IIT ROORKEE

NPTEL ONLINE CERTIFICATION COURSE

QUANTITATIVE INVESTMENT MANAGEMENT

LECTURE 7

Forward Rates, Bond Pricing with Forward Rates

J P Singh

Department of Management Studies

BONOD VALUATION WITH FORWARD RATES

FORWARD RATES

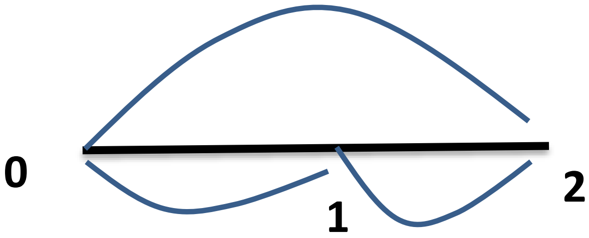
- **Forward rates are yields for future periods.**
- **A forward rate is a borrowing/lending rate for a loan to be made at some future date.**

NOTATION FOR FORWARD RATES

- The notation used must identify both
- when in the future the money will be loaned/borrowed and
- the length of the ending/borrowing period.
- Thus, f_{12} is the rate for a 1-year loan one year from now; f_{23} is the rate for a 1-year loan to be made two years from now; f_{35} is the 2-year forward rate three years from now; and so on.

RELATIONSHIP BETWEEN SPOT & FORWARD RATES

- **To avoid arbitrage, *borrowing for T years at the T-year spot rate, or borrowing for one-year periods in T successive years, should have the same cost.***



$$A = P_0\left(1+S_{02}\right)^2$$

$$A^* = P_0\left(1+S_{01}\right)\left(1+f_{12}\right)$$

$$\text{For no arbitrage}: A = A^*$$

$$\left(1+S_{02}\right)^2 = \left(1+S_{01}\right)\left(1+f_{12}\right)$$

$$\left(1+S_{0T}\right)^{T}=\left(1+f_{01}\right)\left(1+f_{12}\right)\left(1+f_{23}\right)\ldots\left(1+f_{T,T+1}\right)$$

$$=\left(1+S_{01}\right)\left(1+f_{12}\right)\left(1+f_{23}\right)\ldots\left(1+f_{T,T+1}\right)$$

$$=\prod_{t=0}^{T}\left(1+f_{t,t+1}\right)=\left(1+S_{01}\right)\prod_{t=1}^{T}\left(1+f_{t,t+1}\right)$$

$$=\left(1+S_{01}\right)\left(1+f_{12}\right)\prod_{t=2}^{T}\left(1+f_{t,t+1}\right)=\left(1+S_{02}\right)^{2}\prod_{t=2}^{T}\left(1+f_{t,t+1}\right)$$

$$=\left(1+S_{0H}\right)^{H}\prod_{t=H}^{T}\left(1+f_{t,t+1}\right)$$

BOND VALUATION WITH FORWARD RATES

BOND VALUATION WITH FORWARD RATES

- Consider a T-year maturity bond with cash flows C_t in year $t, t = 1, 2, 3, \dots, T$.
- The arbitrage free value of the bond is given by:
- $V_0 = \frac{C_1}{1+S_{01}} + \frac{C_2}{(1+S_{02})^2} + \cdots + \frac{C_T}{(1+S_{0T})^T}$
- $= \frac{C_1}{1+S_{01}} + \frac{C_2}{(1+S_{01})(1+f_{12})} + \cdots + \frac{C_T}{(1+S_{01})(1+f_{12})\dots(1+f_{T-1,T})}$
- These formula hold only if the forward rates are determined by the condition of arbitrage free pricing.

FORWARD PRICE OF A BOND

- The forward price of the above T-year bond for delivery at $t = 1$ year from now (after the receipt of the cash flow C_1) is:
- $F(0,1,T) = V_0(1 + S_{01}) - C_1$
- $= \frac{C_2}{(1+f_{12})} + \cdots + \frac{C_T}{(1+f_{12})\ldots(1+f_{T-1,T})}$
- Similarly,
- $F(0,2,T) = V_0(1 + S_{01})(1 + f_{12}) - C_1(1 + f_{12}) - C_2$
- $= \frac{C_3}{(1+f_{23})} + \cdots + \frac{C_T}{(1+f_{23})\ldots(1+f_{T-1,T})}$ etc.

FUTURE VALUE AT t=1

PV OF FUTURE CF AT t=1

EXAMPLE

- Consider a 20% annual bond of face value 100 with a maturity of three years. Calculate the forward price of the bond at the end of the first year and second year immediately after interest payments are made. The spectrum of interest rates is as follows: $S_{01} = 6\%; f_{12} = 7\%; f_{23} = 8\%$.

SOLUTION

- **Spot price of the bond:**
- $$V_0 = \frac{C_1}{(1+S_{01})} + \frac{C_2}{(1+S_{01})(1+f_{12})} + \frac{C_3}{(1+S_{01})(1+f_{12})(1+f_{23})}$$
- $$\frac{20}{1.06} + \frac{20}{1.06 \times 1.07} + \frac{120}{1.06 \times 1.07 \times 1.08} = 134.4658$$

- **1 year forward price:**
- $F(0,1,3) = V_0(1 + S_{01}) - C_1$
- $= 134.4658 \times 1.06 - 20 = 122.5337$
- $= \frac{C_2}{(1+f_{12})} + \frac{C_3}{(1+f_{12})(1+f_{23})}$
- $= \frac{20}{1.07} + \frac{120}{1.07 \times 1.08} = 122.5337$

THE CASE OF ZCBs

- For the case of a T year ZCB, the above formulae simplify to (because there are no intermediate cash flows i.e. coupon payments):
- $V_0 = \frac{C_T}{(1+S_{0T})^T} = \frac{C_T}{(1+S_{01})(1+f_{12})\ldots(1+f_{T-1,T})}$
- $F(0,1,T) = V_0(1+S_{01}) = \frac{C_T}{(1+f_{12})\ldots(1+f_{T-1,T})}$
- $F(0,2,T) = V_0(1+S_{02})^2 = V_0(1+S_{01})(1+f_{12})$
- $= \frac{C_T}{(1+f_{23})\ldots(1+f_{T-1,T})}$

BOND VALUATION BY BINOMIAL MODEL

NEED FOR BINOMIAL MODEL

- While we can value option-free bonds with a simple spot rate curve, for bonds with embedded options, changes in future rates will affect the probability of the option being exercised and the underlying future cash flows.
- Thus, we need a model that allows both rates and the underlying cash flows to vary to value bonds with embedded options.
- One such model is the binomial interest rate tree framework.