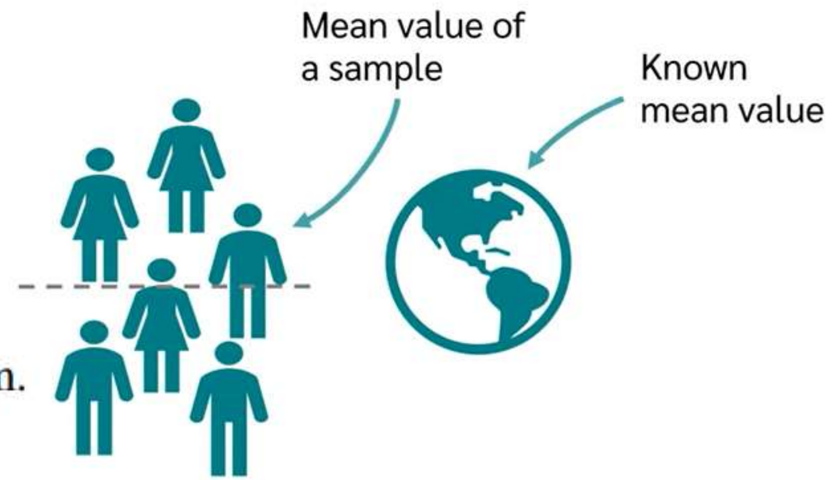
1. One-Sample t-Test:

Purpose: To test if the mean of a single sample is significantly different from a known or hypothesized population mean.
When to Use: When you want to compare the mean of one sample to a known population mean, but the population standard deviation is unknown.
Example: Testing whether the average weight of apples in a sample differs from a hypothesized population mean of 150 grams.

Formula

$$t = \frac{\bar{x} - \mu_0}{\frac{s}{\sqrt{n}}}$$



t is the test statistic and has $n - 1$ degrees of freedom.

$\bar{x}$ is the sample mean

μ_0 is the population mean under the null hypothesis.

s is the sample standard deviation

n is the sample size

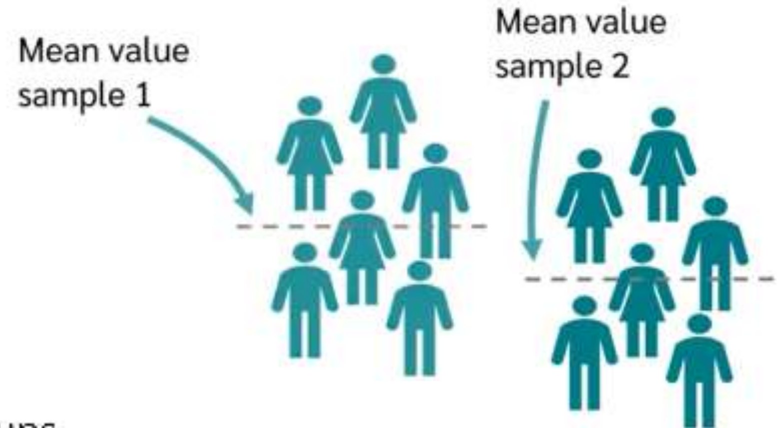
$\frac{s}{\sqrt{n}}$ is the estimated standard error

2. **Independent Two-Sample t-Test**:

•**Purpose**: To compare the means of two independent groups to determine if there is a statistically significant difference between them.
•**When to Use**: When you have two independent samples (e.g., different groups of people) and want to test whether their means are significantly different.
•**Example**: Comparing the average test scores of two different classes to see if there is a significant difference in performance.

Formula (assuming equal variances)

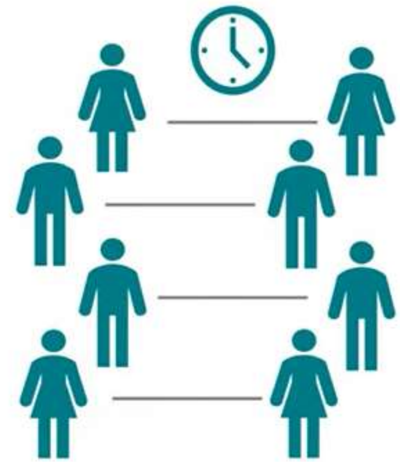
$$t = \frac{\bar{X}_1 - \bar{X}_2}{\sqrt{s_p^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$



