Supplementary Material for the paper "Design Guidelines for Nonlinear Kalman Filters via Covariance Compensation"

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I. USEFUL EQUATIONS APPEARED IN THE MAIN TEXT

The following equations from the main text will be used in the theorems and proofs in the subsequent sections.

Page 2, Equation (1). Consider random vectors $\boldsymbol{x} \in \mathbb{R}^n, \boldsymbol{z} \in \mathbb{R}^m$ and a nonlinear measurable mapping $f : \mathbb{R}^n \to \mathbb{R}^m$ that satisfies:

$$\boldsymbol{x} \sim (0, I_{n\times n}), \boldsymbol{z} = f(\boldsymbol{x}), \tag{1}$$

Page 3, Equation (4). With the conditions in (1), EKF2 estimates $\bar{\boldsymbol{z}}$, P_z, and P_{xz} by [1]:

$$\begin{cases} \bar{\boldsymbol{z}}^{\text{EKF2}} = f(\mathbf{0}) + \frac{1}{2}[\text{tr}(f_i''(\mathbf{0}))]_i \\ P_z^{\text{EKF2}} = f'(\mathbf{0})(f'(\mathbf{0}))^T + \frac{1}{2}[\text{tr}(f_i''(\mathbf{0})f_j''(\mathbf{0}))]_{ij} \\ P_{xz}^{\text{EKF2}} = (f'(\mathbf{0}))^T \end{cases}, \tag{2}$$

Page 4, Equation (12). SKF and CKF both assume that the states follow a uniform discrete distribution. With this assumption, SKF and CKF estimates $\bar{z}$, P_z, and P_{xz} by:

$$\begin{cases} \bar{\boldsymbol{z}}^{\text{SKF/CKF}} = \frac{1}{N}\sum_{i=1}^N f(\boldsymbol{\xi}_i^{\text{SKF/CKF}}) \\ P_z^{\text{SKF/CKF}} = \frac{1}{N}\sum_{i=1}^N (f(\boldsymbol{\xi}_i^{\text{SKF/CKF}}) - \bar{\boldsymbol{z}}^{\text{SKF/CKF}})(\cdot)^T \\ P_{xz}^{\text{SKF/CKF}} = \frac{1}{N}\sum_{i=1}^N \boldsymbol{\xi}_i^{\text{SKF/CKF}}(f(\boldsymbol{\xi}_i^{\text{SKF/CKF}}) - \bar{\boldsymbol{z}}^{\text{SKF/CKF}})^T \end{cases} \tag{3}$$

Page 7, Equation (39). The optimization problem is formulated as:

$$\min_{\beta} \mathbb{E}[\text{tr}(P_{k|k,ac})]. \tag{4}$$

II. PROOF OF EQUATION (6) ON PAGE 3

Proposition 1. *When $\boldsymbol{x}$ satisfies (1) and is evenly distributed on the sphere $||\boldsymbol{x}||_2^2 = n$, the covariance compensation matrix given by the EKF2 should be:*

$$P_{com}^{\text{EKF2, Sphere}} = \frac{n[\text{tr}(f_i''(\mathbf{0})f_j''(\mathbf{0})) - \frac{1}{n}\text{tr}(f_i''(\mathbf{0}))\text{tr}(f_j''(\mathbf{0}))]_{ij}}{2(n+2)}, \tag{5}$$

and $P_{com}^{\text{EKF2, Sphere}}$ is PSD.

Proof. By isotropy on the sphere of radius $\sqrt{n}$, one has $\mathbb{E}[\boldsymbol{x}] = \mathbf{0}$ and $\mathbb{E}[\boldsymbol{x}\boldsymbol{x}^T] = I_n$. Moreover, the fourth moments are (see Lemma 2.22 in [2]):

$$\mathbb{E}[x_a x_b x_c x_d] = \frac{n}{n+2}\big(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}\big), \tag{6}$$

where $a, b, c, d \in \{1, \dots, n\}$ and δ is the Kronecker delta, which satisfies:

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}. \tag{7}$$

EKF2 approximates the system up to the second order. Write the second–order Taylor model componentwise as

$$z_i \approx f_i(\mathbf{0}) + f_i'(\mathbf{0})\boldsymbol{x} + \frac{1}{2}\boldsymbol{x}^T f_i''(\mathbf{0})\boldsymbol{x}. \tag{8}$$

