1998 Calculus BC Scoring Guidelines

3. Let f be a function that has derivatives of all orders for all real numbers. Assume $f(0) = 5$, $f'(0) = -3$, $f''(0) = 1$, and $f'''(0) = 4$.
   (a) Write the third–degree Taylor polynomial for f about $x = 0$ and use it to approximate $f(0.2)$.
   (b) Write the fourth–degree Taylor polynomial for g, where $g(x) = f(x^2)$, about $x = 0$.
   (c) Write the third–degree Taylor polynomial for h, where $h(x) = \int_0^x f(t)\,dt$, about $x = 0$.
   (d) Let h be defined as in part (c). Given that $f(1) = 3$, either find the exact value of $h(1)$ or explain why it cannot be determined.

(a) $P_3(f)(x) = 5 - 3x + \frac{1}{2}x^2 + \frac{2}{3}x^3$

$f(0.2) \approx P_3(f)(0.2) =$

$$5 - 3(0.2) + \frac{0.04}{2} + \frac{2(0.008)}{3} =$$

4.425

$$3 \begin{cases} 2\text{: } 5 - 3x + \frac{1}{2}x^2 + \frac{2}{3}x^3 \\ \quad <-1> \text{ each incorrect term,} \\ \quad \text{extra term, or } + \cdots \\ 1\text{: approximates } f(0.2) \end{cases}$$

$<-1>$ for incorrect use of $=$

(b) $P_4(g)(x) = P_2(f)(x^2) = 5 - 3x^2 + \frac{1}{2}x^4$

2: $P_2(f)(x^2)$

$<-1>$ each incorrect or extra term

(c) $P_3(h)(x) = \int_0^x \left(5 - 3t + \frac{1}{2}t^2\right) dt$

$$= \left[5t - \frac{3}{2}t^2 + \frac{1}{6}t^3\right]_0^x$$

$$= 5x - \frac{3}{2}x^2 + \frac{1}{6}x^3$$

$$2 \begin{cases} 1\text{: } P_3(h)(x) = \int_0^x P_2(f)(t)\,dt \\ 1\text{: answer} \\ \quad 0/1 \text{ if any incorrect or extra terms} \end{cases}$$

(d) $h(1) = \int_0^1 f(t)\,dt$

cannot be determined because $f(t)$ is known only for $t = 0$ and $t = 1$

$$2 \begin{cases} 1\text{: } h(1) \text{ cannot be determined} \\ 1\text{: reason} \end{cases}$$

4. The function f has derivatives of all orders for all real numbers x. Assume $f(2) = -3$, $f'(2) = 5$, $f''(2) = 3$, and $f'''(2) = -8$.
   (a) Write the third–degree Taylor polynomial for f about $x = 2$ and use it to approximate $f(1.5)$.
   (b) The fourth derivative of f satisfies the inequality $|f^{(4)}(x)| \leq 3$ for all x in the closed interval $[1.5, 2]$. Use the Lagrange error bound on the approximation to $f(1.5)$ found in part (a) to explain why $f(1.5) \neq -5$.
   (c) Write the fourth–degree Taylor polynomial, $P(x)$, for $g(x) = f(x^2 + 2)$ about $x = 0$. Use P to explain why g must have a relative minimum at $x = 0$.

(a) $T_3(f,2)(x) = -3 + 5(x-2) + \frac{3}{2}(x-2)^2 - \frac{8}{6}(x-2)^3$

$$f(1.5) \approx T_3(f,2)(1.5)$$
$$= -3 + 5(-0.5) + \frac{3}{2}(-0.5)^2 - \frac{4}{3}(-0.5)^3$$
$$= -4.958\overline{3} = -4.958$$

4 $\begin{cases} 3\text{: } T_3(f,2)(x) \\ \quad <-1> \text{ each error} \\ 1\text{: approximation of } f(1.5) \end{cases}$

(b) Lagrange Error Bound $= \frac{3}{4!}|1.5 - 2|^4 = 0.0078125$

$f(1.5) > -4.958\overline{3} - 0.0078125 = -4.966 > -5$

Therefore, $f(1.5) \neq -5$.

2 $\begin{cases} 1\text{: value of Lagrange Error Bound} \\ 1\text{: explanation} \end{cases}$

(c) $P(x) = T_4(g,0)(x)$

$$= T_2(f,2)(x^2+2) = -3 + 5x^2 + \frac{3}{2}x^4$$

The coefficient of x in $P(x)$ is $g'(0)$. This coefficient is 0, so $g'(0) = 0$.

The coefficient of x^2 in $P(x)$ is $\frac{g''(0)}{2!}$. This coefficient is 5, so $g''(0) = 10$ which is greater than 0.

Therefore, g has a relative minimum at $x = 0$.

3 $\begin{cases} 2\text{: } T_4(g,0)(x) \\ \quad <-1> \text{ each incorrect, missing,} \\ \qquad\qquad \text{or extra term} \\ 1\text{: explanation} \end{cases}$

Note:
$<-1>$ max for improper use of $+\ldots$ or equality

AP Calculus BC–3 2000

The Taylor series about $x = 5$ for a certain function f converges to $f(x)$ for all x in the interval of convergence. The nth derivative of f at $x = 5$ is given by $f^{(n)}(5) = \dfrac{(-1)^n n!}{2^n (n+2)}$, and $f(5) = \dfrac{1}{2}$.

(a) Write the third-degree Taylor polynomial for f about $x = 5$.

(b) Find the radius of convergence of the Taylor series for f about $x = 5$.

(c) Show that the sixth-degree Taylor polynomial for f about $x = 5$ approximates $f(6)$ with error less than $\dfrac{1}{1000}$.

(a) $f'(5) = \dfrac{-1!}{2(3)}$, $f''(5) = \dfrac{2!}{4(4)}$, $f'''(5) = \dfrac{-3!}{8(5)}$

$$P_3(f,5)(x) = \frac{1}{2} - \frac{1}{6}(x-5) + \frac{1}{16}(x-5)^2 - \frac{1}{40}(x-5)^3$$

3 : $P_3(f,5)(x)$
<–1> each error or missing term

Note: <–1> max for improper use of extra terms, equality or +…

(b) $a_n = \dfrac{f^{(n)}(5)}{n!} = \dfrac{(-1)^n}{2^n (n+2)}$

$$\lim_{n\to\infty} \left| \frac{\dfrac{(-1)^{n+1}(x-5)^{n+1}}{2^{n+1}(n+3)}}{\dfrac{(-1)^n (x-5)^n}{2^n (n+2)}} \right| = \lim_{n\to\infty} \frac{1}{2}\left(\frac{n+2}{n+3}\right)|x-5|$$

$$= \frac{|x-5|}{2} < 1$$

The radius of convergence is 2.

4 $\begin{cases} 1: & \text{general term} \\ 1: & \text{sets up ratio test} \\ 1: & \text{computes the limit} \\ 1: & \text{applies ratio test to} \\ & \text{get radius of convergence} \end{cases}$

(c) The Taylor series about $x = 5$ for the function f, when evaluated at $x = 6$, is an alternating series with absolute value of terms decreasing to 0. The error in approximating $f(6)$ with the 6th degree Taylor polynomial at $x = 6$ is less than the first omitted term in the series.

$$\left| f(6) - P_6(f,5)(6) \right| \le \frac{1}{2^7(9)} = \frac{1}{1152} < \frac{1}{1000}$$

2 $\begin{cases} 1: & \text{error bound} < \dfrac{1}{1000} \\ 1: & \text{refers to an alternating series} \\ & \text{and indicates the error bound is} \\ & \text{found from the next term} \end{cases}$

Question 6

A function f is defined by

$$f(x) = \frac{1}{3} + \frac{2}{3^2}x + \frac{3}{3^3}x^2 + \cdots + \frac{n+1}{3^{n+1}}x^n + \cdots$$

for all x in the interval of convergence of the given power series.

(a) Find the interval of convergence for this power series. Show the work that leads to your answer.

(b) Find $\displaystyle\lim_{x\to 0} \frac{f(x) - \frac{1}{3}}{x}$.

(c) Write the first three nonzero terms and the general term for an infinite series that represents $\int_0^1 f(x)\,dx$.

(d) Find the sum of the series determined in part (c).

(a) $$\lim_{n\to\infty} \left| \frac{\frac{(n+2)x^{n+1}}{3^{n+2}}}{\frac{(n+1)x^n}{3^{n+1}}} \right| = \lim_{n\to\infty} \left| \frac{(n+2)}{(n+1)} \frac{x}{3} \right| = \left| \frac{x}{3} \right| < 1$$

At $x = -3$, the series is $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{n+1}{3}$, which diverges.

At $x = 3$, the series is $\displaystyle\sum_{n=0}^{\infty} \frac{n+1}{3}$, which diverges.

Therefore, the interval of convergence is $-3 < x < 3$.

