Lesson 2: Single- and Multi-Index Models

Introduction

- Single-index models and correlation structure
- Construction of single-index models
- Portfolio characteristics with single-index models
- Estimation of portfolio beta with single-index models
- Single-index models: simple example

Introduction

- A few words on beta
- Introduction to multi-index models
- Multi-index models: expected return and risk
- Summary and concluding remarks

Single-Index Models and Correlation Structure

INDIAN INSTITUTE OF TECHNOLOGY KANPUR

Single-Index Models and Correlation Structure

These are the equations corresponding to portfolio returns and standard deviation

- $\bar{R}_P = \sum_{i=1}^{N} X_i \bar{R}_i$ (1)
- $\sigma_p^2 = \sum_{i=1}^{N} X_i^2 \sigma_i^2 + \sum_{j=1}^{N} \sum_{k=1}^{N} (X_j X_k \sigma_{jk})$ where *i≠j* (2)
- In order to draw an efficient frontier, three key inputs are required
  - Expected returns from each security
  - Standard deviations from each security
  - Correlations between each possible pair of security

Single-Index Models and Correlation Structure

If an analyst follows 150 stocks, how many estimates he/she requires?

- 150 estimates of expected returns and 150 estimates of standard deviation but, in addition, she also needs $150 \times 149/2 = 11{,}175$ estimates of covariance (or correlations)
- What if one factor or index affected all these 150 securities?
- That means the observed covariances essentially reflected the correlation structure between that index and these securities
- This leads to the genesis of single-index models

Single-Index Models and Correlation Structure

Single-index model assumes a single common influence that affects a large number of securities in a similar manner: $R_i = a_i + \beta_i R_m + e_i$ (3)

- This is a more data-driven model
- Researchers in the early days realized that market movements affect a large number of stocks in a similar manner
- Indices like Nifty affect the returns on a large number of securities

Construction of Single-Index Models

Construction of Single-Index Models

Single-index model: $R_i = a_i + \beta_i R_m + e_i$

- Both R_m and e_i are random variables
- Random variables are defined by a probability distribution with a mean and standard deviation
- Mean of R_m and e_i are $\bar{R}_m$ and 0, whereas standard deviations of R_m and e_i are σ_m and σ_{ei}, respectively
- Here, by definition R_m and e_i are uncorrelated:

$$\mathrm{Cov}(e_i, R_m) = E[(e_i - 0)(R_m - \bar{R}_m)] = 0$$

- The model is generally estimated using regression analysis

Construction of Single-Index Models

Single-index model also assumes that e_i is independent of all e_js: More formally, $E(e_i e_j) = 0$

- This means that the only reason two stocks commove is because of market; no other effects, such as industry
- This is not ensured by the regression analysis
- Thus, the performance of the model depends how good this assumption is
- $R_i = a_i + \beta_i R_m + e_i$ under the assumption of single-index model is assumed to represent the return dynamics for all the stocks, where $i = 1,2,3,\ldots,N$.

Construction of Single-Index Models

Under the assumption of a single-index model, this equation is assumed to represent the return dynamics for all the stocks, where *i* =1,2,3,…,*N* : $\boldsymbol{R_i = a_i + \beta_i(R_m) + e_i}$

- By the design (or construction) of the regression model. Mean of e_i, i.e., $E(e_i)$ = 0.
- By assumption, index (market) is unrelated to the idiosyncratic-specific component (e_i), that is, $E[e_i(R_m - R_m)] = 0$
- By assumption, securities are only related to each other through the index (market). That is, $E\left[\left(e_i e_j\right)\right] = 0$
- By definition, Variance of $e_i = E(e_i)^2 = \sigma_{ei}^2$
- By definition, Variance of $R_m = E(R_m - \bar{R}_m)^2 = \sigma_m^2$

Construction of Single-Index Models

Now that we have boundary conditions, we can derive the expressions for expected return, standard deviation, and covariance

- Expected returns:
  $E(R_i) = E[a_i + \beta_i R_m + e_i] = E(a_i) + E(\beta_i R_m) + E(e_i)$
- $E(e_i) = 0$, and that a_i and β_i are constants
- $E(R_i) = \bar{R}_i = a_i + \beta_i \bar{R}_m$

Construction of Single-Index Models

Standard deviation (σ_i^2)

