SUPPLEMENTARY of “Toward Consistent and Efficient Map-based Visual-inertial Localization: Theory Framework and Filter Design”

***Abstract*—This document provides the theoretical proofs mentioned in the main paper “Toward Consistent and Efficient Map-based Visual-inertial Localization: Theory Framework and Filter Design”. All the notations, equation numbers, reference numbers and section numbers are identical to those in the main paper.**

APPENDIX A
DERIVATION OF ERROR PROPAGATION FUNCTION

From (8) and (12), we need to derive the linearized error kinematics:

$$\frac{d}{dt}\mathbf{e}_t = \left(\frac{d}{dt}\boldsymbol{\eta}_{A_t}, \frac{d}{dt}\boldsymbol{\xi}_{B_t}\right). \tag{100}$$

To derive $\frac{d}{dt}\boldsymbol{\eta}_{A_t}$, recall the definition of the *right invariant error* (11), we have:

$$\begin{aligned}
\frac{d}{dt}({}^L\hat{\mathbf{R}}_{I_t}{}^L\mathbf{R}_{I_t}^{-1}) =& {}^L\hat{\mathbf{R}}_{I_t}(\boldsymbol{\omega}_t-\hat{\mathbf{b}}_{g_t})_\times {}^L\mathbf{R}_{I_t}^\top \\
&+{}^L\hat{\mathbf{R}}_{I_t}({}^L\mathbf{R}_{I_t}(\boldsymbol{\omega}_t-\mathbf{b}_{g_t}-\mathbf{w}_{g_t})_\times)^\top \\
\frac{d}{dt}({}^L\hat{\mathbf{R}}_{I_t}{}^L\mathbf{R}_{I_t}^{-1}) =& {}^L\hat{\mathbf{R}}_{I_t}(\boldsymbol{\omega}_t-\hat{\mathbf{b}}_{g_t})_\times {}^L\hat{\mathbf{R}}_{I_t}^\top\boldsymbol{\eta}_R \\
&-{}^L\hat{\mathbf{R}}_{I_t}(\boldsymbol{\omega}_t-\mathbf{b}_{g_t}-\mathbf{w}_{g_t})_\times {}^L\hat{\mathbf{R}}_{I_t}^\top\boldsymbol{\eta}_R \\
\frac{d}{dt}({}^L\hat{\mathbf{R}}_{I_t}{}^L\mathbf{R}_{I_t}^{-1}) =& {}^L\hat{\mathbf{R}}_{I_t}(-\boldsymbol{\xi}_{B_{gt}}+\mathbf{w}_{g_t})_\times {}^L\hat{\mathbf{R}}_{I_t}^\top\boldsymbol{\eta}_R \\
\frac{d}{dt}({}^L\hat{\mathbf{R}}_{I_t}{}^L\mathbf{R}_{I_t}^{-1}) =& ({}^L\hat{\mathbf{R}}_{I_t}(-\boldsymbol{\xi}_{B_{gt}}+\mathbf{w}_{g_t}))_\times\boldsymbol{\eta}_R \\
\frac{d}{dt}(\mathbf{I}+(\boldsymbol{\xi}_{\boldsymbol{\theta}_{LI_t}})_\times) \approx& ({}^L\hat{\mathbf{R}}_{I_t}(-\boldsymbol{\xi}_{B_{gt}}+\mathbf{w}_{g_t}))_\times(\mathbf{I}+(\tilde{\boldsymbol{\theta}}_{LI_t})_\times) \\
\frac{d}{dt}((\boldsymbol{\xi}_{\boldsymbol{\theta}_{LI_t}})_\times) \approx& ({}^L\hat{\mathbf{R}}_{I_t}(-\boldsymbol{\xi}_{B_{gt}}+\mathbf{w}_{g_t}))_\times \\
\frac{d}{dt}\boldsymbol{\xi}_{\boldsymbol{\theta}_{LI_t}} \approx& {}^L\hat{\mathbf{R}}_{I_t}(-\boldsymbol{\xi}_{B_{gt}}+\mathbf{w}_{g_t}).
\end{aligned} \tag{101}$$

$$\begin{aligned}
\frac{d}{dt}({}^L\hat{\mathbf{v}}_{I_t}-{}^L\hat{\mathbf{R}}_{I_t}{}^L\mathbf{R}_{I_t}^{-1}{}^L\mathbf{v}_{I_t}) =& {}^L\hat{\mathbf{R}}_{I_t}(\mathbf{a}_t-\hat{\mathbf{b}}_{a_t})+\mathbf{g}- \\
&\left(\frac{d}{dt}({}^L\hat{\mathbf{R}}_{I_t}{}^L\mathbf{R}_{I_t}^{-1}){}^L\mathbf{v}_{I_t}+\right. \\
&\left.{}^L\hat{\mathbf{R}}_{I_t}{}^L\mathbf{R}_{I_t}^{-1}({}^L\mathbf{R}_{I_t}(\mathbf{a}_t-\mathbf{b}_{a_t}-\mathbf{w}_{a_t})+\mathbf{g})\right) \\
\frac{d}{dt}(\boldsymbol{\xi}_{\mathbf{v}_{LI_t}}) =& {}^L\hat{\mathbf{R}}_{I_t}(\mathbf{a}_t-\hat{\mathbf{b}}_{a_t})+\mathbf{g}-({}^L\hat{\mathbf{R}}_{I_t}(-\boldsymbol{\xi}_{B_{gt}}+\mathbf{w}_{g_t}))_\times \\
&\boldsymbol{\eta}_R{}^L\mathbf{v}_{I_t}-{}^L\hat{\mathbf{R}}_{I_t}(\mathbf{a}_t-\mathbf{b}_{a_t}-\mathbf{w}_{a_t})-\boldsymbol{\eta}_R\mathbf{g} \\
\frac{d}{dt}(\boldsymbol{\xi}_{\mathbf{v}_{LI_t}}) =& {}^L\hat{\mathbf{R}}_{I_t}(-\boldsymbol{\xi}_{B_{at}}+\mathbf{w}_{a_t})+\mathbf{g}-\boldsymbol{\eta}_R\mathbf{g}- \\
&({}^L\hat{\mathbf{R}}_{I_t}(-\boldsymbol{\xi}_{B_{gt}}+\mathbf{w}_{g_t}))_\times({}^L\hat{\mathbf{v}}_{I_t}-\boldsymbol{\eta}_v) \\
\frac{d}{dt}(\boldsymbol{\xi}_{\mathbf{v}_{LI_t}}) \approx& {}^L\hat{\mathbf{R}}_{I_t}(-\boldsymbol{\xi}_{B_{at}}+\mathbf{w}_{a_t})+\mathbf{g}-(\mathbf{I}+(\tilde{\boldsymbol{\theta}}_{LI_t})_\times)\mathbf{g}- \\
&({}^L\hat{\mathbf{R}}_{I_t}(-\boldsymbol{\xi}_{B_{gt}}+\mathbf{w}_{g_t}))_\times({}^L\hat{\mathbf{v}}_{I_t}-\boldsymbol{\xi}_{\mathbf{v}_{LI_t}}) \\
\frac{d}{dt}(\boldsymbol{\xi}_{\mathbf{v}_{LI_t}}) \approx& {}^L\hat{\mathbf{R}}_{I_t}(-\boldsymbol{\xi}_{B_{at}}+\mathbf{w}_{a_t})+(\mathbf{g})_\times\boldsymbol{\xi}_{\boldsymbol{\theta}_{LI_t}}+ \\
&({}^L\hat{\mathbf{v}}_{I_t})_\times{}^L\hat{\mathbf{R}}_{I_t}(-\boldsymbol{\xi}_{B_{gt}}+\mathbf{w}_{g_t}).
\end{aligned} \tag{102}$$

