NUS National University of Singapore | School of Computing

CS4277 / CS5477
3D Computer Vision

Lecture 2:
Circular Points and 3D Projective Geometry

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AY 2019/20

Semester 2

Course Schedule

Week	Date	Topic	Assignments
1	15 Jan	2D and 1D projective geometry	
2	22 Jan	Circular points and 3D projective geometry	
3	29 Jan	Absolute conic and robust homography estimation	**Assignment 1**: Panoramic stitching (15%)
4	05 Feb	Camera models and calibration	
5	12 Feb	Single view metrology	**Due:** Assignment 1 **Assignment 2**: Camera calibration (15%)
6	19 Feb	The fundamental and essential matrices	
-	26 Feb	Semester Break	No lecture **Due:** Assignment 2
7	04 Mar	Multiple-view geometry from points and/or lines	**Assignment 3**: Relative and absolute pose estimation (20%)
8	11 Mar	Absolute pose estimation from points and/or lines	
9	18 Mar	Structure-from-Motion (SfM) and Visual Simultaneous Localization and Mapping (vSLAM)	**Due:** Assignment 3
10	25 Mar	Two-view and multi-view stereo	**Assignment 4:** Dense 3D model from multi-view stereo (20%)
11	01 Apr	Generalized cameras	
12	08 Apr	Factorization and non-rigid structure-from-motion	**Due:** Assignment 4
13	15 Apr	Auto-Calibration	

Learning Outcomes

- Students should be able to:

1. Use line at infinity and/or circular points to remove affine and/or projective distortions.
2. Explain the projective mapping of a line and point with conics, i.e. pole-polar relationship.
3. Represent points and plane in $\mathbb{p}^3$, and describe the point-plane duality.
4. Describe a line in $\mathbb{p}^3$ using null space and span matrix, Plücker matrix and Plücker coordinates.
5. Extend the $\mathbb{p}^2$ conics properties to quadric in $\mathbb{p}^3$.

Acknowledgements

- A lot of slides and content of this lecture are adopted from:

1. R. Hartley, and Andrew Zisserman: “Multiple view geometry in computer vision”, Chapter 2 and 3.

2. Y. Ma, S. Soatto, J. Kosecka, S. S. Sastry, “ An invitation to 3-D vision”, Chapter 2.

Line at Infinity and Circular Points

- In the following, it will be shown that:

1. The projective distortion may be removed once the image of $\mathbf{l}_\infty$ is specified;

2. And the affine distortion removed once the image of the circular points is specified.

- Then the only remaining distortion is a similarity.

The Line at Infinity

- The line at infinity, $\mathbf{l}_\infty$, is a fixed line under the projective transformation H if and only if H is an affinity, i.e.,

$$\mathbf{l}'_\infty = \mathtt{H}_\mathtt{A}^{-\mathsf{T}}\mathbf{l}_\infty = \begin{bmatrix} \mathtt{A}^{-\mathsf{T}} & \mathbf{0} \\ -\mathbf{t}^\mathsf{T}\mathtt{A}^{-\mathsf{T}} & 1 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \mathbf{l}_\infty .$$

- An affinity is the most general linear transformation with $H_{31} = H_{32} = 0$ for the relationship to be true.
- We will see that identifying $\mathbf{l}_\infty$ allows the recovery of affine properties (parallelism, ratio of lengths).

The Line at Infinity

- Contrast this with projective transformation, where an ideal point and line at infinity might not remain at infinity.

$$H_p\mathbf{x} = \mathbf{x}' \quad \Rightarrow \quad \begin{bmatrix} \mathtt{A} & \mathbf{t} \\ \mathbf{v}^\mathsf{T} & v \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathtt{A}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ \boxed{v_1x_1 + v_2x_2} \end{pmatrix}$$

Might not be 0 since v_1 and v_2 are not 0.

$$H_p^{-T}\mathbf{l} = \mathbf{l}' \quad \Rightarrow \quad \begin{bmatrix} \mathrm{A} & \mathbf{t} \\ \mathbf{v}^T & v \end{bmatrix}^{-T} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \boxed{\begin{matrix} a_{21}v_2 - a_{22}v_1 \\ -a_{11}v_2 + a_{12}v_1 \end{matrix}} \\ a_{11}a_{22} - a_{12}a_{21} \end{pmatrix}$$

Might not be 0 since v_1 and v_2 are not 0.

The Line at Infinity

- Interestingly, $\mathbf{l}_\infty$ is not fixed pointwise under an affine transformation.

- In general, under an affinity , a point on $\mathbf{l}_\infty$ (an ideal point) is mapped to another point on $\mathbf{l}_\infty$:

$$\begin{bmatrix} \mathtt{A} & \mathbf{t} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathtt{A}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ 0 \end{pmatrix}.$$

- Nonetheless, it would be the same point when:

$$\mathtt{A}(x_1, x_2)^\mathsf{T} = k(x_1, x_2)^\mathsf{T}.$$

Recovery of Affine Properties from Images

Affine Rectification: imaged line at infinity can be used to remove projective distortion.

H_p maps ideal points and l_∞ to finite

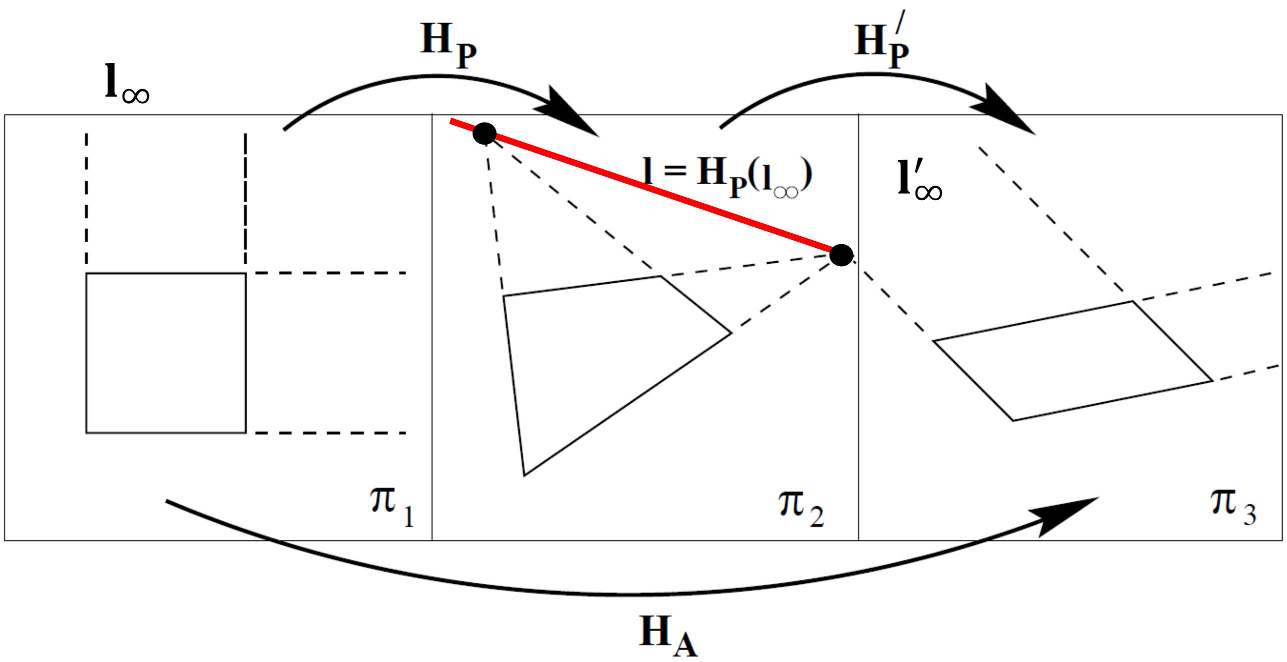


Image source: "Multiple View Geometry in Computer Vision", Richard Hartley and Andrew Zisserman

Recovery of Affine Properties from Images

Problem:

Given $\mathbf{l} = (l_1, l_2, l_3)^T$ where $l_3 \neq 0$, find H_p' that can be used to remove the projective distortion.

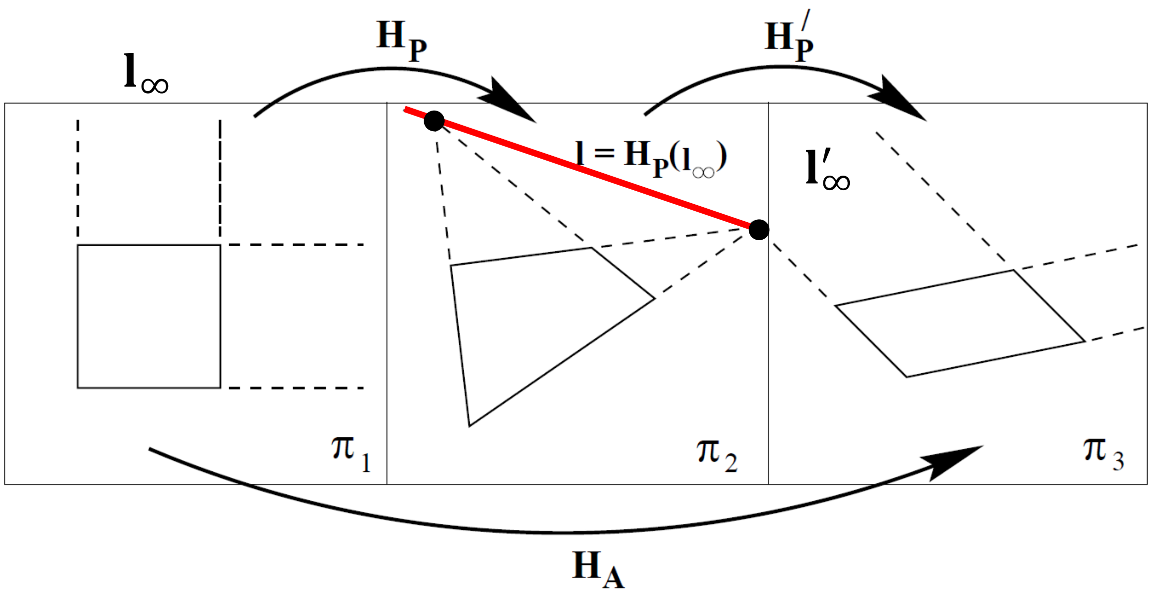


Image source: "Multiple View Geometry in Computer Vision", Richard Hartley and Andrew Zisserman

Recovery of Affine Properties from Images

Solution:

Since $\mathbf{l} = H_p^{-T}\mathbf{l}_\infty$, $\mathbf{l}'_\infty = {H'_p}^{-T}\mathbf{l}$ and $\mathbf{l}'_\infty = H_A^{-T}\mathbf{l}_\infty$, we get

$\mathbf{l}'_\infty = {H'_p}^{-T}\mathbf{l} = \underbrace{\boxed{{H'_p}^{-T}H_p^{-T}}}_{= H_A^{-T}}\mathbf{l}_\infty \Rightarrow\ H_A = H'_p H_P$, hence $H'_p = H_A H_P^{-1}$.

Substituting back to $\mathbf{l}'_\infty = {H'_p}^{-T}\mathbf{l}$, we get $\mathbf{l}'_\infty = (H_A H_P^{-1})^{-T}\mathbf{l}$.

Furthermore, $\mathbf{l}'_\infty = \mathbf{l}_\infty = (0,0,1)^T$ and $\mathbf{l} = (l_1, l_2, l_3)^T$,

$$\Rightarrow (0,0,1)^T = (H_A H_P^{-1})^{-T}\mathbf{l} = H_A^{-T}\underbrace{\boxed{H_P^T(l_1, l_2, l_3)^T}}_{= (0,0,1)^T}.$$

We solve for $H_P^T(l_1, l_2, l_3)^T = (0,0,1)^T$ since $\mathbf{l}_\infty = H_A^{-T}\mathbf{l}_\infty$,

$$\Rightarrow H_p^T = \begin{pmatrix} 1 & 0 & -l_1/l_3 \\ 0 & 1 & -l_2/l_3 \\ 0 & 0 & 1/l_3 \end{pmatrix}, \text{ hence we get } \boxed{H'_p = H_A \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l_1 & l_2 & l_3 \end{pmatrix}}_{H_p^{-1} = (H_P^T)^{-T}}},$$

where H_A can be any affinity transformation.