$4: \begin{cases} 1: \text{sets up ratio test} \\ 1: \text{computes limit} \\ 1: \text{conclusion of ratio test} \\ 1: \text{endpoint conclusion} \end{cases}$

(b) $$\lim_{x\to 0} \frac{f(x) - \frac{1}{3}}{x} = \lim_{x\to 0} \left(\frac{2}{3^2} + \frac{3}{3^3}x + \frac{4}{3^4}x^2 + \cdots \right) = \frac{2}{9}$$

1 : answer

(c) $$\int_0^1 f(x)\,dx = \int_0^1 \left(\frac{1}{3} + \frac{2}{3^2}x + \frac{3}{3^3}x^2 + \cdots + \frac{n+1}{3^{n+1}}x^n + \cdots \right) dx$$

$$= \left(\frac{1}{3}x + \frac{1}{3^2}x^2 + \frac{1}{3^3}x^3 + \cdots + \frac{1}{3^{n+1}}x^{n+1} + \cdots \right) \Bigg|_{x=0}^{x=1}$$

$$= \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots + \frac{1}{3^{n+1}} + \cdots$$

$3: \begin{cases} 1: \text{antidifferentiation of series} \\ 1: \text{first three terms for definite integral series} \\ 1: \text{general term} \end{cases}$

(d) The series representing $\int_0^1 f(x)\,dx$ is a geometric series.

Therefore, $$\int_0^1 f(x)\,dx = \frac{\frac{1}{3}}{1 - \frac{1}{3}} = \frac{1}{2}.$$

1 : answer

Question 6

The Maclaurin series for the function f is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{(2x)^{n+1}}{n+1} = 2x + \frac{4x^2}{2} + \frac{8x^3}{3} + \frac{16x^4}{4} + \cdots + \frac{(2x)^{n+1}}{n+1} + \cdots$$

on its interval of convergence.

(a) Find the interval of convergence of the Maclaurin series for f. Justify your answer.

(b) Find the first four terms and the general term for the Maclaurin series for $f'(x)$.

(c) Use the Maclaurin series you found in part (b) to find the value of $f'\left(-\frac{1}{3}\right)$.

(a) $$\lim_{n\to\infty}\left|\frac{\frac{(2x)^{n+2}}{n+2}}{\frac{(2x)^{n+1}}{n+1}}\right| = \lim_{n\to\infty}\left|\frac{(n+1)}{(n+2)}2x\right| = \left|2x\right|$$

$\left|2x\right| < 1$ for $-\frac{1}{2} < x < \frac{1}{2}$

At $x = \frac{1}{2}$, the series is $\sum_{n=0}^{\infty} \frac{1}{n+1}$ which diverges since this is the harmonic series.

At $x = -\frac{1}{2}$, the series is $\sum_{n=0}^{\infty} (-1)^{n+1}\frac{1}{n+1}$ which converges by the Alternating Series Test.

Hence, the interval of convergence is $-\frac{1}{2} \le x < \frac{1}{2}$.

5 {
- 1 : sets up ratio
- 1 : computes limit of ratio
- 1 : identifies interior of interval of convergence
- 2 : analysis/conclusion at endpoints
  - 1 : right endpoint
  - 1 : left endpoint
- $< -1 >$ if endpoints not $x = \pm\frac{1}{2}$
- $< -1 >$ if multiple intervals

(b) $f'(x) = 2 + 4x + 8x^2 + 16x^3 + \ldots + 2(2x)^n + \ldots$

2 {
- 1 : first 4 terms
- 1 : general term

(c) The series in (b) is a geometric series.

$$f'\left(-\frac{1}{3}\right) = 2 + 4\left(-\frac{1}{3}\right) + 8\left(-\frac{1}{3}\right)^2 + \ldots + 2\left(2\cdot\left(-\frac{1}{3}\right)\right)^n + \ldots$$

$$= 2 - \frac{4}{3} + \frac{8}{9} - \frac{16}{27} + \ldots + 2\left(-\frac{2}{3}\right)^n + \ldots$$

$$= \frac{2}{1+\frac{2}{3}} = \frac{6}{5}$$

OR

$f'(x) = \frac{2}{1-2x}$ for $-\frac{1}{2} < x < \frac{1}{2}$. Therefore,

$$f'\left(-\frac{1}{3}\right) = \frac{2}{1+\frac{2}{3}} = \frac{6}{5}$$

2 {
- 1 : substitutes $x = -\frac{1}{3}$ into infinite series from (b) or expresses series from (b) in closed form
- 1 : answer for student's series

Question 6

The Maclaurin series for $\ln\left(\frac{1}{1-x}\right)$ is $\sum_{n=1}^{\infty} \frac{x^n}{n}$ with interval of convergence $-1 \le x < 1$.

(a) Find the Maclaurin series for $\ln\left(\frac{1}{1+3x}\right)$ and determine the interval of convergence.

(b) Find the value of $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$.

(c) Give a value of p such that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ converges, but $\sum_{n=1}^{\infty} \frac{1}{n^{2p}}$ diverges. Give reasons why your value of p is correct.

(d) Give a value of p such that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges, but $\sum_{n=1}^{\infty} \frac{1}{n^{2p}}$ converges. Give reasons why your value of p is correct.

(a) $\ln\left(\frac{1}{1+3x}\right) = \ln\left(\frac{1}{1-(-3x)}\right)$

$= \sum_{n=1}^{\infty} \frac{(-3x)^n}{n}$ or $\sum_{n=1}^{\infty} (-1)^n \frac{3^n}{n} x^n$

We must have $-1 \le -3x < 1$, so interval of convergence is $-\frac{1}{3} < x \le \frac{1}{3}$.

$2 \begin{cases} 1 : \text{series} \\ 1 : \text{interval of convergence} \end{cases}$

(b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = \ln\left(\frac{1}{1-(-1)}\right) = \ln\left(\frac{1}{2}\right)$

1 : answer

(c) Some p such that $0 < p \le \frac{1}{2}$ because $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ converges by AST, but the p-series $\sum_{n=1}^{\infty} \frac{1}{n^{2p}}$ diverges for $2p \le 1$.

$3 \begin{cases} 1 : \text{correct } p \\ 1 : \text{reason why } \sum \frac{(-1)^n}{n^p} \text{ converges} \\ 1 : \text{reason why } \sum \frac{1}{n^{2p}} \text{ diverges} \end{cases}$

(d) Some p such that $\frac{1}{2} < p \le 1$ because the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges for $p \le 1$ and the p-series $\sum_{n=1}^{\infty} \frac{1}{n^{2p}}$ converges for $2p > 1$.

$3 \begin{cases} 1 : \text{correct } p \\ 1 : \text{reason why } \sum \frac{1}{n^p} \text{ diverges} \\ 1 : \text{reason why } \sum \frac{1}{n^{2p}} \text{ converges} \end{cases}$

Question 6

The function f is defined by the power series

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots + \frac{(-1)^n x^{2n}}{(2n+1)!} + \cdots$$

for all real numbers x.

(a) Find $f'(0)$ and $f''(0)$. Determine whether f has a local maximum, a local minimum, or neither at $x = 0$. Give a reason for your answer.

(b) Show that $1 - \frac{1}{3!}$ approximates $f(1)$ with error less than $\frac{1}{100}$.

(c) Show that $y = f(x)$ is a solution to the differential equation $xy' + y = \cos x$.

(a) $f'(0) =$ coefficient of x term $= 0$

$f''(0) = 2$ (coefficient of x^2 term) $= 2\left(-\frac{1}{3!}\right) = -\frac{1}{3}$

f has a local maximum at $x = 0$ because $f'(0) = 0$ and $f''(0) < 0$.

$$4 : \begin{cases} 1 : f'(0) \\ 1 : f''(0) \\ 1 : \text{critical point answer} \\ 1 : \text{reason} \end{cases}$$

(b) $f(1) = 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots + \frac{(-1)^n}{(2n+1)!} + \cdots$

This is an alternating series whose terms decrease in absolute value with limit 0. Thus, the error is less than the first omitted term, so $\left| f(1) - \left(1 - \frac{1}{3!}\right) \right| \le \frac{1}{5!} = \frac{1}{120} < \frac{1}{100}$.

$1 :$ error bound $< \frac{1}{100}$

(c)

$$y' = -\frac{2x}{3!} + \frac{4x^3}{5!} - \frac{6x^5}{7!} + \cdots + \frac{(-1)^n 2nx^{2n-1}}{(2n+1)!} + \cdots$$

$$xy' = -\frac{2x^2}{3!} + \frac{4x^4}{5!} - \frac{6x^6}{7!} + \cdots + \frac{(-1)^n 2nx^{2n}}{(2n+1)!} + \cdots$$

$$\begin{aligned} xy' + y &= 1 - \left(\frac{2}{3!} + \frac{1}{3!}\right)x^2 + \left(\frac{4}{5!} + \frac{1}{5!}\right)x^4 - \left(\frac{6}{7!} + \frac{1}{7!}\right)x^6 + \cdots \\ &\quad + (-1)^n\left(\frac{2n}{(2n+1)!} + \frac{1}{(2n+1)!}\right)x^{2n} + \cdots \\ &= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots + \frac{(-1)^n}{(2n)!}x^{2n} + \cdots \\ &= \cos x \end{aligned}$$

$$4 : \begin{cases} 1 : \text{series for } y' \\ 1 : \text{series for } xy' \\ 1 : \text{series for } xy' + y \\ 1 : \text{identifies series as } \cos x \end{cases}$$

OR

$$xy = xf(x) = x - \frac{x^3}{3!} + \cdots + (-1)^n \frac{1}{(2n+1)!}x^{2n+1} + \cdots$$

$$= \sin x$$

$$xy' + y = (xy)' = (\sin x)' = \cos x$$

OR

$$4 : \begin{cases} 1 : \text{series for } xf(x) \\ 1 : \text{identifies series as } \sin x \\ 1 : \text{handles } xy' + y \\ 1 : \text{makes connection} \end{cases}$$

Question 6

The function f has a Taylor series about $x = 2$ that converges to $f(x)$ for all x in the interval of convergence. The nth derivative of f at $x = 2$ is given by $f^{(n)}(2) = \dfrac{(n+1)!}{3^n}$ for $n \ge 1$, and $f(2) = 1$.

