Finding limits at $\pm\infty$ without using L'Hopital's Rule

A few important things must be used/recalled:

I. Use the fact that as $x \to \pm\infty$, $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ behaves like $a_n x^n$

II. If n is a positive integer, then $\sqrt[2n]{x^{2n}} = |x^n|$

$\sqrt{x^2} = |x|$ $\quad\quad$ $\sqrt[4]{x^4} = |x|$ $\quad\quad$ $\sqrt[6]{x^6} = |x|$ $\quad\quad$ $\cdots$

III. Let a and b be non-zero constants.

$\lim\limits_{x\to\infty} \dfrac{ax^n}{bx^m} = 0$ If and only if $m > n$

$\lim\limits_{x\to\infty} \dfrac{ax^n}{bx^m} = \dfrac{a}{b}$ If and only if $m = n$

$\lim\limits_{x\to\infty} \dfrac{ax^n}{bx^m} \to \pm\infty$ If and only if $n > m$

IV. $|x| = \begin{cases} x & \text{if } x \ge 0 \\ -x & \text{if } x < 0 \end{cases}$

V. $-1 \le \sin(x) \le 1$ and $-1 \le \cos(x) \le 1$

VI. $\dfrac{\infty}{\infty} \ne 1$ and $\infty - \infty \ne 0$

Examples:

$\lim\limits_{x\to\infty} \dfrac{2x+5}{3x^2+1} \sim \lim\limits_{x\to\infty} \dfrac{2x}{3x^2}$ $= \lim\limits_{x\to\infty} \dfrac{2}{3x}$ $= 0$	$\lim\limits_{x\to\infty} \dfrac{2x^2+5}{3x^2+1} \sim \lim\limits_{x\to\infty} \dfrac{2x^2}{3x^2}$ $= \lim\limits_{x\to\infty} \dfrac{2}{3}$ $= \dfrac{2}{3}$	$\lim\limits_{x\to\infty} \dfrac{2x^2+5}{3x+1} \sim \lim\limits_{x\to\infty} \dfrac{2x^2}{3x}$ $= \lim\limits_{x\to\infty} \dfrac{2x}{3}$ $\to \infty$

$\lim\limits_{x\to\infty} \dfrac{3x-2}{\sqrt{2x^2-1}} \sim \lim\limits_{x\to\infty} \dfrac{3x}{\sqrt{2x^2}}$ $= \lim\limits_{x\to\infty} \dfrac{3x}{\sqrt{2}\sqrt{x^2}}$ $= \lim\limits_{x\to\infty} \dfrac{3x}{\sqrt{2}\,\lvert x\rvert}$ since $x > 0$, $\lvert x\rvert = x$ $= \lim\limits_{x\to\infty} \dfrac{3x}{\sqrt{2}x}$ $= \dfrac{3}{\sqrt{2}}$	$\lim\limits_{x\to\infty} \dfrac{3x-2}{\sqrt{2x^2-1}} \sim \lim\limits_{x\to\infty} \dfrac{3x}{\sqrt{2x^2}}$ $= \lim\limits_{x\to\infty} \dfrac{3x}{\sqrt{2}\sqrt{x^2}}$ $= \lim\limits_{x\to\infty} \dfrac{3x}{\sqrt{2}\,\lvert x\rvert}$ since $x < 0$, $\lvert x\rvert = -x$ $= \lim\limits_{x\to\infty} \dfrac{3x}{\sqrt{2}(-x)}$ $= -\dfrac{3}{\sqrt{2}}$

$\lim_{x\to\infty}\frac{3\sin(2x)}{x} \sim \lim_{x\to\infty}\frac{3(\text{between } -1 \text{ and } 1)}{x}$ $= 0$	$\lim_{x\to\infty}\frac{2}{3x+\cos(x)} \sim \lim_{x\to\infty}\frac{2}{3x}$ $= 0$

In rare instances (not on the AP exam), you will need to rationalize the expression to find the limit.

$$
\begin{aligned}
\lim_{x\to\infty} x-\sqrt{x^2+x} &= \lim_{x\to\infty}\frac{x-\sqrt{x^2+x}}{1}\cdot\frac{x+\sqrt{x^2+x}}{x+\sqrt{x^2+x}} \\
&= \lim_{x\to\infty}\frac{x^2-\left(x^2+x\right)}{x+\sqrt{x^2+x}} \\
&= \lim_{x\to\infty}\frac{-x}{x+\sqrt{x^2+x}} \\
&\sim \lim_{x\to\infty}\frac{-x}{x+\sqrt{x^2}} \\
&= \lim_{x\to\infty}\frac{-x}{x+|x|} \\
\text{since } x>0,\ |x| = x & \\
&= \lim_{x\to\infty}\frac{-x}{x+x} \\
&= \lim_{x\to\infty}\frac{-x}{2x} \\
&= -\frac{1}{2}
\end{aligned}
$$