- $\bar{X}_1$ and $\bar{X}_2$ = sample means of the two groups
- s_p^2 = pooled variance (a weighted average of the two sample variances)
- n_1 and n_2 = sample sizes of the two groups

3. **Paired t-Test**:

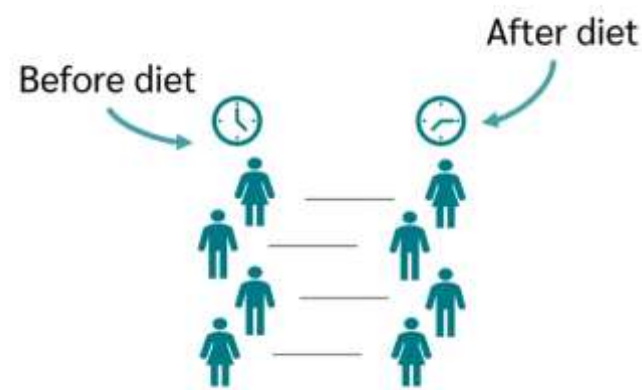
•**Purpose**: To compare the means of two related groups, such as measurements taken from the same group at different times.
•**When to Us**e: When you have paired or matched samples, like before-and-after measurements or the same participants undergoing two treatments.
•**Example**: Measuring the weight of individuals before and after a diet program to test if there is a significant change in weight.



Formula

$$t = \frac{\bar{D}}{\frac{s_D}{\sqrt{n}}}$$

- $\bar{D}$ = mean of the differences between paired observations
- s_D = standard deviation of the differences
- n = number of pairs



- Back in the early 1900's, William Sealy Gosset, a chemist at a brewery in Ireland discovered that when he was working with very small samples, the distributions of the mean differed significantly from the normal distribution.
- He noticed that as his sample sizes changed, the shape of the distribution changed as well.
- He published his results under the pseudonym 'Student' and this concept and the distributions for small sample sizes are now known as "Student's t−distributions."
- The differences between the t-distribution and the normal distribution are more exaggerated when there are fewer data points, and therefore fewer degrees of freedom.
- Degrees of freedom are essentially the number of samples that have the 'freedom' to change without affecting the sample mean.

$$df = n - 1$$

- If you were conducting a two-tailed hypothesis test on a sample of 25 students, your df = 25-1 = 24

Example A

The high school athletic director is asked if football players are doing as well academically as the other student athletes. We know from a previous study that the average GPA for the student athletes is 3.10. After an initiative to help improve the GPA of student athletes, the athletic director randomly samples 20 football players and finds that the average GPA of the sample is 3.18 with a sample standard deviation of 0.54. Is there a significant improvement? Use a 0.05 significance level.

Solution :

Step 1: Cleary state the null and alternative hypotheses.

$$H_0 : \mu = 3.10$$
$$H_a : \mu \neq 3.10$$

Step 2: Identify the appropriate significance level and confirm the test assumptions.

We were told that we should use a 0.05 significance level. The size of the sample also helps here, as we have 20 players. So, we can conclude that the assumptions for the single sample t-test have been met.

Step 3: Analyze the data

We use our t-test formula:

$$t = \frac{\bar{x} - \mu_0}{\frac{s}{\sqrt{n}}} = \frac{3.18 - 3.10}{\frac{0.54}{\sqrt{20}}} = 0.66$$

We know that we have 20 observations, so our degrees of freedom for this test is 19. Nineteen degrees of freedom at the 0.05 significance level gives us a critical value of ± 2.093.