Recovery of Affine Properties from Images

1. The imaged vanishing line of the plane $\mathbf{l}$ is computed from the intersection of two sets of imaged parallel lines.
2. Compute $H_p' = H_A H_P^{-1}$ by choosing an arbitrary affinity H_A.

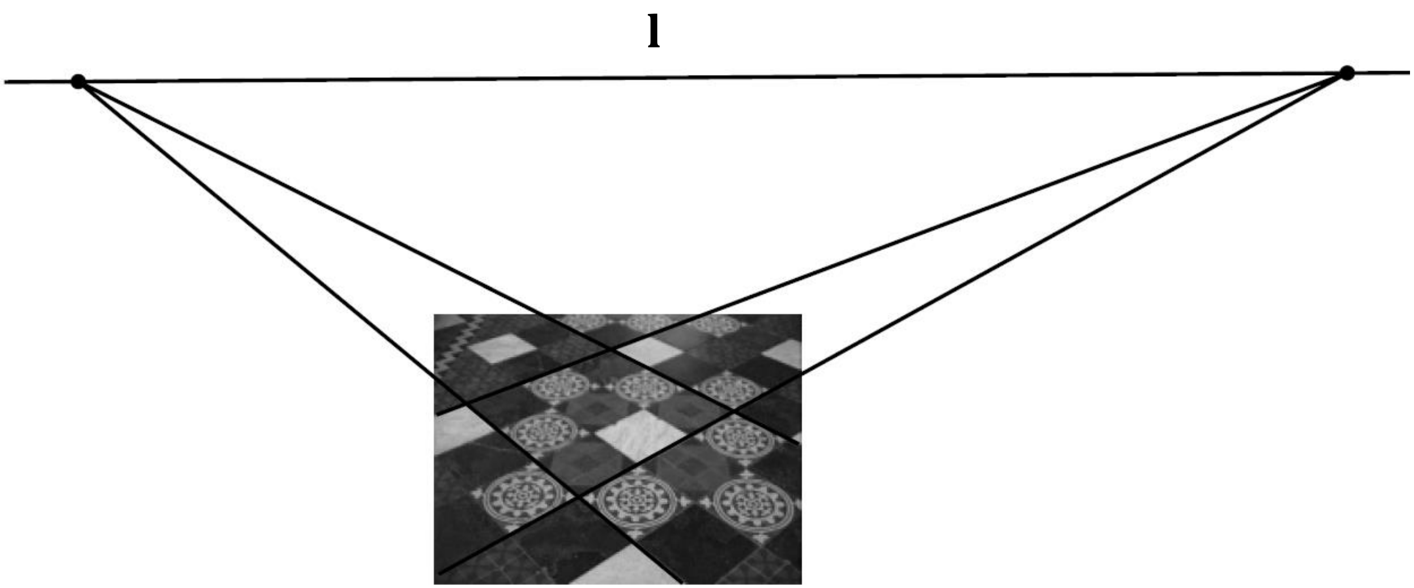


Image source: "Multiple View Geometry in Computer Vision", Richard Hartley and Andrew Zisserman

Recovery of Affine Properties from Images

3. Use H_p' to projectively warp the image to produce the affinely rectified image.

4. Affine properties can be recovered from the affinely rectified image, e.g. parallel lines and ratio of lengths.

5. Note: angles cannot be recovered since image is still affinely distorted.

H_p'

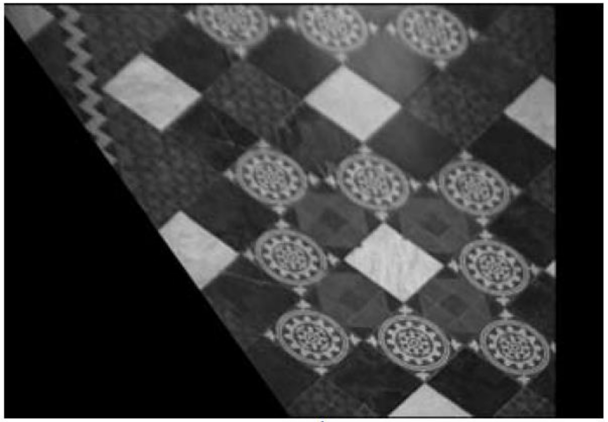
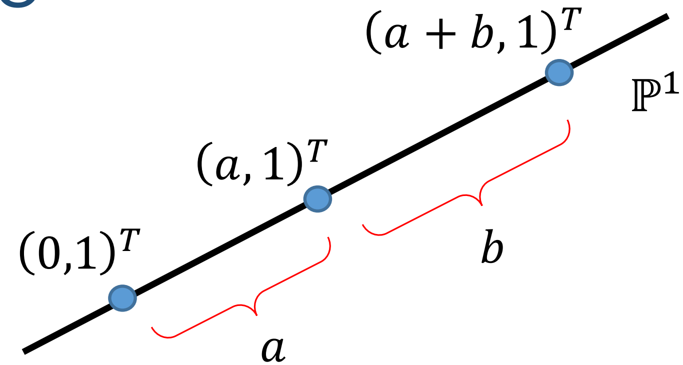


Image source: "Multiple View Geometry in Computer Vision", Richard Hartley and Andrew Zisserman

Computing a Vanishing Point from a Length Ratio

- Conversely, known affine properties may be used to determine points and the line at infinity.
- A typical case is where three points $\mathbf{a}'$, $\mathbf{b}'$ and $\mathbf{c}'$ are identified on a line in an image.
- Suppose $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are the corresponding collinear points on the world line.
- The length ratio $d(\mathbf{a}, \mathbf{b}) : d(\mathbf{b}, \mathbf{c}) = a : b$ **is known**; $d(\mathbf{x}, \mathbf{y})$ is the Euclidean distance between points $\mathbf{x}$ and $\mathbf{y}$.

Computing a Vanishing Point from a Length Ratio



Solution:

i. Measure the distance ratio in the image, $d(\mathbf{a}',\mathbf{b}') : d(\mathbf{b}',\mathbf{c}') = a' : b'$.

ii. Points $\mathbf{a},\mathbf{b}$ and $\mathbf{c}$ may be represented as coordinates 0, a and $a+b$ in a coordinate frame on the line $\langle \mathbf{a},\mathbf{b},\mathbf{c} \rangle$.

These points are represented by homogeneous 2-vectors in $\mathbb{P}^1$, i.e. $(0,1)^T, (a,1)^T$ and $(a+b,1)^T$.

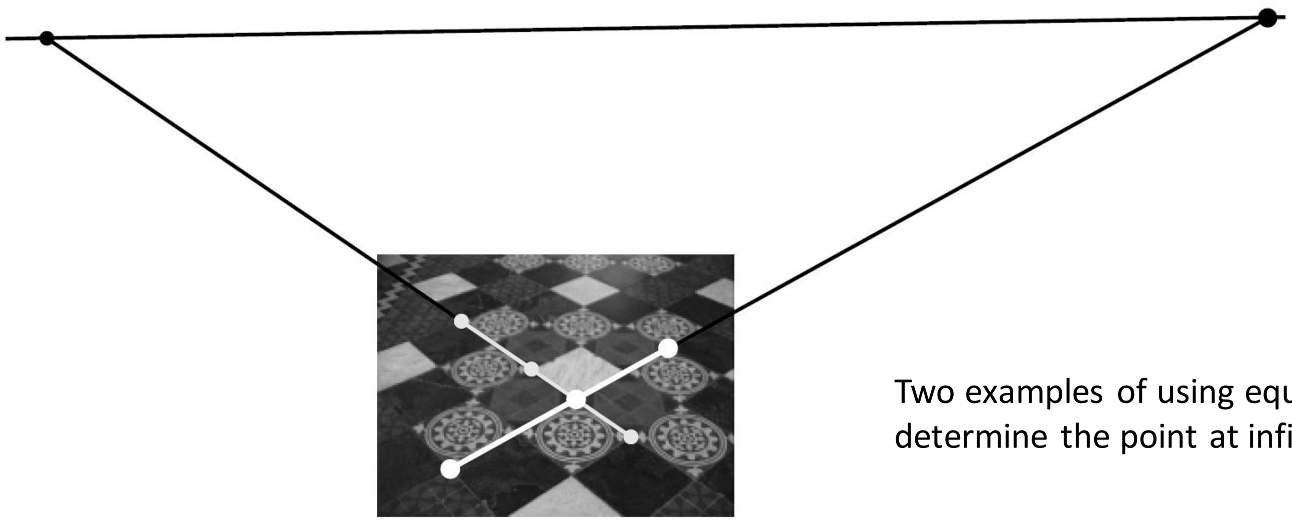
Similarly, $\langle \mathbf{a}',\mathbf{b}',\mathbf{c}' \rangle$ have coordinates $(0,1)^T, (a',1)^T$ and $(a'+b',1)^T$.

Computing a Vanishing Point from a Length Ratio

Solution:

iii. Relative to these coordinate frames, compute the 1D projective transformation $H_{2\times 2}$ mapping $\mathbf{a} \mapsto \mathbf{a}'$, $\mathbf{b} \mapsto \mathbf{b}'$ and $\mathbf{c} \mapsto \mathbf{c}'$.

iv. The image of the point at infinity (with coordinates $(1, 0)^T$) under $H_{2\times 2}$ is the vanishing point on the line $\langle \mathbf{a}', \mathbf{b}', \mathbf{c}' \rangle$.



Two examples of using equal length ratios on a line to determine the point at infinity.

Image source: “Multiple View Geometry in Computer Vision”, Richard Hartley and Andrew Zisserman

Circular Points and Their Dual

- Under any similarity transformation there are two points on $\mathbf{l}_\infty$ which are fixed.
- These are the circular points (also called the absolute points) $\mathbf{I}$, $\mathbf{J}$, with canonical coordinates:

$$\mathbf{I} = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \qquad \mathbf{J} = \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}.$$

- The circular points are a pair of complex conjugate ideal points.

Circular Points and Their Dual

- The circular points, $\mathbf{I}, \mathbf{J}$, are fixed points under the projective transformation H if and only if H is a similarity, i.e.

$$\begin{aligned} \mathbf{I}' &= \mathrm{H_S}\mathbf{I} \\ &= \begin{bmatrix} s\cos\theta & -s\sin\theta & t_x \\ s\sin\theta & s\cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \\ &= se^{-i\theta} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} = \mathbf{I}\,, \quad \text{where } e^{i\theta} = \cos\theta - i\sin\theta. \end{aligned}$$

- with an analogous proof for $\mathbf{J}$.

- The converse is also true, i.e. if the circular points are fixed then the linear transformation is a similarity.

Circular Points and Their Dual

- The name "circular points" arises because every circle intersects $\mathbf{l}_\infty$ at the circular points.

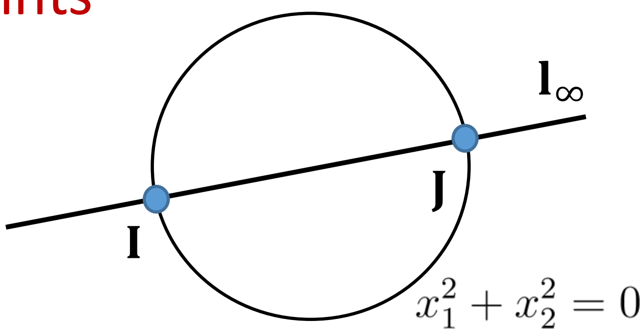
- To see this, we start from the conic equation of a circle, i.e. $a = c$ (we scale to 1) and $b = 0$:

$$x_1^2 + x_2^2 + dx_1x_3 + ex_2x_3 + fx_3^2 = 0$$

- This conic intersects $\mathbf{l}_\infty$ at the ideal points where $x_3 = 0$:

$$x_1^2 + x_2^2 = 0$$

$$\Rightarrow (x_1 + ix_2)(x_1 - ix_2) = 0$$

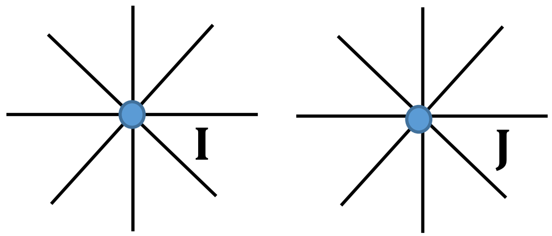


- with solution $\mathbf{I} = (1, \mathrm{i}, 0)^T, \mathbf{J} = (1, -\mathrm{i}, 0)^T$

Circular Points and Their Dual

- The dual to the circular points is the conic:

$$C_\infty^* = \mathbf{I}\mathbf{J}^\mathsf{T} + \mathbf{J}\mathbf{I}^\mathsf{T}$$



- The conic C_∞^* is a degenerate (rank 2) line conic which consists of the two circular points.