Analogously,

$$\begin{aligned}
\frac{d}{dt}({}^L\hat{\mathbf{p}}_{I_t}-{}^L\hat{\mathbf{R}}_{I_t}{}^L\mathbf{R}_{I_t}^{-1}{}^L\mathbf{p}_{I_t}) =& {}^L\hat{\mathbf{v}}_{I_t}- \\
&\left(\frac{d}{dt}({}^L\hat{\mathbf{R}}_{I_t}{}^L\mathbf{R}_{I_t}^{-1}){}^L\mathbf{p}_{I_t}+{}^L\hat{\mathbf{R}}_{I_t}{}^L\mathbf{R}_{I_t}^{-1}{}^L\mathbf{v}_{I_t}\right) \\
\frac{d}{dt}(\boldsymbol{\xi}_{\mathbf{p}_{LI_t}}) \approx& \boldsymbol{\xi}_{\mathbf{v}_{LI_t}}+({}^L\hat{\mathbf{p}}_{I_t})_\times({}^L\hat{\mathbf{R}}_{I_t}(-\boldsymbol{\xi}_{B_{gt}}+\mathbf{w}_{g_t})).
\end{aligned} \tag{103}$$

$$\begin{aligned}
\frac{d}{dt}({}^L\hat{\mathbf{p}}_{f_t}-{}^L\hat{\mathbf{R}}_{I_t}{}^L\mathbf{R}_{I_t}^{-1}{}^L\mathbf{p}_{f_t}) =& -\frac{d}{dt}({}^L\hat{\mathbf{R}}_{I_t}{}^L\mathbf{R}_{I_t}^{-1}){}^L\mathbf{p}_{f_t} \\
\frac{d}{dt}(\boldsymbol{\xi}_{\mathbf{p}_{Lf_t}}) \approx& ({}^L\hat{\mathbf{p}}_{f_t})_\times({}^L\hat{\mathbf{R}}_{I_t}(-\boldsymbol{\xi}_{B_{gt}}+\mathbf{w}_{g_t}))
\end{aligned} \tag{104}$$

$$\begin{aligned}
\frac{d}{dt}({}^L\hat{\mathbf{p}}_{G_t}-{}^L\hat{\mathbf{R}}_{I_t}{}^L\mathbf{R}_{I_t}^{-1}{}^L\mathbf{p}_{G_t}) =& -\frac{d}{dt}({}^L\hat{\mathbf{R}}_{I_t}{}^L\mathbf{R}_{I_t}^{-1}){}^L\mathbf{p}_{G_t} \\
\frac{d}{dt}(\boldsymbol{\xi}_{\mathbf{p}_{LG_t}}) \approx& ({}^L\hat{\mathbf{p}}_{G_t})_\times({}^L\hat{\mathbf{R}}_{I_t}(-\boldsymbol{\xi}_{B_{gt}}+\mathbf{w}_{g_t}))
\end{aligned} \tag{105}$$

So, the linearized error kinematics of be written as (13).

APPENDIX B
DERIVATION OF THE JACOBIAN MATRIX OF THE OBSERVATION FUNCTION

To derive the Jacobian matrix of (19), we have

$$\begin{aligned}
&h({}^L\mathbf{R}_{I_t}^\top({}^L\mathbf{p}_{f_t}-{}^L\mathbf{p}_{I_t}))-h({}^L\hat{\mathbf{R}}_{I_t}^\top({}^L\hat{\mathbf{p}}_{f_t}-{}^L\hat{\mathbf{p}}_{I_t})) \\
&\approx \nabla h|_{\hat{\mathfrak{q}}}(\mathfrak{q}-\hat{\mathfrak{q}}) \\
&= \nabla h|_{\hat{\mathfrak{q}}}\left[{}^L\mathbf{R}_{I_t}^\top({}^L\mathbf{p}_{f_t}-{}^L\mathbf{p}_{I_t})-{}^L\hat{\mathbf{R}}_{I_t}^\top({}^L\hat{\mathbf{p}}_{f_t}-{}^L\hat{\mathbf{p}}_{I_t})\right] \\
&= \nabla h|_{\hat{\mathfrak{q}}}\left[{}^L\hat{\mathbf{R}}_{I_t}^\top\left[{}^L\hat{\mathbf{R}}_{I_t}{}^L\mathbf{R}_{I_t}^\top({}^L\mathbf{p}_{f_t}-{}^L\mathbf{p}_{I_t})-({}^L\hat{\mathbf{p}}_{f_t}-{}^L\hat{\mathbf{p}}_{I_t})\right]\right] \\
&= \nabla h|_{\hat{\mathfrak{q}}}\left[{}^L\hat{\mathbf{R}}_{I_t}^\top\left(\boldsymbol{\xi}_{\mathbf{p}_{LI_t}}-\boldsymbol{\xi}_{\mathbf{p}_{Lf_t}}\right)\right] \\
&= \nabla h|_{\hat{\mathfrak{q}}}{}^L\hat{\mathbf{R}}_{I_t}^\top\begin{bmatrix}\mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3\end{bmatrix}\boldsymbol{\epsilon}_t,
\end{aligned} \tag{106}$$

and the Jacobian matrix of (21) can be derived from

$$\begin{aligned}
&h({}^L\mathbf{R}_{I_t}^\top({}^L\mathbf{R}_{G_t}{}^G\mathbf{p}_F+{}^L\mathbf{p}_{G_t}-{}^L\mathbf{p}_{I_t})) \\
&-h({}^L\hat{\mathbf{R}}_{I_t}^\top({}^L\hat{\mathbf{R}}_{G_t}{}^G\mathbf{p}_F+{}^L\hat{\mathbf{p}}_{G_t}-{}^L\hat{\mathbf{p}}_{I_t})) \\
&\approx \nabla h'|_{\hat{\mathfrak{q}}'}(\mathfrak{q}'-\hat{\mathfrak{q}}') \\
&= \nabla h'|_{\hat{\mathfrak{q}}'}\left[{}^L\hat{\mathbf{R}}_{I_t}^\top({}^L\hat{\mathbf{R}}_{I_t}{}^L\mathbf{R}_{I_t}^\top({}^L\mathbf{R}_{G_t}{}^G\mathbf{p}_F+{}^L\mathbf{p}_{G_t}-{}^L\mathbf{p}_{I_t}))\right. \\
&\left.-{}^L\hat{\mathbf{R}}_{I_t}^\top({}^L\hat{\mathbf{R}}_{G_t}{}^G\mathbf{p}_F+{}^L\hat{\mathbf{p}}_{G_t}-{}^L\hat{\mathbf{p}}_{I_t})\right]
\end{aligned}$$

