A Study on Partially Homomorphic Encryption Schemes

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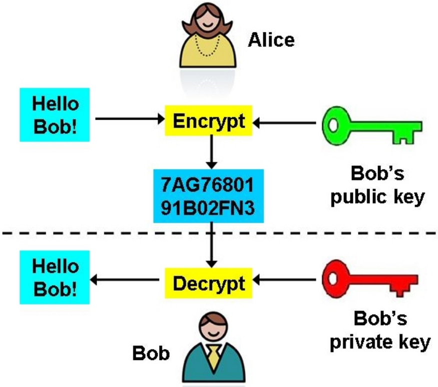
Overview

1. Public key encryption schemes
2. Homomorphic encryption schemes
3. CGS encryption scheme
4. Boosting technique for linearly homomorphic encryption scheme
5. Concluding remarks

Public key encryption scheme

- Symmetric key cryptography:
  $\rightarrow$ one single key used.
- Asymmetric/ Public key cryptography:
  $\rightarrow$ a pair of keys (pk, sk) is used.



Short history of public key encryption

$\rightarrow$ Introduced in 1976, by Diffie and Hellman.
$\rightarrow$ Diffie and Hellman proposed "key-exchange protocol".

RSA scheme: Ron Rivest, Adi Shamir and Leonard Adleman, in 1978

- First public key cryptosystem.
- Based on integer factorization problem.
- Security depends on: Factoring N, computing $\phi(N)$ or computing d.
- Widely used in secure data transmission, mostly in "key agreement" and "digital signature".

ElGamal encryption scheme: Construction

ElGamal scheme: Taher ElGamal, in 1985

- Based on Diffie-Hellman key exchange.
- Implemented on hybrid cryptosystems, PGP, free GNU privacy guard software etc.

KeyGen :

- Input is $(\mathbb{G}, q, g)$.
- Choose a random $a \longleftarrow [1, q-1]$
- Compute g^a
- Outputs are the public key is $\langle \mathbb{G}, q, g, g^a \rangle$ and the private key is $\langle \mathbb{G}, q, g, a \rangle$

ElGamal encryption scheme: Construction

Enc :

- Input a public key $pk = \langle \mathbb{G}, q, g, g^a \rangle$ and a message $m \in \mathbb{G}$
- Choose a random $r \longleftarrow [1, q]$
- Output the ciphertext $(c_1, c_2) := (g^r, (g^a)^r \cdot m)$

Dec :

- Input a private key $sk = \langle \mathbb{G}, q, g, a \rangle$ and a ciphertext (c_1, c_2)
- Output the message $m := c_2 / {c_1}^a$

Security of ElGamal scheme

- Breaking ElGamal $\equiv$ Computational Diffie-Hellman (CDH) problem. (**CDH:** From given (g, g^a, g^b), can we find g^{ab} having no knowledge about a and b?).

- Semantic security of the ElGamal $\equiv$ Decisional Diffie-Hellman (DDH) problem.

  (**DDH:** Can we distinguish between the given tuples (g, g^a, g^b, g^{ab}) and (g, g^a, g^b, g^r), having no knowledge about a and b?).

Effieciency of ElGamal scheme

Table: **Efficiency of ElGamal Encryption Scheme**

Functions	Operations (we denote multiplication by M, pseudo random number generation by PRNG, scalar multiplication by SM, division by D, subtraction by S, addition by A and exponentiation by E)
ElGamal Key Generation	1PRNG + 1E
ElGamal Encrytion	1PRNG + 2E + 1M
ElGamal Decrytion	1E + 1D

Homomorphic encryption scheme

- Certain computations (addition and multiplication) can be performed on the encrypted plaintexts/ ciphertexts.
- Generates a ciphertext such that when decrypted, gives same result from the similar operations performed on the plaintexts.
- Outsourcing computations.
- Implemented in cloud computing, electronic voting protocol, watermarking and fingerprinting, secure multiparty computations etc.
- Message space $\mathcal{M} = (\mathbb{Z}, +, \cdot)$