- In a Euclidean coordinate system it is given by:

$$C_\infty^* = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \begin{pmatrix} 1 & -i & 0 \end{pmatrix} + \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} \begin{pmatrix} 1 & i & 0 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Circular Points and Their Dual

- The conic C_∞^* is fixed under similarity transformations, i.e.

$$C_\infty^{*\prime} = H_S C_\infty^* H_S^\mathsf{T}$$

$$= \begin{pmatrix} s\cos\theta & -s\sin\theta & t_x \\ s\sin\theta & s\cos\theta & t_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} s\cos\theta & s\sin\theta & 0 \\ -s\sin\theta & s\cos\theta & 0 \\ t_x & t_y & 1 \end{pmatrix}$$

$$= \begin{pmatrix} s\cos\theta & -s\sin\theta & 0 \\ s\sin\theta & s\cos\theta & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} s\cos\theta & s\sin\theta & 0 \\ -s\sin\theta & s\cos\theta & 0 \\ t_x & t_y & 1 \end{pmatrix}$$

$$= s \begin{pmatrix} \cos^2\theta + \sin^2\theta & 0 & 0 \\ 0 & \cos^2\theta + \sin^2\theta & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Circular Points and Their Dual

- Some properties of C^*_∞ in any projective frame:

i. C^*_∞ has 4 degrees of freedom:

A 3×3 homogeneous symmetric matrix has 5 degrees of freedom, but the constraint $\det(C^*_\infty) = 0$ reduces the degrees of freedom by 1.

ii. $\mathbf{l}_\infty$ is the null vector of C^*_∞:

This is clear from the definition: the circular points lie on $\mathbf{l}_\infty$, so that $\mathbf{I}^T\mathbf{l}_\infty = \mathbf{J}^T\mathbf{l}_\infty = 0$; then

$$C^*_\infty \mathbf{l}_\infty = (\mathbf{I}\mathbf{J}^\top + \mathbf{J}\mathbf{I}^\top)\mathbf{l}_\infty = \mathbf{I}(\mathbf{J}^\top\mathbf{l}_\infty) + \mathbf{J}(\mathbf{I}^\top\mathbf{l}_\infty) = \mathbf{0}.$$

Angles on the Projective Plane

- In Euclidean geometry, the angle between two lines is given by the inner product of the normals of $\mathbf{l} = (l_1, l_2, l_3)^T$ and $\mathbf{m} = (m_1, m_2, m_3)^T$:

$$\cos\theta = \frac{l_1 m_1 + l_2 m_2}{\sqrt{(l_1^2 + l_2^2)(m_1^2 + m_2^2)}}.$$

- **Problem with this expression**: it is not defined under projective transformation.

- Hence, the expression cannot be applied after an affine or projective transformation of the plane.

Angles on the Projective Plane

- Once the conic C^*_∞ is identified on the projective plane then Euclidean angles may be measured by

$$\cos\theta = \frac{\mathbf{l}^\mathsf{T} \mathtt{C}^*_\infty \mathbf{m}}{\sqrt{(\mathbf{l}^\mathsf{T} \mathtt{C}^*_\infty \mathbf{l})(\mathbf{m}^\mathsf{T} \mathtt{C}^*_\infty \mathbf{m})}},$$

- which is invariant to projective transformation.

Proof: We have $(\mathbf{l}' = \mathtt{H}^{-\mathsf{T}}\mathbf{l})$ and $(\mathtt{C}^{*\prime} = \mathtt{H}\mathtt{C}^*\mathtt{H}^\mathsf{T})$ under the point transformation $\mathbf{x}' = \mathrm{H}\mathbf{x}$, hence the numerator transforms as

$$\mathbf{l}^\mathsf{T} \mathtt{C}^*_\infty \mathbf{m} \mapsto \mathbf{l}^\mathsf{T} \mathtt{H}^{-1} \mathtt{H} \mathtt{C}^*_\infty \mathtt{H}^\mathsf{T} \mathtt{H}^{-\mathsf{T}} \mathbf{m} = \mathbf{l}^\mathsf{T} \mathtt{C}^*_\infty \mathbf{m}.$$

It can be verified that the denominator terms also stay the same, and the scales of $\mathbf{l}$ and $\mathbf{m}$ cancel out. $\square$

Angles on the Projective Plane

- Lines $\mathbf{l}$ and $\mathbf{m}$ are orthogonal if $\mathbf{l}^T C_\infty^* \mathbf{m} = 0$.

Proof:

$$\cos\theta = \frac{\mathbf{l}^\mathsf{T} C_\infty^* \mathbf{m}}{\sqrt{(\mathbf{l}^\mathsf{T} C_\infty^* \mathbf{l})(\mathbf{m}^\mathsf{T} C_\infty^* \mathbf{m})}}$$

This is because $\cos\left(\frac{\pi}{2}\right) = 0$.

$\square$

Metric rectification using C_∞^*

- Once the conic C_∞^* is identified on the projective plane then projective distortion may be rectified up to a similarity.

Proof:

If the point transformation is $\mathbf{x}' = H\mathbf{x}$, we have

$$
\begin{aligned}
{C_\infty^*}' &= (H_P\, H_A\, H_S)\, C_\infty^*\, (H_P\, H_A\, H_S)^\top = (H_P\, H_A)\, \underbrace{\left(H_S\, C_\infty^* H_S^\top\right)}_{= C_\infty^*}\, \left(H_A^\top\, H_P^\top\right) \\
&= (H_P\, H_A)\, C_\infty^*\, \left(H_A^\top\, H_P^\top\right) \\
&= \begin{bmatrix} KK^\top & KK^\top\mathbf{v} \\ \mathbf{v}^\top KK^\top & \mathbf{v}^\top KK^\top\mathbf{v} \end{bmatrix}.
\end{aligned}
$$

It is clear that image of C_∞^* gives the projective ($\mathbf{v}$) and affine (K) components, but not the similarity component. $\square$

Metric rectification using C^*_∞

- Given the identified C^*_∞ in an image, i.e. C'^*_∞, a suitable rectifying homography H can be found from the SVD of C'^*_∞:

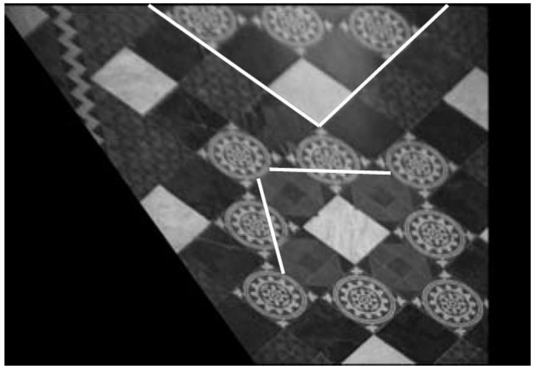
$$C^{*\prime}_\infty = U \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{= C^*_\infty} U^T$$

- where the rectifying projectivity is $H = U$ up to a similarity.

Identifying C^*_∞ in an Image

Example 1: Metric rectification of an affinely rectified image

1. Affine rectification, i.e. removal of projective distortion H_p (seen earlier)
2. Metric rectification, i.e. removal of affine distortion H_A



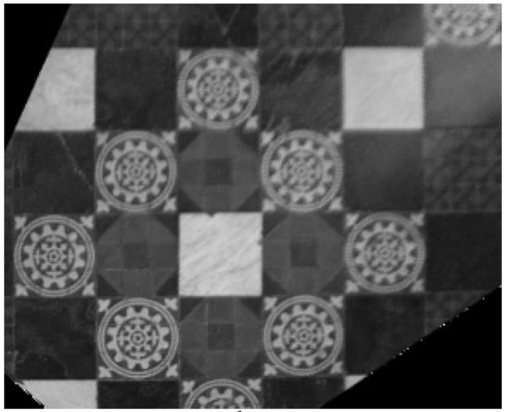


Image source: “Multiple View Geometry in Computer Vision”, Richard Hartley and Andrew Zisserman

Identifying C^*_∞ in an Image

Example 1: Metric rectification of an affinely rectified image

- We have seen that

$$C^{*\prime}_\infty = (H_P H_A H_S) C^*_\infty (H_P H_A H_S)^\mathsf{T} = (H_P H_A) C^*_\infty \left(H_A^\mathsf{T} H_P^\mathsf{T}\right),$$

- which can be written as

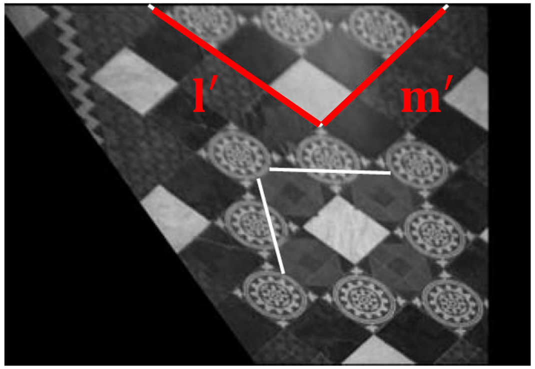
$$\underbrace{H_p^{-1} C'^*_\infty H_p^{-T}}_{= C''^*_\infty} = H_A C^*_\infty H_A^T,$$

- where C''^*_∞ is the image of the conic C^*_∞ after removal of projective distortion.

Identifying C_∞^* in an Image

Example 1: Metric rectification of an affinely rectified image

- We can compute C''^*_∞ and hence H_A from two pairs of orthogonal lines.
- Suppose the lines $\mathbf{l}'$, $\mathbf{m}'$ in the affinely rectified image correspond to an orthogonal line pair $\mathbf{l}$, $\mathbf{m}$ on the world plane, we get:



$$\underbrace{\left(\mathbf{l}^T \mathrm{H}_\mathrm{A}^{-1}\right)}_{\mathbf{l}'} \underbrace{\mathrm{H}_A \mathrm{C}_\infty^* \mathrm{H}_A^T}_{\mathrm{C}''^*_\infty} \underbrace{\left(\mathrm{H}_\mathrm{A}^{-\mathrm{T}} \mathbf{m}^T\right)}_{\mathbf{m}'} = 0\,, \qquad H_A = \begin{bmatrix} \mathrm{K} & 0 \\ 0^T & 1 \end{bmatrix}$$

$$\Rightarrow \begin{pmatrix} l_1' & l_2' & l_3' \end{pmatrix} \begin{bmatrix} \mathrm{KK}^\mathrm{T} & \mathbf{0} \\ \mathbf{0}^\mathrm{T} & 0 \end{bmatrix} \begin{pmatrix} m_1' \\ m_2' \\ m_3' \end{pmatrix} = 0\,,$$

where we write $S_{2\times 2} = \mathrm{KK}^T$ with 3 independent elements.

Image source: "Multiple View Geometry in Computer Vision", Richard Hartley and Andrew Zisserman

Identifying $\mathbf{C}_\infty^*$ in an Image

Example 1: Metric rectification of an affinely rectified image

- Thus the orthogonality constraint can be written as:

$$(l_1' m_1', l_1' m_2' + l_2' m_1', l_2' m_2')\, \mathbf{s} = 0,$$

where $\mathbf{s} = (s_{11}, s_{12}, s_{22})^T$ is S written as a 3-vector.

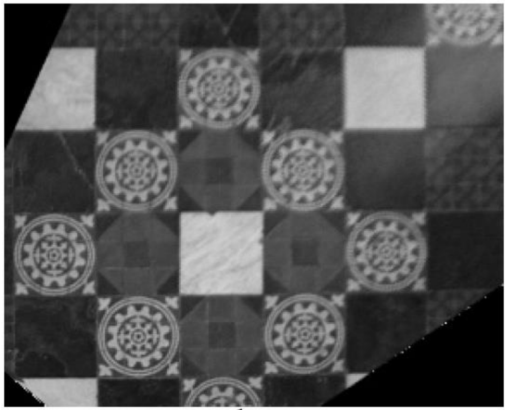
- Two constraints from two orthogonal line pairs which may be stacked to give a 2 × 3 matrix with **s** determined as the null vector.