DF	0.25	0.20	0.15	0.10	0.05	0.025	0.01	0.005	0.001
1	1.000	1.376	1.963	3.078	6.314	12.710	31.820	63.660	318.300
2	0.816	1.061	1.386	1.886	2.920	4.303	6.965	9.925	22.330
3	0.765	0.978	1.250	1.638	2.353	3.182	4.541	5.841	10.210
4	0.741	0.941	1.190	1.533	2.132	2.776	3.747	4.604	7.173
5	0.727	0.920	1.156	1.476	2.015	2.571	3.365	4.032	5.893
6	0.718	0.906	1.134	1.440	1.943	2.447	3.143	3.707	5.208
7	0.711	0.896	1.119	1.415	1.895	2.365	2.998	3.499	4.785
8	0.706	0.889	1.108	1.397	1.860	2.306	2.896	3.355	4.501
9	0.703	0.883	1.100	1.383	1.833	2.262	2.821	3.250	4.297
10	0.700	0.879	1.093	1.372	1.812	2.228	2.764	3.169	4.144
11	0.697	0.876	1.088	1.363	1.796	2.201	2.718	3.106	4.025
12	0.695	0.873	1.083	1.356	1.782	2.179	2.681	3.055	3.930
13	0.694	0.870	1.079	1.350	1.771	2.160	2.650	3.012	3.852
14	0.692	0.868	1.076	1.345	1.761	2.145	2.624	2.977	3.787
15	0.691	0.866	1.074	1.341	1.753	2.131	2.602	2.947	3.733
16	0.690	0.865	1.071	1.337	1.746	2.120	2.583	2.921	3.686
17	0.689	0.863	1.069	1.333	1.740	2.110	2.567	2.898	3.646
18	0.688	0.862	1.067	1.330	1.734	2.101	2.552	2.878	3.610
19	0.688	0.861	1.066	1.328	1.729	2.093	2.539	2.861	3.579
20	0.687	0.860	1.064	1.325	1.725	2.086	2.528	2.845	3.552
21	0.686	0.859	1.063	1.323	1.721	2.080	2.518	2.831	3.527

Step 4: Interpret your results

- Since our calculated t-test value is lower than our t-critical value, we fail to reject the Null Hypothesis.
- Therefore, the average GPA of the sample of football players is not significantly different from the average GPA of student athletes.
- Thus, the athletic director can conclude that the mean academic performance of football players does not differ from the mean performance of other student athletes.

Example B

Duracell manufactures batteries that the CEO claims will last an average of 300 hours under normal use. A researcher randomly selected 20 batteries from the production line and tested these batteries. The tested batteries had a mean life span of 270 hours with a standard deviation of 50 hours. Do we have enough evidence to suggest that the claim of an average lifetime of 300 hours is false?

Solution :

Step 1: Clearly state the Null and Alternative Hypothesis

$$H_0 : \mu = 300$$

$$H_A : \mu \neq 300$$

Step 2: Identify the appropriate significance level and confirm the test assumptions.

We'll use the standard significance level of 0.05, and we assume a normal population distribution.

Step 3: Analyze the data and compute the test statistic

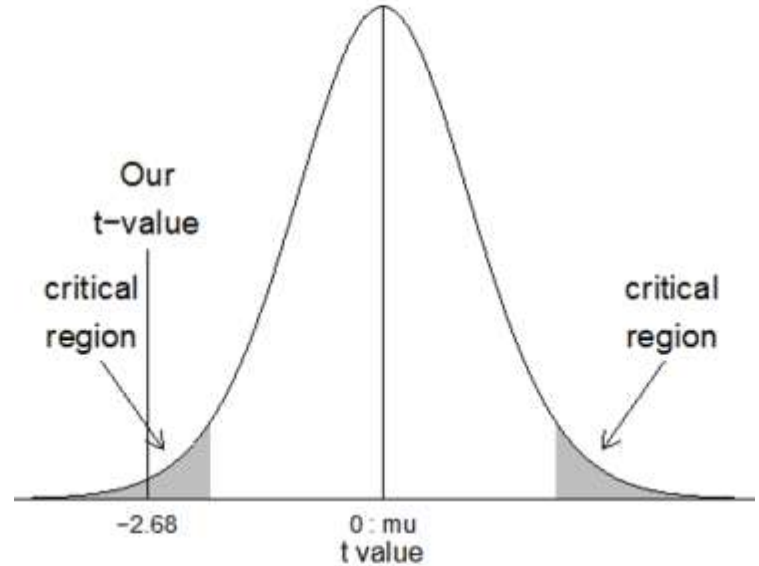
First, we need to calculate the Standard Error:

$$SE_{\bar{x}} = \frac{s}{\sqrt{n}}$$

$$SE_{\bar{x}} = \frac{50}{\sqrt{20}} = 11.18$$

Now, we'll use that value in out t-test formula:

$$t = \frac{\bar{x}-\mu}{SE_{\bar{x}}} = \frac{270-300}{11.18} = -2.68$$



- We know that we have 20 batteries, so our degrees of freedom for this test is (20-1)= 19.