$$= \nabla h'|_{\hat{\mathbf{q}}'} \left[{}^L\hat{\mathbf{R}}_{I_t}^\top \left[{}^L\hat{\mathbf{R}}_{I_t} {}^L\mathbf{R}_{I_t}^\top ({}^L\mathbf{R}_{G_t} {}^G\mathbf{p}_F + {}^L\mathbf{p}_{G_t} - {}^L\mathbf{p}_{I_t}) - {}^L\hat{\mathbf{R}}_{G_t} {}^G\mathbf{p}_F - {}^L\hat{\mathbf{p}}_{G_t} + {}^L\hat{\mathbf{p}}_{I_t} \right]\right]$$
$$\approx \nabla h'|_{\hat{\mathbf{q}}'} \left[{}^L\hat{\mathbf{R}}_{I_t}^\top \left[(\mathbf{I} + (\tilde{\boldsymbol{\theta}}_{LI_t})_\times)({}^L\mathbf{R}_{G_t} {}^G\mathbf{p}_F + {}^L\mathbf{p}_{G_t} - {}^L\mathbf{p}_{I_t}) - {}^L\hat{\mathbf{R}}_{G_t} {}^G\mathbf{p}_F - {}^L\hat{\mathbf{p}}_{G_t} + {}^L\hat{\mathbf{p}}_{I_t} \right]\right]$$
$$= \nabla h'|_{\hat{\mathbf{q}}'} {}^L\hat{\mathbf{R}}_{I_t}^\top \left[(\mathbf{I} + (\tilde{\boldsymbol{\theta}}_{LI_t})_\times) {}^L\mathbf{R}_{G_t} {}^G\mathbf{p}_F - {}^L\hat{\mathbf{R}}_{G_t} {}^G\mathbf{p}_F - \boldsymbol{\xi}_{\mathbf{p}_{LG_t}} + \boldsymbol{\xi}_{\mathbf{p}_{LI_t}} \right]$$
$$\approx \nabla h'|_{\hat{\mathbf{q}}'} {}^L\hat{\mathbf{R}}_{I_t}^\top \left[(\mathbf{I} + (\tilde{\boldsymbol{\theta}}_{LI_t})_\times)(\mathbf{I} - (\tilde{\boldsymbol{\theta}}_{LG_t})_\times) {}^L\hat{\mathbf{R}}_{G_t} {}^G\mathbf{p}_F - {}^L\hat{\mathbf{R}}_{G_t} {}^G\mathbf{p}_F - \boldsymbol{\xi}_{\mathbf{p}_{LG_t}} + \boldsymbol{\xi}_{\mathbf{p}_{LI_t}} \right]$$
$$\approx \nabla h'|_{\hat{\mathbf{q}}'} {}^L\hat{\mathbf{R}}_{I_t}^\top \left[({}^L\hat{\mathbf{R}}_{G_t} {}^G\mathbf{p}_F)_\times \tilde{\boldsymbol{\theta}}_{LG_t} - ({}^L\hat{\mathbf{R}}_{G_t} {}^G\mathbf{p}_F)_\times \tilde{\boldsymbol{\theta}}_{LI_t} - \boldsymbol{\xi}_{\mathbf{p}_{LG_t}} + \boldsymbol{\xi}_{\mathbf{p}_{LI_t}} \right]$$
$$= \nabla h'|_{\hat{\mathbf{q}}'} {}^L\hat{\mathbf{R}}_{I_t}^\top \left[-({}^L\hat{\mathbf{R}}_{G_t} {}^G\mathbf{p}_F)_\times \quad \mathbf{0}_3 \quad \mathbf{I}_3 \quad \mathbf{0}_3 \quad -\mathbf{I}_3 \quad ({}^L\hat{\mathbf{R}}_{G_t} {}^G\mathbf{p}_F)_\times \quad \mathbf{0}_3 \quad \mathbf{0}_3 \right] \boldsymbol{\epsilon}_t. \tag{107}$$

Further, if the map information is considered as a part of the system state (cf. Sec.V), the observation function of a map landmark ${}^G\mathbf{p}_{F_j,t}$ is given by (27) and (28). The Jacobian matrix of (27) can be derived by:

$$h({}^L\mathbf{R}_{I_t}^\top ({}^L\mathbf{R}_{G_t} {}^G\mathbf{p}_{F_j,t} + {}^L\mathbf{p}_{G_t} - {}^L\mathbf{p}_{I_t})) - h({}^L\hat{\mathbf{R}}_{I_t}^\top ({}^L\hat{\mathbf{R}}_{G_t} {}^G\hat{\mathbf{p}}_{F_j,t} + {}^L\hat{\mathbf{p}}_{G_t} - {}^L\hat{\mathbf{p}}_{I_t}))$$
$$\approx \nabla h'|_{\hat{\mathbf{q}}'} {}^L\hat{\mathbf{R}}_{I_t}^\top \left[{}^L\hat{\mathbf{R}}_{I_t} {}^L\mathbf{R}_{I_t}^\top ({}^L\mathbf{R}_{G_t} {}^G\mathbf{p}_{F_j,t}) - {}^L\hat{\mathbf{R}}_{G_t} {}^G\hat{\mathbf{p}}_{F_j,t} - \boldsymbol{\xi}_{\mathbf{p}_{LG_t}} + \boldsymbol{\xi}_{\mathbf{p}_{LI_t}} \right]$$
$$\approx \nabla h'|_{\hat{\mathbf{q}}'} {}^L\hat{\mathbf{R}}_{I_t}^\top \left[(\mathbf{I} + (\tilde{\boldsymbol{\theta}}_{LI_t})_\times)(\mathbf{I} - (\tilde{\boldsymbol{\theta}}_{LG_t})_\times) {}^L\hat{\mathbf{R}}_{G_t} {}^G\mathbf{p}_{F_j,t} - {}^L\hat{\mathbf{R}}_{G_t} {}^G\hat{\mathbf{p}}_{F_j,t} - \boldsymbol{\xi}_{\mathbf{p}_{LG_t}} + \boldsymbol{\xi}_{\mathbf{p}_{LI_t}} \right]$$
$$\approx \nabla h'|_{\hat{\mathbf{q}}'} {}^L\hat{\mathbf{R}}_{I_t}^\top \left[{}^L\hat{\mathbf{R}}_{G_t} ({}^G\mathbf{p}_{F_j,t} - {}^G\hat{\mathbf{p}}_{F_j,t}) - ({}^L\hat{\mathbf{R}}_{G_t} {}^G\hat{\mathbf{p}}_{F_j,t})_\times \tilde{\boldsymbol{\theta}}_{LI_t} + ({}^L\hat{\mathbf{R}}_{G_t} {}^G\hat{\mathbf{p}}_{F_j,t})_\times \tilde{\boldsymbol{\theta}}_{LG_t} - \boldsymbol{\xi}_{\mathbf{p}_{LG_t}} + \boldsymbol{\xi}_{\mathbf{p}_{LI_t}} \right]$$
$$= \nabla h'|_{\hat{\mathbf{q}}'} {}^L\hat{\mathbf{R}}_{I_t}^\top \left[-({}^L\hat{\mathbf{R}}_{G_t} {}^G\hat{\mathbf{p}}_{F_j,t})_\times \quad \mathbf{0}_3 \quad \mathbf{I}_3 \quad \mathbf{0}_3 \quad -\mathbf{I}_3 \quad ({}^L\hat{\mathbf{R}}_{G_t} {}^G\hat{\mathbf{p}}_{F_j,t})_\times \quad \mathbf{0}_3 \quad \mathbf{0}_3 | \cdots \mathbf{0}_3 \quad \mathbf{0}_3 \cdots - {}^L\hat{\mathbf{R}}_{G_t} \cdots \right] \boldsymbol{\epsilon}_t^*, \tag{108}$$

