individual portfolios to examine their sensitivities (b_{ij}) towards these macroeconomic variables
- $R_i = a_i + b_{i1}I_1 + b_{i2}I_2 + \ \ldots\ldots + b_{ij}I_j + e_i$
- In the second stage, cross-sectional regressions are performed using all the portfolios to determine the market price of risk (λ_j)
- $\bar{R}_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2} + \cdots + \lambda_j b_{ij}$

Testing the APT

Specifying the influences (factors) affecting the return-generating process

- For example, ONGC will be definitely affected by the crude-oil prices
- So, a crude oil price index or any broad energy index can provide one risk factor, that is I_j
- Using these indices, the return-generating process can be employed to estimate the betas (b_{ij})
- Once the betas are obtained, the APT model can be used to obtain risk premiums (λ_j: $Rj - Rf$)

Testing the APT

Specifying a set of portfolios that capture the return-generating process

- Another option is to construct a set of portfolios that capture the influence of risk factors affecting the return-generating process. For example
  - Difference in the returns on small and large stock portfolios
  - Difference in returns on the high book-to-market and low book-to-market stocks
  - Difference in the returns on long-term corporate and long-term government bonds

APT and CAPM: Single Market Index

APT and CAPM

Does CAPM become inconsistent in the presence of APT?

- We start with a simple single-index case, where this index is a market portfolio (or market index like Nifty-50)
- The return-generating process is of the following form
- $R_i = a_i + \beta_i R_m + e_i$

APT and CAPM

Now, refer to our earlier discussions on APT, where we said that the above return-generating process could be written in terms of sensitivities of the securities to index and the price of risk in the following form

- $\bar{R}_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2} + \cdots + \lambda_j b_{ij}$ with $\lambda_0 = R_F$ and $\lambda_j = \bar{R}_j - R_F$

APT and CAPM

$\bar{R}_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2} + \cdots + \lambda_j b_{ij}$ with $\lambda_0 = R_F$ and $\lambda_j = \bar{R}_j - R_F$

For a single index case, that is, market index, and in the presence of a risk-free rate, the above expression becomes

$\bar{R}_i = R_F + \beta_i(\bar{R}_m - R_F)$: this is the expected return form provided by CAPM

This suggests when a single-index return-generating process is true depiction, the CAPM is clearly consistent

But what about multi-indices?

APT and CAPM: Multi-Index

APT and CAPM

The return-generating process in the context of two indices becomes

- $R_i = a_i + b_{i1}I_1 + b_{i2}I_2 + e_i$
- The equilibrium model for this return-generating process with a risk-less asset becomes: $\bar{R}_i = R_F + \lambda_1 b_{i1} + \lambda_2 b_{i2}$
- Recall that λ_j is the price of risk for a portfolio that has b_{ij}=1 for one index and zero for all the other indices: $\lambda_j = \bar{R}_j - R_F$
- If we say that CAPM holds, it holds for all the securities as well as portfolios

APT and CAPM

If we say that CAPM holds, it holds for all the securities as well as portfolios

- Therefore, this industry portfolio may have some sensitivity to the market portfolio, that is, β_{λ_j}
- Recall that the risk premium was $\beta_i(\bar{R}_m - R_F)$ when the sensitivity to the market was β_i
- Then, the effective risk premium for this index λ_j becomes $\beta_{\lambda_j}(\bar{R}_m - R_F)$

APT and CAPM

The return-generating process in the context of two indices becomes

- $R_i = a_i + b_{i1}I_1 + b_{i2}I_2 + e_i$
- The equilibrium model for this return-generating process with a risk-less asset becomes: $\bar{R}_i = R_F + \lambda_1 b_{i1} + \lambda_2 b_{i2}$; $\lambda_1 = \bar{R}_1 - R_F$
- But if you believe in CAPM, then
- $\bar{R}_1 - R_F = \beta_{\lambda_1}(\bar{R}_m - R_F)$ for index I_1 and $\bar{R}_2 - R_F = \beta_{\lambda_2}(\bar{R}_m - R_F)$ for index I_2