- Thus S, and hence K (therefore H_A), is obtained up to scale by Cholesky decomposition.

Identifying C_∞^* in an Image

Example 2: Metric rectification of perspective image of the plane (not affinely rectified).

Removal of projective and affine distortion $H_p H_A$



This can be achieved by identifying C_∞^* on the perspective image, i.e.

$$C_\infty^{*\prime} = (H_P H_A) C_\infty^* (H_A^T H_P^T) = \begin{bmatrix} KK^T & KK^T\mathbf{v} \\ \mathbf{v}^T KK^T & \mathbf{v}^T KK^T \mathbf{v} \end{bmatrix}.$$

Image source: "Multiple View Geometry in Computer Vision", Richard Hartley and Andrew Zisserman

Identifying C^*_∞ in an Image

Example 2: Metric rectification of perspective image of the plane (not affinely rectified).

- Each orthogonal pair of lines $\mathbf{l}', \mathbf{m}'$ on the perspective image gives the constraint:

$$(l'_1 m'_1, (l'_1 m'_2 + l'_2 m'_1)/2, l'_2 m'_2, (l'_1 m'_3 + l'_3 m'_1)/2, (l'_2 m'_3 + l'_3 m'_2)/2, l'_3 m'_3)\, \mathbf{c} = 0$$

- where $\mathbf{c} = (a, b, c, d, e, f)^T$ is C'^*_∞ written as a 6-vector.

- Five such constraints can be stacked to form a 5×6 matrix, and $\mathbf{c}$, and hence C'^*_∞ (therefore $H_p H_A$), is obtained as the null vector.

Stratification

- Note the two-step (remove projective then affine) and one-step (remove both) difference between example 1 and 2.

- The two-step approach is termed stratified.

The Pole–Polar Relationship

The polar line $\mathbf{l} = \mathrm{C}\mathbf{x}$ of the point $\mathbf{x}$ with respect to a conic C intersects the conic in two points. The two lines tangent to C at these points intersect at $\mathbf{x}$.

Note: Point $\mathbf{x}$ does not lie on C implies $\mathbf{x}^T\mathrm{C}\mathbf{x} \neq 0$.

Remark: If the point $\mathbf{x}$ is on C then the polar is the tangent line to the conic at $\mathbf{x}$.

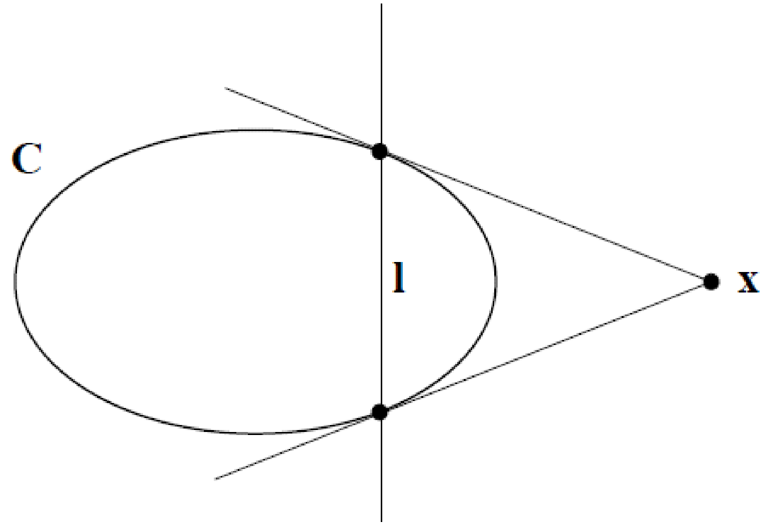
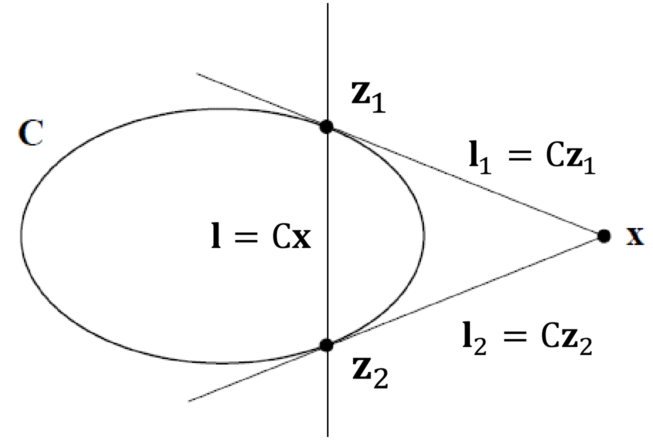


Image source: "Multiple View Geometry in Computer Vision", Richard Hartley and Andrew Zisserman

The Pole–Polar Relationship

Proof:

Consider two points $\mathbf{z}_1$ and $\mathbf{z}_2$ on the conics, the tangent lines are given as $\mathbf{l}_1 = \mathrm{C}\mathbf{z}_1$ and $\mathbf{l}_2 = \mathrm{C}\mathbf{z}_2$, respectively.



The point $\mathbf{x} = \mathbf{l_1} \times \mathbf{l_2}$ is the intersection of lines $\mathbf{l}_1$ and $\mathbf{l}_2$. Putting $\mathbf{l}_1 = \mathrm{C}\mathbf{z}_1$ and $\mathbf{l}_2 = \mathrm{C}\mathbf{z}_2$ into $\mathbf{x} = \mathbf{l_1} \times \mathbf{l_2}$, we get:

$$\mathbf{x} = \mathbf{l_1} \times \mathbf{l_2} = (\mathrm{C}\mathbf{z}_1) \times (\mathrm{C}\mathbf{z}_2) = \det(\mathrm{C})(\mathrm{C}^{-1})^{\mathrm{T}}(\mathbf{z}_1 \times \mathbf{z}_2),$$

where $(\mathrm{C}^{-1})^{\mathrm{T}} = \mathrm{C}^{-1}$ since C is symmetric and $\mathbf{l} = \mathbf{z}_1 \times \mathbf{z}_2$, i.e.

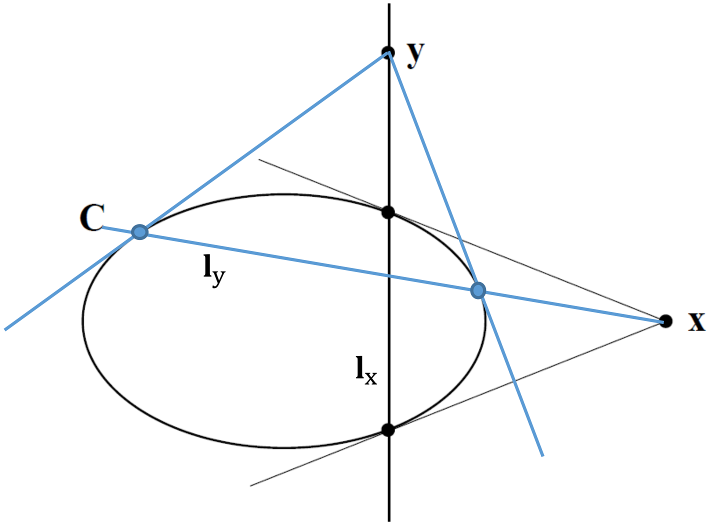
$$\mathbf{x} = \det(\mathrm{C})\mathrm{C}^{-1}\mathbf{l} = k\mathrm{C}^{-1}\mathbf{l} \;\; \Rightarrow \mathbf{l} = \mathrm{C}\mathbf{x}.$$

Taking $\det(\mathrm{C})$ constant scale k, we get the relation $\mathbf{l} = \mathrm{C}\mathbf{x}$. □

Image source: “Multiple View Geometry in Computer Vision”, Richard Hartley and Andrew Zisserman

Conjugate Points

- If the point $\mathbf{y}$ is on the line $\mathbf{l_x} = \mathrm{C}\mathbf{x}$, then $\mathbf{y}^\mathrm{T}\mathbf{l_x} = \mathbf{y}^\mathrm{T}\mathrm{C}\mathbf{x} = 0$.
- Any two points $\mathbf{x}$ and $\mathbf{y}$ satisfying $\mathbf{y}^\mathrm{T}\mathrm{C}\mathbf{x} = 0$ are conjugate with respect to the conic C.
- The conjugacy relation is symmetric: If $\mathbf{x}$ is on the polar of $\mathbf{y}$ then $\mathbf{y}$ is on the polar of $\mathbf{x}$.

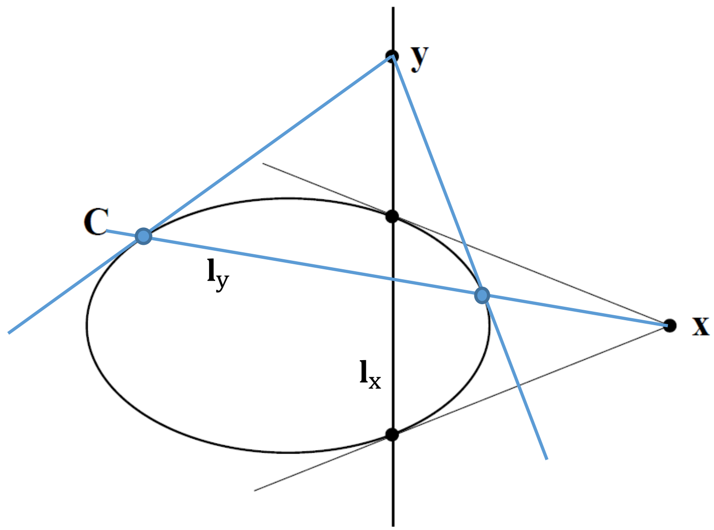


$(\mathbf{x}, \mathbf{l_x})$ and $(\mathbf{y}, \mathbf{l_y})$ are two pairs of pole-polar, where $\mathbf{x}$ and $\mathbf{y}$ are conjugate.

Conjugate Points

Proof:

The point $\mathbf{x}$ is on the polar of $\mathbf{y}$ if $\mathbf{x}^{\mathrm{T}}\mathrm{C}\mathbf{y} = 0$, and the point $\mathbf{y}$ is on the polar of $\mathbf{x}$ if $\mathbf{y}^{\mathrm{T}}\mathrm{C}\mathbf{x} = 0$. Since $\mathbf{x}^{\mathrm{T}}\mathrm{C}\mathbf{y} = \mathbf{y}^{\mathrm{T}}\mathrm{C}\mathbf{x}$, if one form is zero, then so is the other.



$\square$

Remark: There is a dual conjugacy relationship for lines: two lines $\mathbf{l}$ and $\mathbf{m}$ are conjugate if $\mathbf{l}^{\mathrm{T}}\mathrm{C}^{*}\mathbf{m} = 0$.

Projective Geometry and Transformations of 3D

- Many properties and entities of $\mathbb{P}^3$ are straightforward generalizations of those of $\mathbb{P}^2$.

  Example:
  In $\mathbb{P}^3$ Euclidean 3-space is augmented with a set of ideal points which are on a plane at infinity, $\boldsymbol{\pi}_\infty$; this is analogous of $\mathbf{l}_\infty$ in $\mathbb{P}^2$.

- However, additional properties appear by virtue of the extra dimension.

  Example:
  Two lines always intersect on the projective plane, but they need not intersect in 3-space.

Points in $\mathbb{P}^3$

- A point $\mathbf{X}$ in 3-space is represented in homogeneous coordinates as a 4-vector, i.e.

$$\mathbf{X} = (\mathrm{X}_1, \mathrm{X}_2, \mathrm{X}_3, \mathrm{X}_4)^{\mathrm{T}} \text{ with } \mathrm{X}_4 \neq 0$$

- represents the point $(\mathrm{X}, \mathrm{Y}, \mathrm{Z})^T$ of $\mathbb{R}^3$ with inhomogeneous coordinates

$$\mathrm{X} = \mathrm{X}_1/\mathrm{X}_4,\ \mathrm{Y} = \mathrm{X}_2/\mathrm{X}_4,\ \mathrm{Z} = \mathrm{X}_3/\mathrm{X}_4.$$

- Homogeneous points with $\mathrm{X}_4 = 0$ represent points at infinity.