- $\sigma_i^2 = E(R_i - \bar{R}_i)^2 = E\left[\left((a_i + \beta_i R_m + e_i) - (a_i + \beta_i \bar{R}_m)\right)\right]^2$
- $\sigma_i^2 = E[\beta_i(R_m - \bar{R}_m) + e_i]^2 = \beta_i^2 E([(R_m - \bar{R}_m)]^2 + E(e_i)^2 + 2\beta_i E[e_i(R_m - \bar{R}_m)]$
- $\sigma_i^2 = \beta_i^2 E([(R_m - \bar{R}_m)]^2 + E(e_i)^2 \text{ because } E[e_i(R_m - \bar{R}_m)] = 0$
- $\sigma_i^2 = \beta_i^2 \sigma_m^2 + \sigma_{ei}^2$

Construction of Single-Index Models

Covariance (σ_{ij})

$$\sigma_{ij} = E\left[(R_i - \bar{R}_i)\left(R_j - \bar{R}_j\right)\right]$$

$$\sigma_{ij} = E[\left((a_i + \beta_i R_m + e_i) - (a_i + \beta_i \bar{R}_m)\right)\left((a_j + \beta_j R_m + e_j) - (a_j + \beta_j \bar{R}_m)\right)]$$

$$\sigma_{ij} = E[(\beta_i(R_m - \bar{R}_m) + e_i)(\beta_j(R_m - \bar{R}_m) + e_j)]$$

$$\sigma_{ij} = \beta_i\beta_j E[(R_m - \bar{R}_m)^2] + \beta_j E[e_i(R_m - \bar{R}_m)] + \beta_i E\left[e_j(R_m - \bar{R}_m)\right] + E(e_i e_j)$$

The last three terms on RHS of the above equation are ‘0’ by definition

$$\boldsymbol{\sigma_{ij} = \beta_i\beta_j E\left[(R_m - \bar{R}_m)^2\right] = \beta_i\beta_j\sigma_m^2}$$

Single-Index Models: Example

Example

Consider the following example, where we are given the actual returns and the beta (1.5) of the stock and the market returns. We compute the average expected returns for the stock and market. Now using the following Equation: $\boldsymbol{\bar{R}_i = a_i + \beta_i \bar{R}_m}$, we can estimate $\boldsymbol{a_i}$. 8= a_i+1.5*4, i.e., a_i=2. Now that we have a_i, we can estimate the values of e_i for each period.

Example

Here, one can confirm that $E(e_i) = 0$. Also, note that $\sigma_{ei}^2 = 14$. Using the eq. $\boldsymbol{\sigma_i^2 = \beta_i^2 \sigma_m^2 + \sigma_{ei}^2}$, we get the value σ_i^2= 1.5*1.5*8+2.8=20.8. This variance is same as that directly calculated from the table.

Period	Ri and $\beta_i = 1.5$	Rm	$e_i = R_i - a_i - \beta_i R_m$
1	10	4	10-2-1.5*4=2
2	3	2	3-2-1.5*2=-2
3	15	8	15-2-1.5*8=1
4	9	6	9-2-1.5*6=-2
5	3	0	3-2-1.5*0-3=1
Average	8	4	0
Variance	20.8	8	2.8

Portfolio Characteristics with Single-Index Models

Portfolio Characteristics with Single-Index Models

With the assumption that a single-index model holds, let us examine its impact on portfolio returns and standard deviation

Expected return

- $\bar{R}_p = \sum_{i=1}^{N} X_i \bar{R}_i$; substitute the single-index model $\bar{R}_i = a_i + \beta_i \bar{R}_m$
- $\bar{R}_p = \sum_{i=1}^{N} X_i a_i + \sum_{i=1}^{N} X_i \beta_i \bar{R}_m$

Standard deviation

- $\sigma_p^2 = \sum_{i=1}^{N} X_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ i \neq j}}^{N} X_i X_j \sigma_{ij}$; substituting the expression for variance and covariance
- $\sigma_p^2 = \sum_{i=1}^{N} X_i^2 \beta_i^2 \sigma_m^2 + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ i \neq j}}^{N} X_i X_j \beta_i \beta_j \sigma_m^2 + \sum_{i=1}^{N} X_i^2 \sigma_{ei}^2$

