Assignment 2
Geometric Motion Planning

Due on **6 March 2020**

Name ____________________________

Matriculation Number ____________________________

Instructions

1. Attach this **cover sheet** to the work that you hand in.
2. Please complete the feedback questions and submit separately.
3. Type up your solutions, or write nearly as **legibly** as the printer. You may write the mathematical formulas and draw pictures by hand.

Grading

Problem	Max Points	Points
1	10	
2	10	
3	15	
4	10	
5	10	
6	20	
7	10	
8	15	
Total	100	

Feedback

1. How much time did you spend on this homework? ______________ hours.
2. How difficult do you find the homework? If you choose D, please give at least one example of which problem it is.

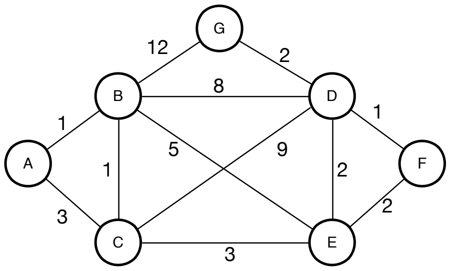
   A. *Easy*: I can finish all problems with ease.

   B. *Challenging*: This is the right level of difficulty. With some efforts, I can finish most problems successfully.

   C. *Difficult*: After significant efforts, I still have difficulty and cannot finish several problems successfully.

   D. *Unsuitably difficult*: The problems are too difficult. I do not have sufficient information from the course materials to work on them. For example, problem ______________ .

1. Find the shortest path to node A from every other node in the weighted graph below.



   (*a*) Apply the backward dynamic programming algorithm. Let $V^*(s)$ be the shortest-path length from node s to A and $V_i(s)$ be the estimated shortest-path length in the i'th iteration of the dynamic programming algorithm. Show the values for $V_0(s)$, $V_1(s)$, and $V_2(s)$ as well as $V^*(s)$ for all nodes in the graph.

   (*b*) Apply the Dijstra's algorithm. The shortest paths form a tree. Draw the shortest-path tree.

2. Let M be a 3×3 orthonormal matrices with determinant $+1$. We have discussed in the class that every such matrix corresponds to a rotation in 3-D space, and vice versa. This implies that M has only 3 *independent* degrees of freedom (DOFs); however, M contains 9 parameters.

   (*a*) What are the constraints on the 9 parameters that reduce the number of DOFs of M to 3?

   (*b*) M is required to have determinant $+1$. Does this constraint reduce the number of DOFs of M? Why or why not?

3. Give the dimension of the configuration space for the following systems. Briefly justify your answer.

   (*a*) Two mobile robots freely translate and rotate in the plane.

   (*b*) Two mobile robots translate and rotate in the plane, but they are connected by a flexible rope.

   (*c*) Two mobile robots translate and rotate in the plane. The two robots are connected together by a metal bar. The metal bar can hinge at the connection points.

4. This problem examines the relationship between distance in the configuration space and distance in the workspace. Specifically, if a robot moves by a certain amount in the configuration space, how much can a point on the robot move in the workspace? Consider a planar robot arm with n sequential links. Each link is a straight-line segment of length L. One endpoint of the link is called the *origin*, and the other is called the *extremity*. The origin of the first link is fixed. The origin of the ith link ($2 \leq i \leq n$) coincides with the extremity of the ($i - 1$)th link at a point called a *joint*. A link can rotate freely about the joint.