(a) Write the first four terms and the general term of the Taylor series for f about $x = 2$.

(b) Find the radius of convergence for the Taylor series for f about $x = 2$. Show the work that leads to your answer.

(c) Let g be a function satisfying $g(2) = 3$ and $g'(x) = f(x)$ for all x. Write the first four terms and the general term of the Taylor series for g about $x = 2$.

(d) Does the Taylor series for g as defined in part (c) converge at $x = -2$? Give a reason for your answer.

(a) $f(2) = 1$; $f'(2) = \dfrac{2!}{3}$; $f''(2) = \dfrac{3!}{3^2}$; $f'''(2) = \dfrac{4!}{3^3}$

$$f(x) = 1 + \frac{2}{3}(x-2) + \frac{3!}{2!3^2}(x-2)^2 + \frac{4!}{3!3^3}(x-2)^3 + \cdots + \frac{(n+1)!}{n!3^n}(x-2)^n + \cdots$$

$$= 1 + \frac{2}{3}(x-2) + \frac{3}{3^2}(x-2)^2 + \frac{4}{3^3}(x-2)^3 + \cdots + \frac{n+1}{3^n}(x-2)^n + \cdots$$

$3: \begin{cases} 1: \text{coefficients } \dfrac{f^{(n)}(2)}{n!} \text{ in first four terms} \\ 1: \text{powers of } (x-2) \text{ in first four terms} \\ 1: \text{general term} \end{cases}$

(b) $$\lim_{n\to\infty} \left| \frac{\frac{n+2}{3^{n+1}}(x-2)^{n+1}}{\frac{n+1}{3^n}(x-2)^n} \right| = \lim_{n\to\infty} \frac{n+2}{n+1} \cdot \frac{1}{3}|x-2|$$

$= \dfrac{1}{3}|x-2| < 1$ when $|x-2| < 3$

The radius of convergence is 3.

$3: \begin{cases} 1: \text{sets up ratio} \\ 1: \text{limit} \\ 1: \text{applies ratio test to conclude radius of convergence is 3} \end{cases}$

(c) $g(2) = 3$; $g'(2) = f(2)$; $g''(2) = f'(2)$; $g'''(2) = f''(2)$

$$g(x) = 3 + (x-2) + \frac{1}{3}(x-2)^2 + \frac{1}{3^2}(x-2)^3 + \cdots + \frac{1}{3^n}(x-2)^{n+1} + \cdots$$

$2: \begin{cases} 1: \text{first four terms} \\ 1: \text{general term} \end{cases}$

(d) No, the Taylor series does not converge at $x = -2$ because the geometric series only converges on the interval $|x-2| < 3$.

1 : answer with reason

Question 6

Let f be the function given by $f(x) = \sin\left(5x + \frac{\pi}{4}\right)$, and let $P(x)$ be the third-degree Taylor polynomial for f about $x = 0$.

(a) Find $P(x)$.

(b) Find the coefficient of x^{22} in the Taylor series for f about $x = 0$.

(c) Use the Lagrange error bound to show that $\left|f\left(\frac{1}{10}\right) - P\left(\frac{1}{10}\right)\right| < \frac{1}{100}$.

(d) Let G be the function given by $G(x) = \int_0^x f(t)\,dt$. Write the third-degree Taylor polynomial for G about $x = 0$.

(a)
$$f(0) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$
$$f'(0) = 5\cos\left(\frac{\pi}{4}\right) = \frac{5\sqrt{2}}{2}$$
$$f''(0) = -25\sin\left(\frac{\pi}{4}\right) = -\frac{25\sqrt{2}}{2}$$
$$f'''(0) = -125\cos\left(\frac{\pi}{4}\right) = -\frac{125\sqrt{2}}{2}$$
$$P(x) = \frac{\sqrt{2}}{2} + \frac{5\sqrt{2}}{2}x - \frac{25\sqrt{2}}{2(2!)}x^2 - \frac{125\sqrt{2}}{2(3!)}x^3$$

4 : $P(x)$

$\langle -1 \rangle$ each error or missing term

deduct only once for $\sin\left(\frac{\pi}{4}\right)$ evaluation error

deduct only once for $\cos\left(\frac{\pi}{4}\right)$ evaluation error

$\langle -1 \rangle$ max for all extra terms, $+\cdots$, misuse of equality

(b)
$$\frac{-5^{22}\sqrt{2}}{2(22!)}$$

2 : $\begin{cases} 1 : \text{magnitude} \\ 1 : \text{sign} \end{cases}$

(c)
$$\left|f\left(\frac{1}{10}\right) - P\left(\frac{1}{10}\right)\right| \le \max_{0 \le c \le \frac{1}{10}} \left|f^{(4)}(c)\right| \left(\frac{1}{4!}\right)\left(\frac{1}{10}\right)^4$$
$$\le \frac{625}{4!}\left(\frac{1}{10}\right)^4 = \frac{1}{384} < \frac{1}{100}$$

1 : error bound in an appropriate inequality

(d) The third-degree Taylor polynomial for G about $x = 0$ is
$$\int_0^x \left(\frac{\sqrt{2}}{2} + \frac{5\sqrt{2}}{2}t - \frac{25\sqrt{2}}{4}t^2\right) dt$$
$$= \frac{\sqrt{2}}{2}x + \frac{5\sqrt{2}}{4}x^2 - \frac{25\sqrt{2}}{12}x^3$$

2 : third-degree Taylor polynomial for G about $x = 0$

$\langle -1 \rangle$ each incorrect or missing term

$\langle -1 \rangle$ max for all extra terms, $+\cdots$, misuse of equality

Question 2

Let f be a function having derivatives of all orders for all real numbers. The third-degree Taylor polynomial for f about $x = 2$ is given by $T(x) = 7 - 9(x-2)^2 - 3(x-2)^3$.

(a) Find $f(2)$ and $f''(2)$.

(b) Is there enough information given to determine whether f has a critical point at $x = 2$? If not, explain why not. If so, determine whether $f(2)$ is a relative maximum, a relative minimum, or neither, and justify your answer.

(c) Use $T(x)$ to find an approximation for $f(0)$. Is there enough information given to determine whether f has a critical point at $x = 0$? If not, explain why not. If so, determine whether $f(0)$ is a relative maximum, a relative minimum, or neither, and justify your answer.

(d) The fourth derivative of f satisfies the inequality $\left|f^{(4)}(x)\right| \le 6$ for all x in the closed interval $[0, 2]$. Use the Lagrange error bound on the approximation to $f(0)$ found in part (c) to explain why $f(0)$ is negative.

(a) $f(2) = T(2) = 7$ $\dfrac{f''(2)}{2!} = -9$ so $f''(2) = -18$	$2: \begin{cases} 1: f(2) = 7 \\ 1: f''(2) = -18 \end{cases}$
(b) Yes, since $f'(2) = T'(2) = 0$, f does have a critical point at $x = 2$. Since $f''(2) = -18 < 0$, $f(2)$ is a relative maximum value.	$2: \begin{cases} 1: \text{states } f'(2) = 0 \\ 1: \text{declares } f(2) \text{ as a relative maximum because } f''(2) < 0 \end{cases}$
(c) $f(0) \approx T(0) = -5$ It is not possible to determine if f has a critical point at $x = 0$ because $T(x)$ gives exact information only at $x = 2$.	$3: \begin{cases} 1: f(0) \approx T(0) = -5 \\ 1: \text{declares that it is not possible to determine} \\ 1: \text{reason} \end{cases}$
(d) Lagrange error bound $= \dfrac{6}{4!}\lvert 0 - 2\rvert^4 = 4$ $f(0) \le T(0) + 4 = -1$ Therefore, $f(0)$ is negative.	$2: \begin{cases} 1: \text{value of Lagrange error bound} \\ 1: \text{explanation} \end{cases}$

Question 6

Let f be a function with derivatives of all orders and for which $f(2) = 7$. When n is odd, the nth derivative of f at $x = 2$ is 0. When n is even and $n \geq 2$, the nth derivative of f at $x = 2$ is given by $f^{(n)}(2) = \frac{(n-1)!}{3^n}$.

(a) Write the sixth-degree Taylor polynomial for f about $x = 2$.

(b) In the Taylor series for f about $x = 2$, what is the coefficient of $(x-2)^{2n}$ for $n \geq 1$?

(c) Find the interval of convergence of the Taylor series for f about $x = 2$. Show the work that leads to your answer.