Portfolio Characteristics with Single-Index Models

Assume we have a portfolio of 150 stocks and we require estimates of

(1) a_i, β_i, and σ_{ei} for each of the stock

(2) $\bar{R}_m$ and σ_m^2 for the market

That is, 150 × 3 + 2 = 452 estimates are needed (as compared to 11,485 estimates in the absence of a single-index model)

Portfolio Characteristics with Single-Index Models

Portfolio expected return

- $\bar{R}_p = \sum_{i=1}^{N} X_i a_i + \sum_{i=1}^{N} X_i \beta_i \bar{R}_m$
- $\beta_p = \sum_{i=1}^{N} X_i \beta_i$; $a_p = \sum_{i=1}^{N} X_i a_i$
- $\bar{R}_p = a_p + \beta_p \bar{R}_m$
- Please note that if the portfolio under consideration is the market portfolio, then a_p = 0 and β_p = 1. Then, $\bar{R}_p$ = $\bar{R}_m$. Thus, stocks with β_p>1 are said to be riskier than the market, and stocks with β_p < 1 are said to be less risky than the market

Portfolio Characteristics with Single-Index Models

Portfolio standard deviation

- $\sigma_p^2 = \sum_{i=1}^{N} X_i^2 \beta_i^2 \sigma_m^2 + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ i \neq j}}^{N} X_i X_j \beta_i \beta_j \sigma_m^2 + \sum_{i=1}^{N} X_i^2\, \sigma_{ei}^2$
- $\sigma_p^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} X_i X_j \beta_i \beta_j \sigma_m^2 + \sum_{i=1}^{N} X_i^2\, \sigma_{ei}^2$
- $\sigma_p^2 = (\sum_{i=1}^{N} X_i \beta_i)(\sum_{j=1}^{N} X_j \beta_j)\sigma_m^2 + \sum_{i=1}^{N} X_i^2\, \sigma_{ei}^2$
- But $(\sum_{j=1}^{N} X_i \beta_i) = (\sum_{j=1}^{N} X_j \beta_j) = \beta_p$
- $\sigma_p^2 = \beta_p^2 \sigma_m^2 + \sum_{i=1}^{N} X_i^2\, \sigma_{ei}^2$

Characteristics of Single-Index Model

Portfolio standard deviation

- $\sigma_p^2 = \beta_p^2 \sigma_m^2 + \sum_{i=1}^{N} X_i^2 \, \sigma_{ei}^2$
- Consider equal investments in the securities so that $X_1 = X_2 = \cdots = X_N = \frac{1}{N}$
- If there are a large number of securities, then the term $\frac{\sigma_{ei}^2}{N}$, which represents the residual (or specific risk), approaches to zero
- $\sigma_p^2 = \beta_p^2 \sigma_m^2$

Single-Index Models: Market Model

Market Model

- Consider $\bar{R}_i = a_i + \beta_i \bar{R}_m$ index model. When the assumption of $\mathrm{Cov}(e_i e_j) = 0$ is waived, then it becomes the market model
- This allows for comovement across securities because of factors other than the market
- This means it is a less restrictive form of index model family
- It suggests that there are additional systematic marketwide factors that can also affect the individual securities
- What can be these factors?