which is identical to the Jacobian matrix given in (29).

The Jacobian matrix of (30) is derived from

$$h\left[{}^G\mathbf{R}_{KF_i,t}^\top ({}^G\mathbf{p}_{F_j,t} - {}^G\mathbf{p}_{KF_i,t}) \right] - h\left[{}^G\hat{\mathbf{R}}_{KF_i,t}^\top ({}^G\hat{\mathbf{p}}_{F_j,t} - {}^G\hat{\mathbf{p}}_{KF_i,t}) \right]$$
$$\approx \nabla h^*|_{\hat{\mathbf{q}}^*} {}^G\hat{\mathbf{R}}_{KF_i,t}^\top \left[{}^G\hat{\mathbf{R}}_{KF_i,t} {}^G\mathbf{R}_{KF_i,t}^\top ({}^G\mathbf{p}_{F_j,t} - {}^G\mathbf{p}_{KF_i,t}) + {}^G\hat{\mathbf{p}}_{KF_i,t} - {}^G\hat{\mathbf{p}}_{F_j,t} \right]$$
$$= \nabla h^*|_{\hat{\mathbf{q}}^*} {}^G\hat{\mathbf{R}}_{KF_i,t}^\top \left[{}^G\hat{\mathbf{R}}_{KF_i,t} {}^G\mathbf{R}_{KF_i,t}^\top {}^G\mathbf{p}_{F_j,t} - {}^G\hat{\mathbf{p}}_{F_j,t} + \boldsymbol{\xi}_{\mathbf{p}_{KF_i,t}} \right]$$
$$\approx \nabla h^*|_{\hat{\mathbf{q}}^*} {}^G\hat{\mathbf{R}}_{KF_i,t}^\top \left[(\mathbf{I} + (\tilde{\boldsymbol{\theta}}_{KF_i,t})_\times) {}^G\mathbf{p}_{F_j,t} - {}^G\hat{\mathbf{p}}_{F_j,t} + \boldsymbol{\xi}_{\mathbf{p}_{KF_i,t}} \right]$$
$$\approx \nabla h^*|_{\hat{\mathbf{q}}^*} {}^G\hat{\mathbf{R}}_{KF_i,t}^\top \left[-({}^G\hat{\mathbf{p}}_{F_j,t})_\times \tilde{\boldsymbol{\theta}}_{KF_i,t} - \boldsymbol{\xi}_{\mathbf{p}_{F_j,t}} + \boldsymbol{\xi}_{\mathbf{p}_{KF_i,t}} \right]$$
$$= \nabla h^*|_{\hat{\mathbf{q}}^*} {}^G\hat{\mathbf{R}}_{KF_i,t}^\top \left[\mathbf{0}_3 \quad \mathbf{0}_3 \quad \mathbf{0}_3 \quad \mathbf{0}_3 \quad \mathbf{0}_3 \quad \mathbf{0}_3 \quad \mathbf{0}_3 \quad \mathbf{0}_3 | \cdots - ({}^G\hat{\mathbf{p}}_{F_j,t})_\times \quad \mathbf{I}_3 \cdots - \mathbf{I}_3 \cdots \right] \boldsymbol{\epsilon}_t^*. \tag{109}$$

APPENDIX C
PROOF OF LEMMA 3 AND THEOREM 4

For brevity, in the following derivation, all the $\mathbf{0}$ and $\mathbf{I}$ without the subscript are the 3×3 matrices.

Propagation matrix: With the state definition of (31), and the propagation functions (12). Similar to the derivation of [10], we can get that the state transition matrix is:

$$ {}^{st}\boldsymbol{\Phi}_{t+1|t} = \begin{bmatrix} {}^{st}\boldsymbol{\Phi}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ {}^{st}\boldsymbol{\Phi}_2 & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ {}^{st}\boldsymbol{\Phi}_3 & \mathbf{I}\Delta & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix}, \tag{110}$$

where

$$ {}^{st}\boldsymbol{\Phi}_1 = {}^L\hat{\mathbf{R}}_{I_{t+1}}^\top {}^L\hat{\mathbf{R}}_{I_t}, $$
$$ {}^{st}\boldsymbol{\Phi}_2 = -({}^L\hat{\mathbf{v}}_{I_{t+1}} - {}^L\hat{\mathbf{v}}_{I_t} + \mathbf{g}\Delta)_\times {}^L\hat{\mathbf{R}}_{I_t}, $$
$$ {}^{st}\boldsymbol{\Phi}_3 = -({}^L\hat{\mathbf{p}}_{I_{t+1}} - {}^L\hat{\mathbf{p}}_{I_t} - {}^L\hat{\mathbf{v}}_{I_t}\Delta + \frac{1}{2}\mathbf{g}\Delta^2)_\times {}^L\hat{\mathbf{R}}_{I_t}. $$

Δ is one time step from timestamp t to timestamp $t+1$.