Properties of homomorphic encryption scheme

Additively homomorphic:

$$\text{Enc}(m_1 + m_2) = \text{Enc}(m_1) \boxplus \text{Enc}(m_2)$$

Multiplicatively homomorphic:

$$\text{Enc}(m_1 \cdot m_2) = \text{Enc}(m_1) \boxdot \text{Enc}(m_2)$$

Scaler multiplication property:

$$\begin{aligned}
\text{Enc}(s \cdot m) &= \text{Enc}(m + m + \cdots + m) \\
&= \text{Enc}(m) \boxplus \text{Enc}(m) \boxplus \cdots \boxplus \text{Enc}(m) \\
&= s \boxdot \text{Enc}(m)
\end{aligned}$$

Different types of homomorphic scheme

Partially homomorphic scheme:

- Allows only one homomorphic property (addition or multiplication but not both).

Fully homomorphic scheme:

- Allows both the homomorphic properties (arbitrary number of additions and multiplications).

Somewhat homomorphic scheme:

- More than partilly homomorphic.
- But Not fully homomorphic.

Examples of homomorphic schemes

- **RSA is partially (multiplicative) homomorphic**
- **ElGamal is partially (multiplicative) homomorphic**

$$\begin{aligned} E(m_1) \boxdot E(m_2) &= (g^{r_1}, (g^a)^{r_1} \cdot m_1) \boxdot (g^{r_2}, (g^a)^{r_2} \cdot m_2) \\ &= (g^{r_1+r_2}, (g^a)^{r_1+r_2} \cdot m_1 \cdot m_2) \\ &= (g^r, (g^a)^r \cdot m_1 \cdot m_2) \\ &= E(m_1 \cdot m_2) \end{aligned}$$

for some $r = r_1 + r_2$

CGS homomorphic encryption scheme

- Cramer, Genarro and Schoenmakers, in 1997.
- Presented as a variant on the ElGamal scheme.
- Consists of four faces: key generation, encryption, evaluation functions and decryption.
  $\mathsf{CGS} = (\mathsf{KeyGen}_{\mathsf{CGS}}, \mathsf{Enc}_{\mathsf{CGS}}, \mathsf{Dec}_{\mathsf{CGS}}, \mathsf{Eval}_{\mathsf{CGS}})$
- Linearly homomorphic scheme.

CGS encryption scheme: Construction

$\text{KeyGen}_{\text{CGS}}$:

- Inputs are security parameter 1^n, group $\mathbb{G}$ and element $g \in \mathbb{G}$.
- Choose a random $a \longleftarrow [1, q-1]$
- Compute g^a
- Outputs are the private key $sk = a$, public key $pk = g^a$

Enc_{CGS} :

- Inputs are public key $G = g^a$, message $m \in [-B, B]$
- Choose a random number $r \in_R [1, q-1]$
- If r is prime then
  compute $x := g^r$
  compute $y := G^r * G^m$
- Output the ciphertext $c = [x, y]$

CGS encryption scheme: Construction

Dec$_{CGS}$:

- Inputs are secret key a, the ciphertext $c = [x, y]$
- Compute $k_1 := x^a$
- Compute $k_2 := y/k_1$
- For $i \in [-B, B]$
  If $G^i == k_2$ then $m = i$
  Otherwise return "error"
- Output the message m

CGS encryption scheme: Homomorphic properties (Eval_{CGS})

Addition (Add_{CGS})

- Inputs are the ciphertext pair $c_1 = \text{Enc}_{\text{CGS}}(m_1) = [x_1, y_1]$ and $c_2 = \text{Enc}_{\text{CGS}}(m_2) = [x_2, y_2]$
- Compute $x := x_1 \cdot x_2$
- Compute $y := y_1 \cdot y_2$
- Output the ciphertext $c = c_1 \boxplus c_2 = [x, y] = \text{Enc}_{\text{CGS}}(m_1 + m_2)$

Scaler multiplication ($SMult_{CGS}$)

- Inputs are the ciphertext $c_1 = [x_1, y_1]$ and a scaler $s \in \mathcal{M}$
- Compute $x := x_1^s$
- Compute $y := y_1^s$
- Output is the ciphertext $s \boxdot c_1 = [x, y] = \text{Enc}_{\text{CGS}}(s \cdot m_1)$