Since $\boldsymbol{x}$ has a distribution symmetric about the origin, all third moments vanish. Hence, for any i, j,

$$\text{Cov}\big(f_i'(\mathbf{0})\boldsymbol{x},\, \boldsymbol{x}^T f_j''(\mathbf{0})\boldsymbol{x}\big) = 0 \tag{9}$$

Therefore, the covariance compensation coming from the quadratic part is

$$\begin{aligned} P_{\text{com},ij}^{\text{EKF2, Sphere}} &= \text{Cov}\Big(\tfrac{1}{2}\boldsymbol{x}^T f_i''(\mathbf{0})\boldsymbol{x},\, \tfrac{1}{2}\boldsymbol{x}^T f_j''(\mathbf{0})\boldsymbol{x}\Big) \\ &= \frac{1}{4}\Big(\mathbb{E}\big[(\boldsymbol{x}^T f_i''(\mathbf{0})\boldsymbol{x})(\boldsymbol{x}^T f_j''(\mathbf{0})\boldsymbol{x})\big] \\ &\quad - \mathbb{E}[\boldsymbol{x}^T f_i''(\mathbf{0})\boldsymbol{x}]\,\mathbb{E}[\boldsymbol{x}^T f_j''(\mathbf{0})\boldsymbol{x}]\Big). \end{aligned} \tag{10}$$

Using (6) and standard index gymnastics,

$$\begin{aligned} \mathbb{E}\big[(\boldsymbol{x}^T A\boldsymbol{x})(\boldsymbol{x}^T B\boldsymbol{x})\big] &= \sum_{a,b,c,d} A_{ab}B_{cd}\,\mathbb{E}[x_a x_b x_c x_d] \\ &= \frac{n}{n+2}\Big(\text{tr}(A)\text{tr}(B) + 2\,\text{tr}(AB)\Big), \end{aligned} \tag{11}$$

for any $A, B \in \mathbb{R}^{n\times n}$. Also,

$$\mathbb{E}[\boldsymbol{x}^T A\boldsymbol{x}] = \text{tr}(A\mathbb{E}[\boldsymbol{x}\boldsymbol{x}^T]) = \text{tr}(A). \tag{12}$$

Applying these with $A = f_i''(\mathbf{0})$, $B = f_j''(\mathbf{0})$ gives

$$\begin{aligned} P_{\text{com},ij}^{\text{EKF2,Sphere}} &= \frac{1}{4}\Bigg(\frac{n}{n+2}\Big(\text{tr}(f_i''(\mathbf{0}))\,\text{tr}(f_j''(\mathbf{0})) + 2\,\text{tr}(f_i''(\mathbf{0})f_j''(\mathbf{0}))\Big) \\ &\quad - \text{tr}(f_i''(\mathbf{0}))\,\text{tr}(f_j''(\mathbf{0}))\Bigg) \\ &= \frac{1}{2(n+2)}\Big(n\,\text{tr}(f_i''(\mathbf{0})f_j''(\mathbf{0})) - \text{tr}(f_i''(\mathbf{0}))\text{tr}(f_j''(\mathbf{0}))\Big) \\ &= \frac{n}{2(n+2)}\left(\text{tr}(f_i''(\mathbf{0})f_j''(\mathbf{0})) - \frac{1}{n}\text{tr}(f_i''(\mathbf{0}))\text{tr}(f_j''(\mathbf{0}))\right). \end{aligned} \tag{13}$$

This proves the stated formula (5). The rest is to prove that (5) is PSD. Define the "trace–centered" matrices

$$H_i := f_i''(\mathbf{0}) - \frac{\text{tr}(f_i''(\mathbf{0}))}{n} I_n. \tag{14}$$

Then

$$\text{tr}(H_i H_j) = \text{tr}(f_i''(\mathbf{0})f_j''(\mathbf{0})) - \frac{1}{n}\text{tr}(f_i''(\mathbf{0}))\text{tr}(f_j''(\mathbf{0})). \tag{15}$$

Hence the $m \times m$ matrix with (i, j) entry inside the bracket of (5) is the Gram matrix $G = [\langle H_i, H_j\rangle_F]_{ij}$ under the Frobenius inner product, so (5) is PSD. $\square$

III. PROOFS OF THE THEOREMS

Theorem 1. *With the conditions in (1), and when the nonlinear mapping f is real analytic at the origin, we have $P_{com}^{\text{CKF}} \succeq 0$.*