   (*a*) A configuration q of this robot can be represented by the joint angles $(\theta_1, \theta_2, \ldots, \theta_n)$. The metric d_c in the robot's configuration space is defined as

   $$d_c(q, q') = \max_{1 \leq i \leq n} |\theta_i - \theta'_i|$$

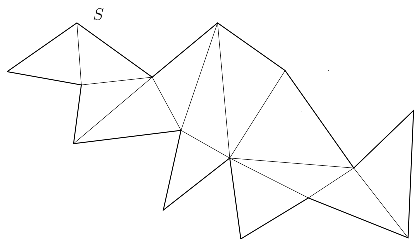
   for two configurations q and q'. Suppose that the robot moves from a configuration $q = (\theta_1, \theta_2, \ldots, \theta_n)$ to a configuration $q' = (\theta'_1, \theta'_2, \ldots, \theta'_n)$ along the straight-line segment joining q and q' in the Cartesian space R^n. In other words, the robot moves along the path $(1 - \lambda)q + \lambda q'$ for $0 \leq \lambda \leq 1$. Show that no point on the robot traces a path longer than $\alpha d_c(q, q')$ for some positive constant α. Give a bound of α in terms of L, the link length, and n, the number of links.

   (*b*) Let $d_w(q, B)$ denote the minimum distance between the robot placed at a configuration q and a (workspace) obstacle B, *i.e.*, the distance between the closest pair of points on the robot placed at q and B. Using the result from part (*a*), calculate the radius ρ of the neighborhood

   $$N(q) = \{q' \mid d_c(q, q') \leq \rho\}$$

   in which the robot is guaranteed to move freely without colliding with B. Express your answer in terms of α and $d_w(q, B)$.

5. The problem investigates sampling a triangle and triangulated free space, *i.e.*, free space consisting of triangles.
   (*a*) Let T be a triangle with vertices a, b, and c. Give a simple method to sample n points uniformly at random inside T. Justify that your method samples T uniformly and show that the total running time for yourmethod is $O(n)$.
   (*b*) Suppose that S is a triangulated surface, *i.e.*, S consists of a set of triangles. How would you sample n points uniformly at random from S? Make your method as efficient as possible.



6. The easiest method to sample a point from a unit circle is to pick an angle θ uniformly at random from the range $[0, 2\pi]$, and take the point $(\cos\theta, \sin\theta)$. However, this method cannot be easily generalized to higher dimensions. An alternative is to pick a point p from the square $[-1, 1] \times [-1, 1]$. If the distance from p to the origin is greater than 1, then discard p. Otherwise, take the point $p' = p/\|p\|$, where $\|p\|$ denotes the distance of p to the origin. Convince yourself that this method indeed generates a point uniformly at random on the unit circle.
   (*a*) Let n be the total number of points sampled from the square, and r be the number of points discarded. The ratio r/n is called the rejection ratio. What is the rejection ratio for sampling a unit circle using the method described above.
   (*b*) Can you generalize this method to sample a point uniformly at random from a unit sphere in three dimensions? What is the rejection ratio in this case?
   (*c*) Can you generalize this method to sample a point uniformly at random from a unit sphere in d dimensions? What can you say about the rejection ratio as $d \to \infty$?
   (*d*) Is it a good idea to use such a sampling method in PRM planning? Why or why not?

7. Suppose that the configuration space $\mathcal{C}$ is the unit square $[0, 1] \times [0, 1]$. The multi-query PRM algorithm, first samples n collision-free configurations and then tries to connect these milestones by calling LINK.
   (*a*) If the algorithm calls LINK for every pair of roadmap nodes, give an asymptotic bound on the number of calls to LINK.
   (*b*) Suppose that the algorithms calls LINK only if the Euclidean distance distance between two milestones is smaller than a threshold t. Give an asymptotic bound on the number of calls to LINK, if $t = O(1/\sqrt{n})$. You may assume that the milestones are distributed roughly uniformly in $\mathcal{C}$.

8. Among the motion planning algorithms covered in the class (dynamic programming, A*, PRM, EST, and RRT), which algorithms would you choose for motion planning in the following robot tasks? Justify your choice by considering the configuration space dimensions and environment characteristics.
   (*a*) A vacuum cleaning robot is deployed to a factory workshop to clean the floor at night after workers leave.
   (*b*) In Changi hospital, a mobile robot cart delivers lunches from the basement kitchen to the ward on the 7th floor.
   (*c*) A hyper-redundant robot manipulator arm with 30 DoFs is mounted with a camera at its end-effector to perform safety inspection on various bridges in a large city.