Definition:

vector space: a set V along with an addition and scalar multiplication on V such that commutativity, associativity, additive identity, additive inverse, multiplicative identity, distributive properties

P_m(F): VS of polynomial of degree at most m

subspace: additive identity, closed under addition, closed under scalar multiplication

direct sum: V is the direct sum of subspaces U_1, ..., U_m if each element of v can be written uniquely as a sum u_1 + ... + u_m

linearly independent (including empty list)

basis: a list of linearly independent vectors that spans V

linear map: additivity, homogeneity (aT(v) = T(av))

operator: a linear map from a vector space to itself L(V)

invariant subspace under T: T(U) subset of U (condition that restriction is an operator)

eigenvalue: exists a *nonzero* vector s.t. ...

M(T, (v1, ..., vn)): default: standard basis

P_{U,W}(v) = u: v = u + w; V = U oplus W

inner product: positivity (v,v), definiteness (v,v), additivity in first slot, homogeneity in first slot, conjugate symmetry

Euclidean inner product on F^n: sum ui * bar(wi)

orthogonal complement: U^perp

[U subspace of V] V = U oplus U^perp

P_U is called the orthogonal projection of V onto U

unitary / orthogonal matrix: matrix over C / R s.t. columns form an orthonormal basis of R^n with the Euclidean inner product

adjoint: T in L(V, w), T^* maps from W to V s.t. <Tv, w> = <v, T^*w>.

self-adjoint: operator T in L(V) s.t. T = T^*

normal: T operator in L(V), TT^* = T^*T

positive operator: (V real && T self-adjoint || V complex) and <Tv, v> >= 0 for all v in V

isometry: norm(Tv) = norm(v) for all v in V

generalized eigenvector: v generalized eigenvector if (T-λ I)^j v = 0 for some positive integer j

nilpotent: N^j = 0 for some j > 0

multiplicity: dim null(T-λ*I)^dim

characteristic polynomial: p(z) = (z-λ_1)^d1 * ... * (z-λ_m)^dm [V complex, T in L(V), λ_1, ..., λ_m distinct eigenvalues of T, dj = multiplicity of λ_j].

singular value: eigenvalue of sqrt(T^*T)

Jordan basis: if with respect to this basis T has a block diagonal matrix Diagonal(A_1, ..., A_m), where each A_j is an upper-triangular matrix with diagonal filled with some eigenvalue λ_j of T, and the line directly above it filled with 1's, and all other 0's.

m(v): [N nilpotent] the largest nonnegative integer such that N^(m(v)) v != 0

Proposition / Lemma:

V = U1 oplus U2 oplus ... oplus Um iff

V = U1 + ... + Um

and the only way to write 0 is by taking all ui 0

If (v1, ... ,vm) is linearly dependent in V and v1 != 0, then exists j in {2, ..., m} such that:

vj in Span(v1, ..., v_{j-1})

if the jth term is removed from (v1, ..., vm), the span of the remaining list still equals span(v1, ..., vm)

Suppose V is finite dimensional and U is a subspace of V. Then there is a subspace W of V such that V = U direct W.

Proof by extending a basis of U to a basis of V

Suppose U1, ..., Um subspaces s.t. V = U1 + ... + Um and dim V = dim U1 + ... + dim Um, then V = U1 oplus ... oplus Um

A linear map is invertible (exists S such that ST = TS = I) iff it is injective and surjective

Suppose that (v1, ... , vn) is a basis of V and (w1, ... ,wm) is a basis of W. Then M is an invertible linear map between L(V , W) and Mat(m, n, F).