(a) $P_6(x) = 7 + \frac{1!}{3^2} \cdot \frac{1}{2!}(x-2)^2 + \frac{3!}{3^4} \cdot \frac{1}{4!}(x-2)^4 + \frac{5!}{3^6} \cdot \frac{1}{6!}(x-2)^6$

$$3 : \begin{cases} 1 : \text{polynomial about } x = 2 \\ 2 : P_6(x) \\ \quad \langle -1 \rangle \text{ each incorrect term} \\ \quad \langle -1 \rangle \text{ max for all extra terms,} \\ \qquad + \cdots, \text{ misuse of equality} \end{cases}$$

(b) $\frac{(2n-1)!}{3^{2n}} \cdot \frac{1}{(2n)!} = \frac{1}{3^{2n}(2n)}$

1 : coefficient

(c) The Taylor series for f about $x = 2$ is

$$f(x) = 7 + \sum_{n=1}^{\infty} \frac{1}{2n \cdot 3^{2n}}(x-2)^{2n}.$$

$$L = \lim_{n\to\infty} \left| \frac{\frac{1}{2(n+1)} \cdot \frac{1}{3^{2(n+1)}}(x-2)^{2(n+1)}}{\frac{1}{2n} \cdot \frac{1}{3^{2n}}(x-2)^{2n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{2n}{2(n+1)} \cdot \frac{3^{2n}}{3^2 3^{2n}}(x-2)^2 \right| = \frac{(x-2)^2}{9}$$

$L < 1$ when $|x-2| < 3$.

Thus, the series converges when $-1 < x < 5$.

When $x = 5$, the series is $7 + \sum_{n=1}^{\infty} \frac{3^{2n}}{2n \cdot 3^{2n}} = 7 + \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}$,

which diverges, because $\sum_{n=1}^{\infty} \frac{1}{n}$, the harmonic series, diverges.

When $x = -1$, the series is $7 + \sum_{n=1}^{\infty} \frac{(-3)^{2n}}{2n \cdot 3^{2n}} = 7 + \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}$,

which diverges, because $\sum_{n=1}^{\infty} \frac{1}{n}$, the harmonic series, diverges.

The interval of convergence is $(-1, 5)$.

$$5 : \begin{cases} 1 : \text{sets up ratio} \\ 1 : \text{computes limit of ratio} \\ 1 : \text{identifies interior of} \\ \quad \text{interval of convergence} \\ 1 : \text{considers both endpoints} \\ 1 : \text{analysis/conclusion for} \\ \quad \text{both endpoints} \end{cases}$$

Question 3

The Taylor series about $x = 0$ for a certain function f converges to $f(x)$ for all x in the interval of convergence. The nth derivative of f at $x = 0$ is given by

$$f^{(n)}(0) = \frac{(-1)^{n+1}(n+1)!}{5^n (n-1)^2} \text{ for } n \ge 2.$$

The graph of f has a horizontal tangent line at $x = 0$, and $f(0) = 6$.

(a) Determine whether f has a relative maximum, a relative minimum, or neither at $x = 0$. Justify your answer.

(b) Write the third-degree Taylor polynomial for f about $x = 0$.

(c) Find the radius of convergence of the Taylor series for f about $x = 0$. Show the work that leads to your answer.

(a) f has a relative maximum at $x = 0$ because $f'(0) = 0$ and $f''(0) < 0$.

$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{reason} \end{cases}$

(b) $f(0) = 6,\ f'(0) = 0$

$$f''(0) = -\frac{3!}{5^2\, 1^2} = -\frac{6}{25},\quad f'''(0) = \frac{4!}{5^3\, 2^2}$$

$$P(x) = 6 - \frac{3!x^2}{5^2\, 2!} + \frac{4!x^3}{5^3\, 2^2\, 3!} = 6 - \frac{3}{25}x^2 + \frac{1}{125}x^3$$

3 : $P(x)$
$\langle -1 \rangle$ each incorrect term
Note: $\langle -1 \rangle$ max for use of extra terms

(c) $$u_n = \frac{f^{(n)}(0)}{n!}x^n = \frac{(-1)^{n+1}(n+1)}{5^n (n-1)^2}x^n$$

$$\left|\frac{u_{n+1}}{u_n}\right| = \left| \frac{\dfrac{(-1)^{n+2}(n+2)}{5^{n+1}n^2}x^{n+1}}{\dfrac{(-1)^{n+1}(n+1)}{5^n (n-1)^2}x^n} \right|$$

$$= \left(\frac{n+2}{n+1}\right)\left(\frac{n-1}{n}\right)^2 \frac{1}{5}|x|$$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \frac{1}{5}|x| < 1 \text{ if } |x| < 5.$$

The radius of convergence is 5.

$4 : \begin{cases} 1 : \text{general term} \\ 1 : \text{sets up ratio} \\ 1 : \text{computes limit} \\ 1 : \text{applies ratio test to get radius of convergence} \end{cases}$

Question 6

The function f is defined by the power series

$$f(x) = -\frac{x}{2} + \frac{2x^2}{3} - \frac{3x^3}{4} + \cdots + \frac{(-1)^n nx^n}{n+1} + \cdots$$

for all real numbers x for which the series converges. The function g is defined by the power series

$$g(x) = 1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \cdots + \frac{(-1)^n x^n}{(2n)!} + \cdots$$

for all real numbers x for which the series converges.

(a) Find the interval of convergence of the power series for f. Justify your answer.

(b) The graph of $y = f(x) - g(x)$ passes through the point $(0, -1)$. Find $y'(0)$ and $y''(0)$. Determine whether y has a relative minimum, a relative maximum, or neither at $x = 0$. Give a reason for your answer.

(a) $$\left|\frac{(-1)^{n+1}(n+1)x^{n+1}}{n+2} \cdot \frac{n+1}{(-1)^n nx^n}\right| = \frac{(n+1)^2}{(n+2)(n)} \cdot |x|$$

$$\lim_{n\to\infty} \frac{(n+1)^2}{(n+2)(n)} \cdot |x| = |x|$$

The series converges when $-1 < x < 1$.

When $x = 1$, the series is $-\frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \cdots$

This series does not converge, because the limit of the individual terms is not zero.

When $x = -1$, the series is $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots$

This series does not converge, because the limit of the individual terms is not zero.

Thus, the interval of convergence is $-1 < x < 1$.

$$5: \begin{cases} 1: \text{sets up ratio} \\ 1: \text{computes limit of ratio} \\ 1: \text{identifies radius of convergence} \\ 1: \text{considers both endpoints} \\ 1: \text{analysis/conclusion for both endpoints} \end{cases}$$

(b) $f'(x) = -\frac{1}{2} + \frac{4}{3}x - \frac{9}{4}x^2 + \cdots$ and $f'(0) = -\frac{1}{2}$.

$g'(x) = -\frac{1}{2!} + \frac{2}{4!}x - \frac{3}{6!}x^2 + \cdots$ and $g'(0) = -\frac{1}{2}$.

$y'(0) = f'(0) - g'(0) = 0$

$f''(0) = \frac{4}{3}$ and $g''(0) = \frac{2}{4!} = \frac{1}{12}$.

Thus, $y''(0) = \frac{4}{3} - \frac{1}{12} > 0$.

Since $y'(0) = 0$ and $y''(0) > 0$, y has a relative minimum at $x = 0$.

$$4: \begin{cases} 1: y'(0) \\ 1: y''(0) \\ 1: \text{conclusion} \\ 1: \text{reasoning} \end{cases}$$

Question 6

The function f is defined by $f(x) = \dfrac{1}{1 + x^3}$. The Maclaurin series for f is given by

$$1 - x^3 + x^6 - x^9 + \cdots + (-1)^n x^{3n} + \cdots,$$

which converges to $f(x)$ for $-1 < x < 1$.

(a) Find the first three nonzero terms and the general term for the Maclaurin series for $f'(x)$.

(b) Use your results from part (a) to find the sum of the infinite series $-\dfrac{3}{2^2} + \dfrac{6}{2^5} - \dfrac{9}{2^8} + \cdots + (-1)^n \dfrac{3n}{2^{3n-1}} + \cdots$.

(c) Find the first four nonzero terms and the general term for the Maclaurin series representing $\int_0^x f(t)\, dt$.

(d) Use the first three nonzero terms of the infinite series found in part (c) to approximate $\int_0^{1/2} f(t)\, dt$. What are the properties of the terms of the series representing $\int_0^{1/2} f(t)\, dt$ that guarantee that this approximation is within $\dfrac{1}{10{,}000}$ of the exact value of the integral?

