1 Model

Target Tensor $\boldsymbol{\mathcal{Y}} \in \mathbb{R}^{T\times K\times N_Y}$. Observed $\boldsymbol{Y} \in \mathbb{R}^{T\times N_Y}$.

Auxiliary tensor $\boldsymbol{\mathcal{X}} \in \mathbb{R}^{T\times K\times N_X}$ is fully observed.

Suppose

$$\boldsymbol{Y} = \sum_{i=1}^{r} \boldsymbol{g}_i \circ \boldsymbol{\Lambda}_i^Y \tag{1.1}$$

$$\boldsymbol{\mathcal{X}} = \sum_{i=1}^{r} \boldsymbol{g}_i \circ \boldsymbol{w}_i \circ \boldsymbol{\Lambda}_i^X \tag{1.2}$$

2 Estimation

Consider loss function

$$\left\|\boldsymbol{\mathcal{X}} - \sum_{i=1}^{r} \boldsymbol{g}_i \circ \boldsymbol{w}_i \circ \boldsymbol{\Lambda}_i^X\right\|_F + \gamma\left\|\boldsymbol{Y} - \sum_{i=1}^{r} \boldsymbol{g}_i \circ \boldsymbol{\Lambda}_i^Y\right\|_F \tag{2.1}$$

$$\left\|\boldsymbol{\mathcal{X}}_{(1)} - \sum_{i=1}^{r} \boldsymbol{g}_i \circ (\boldsymbol{\Lambda}_i^X \otimes \boldsymbol{w}_i)\right\|_F + \gamma\left\|\boldsymbol{Y} - \sum_{i=1}^{r} \boldsymbol{g}_i \circ \boldsymbol{\Lambda}_i^Y\right\|_F \tag{2.2}$$

Let $\boldsymbol{H}_i = \boldsymbol{\Lambda}_i^X \otimes \boldsymbol{w}_i$, $\boldsymbol{\Lambda}_{(\gamma)}^T = (\boldsymbol{\Lambda}_Y^T, \boldsymbol{H}^T)$. Identification condition $\frac{1}{N_Y+KN_X}\boldsymbol{\Lambda}_{(\gamma)T}\boldsymbol{\Lambda}_{(\gamma)} = \boldsymbol{I}_r$.

Estimate $\boldsymbol{\Lambda}^X$ and $\boldsymbol{W}$ with $\sum_{i=1}^{r}\|\boldsymbol{H}_i - \boldsymbol{\Lambda}_i^X \otimes \boldsymbol{w}_i\|_F$. Fold $\boldsymbol{H}_i$ to $\boldsymbol{H}_i^M \in \mathbb{R}^{N_Y\times K}$, then the problem becomes $\sum_{i=1}^{r}\|\boldsymbol{H}_i^M - \boldsymbol{\Lambda}_i^X \circ \boldsymbol{w}_i\|_F$. Assume $\boldsymbol{H}_i^{M*} = \boldsymbol{\Lambda}_i^{X*} \circ \boldsymbol{w}_i^*$, then $\boldsymbol{H}_i^{M*}$ becomes a rank-one matrix. The identification condition of $\boldsymbol{\Lambda}_i^{X*}$ and $\boldsymbol{w}_i^*$ is $\|\boldsymbol{\Lambda}_i^{X*}\| = 1$, since the SVD of $\boldsymbol{H}_i^{M*}$ is unique, and $\boldsymbol{\Lambda}_i^{X*}$ is equivalent to the singular factor of $\boldsymbol{H}_i^{M*}$.