Proof by injectivity and surjectivity

- **dim L(V,W)** = (dimV)(dimW)
- A root of p (deg m) iff exists q (deg m-1) s.t. p(z) = (z-λ) * q(z)
- root => complex conjugate also root
- operator at most dim V eigenvalues
- T in L(V), (v1, ..., vn) basis then
  - matrix upper triangular
  - <=> Tvk in Span(v1, ..., vk)
  - <=> span(v1, ..., vk) invariant under T
- Suppose T in L(V) has an upper triangular matrix w.r.t some basis of V. T invertible <=> all entries on diagonal of u.t.m. nonzero.
  - proof technique: prove contrapositive.
    - <=: if exists zero entry, Tvk in Span(v1, ..., vk-1). Construct linear map S: Span(v1,...,vk) -> Span(v1,...,vk-1) by Sv=Tv. not injective => exists v Tv = 0 => T not invertible
    - =>: if not invertible, exists v, Tv = 0 => T(a1v1+...+akvk) = 0, ak!=0 => Tvk in Span(v1,...,vk-1) => λ_k must be 0
- M(ST, (u1,...), (w1...)) = M(S, (v1...), (w1...))M(T, (u1...), (v1...))
- **Q unitary / orthogonal** <=> T(x) = Qx isometry <=> QQ* = I <=> Q* unitary / orthogonal <=> rows of Q form an orthonormal basis
- **adjoint**
  - (aT)* = conjugate(a)T^*
  - null T^* = (range T)^perp
  - range T^* = (null T)^perp
  - [both orthonormal basis] A^* is conjugate transpose of A
  - T injective <=> T* surjective
- **self-adjoint**
  - every eigenvalue real
  - [V complex, <Tv, v> = 0 for all v in V] => T = 0
    - proof technique: <Tu, w> = (<T(u+w), u+w> - <T(u-w), u-w>) / 4 + i(<T(i((u+w)), i(u+w)> - <T(i((u-w)), i(u-w)>) / 4
  - [V real, T self-adjoint, <Tv, v> = 0 for all v in V] => T = 0
    - proof technique: <Tu, w> = (<T(u+w), u+w> - <T(u-w), u-w>) / 4
  - [V complex] T self-adjoint <=> <Tv, v> in R for all v in V
    - proof technique: <Tv, v> - conjugate(<Tv, v>) = <(T - T^*)v, v>
- <u, v> = 0 iff norm(u) <= norm(u+av) for all a in F
- [P in L(V), P^2 = P]
  - P orthogonal projection <=> P self-adjoint
  - norm(Pv) <= norm(v) for every v in V => P orthogonal projection
  - every vector in null P orthogonal to every vector in range P => P orthogonal projection
- **normal**
  - T normal <=> norm(Tv) = norm(T^*v) for all v in V
  - T normal => [v in V eigenvector of T with eigenvalue λ <=> eigenvector of T^* with eigenvalue conjugate(λ)]
  - T normal => eigenvectors of T corresponding to distinct eigenvalues are orthogonal
  - [T normal, U invariant subspace] =>
    - U^perp invariant under T
    - U invariant under T^*
    - (T|U)^* = (T^*)|U
    - T|U normal operator on U
    - T|U^perp normal operator on U^perp
  - [T normal, V complex inner-product space] T self-adjoint <=> eigenvectors all real
- [F = R && T self-adjoint || F = C && T normal], λ_1 .. λ_n distinct eigenvalues of T => V = null(T-λ_1 I) oplus ... oplus null(T-λ_n I). Furthermore, each vector in each null(T-λ_j I) is orthogonal to all vectors in the other subspaces of this decomposition
- **T is positive** <=>
  - T is self-adjoint and all the eigenvalues of T are nonnegative <=>
  - T has a positive square root <=>
  - T has a self-adjoint square root <=>
  - exists S in L(V) s.t. T = S*S
- **T positive** => has unique positive square root

S is an isometry <=> <Su, Sv> = <u, v> for all u, v in V <=> S*S = I <=> (Se1, ..., Sen) is orthonormal whenever (e1, ..., en) orthonormal in V <=> exists (e1, ..., en) s.t. (Se1, ..., Sen) orthonormal <=> S* isometry

isometry => normal

[V complex, S in L(V)]. S isometry <=> exists orthonormal basis of V consisting of eigenvectors of S all of whose corresponding eigenvalues have absolute value 1

