S631 Problem Set 1

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1. Assume that the experiement is to toss three fair coins: one penny, one dime, and one quarter. In addition, let $X : S \to \{0, 1, 2, 3\}$ a random variable that is defined as the number of tails for each experiental outcome.

   a. What is the sample space, S?

      Answer: $S = \{HHH, THH, HTH, HHT, TTH, THT, HTT, TTT\}$.

   b. Obtain the cumulative distribution function of X.

      Answer: Since $P(X = 0) = \frac{1}{8}$, $P(X = 1) = \frac{3}{8}$, $P(X = 2) = \frac{3}{8}$, $P(X = 3) = \frac{1}{8}$, the cdf of X is:

$$F(y) = \begin{cases} 0, & y < 0 \\ \frac{1}{8}, & 0 \leqslant y < 1 \\ \frac{4}{8}, & 1 \leqslant y < 2 \\ \frac{7}{8}, & 2 \leqslant y < 3 \\ 1, & y \geqslant 3. \end{cases}$$

   c. What is the probability that, if the experiement is performed, only one tail appears?

      Answer: $P(X = 1) = \frac{3}{8}$.

2. An experiement consists in tossing a fiar coin as many times as needed until the first *heads* appears. Let X be a randome variable that counts the number of tosses.

   a. Describe the sample space, S, and provide at least three possible outcomes?

      Answer: $S = \{H, TH, TTH, TTTH, ...\}$.

      Possible outcomes: get *heads* at the first toss: H;
      get first *heads* at the second toss: TH;
      get first *heads* at the third toss: TTH.

   b. Write down the image or range of X, $X(S)$; i.e., the set of numbers assigned to all possible outcomes.

      Answers: $X(S) = \{1, 2, 3, 4, ...\}$, i.e., $X(S) = \mathbb{Z}^+$.

c. Is X finite? Is X discrete? Explain.

Answer: X is not finite as it may take infinite tosses of coins before the first head appears. X is discrete because it only takes positive integers.

d. What is $P(X \leqslant 3)$?

Answer: $P(X \leqslant 3) = P(X = 1) + P(X = 2) + P(X = 3) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$

e. Find $g()$ such that

$$F(y) = \sum_{i=1}^{\lfloor y \rfloor} g(i)$$

where $F()$ is the CDF of X and $\lfloor y \rfloor$ is the greatest integer less than or equal to y (e.g. $\lfloor \pi \rfloor = 3$).

Answer:

$$\begin{aligned}
F(y) &= P(X \leqslant y) \\
&= P(X = 1) + P(X = 2) + P(X = 3) ... + P(X = \lfloor y \rfloor) \\
&= \frac{1}{2} + (\frac{1}{2})^2 + (\frac{1}{2})^3 + ... + (\frac{1}{2})^{\lfloor y \rfloor} \\
&= \sum_{i=1}^{\lfloor y \rfloor} (\frac{1}{2})^i
\end{aligned}$$

Therefore, $g(i) = (\frac{1}{2})^i$.

3. Reading Assignment.