Estimation of Portfolio Beta with Single-Index Models

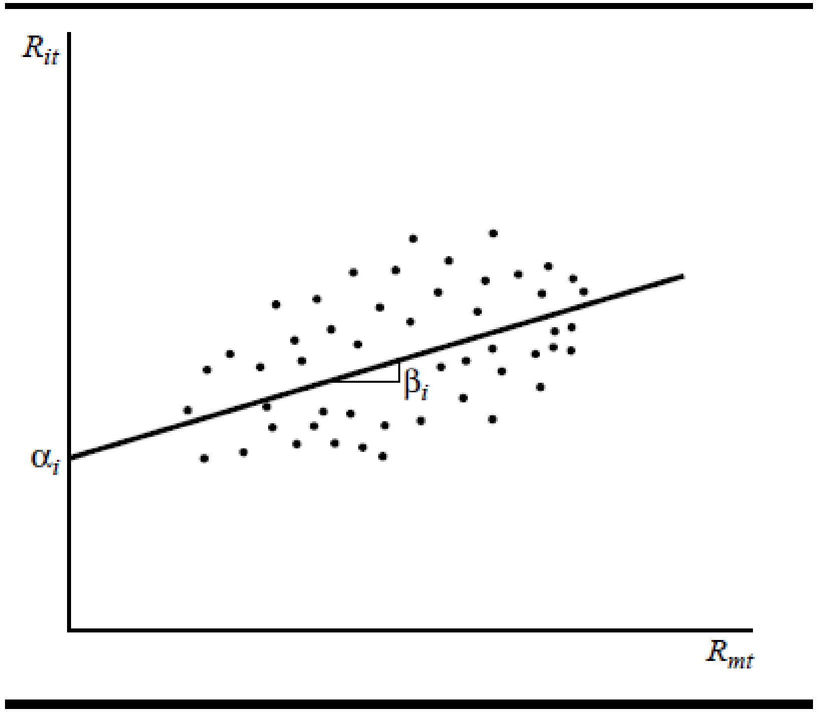
Estimation of Portfolio Beta

We had discussion about the estimation of betas!

$$R_{it} = a_i + \beta_i R_{mt} + e_{it}$$

Here, we are fitting a line across the scatter points of R_i and R_m observations, available over time

The slope of this line is the best estimate of the beta over the period of examination



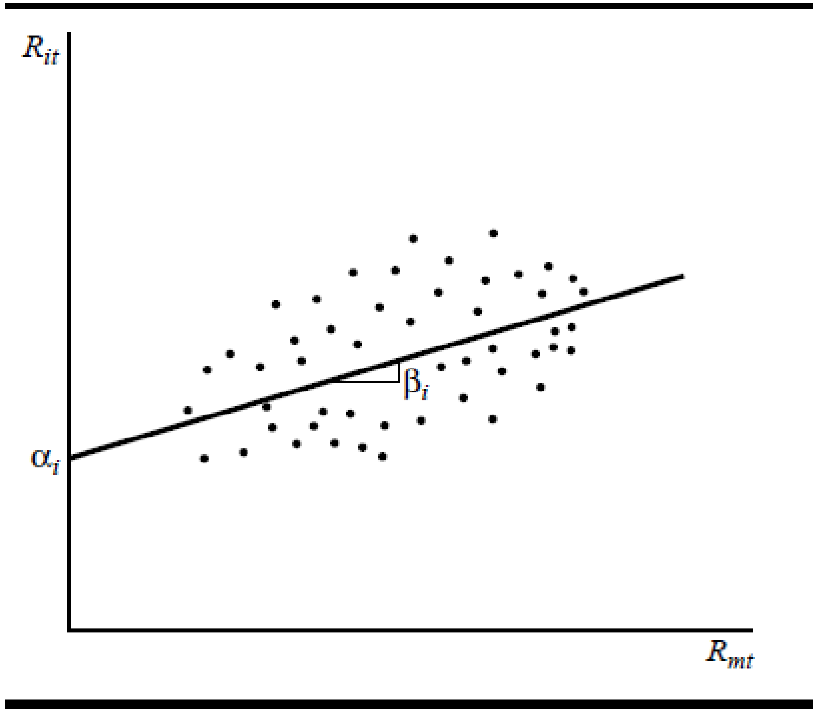
Estimation of Portfolio Beta

$$R_i = a_i + \beta_i R_m + e_i$$

Here, we are fitting a line across the scatter points of R_i and R_m observations, available over time

The slope of this line is the best estimate of the beta over the period of examination

$$\beta_i = \frac{\sigma_{im}}{\sigma_m^2} = \frac{\sum_{t=1}^{T}[(R_{it}-\bar{R}_{it})(R_{mt}-\bar{R}_{mt})]}{\sum_{t=1}^{T}(R_{mt}-\bar{R}_{mt})^2}$$



Estimation of Portfolio Beta

- But beta estimates are also subject to estimation errors
- Also, firm betas change over time (changes in capital structure, industry, etc.)
- Therefore, analysts estimate betas of industry portfolios
- These are less noisy and more reliable estimates
- The random variation in one security (upwards) and the other security (downwards) tend to cancel out each other