APT and CAPM

$\bar{R}_i = R_F + \lambda_1 b_{i1} + \lambda_2 b_{i2}$ can be effectively written as

- $\bar{R}_i = R_F + b_{i1}\beta_{\lambda_1}(\bar{R}_m - R_F) + b_{i2}\beta_{\lambda_2}(\bar{R}_m - R_F)$
- $\bar{R}_i = R_F + (b_{i1}\beta_{\lambda_1} + b_{i2}\beta_{\lambda_2})(\bar{R}_m - R_F)$
- Define $\beta_i = (b_{i1}\beta_{\lambda_1} + b_{i2}\beta_{\lambda_2})$
- Then, we obtain the CAPM form as follows: $\bar{R}_i = R_F + \beta_i(\bar{R}_m - R_F)$
- This can be extended to multiple factors (indices) as well

APT and CAPM

Therefore, the APT solution, even with multiple factors, is consistent with CAPM

This means that despite the fact that multiple indices (risk factors) explain the covariance between the returns, the CAPM holds

Application of Asset Pricing Models: Passive Management

Passive Asset Management

Passive management

- A simple application of APT is to construct a portfolio of stocks that closely tracks an index
- The index that represents a risk factor (Bank Nifty represents the risk of banking stocks)

Passive Asset Management

Passive management

- The attempt is made to use a rather lesser number of stocks
- A large number of stocks would incur significant transaction costs
- In order to track the market index (market portfolio), one cannot hold all the stocks in the markets

Passive Asset Management

Passive management

- One attempts to hold only to the extent the diversifiable risk can be offset
- Those indices for which portfolio sensitivity is not matched, if receive unexpected shocks (like oil price shock), may appear as the residual risk in the model
- That is, our portfolio may be exposed to these changes

Passive Asset Management

Passive management

- The benefit of using multi-indices instead of a single market index can be explained here as follows
- Consider five indices, including the energy portfolio, banking, inflation, cyclical stocks, and government bond portfolio

Passive Asset Management

Passive management

- Compare this to holding only the market portfolio (Nifty)
- Both of these strategies will capture the sensitivity to market risk, as all the portfolios (except government bonds) may reflect, to some extent, the risk of market

Passive Asset Management

Passive management

- However, if there is a certain oil price shock or unexpected changes in inflation, the market portfolio with its sensitivity matched to Nifty may not be very efficient in tracking the index
- This is because one is indifferent to holding stock from different industries (e.g., oil stocks) in constructing the Nifty, as long as she is able to replicate a market portfolio with no diversifiable risk

Passive Asset Management

Passive management

However, this portfolio's sensitivity to oil shocks can be very different to that of a multi-index (APT) model that is explicitly matched to the sensitivity of the oil price index

Application of Asset Pricing Models: Active Management

Active Asset Management

Active management

- In active management, one continuously holds on the market portfolio and makes calculated bets on different risk factors
- For example, if one believes that oil prices can go up – this means that currently the stocks that are sensitive to this risk are underpriced and will go up in future

Active Asset Management

Active management

- Then, one can increase the sensitivity of his portfolio by adding additional stocks from oil companies and others to the extent that increases the sensitivity to this risk index
- Once the price increase has materialized, one can go back to holding the market portfolio by selling the additional stocks and realizing the gains

Summary and Concluding Remarks

Summary and Concluding Remarks

- While CAPM has its genesis in the mean-variance framework, APT relies on the arbitrage argument
- APT utilizes the return-generating process provided by the single- and multi-index models to generate the equilibrium asset pricing model
- Under the APT, risk-less arbitrage drives prices towards the equilibrium plane
- The equation of this plane is determined by the systematic risk influences affecting the set of securities under consideration

Summary and Concluding Remarks

- APT can be tested with the help of (a) factor analysis, (b) specifying the attributes, (c) specifying a set of systematic influences, and (d) specifying a set of portfolios
- In the presence of APT, CAPM does not necessarily become invalid as long as the APT factors are influenced by the market factor (have a well-specified beta with respect to the market factor)
- Some of the most widely employed applications of asset pricing models include active and passive asset management and factor investing

Thanks!