Projective Transformation of Points in $\mathbb{P}^3$

- A projective transformation acting on $\mathbb{P}^3$ is a linear transformation on $\mathbf{X}$ by a non-singular 4×4 matrix:

$$\mathbf{X}' = \mathrm{H}\mathbf{X}.$$

- The matrix H is homogeneous and has 15 degrees of freedom: 16 elements less one for scaling.

- As in $\mathbb{P}^2$, the map is a collineation (lines are mapped to lines),

- which preserves incidence relations such as the intersection point of a line with a plane, and order of contact.

Planes in $\mathbb{P}^3$

- A plane in 3-space may be written as:

$$\pi_1 \mathrm{X} + \pi_2 \mathrm{Y} + \pi_3 \mathrm{Z} + \pi_4 = 0.$$

- Homogenizing by $\mathrm{X} \mapsto \mathrm{X}_1/\mathrm{X}_4,\ \mathrm{Y} \mapsto \mathrm{X}_2/\mathrm{X}_4,\ \mathrm{Z} \mapsto \mathrm{X}_3/\mathrm{X}_4$ gives

$$\pi_1 \mathrm{X}_1 + \pi_2 \mathrm{X}_2 + \pi_3 \mathrm{X}_3 + \pi_4 \mathrm{X}_4 = 0 \quad \text{or} \quad \boldsymbol{\pi}^{\mathsf{T}} \mathbf{X} = 0,$$

which expresses that the point $\mathbf{X}$ is on the plane $\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3, \pi_4)^T$.

- Only three independent ratios $\{\pi_1 : \pi_2 : \pi_3 : \pi_4\}$ of the plane coefficients are significant, i.e. 3 degrees of freedom.

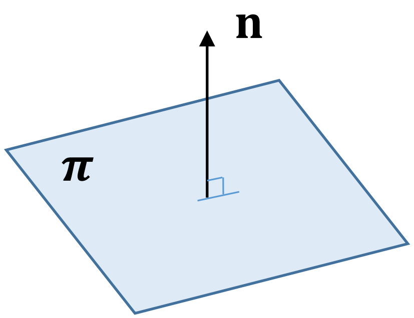
Planes in $\mathbb{P}^3$

- The first 3 components of $\boldsymbol{\pi}$ correspond to the plane normal of Euclidean geometry, i.e. $\mathbf{n} = (\pi_1, \pi_2, \pi_3)^T$.

- Using inhomogenous notation to rewrite $\boldsymbol{\pi}^\mathrm{T}\mathrm{X} = 0$ as:

$$\mathbf{n}.\widetilde{\mathbf{X}} + d = 0,$$

where $\mathbf{X} = (\mathrm{X}, \mathrm{Y}, \mathrm{Z})^T$, $\mathrm{X}_4 = 1$ and $d = \pi_4$.



- In this form $d/||\mathbf{n}||$ is the distance of the plane from the origin.

- Under the point transformation $\mathbf{X}' = \mathrm{H}\mathbf{X}$, a plane transforms as: $\boldsymbol{\pi}' = \mathrm{H}^{-\mathrm{T}}\boldsymbol{\pi}$.

Planes in $\mathbb{P}^3$: Join and Incidence Relations

In $\mathbb{P}^3$ there are numerous geometric relations between planes and points and lines:

1. A plane is defined uniquely by the join of three points (not collinear), or the join of a line and point (not incident), in general position.

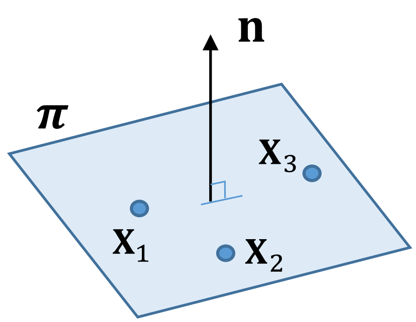
2. Two distinct planes intersect in a unique line.

3. Three distinct planes intersect in a unique point.

Three Points Define a Plane

- Suppose three points $\mathbf{X}_i$ are incident with the plane $\boldsymbol{\pi}$, where each point satisfies $\boldsymbol{\pi}^T \mathbf{X}_i = 0$ for $i = 1,2,3$, i.e.

$$\begin{bmatrix} \mathbf{X}_1^\mathsf{T} \\ \mathbf{X}_2^\mathsf{T} \\ \mathbf{X}_3^\mathsf{T} \end{bmatrix} \boldsymbol{\pi} = \mathbf{0}.$$



- The 3×4 matrix $[\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3]^T$ has rank 3 when the points are in general positions, i.e. linearly independent.

- The plane $\boldsymbol{\pi}$ defined by the points is obtained uniquely (up to scale) as the 1-dimensional (right) null-space.

Three Points Define a Plane

- If the matrix $[\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_2]^T$ has only a rank of 2, and consequently the null-space is 2-dimensional.

- Then the points are collinear, and define a pencil of planes with the line of collinear points as axis.

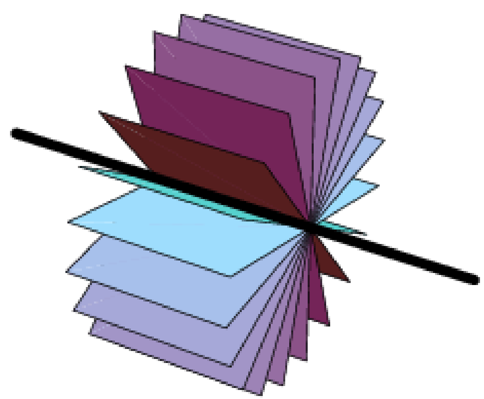
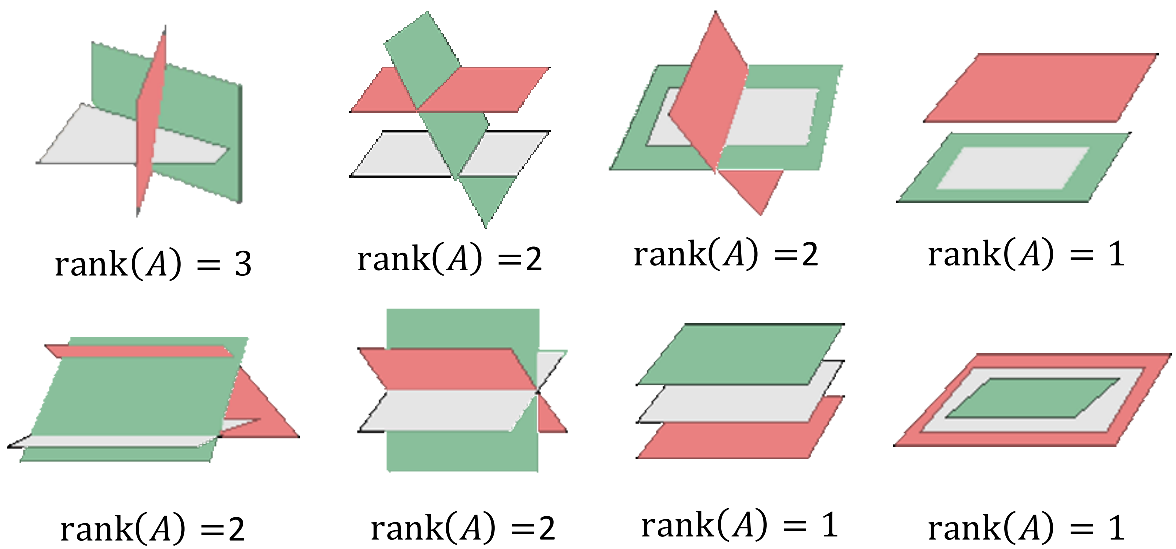


Image source: https://en.wikipedia.org/wiki/Sheaf_of_planes

Three Planes Define a Point

- The intersection point $\mathbf{X}$ of three planes $\boldsymbol{\pi}_i$ can be computed as the (right) null-space of the 3×4 matrix composed of the planes as rows:

$$\underbrace{\begin{bmatrix} \boldsymbol{\pi}_1^\mathsf{T} \\ \boldsymbol{\pi}_2^\mathsf{T} \\ \boldsymbol{\pi}_3^\mathsf{T} \end{bmatrix}}_{A} \mathbf{X} = \mathbf{0}.$$



- The development here is dual to the case of three points defining a plane and it shows the point-plane duality.

Image source: https://www.ditutor.com/space/three_planes.html
Refer to link for details of the eight possibilities.

Parametrized Points on a Plane

- The points $\mathbf{X}$ on the plane $\boldsymbol{\pi}$ may be written as

$$\mathbf{X} = \mathtt{M}\mathbf{x}\,.$$

- The columns of the 4×3 matrix M generate the rank 3 null-space of $\boldsymbol{\pi}^T$, i.e. $\boldsymbol{\pi}^T \mathrm{M} = \mathbf{0}$, and the 3-vector $\mathbf{x}$ parametrizes points on the plane $\boldsymbol{\pi}$.

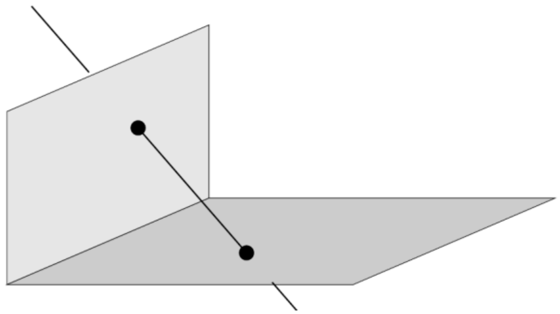
- M is not unique, suppose the plane is $\boldsymbol{\pi} = (a, b, c, d)^T$ and a is non-zero, then M^T can be written as

$$\mathrm{M}^T = [\mathbf{p} \mid I_{3\times3}],$$

where $\mathbf{p} = \left(-\frac{b}{a}, -\frac{c}{a}, -\frac{d}{a}\right)^T$.

Lines in $\mathbb{P}^3$

- A line is defined by the join of two points or the intersection of two planes.



- Lines have 4 degrees of freedom in 3-space.

Sketch of Proof: A line may be specified by its points of intersection with two orthogonal planes. Each intersection point has 2 degrees of freedom, hence 4 degrees of freedom.

- Awkward to represent 3-space line with a homogeneous 5-vector, we will look at three alternatives representations.

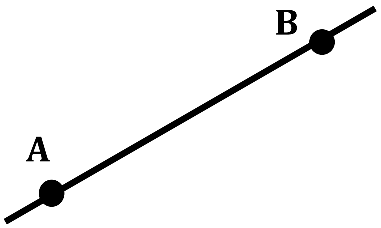
Image source: “Multiple View Geometry in Computer Vision”, Richard Hartley and Andrew Zisserman

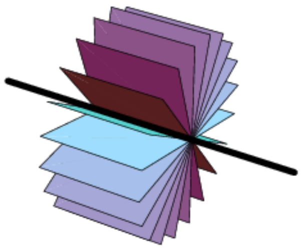
Lines in $\mathbb{P}^3$: Null-Space and Span Representation

- Suppose $\mathbf{A}, \mathbf{B}$ are two (non-coincident) space points.

- The line joining these points (6 dofs, i.e. overparameterized) is represented by the span of the row space of the 2×4 matrix W composed of $\mathbf{A}^T$ and $\mathbf{B}^T$ as rows:

$$\mathtt{W} = \begin{bmatrix} \mathbf{A}^\mathsf{T} \\ \mathbf{B}^\mathsf{T} \end{bmatrix}.$$





- Then:

1. The span of W^T is the pencil of points $\lambda\mathbf{A} + \mu\mathbf{B}$ on the line.
2. The span of the 2-dimensional right null-space of W is the pencil of planes with the line as axis.

Image source: https://en.wikipedia.org/wiki/Sheaf_of_planes

Lines in $\mathbb{P}^3$: Null-Space and Span Representation

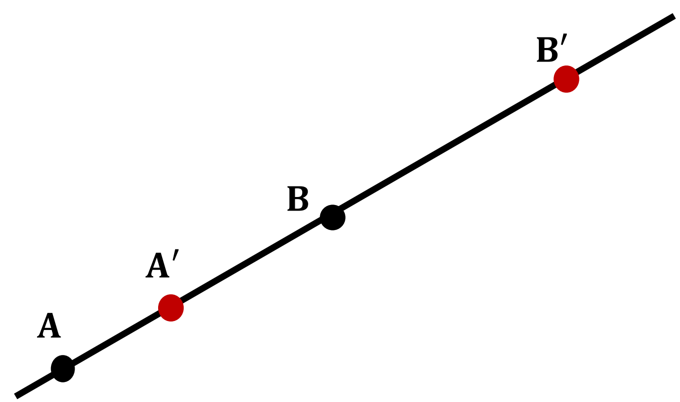
Remarks on (1):

- It is evident that two other points, $\mathbf{A}'^T$ and $\mathbf{B}'^T$, on the line will generate a matrix W' with the same span as W.
- Hence, the representation is independent of the particular points used to define it.