Single-Index Models: Beta Example

Beta Example

Period	R_{it}	R_{mt}	$(R_{it} - \bar{R}_{it})(R_{mt} - \bar{R}_{mt})$	Value
1	10	4	$(10 - 8) \times (4 - 4)$	0
2	3	2	$(3 - 8) \times (2 - 4)$	10
3	15	8	$(15 - 8) \times (8 - 4)$	28
4	9	6	$(9 - 8) \times (6 - 4)$	2
5	3	0	$(3 - 8) \times (0 - 4)$	20
Average	8	4	Total	60
Variance	20.8	$\sigma_m^2 = 8$	Covariance (σ_{im})	= 60/5 = 12

- $\beta_i = \frac{\sigma_{im}}{\sigma_m^2}$=1.5

A Few Words on Beta

A Few Words on Beta

- Beta is a risk measure that is estimated from the relationship between the return of a security and that of the market
- Some of the well-known fundamental variables that affect the risk of stock are dividend payout, asset growth, leverage, liquidity, asset size, and earnings variability

A Few Words on Beta

- Firm beta and dividends: Firms that pay more dividends have positive future expectations and are considered to be less risky: low beta
- Firm beta and growth: High-growth firms are generally young firms with high capital requirements and are considered to be riskier: high beta
- Firm beta and liquidity: Firms with high liquidity are considered to be less risky: low beta

A Few Words on Beta

- **Size and beta:** Large firms are considered to be less risky than smaller firms: **large firms have low beta**
- **Earnings variability and beta:** A firm with high earnings variability (earnings beta) is considered riskier: **positive beta**

Introduction to Multi-Index Models

Introduction to Multi-Index models

- An improvement over single-index models is a multi-index model
- These models aim to capture the non-market influences that may cause securities to move together
- These multi-index models aim to capture the economic factors or structural groups (e.g., industrial effects)

Introduction to Multi-index models

The generalized multi-index models can be written in the following form

- $R_i = a_i^* + b_{i1}^* I_1^* + b_{i2}^* I_2^* + b_{i3}^* I_3^* + \cdots + b_{iL}^* I_L^* + c_i$
- What is the interpretation of a_i^*, b_{i1}^*, c_i?

Introduction to Multi-Index Models

The indices (I_j^*s) would capture the influence of market returns, level of interest rate, and various industry effects.

- However, this model faces one major challenge
- Some of the indices employed in the model may be correlated
- This vitiates the estimation, as the regression estimations of this kind require the independent variables to be uncorrelated
- When the variables are correlated, it is difficult to segregate their respective effects (b_{ij}^*'s) on the security

Introduction to Multi-Index Models

Researchers often perform a procedure called orthogonalization to remove the correlated portion from the respective indices and create orthogonalized indices

- The new transformed equation is provided below
- $R_i = a_i + b_{i1}\, I_1 + b_{i2}\, I_2 + b_{i3}\, I_3 + \cdots + b_{iL}\, I_L + c_i$

Introduction to Multi-Index Models

Multi-index model: $R_i = a_i + b_{i1}\, I_1 + b_{i2}\, I_2 + b_{i3}\, I_3 + \cdots + b_{iL}\, I_L + c_i$

- The new indices are so constructed as they have no correlation
- Also, the error term (c_i) is not correlated with indices, i.e., $E[c_i(I_j - \bar{I_j})] = 0$
- However, the economic interpretation of new indices is slightly difficult

Design of Multi-Index Models

Introduction to Multi-Index Models: Basic Equation

$\boldsymbol{R_i = a_i + b_{i1}\, I_1 + b_{i2}\, I_2 + b_{i3}\, I_3 + \cdots + b_{iL}\, I_L + c_i}$; for all stocks *i* = 1, 2, 3,....,*N*, and indices *j* = 1,2,3,….,*L*

By definition

- Residual variance of stock *i* equals σ_{ci}^2
- Variance of index I_j equals σ_{Ij}^2

Introduction to Multi-Index Models: Basic Equation

By construction

- Mean of c_i equals $E(c_i) = 0$
- Covariance between indexes *j* and *k* equals $E[(I_j - \bar{I}_j)\,(I_k - \bar{I}_k)] = 0$
- Covariance between residuals for stock *i* and index *j* equals $E[c_i(I_j - \bar{I}_j)] = 0$