- Nineteen degrees of freedom at the 0.05 significance level gives us a critical value of
  ± 2.093.

Step 4: Interpret your results

Since our calculated t-test value is greater than our t-critical value, it lies in the critical region therefore, we reject the Null Hypothesis.
The average battery life of the sample is significantly different from the average battery life claim by the CEO. Therefore, the claim of an average lifetime of 300 hours is false

Example : Independent Two Sample Test

A researcher wants to determine if two different diets have different effects on weight loss. The researcher takes a random sample of 10 people from each group:
Group 1 (Diet A): Their weight losses in pounds are: 5, 7, 6, 9, 8, 4, 7, 5, 6, 8
Group 2 (Diet B): Their weight losses in pounds are: 8, 10, 6, 9, 12, 11, 9, 10, 8, 11
The researcher wonders if there is a significant difference in the average weight loss between Diet A and Diet B.

Solution :

Null hypothesis (H_0): There is no difference in the means of the two groups.
$H0: \mu 1 = \mu 2$

Alternative hypothesis (H_1): There is a difference in the means of the two groups.
$H1: \mu 1 \neq \mu 2$

This is a **two-tailed test** since we are testing for any difference between the means, not specifically an increase or decrease.

Step 2: Calculate the sample means and standard deviations

- Group 1 (Diet A):
  - $\bar{x}_1$ = sample mean of Group 1

$$\bar{x}_1 = \frac{5+7+6+9+8+4+7+5+6+8}{10} = 6.5$$

  - Sample standard deviation s_1 for Group 1

$$s_1 = \sqrt{\frac{\sum(x_i - \bar{x}_1)^2}{n-1}} = \sqrt{\frac{(5-6.5)^2 + (7-6.5)^2 + \dots}{10-1}} = 1.58$$

- Group 2 (Diet B):
  - $\bar{x}_2$ = sample mean of Group 2

$$\bar{x}_2 = \frac{8+10+6+9+12+11+9+10+8+11}{10} = 9.4$$

  - Sample standard deviation s_2 for Group 2

$$s_2 = \sqrt{\frac{\sum(x_i - \bar{x}_2)^2}{n-1}} = \sqrt{\frac{(8-9.4)^2 + (10-9.4)^2 + \dots}{10-1}} = 1.65$$

Step 3: Compute the t-statistic

The formula for the t-statistic in a two-sample t-test is:

$$t = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

Substituting the values:

$$t = \frac{6.5 - 9.4}{\sqrt{\frac{1.58^2}{10} + \frac{1.65^2}{10}}} = \frac{-2.9}{\sqrt{\frac{2.4964}{10} + \frac{2.7225}{10}}} = \frac{-2.9}{0.7225} = -4.01$$

Step 4: Degrees of Freedom

The degrees of freedom df for the two-sample t-test are calculated using:

$$df = \frac{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)^2}{\frac{\left(\frac{s_1^2}{n_1}\right)^2}{n_1 - 1} + \frac{\left(\frac{s_2^2}{n_2}\right)^2}{n_2 - 1}}$$

Substituting the values:

$$df = \frac{\left(\frac{1.58^2}{10} + \frac{1.65^2}{10}\right)^2}{\frac{\left(\frac{1.58^2}{10}\right)^2}{9} + \frac{\left(\frac{1.65^2}{10}\right)^2}{9}} = \frac{(0.24964 + 0.27225)^2}{\frac{0.24964^2}{9} + \frac{0.27225^2}{9}} = \frac{(0.52189)^2}{\frac{0.06232}{9} + \frac{0.07412}{9}} = 17.7 \text{ (approximately)}$$

Step 5: Determine the critical t-value or p-value

Using a t-distribution table or software, look up the critical t-value for a two-tailed test with $df = 17.7$ and a significance level $\alpha = 0.05$.

For $df \approx 18$ and $\alpha = 0.05$, the critical t-value is approximately ±2.101.

Since |t|=4.01 is greater than the critical value of 2.101, we **reject the null hypothesis**.
There is significant evidence to suggest that the average weight loss between the two diets is different.