Jacobian matrix of local feature measurements: For the local feature based observation function (19), the Jacobian matrix is given as

$$ {}^{st}\mathbf{H}_{L_t} = {}^{st}\mathbf{H}_{\pi_t} \left[{}^{st}\mathbf{H}_{L1} \quad \mathbf{0} \quad -{}^L\hat{\mathbf{R}}_{I_t}^\top \quad {}^L\hat{\mathbf{R}}_{I_t}^\top \quad \mathbf{0} \quad \mathbf{0} \right], \tag{111}$$

where ${}^{st}\mathbf{H}_{\pi_t}$ is the Jacobians of the projection function and

$$ {}^{st}\mathbf{H}_{L1} = ({}^L\hat{\mathbf{R}}_{I_t}^\top ({}^L\hat{\mathbf{p}}_{f_t} - {}^L\hat{\mathbf{p}}_{I_t}))_\times. $$

Jacobian matrix of map based measurements: For the map based observation function (21), the Jacobian matrix is given as

$$ {}^{st}\mathbf{H}_{G_t} = {}^{st}\mathbf{H}'_{\pi_t} \left[{}^{st}\mathbf{H}_{G1} \quad \mathbf{0} \quad -{}^L\hat{\mathbf{R}}_{I_t}^\top \quad \mathbf{0} \quad {}^{st}\mathbf{H}_{G2} \quad {}^L\hat{\mathbf{R}}_{I_t}^\top \right], \tag{112}$$

where ${}^{st}\mathbf{H}'_{\pi_t}$ is the Jacobians of the projection function, the superscript $'$ is just for distinguishing from local observation, and

$$ {}^{st}\mathbf{H}_{G1} = \left[{}^L\hat{\mathbf{R}}_{I_t}^\top ({}^L\hat{\mathbf{R}}_{G_t} {}^G\mathbf{p}_F + {}^L\hat{\mathbf{p}}_{G_t} - {}^L\hat{\mathbf{p}}_{I_t}) \right]_\times, $$
$$ {}^{st}\mathbf{H}_{G2} = -{}^L\hat{\mathbf{R}}_{I_t}^\top {}^L\hat{\mathbf{R}}_{G_t} ({}^G\mathbf{p}_F)_\times. $$

Observability matrix: Assume at time step $t-1$, we have a state denoted as $\mathbf{X}_{t-1|t-1}$. After propagation, we get the state $\mathbf{X}_{t|t-1}$. After update, we get the state at time step t, i.e. $\mathbf{X}_{t|t}$.

The following derivation fall into two cases: (a) one is that the prediction state $\mathbf{X}_{t|t-1}$ is equal to update state $\mathbf{X}_{t|t}$, which is ideal situation. In this situation, we denote both $\mathbf{X}_{t|t-1}$ and $\mathbf{X}_{t|t}$ as $\mathbf{X}_t$. (b) The other one is that the prediction state $\mathbf{X}_{t|t-1}$ is not equal to update state $\mathbf{X}_{t|t}$.

- For case (a):

$$
\begin{aligned}
{}^{st}\mathbf{\Phi}_{t|0} &= {}^{st}\mathbf{\Phi}_{t|t-1}\cdots{}^{st}\mathbf{\Phi}_{2|1}{}^{st}\mathbf{\Phi}_{1|0}\\
&= \begin{bmatrix} {}^{st}\mathbf{\Phi}(1) & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ {}^{st}\mathbf{\Phi}(2) & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ {}^{st}\mathbf{\Phi}(3) & \mathbf{I}\Delta_t & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix},
\end{aligned}
\tag{113}
$$

where Δ_t is t time steps from timestamp 0 to timestamp t,

$$
\begin{aligned}
&{}^{st}\mathbf{\Phi}(1) = {}^{L}\hat{\mathbf{R}}_{I_t}^{\top}{}^{L}\hat{\mathbf{R}}_{I_0},\\
&{}^{st}\mathbf{\Phi}(2) = -({}^{L}\hat{\mathbf{v}}_{I_t} - {}^{L}\hat{\mathbf{v}}_{I_0} + \mathbf{g}\Delta_t)_{\times}{}^{L}\hat{\mathbf{R}}_{I_0},\\
&{}^{st}\mathbf{\Phi}(3) = -({}^{L}\hat{\mathbf{p}}_{I_t} - {}^{L}\hat{\mathbf{p}}_{I_0} - {}^{L}\hat{\mathbf{v}}_{I_0}\Delta_t + \frac{1}{2}\mathbf{g}\Delta_t^2)_{\times}{}^{L}\hat{\mathbf{R}}_{I_0}.
\end{aligned}
$$

Taking (111),(112) and (113) into (33), we have the following expressions for $i = 1, \cdots, t$:

$$
\begin{aligned}
{}^{st}\mathbf{M}_{L_i} = {}^{st}\mathbf{H}_{\pi_i}[&{}^{st}\mathbf{M}_{L_i}(1) \quad -{}^{L}\hat{\mathbf{R}}_{I_i}^{\top}\Delta_i \quad -{}^{L}\hat{\mathbf{R}}_{I_i}^{\top} \quad {}^{L}\hat{\mathbf{R}}_{I_i}^{\top}\\
&\mathbf{0} \quad \mathbf{0}],
\end{aligned}
\tag{114}
$$

$$
\begin{aligned}
{}^{st}\mathbf{M}_{G_i} = {}^{st}\mathbf{H}_{\pi_i}^{'}[&{}^{st}\mathbf{M}_{G_i}(1) \quad -{}^{L}\hat{\mathbf{R}}_{I_i}^{\top}\Delta_i \quad -{}^{L}\hat{\mathbf{R}}_{I_i}^{\top} \quad \mathbf{0}\\
&{}^{st}\mathbf{M}_{G_i}(2) \quad {}^{L}\hat{\mathbf{R}}_{I_i}^{\top}],
\end{aligned}
\tag{115}
$$

where

$$
\begin{aligned}
{}^{st}\mathbf{M}_{L_i}(1) &= {}^{L}\hat{\mathbf{R}}_{I_i}^{\top}({}^{L}\hat{\mathbf{p}}_{f_i} - {}^{L}\hat{\mathbf{p}}_{I_0} - {}^{L}\hat{\mathbf{v}}_{I_0}\Delta_i + \frac{1}{2}\mathbf{g}\Delta_i^2)_{\times}{}^{L}\hat{\mathbf{R}}_{I_0},\\
{}^{st}\mathbf{M}_{G_i}(1) &= {}^{L}\hat{\mathbf{R}}_{I_i}^{\top}({}^{L}\hat{\mathbf{R}}_{G_i}{}^{G}\mathbf{p}_F + {}^{L}\hat{\mathbf{p}}_{G_i} - {}^{L}\hat{\mathbf{p}}_{I_0} - {}^{L}\hat{\mathbf{v}}_{I_0}\Delta_i\\
&\quad + \frac{1}{2}\mathbf{g}\Delta_i^2)_{\times}{}^{L}\hat{\mathbf{R}}_{I_0},\\
{}^{st}\mathbf{M}_{G_i}(2) &= -{}^{L}\hat{\mathbf{R}}_{I_i}^{\top}{}^{L}\hat{\mathbf{R}}_{G_i}({}^{G}\mathbf{p}_F)_{\times}.
\end{aligned}
$$

Noting that in case (a), for each timestamp, the state value after propagation is the same as the state value after update, i.e. for the state elements whose differential equation equal to zero, i.e.,${}^{L}\mathbf{p}_f$, ${}^{L}\mathbf{R}_G$, and ${}^{L}\mathbf{p}_G$, their values are always equal to their initial values. Therefore, in the observability matrix above, ${}^{L}\hat{\mathbf{R}}_{G_0} = {}^{L}\hat{\mathbf{R}}_{G_1} = \cdots = {}^{L}\hat{\mathbf{R}}_{G_t} \triangleq {}^{L}\mathbf{R}_G$, so as ${}^{L}\mathbf{p}_f$, ${}^{L}\mathbf{p}_G$.