Proof. Consider the Taylor expansion of the nonlinear mapping $f(\boldsymbol{a})$ at the origin as a sum of homogeneous parts. For any $\boldsymbol{a}$ in the domain of $\boldsymbol{x}$, we have:

$$f(\boldsymbol{a}) = \sum_{k=0}^{\infty} f^{(k)}(\boldsymbol{a}), \tag{16}$$

where $f^{(k)}$ represents homogeneous of degree k. Specifically, $f^{(0)}(\boldsymbol{a}) = f(\boldsymbol{0}), f^{(1)}(\boldsymbol{a}) = f'(\boldsymbol{0})\boldsymbol{a}$. Split $f(\boldsymbol{a})$ into odd and even components:

$$f_{\text{odd}}(\boldsymbol{a}) = \sum_{j\geq 0} f^{(2j+1)}(\boldsymbol{a}), f_{\text{even}}(\boldsymbol{a}) = \sum_{j\geq 1} f^{(2j)}(\boldsymbol{a}). \tag{17}$$

Recall the equation for $\bar{\boldsymbol{z}}^{\text{CKF}}$ in (3). Because the point set is symmetric, all odd contributions vanish, and only even parts remain in the mean. Therefore, we can define the centered residuals for an arbitrary sigma point $\boldsymbol{\xi}_r^{\text{CKF}}$ (also simply written as $\boldsymbol{\xi}_r$ in this proof) by:

$$\delta f(\boldsymbol{\xi}_r) = f(\boldsymbol{\xi}_r) - \bar{\boldsymbol{z}}^{\text{CKF}} = f_{\text{odd}}(\boldsymbol{\xi}_r) + \big(f_{\text{even}}(\boldsymbol{\xi}_r) - \sum_{i=1}^{2n} \frac{f_{\text{even}}(\boldsymbol{\xi}_i)}{2n}\big). \tag{18}$$

By symmetry, cross terms between odd and even parts vanish in the weighted sum. Hence

$$P_z^{\text{CKF}} = \sum_r w_r\, \delta f(\boldsymbol{\xi}_r)\, \delta f(\boldsymbol{\xi}_r)^T = P_{\text{odd}} + \Sigma_{\text{even}}, \tag{19}$$

where both summands are PSD, and Σ_{even} is defined as:

$$\Sigma_{\text{even}} = \big(f_{\text{even}}(\boldsymbol{\xi}_r) - \sum_{i=1}^{2n} \frac{f_{\text{even}}(\boldsymbol{\xi}_i)}{2n}\big)\big(\,\cdot\,\big)^T. \tag{20}$$

Now expand the odd part:

$$f_{\text{odd}}(\boldsymbol{a}) = f'(\boldsymbol{0})\boldsymbol{a} + \sum_{j\geq 1} f^{(2j+1)}(\boldsymbol{a}). \tag{21}$$

Since the cross terms vanish by central symmetry and the covariance of the sigma points is the identity matrix, we have:

$$P_{\text{odd}} = f'(\boldsymbol{0})f'(\boldsymbol{0})^T + \sum_{j,\ell\geq 1}\sum_{r=1}^{2n} \frac{f^{(2j+1)}(\boldsymbol{\xi}_r) f^{(2\ell+1)}(\boldsymbol{\xi}_r)^T}{2n}. \tag{22}$$

Notice that the first part of P_{odd} is equal to P_z^{EKF}, and the second part can be denoted as a PSD matrix, $\Sigma_{\text{odd},\geq 3}$. Therefore,

$$P_{com}^{\text{CKF}} = \underbrace{\Sigma_{\text{odd},\geq 3}}_{\succeq 0} + \underbrace{\Sigma_{\text{even}}}_{\succeq 0} \succeq 0. \tag{23}$$

□

Theorem 2. *Consider the conditions in (1), and further assume that $\boldsymbol{x}$ follows a radially symmetric distribution, i.e., its probability density depends only on $\|\boldsymbol{x}\|_2^2$. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a vector of quadratic functions of the form*

$$\boldsymbol{z} = f(\boldsymbol{x}) = \boldsymbol{c} + f'(\boldsymbol{0})\boldsymbol{x} + \tfrac{1}{2}[\boldsymbol{x}^T f_i''(\boldsymbol{0})\boldsymbol{x}]_i. \tag{24}$$

