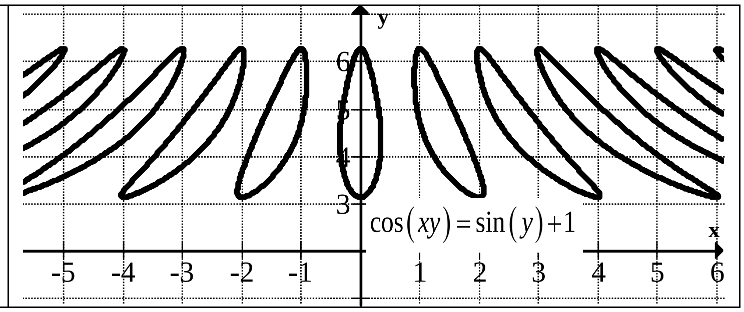
<u>Implicit Differentiation</u>

A graph can be expressed as an explicit function of *x* or an implicit function of *x*:

Explicitly in terms of *x*	**Implicitly in terms of *x***
y is written explicitly in terms of *x* $y = \sqrt{x^2 + 1}$ $y = x^2 + 2x + 1$ $\vdots$	$y = \frac{1}{x}$ can be written implicitly in terms of *x* as $xy = 1$ The function $y = f(x)$ is implied by the equation. $xy = 1$ $x^2 + y^2 = 25$ $\vdots$

It is oftentimes difficult to define *y* in terms of *x* explicitly. To find the derivative of such a relation, we must use ***<u>implicit differentiation.</u>***



In implicit differentiation, we differentiate with respect to *x*, however *<u>we treat each y as</u>* $f(x)$, *<u>and use chain rule on each</u>* $y \leftrightarrow f(x)$.

A few things to remember when executing implicit differentiation:

- ✓ Remember that $y = f(x)$. You have to apply chain rule when differentiating *y*.

$f(x)$	y	$\cos(f(x))$	$\cos(y)$	$[f(x)]^5$	y^5
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$f'(x)$	y'	$-\sin(f(x)) \cdot f'(x)$	$-\sin(y) \cdot y'$	$5[f(x)]^4 \cdot f'(x)$	$5y^4 \cdot y'$

  - o Differentiate a component involving *y* just like *x*, however, you multiply the result of that component by y'
- ✓ If an "*x*" is multiplying a "*y*" then product rule must be used.
  - o
$$xy = 3y^2 \cos(x)$$
$$\downarrow$$
$$1 \cdot y + x \cdot y' = 6y \cdot y' \cos(x) + 3y^2 [-\sin(x)]$$
- ✓ When solving for y' a.k.a. $\frac{dy}{dx}$, or y'' a.k.a. $\frac{d^2y}{dx^2}$ *<u>you must solve in terms of x and y only</u>*.
  - o When finding y'', if $y' = [\cdots]$,then replace every y' with $[\cdots]$ in the expression for y''

When solving for y'

I. Differentiate, and after differentiating distribute wherever possible.
II. Move all terms with y' to one side, and all terms without y' to the other.
III. Factor out the common factor of y'
IV. Divide both sides by the factor of y'

Find y' in terms of x and y only: $\cos(xy) = \sin(y) + 1$

Then find $y'\big|_{(0,0)}$.

$$\cos(xy) = \sin(y) + 1$$
$$\frac{d}{dx}\left[\cos(xy)\right] = \frac{d}{dx}\left[\sin(y) + 1\right]$$
$$-\sin(xy)\cdot(y + x\cdot y') = \cos(y)\cdot y'$$
$$-\sin(xy)\cdot y - \sin(xy)\, x\cdot y' = \cos(y)\cdot y'$$
$$-\sin(xy)\cdot x\cdot y' - \cos(y)\cdot y' = \sin(xy)\cdot y$$
$$y'\left[-\sin(xy)\cdot x - \cos(y)\right] = \sin(xy)\cdot y$$
$$y' = \frac{\sin(xy)\cdot y}{-\sin(xy)\cdot x - \cos(y)}$$

$$y'\big|_{(0,0)} = \frac{\sin(0\cdot 0)\cdot 0}{-\sin(0\cdot 0)\cdot 0 - \cos(0)}$$
$$= \frac{0}{0-1}$$
$$= 0$$

Find $\dfrac{d^2y}{dx^2}$ in terms of x and y only: $x^2 + xy + y^2 = 1$

$$x^2 + xy + y^2 = 1$$
$$\frac{d}{dx}\left[x^2 + xy + y^2\right] = \frac{d}{dx}[1]$$
$$2x + y + x\cdot y' + 2y\cdot y' = 0$$
$$x\cdot y' + 2y\cdot y' = -2x - y$$
$$y'\cdot[x + 2y] = -2x - y$$
$$y' = \frac{-2x - y}{x + 2y}$$

$$y' = \frac{-2x - y}{x + 2y}$$
$$\downarrow$$
$$y'' = \frac{(-2 - y')(x + 2y) - (-2x - y)(1 + 2y')}{(x + 2y)^2}$$
$$y'' = \frac{\left(-2 - \left[\dfrac{-2x - y}{x + 2y}\right]\right)(x + 2y) - (-2x - y)\left(1 + 2\left[\dfrac{-2x - y}{x + 2y}\right]\right)}{(x + 2y)^2}$$