From above observability matrix, we can find that its null space would be

$$
\mathrm{Null}(\mathbf{M}) = \begin{bmatrix} {}^{L}\hat{\mathbf{R}}_{I_0}^{\top}\mathbf{g} & \mathbf{0}_3\\ -({}^{L}\hat{\mathbf{v}}_{I_0})_{\times}\mathbf{g} & \mathbf{0}_3\\ -({}^{L}\hat{\mathbf{p}}_{I_0})_{\times}\mathbf{g} & \mathbf{I}_3\\ -({}^{L}\hat{\mathbf{p}}_f)_{\times}\mathbf{g} & \mathbf{I}_3\\ {}^{L}\mathbf{R}_G^{\top}\mathbf{g} & \mathbf{0}_3\\ -({}^{L}\mathbf{p}_G)_{\times}\mathbf{g} & \mathbf{I}_3 \end{bmatrix},
\tag{116}
$$

whose dimension is four, so that **Lemma 3** is proved.

It is worth mentioning that from (114) and (115), readers can find another right null space of ${}^{st}\mathbf{M}$ except (116):

$$
\begin{bmatrix} \mathbf{0}_3\\ \mathbf{0}_3\\ \mathbf{0}_3\\ \mathbf{0}_3\\ \mathbf{I}_3\\ {}^{L}\mathbf{R}_G({}^{G}\mathbf{p}_F)_{\times} \end{bmatrix}.
\tag{117}
$$

This is because we only consider one map feature ${}^{G}\mathbf{p}_F$. However, in practice, a map keyframe will observe many features. If there are three or more non-collinear features observed by the map keyframe, the right null space given by (117) vanishes. The similar situation will be encountered in the following analyses in Appendix D, Appendix E and Appendix F. Therefore, in the following context, we assume there are three or more non-collinear map features when deriving the right null space of the observability matrix, whereas only one map feature is considered when deriving Jacobian matrices.

- For case (b): In this situation, the predicted state value is usually not equal with the updated state value. Therefore, the first three columns of the transition matrix $\mathbf{\Phi}_{t|0}$ do not have the elegant form as (113). On the other hand, for case (b), the estimated ${}^{L}\hat{\mathbf{R}}_{G_0} \neq {}^{L}\hat{\mathbf{R}}_{G_1} \neq {}^{L}\hat{\mathbf{R}}_{G_t}$, so as ${}^{L}\hat{\mathbf{p}}_{G_t}$ and ${}^{L}\hat{\mathbf{p}}_{f_t}$. Therefore, the first column of (116) will not be the null space of the observability matrix, and the dimension of the right null space of the observability matrix is three, which is in accordance with **Theorem 4**.

APPENDIX D
PROOF OF THEOREM 5

With (13), the discrete-time state transition matrix can be represented by the matrix exponential map:

$$
\begin{aligned}
{}^{in}\mathbf{\Phi}_{t+1|t} = \exp_m(\mathbf{A}_{\mathbf{X}_{A_t}}\Delta) = \mathbf{I} + \mathbf{A}_{\mathbf{X}_{A_t}}\Delta + \frac{1}{2!}\mathbf{A}_{\mathbf{X}_{A_t}}^2\Delta^2\\
+ \frac{1}{3!}\mathbf{A}_{\mathbf{X}_{A_t}}^3\Delta^3 + \cdots,
\end{aligned}
\tag{118}
$$

where Δ is one time step from timestamp t to timestamp $t+1$.

Fortunately, $\mathbf{A}_{\mathbf{X}_{A_t}}$ is a nilpotent matrix with the degree of three, i.e. $\mathbf{A}_{\mathbf{X}_{A_t}}^3 = \mathbf{0}$, so that we have:

$$
{}^{in}\mathbf{\Phi}_{t+1|t} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ (\mathbf{g})_{\times}\Delta & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \frac{1}{2}(\mathbf{g})_{\times}\Delta^2 & \mathbf{I}\Delta & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix}.
\tag{119}
$$

All the $\mathbf{0}$ and $\mathbf{I}$ in (119) are 3×3 matrices. Similarly, $\mathbf{A}_t$ is a nilpotent matrix with the degree of four.

Substituting (119) into (32), we have

$$
{}^{in}\mathbf{\Phi}_{t|0} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ (\mathbf{g})_{\times}\Delta_t & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \frac{1}{2}(\mathbf{g})_{\times}\Delta_t^2 & \mathbf{I}\Delta_t & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix}.
\tag{120}
$$

For the Jacobian matrix of observation function, with (20) and (22), neglecting the IMU bias parts, we have

$$
\begin{aligned}
{}^{in}\mathbf{H}_{L_t} &= \mathbf{H}_{L_t},\\
{}^{in}\mathbf{H}_{G_t} &= \mathbf{H}_{G_t}.
\end{aligned}
\tag{121}
$$

Taking (120) and (121) into (33), we have the following expressions for $i = 1, \cdots, t$:

$$
\begin{aligned}
{}^{in}\mathbf{M}_{L_i} &= -\nabla h|_{\hat{\mathrm{q}}}\, {}^{L}\hat{\mathbf{R}}_{I_i}^{\top}[\frac{1}{2}(\mathbf{g})_{\times}\Delta_i{}^2 \quad \mathbf{I}_3\Delta_i \quad \mathbf{I}_3 \quad -\mathbf{I}_3 \\
&\qquad \mathbf{0}_3 \quad \mathbf{0}_3], \\
{}^{in}\mathbf{M}_{G_i} &= -\nabla h'|_{\hat{\mathrm{q}}'}\, {}^{L}\hat{\mathbf{R}}_{I_i}^{\top}[-({}^{L}\hat{\mathbf{R}}_{G_i}{}^{G}\mathbf{p}_F)_{\times} + \frac{1}{2}(\mathbf{g})_{\times}\Delta_i^2 \quad \mathbf{I}_3\Delta_i \\
&\qquad \mathbf{I}_3 \quad \mathbf{0}_3 \quad -\mathbf{I}_3 \quad ({}^{L}\hat{\mathbf{R}}_{G_i}{}^{G}\mathbf{p}_F)_{\times}].
\end{aligned} \tag{122}
$$

Readers can verify that (36) is the right null space of ${}^{in}\mathbf{M}$, which is independent from the state estimation. Therefore, our proposed invariant EKF can naturally maintain the correct dimension of the unobservable subspace of the *perfect augmented system*.