Same line!

$$\mathrm{W} = \begin{bmatrix} \mathrm{A}^T \\ \mathrm{B}^T \end{bmatrix}$$

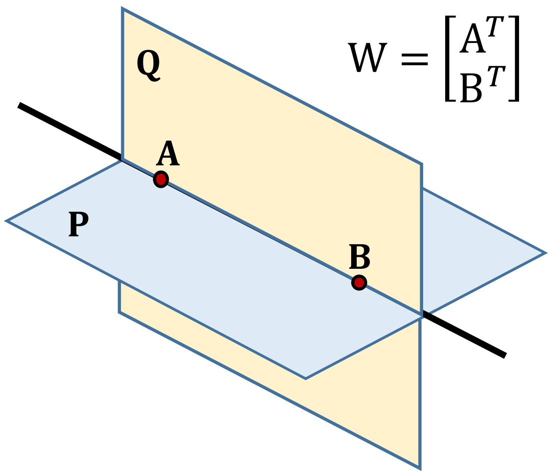
$$\mathrm{W}' = \begin{bmatrix} \mathrm{A}'^T \\ \mathrm{B}'^T \end{bmatrix}$$



Lines in $\mathbb{P}^3$: Null-Space and Span Representation

Remarks on (2):

- Suppose that $\mathbf{P}$ and $\mathbf{Q}$ are a basis for the null-space, then $\mathrm{W}\mathbf{P} = \mathbf{0}$ and consequently $\mathbf{A}^T\mathbf{P} = \mathbf{B}^T\mathbf{P} = 0$, so that $\mathbf{P}$ is a plane containing the points $\mathbf{A}$ and $\mathbf{B}$.



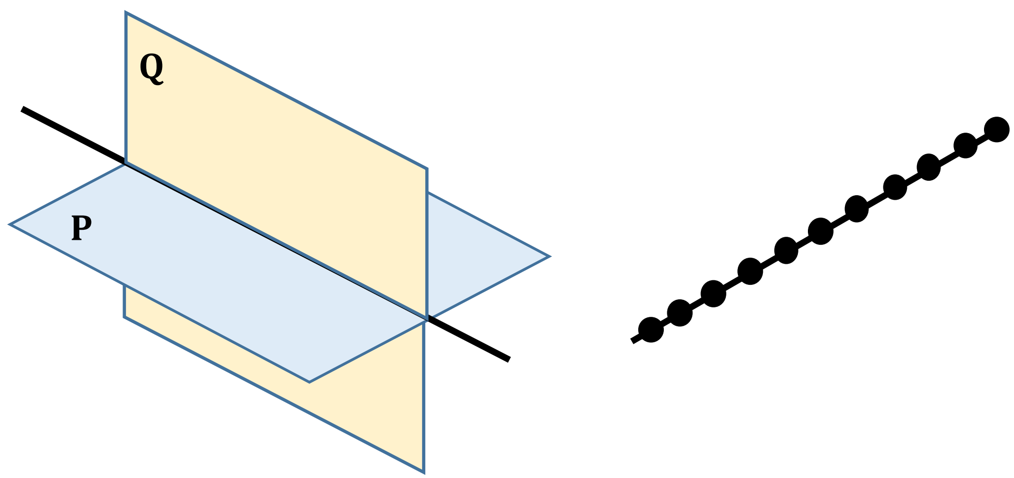
- Similarly, $\mathbf{Q}$ is a distinct plane also containing the points $\mathbf{A}$ and $\mathbf{B}$.
- $\mathbf{A}$ and $\mathbf{B}$ lie on both the (linearly independent) planes $\mathbf{P}$ and $\mathbf{Q}$, so the line defined by W is the plane intersection.
- Any plane of the pencil, with the line as axis, is given by the span $\lambda\mathbf{P} + \mu\mathbf{Q}$.

Lines in $\mathbb{P}^3$: Null-Space and Span Representation

- The dual representation of a line as the intersection of two planes, $\mathbf{P}, \mathbf{Q}$, follows in a similar manner.

- The line is represented as the span (of the row space) of the 2×4 matrix W^* composed of $\mathbf{P}^T$ and $\mathbf{Q}^T$ as rows:

$$\mathtt{W}^* = \begin{bmatrix} \mathbf{P}^\mathsf{T} \\ \mathbf{Q}^\mathsf{T} \end{bmatrix}$$



Lines in $\mathbb{P}^3$: Null-Space and Span Representation

- With the properties:

1. The span of W^{*T} is the pencil of planes $\lambda\mathbf{P} + \mu\mathbf{Q}$ with the line as axis.

2. The span of the 2-dimensional null-space of W^* is the pencil of points on the line.

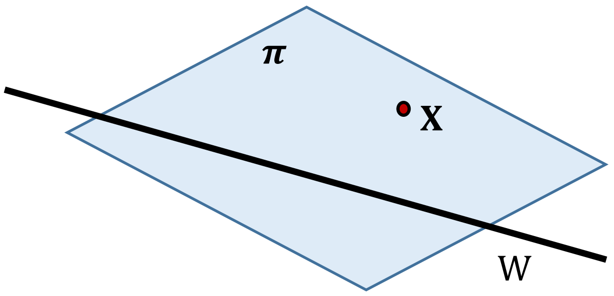
- The two representations are related by $\mathrm{W}^*\mathrm{W}^T = \mathrm{W}\mathrm{W}^{*T} = 0_{2\times 2}$, where $0_{2\times 2}$ is a 2×2 null matrix.

Lines in $\mathbb{P}^3$: Null-Space and Span Representation

- Join and incidence relations are also computed from null-spaces:

1. The plane $\boldsymbol{\pi}$ defined by the join of the point $\mathbf{X}$ and line W is obtained from the null-space of

$$\mathrm{M} = \begin{bmatrix} \mathrm{W} \\ \mathbf{X}^\top \end{bmatrix}.$$



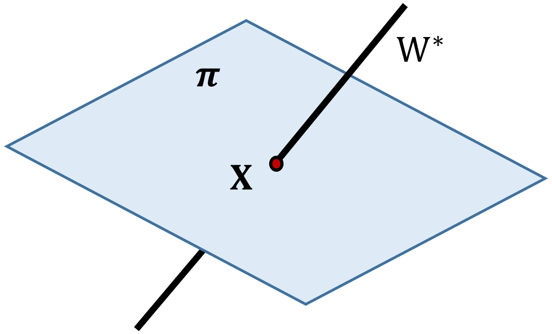
If the null-space of M is 2-dimensional then $\mathbf{X}$ is on W, otherwise $\mathrm{M}\boldsymbol{\pi} = \mathbf{0}$.

Lines in $\mathbb{P}^3$: Null-Space and Span Representation

- Join and incidence relations are also computed from null-spaces:

2. The point $\mathbf{X}$ defined by the intersection of the line W with the plane $\boldsymbol{\pi}$ is obtained from the null-space of

$$\mathtt{M} = \begin{bmatrix} \mathtt{W}^* \\ \boldsymbol{\pi}^\top \end{bmatrix}.$$



If the null-space of M is 2-dimensional then the line W is on $\boldsymbol{\pi}$, otherwise $\mathrm{M}\mathbf{X} = 0$.

Lines in $\mathbb{P}^3$: Plücker Matrices

- The line joining the two points $\mathbf{A} = (\mathrm{A}_1, \mathrm{A}_2, \mathrm{A}_3, \mathrm{A}_4)^T, \mathbf{B} = (\mathrm{B}_1, \mathrm{B}_2, \mathrm{B}_3, \mathrm{B}_4)^T$ is represented by the 4×4 skew-symmetric homogeneous matrix:

$\mathrm{L} = \mathbf{A}\mathbf{B}^\mathsf{T} - \mathbf{B}\mathbf{A}^\mathsf{T}$, with elements $l_{ij} = A_i B_j - B_i A_j$, i.e.

$$\mathrm{L} = \begin{pmatrix} 0 & -l_{12} & -l_{13} & -l_{14} \\ l_{12} & 0 & -l_{23} & -l_{24} \\ l_{13} & l_{23} & 0 & -l_{34} \\ l_{14} & l_{24} & l_{34} & 0 \end{pmatrix}$$

- We will see later that this rearranges to the Plücker line:

$$\mathcal{L} = \{l_{12}, l_{13}, l_{14}, l_{23}, l_{42}, l_{34}\}.$$

Note: The indices of l_{ij} is for $\mathbf{A} = (\mathrm{A}_1, \mathrm{A}_2, \mathrm{A}_3, \mathrm{A}_4)^T = (w, x, y, z)^T$.

Lines in $\mathbb{P}^3$: Plücker Matrices

- Several properties of L:

1. $\mathrm{Rank}(\mathrm{L}) = 2$, its 2-dimensional null-space is spanned by the pencil of planes with the line as axis.

   Remarks: In fact $\mathrm{L}\mathrm{W}^{*T} = 0$, with 0 a 4 × 2 null-matrix.

2. The representation has the required 4 degrees of freedom for a line.

   Remarks: 6 non-zero elements less $\det(\mathrm{L}) = 0$ and 5 ratios are significant.

3. Under the point transformation $\mathbf{X}' = H\mathbf{X}$, the matrix transforms as $\mathrm{L}' = H\mathrm{L}H^T$.

Lines in $\mathbb{P}^3$: Plücker Matrices

4. The relation $\mathrm{L} = \mathbf{A}\mathbf{B}^T - \mathbf{B}\mathbf{A}^T$ is the generalization to 4-space of the vector product formula $\mathbf{l} = \mathbf{x} \times \mathbf{y}$ of $\mathbb{P}^2$ for a line $\mathbf{l}$ defined by two points $\mathbf{x}, \mathbf{y}$ represented by 3-vectors.

5. The matrix L is independent of the points $\mathbf{A}, \mathbf{B}$ used to define it.

Proof:

Since if a different point $\mathbf{C}$ on the line is used, with $\mathbf{C} = \mathbf{A} + \mu\mathbf{B}$, then the resulting matrix is:

$$\begin{aligned}\hat{\mathrm{L}} &= \mathbf{A}\mathbf{C}^\mathsf{T} - \mathbf{C}\mathbf{A}^\mathsf{T} = \mathbf{A}(\mathbf{A}^\mathsf{T} + \mu\mathbf{B}^\mathsf{T}) - (\mathbf{A} + \mu\mathbf{B})\mathbf{A}^\mathsf{T} \\ &= \mathbf{A}\mathbf{B}^\mathsf{T} - \mathbf{B}\mathbf{A}^\mathsf{T} = \mathrm{L}.\end{aligned}$$

□

Lines in $\mathbb{P}^3$: Dual Plücker Matrices

- A dual Plücker representation L^* (similar properties to L) is obtained for a line formed by the intersection of two planes $\mathbf{P}, \mathbf{Q}$,

$$\mathrm{L}^* = \mathbf{P}\mathbf{Q}^\mathsf{T} - \mathbf{Q}\mathbf{P}^\mathsf{T}.$$

- Under the point transformation $\mathbf{X}' = H\mathbf{X}$, the matrix L^* transforms as $\mathrm{L}^* = H^{-T}\mathrm{L}H^{-1}$.