CGS encryption scheme: Homomorphic properties (Eval_{CGS})

Linear Combination ($LinComb_{CGS}$)

- Inputs are a pair of sets (s, c) where $s = [s_1, s_2 .....]$ and $c = [c_1, c_2, ...]$ where each $c_i = [x_i, y_i]$
- Define $k := \#s$
- Choose $x := 1$
- Choose $y := 1$
- For i=1 to k
  Compute $x := x.x_i^{s_i}$
  Compute $y := y.y_i^{s_i}$
- Output is the ciphertext $c = [x, y] = \text{Enc}_{\text{CGS}}(\sum_i s_i \cdot m_i)$

Security of CGS scheme

- Discrete logarithm problem is required to be intractable.
- Computational Diffie-Hellman problem has to be intractable.
- Semantic security of the CGS encryption scheme requires the intractability of the decisional Diffie-Hellman(DDH) problem.
- Not known whether the security of ElGamal and CGS schemes are equivalent or not.

Efficiency of CGS scheme

Table: **Efficiency of CGS Encryption Scheme**

Functions	Operations
CGS Key generation	1 PRNG + 1E
CGS Encrytion	1 PRPNG + 3E + 1M
CGS Decrytion	2E + 1D
CGS Addition	2M
CGS Scalar multiplication	2E
CGS Linear combination ($k = \#s$)	2k E + 2k M

Boosting linearly homomorphic encryption scheme

- Dario Catalano and Dario Fiore, 2015.
- Converts a public-space LHE scheme $\widehat{\mathsf{HE}} = (\widehat{\mathsf{KeyGen}}, \widehat{\mathsf{Enc}}, \widehat{\mathsf{Eval}}, \widehat{\mathsf{Dec}})$ to a HE scheme supporting one multiplication, denoted by $\mathsf{HE_B} = (\mathsf{KeyGen_B}, \mathsf{Enc_B}, \mathsf{Eval_B}, \mathsf{Dec_B})$.
- The message space $\rightarrow$ public ring.
- Claimed to work on virtually all the existing number theoretic LHE such as Paillier, ElGamal or Goldwasser-Micalli.

Boosting LHE scheme: Construction

$\mathsf{KeyGen_B}$:

$\widehat{\mathsf{KeyGen}} = \mathsf{KeyGen_B}$.

$\mathsf{Enc_B}$:

- Inputs are public key $pk = G$, message m
- Choose a random number $b \in_R \mathcal{M}$
- Compute $u = m - b$
- Compute $\beta = \widehat{\mathsf{Enc}}(b)$
- Output the ciphertext $c = [u, \beta]$

Boosting LHE scheme: Construction

Evaluation functions for the Boosted-LHE scheme ($\mathsf{Eval_B}$)

- Ciphertexts are of two levels:
  $\rightarrow$ **Level 1 ciphertext** : encode "fresh" messages/ linear combinations of "fresh" messages.

  $\rightarrow$ **Level 2 ciphertexts** : "multiplied" level 1 ciphertexts.

- Five different evaluation functions:

  $\rightarrow$ Add_1: Addition between two level 1 ciphertexts.

  $\rightarrow$ Mult_1: Multiplication between two level 1 ciphertexts.

  $\rightarrow$ Add_2: Addition between two level 2 ciphertexts.

  $\rightarrow$ SMult_1: Scalar multiplication over a single level 1 ciphertext.

  $\rightarrow$ SMult_2: Scalar multiplication over a single level 2 ciphertext.

Boosting LHE scheme: Homomorphic properties ($\mathsf{Eval}_{\mathsf{B}}$)

Boosted-LHE multiplication function, level 1 (Mult_1)

- Inputs are ciphertexts $c_1 = [u_1, \beta_1]$ and $c_2 = [u_2, \beta_2]$ where $u_1, u_2 \in \mathcal{M}$ and $\beta_1, \beta_2 \in \widehat{C}$
- Compute $\alpha := \widehat{\mathsf{Enc}}(u_1 \cdot u_2) \boxplus (u_1 \boxdot \beta_2) \boxplus (u_2 \boxdot \beta_1)$
- Compute $\beta := \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix}$
- Output is the ciphertext $c = [\alpha, \beta] = \mathsf{Enc}_{\mathsf{B}}(m_1 \cdot m_2)$