By assumption

- Covariance between c_i and c_j is zero, i.e., $E[c_i c_j] = 0$

Introduction to Multi-Index Models: Basic Equation

By assumption: Covariance between c_i and c_j is zero, i.e., $E[c_i c_j] = 0$

- This last assumption suggests that the only reason stocks vary together is because of their common relationship with the indexes specified in the model
- There is no other reason that two stocks (i,j) should have a correlation
- However, there is nothing in the model estimation that forces this to be true
- This is only an approximation, and the performance of the model will be as good as the approximation

Multi-Index Models: Expected Return and Risk

Multi-Index Models: Expected Return and Risk

Expected return

- $\bar{R}_i = a_i + b_{i1}\bar{I}_1 + b_{i2}\bar{I}_2 + \cdots + b_{iL}\bar{I}_L$

Variance of return

- $\sigma_i^2 = b_{i1}^2\sigma_{I1}^2 + b_{i2}^2\sigma_{I2}^2 + \cdots + b_{iL}^2\sigma_{IL}^2 + \sigma_{ci}^2$

Covariance between security *i* and *j*

- $\sigma_{ij} = b_{i1}b_{j1}\sigma_{I1}^2 + b_{i2}b_{j2}\sigma_{I2}^2 + \cdots + b_{iL}b_{jL}\sigma_{IL}^2$

Multi-Index Models: Expected Return and Risk

To estimate the expected return and risk, the following estimates are required

- a_i and σ_{ci}^2 for each stock
- b_{ik} between each stock and index
- An estimate of index mean ($\bar{I}_{\mathrm{j}}$) and variance σ_{Ij}^2 of each index

Assuming *N* securities and *L* indices, this is a total 2*N* + *LN* + 2*L* estimates

An analyst following 150 stocks having 10 indices, this means 1820 inputs

This structure, although more complex than single-index models, is still less complex when no simplifying correlation structure is assumed

Multi-Index Models: Expected Return and Risk

- Researchers often derive indices from the available data using quantitative techniques (e.g., principal component analysis and factor analysis)
- One can add more indices to increase the explanatory power of the model
- However, with more indices model becomes less efficient and more complex
- Therefore, it is a sort of trade-off between the complexity, efficiency, and explanatory power of the model

Multi-Index Models: 3-Factor Fama–French Model

3-Factor Fama–French Model

$\bar{R}_i = a_i + b_{iM}(\bar{R}_M - \bar{R}_f) + b_{iSMB}\bar{R}_{SMB} + b_{iHML}\bar{R}_{HML}$ Or

$\bar{R}_i - \bar{R}_f = a_i^* + b_{iM}(\bar{R}_M - \bar{R}_f) + b_{iSMB}\bar{R}_{SMB} + b_{iHML}\bar{R}_{HML}$

$(\bar{\boldsymbol{R}}_{\boldsymbol{M}} - \bar{\boldsymbol{R}}_{\boldsymbol{f}})$ **Market**: is the market index indicating the excess returns over risk-free rate

$\bar{\boldsymbol{R}}_{\boldsymbol{SMB}}$ (small minus big): indicates the excess return on a portfolio of small stocks over large stocks. The excess returns by small stocks capture the fact that they are riskier than large stocks

$\bar{\boldsymbol{R}}_{\boldsymbol{HML}}$ (high minus low): indicates the excess return on a portfolio of high book-to-market (BTM) stocks (value stocks) over that of low (BTM) stocks (growth stocks)

Summary and Concluding Remarks

Summary and Concluding Remarks

- Introduction of single- and multi-index models considerably simplifies security analysis
- In particular, the complex correlation structure between two securities is replaced by the common influence of the index on each of the security
- With the application of these index models, portfolio analysis is considerably simplified

Summary and Concluding Remarks

- Portfolio betas are often less noisy and more informationally efficient than betas of individual securities
- Construction of multi-index models broadly employ similar theoretical underpinnings, except that they employ multiple indices
- Construction of these index models requires certain assumptions, some of which are held by the assumption, design, or definition of the respective model

Summary and Concluding Remarks

Some of these key assumptions in these index models are as follows

- Idiosyncratic error terms are not correlated with indices that are more systematic influence: $E[c_i(I_j - \bar{I}_j)] = 0$
- These indices are not correlated across each other: $E[(I_j - \bar{I}_j)(I_k - \bar{I}_k)] = 0$
- The error terms are not correlated with each other: $E[c_i c_j] = 0$

Thanks!