- The matrix L^* can be obtained directly from L by a simple rewrite rule:

$$l_{12} : l_{13} : l_{14} : l_{23} : l_{42} : l_{34} = l^*_{34} : l^*_{42} : l^*_{23} : l^*_{14} : l^*_{13} : l^*_{12}.$$

Lines in $\mathbb{P}^3$: (Dual) Plücker Matrices

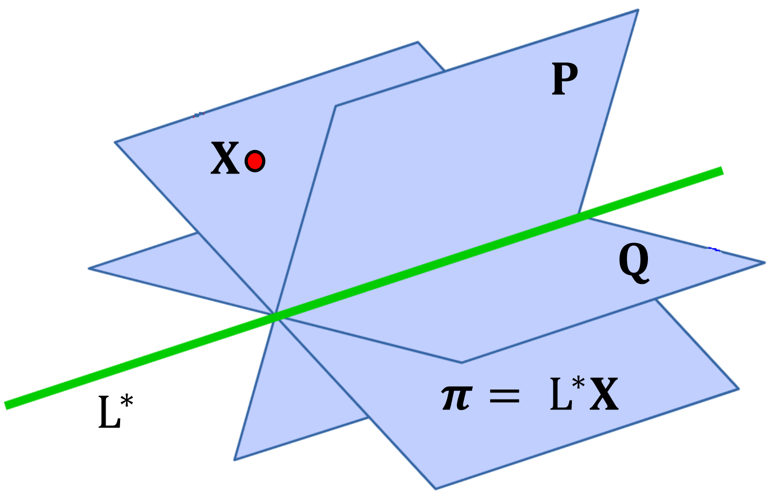
- Join and incidence properties are very nicely represented in the Plücker Matrix and its dual.

1. The plane defined by the join of the point $\mathbf{X}$ and line L is:
   $\boldsymbol{\pi} = \mathrm{L}^*\mathbf{X}$ and $\mathrm{L}^*\mathbf{X} = \mathbf{0}$, if and only if $\mathbf{X}$ is on L.

Proof: $\boldsymbol{\pi} = \mathrm{L}^*\mathbf{X} = \mathbf{P}\underbrace{\mathbf{Q}^\mathrm{T}\mathbf{X}}_{\alpha} - \mathbf{Q}\underbrace{\mathbf{P}^\mathrm{T}\mathbf{X}}_{\beta}$

The plane π contains L^* since it is a linear combination of planes $\mathbf{P}$ and $\mathbf{Q}$. It also contains $\mathbf{X}$ since:

$$\mathbf{X}^T\boldsymbol{\pi} = \underbrace{\mathbf{X}^T\mathbf{P}}_{\beta}\underbrace{\mathbf{Q}^\mathrm{T}\mathbf{X}}_{\alpha} - \underbrace{\mathbf{X}^T\mathbf{Q}}_{\alpha}\underbrace{\mathbf{P}^\mathrm{T}\mathbf{X}}_{\beta} = 0$$



Lines in $\mathbb{P}^3$: (Dual) Plücker Matrices

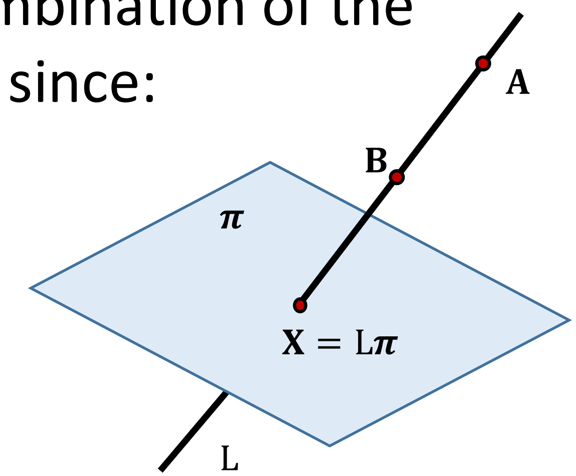
2. The point defined by the intersection of the line L with the plane $\boldsymbol{\pi}$ is:

$\mathbf{X} = \mathrm{L}\boldsymbol{\pi}$ and $\mathrm{L}\boldsymbol{\pi} = \mathbf{0}$ if, and only if, L is on $\boldsymbol{\pi}$.

Proof: $$\mathbf{X} = \mathrm{L}\boldsymbol{\pi} = \mathbf{A}\underbrace{\mathbf{B}^{\mathrm{T}}\boldsymbol{\pi}}_{\alpha} - \mathbf{B}\underbrace{\mathbf{A}^{\mathrm{T}}\boldsymbol{\pi}}_{\beta}$$

The point $\mathbf{X}$ lies on L since it is a linear combination of the points $\mathbf{A}$ and $\mathbf{B}$. It also lies on the plane $\boldsymbol{\pi}$ since:

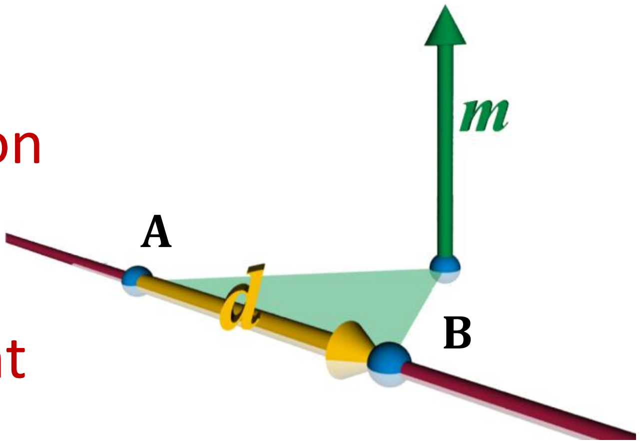
$$\boldsymbol{\pi}^T\mathbf{X} = \underbrace{\boldsymbol{\pi}^T\mathbf{A}}_{\beta}\underbrace{\mathbf{B}^{\mathrm{T}}\boldsymbol{\pi}}_{\alpha} - \underbrace{\boldsymbol{\pi}^T\mathbf{B}}_{\alpha}\underbrace{\mathbf{A}^{\mathrm{T}}\boldsymbol{\pi}}_{\beta} = 0$$



Lines in $\mathbb{P}^3$: Plücker Line Coordinates

- The Plücker line coordinates are the six non-zero elements of the 4 × 4 skew-symmetric Plücker matrix L, i.e.

$$\mathcal{L} = \{\underbrace{l_{12}, l_{13}, l_{14}}_{\mathbf{d}}, \underbrace{l_{23}, l_{42}, l_{34}}_{\mathbf{m}}\}.$$



- The first three elements are the direction vector of $\mathbf{A}$ and $\mathbf{B}$, i.e. $\mathbf{d} = \mathbf{B} - \mathbf{A}$.

- The last three elements are the moment vector of $\mathbf{A}$ and $\mathbf{B}$, i.e. $\mathbf{m} = \mathbf{A} \times \mathbf{B}$.

- $\det(\mathrm{L}) = 0$ means that the coordinates satisfy:

$$l_{12}l_{34} - l_{13}l_{42} + l_{14}l_{23} = 0.$$

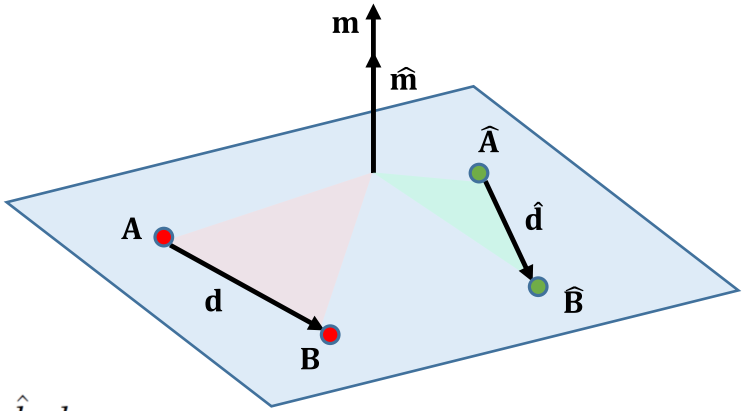
Image source: https://en.wikipedia.org/wiki/Pl%C3%BCcker_coordinates

Lines in $\mathbb{P}^3$: Plücker Line Coordinates

- Suppose two lines $\mathcal{L}, \hat{\mathcal{L}}$ are the joins of the points $\mathbf{A}, \mathbf{B}$ and $\widehat{\mathbf{A}}, \widehat{\mathbf{B}}$ respectively.
- The lines intersect if and only if the four points are coplanar, a necessary and sufficient condition for this is that

$$\begin{aligned} \det[\mathbf{A}, \mathbf{B}, \widehat{\mathbf{A}}, \widehat{\mathbf{B}}] &= l_{12}\hat{l}_{34} + \hat{l}_{12}l_{34} + l_{13}\hat{l}_{42} + \hat{l}_{13}l_{42} + l_{14}\hat{l}_{23} + \hat{l}_{14}l_{23} \\ &= (\mathcal{L}|\hat{\mathcal{L}}) = 0. \end{aligned}$$

Proof: When the four points $\mathbf{A}, \mathbf{B}$ and $\widehat{\mathbf{A}}, \widehat{\mathbf{B}}$ are coplanar, the direction vector of $\mathbf{A}, \mathbf{B}$ is perpendicular to the moment vector of $\widehat{\mathbf{A}}, \widehat{\mathbf{B}}$ and vice-versa, i.e.



$$\begin{aligned} \mathbf{d}^T\widehat{\mathbf{m}} + \mathbf{m}^T\hat{\mathbf{d}} &= l_{12}\hat{l}_{34} + \hat{l}_{12}l_{34} + l_{13}\hat{l}_{42} + \hat{l}_{13}l_{42} + l_{14}\hat{l}_{23} + \hat{l}_{14}l_{23} \\ &= \det[\mathbf{A}, \mathbf{B}, \widehat{\mathbf{A}}, \widehat{\mathbf{B}}] = 0. \end{aligned}$$

Lines in $\mathbb{P}^3$: Plücker Line Coordinates

- Similarly, suppose two lines $\mathcal{L}$, $\hat{\mathcal{L}}$ are the intersections of the planes $\mathbf{P}$, $\mathbf{Q}$ and $\widehat{\mathbf{P}}$, $\widehat{\mathbf{Q}}$, respectively, then

$$(\mathcal{L}|\hat{\mathcal{L}}) = \det[\mathbf{P}, \mathbf{Q}, \widehat{\mathbf{P}}, \widehat{\mathbf{Q}}] = 0 \ ,$$

  if and only if the lines intersects.

- If $\mathcal{L}$ is the intersection of two planes $\mathbf{P}$ and $\mathbf{Q}$ and $\hat{\mathcal{L}}$ is the join of two points $\mathbf{A}$ and $\mathbf{B}$, then

$$(\mathcal{L}|\hat{\mathcal{L}}) = (\mathbf{P}^{\mathsf{T}}\mathbf{A})(\mathbf{Q}^{\mathsf{T}}\mathbf{B}) - (\mathbf{Q}^{\mathsf{T}}\mathbf{A})(\mathbf{P}^{\mathsf{T}}\mathbf{B}) = 0 \ ,$$

  if and only if the lines intersects.

Quadrics and Dual Quadrics

- A quadric is a surface in $\mathbb{P}^3$ defined by the equation

$$\mathbf{X}^{\mathsf{T}}\mathtt{Q}\mathbf{X} = 0$$

- where Q is a symmetric 4 × 4 matrix.

- Often the matrix Q and the quadric surface it defines are not distinguished, and we will simply refer to the quadric Q.

Quadrics and Dual Quadrics

- Many of the properties of quadrics follow directly from those of conics:

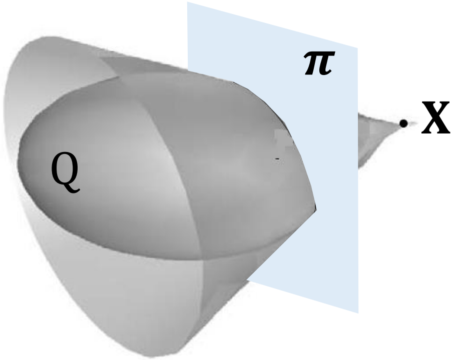
1. A quadric has 9 degrees of freedom. These correspond to the ten independent elements of a 4×4 symmetric matrix less one for scale.

2. Nine points in general position define a quadric.

3. If the matrix Q is singular, then the quadric is degenerate, and may be defined by fewer points.

Quadrics and Dual Quadrics

4. A quadric defines a polarity between a point and a plane, in a similar manner to the polarity defined by a conic between a point and a line.