Correctness of Mult_1

Theorem

Assume that m_1, m_2 are messages from the message space $\mathcal{M}$ and b_1, b_2 are randomly picked numbers from $\mathcal{M}$. If $c_1 = [u_1, \beta_1] = \mathsf{Enc_B}(m_1)$, $c_2 = [u_2, \beta_2] = \mathsf{Enc_B}(m_2)$ and c is the output of $\mathsf{Mult}_1(c_1, c_2)$, then one can decrypt c and recover $m_1 \cdot m_2$.

Proof.

$$c = \mathsf{Mult}_1(c_1, c_2) = [\alpha, \beta]$$

where

$$\alpha := \widehat{\mathsf{Enc}}(u_1 \cdot u_2) \boxplus (u_1 \boxdot \beta_2) \boxplus (u_2 \boxdot \beta_1)$$
$$\beta := \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix}$$

Correctness of Mult_1

Proof.

$$
\begin{aligned}
\alpha &= \widehat{\text{Enc}}((m_1 - b_1) \cdot (m_2 - b_2)) \boxplus ((m_1 - b_1) \boxdot \widehat{\text{Enc}}(b_2)) \\
&\boxplus ((m_2 - b_2) \boxdot \widehat{\text{Enc}}(b_1)) \\
&= \widehat{\text{Enc}}((m_1 - b_1) \cdot (m_2 - b_2)) \boxplus \widehat{\text{Enc}}((m_1 - b_1) \cdot b_2) \\
&\boxplus \widehat{\text{Enc}}((m_2 - b_2) \cdot b_1)) \\
&= \widehat{\text{Enc}}((m_1 - b_1) \cdot (m_2 - b_2) + ((m_1 - b_1) \cdot b_2) + ((m_2 - b_2) \cdot b_1)) \\
&= \widehat{\text{Enc}}(m_1 m_2 - b_1 m_2 - b_2 m_1 + b_1 b_2 + m_1 b_2 - b_1 b_2 + m_2 b_1 - b_1 b_2) \\
&= \widehat{\text{Enc}}(m_1 m_2 - b_1 b_2)
\end{aligned}
$$

and

$$\beta = (\widehat{\text{Enc}}(b_1), \widehat{\text{Enc}}(b_2))^T$$

Correctness of Mult_1

Proof.

Hence, one can recover $m_1 m_2$ as follows:

$$\begin{aligned}\widehat{\mathsf{Dec}}(\alpha) + \widehat{\mathsf{Dec}}(\beta_1) \cdot \widehat{\mathsf{Dec}}(\beta_2) &= \widehat{\mathsf{Dec}}(\alpha) + \widehat{\mathsf{Dec}}(\widehat{\mathsf{Enc}}(b_1)) \cdot \widehat{\mathsf{Dec}}(\widehat{\mathsf{Enc}}(b_2)) \\ &= (m_1 m_2 - b_1 b_2) + (b_1 \cdot b_2) \\ &= m_1 m_2 - b_1 b_2 + b_1 b_2 \\ &= m_1 m_2\end{aligned}$$

$\square$

Boosting LHE scheme: Homomorphic properties ($\mathsf{Eval_B}$)

Boosted-LHE addition function, level 2 (Add_2)