T1, T2 same singular values <=> exists isometries S1, S2 s.t. T1 = S1T2S2

N in L(V) nilpotent => N^(dim V) = 0

[T in L(V), λ in F]. foreach basis s.t. T has upper-tri matrix, λ appears on diagonal dim(null((T - λ I)^(dim V))) times.

proof technique: WLOG only consider λ = 0. induction on dimension. break down into cases λ_n != 0, λ_n = 0

V complex vector space. sum of **multiplicities** of all the eigenvalues of T equals V

[N **nilpotent** operator on V] => exists basis of V with respect to which the matrix of N has all entries on and below diagonal 0's.

If N in L(V) is **nilpotent**, then there exists vectors v1, ..., vk s.t.

(a) (v1, Nv1, ..., N^(m(v1))v1, ..., vk, Nvk, ..., N^(m(vk))vk) is a basis of V

(b) (N^(m(v1))v1, ..., N^(m(vk))vk) is a basis of null N

proof technique: induction on dim. find basis for range(N) and null(N) cap range(N). choose basis for W where null(N) = (null(N) cap range(N)) oplus W

ST **nilpotent** <=> TS nilpotent

N **nilpotent** => only eigenvalue 0

V complex, N only eigenvalue 0 => N **nilpotent**

N **self-adjoint & nilpotent** => N = 0

det(-A) = (-1)^n det(A), A=M_{n*n}

Theorem:

In a finite-dimensional vector space, the length of every linearly independent list of vectors is less than or equal to the length of every spanning list of vectors.

Proof by adjoining vi to wi

Every spanning list in a vector space can be reduced to a basis of the vector space.

Proof by discarding vectors not in the span of previous vectors

Every linearly independent list of vectors in a finite-dimensional vector space can be extended to a basis of the vector space.

If U1 U2 subspaces, then dim(U1+U2) = dim U1 + dim U2 - dim(U1 intersect U2)

V finite and T linear map form V to W, then dim V = dim null T + dim range T

Suppose V is finite dimensional. If T in L(V), then the following are equivalent:

(a) T is invertible; (b) T is injective; (c) T is surjective.

Fundamental Theorem of Algebra: every nonconstant polynomial with complex coefficients has a root

p real polynomial nonconstant => unique factorization into deg <= 2 polynomials

l1, ..., lm distinct eigenvalues, v1, ..., vm nonzero eigenvectors => linearly independent

proof technique: contradiction. pick the first vk in span(v1,...,vk-1)

Every operator on a finite dimensional, nonzero, complex vector space has an eigenvalue

proof technique: pick nonzero v, construct (v, Tv, ..., T^n(v)). must be linearly dependent

factor the operator polynomial (at least one factor not injective) and get a root => eigenvalue

V complex vector space, T in L(V), then T has an upper triangular matrix w.r.t some basis of V

proof technique: use induction. partition V by U = range(T - λ I), λ eigenvalue. U invariant.

dim U < dim V. extend basis of U to V. proof property of upper triangular matrix

Every operator on a finite-dimensional, nonzero, real vector space has an invariant subspace of dimension 1 or 2

proof technique: if quadratic, consider Span(u, Tu)

Every operator on an odd-dimensional real vector space has an eigenvalue

proof technique: induction on dimension. T = U oplus W, if U dim 2 , W not an invariant subspace define Sw = P_{W,U} T(w). S has eigenvalue λ. goal: find eigenvector in U+span(w)

consider u + aw (l eigenvalue of U): (T-lI)(u+aw) = Tu-lu + a(Tw-lw) = Tu-lu + a(P_{U,W}T(w)+P_{W,U}T(w)-lw) = Tu-lu + aP_{U,W}(Tw) in U. So maps U+span(w) to U. not injective