(a) $f'(x) = -3x^2 + 6x^5 - 9x^8 + \cdots + 3n(-1)^n x^{3n-1} + \cdots$	2 : { 1 : first three terms; 1 : general term }
(b) The given series is the Maclaurin series for $f'(x)$ with $x = \frac{1}{2}$. $f'(x) = -\left(1 + x^3\right)^{-2}\left(3x^2\right)$. Thus, the sum of the series is $f'\left(\frac{1}{2}\right) = -\dfrac{3\left(\frac{1}{4}\right)}{\left(1 + \frac{1}{8}\right)^2} = -\dfrac{16}{27}$.	2 : { 1 : $f'(x)$; 1 : $f'\left(\frac{1}{2}\right)$ }
(c) $\displaystyle\int_0^x \frac{1}{1 + t^3}\, dt = x - \frac{x^4}{4} + \frac{x^7}{7} - \frac{x^{10}}{10} + \cdots + \frac{(-1)^n x^{3n+1}}{3n + 1} + \cdots$	2 : { 1 : first four terms; 1 : general term }
(d) $\displaystyle\int_0^{1/2} \frac{1}{1 + t^3}\, dt \approx \frac{1}{2} - \frac{\left(\frac{1}{2}\right)^4}{4} + \frac{\left(\frac{1}{2}\right)^7}{7}$. The series in part (c) with $x = \frac{1}{2}$ has terms that alternate, decrease in absolute value, and have limit 0. Hence the error is bounded by the absolute value of the next term. $\displaystyle\left\| \int_0^{1/2} \frac{1}{1 + t^3}\, dt - \left(\frac{1}{2} - \frac{\left(\frac{1}{2}\right)^4}{4} + \frac{\left(\frac{1}{2}\right)^7}{7} \right) \right\| < \frac{\left(\frac{1}{2}\right)^{10}}{10} = \frac{1}{10240} < 0.0001$	3 : { 1 : approximation; 1 : properties of terms; 1 : absolute value of fourth term < 0.0001 }

Question 6

Let f be the function given by $f(x) = e^{-x^2}$.

(a) Write the first four nonzero terms and the general term of the Taylor series for f about $x = 0$.

(b) Use your answer to part (a) to find $\lim\limits_{x\to 0} \dfrac{1 - x^2 - f(x)}{x^4}$.

(c) Write the first four nonzero terms of the Taylor series for $\int_0^x e^{-t^2}\,dt$ about $x = 0$. Use the first two terms of your answer to estimate $\int_0^{1/2} e^{-t^2}\,dt$.

(d) Explain why the estimate found in part (c) differs from the actual value of $\int_0^{1/2} e^{-t^2}\,dt$ by less than $\dfrac{1}{200}$.

(a) $$e^{-x^2} = 1 + \frac{\left(-x^2\right)}{1!} + \frac{\left(-x^2\right)^2}{2!} + \frac{\left(-x^2\right)^3}{3!} + \cdots + \frac{\left(-x^2\right)^n}{n!} + \cdots$$
$$= 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \cdots + \frac{(-1)^n x^{2n}}{n!} + \cdots$$

$3 : \begin{cases} 1 : \text{two of } 1, -x^2, \dfrac{x^4}{2}, -\dfrac{x^6}{6} \\ 1 : \text{remaining terms} \\ 1 : \text{general term} \end{cases}$

(b) $$\frac{1 - x^2 - f(x)}{x^4} = -\frac{1}{2} + \frac{x^2}{6} + \sum_{n=4}^{\infty} \frac{(-1)^{n+1} x^{2n-4}}{n!}$$

Thus, $\lim\limits_{x\to 0}\left(\dfrac{1 - x^2 - f(x)}{x^4}\right) = -\dfrac{1}{2}$.

1 : answer

(c) $$\int_0^x e^{-t^2}\,dt = \int_0^x \left(1 - t^2 + \frac{t^4}{2} - \frac{t^6}{6} + \cdots + \frac{(-1)^n t^{2n}}{n!} + \cdots\right) dt$$
$$= x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \cdots$$

Using the first two terms of this series, we estimate that
$\int_0^{1/2} e^{-t^2}\,dt \approx \dfrac{1}{2} - \left(\dfrac{1}{3}\right)\left(\dfrac{1}{8}\right) = \dfrac{11}{24}$.

$3 : \begin{cases} 1 : \text{two terms} \\ 1 : \text{remaining terms} \\ 1 : \text{estimate} \end{cases}$

(d) $\left|\int_0^{1/2} e^{-t^2}\,dt - \dfrac{11}{24}\right| < \left(\dfrac{1}{2}\right)^5 \cdot \dfrac{1}{10} = \dfrac{1}{320} < \dfrac{1}{200}$, since

$\int_0^{1/2} e^{-t^2}\,dt = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{2}\right)^{2n+1}}{n!(2n+1)}$, which is an alternating series with individual terms that decrease in absolute value to 0.

$2 : \begin{cases} 1 : \text{uses the third term as the error bound} \\ 1 : \text{explanation} \end{cases}$

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Question 6

Let f be the function given by $f(x) = 6e^{-x/3}$ for all x.

(a) Find the first four nonzero terms and the general term for the Taylor series for f about $x = 0$.

(b) Let g be the function given by $g(x) = \int_0^x f(t)\,dt$. Find the first four nonzero terms and the general term for the Taylor series for g about $x = 0$.

(c) The function h satisfies $h(x) = k\,f'(ax)$ for all x, where a and k are constants. The Taylor series for h about $x = 0$ is given by

$$h(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots.$$

Find the values of a and k.

(a)
$$f(x) = 6\left[1 - \frac{x}{3} + \frac{x^2}{2!3^2} - \frac{x^3}{3!3^3} + \cdots + \frac{(-1)^n x^n}{n!3^n} + \cdots\right]$$
$$= 6 - 2x + \frac{x^2}{3} - \frac{x^3}{27} + \cdots + \frac{6(-1)^n x^n}{n!3^n} + \cdots$$

$3 : \begin{cases} 1 : \text{two of } 6, -2x, \frac{x^2}{3}, -\frac{x^3}{27} \\ 1 : \text{remaining terms} \\ 1 : \text{general term} \end{cases}$

$\langle -1 \rangle$ missing factor of 6

(b) $g(0) = 0$ and $g'(x) = f(x)$, so
$$g(x) = 6\left[x - \frac{x^2}{6} + \frac{x^3}{3!3^2} - \frac{x^4}{4!3^3} + \cdots + \frac{(-1)^n x^{n+1}}{(n+1)!3^n} + \cdots\right]$$
$$= 6x - x^2 + \frac{x^3}{9} - \frac{x^4}{4(27)} + \cdots + \frac{6(-1)^n x^{n+1}}{(n+1)!3^n} + \cdots$$

$3 : \begin{cases} 1 : \text{two terms} \\ 1 : \text{remaining terms} \\ 1 : \text{general term} \end{cases}$

$\langle -1 \rangle$ missing factor of 6

(c) $f'(x) = -2e^{-x/3}$, so $h(x) = -2k\,e^{-ax/3}$

$$h(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots = e^x$$

$-2k\,e^{-ax/3} = e^x$

$\frac{-a}{3} = 1$ and $-2k = 1$

$a = -3$ and $k = -\frac{1}{2}$

OR

$f'(x) = -2 + \frac{2}{3}x + \cdots$, so

$h(x) = kf'(ax) = -2k + \frac{2}{3}akx + \cdots$

$h(x) = 1 + x + \cdots$

$-2k = 1$ and $\frac{2}{3}ak = 1$

$k = -\frac{1}{2}$ and $a = -3$

$3 : \begin{cases} 1 : \text{computes } k\,f'(ax) \\ 1 : \text{recognizes } h(x) = e^x, \\ \qquad \text{or} \\ \quad \text{equates 2 series for } h(x) \\ 1 : \text{values for } a \text{ and } k \end{cases}$

Question 3

x	$h(x)$	$h'(x)$	$h''(x)$	$h'''(x)$	$h^{(4)}(x)$
1	11	30	42	99	18
2	80	128	$\frac{488}{3}$	$\frac{448}{3}$	$\frac{584}{9}$
3	317	$\frac{753}{2}$	$\frac{1383}{4}$	$\frac{3483}{16}$	$\frac{1125}{16}$

Let h be a function having derivatives of all orders for $x > 0$. Selected values of h and its first four derivatives are indicated in the table above. The function h and these four derivatives are increasing on the interval $1 \le x \le 3$.

(a) Write the first-degree Taylor polynomial for h about $x = 2$ and use it to approximate $h(1.9)$. Is this approximation greater than or less than $h(1.9)$? Explain your reasoning.

(b) Write the third-degree Taylor polynomial for h about $x = 2$ and use it to approximate $h(1.9)$.

(c) Use the Lagrange error bound to show that the third-degree Taylor polynomial for h about $x = 2$ approximates $h(1.9)$ with error less than 3×10^{-4}.

(a) $P_1(x) = 80 + 128(x - 2)$, so $h(1.9) \approx P_1(1.9) = 67.2$

$P_1(1.9) < h(1.9)$ since h' is increasing on the interval $1 \le x \le 3$.

$$4 : \begin{cases} 2 : P_1(x) \\ 1 : P_1(1.9) \\ 1 : P_1(1.9) < h(1.9) \text{ with reason} \end{cases}$$

(b) $P_3(x) = 80 + 128(x - 2) + \frac{488}{6}(x - 2)^2 + \frac{448}{18}(x - 2)^3$

$h(1.9) \approx P_3(1.9) = 67.988$

$$3 : \begin{cases} 2 : P_3(x) \\ 1 : P_3(1.9) \end{cases}$$

(c) The fourth derivative of h is increasing on the interval $1 \le x \le 3$, so $\max\limits_{1.9 \le x \le 2} \left|h^{(4)}(x)\right| = \frac{584}{9}$.

