Problem 5. Expand the following functions about the given center x_0. Find the radius of convergence of each of the series.

a. $f(x) = sin(2x)$ and $x_0 = 0$.

$$sin(2x) = sin(0) + cos(0)(x-0) - \frac{sin(0)(x-0)^2}{2!} - \frac{cos(0)(x-0)^3}{3!} + \frac{sin(0)(x-0)^4}{4!} + ...$$
$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ... = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

b. $f(x) = ln(2x)$ and $x_0 = 1$. Hint: Use properties of logarithms on this problem to make it easier.

$$ln(2x) = ln(1) + \frac{x-1}{1} - \frac{(x-1)^2}{1^2} + 2!\frac{(x-1)^3}{1^3} - 3!\frac{(x-1)^4}{1^4} + ... =$$
$$0!(x-1) - 1!(x-1)^2 + 2!(x-1)^3 - 3!(x-1)^4 + ... = \sum_{n=0}^{\infty} n!(-1)^n(x-1)^{n+1}$$

c. $f(x) = e^{2x}$ and $x_0 = 1$

$$e^{2x} = e^2 + 2e^2(x-1) + \frac{4}{2!}e^2(x-1)^2 + \frac{8}{3!}e^2(x-1)^3 + ...$$
$$= e^2 \sum_{n=0}^{\infty} \tfrac{2^n}{n!}(x-1)^n$$

d. $f(x) = 3x^2 - 2x + 5$ and $x_0 = 0$

$$f(x) = 3x^2 - 2x + 5 = 5 + (6(0) - 2)(x-0) + \tfrac{6}{2!}(x-0)^2 = 5 - 2x + 3x^2$$

e. $f(x) = 3x^2 - 2x + 5$ and $x_0 = 1$

$$f(x) = 3x^2 - 2x + 5 = (3 - 2 + 5) + (6(1) - 2)(x-1) + \tfrac{6}{2!}(x-1)^2$$
$$= 6 + 4x - 4 + 3(x^2 - 2x + 1) = 3x^2 - 2x + 5$$

f. $f(x) = (3x^2 - 2x + 5)^{-1}$ and $x_0 = 1$

$$f(x) = \tfrac{1}{2} - \frac{6(1) - 2}{3(1)^2 - 2(1) + 5}(x-1) + 2\frac{(3(1)^2 - 2(1) + 5)(6) - (6(1) - 2)^2}{2!(3(1)^2 - 2(1) + 5)^2}(x-1)^2 + ...$$
$$= \tfrac{1}{2} - \tfrac{4}{6}(x-1) + \tfrac{20}{36}(x-1)^2 = \tfrac{1}{2} - \tfrac{2}{3}(x-1) + \tfrac{5}{9}(x-1)^2 + ...$$

g. $f(x) = cosh(x-3)$ and $x_0 = 1$

$cosh(x-3) = cosh(-2) + sinh(-2)(x-1) + \frac{1}{2!}cosh(-2)(x-1)^2 + \frac{1}{3!}sinh(-2)(x-1)^3 + \frac{1}{4!}cosh(-2)(x-1)^4 + ...$

h. $f(x)$ and $x_0 = a$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4 + ...$$

i. $f(a)$ and $x_0 = x$

$$f(a) = f(x) + f'(x)(a-x) + \frac{f''(x)}{2!}(a-x)^2 + \frac{f^{(3)}(x)}{3!}(a-x)^3 + \frac{f^{(4)}(x)}{4!}(a-x)^4 + ...$$

j. $f(a+h)$ and $x_0 = a$

$$f(a+h) = f(a) + f'(a)h + \frac{f''(a)}{2!}h^2 + \frac{f^{(3)}(a)}{3!}h^3 + \frac{f^{(4)}(a)}{4!}h^4 + ...$$

Problem 6. Compute the following antiderivatives.

a. $\int xsin(2x)dx$ (by parts)

Let $u = x$ and $v = -\frac{1}{2}cos(2x)$ so that $dv = sin(2x)dx$. Now integrate by parts: $\int udv = \int xsin(2x)dx = -\frac{x}{2}cos(2x) + \int \frac{1}{2}cos(2x)dx = -\frac{x}{2}cos(2x) + \frac{1}{4}sin(2x) + C$.

b. $\int xe^{x^2}dx$ (by substitution)

Let $u = e^{x^2}$ and derivate to obtain $du = 2xe^{x^2}dx$ and note that $xe^{x^2}dx = \frac{1}{2}du$. Now substitute: $\int xe^{x^2}dx = \int \frac{1}{2}du = \frac{u}{2} + C = \dfrac{e^{x^2}}{2} + C$.