Suppose T in L(V). (u1..), (v1..) bases of V. Let A = M(I, (u1..), (v1..)). Then M(T, (u1..)) = A-1M(T, (v1..))A

ST = I => T injective & S surjective

Triangle Inequality (prove by squaring both sides and apply Cauchy), Pythagorean Theorem

Cauchy-Schwarz Inequality (|<u,v>| <= |u||v|, eq holds iff one is scalar multiple of the other) prove by orthogonal decomposition

parallelogram equality: |u+v|^2 + |u-v|^2 = 2(|u|^2 + |v|^2)

orthonormal list of vectors => linearly independent (prove by checking norm)

(e1...) orthonormal basis => v = <v,e1>e1 + ..., |v|^2 = |<v,e1>|^2 + ...

Gram-Schmidt: (v1,...) l.i. list of vectors => exists orthonormal (e1,...) s.t. span equal

ej = orthonormalize(vj - <vj,e1>e1 - ... - <vj, e{j-1}>e{j-1})

every orthonormal list of vectors can be extended to an orthonormal basis (first extend & then G-S)

T has u.t.m => T has u.t.m. w.r.t. some orthonormal basis

orthogonal projection: |v - P_U v| <= |v - u| for all u in U, eq holds iff u = P_U v

Spectral Theorem: [V complex / real], V has an orthonormal basis consisting of eigenvectors of T <=> T is normal / self-adjoint

proof technique:

if V complex, exists orthonormal basis s.t. M(T) upper triangular, argue actually diagonal

if V real, prove by induction on dimension of V, pick any eigenvector with norm 1, argue T|_({v}^perp) self-adjoint

Lemma: [T in L(V) self-adjoint], a, b in R s.t. a^2 < 4b => T^2 + aT + bI invertible

Lemma: [T in L(V) self-adjoint] => T has an eigenvalue

Polar Decomposition: [T in L(V)]. exists isometry S in L(V) s.t. T = S*sqrt(T^*T)

proof technique: observe that norm(Tv) = norm(sqrt(T^*T)v) for all v. define S1: range(sqrt(T^*T)) -> range(T), S2: range(sqrt(T^*T))^perp -> range(T)^perp. define S: Sv = S1u + S2w where v = u + w

Singular-Value Decomposition: suppose T in L(V) has singular values s1, ..., sn. Then there exist orthonormal bases (e1, ..., en) and (f1, ..., fn) of V such that Tv = s1<v, e1>f1 + ... + sn<v, en>fn for every v in V

Cayley-Hamilton Theorem: [V complex, T in L(V)]. q := characteristic polynomial of T. => q(T) = 0

prove by induction on j: (T-l1I)...(T-ljI)vj = 0. Note that (T-ljI)vj in span(v1,...,vj-1)

[V complex, T in L(V), λ_1, ..., λ_m distinct eigenvalues of T, U_1...U_m corresponding subspaces of generalized eigenvectors]

V = U_1 oplus ... oplus U_m <=>

each U_j is invariant under T <=>

each (T - λ_j I)|U_j is nilpotent

remark: null T^n = generalized eigenspace of eigenval 0; range T^n rest.

[V complex, T in L(V)] => exists basis of V consisting of generalized eigenvectors of T

[V complex, T in L(V), λ_1, ..., λ_m distinct eigenvalues of T] => exists basis of V w.r.t which T has a block diagonal matrix of the form Diagonal(A1, ..., Am) where each A_j is an upper triangular matrix, with λ_j along the diagonal.

[V complex, T in L(V)]. exists a basis of V that is a Jordan basis for T

Problem tips:

Consider the special case of zero space.

Example of real linear operator with no real eigenvalue (R^4): T(z1,z2,z3,z4) = (z2,-z1,z4,-z3)

Prove / disprove existence of inner product: try parallelogram equality

Example of singular_value(T^2) != square(singular_value(T)): T = ((0,1),(2,0))