Remarks:

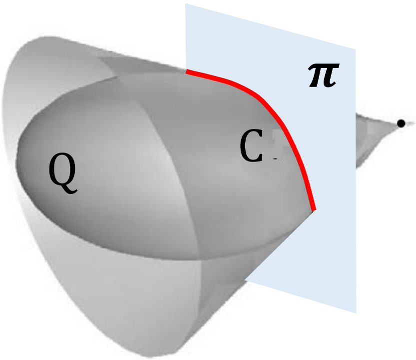
The plane $\boldsymbol{\pi} = \mathrm{Q}\mathbf{X}$ is the polar plane of $\mathbf{X}$ with respect to Q.

i. In the case that Q is non-singular and $\mathbf{X}$ is outside the quadric, the polar plane is defined by the points of contact with Q of the cone of rays through $\mathbf{X}$ tangent to Q.

ii. If $\mathbf{X}$ lies on Q, then $\mathrm{Q}\mathbf{X}$ is the tangent plane to Q at $\mathbf{X}$.

Quadrics and Dual Quadrics

5. The intersection of a plane $\boldsymbol{\pi}$ with a quadric Q is a conic C.



Remarks:

- Recall that a coordinate system for the plane can be defined by the complement space to $\boldsymbol{\pi}$ as $\mathbf{X} = \mathrm{M}\mathbf{x}$.
- Points on $\boldsymbol{\pi}$ are on Q if $\mathbf{X}^T \mathrm{Q}\mathbf{X} = \mathbf{x}^T \mathrm{M}^T \mathrm{Q}\mathrm{M}\mathbf{x} = 0$.
- These points lie on a conic C, since $\mathbf{x}^T \mathrm{C}\mathbf{x} = 0$, with $\mathrm{C} = \mathrm{M}^T \mathrm{Q}\mathrm{M}$.

Quadrics and Dual Quadrics

5. Under the point transformation $\mathbf{X}' = H\mathbf{X}$, a (point) quadric transforms as:

$$\mathtt{Q}' = \mathtt{H}^{-\mathsf{T}}\mathtt{Q}\mathtt{H}^{-1}.$$

Note:

Adjoint of a matrix A:

$$\mathrm{adj}(\mathbf{A}) = \mathbf{C}^{\mathsf{T}}.$$

C is the cofactor of A:

$$\mathbf{C} = \left((-1)^{i+j}\mathbf{M}_{ij}\right)_{1\le i,j\le n}.$$

- The dual of a quadric is also a quadric.
- Dual quadrics are equations on planes: the tangent planes $\boldsymbol{\pi}$ to the point quadric Q satisfy $\boldsymbol{\pi}^T \mathrm{Q}^* \boldsymbol{\pi} = 0$, where Q^* = **adjoint** Q, or Q^{-1} if Q is invertible.
- Under the point transformation $\mathbf{X}' = H\mathbf{X}$, a dual quadric transforms as $\mathtt{Q}^{*\prime} = \mathtt{H}\mathtt{Q}^*\mathtt{H}^{\mathsf{T}}$.

Common Quadric Surfaces

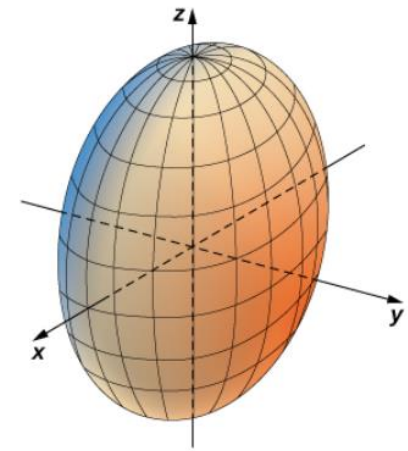
Ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Traces
In plane $z = p$: an ellipse
In plane $y = q$: an ellipse
In plane $x = r$: an ellipse

If $a = b = c$, then this surface is a sphere.

$\text{Rank}(\text{Q}) = 4$



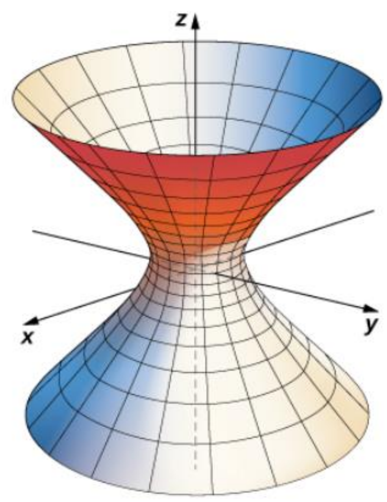
Hyperboloid of One Sheet

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

Traces
In plane $z = p$: an ellipse
In plane $y = q$: a hyperbola
In plane $x = r$: a hyperbola

In the equation for this surface, two of the variables have positive coefficients and one has a negative coefficient. The axis of the surface corresponds to the variable with the negative coefficient.

$\text{Rank}(\text{Q}) = 4$



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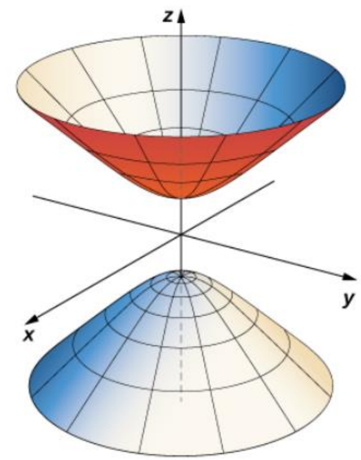
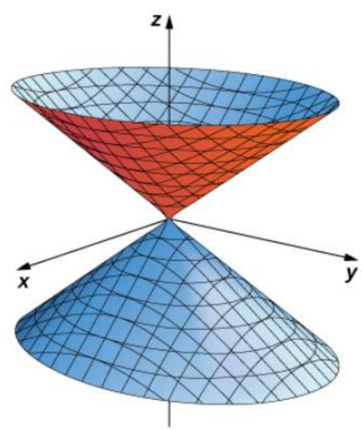
Common Quadric Surfaces

Hyperboloid of Two Sheets $$\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$ *Traces* In plane $z = p$: an ellipse or the empty set (no trace) In plane $y = q$: a hyperbola In plane $x = r$: a hyperbola In the equation for this surface, two of the variables have negative coefficients and one has a positive coefficient. The axis of the surface corresponds to the variable with a positive coefficient. The surface does not intersect the coordinate plane perpendicular to the axis. $\text{Rank}(Q) = 4$	
Elliptic Cone $$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$$ *Traces* In plane $z = p$: an ellipse In plane $y = q$: a hyperbola In plane $x = r$: a hyperbola In the xz – plane: a pair of lines that intersect at the origin In the yz – plane: a pair of lines that intersect at the origin The axis of the surface corresponds to the variable with a negative coefficient. The traces in the coordinate planes parallel to the axis are intersecting lines. $\text{Rank}(Q) = 1$	

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Common Quadric Surfaces

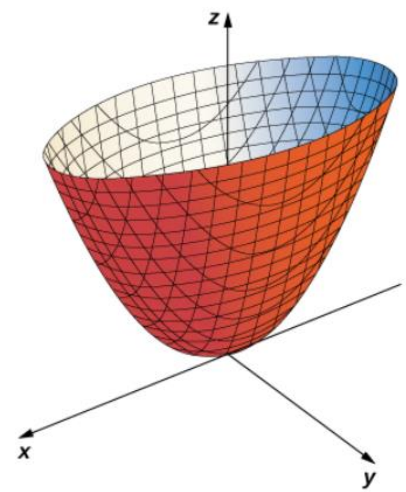
Elliptic Paraboloid

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

Traces
In plane $z = p$: an ellipse
In plane $y = q$: a parabola
In plane $x = r$: a parabola

The axis of the surface corresponds to the linear variable.

$\text{Rank}(\text{Q}) = 4$



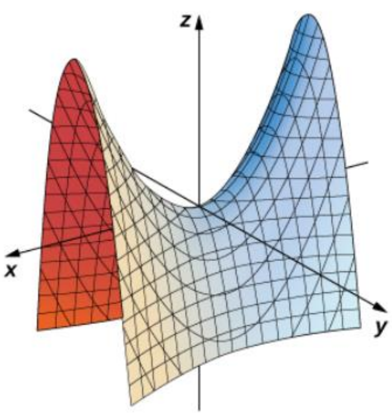
Hyperbolic Paraboloid

$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

Traces
In plane $z = p$: a hyperbola
In plane $y = q$: a parabola
In plane $x = r$: a parabola

The axis of the surface corresponds to the linear variable.

$\text{Rank}(\text{Q}) = 4$



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3D Hierarchy of Transformations

$$R = \begin{bmatrix} c_1 c_2 & c_1 s_2 s_3 - c_3 s_1 & s_1 s_3 + c_1 c_3 s_2 \\ c_2 s_1 & c_1 c_3 + s_1 s_2 s_3 & c_3 s_1 s_2 - c_1 s_3 \\ -s_2 & c_2 s_3 & c_2 c_3 \end{bmatrix}$$, 3x3 rotation matrix (see next slide)

$t = (t_x, t_y, t_z)^T$, 3x1 translation vector

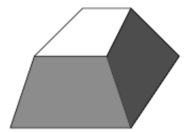
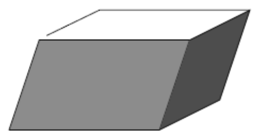
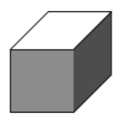
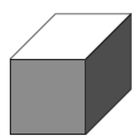
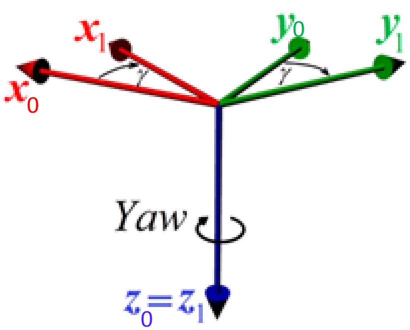
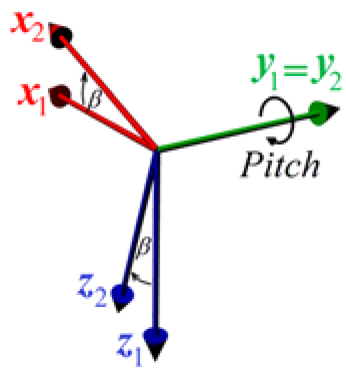
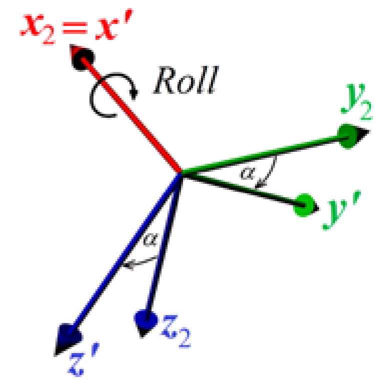
Group	Matrix	Distortion	Invariant properties
Projective 15 dof	$\begin{bmatrix} \mathtt{A} & \mathbf{t} \\ \mathbf{v}^{\mathsf{T}} & v \end{bmatrix}$		Intersection and tangency of surfaces in contact.
Affine 12 dof	$\begin{bmatrix} \mathtt{A} & \mathbf{t} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix}$		Parallelism of planes, volume ratios, centroids. The plane at infinity, π_∞, (Next lecture)
Similarity 7 dof	$\begin{bmatrix} s\mathtt{R} & \mathbf{t} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix}$		The absolute conic, Ω_∞, (Next lecture)
Euclidean 6 dof	$\begin{bmatrix} \mathtt{R} & \mathbf{t} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix}$		Volume.

Table source: "Multiple View Geometry in Computer Vision", Richard Hartley and Andrew Zisserman

Euler Angles to Rotation Matrix







$$R_z(\gamma) = R_1^0 = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow X_0 = R_1^0 X_1$$

$$R_y(\beta) = R_2^1 = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$$

$$\Rightarrow X_1 = R_2^1 X_2$$

$$R_x(\alpha) = R_3^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}$$

$$\Rightarrow X_3 = R_2^3 X_2$$

$$R_3^0 = R_1^0 R_2^1 R_3^2 = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}$$

$$\Rightarrow X_0 = R_3^0 X_3$$

Image Source: http://www.mdpi.com/1424-8220/15/3/7016/htm

Summary

- We have looked at how to:

1. Use line at infinity and/or circular points to remove affine and/or projective distortions.
2. Explain the projective mapping of a line and point with conics, i.e. pole-polar relationship.
3. Represent points and plane in $\mathbb{P}^3$, and describe the point-plane duality.
4. Describe a line in $\mathbb{P}^3$ using null space and span matrix, Plücker matrix and Plücker coordinates.
5. Extend the $\mathbb{P}^2$ conics properties to quadric in $\mathbb{P}^3$.