

CS 61B: Lecture 20  
Monday, March 5, 2012

Today's reading: Goodrich & Tamassia, Chapter 4 (especially 4.2 and 4.3).

#### ASYMPTOTIC ANALYSIS (bounds on running time or memory)

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Suppose an algorithm for processing a retail store's inventory takes:

- 10,000 milliseconds to read the initial inventory from disk, and then
- 10 milliseconds to process each transaction (items acquired or sold).

Processing  $n$  transactions takes  $(10,000 + 10n)$  ms. Even though  $10,000 \gg 10$ , we sense that the " $10n$ " term will be more important if the number of transactions is very large.

We also know that these coefficients will change if we buy a faster computer or disk drive, or use a different language or compiler. We want a way to express the speed of an algorithm independently of a specific implementation on a specific machine--specifically, we want to ignore constant factors (which get smaller and smaller as technology improves).

#### Big-Oh Notation (upper bounds on a function's growth)

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Big-Oh notation compares how quickly two functions grow as  $n \rightarrow \infty$ .

Let  $n$  be the size of a program's `_input_` (in bits or data words or whatever). Let  $T(n)$  be a function. For now,  $T(n)$  is the algorithm's precise running time in milliseconds, given an input of size  $n$  (usually a complicated expression). Let  $f(n)$  be another function--preferably a simple function like  $f(n) = n$ .

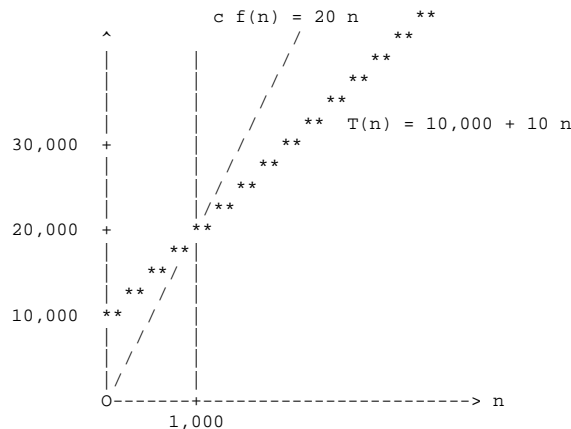
We say that  $T(n)$  is in  $O(f(n))$  IF AND ONLY IF  $T(n) \leq c f(n)$  WHENEVER  $n$  IS BIG, FOR SOME LARGE CONSTANT  $c$ .

- \* HOW BIG IS "BIG"? Big enough to make  $T(n)$  fit under  $c f(n)$ .
- \* HOW LARGE IS  $c$ ? Large enough to make  $T(n)$  fit under  $c f(n)$ .

#### EXAMPLE: Inventory

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Let's consider the function  $T(n) = 10,000 + 10n$ , from our example above. Let's try out  $f(n) = n$ , because it's simple. We can choose  $c$  as large as we want, and we're trying to make  $T(n)$  fit underneath  $c f(n)$ , so pick  $c = 20$ .



As these functions extend forever to the right, their asymptotes will never cross again. For large  $n$ --any  $n$  bigger than 1,000, in fact-- $T(n) \leq c f(n)$ .  
\*\*\* THEREFORE,  $T(n)$  is in  $O(f(n))$ . \*\*\*

Next, you must learn how to express this idea rigorously. Here is the central lesson of today's lecture, which will bear on your entire career as a professional computer scientist, however abstruse it may seem now:

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| FORMALY:  O(f(n)) is the SET of ALL functions T(n) that satisfy:
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|   There exist positive constants c and N such that, for all n >= N,
|                                   T(n) <= c f(n)
|=====
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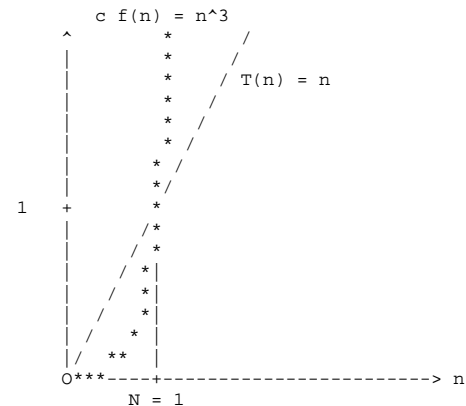
Pay close attention to  $c$  and  $N$ . In the graph above,  $c = 20$ , and  $N = 1,000$ .

Think of it this way: if you're trying to prove that one function is asymptotically bounded by another [ $f(n)$  is in  $O(g(n))$ ], you're allowed to multiply them by positive constants in an attempt to stuff one underneath the other. You're also allowed to move the vertical line ( $N$ ) as far to the right as you like (to get all the crossings onto the left side). We're only interested in how the functions behave as  $n$  shoots off toward infinity.