Appendix E
Proof of Theorem 6

For ease of understanding, it is recommended that the readers read the following paragraphs in comparison with Appendix C.

As the map information is supposed to be static, the corresponding part of the propagation matrix should be identity. Therefore, we can get the state transition matrix as

$$
{}^{st}\mathbf{\Phi}^*_{t+1|t} = \left[\begin{array}{cccccc|ccc}
{}^{st}\mathbf{\Phi}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
{}^{st}\mathbf{\Phi}_2 & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
{}^{st}\mathbf{\Phi}_3 & \mathbf{I}\Delta & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\hline
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}
\end{array}\right], \tag{123}
$$

where ${}^{st}\mathbf{\Phi}_1, {}^{st}\mathbf{\Phi}_2, {}^{st}\mathbf{\Phi}_3$ are defined in (110).

As the local observation function is not involved with map information, the Jacobian matrix of the local observation function is identical to that in Appendix C. For the global observation, there are two kinds of functions (cf. (27) and (28))[1].

The Jacobian matrix of (27) with standard EKF is given by

$$
\begin{aligned}
{}^{st}\mathbf{H}^*_{G_t,C} ={}^{st}\mathbf{H}'_{\pi_t,C}\Big[&{}^{st}\mathbf{H}^*_{G1} \quad \mathbf{0} \quad -{}^{L}\hat{\mathbf{R}}_{I_t}^{\top} \quad \mathbf{0} \quad {}^{st}\mathbf{H}^*_{G2} \\
&{}^{L}\hat{\mathbf{R}}_{I_t}^{\top}| \quad \mathbf{0} \quad \mathbf{0} \quad {}^{L}\hat{\mathbf{R}}_{I_t}^{\top}\,{}^{L}\hat{\mathbf{R}}_{G_t}\Big],
\end{aligned} \tag{124}
$$

where ${}^{st}\mathbf{H}'_{\pi_t,C}$ is the Jacobians of the projection function. The superscript $'$ is just for distinguishing from local observation.

$$
\begin{aligned}
{}^{st}\mathbf{H}^*_{G1} &= \left[{}^{L}\hat{\mathbf{R}}_{I_t}^{\top}({}^{L}\hat{\mathbf{R}}_{G_t}{}^{G}\hat{\mathbf{p}}_{F,t} + {}^{L}\hat{\mathbf{p}}_{G_t} - {}^{L}\hat{\mathbf{p}}_{I_t})\right]_{\times}, \\
{}^{st}\mathbf{H}^*_{G2} &= -{}^{L}\hat{\mathbf{R}}_{I_t}^{\top}\,{}^{L}\hat{\mathbf{R}}_{G_t}({}^{G}\hat{\mathbf{p}}_{F,t})_{\times}.
\end{aligned}
$$

[1]Here, we neglect the IMU bias parts of (29) and (30) for simplicity. Besides, only one map keyframe and one map landmark are considered, such that ${}^{G}\hat{\mathbf{p}}_{F_j,t}$ in (29) and (30) is simplified as ${}^{G}\hat{\mathbf{p}}_{F,t}$, and ${}^{G}\hat{\mathbf{R}}_{KF_i,t}$ is simplified as ${}^{G}\hat{\mathbf{R}}_{KF,t}$

The Jacobian matrix of (28) with standard EKF is given by

$$
\begin{aligned}
{}^{st}\mathbf{H}^*_{G_t,KF} ={}^{st}\mathbf{H}'_{\pi_t,KF}\,&[\mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0}| \\
&\left[{}^{G}\hat{\mathbf{R}}_{KF,t}^{\top}({}^{G}\hat{\mathbf{p}}_{F,t} - {}^{G}\hat{\mathbf{p}}_{KF,t})\right]_{\times} \quad -{}^{G}\hat{\mathbf{R}}_{KF,t}^{\top} \quad {}^{G}\hat{\mathbf{R}}_{KF,t}^{\top}\Big].
\end{aligned} \tag{125}
$$

Therefore the Jacobian matrix of map based measurements is

$$
{}^{st}\mathbf{H}^*_{G_t} = \begin{bmatrix} {}^{st}\mathbf{H}^*_{G_t,C} \\ {}^{st}\mathbf{H}^*_{G_t,KF} \end{bmatrix}. \tag{126}
$$

Follow the procedure of Appendix C.

- For the ideal case, the components of the observability matrix ${}^{st}\mathbf{M}^*$ are given by

$$
\begin{aligned}
{}^{st}\mathbf{M}^*_{L_i} &= {}^{st}\mathbf{H}_{\pi_i}[{}^{st}\mathbf{M}^*_{L_i}(1) \quad -{}^{L}\hat{\mathbf{R}}_{I_i}^{\top}\Delta_i \quad -{}^{L}\hat{\mathbf{R}}_{I_i}^{\top} \quad {}^{L}\hat{\mathbf{R}}_{I_i}^{\top} \\
&\qquad \mathbf{0} \quad \mathbf{0}| \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0}],
\end{aligned}
$$

$$
\begin{aligned}
{}^{st}\mathbf{M}^*_{G_i,C} ={}^{st}\mathbf{H}'_{\pi_i,C}[&{}^{st}\mathbf{M}^*_{G_i}(1) \quad -{}^{L}\hat{\mathbf{R}}_{I_i}^{\top}\Delta_i \quad -{}^{L}\hat{\mathbf{R}}_{I_i}^{\top} \quad \mathbf{0} \\
&{}^{st}\mathbf{M}^*_{G_i}(2) \quad {}^{L}\hat{\mathbf{R}}_{I_i}^{\top}| \quad \mathbf{0} \quad \mathbf{0} \quad {}^{L}\hat{\mathbf{R}}_{I_i}^{\top}\,{}^{L}\hat{\mathbf{R}}_{G_i}],
\end{aligned}
$$

$$
\begin{aligned}
{}^{st}\mathbf{M}^*_{G_i,KF} ={}^{st}\mathbf{H}'_{\pi_i,KF}[&\mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0}| \\
&{}^{st}\mathbf{M}^*_{G_i}(3) \quad -{}^{G}\hat{\mathbf{R}}_{KF,i}^{\top} \quad {}^{G}\hat{\mathbf{R}}_{KF,i}^{\top}],
\end{aligned}
$$

$$
{}^{st}\mathbf{M}^*_{G_i} = \begin{bmatrix} {}^{st}\mathbf{M}^*_{G_i,C} \\ {}^{st}\mathbf{M}^*_{G_i,KF} \end{bmatrix},
$$

