Distribution and random numbers

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Today we are going to learn...

1. Basic concepts of random numbers
2. Continuous random variables
3. Likelihood Function
4. Walking APP example
5. Discrete random variables

Preliminary

- **Pseudo random numbers**
  - an algorithm for generating a sequence of numbers that approximates the properties of random numbers.
  - The sequence is not truly random in that it is completely determined by a relatively small set of initial values, called the PRNG's state.
  - Pseudo random numbers are important in practice for their **speed** in number generation and their **reproducibility**.
- **Random seed**
  A random seed (or seed state, or just seed) is a number (or vector) used to initialize a pseudo random number generator.
- The most important random numbers are from uniform distributed numbers.
  `> runif(n,a,b)`
- Numbers selected from a non-uniform probability distribution can be generated using a uniform distribution PRNG and a function that relates the two distributions.
- Assume you have uniformly distributed random numbers from [0, 1], how do you extend it to [a, b]?

Normal Distribution

- **The normal density function**

$$f(x,\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

```
> dnorm(x,mu,sigma)
> dnorm(x,mu,sigma, log=TRUE)
```

  - In theory, `dnorm(x,mu,sigma, log=TRUE)==log(dnorm(x,mu,sigma))` but `dnorm(x,mu,sigma, log=TRUE)` but is more stable for very large values. Why?
  - **We love logs**.

- **The CDF** (cumulate desity function)

$$\Phi(x) = \int_{-\infty}^{x} f(t,\mu,\sigma)dt$$

```
> pnorm(q,mu,sigma)
```

- **The quantile** (Given CDF, what is x?), i.e. $\Phi^{-1}(p)$

```
> qnorm(p,mu,sigma)
```

- **Random numbers from normal distribution**

```
> rnorm(n,mu,sigma)
```

Likelihood function

- Given that $x_i \sim N(\mu, \sigma)$ for $i = 1, ..., n$, the **likelihood function** is

$$\prod_{i=1}^{n} f(x_i, \mu, \sigma)$$

- However the **log likelihood function** is more often used

$$\sum_{i=1}^{n} \log f(x_i, \mu, \sigma)$$

- Do you know why?

How long do you walk every day?

- Here is a list about my past six days walking statistics. Can you estimate how long do I walk everyday? and what is the variation?

The likelihood function

- We assume everyday's walking steps (x_i) are independent, and x_i follows standard normal distribution $\sim N(\mu, \sigma)$, the corresponding likelihood function is

  $$\prod_{i=1}^{n} f(x_i, \mu, \sigma)$$

  which can be easily written in R as

  ```
  logNormLike <- function(mu, sigma, data)
    {
      out = sum(dnorm(x = data, mean = mu, sd = sigma,log = TRUE))
      return(out)
    }
  ```

- **The scope** Find a proper combination of μ and σ that maximizes the loglikelihood function.

Conditional likelihood function I

- Fix other parameters

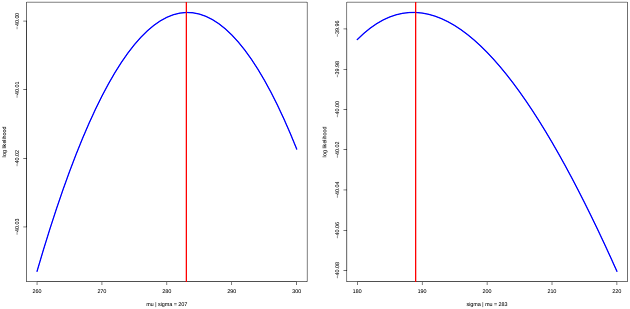
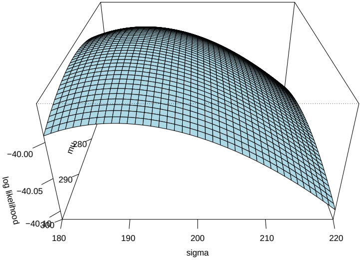


Figure: Left: fix variance to allow μ to change with likelihood function. Right: fix mean to allow σ to change with likelihood function.

- Are μ and σ we obtained the best combination?

Conditional likelihood function II



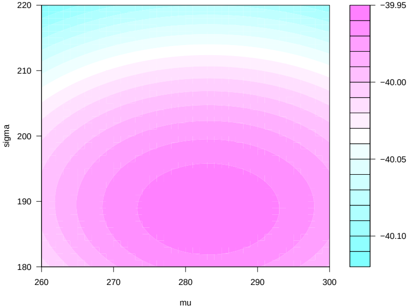


Figure: 2D and 3D loglikelihood function

Likelihood function for linear regression

- Assume you want to make a regression model

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i$$

  where $\epsilon_i \sim N(0, \sigma^2)$
- What is the (log) likelihood function?
- What are the unknown parameters?
- How do we estimate the parameters?
  - Write down a likelihood function with respect to the unknown parameters.
  - Use an optimization algorithm to find the estimates of the unknown parameters.

The Likelihood function

```
logNormLikelihood <- function(par, y, x)
    {
        beta0 <- par[1]
        beta1 <- par[2]
        sigma <- par[3]

        mean <- beta0 + x*beta1

        logDens <- dnorm(x = y, mean = mean,
                         sd = sigma, log = TRUE)
        loglikelihood <- sum(logDens)

        return(loglikelihood)
    }
```

Other types of continuous distribution

- ```
  Distribution                      Function in R
  -------------------------------------------
  Student t                         {p,d,q}t
  Chi squared                       {p,d,q}chi
  Gamma                             {p,d,q}gamma
  Exponential                       {p,d,q}exp
  -------------------------------------------
  ```
- For a significance test, what distribution do you use?
  ```

Discrete random variables

- ```
  Distribution                     Function in R
  ------------------------------------------------
  Binomial                         {p,d,q}binorm
  Negative binomial                {p,d,q}nbinom
  Poisson                          {p,d,q}pois
  Geometric                        {p,d,q}geom
  ------------------------------------------------
  ```
- Bernoulli distribution is a special case of binomial distribution.
  ```

Suggested reading

- Jones (2009): **Chapter 14, 15, 16**