3 Algorithm

1. Combine $\boldsymbol{\mathcal{X}}_{(1)}$ and $\boldsymbol{Y}$

2. Estimate $\widehat{\boldsymbol{\Lambda}}_{(\gamma)}$ by PCA.

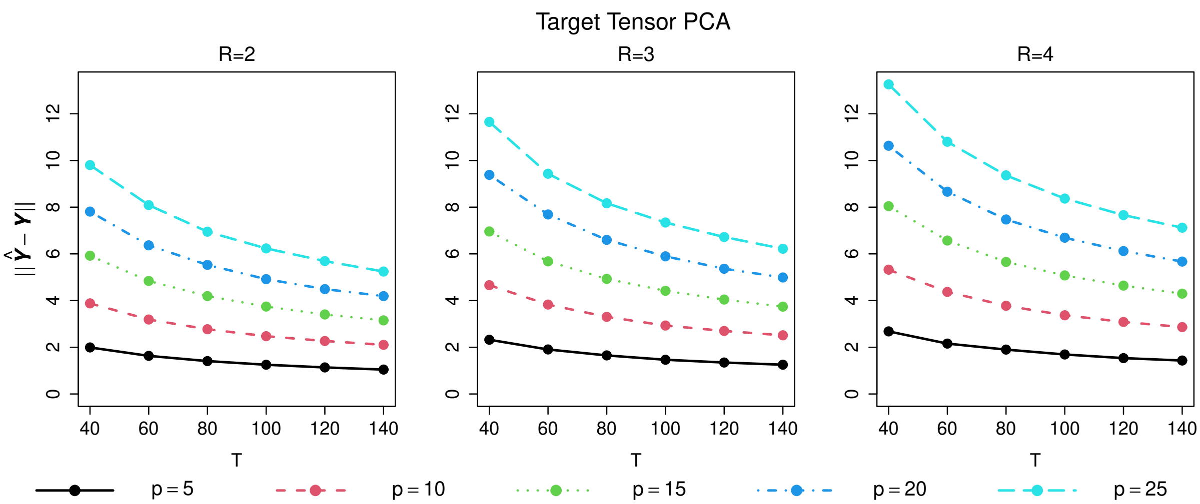
3. Estimate $\widehat{\boldsymbol{G}}$ by weighted regression.

4. Estimate $\boldsymbol{\Lambda}_i^X$ and $\boldsymbol{W}_i$ for each $\boldsymbol{H}_i$ by rank-1 SVD.

5. Orthogonalize $\mathbf{\Lambda}^X$ with SVD. Multiply the same orthogonalization matrix to $\boldsymbol{W}$

4 Simulation

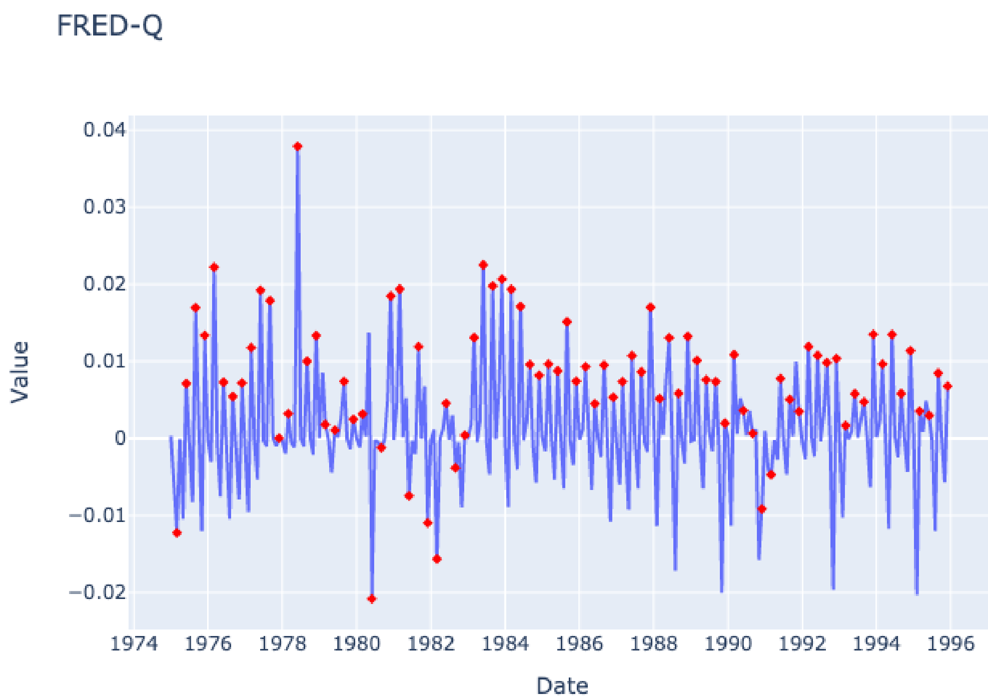
We set $N_X = N_Y = K \in \{5, 10, 15, 20, 25\}$ and $r \in \{2, 3, 4\}$.



5 FRED-Q

This data set contains low-frequency quarterly data from FRED-QD, which records 179 indexes classified by NIPA, Industrial Production, Employment & Unemployment, Housing, Prices, Interest Rates, Money & Credit, and others; high-frequency quarterly data from FRED-MD, which records 112 indexes classified by Output & Income, Trade, Labor Market, Housing, Money & Credit, Interest & Exchange Rates and Prices. We take the data from 1975 to 1995, in total of 20 years.

We take $r = 5$, $\gamma = 1.8$. The estimation to the missed GDP in FRED-Q is given as below.


FRED-Q
0.04
0.03
0.02
0.01
0
−0.01
−0.02
Value
1974
1976
1978
1980
1982
1984
1986
1988
1990
1992
1994
1996
Date