where $i = 1, \cdots, t$, and

$$
\begin{aligned}
{}^{st}\mathbf{M}^*_{L_i}(1) &= {}^{L}\hat{\mathbf{R}}_{I_i}^{\top}({}^{L}\hat{\mathbf{p}}_{f_i} - {}^{L}\hat{\mathbf{p}}_{I_0} - {}^{L}\hat{\mathbf{v}}_{I_0}\Delta_i + \frac{1}{2}\mathbf{g}\Delta_i^2)_{\times}\,{}^{L}\hat{\mathbf{R}}_{I_0}, \\
{}^{st}\mathbf{M}^*_{G_i}(1) &= {}^{L}\hat{\mathbf{R}}_{I_i}^{\top}({}^{L}\hat{\mathbf{R}}_{G_i}{}^{G}\hat{\mathbf{p}}_{F,i} + {}^{L}\hat{\mathbf{p}}_{G_i} - {}^{L}\hat{\mathbf{p}}_{I_0} - {}^{L}\hat{\mathbf{v}}_{I_0}\Delta_i \\
&\qquad + \frac{1}{2}\mathbf{g}\Delta_i^2)_{\times}\,{}^{L}\hat{\mathbf{R}}_{I_0}, \\
{}^{st}\mathbf{M}^*_{G_i}(2) &= -{}^{L}\hat{\mathbf{R}}_{I_i}^{\top}\,{}^{L}\hat{\mathbf{R}}_{G_i}({}^{G}\hat{\mathbf{p}}_{F,i})_{\times}, \\
{}^{st}\mathbf{M}^*_{G_i}(3) &= \left[{}^{G}\hat{\mathbf{R}}_{KF,i}^{\top}({}^{G}\hat{\mathbf{p}}_{F,i} - {}^{G}\hat{\mathbf{p}}_{KF,i})\right]_{\times}.
\end{aligned}
$$

Noting that for the ideal case, all the map-related variables do not change over timestamp. Therefore, ${}^{G}\hat{\mathbf{R}}_{KF,0} ={}^{G}\hat{\mathbf{R}}_{KF,1} = \cdots = {}^{G}\hat{\mathbf{R}}_{KF,t} \triangleq {}^{G}\mathbf{R}_{KF}$, so as ${}^{G}\mathbf{p}_{KF}, {}^{G}\mathbf{p}_F$. At this point, readers can verify that (38) is the right null space of the observability matrix ${}^{st}\mathbf{M}^*$ for the ideal case.

Appendix F
Proof of Theorem 8

Following the derivation of the Appendix D, we can get the state transition matrix as

$$
{}^{in}\mathbf{\Phi}^*_{t+1|t} = \left[\begin{array}{cccccc|ccc}
\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
(\mathbf{g})_{\times}\Delta & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\frac{1}{2}(\mathbf{g})_{\times}\Delta^2 & \mathbf{I}\Delta & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\hline
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}
\end{array}\right], \tag{127}
$$

and

$$
{}^{in}\mathbf{\Phi}^{*}_{t|0}=\left[\begin{array}{cccccc|ccc}
\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\
(\mathbf{g})_{\times}\Delta_t & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\
\frac{1}{2}(\mathbf{g})_{\times}\Delta_t^2 & \mathbf{I}\Delta_t & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\
\hline
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0}\\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0}\\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}
\end{array}\right], \tag{128}
$$

With the Jacobian matrices of local/map feature based observation functions given by (20), (29) and (30)[2], the components of the observability matrix ${}^{in}\mathbf{M}^{*}$ is given by

$$
\begin{aligned}
&{}^{in}\mathbf{M}^{*}_{L_i}=-\nabla h|_{\hat{\mathfrak{q}}}\,{}^{L}\hat{\mathbf{R}}^{\top}_{I_i}[\frac{1}{2}(\mathbf{g})_{\times}\Delta_i{}^2 \quad \mathbf{I}_3\Delta_i \quad \mathbf{I}_3 \quad -\mathbf{I}_3\\
&\qquad \mathbf{0}_3 \quad \mathbf{0}_3| \quad \mathbf{0}_3 \quad \mathbf{0}_3 \quad \mathbf{0}_3],\\
&{}^{in}\mathbf{M}^{*}_{G_i,C}=-\nabla h'|_{\hat{\mathfrak{q}}'}\,{}^{L}\hat{\mathbf{R}}^{\top}_{I_i}[-({}^{L}\hat{\mathbf{R}}_{G_i}\,{}^{G}\hat{\mathbf{p}}_{F,i})_{\times}+\frac{1}{2}(\mathbf{g})_{\times}\Delta_i^2\\
&\quad \mathbf{I}_3\Delta_i \quad \mathbf{I}_3 \quad \mathbf{0}_3 \quad -\mathbf{I}_3 \quad ({}^{L}\hat{\mathbf{R}}_{G_i}\,{}^{G}\hat{\mathbf{p}}_{F,i})_{\times}| \quad \mathbf{0}_3 \quad \mathbf{0}_3 \quad -{}^{L}\hat{\mathbf{R}}_{G_i}],\\
&{}^{in}\mathbf{M}^{*}_{G_i,KF}=-\nabla h^{*}|_{\hat{\mathfrak{q}}^{*}}\,{}^{G}\hat{\mathbf{R}}^{\top}_{KF,i}\left[\mathbf{0}_3 \quad \mathbf{0}_3 \quad \mathbf{0}_3 \quad \mathbf{0}_3 \quad \mathbf{0}_3\right.\\
&\left.\mathbf{0}_3 \quad \mathbf{0}_3 \quad \mathbf{0}_3| -({}^{G}\hat{\mathbf{p}}_{F_i})_{\times} \quad \mathbf{I}_3 \quad -\mathbf{I}_3\right].
\end{aligned} \tag{129}
$$

Again, for the ideal case, all the map-related variables do not change over timestamp. Therefore, ${}^{G}\hat{\mathbf{R}}_{KF,0}={}^{G}\hat{\mathbf{R}}_{KF,1}=\cdots={}^{G}\hat{\mathbf{R}}_{KF,t}\triangleq{}^{G}\mathbf{R}_{KF}$, so as ${}^{G}\mathbf{p}_{KF}, {}^{G}\mathbf{p}_{F}$. At this point, readers can verify that (40) is the right null space of the observability matrix ${}^{in}\mathbf{M}^{*}$ for the ideal case.

[2]Here, we neglect the IMU bias parts of (20), (29) and (30) for simplicity. Besides, only one map keyframe and one map landmark are considered, such that ${}^{G}\hat{\mathbf{p}}_{F_j,t}$ in (29) and (30) becomes ${}^{G}\hat{\mathbf{p}}_{F,t}$, and ${}^{G}\hat{\mathbf{R}}_{KF_i,t}$ becomes ${}^{G}\hat{\mathbf{R}}_{KF,t}$