Therefore, $\left|h(1.9) - P_3(1.9)\right| \le \frac{584}{9}\frac{|1.9 - 2|^4}{4!}$
$= 2.7037 \times 10^{-4}$
$< 3 \times 10^{-4}$

$$2 : \begin{cases} 1 : \text{form of Lagrange error estimate} \\ 1 : \text{reasoning} \end{cases}$$

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Question 6

Let f be the function given by $f(x) = \dfrac{2x}{1+x^2}$.

(a) Write the first four nonzero terms and the general term of the Taylor series for f about $x = 0$.

(b) Does the series found in part (a), when evaluated at $x = 1$, converge to $f(1)$? Explain why or why not.

(c) The derivative of $\ln\left(1+x^2\right)$ is $\dfrac{2x}{1+x^2}$. Write the first four nonzero terms of the Taylor series for $\ln\left(1+x^2\right)$ about $x = 0$.

(d) Use the series found in part (c) to find a rational number A such that $\left|A - \ln\left(\dfrac{5}{4}\right)\right| < \dfrac{1}{100}$. Justify your answer.

(a) $$\frac{1}{1-u} = 1 + u + u^2 + \cdots + u^n + \cdots$$
$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots + \left(-x^2\right)^n + \cdots$$
$$\frac{2x}{1+x^2} = 2x - 2x^3 + 2x^5 - 2x^7 + \cdots + (-1)^n 2x^{2n+1} + \cdots$$

$3 : \begin{cases} 1 : \text{two of the first four terms} \\ 1 : \text{remaining terms} \\ 1 : \text{general term} \end{cases}$

(b) No, the series does not converge when $x = 1$ because when $x = 1$, the terms of the series do not converge to 0.

1 : answer with reason

(c) $$\ln\left(1+x^2\right) = \int_0^x \frac{2t}{1+t^2}\,dt$$
$$= \int_0^x \left(2t - 2t^3 + 2t^5 - 2t^7 + \cdots\right) dt$$
$$= x^2 - \frac{1}{2}x^4 + \frac{1}{3}x^6 - \frac{1}{4}x^8 + \cdots$$

$2 : \begin{cases} 1 : \text{two of the first four terms} \\ 1 : \text{remaining terms} \end{cases}$

(d) $$\ln\left(\frac{5}{4}\right) = \ln\left(1+\frac{1}{4}\right) = \left(\frac{1}{2}\right)^2 - \frac{1}{2}\left(\frac{1}{2}\right)^4 + \frac{1}{3}\left(\frac{1}{2}\right)^6 - \frac{1}{4}\left(\frac{1}{2}\right)^8 + \cdots$$

Let $A = \left(\dfrac{1}{2}\right)^2 - \left(\dfrac{1}{2}\right)\left(\dfrac{1}{2}\right)^4 = \dfrac{7}{32}$.

Since the series is a converging alternating series and the absolute values of the individual terms decrease to 0,

$$\left|A - \ln\left(\frac{5}{4}\right)\right| < \left|\frac{1}{3}\left(\frac{1}{2}\right)^6\right| = \frac{1}{3}\cdot\frac{1}{64} < \frac{1}{100}.$$

$3 : \begin{cases} 1 : \text{uses } x = \frac{1}{2} \\ 1 : \text{value of } A \\ 1 : \text{justification} \end{cases}$

Question 6

The Maclaurin series for e^x is $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^n}{n!} + \cdots$. The continuous function f is defined by $f(x) = \frac{e^{(x-1)^2} - 1}{(x-1)^2}$ for $x \neq 1$ and $f(1) = 1$. The function f has derivatives of all orders at $x = 1$.

(a) Write the first four nonzero terms and the general term of the Taylor series for $e^{(x-1)^2}$ about $x = 1$.

(b) Use the Taylor series found in part (a) to write the first four nonzero terms and the general term of the Taylor series for f about $x = 1$.

(c) Use the ratio test to find the interval of convergence for the Taylor series found in part (b).

(d) Use the Taylor series for f about $x = 1$ to determine whether the graph of f has any points of inflection.

(a) $1 + (x-1)^2 + \frac{(x-1)^4}{2} + \frac{(x-1)^6}{6} + \cdots + \frac{(x-1)^{2n}}{n!} + \cdots$

$2: \begin{cases} 1: \text{first four terms} \\ 1: \text{general term} \end{cases}$

(b) $1 + \frac{(x-1)^2}{2} + \frac{(x-1)^4}{6} + \frac{(x-1)^6}{24} + \cdots + \frac{(x-1)^{2n}}{(n+1)!} + \cdots$

$2: \begin{cases} 1: \text{first four terms} \\ 1: \text{general term} \end{cases}$

(c) $\lim_{n\to\infty} \left| \frac{\frac{(x-1)^{2n+2}}{(n+2)!}}{\frac{(x-1)^{2n}}{(n+1)!}} \right| = \lim_{n\to\infty} \frac{(n+1)!}{(n+2)!}(x-1)^2 = \lim_{n\to\infty} \frac{(x-1)^2}{n+2} = 0$

Therefore, the interval of convergence is $(-\infty, \infty)$.

$3: \begin{cases} 1: \text{sets up ratio} \\ 1: \text{computes limit of ratio} \\ 1: \text{answer} \end{cases}$

(d) $f''(x) = 1 + \frac{4\cdot 3}{6}(x-1)^2 + \frac{6\cdot 5}{24}(x-1)^4 + \cdots + \frac{2n(2n-1)}{(n+1)!}(x-1)^{2n-2} + \cdots$

Since every term of this series is nonnegative, $f''(x) \geq 0$ for all x. Therefore, the graph of f has no points of inflection.

$2: \begin{cases} 1: f''(x) \\ 1: \text{answer} \end{cases}$

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2009 SCORING GUIDELINES (Form B)

Question 6

The function f is defined by the power series

$$f(x) = 1 + (x+1) + (x+1)^2 + \cdots + (x+1)^n + \cdots = \sum_{n=0}^{\infty} (x+1)^n$$

for all real numbers x for which the series converges.

(a) Find the interval of convergence of the power series for f. Justify your answer.

(b) The power series above is the Taylor series for f about $x = -1$. Find the sum of the series for f.

(c) Let g be the function defined by $g(x) = \int_{-1}^{x} f(t)\,dt$. Find the value of $g\left(-\frac{1}{2}\right)$, if it exists, or explain why $g\left(-\frac{1}{2}\right)$ cannot be determined.

(d) Let h be the function defined by $h(x) = f\left(x^2 - 1\right)$. Find the first three nonzero terms and the general term of the Taylor series for h about $x = 0$, and find the value of $h\left(\frac{1}{2}\right)$.

(a) The power series is geometric with ratio $(x+1)$. The series converges if and only if $\lvert x+1 \rvert < 1$. Therefore, the interval of convergence is $-2 < x < 0$.	$3: \begin{cases} 1: \text{identifies as geometric} \\ 1: \lvert x+1 \rvert < 1 \\ 1: \text{interval of convergence} \end{cases}$
OR	OR
$\lim_{n\to\infty} \left\lvert \frac{(x+1)^{n+1}}{(x+1)^n} \right\rvert = \lvert x+1 \rvert < 1$ when $-2 < x < 0$ At $x = -2$, the series is $\sum_{n=0}^{\infty} (-1)^n$, which diverges since the terms do not converge to 0. At $x = 0$, the series is $\sum_{n=0}^{\infty} 1$, which similarly diverges. Therefore, the interval of convergence is $-2 < x < 0$.	$3: \begin{cases} 1: \text{sets up limit of ratio} \\ 1: \text{radius of convergence} \\ 1: \text{interval of convergence} \end{cases}$
(b) Since the series is geometric, $f(x) = \sum_{n=0}^{\infty} (x+1)^n = \frac{1}{1-(x+1)} = -\frac{1}{x}$ for $-2 < x < 0$.	1 : answer
(c) $g\left(-\frac{1}{2}\right) = \int_{-1}^{-\frac{1}{2}} -\frac{1}{x}\,dx = -\ln\lvert x \rvert \Big\rvert_{x=-1}^{x=-\frac{1}{2}} = \ln 2$	$2: \begin{cases} 1: \text{antiderivative} \\ 1: \text{value} \end{cases}$
(d) $h(x) = f\left(x^2 - 1\right) = 1 + x^2 + x^4 + \cdots + x^{2n} + \cdots$ $h\left(\frac{1}{2}\right) = f\left(-\frac{3}{4}\right) = \frac{4}{3}$	$3: \begin{cases} 1: \text{first three terms} \\ 1: \text{general term} \\ 1: \text{value of } h\left(\frac{1}{2}\right) \end{cases}$

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Question 6

$$f(x) = \begin{cases} \dfrac{\cos x - 1}{x^2} & \text{for } x \neq 0 \\ -\dfrac{1}{2} & \text{for } x = 0 \end{cases}$$

The function f, defined above, has derivatives of all orders. Let g be the function defined by $g(x) = 1 + \int_0^x f(t)\,dt.$

(a) Write the first three nonzero terms and the general term of the Taylor series for $\cos x$ about $x = 0$. Use this series to write the first three nonzero terms and the general term of the Taylor series for f about $x = 0$.

(b) Use the Taylor series for f about $x = 0$ found in part (a) to determine whether f has a relative maximum, relative minimum, or neither at $x = 0$. Give a reason for your answer.

(c) Write the fifth-degree Taylor polynomial for g about $x = 0$.