- Inputs are ciphertexts $c_1 = [\alpha_1, \beta_1] = \mathsf{Enc_B}(m_1)$ and $c_2 = [\alpha_2, \beta_2] = \mathsf{Enc_B}(m_2)$ where $m_1, m_2 \in \mathcal{M}$; $\alpha_1, \alpha_2 \in \widehat{C}$, $\beta_1 \in \widehat{C}^{2l_1}$ and $\beta_2 \in \widehat{C}^{2l_2}$; $\alpha_i = \widehat{\mathsf{Enc}}(m_i - b_i)$; $\beta_i := \begin{pmatrix} {\beta_{11}}^{(i)} & {\beta_{12}}^{(i)} & \dots & {\beta_{1l_i}}^{(i)} \\ {\beta_{21}}^{(i)} & {\beta_{22}}^{(i)} & \dots & {\beta_{2l_i}}^{(i)} \end{pmatrix}$ where ${\beta_{1k}}^{(i)} = \widehat{\mathsf{Enc}}(b_{1k}^{(i)})$, also ${\beta_{2k}}^{(i)} = \widehat{\mathsf{Enc}}(b_{2k}^{(i)})$ for some $b_{1k}^{(i)}, b_{2k}^{(i)} \in \mathcal{M}$ with $1 \le k \le l_i$ and $\sum_{k=1}^{l_i}[{b_{1,k}}^{(i)} \cdot {b_{2,k}}^{(i)}] = b_i$
- Compute $\alpha := \alpha_1 \boxplus \alpha_2$
- Compute $\beta := (\beta_1 || \beta_2) =$ $\begin{pmatrix} {\beta_{11}}^{(1)} & {\beta_{12}}^{(1)} & \dots & {\beta_{1l_1}}^{(1)} & {\beta_{11}}^{(2)} & {\beta_{12}}^{(2)} & \dots & {\beta_{1l_2}}^{(2)} \\ {\beta_{21}}^{(1)} & {\beta_{22}}^{(1)} & \dots & {\beta_{2l_1}}^{(1)} & {\beta_{21}}^{(2)} & {\beta_{22}}^{(2)} & \dots & {\beta_{2l_2}}^{(2)} \end{pmatrix}$
- Output is the ciphertext $c = [\alpha, \beta] = \mathsf{Enc_B}(m_1 + m_2)$

Boosted-LHE addition function, level 2 (Add_2)

Example: Inputs are ciphertexts $c_1 = [\alpha_1, \beta_1] = \mathrm{Enc}_{\mathrm{B}}(m_1)$ and $c_2 = [\alpha_2, \beta_2] = \mathrm{Enc}_{\mathrm{B}}(m_2)$ where $m_1, m_2 \in \mathcal{M}$; $\alpha_1, \alpha_2 \in \widehat{C}$, $\beta_1 \in \widehat{C}^{2l_1}$ and $\beta_2 \in \widehat{C}^{2l_2}$; $\alpha_i = \widehat{\mathrm{Enc}}(m_i - b_i)$;

Here $l_1 = l_2 = 1$. $\beta_1 := \begin{pmatrix} \beta_{11} \\ \beta_{21} \end{pmatrix}$ and $\beta_2 := \begin{pmatrix} \beta_{12} \\ \beta_{22} \end{pmatrix}$ where each $\beta_{jk} = \widehat{\mathrm{Enc}}(b_{jk})$

- Compute $\alpha := \alpha_1 \boxplus \alpha_2$
- Compute $\beta := (\beta_1 || \beta_2) = \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix}$
- Output is the ciphertext $c = [\alpha, \beta] = \mathrm{Enc}_{\mathrm{B}}(m_1 + m_2)$

Boosted-LHE addition function, level 2 (Add_2)

Observe that

$$\begin{aligned}\alpha = \alpha_1 \boxplus \alpha_2 &= \widehat{\text{Enc}}((m_1 - b_1) + (m_2 - b_2)) \\ &= \widehat{\text{Enc}}((m_1 + m_2) - (b_1 + b_2))\end{aligned}$$

where $b_1 + b_2 = [b_{11} \cdot b_{21}] + [b_{12} \cdot b_{22}]$.

Hence, one can recover $m_1 + m_2$ as follows:

$$\begin{aligned}\widehat{\text{Dec}}(\alpha) + \sum_{k=1}^{l_1+l_2} [\widehat{\text{Dec}}(\beta_{1k}) \cdot \widehat{\text{Dec}}(\beta_{2k})] &= \widehat{\text{Dec}}(\alpha) + [\widehat{\text{Dec}}(\beta_{11}) \cdot \widehat{\text{Dec}}(\beta_{21})] \\ &\quad + [\widehat{\text{Dec}}(\beta_{11}) \cdot \widehat{\text{Dec}}(\beta_{21})] \\ &= ((m_1 + m_2) - (b_1 + b_2)) + [b_{11} \cdot b_{21}] + [b_{12} \cdot b_{22}] \\ &= (m_1 + m_2) - (b_1 + b_2) + (b_1 + b_2) \\ &= m_1 + m_2.\end{aligned}$$