c. $\int xe^x dx$ (by parts)

Let $u = x$ and $v = e^x$ so that $dv = e^x dx$. Now integrate by parts: $\int udv = \int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C$.

d. $\int e^{x^2}dx$ (expand integrand in a Taylor series)

Note the Taylor series expansion $e^{x^2} = e^2 \sum_{n=0}^{\infty} \frac{2^n}{n!}(x-1)^n$ from Problem 5.c. Now integrate: $\int e^{x^2}dx = \int e^2 \sum_{n=0}^{\infty} \frac{2^n}{n!}(x-1)^n dx = e^2 \sum_{n=0}^{\infty} \int \frac{2^n}{n!}(x-1)^n dx$
$= e^2 \sum_{n=0}^{\infty} \frac{2^n}{(n+1)!}(x-1)^{n+1} + C$

e. $\int x\sqrt{1+x}dx$

Let $u = x$ and $v = \frac{2}{3}(1+x)^{\frac{3}{2}}$ so that $dv = \sqrt{1+x}dx$. Now integrate by parts: $\int udv = \int x\sqrt{1+x}dx = \dfrac{2x}{3}(1+x)^{\frac{3}{2}} - \int \frac{2}{3}(1+x)^{\frac{3}{2}}dx = \dfrac{2x}{3}(1+x)^{\frac{3}{2}} - \dfrac{4}{15}(1+x)^{\frac{5}{2}} + C$.

f. $\int sec(\theta)d\theta$

Let $u = sec(\theta) + tan(\theta)$ so that $du = (sec(\theta)tan(\theta) + sec^2(\theta))d\theta$. Note $\int sec(\theta)d\theta = \int \dfrac{(sec^2(\theta) + sec(\theta)tan(\theta))d\theta}{sec(\theta) + tan(\theta)} = \int \dfrac{du}{u} = ln|u| + C = ln|sec(\theta) + tan(\theta)| + C$.

g. $\int sec^2(\theta)d\theta$

$\int sec^2(\theta)d\theta = tan(\theta) + C$.

h. $\int sech^2(\theta)d\theta$

$\int sech^2(\theta)d\theta = tanh(x) + C$.

i. $\int \dfrac{x^2+2}{7-x^2}dx$

First note $\int \frac{x^2+2}{7-x^2}dx = -\int \frac{x^2+2}{x^2-7}dx = -\int(\frac{9}{x^2-7}+1)dx = -\int \frac{9}{x^2-7}dx - \int 1dx = -\frac{9}{7}\int \frac{1}{1-x^2/7} - x$. Now let $u = \frac{x}{\sqrt{7}}$, which implies $du = \frac{1}{\sqrt{7}}dx$ so that we have $-\frac{9}{7}\int \frac{1}{1-x^2/7}dx - x = -\frac{9}{\sqrt{7}}\int \frac{1}{1-u^2}du - x = -\frac{9}{7}\frac{tanh^{-1}(u)}{\sqrt{7}} - x = -\frac{9}{7}\frac{tanh^{-1}(\frac{x}{\sqrt{7}})}{\sqrt{7}} - x + C$.

j. $\int \frac{1}{ap-bp^2}dp$

First note that by completing the square we get $\frac{1}{ap-bp^2}dp = \frac{1}{\frac{a^2}{4b} - (\sqrt{b}p - \frac{a}{2\sqrt{b}})^2}$. Now let $u = \sqrt{b}p - \frac{a}{2\sqrt{b}}$ so that $du = \sqrt{b}dp$, or $dp = \frac{du}{\sqrt{b}}$. Hence, $\int \frac{1}{ap-bp^2}dp = \frac{1}{\sqrt{b}}\int \frac{1}{\frac{a^2}{4b} - u^2}du = \frac{4\sqrt{b}}{a^2}\int \frac{1}{1-\frac{4bu^2}{a^2}}du$. Finally let $v = \frac{2i\sqrt{b}u}{a}$ so that $dv = \frac{2i\sqrt{b}}{a}du$ and $du = \frac{a}{2i\sqrt{b}}dv$, then substitute to obtain $\int \frac{1}{ap-bp^2}dp = -\frac{2i}{a}\int \frac{1}{1+v^2}dv$. Now integrate and back-substitute to obtain $\int \frac{1}{ap-bp^2}dp = -\frac{2i}{a}\int \frac{1}{1+v^2}dv = \frac{-2i}{a}tan^{-1}(v) + C = \frac{2}{a}tanh^{-1}\left(\frac{2\sqrt{b}u}{a}\right) + C = \frac{2}{a}tanh^{-1}\left(\frac{2\sqrt{b}(\sqrt{b}p - \frac{a}{2\sqrt{b}})}{a}\right) + C = \frac{2}{a}tanh^{-1}\left(\frac{2bp-a}{a}\right) + C$