(d) The Taylor series for g about $x = 0$, evaluated at $x = 1$, is an alternating series with individual terms that decrease in absolute value to 0. Use the third-degree Taylor polynomial for g about $x = 0$ to estimate the value of $g(1)$. Explain why this estimate differs from the actual value of $g(1)$ by less than $\frac{1}{6!}$.

(a) $\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots$ $f(x) = -\frac{1}{2} + \frac{x^2}{4!} - \frac{x^4}{6!} + \cdots + (-1)^{n+1} \frac{x^{2n}}{(2n+2)!} + \cdots$	$3 : \begin{cases} 1 : \text{terms for } \cos x \\ 2 : \text{terms for } f \\ \quad 1 : \text{first three terms} \\ \quad 1 : \text{general term} \end{cases}$
(b) $f'(0)$ is the coefficient of x in the Taylor series for f about $x = 0$, so $f'(0) = 0$. $\frac{f''(0)}{2!} = \frac{1}{4!}$ is the coefficient of x^2 in the Taylor series for f about $x = 0$, so $f''(0) = \frac{1}{12}$. Therefore, by the Second Derivative Test, f has a relative minimum at $x = 0$.	$2 : \begin{cases} 1 : \text{determines } f'(0) \\ 1 : \text{answer with reason} \end{cases}$
(c) $P_5(x) = 1 - \frac{x}{2} + \frac{x^3}{3 \cdot 4!} - \frac{x^5}{5 \cdot 6!}$	$2 : \begin{cases} 1 : \text{two correct terms} \\ 1 : \text{remaining terms} \end{cases}$
(d) $g(1) \approx 1 - \frac{1}{2} + \frac{1}{3 \cdot 4!} = \frac{37}{72}$ Since the Taylor series for g about $x = 0$ evaluated at $x = 1$ is alternating and the terms decrease in absolute value to 0, we know $\left\lvert g(1) - \frac{37}{72} \right\rvert < \frac{1}{5 \cdot 6!} < \frac{1}{6!}$.	$2 : \begin{cases} 1 : \text{estimate} \\ 1 : \text{explanation} \end{cases}$

Question 6

The Maclaurin series for the function f is given by $f(x) = \sum_{n=2}^{\infty} \frac{(-1)^n (2x)^n}{n-1}$ on its interval of convergence.

(a) Find the interval of convergence for the Maclaurin series of f. Justify your answer.

(b) Show that $y = f(x)$ is a solution to the differential equation $xy' - y = \frac{4x^2}{1+2x}$ for $|x| < R$, where R is the radius of convergence from part (a).

(a) $$\lim_{n\to\infty}\left|\frac{\frac{(2x)^{n+1}}{(n+1)-1}}{\frac{(2x)^n}{n-1}}\right| = \lim_{n\to\infty}\left|2x\cdot\frac{n-1}{n}\right| = \lim_{n\to\infty}\left|2x\cdot\frac{n-1}{n}\right| = |2x|$$

$|2x| < 1$ for $|x| < \frac{1}{2}$

Therefore the radius of convergence is $\frac{1}{2}$.

When $x = -\frac{1}{2}$, the series is $\sum_{n=2}^{\infty}\frac{(-1)^n(-1)^n}{n-1} = \sum_{n=2}^{\infty}\frac{1}{n-1}$.

This is the harmonic series, which diverges.

When $x = \frac{1}{2}$, the series is $\sum_{n=2}^{\infty}\frac{(-1)^n 1^n}{n-1} = \sum_{n=2}^{\infty}\frac{(-1)^n}{n-1}$.

This is the alternating harmonic series, which converges.

The interval of convergence for the Maclaurin series of f is $\left(-\frac{1}{2}, \frac{1}{2}\right]$.

$$5 : \begin{cases} 1 : \text{sets up ratio} \\ 1 : \text{limit evaluation} \\ 1 : \text{radius of convergence} \\ 1 : \text{considers both endpoints} \\ 1 : \text{analysis and interval of convergence} \end{cases}$$

(b) $$y = \frac{(2x)^2}{1} - \frac{(2x)^3}{2} + \frac{(2x)^4}{3} - \cdots + \frac{(-1)^n(2x)^n}{n-1} + \cdots$$

$$= 4x^2 - 4x^3 + \frac{16}{3}x^4 - \cdots + \frac{(-1)^n(2x)^n}{n-1} + \cdots$$

$$y' = 8x - 12x^2 + \frac{64}{3}x^3 - \cdots + \frac{(-1)^n n(2x)^{n-1}\cdot 2}{n-1} + \cdots$$

$$xy' = 8x^2 - 12x^3 + \frac{64}{3}x^4 - \cdots + \frac{(-1)^n n(2x)^n}{n-1} + \cdots$$

$$xy' - y = 4x^2 - 8x^3 + 16x^4 - \cdots + (-1)^n(2x)^n + \cdots$$

$$= 4x^2\left(1 - 2x + 4x^2 - \cdots + (-1)^n(2x)^{n-2} + \cdots\right)$$

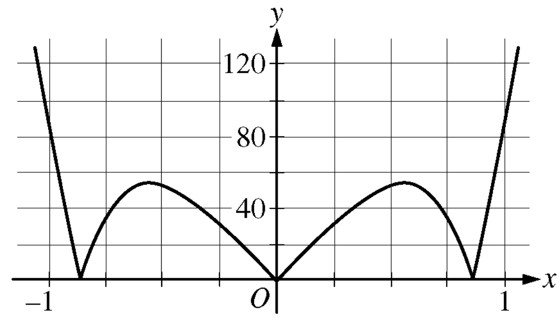
The series $1 - 2x + 4x^2 - \cdots + (-1)^n(2x)^{n-2} + \cdots = \sum_{n=0}^{\infty}(-2x)^n$ is a geometric series that converges to $\frac{1}{1+2x}$ for $|x| < \frac{1}{2}$. Therefore

$xy' - y = 4x^2\cdot\frac{1}{1+2x}$ for $|x| < \frac{1}{2}$.

$$4 : \begin{cases} 1 : \text{series for } y' \\ 1 : \text{series for } xy' \\ 1 : \text{series for } xy' - y \\ 1 : \text{analysis with geometric series} \end{cases}$$

Question 6

Let $f(x) = \sin\left(x^2\right) + \cos x$. The graph of $y = \left|f^{(5)}(x)\right|$ is shown above.



Graph of $y = \left|f^{(5)}(x)\right|$

(a) Write the first four nonzero terms of the Taylor series for $\sin x$ about $x = 0$, and write the first four nonzero terms of the Taylor series for $\sin\left(x^2\right)$ about $x = 0$.

(b) Write the first four nonzero terms of the Taylor series for $\cos x$ about $x = 0$. Use this series and the series for $\sin\left(x^2\right)$, found in part (a), to write the first four nonzero terms of the Taylor series for f about $x = 0$.

(c) Find the value of $f^{(6)}(0)$.

(d) Let $P_4(x)$ be the fourth-degree Taylor polynomial for f about $x = 0$. Using information from the graph of $y = \left|f^{(5)}(x)\right|$ shown above, show that $\left|P_4\left(\frac{1}{4}\right) - f\left(\frac{1}{4}\right)\right| < \frac{1}{3000}$.

(a) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ $\sin\left(x^2\right) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots$	$3 : \begin{cases} 1 : \text{series for } \sin x \\ 2 : \text{series for } \sin\left(x^2\right) \end{cases}$
(b) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$ $f(x) = 1 + \frac{x^2}{2} + \frac{x^4}{4!} - \frac{121x^6}{6!} + \cdots$	$3 : \begin{cases} 1 : \text{series for } \cos x \\ 2 : \text{series for } f(x) \end{cases}$
(c) $\frac{f^{(6)}(0)}{6!}$ is the coefficient of x^6 in the Taylor series for f about $x = 0$. Therefore $f^{(6)}(0) = -121$.	1 : answer
(d) The graph of $y = \left|f^{(5)}(x)\right|$ indicates that $\max\limits_{0 \le x \le \frac{1}{4}} \left|f^{(5)}(x)\right| < 40$. Therefore $\left|P_4\left(\frac{1}{4}\right) - f\left(\frac{1}{4}\right)\right| \le \frac{\max\limits_{0 \le x \le \frac{1}{4}} \left|f^{(5)}(x)\right|}{5!} \cdot \left(\frac{1}{4}\right)^5 < \frac{40}{120 \cdot 4^5} = \frac{1}{3072} < \frac{1}{3000}$.	$2 : \begin{cases} 1 : \text{form of the error bound} \\ 1 : \text{analysis} \end{cases}$

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Question 6

Let $f(x) = \ln\left(1 + x^3\right)$.

(a) The Maclaurin series for $\ln(1 + x)$ is $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1} \cdot \frac{x^n}{n} + \cdots$. Use the series to write the first four nonzero terms and the general term of the Maclaurin series for f.

(b) The radius of convergence of the Maclaurin series for f is 1. Determine the interval of convergence. Show the work that leads to your answer.

(c) Write the first four nonzero terms of the Maclaurin series for $f'\left(t^2\right)$. If $g(x) = \int_0^x f'\left(t^2\right) dt$, use the first two nonzero terms of the Maclaurin series for g to approximate $g(1)$.