Then the following equations hold:

$$\begin{cases} \bar{\boldsymbol{z}} = f(\boldsymbol{0}) + \frac{1}{2}\left[\text{tr}(f_i''(\boldsymbol{0}))\right]_i \\ P_z \succeq f'(\boldsymbol{0})(f'(\boldsymbol{0}))^T + P_{com}^{\text{EKF2, Sphere}} \\ P_{xz} = (f'(\boldsymbol{0}))^T \end{cases}. \tag{25}$$

Proof. The two equalities in (25) have appeared in (2), and have been proved in [1]. Therefore, we only need to prove the inequality. Decompose

$$\boldsymbol{z} = f'(\boldsymbol{0})\boldsymbol{x} + \boldsymbol{q}(\boldsymbol{x}), \quad \boldsymbol{q}(\boldsymbol{x}) = \tfrac{1}{2}[\boldsymbol{x}^T f_i''(\boldsymbol{0})\boldsymbol{x}]_i + \boldsymbol{c}. \tag{26}$$

Since the cross term $\mathbb{E}[(f'(\boldsymbol{0})\boldsymbol{x})(\boldsymbol{q}(\boldsymbol{x}) - \mathbb{E}[\boldsymbol{q}(\boldsymbol{x})])^T] = 0$ by oddness, we have

$$P_z = f'(\boldsymbol{0})(f'(\boldsymbol{0}))^T + \text{Cov}(\boldsymbol{q}(\boldsymbol{x})). \tag{27}$$

Consider the radial representation $\boldsymbol{x} = R\boldsymbol{u}$, where R is defined by $\sqrt{\boldsymbol{x}^T\boldsymbol{x}}$, and $\boldsymbol{u}$ is uniformly distributed on the unit sphere $\mathbb{S}^{n-1} := \{\boldsymbol{v} \in \mathbb{R}^n : \boldsymbol{v}^T\boldsymbol{v} = 1\}$. Since $\boldsymbol{x}$ follows a radially symmetric distribution, R and $\boldsymbol{u}$ are independent. Therefore, $\mathbb{E}[R^2] = n$, and

$$\text{Cov}(\boldsymbol{q}(\boldsymbol{x})) = \mathbb{E}[R^4]\,\text{Cov}(\boldsymbol{q}(\boldsymbol{v})) \succeq n^2\,\text{Cov}(\boldsymbol{q}(\boldsymbol{v})). \tag{28}$$

From (5), we already know that

$$\text{Cov}(\boldsymbol{q}(\boldsymbol{v})) = \frac{[\text{tr}(f_i''(\boldsymbol{0})f_j''(\boldsymbol{0})) - \frac{1}{n}\text{tr}(f_i''(\boldsymbol{0}))\text{tr}(f_j''(\boldsymbol{0}))]_{ij}}{2n(n+2)}. \tag{29}$$

Therefore

$$P_z \succeq f'(\boldsymbol{0})(f'(\boldsymbol{0}))^T + P_{com}^{\text{EKF2, Sphere}}, \tag{30}$$

establishing the second line of (25). Equality holds when $\|\boldsymbol{x}\|^2 = n$ almost surely. □

Theorem 3. *(i) In case 1, the global minimizer β^* of (4) exists (finite if $\bar{P} \neq 0$, and infinite otherwise) and satisfies $\beta^* \geq \beta_0$. Additionally, when $\|\bar{P}\|_F^2 \geq 0$, $\min f$ and β^* both increase monotonically as $\mathbb{E}[\|\Delta P\|_F^2]$ increases. (ii) In case 2, the global minimizer β^* of (4) exists (possibly infinite) and satisfies $\beta^* \geq \beta_0$.*

Proof. **Case 1**. Let P be a random matrix with mean $\bar{P} = \mathbb{E}[P]$. Define

$$f(\beta) := \mathbb{E}\left[\text{tr}\left(\left(\tfrac{P}{1+\beta-\beta_0} - \bar{P}\right)\left(\tfrac{P}{1+\beta-\beta_0} - \bar{P}\right)^T\right)\right]. \tag{31}$$

Note that in case 1, we have $\mathbb{E}[\text{tr}(P_{k|k,ac})] = f(\beta) + \text{tr}(P_{k|k-1} - \bar{P}\bar{P}^T)$. Therefore, the minimizer of f is also the solution to (4) in Case 1. We only need to prove that the global minimizer β^* of f exists (finite if $\bar{P} \neq 0$, and infinite otherwise) and satisfies $\beta^* \geq \beta_0$. Specifically, when $H := \|\bar{P}\|_F^2 > 0$, we want to prove that:

$$\beta^* = \beta_0 + \frac{\sigma^2}{H}, \quad \min f = \frac{\sigma^2 H}{\sigma^2 + H}, \tag{32}$$

where $\sigma^2 := \mathbb{E}[\|\Delta P\|_F^2] \geq 0$ and $\Delta P := P - \bar{P}$.