Problem 8.

a. $\frac{dx}{dt} = 3x$ and $x(0) = 1.0$

First write $\frac{dx}{dt} - 3x = 0$. Now we have $p(t) = -3$ so that $\mu(t) = e^{\int -3dt} = e^{-3t}$. Now multiply by $\mu(t)$ to obtain $e^{-3t}\frac{dx}{dt} - 3xe^{-3t} = 0$ and note that this is equivalent to $(xe^{-3t})' = 0$. Now integrate: $\int(xe^{-3t})' = xe^{-3t} = C$ and solve: $x(t) = Ce^{3t}$. Now using the initial value data we see $x(0) = 1.0 = Ce^{3(0)} = C$ so that $C = 1.0$. Therefore our final solution is $x(t) = e^{3t}$.

b. $\frac{dx}{dt} = 3tx$ and $x(0) = 1.0$

Write $\frac{dx}{dt} - 3tx = 0$. Now we have $p(t) = -3t$ so that $\mu(t) = e^{\int -3tdt} = e^{-3/2t^2}$. Now multiply by $\mu(t)$ to obtain $e^{-3/2t^2}\frac{dx}{dt} - 3txe^{-3/2t^2} = 0$ and note that this is equivalent to $(xe^{-3/2t^2})' = 0$. Integrate to obtain $\int(xe^{-3/2t^2})' = xe^{-3/2t^2} = C$ and solve to obtain $x(t) = Ce^{3/2t^2}$. Substitute our initial value to find C: $x(0) = Ce^0 = C = 1.0$. This gives our final solution of $x(t) = e^{3/2t^2}$.

c. $\frac{dx}{dt} = 0.1x - 0.003x^2$ and $x(0) = 4$

Note $\frac{dx}{dt} = 0.1x - 0.003x^2$ is equivalent to $\frac{dx}{dt} - \frac{x}{10} = -\frac{3x^2}{1000}$. Now divide each side by $-x^2$ to obtain $-\dfrac{\frac{dx}{dx}}{x^2} + \frac{1}{10x} = \frac{3}{1000}$. Let $v = \dfrac{1}{x}$ so that $\frac{dv}{dt} = -\dfrac{\frac{dx}{dt}}{x^2}$, which gives $\frac{dv}{dt} + \frac{v}{10} = \frac{3}{1000}$. Now multiply each side of the equation by $\mu(t) = e^{\int 1/10dt} = e^{t/10}$ to obtain $e^{t/10}\frac{dv}{dt} + \frac{1}{10}e^{t/10}v = \frac{3e^{t/10}}{1000}$ which is equivalent to $e^{t/10}\frac{dv}{dt} + \frac{d}{dt}(e^{t/10})v = \frac{3e^{t/10}}{1000}$ and finally note this is equivalent to $\frac{d}{dt}(e^{t/10}v) = \frac{3e^{e^t/10}}{1000}$. We can then integrate this with respect to t: $\int \frac{d}{dt}(e^{t/10}v)dt = \int \frac{3e^{t/10}}{1000}dt \Rightarrow e^{t/10}v = \frac{3e^{t/10}}{100} + C$. Now divide by $e^{t/10}$ to obtain $v = \frac{3}{100} + C$ which gives $\frac{1}{v} = x = \dfrac{100e^{t/10}}{3e^{t/10} + C}$.

From here we can use the initial condition to obtain $x(0) = 4 = \dfrac{100e^{0/10}}{3e^{0/10} + C} = \dfrac{100}{3 + C}$ which gives $C = 100/4 - 3 = 22$. Hence, a solution to $\frac{dx}{dt} = 0.1x - 0.003x^2$ and $x(0) = 4$ is $x(t) = \dfrac{100e^{t/10}}{3e^{t/10} + 22}$.

d. $\frac{dx}{dt} = 0.1x - 0.003x^2$ and $x(0) = 400$

Using the first paragraph of Part c., we can simply substitute $x(0) = 400$ into the same general solution: $x(0) = 400 = \dfrac{100}{3 + C}$ which gives $C = 100/400 - 3 = 2.75$. Hence, a solution to $\frac{dx}{dt} = 0.1x - 0.003x^2$ and $x(0) = 400$ is $x(t) = \dfrac{100e^{t/10}}{3e^{t/10} - 2.75}$.