Correctness of Add_2

Theorem

If $c_i = [\alpha_i, \beta_i]$ *such that* $\alpha_i = \widehat{\mathrm{Enc}}(m_i - b_i)$ *for some* $b_i \in \mathcal{M}$,
$\beta_i := \begin{pmatrix} {\beta_{11}}^{(i)} & {\beta_{12}}^{(i)} & \dots & {\beta_{1l_i}}^{(i)} \\ {\beta_{21}}^{(i)} & {\beta_{22}}^{(i)} & \dots & {\beta_{2l_i}}^{(i)} \end{pmatrix}$ *where* ${\beta_{1k}}^{(i)} = \widehat{\mathrm{Enc}}(b_{1k}^{(i)})$, *also*
${\beta_{2k}}^{(i)} = \widehat{\mathrm{Enc}}(b_{2k}^{(i)})$ *for some* $b_{1k}^{(i)}, b_{2k}^{(i)} \in \mathcal{M}$ *with* $1 \le k \le l_i$ *and*
$\sum_{k=1}^{l_i} [{b_{1,k}}^{(i)} \cdot {b_{2,k}}^{(i)}] = b_i$, *then* c *can be computed (knowing* pk*) and given* c*, one can decrypt* c *and recover* $m_1 + m_2$ *(knowing* sk*).*

Proof.

$$c = \mathrm{Add}_2(c_1, c_2) = [\alpha, \beta]$$

where

□

Correctness of Add_2

Proof.

$$\alpha := \alpha_1 \boxplus \alpha_2$$

$$\begin{aligned}\beta &:= (\beta_1 || \beta_2) \\ &= \begin{pmatrix} {\beta_{11}}^{(1)} & {\beta_{12}}^{(1)} & \dots & {\beta_{1l_1}}^{(1)} & {\beta_{11}}^{(2)} & {\beta_{12}}^{(2)} & \dots & {\beta_{1l_2}}^{(2)} \\ {\beta_{21}}^{(1)} & {\beta_{22}}^{(1)} & \dots & {\beta_{2l_1}}^{(1)} & {\beta_{21}}^{(2)} & {\beta_{22}}^{(2)} & \dots & {\beta_{2l_2}}^{(2)} \end{pmatrix}\end{aligned}$$

Observe that

$$\begin{aligned}\alpha &= \alpha_1 \boxplus \alpha_2 \\ &= \widehat{\text{Enc}}((m_1 - b_1) + (m_2 - b_2)) \\ &= \widehat{\text{Enc}}((m_1 + m_2) - (b_1 + b_2))\end{aligned}$$

where $b_1 + b_2 = \sum_{k=1}^{l_1}[{b_{1,k}}^{(1)}.{b_{2,k}}^{(1)}] + \sum_{k=1}^{l_2}[{b_{1,k}}^{(2)}.{b_{2,k}}^{(2)}]$. □

Correctness of Add_2

Proof.

Hence, one can recover $m_1 + m_2$ as follows:

$$\widehat{\text{Dec}}(\alpha) + \sum_{k=1}^{l_1+l_2} [\widehat{\text{Dec}}(\beta_{1k}) \cdot \widehat{\text{Dec}}(\beta_{2k})]$$

$$= \widehat{\text{Dec}}(\alpha) + \sum_{k=1}^{l_1} [\widehat{\text{Dec}}({\beta_{1k}}^{(1)}) \cdot \widehat{\text{Dec}}({\beta_{2k}}^{(1)})] + \sum_{k=1}^{l_2} [\widehat{\text{Dec}}({\beta_{1k}}^{(2)}) \cdot \widehat{\text{Dec}}({\beta_{2k}}^{(2)})]$$

$$= ((m_1 + m_2) - (b_1 + b_2)) + \sum_{k=1}^{l_1} [{b_{1,k}}^{(1)} . {b_{2,k}}^{(1)}] + \sum_{k=1}^{l_2} [{b_{1,k}}^{(2)} . {b_{2,k}}^{(2)}]$$

$$= ((m_1 + m_2) - (b_1 + b_2)) + b_1 + b_2$$

$$= m_1 + m_2$$

$\square$

Boosting LHE encryption scheme: Construction

Decryption functions for the Boosted-LHE scheme ($\mathrm{Dec_B}$)