To prove (32), write $b := \beta - \beta_0$. Then

$$\frac{P}{1+\beta-\beta_0} - \bar{P} = \frac{P}{1+b} - \bar{P} = \frac{\Delta P - b\bar{P}}{1+b}. \tag{33}$$

Therefore,

$$\begin{aligned} f(b) &= \mathbb{E}\left[\left\|\frac{\Delta P - b\bar{P}}{1+b}\right\|_F^2\right] \\ &= \frac{1}{(1+b)^2}\,\mathbb{E}\big[\|\Delta P\|_F^2 - 2b\langle \Delta P, \bar{P}\rangle_F + b^2\|\bar{P}\|_F^2\big]\,. \end{aligned} \tag{34}$$

Since $\mathbb{E}[\langle \Delta P, \bar{P}\rangle_F] = 0$,

$$f(b) = \frac{\sigma^2 + b^2 H}{(1+b)^2}, \quad b \neq -1. \tag{35}$$

First, assume that $H > 0$. Take the derivative of $f(b)$:

$$f'(b) = \frac{2(bH - \sigma^2)}{(1+b)^3}. \tag{36}$$

Thus the unique stationary point is at $b^* = \sigma^2/H$, which lies in $(-1, \infty)$. Equation (32) can be derived by substituting b^* into the cost function. Note that this critical point is the unique global minimizer because $f(b) \to +\infty$ when $b \to -1^{\pm}$, and $f(b) \to H > f(b^*)$ as $b \to \pm\infty$.

Then, consider the situation where $H = 0$. In this case, $\bar{P} = 0$ and $f(b)$ is strictly decreasing for $b > -1$ and tends to 0 as $b \to +\infty$. Thus, the infimum 0 is achieved only in the limit $b^* \to +\infty$, which is again consistent with $b^* \geq 0$. Therefore, in all cases, a global minimizer exists (possibly at infinity) and satisfies $\beta^* \geq \beta_0$.

Case 2. Let $c := (1+\beta-\beta_0)^{-1} > 0$. Define

$$f(c) := c^2\,\mathrm{tr}\big(\bar{P}\,\mathbb{E}[\Delta S^{-2}]\,\bar{P}^T\big) - 2c\,\mathrm{tr}\big(\bar{P}\,\mathbb{E}[\Delta S^{-1}]\,\bar{P}^T\big)\,. \tag{37}$$

Note that in case 2, we have $\mathbb{E}[\mathrm{tr}(P_{k|k,ac})] = f(c) + \mathrm{tr}(P_{k|k-1})$. Therefore, the minimizer of f is also the solution to (4) in Case 2. Note that (37) is a convex quadratic in $c > 0$. The unique minimizer c^* is:

$$c^\star = \frac{\mathrm{tr}\big(\bar{P}\,\mathbb{E}[\Delta S^{-1}]\,\bar{P}^T\big)}{\mathrm{tr}\big(\bar{P}\,\mathbb{E}[\Delta S^{-2}]\,\bar{P}^T\big)}. \tag{38}$$

Since $X \mapsto X^{-1}$ is operator convex on the cone of positive–definite matrices, Jensen's inequality yields

$$\mathbb{E}[\Delta S^{-1}] \succeq \big(\mathbb{E}[\Delta S]\big)^{-1} = I. \tag{39}$$

Therefore,

$$\mathbb{E}[\Delta S^{-2}] \succeq (\mathbb{E}[\Delta S^{-1}])^2 \succeq \mathbb{E}[\Delta S^{-1}] \tag{40}$$

Hence $c^* \leq 1$. Therefore,

$$\beta^\star = \frac{1}{c} - 1 + \beta_0 \geq \beta_0. \tag{41}$$

□

References

[1] A. Gelb, *Applied Optimal Estimation*. MIT Press, 1974.

[2] E. S. Meckes, *The random matrix theory of the classical compact groups*. Cambridge University Press, 2019, vol. 218.