#### EXAMPLES: Some Important Corollaries

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- [1]  $1,000,000n$  is in  $O(n)$ . Proof: set  $c = 1,000,000$ ,  $N = 0$ .  
-> Therefore, Big-Oh notation doesn't care about (most) constant factors. We generally leave constants out; it's unnecessary to write  $O(2n)$ , because  $O(2n) = O(n)$ . (The 2 is not wrong; just unnecessary.)
- [2]  $n$  is in  $O(n^3)$ . [That's  $n$  cubed]. Proof: set  $c = 1$ ,  $N = 1$ .  
-> Therefore, Big-Oh notation can be misleading. Just because an algorithm's running time is in  $O(n^3)$  doesn't mean it's slow; it might also be in  $O(n)$ . Big-Oh notation only gives us an UPPER BOUND on a function.



- [3]  $n^3 + n^2 + n$  is in  $O(n^3)$ . Proof: set  $c = 3$ ,  $N = 1$ .  
-> Big-Oh notation is usually used only to indicate the dominating (largest and most displeasing) term in the function. The other terms become insignificant when  $n$  is really big.

Here's a table to help you figure out the dominating term.

## Table of Important Big-Oh Sets

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 Arranged from smallest to largest, happiest to saddest, in order of increasing domination:

function	common name
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$O(1)$	:: constant
is a subset of $O(\log n)$	:: logarithmic
is a subset of $O(\log^2 n)$	:: log-squared [that's $(\log n)^2$ ]
is a subset of $O(\sqrt{n})$	:: root-n [that's the square root]
is a subset of $O(n)$	:: linear
is a subset of $O(n \log n)$	:: $n \log n$
is a subset of $O(n^2)$	:: quadratic
is a subset of $O(n^3)$	:: cubic
is a subset of $O(n^4)$	:: quartic
is a subset of $O(2^n)$	:: exponential
is a subset of $O(e^n)$	:: exponential (but more so)

Algorithms that run in  $O(n \log n)$  time or faster are considered efficient. Algorithms that take  $n^7$  time or more are usually considered useless. In the region between  $n \log n$  and  $n^7$ , the usefulness of an algorithm depends on the typical input sizes and the associated constants hidden by the Big-Oh notation.

If you're not thoroughly comfortable with logarithms, read Sections 4.1.2 and 4.1.7 of Goodrich & Tamassia carefully. Computer scientists need to know logarithms in their bones.

## Warnings

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 [1] Here's a fallacious proof:

$n^2$  is in  $O(n)$ , because if we choose  $c = n$ , we get  $n^2 \leq n^2$ .  
 -> WRONG!  $c$  must be a constant; it cannot depend on  $n$ .

[2] The big-Oh notation expresses a relationship between functions. IT DOES NOT SAY WHAT THE FUNCTIONS MEAN. In particular, the function on the left does not need to be the worst-case running time, though it often is. The number of emails you send to your Mom as a function of time might be in  $O(t^2)$ . In that case, not only are you a very good child; you're an increasingly good child.

In binary search on an array,

- the worst-case running time is in  $O(\log n)$ ,
- the best-case running time is in  $O(1)$ ,
- the memory use is in  $O(n)$ , and
- $47 + 18 \log n - 3/n$  is in  $O(\text{the worst-case running time})$ .

Every semester, a few students get the wrong idea that "big-Oh" always means "worst-case running time." Their brains short out when an exam question uses it some other way.

[3] " $e^3n$  is in  $O(e^n)$  because constant factors don't matter."  
 " $10^n$  is in  $O(2^n)$  because constant factors don't matter."  
 -> WRONG! I said that Big-Oh notation doesn't care about (most) constant factors. Here are some of the exceptions. A constant factor in an exponent is not the same as a constant factor in front of a term.  $e^3n$  is not bigger than  $e^n$  by a constant factor; it's bigger by a factor of  $e^3n$ , which is damn big. Likewise,  $10^n$  is bigger than  $2^n$  by a factor of  $5^n$ .

[4] Big-Oh notation doesn't tell the whole story, because it leaves out the constants. If one algorithm runs in time  $T(n) = n \log_2 n$ , and another algorithm runs in time  $U(n) = 100n$ , then Big-Oh notation suggests you should use  $U(n)$ , because  $T(n)$  dominates  $U(n)$  asymptotically. However,  $U(n)$  is only faster than  $T(n)$  in practice if your input size is greater than current estimates of the number of subatomic particles in the universe. The base-two logarithm  $\log_2 n < 50$  for any input size  $n$  you are ever likely to encounter.

Nevertheless, Big-Oh notation is still a good rule of thumb, because the hidden constants in real-world algorithms usually aren't that big.