Boosted-LHE Decryption Level 1(Dec1)

- Inputs are ciphertext c, secret key $sk = a$
- Compute $m := u + \widehat{\mathrm{Dec}}(\beta)$
- Output the message m

Boosted-LHE Decryption Level 2(Dec2)

- Inputs are ciphertext c, secret key $sk = a$
- Compute $m := \widehat{\mathrm{Dec}}(\alpha) + \sum_{i=1}^{l}(\widehat{\mathrm{Dec}}(\beta_{1i}).\widehat{\mathrm{Dec}}(\beta_{2i}))$
- Output the message m

Correctness of Dec2

Theorem

If a level 2 ciphertext $c = [\alpha, \beta] \in C$ *is an encryption of* $m \in \mathcal{M}$*, then* $\mathrm{Dec2}(c) = m$*.*

Proof.

$$\mathrm{Dec2}([\widehat{\mathrm{Enc}}(m-b), \begin{pmatrix} \beta_{11} & \beta_{12} & \cdots & \beta_{1l} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2l} \end{pmatrix}])$$

where for each $\beta_{ik} = \widehat{\mathrm{Enc}}(b_{ik})$

$$\widehat{\mathrm{Dec}}(\alpha) + \sum_{i=1}^{l} [\widehat{\mathrm{Dec}}(\beta_{1i}).\widehat{\mathrm{Dec}}(\beta_{2i})]$$

□

Correctness of Dec2

Proof.

$$= m - b + \sum_{i=1}^{l} [\widehat{\mathrm{Dec}}(\widehat{\mathrm{Enc}}(b_{1i})).\widehat{\mathrm{Dec}}(\widehat{\mathrm{Enc}}(b_{2i}))]$$

$$= m - b + \sum_{i=1}^{l} [b_{1i} \cdot b_{2i}]$$

which finally yields $m - b + b$ and thus m. Hence we have, $\mathrm{Dec2}(c) = m$. □

Security of Boosted-LHE scheme

- Semantic security of HE_B depends on the semantic security of the scheme $\widehat{\mathsf{HE}}$.
- If $\widehat{\mathsf{HE}}$ is circuit private, then HE_B is also a leveled circuit private homomorphic encryption.

Efficiency of Boosted-LHE scheme

Table: **Efficiency of Boosted-LHE Encryption Scheme**

Functions	Operations
B-LHE Key generation	same as underlying LHE
B-LHE Encryption	1PRNG + 1S+ 1 LHE encryption
B-LHE Decryption	1A + 1 LHE decryption (for Dec1) and $(2l+1)$ LHE decryption + l M+ l A (for Dec2)
B-LHE Add_1	1A + 1 LHE A
B-LHE Mult_1	1M+ 2 LHE SM + 2 LHE A + 1 LHE encryption
B-LHE Add_2	1 LHE A
B-LHE SMult_1	1M+ 2 LHE SM
B-LHE SMult_2	$(l+1)$ LHE SM

Concluding remarks:

- We studied public key homomorphic encryption schemes: RSA, ElGamal, CGS.
- We studied a boosting technique for linearly homomorphic encryption schemes.
- We have full proofs of correctness.
- We implemented this boosting technique on the CGS scheme.
- We provided MAGMA source codes for CGS scheme and Boosted-CGS schemes.

Future works:

- Fully homomorphic enryption scheme, Gentry, in 2009.
- Full implementations:

  $\rightarrow$ to allow arbitrary multiplications.

  $\rightarrow$ to allow arbitrary additions on higher degree polynomials.
- Boosting multiplicative homomorphic schemes to allow additions.

References

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A secure and Optimally Efficient Multi-authority election Scheme
EUROCRYPT, Lecture notes in Computer Science, Springer-Verlag 1233, 103–118.

Dario Catalano and Dario Fiore
Boosting Linearly-Homomorphic Encryption to Evaluate Degree-2 Functions on Encrypted Data

Thank you

Questions?