(d) The Maclaurin series for g, evaluated at $x = 1$, is a convergent alternating series with individual terms that decrease in absolute value to 0. Show that your approximation in part (c) must differ from $g(1)$ by less than $\frac{1}{5}$.

(a) $x^3 - \frac{x^6}{2} + \frac{x^9}{3} - \frac{x^{12}}{4} + \cdots + (-1)^{n+1} \cdot \frac{x^{3n}}{n} + \cdots$

$2 : \begin{cases} 1 : \text{first four terms} \\ 1 : \text{general term} \end{cases}$

(b) The interval of convergence is centered at $x = 0$.
At $x = -1$, the series is $-1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \cdots - \frac{1}{n} - \cdots$, which diverges because the harmonic series diverges.
At $x = 1$, the series is $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + (-1)^{n+1} \cdot \frac{1}{n} + \cdots$, the alternating harmonic series, which converges.

Therefore the interval of convergence is $-1 < x \le 1$.

2 : answer with analysis

(c) The Maclaurin series for $f'(x)$, $f'\left(t^2\right)$, and $g(x)$ are

$$f'(x) : \sum_{n=1}^{\infty} (-1)^{n+1} \cdot 3x^{3n-1} = 3x^2 - 3x^5 + 3x^8 - 3x^{11} + \cdots$$

$$f'\left(t^2\right) : \sum_{n=1}^{\infty} (-1)^{n+1} \cdot 3t^{6n-2} = 3t^4 - 3t^{10} + 3t^{16} - 3t^{22} + \cdots$$

$$g(x) : \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{3x^{6n-1}}{6n-1} = \frac{3x^5}{5} - \frac{3x^{11}}{11} + \frac{3x^{17}}{17} - \frac{3x^{23}}{23} + \cdots$$

Thus $g(1) \approx \frac{3}{5} - \frac{3}{11} = \frac{18}{55}$.

$4 : \begin{cases} 1 : \text{two terms for } f'\left(t^2\right) \\ 1 : \text{other terms for } f'\left(t^2\right) \\ 1 : \text{first two terms for } g(x) \\ 1 : \text{approximation} \end{cases}$

(d) The Maclaurin series for g evaluated at $x = 1$ is alternating, and the terms decrease in absolute value to 0.
Thus $\left| g(1) - \frac{18}{55} \right| < \frac{3 \cdot 1^{17}}{17} = \frac{3}{17} < \frac{1}{5}$.

1 : analysis

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Question 4

x	1	1.1	1.2	1.3	1.4
$f'(x)$	8	10	12	13	14.5

The function f is twice differentiable for $x > 0$ with $f(1) = 15$ and $f''(1) = 20$. Values of f', the derivative of f, are given for selected values of x in the table above.

(a) Write an equation for the line tangent to the graph of f at $x = 1$. Use this line to approximate $f(1.4)$.

(b) Use a midpoint Riemann sum with two subintervals of equal length and values from the table to approximate $\int_1^{1.4} f'(x)\,dx$. Use the approximation for $\int_1^{1.4} f'(x)\,dx$ to estimate the value of $f(1.4)$. Show the computations that lead to your answer.

(c) Use Euler's method, starting at $x = 1$ with two steps of equal size, to approximate $f(1.4)$. Show the computations that lead to your answer.

(d) Write the second-degree Taylor polynomial for f about $x = 1$. Use the Taylor polynomial to approximate $f(1.4)$.

(a) $f(1) = 15, \quad f'(1) = 8$

An equation for the tangent line is $y = 15 + 8(x - 1)$.

$f(1.4) \approx 15 + 8(1.4 - 1) = 18.2$

$2 : \begin{cases} 1 : \text{tangent line} \\ 1 : \text{approximation} \end{cases}$

(b) $\int_1^{1.4} f'(x)\,dx \approx (0.2)(10) + (0.2)(13) = 4.6$

$f(1.4) = f(1) + \int_1^{1.4} f'(x)\,dx$

$f(1.4) \approx 15 + 4.6 = 19.6$

$3 : \begin{cases} 1 : \text{midpoint Riemann sum} \\ 1 : \text{Fundamental Theorem of Calculus} \\ 1 : \text{answer} \end{cases}$

(c) $f(1.2) \approx f(1) + (0.2)(8) = 16.6$

$f(1.4) \approx 16.6 + (0.2)(12) = 19.0$

$2 : \begin{cases} 1 : \text{Euler's method with two steps} \\ 1 : \text{answer} \end{cases}$

(d) $$T_2(x) = 15 + 8(x - 1) + \frac{20}{2!}(x - 1)^2 = 15 + 8(x - 1) + 10(x - 1)^2$$

$f(1.4) \approx 15 + 8(1.4 - 1) + 10(1.4 - 1)^2 = 19.8$

$2 : \begin{cases} 1 : \text{Taylor polynomial} \\ 1 : \text{approximation} \end{cases}$

Question 6

The function g has derivatives of all orders, and the Maclaurin series for g is

$$\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+3} = \frac{x}{3} - \frac{x^3}{5} + \frac{x^5}{7} - \cdots.$$

(a) Using the ratio test, determine the interval of convergence of the Maclaurin series for g.

(b) The Maclaurin series for g evaluated at $x = \frac{1}{2}$ is an alternating series whose terms decrease in absolute value to 0. The approximation for $g\left(\frac{1}{2}\right)$ using the first two nonzero terms of this series is $\frac{17}{120}$. Show that this approximation differs from $g\left(\frac{1}{2}\right)$ by less than $\frac{1}{200}$.

(c) Write the first three nonzero terms and the general term of the Maclaurin series for $g'(x)$.

(a) $$\left|\frac{x^{2n+3}}{2n+5} \cdot \frac{2n+3}{x^{2n+1}}\right| = \left(\frac{2n+3}{2n+5}\right) \cdot x^2$$

$$\lim_{n\to\infty}\left(\frac{2n+3}{2n+5}\right) \cdot x^2 = x^2$$

$x^2 < 1 \;\Rightarrow\; -1 < x < 1$

The series converges when $-1 < x < 1$.

When $x = -1$, the series is $-\frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$

This series converges by the Alternating Series Test.

When $x = 1$, the series is $\frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \cdots$

This series converges by the Alternating Series Test.

Therefore, the interval of convergence is $-1 \le x \le 1$.

$$5 : \begin{cases} 1 : \text{sets up ratio} \\ 1 : \text{computes limit of ratio} \\ 1 : \text{identifies interior of interval of convergence} \\ 1 : \text{considers both endpoints} \\ 1 : \text{analysis and interval of convergence} \end{cases}$$

(b) $$\left|g\left(\frac{1}{2}\right) - \frac{17}{120}\right| < \frac{\left(\frac{1}{2}\right)^5}{7} = \frac{1}{224} < \frac{1}{200}$$

$$2 : \begin{cases} 1 : \text{uses the third term as an error bound} \\ 1 : \text{error bound} \end{cases}$$

(c) $$g'(x) = \frac{1}{3} - \frac{3}{5}x^2 + \frac{5}{7}x^4 + \cdots + (-1)^n\left(\frac{2n+1}{2n+3}\right)x^{2n} + \cdots$$

$$2 : \begin{cases} 1 : \text{first three terms} \\ 1 : \text{general term} \end{cases}$$

Question 6

A function f has derivatives of all orders at $x = 0$. Let $P_n(x)$ denote the nth-degree Taylor polynomial for f about $x = 0$.

(a) It is known that $f(0) = -4$ and that $P_1\left(\frac{1}{2}\right) = -3$. Show that $f'(0) = 2$.

(b) It is known that $f''(0) = -\frac{2}{3}$ and $f'''(0) = \frac{1}{3}$. Find $P_3(x)$.

(c) The function h has first derivative given by $h'(x) = f(2x)$. It is known that $h(0) = 7$. Find the third-degree Taylor polynomial for h about $x = 0$.

(a) $P_1(x) = f(0) + f'(0)x = -4 + f'(0)x$

$$P_1\left(\frac{1}{2}\right) = -4 + f'(0) \cdot \frac{1}{2} = -3$$
$$f'(0) \cdot \frac{1}{2} = 1$$
$$f'(0) = 2$$

$2 : \begin{cases} 1 : \text{uses } P_1(x) \\ 1 : \text{verifies } f'(0) = 2 \end{cases}$

(b) $$P_3(x) = -4 + 2x + \left(-\frac{2}{3}\right) \cdot \frac{x^2}{2!} + \frac{1}{3} \cdot \frac{x^3}{3!}$$
$$= -4 + 2x - \frac{1}{3}x^2 + \frac{1}{18}x^3$$

$3 : \begin{cases} 1 : \text{first two terms} \\ 1 : \text{third term} \\ 1 : \text{fourth term} \end{cases}$

(c) Let $Q_n(x)$ denote the Taylor polynomial of degree n for h about $x = 0$.

$$h'(x) = f(2x) \Rightarrow Q_3'(x) = -4 + 2(2x) - \frac{1}{3}(2x)^2$$
$$Q_3(x) = -4x + 4 \cdot \frac{x^2}{2} - \frac{4}{3} \cdot \frac{x^3}{3} + C;\ C = Q_3(0) = h(0) = 7$$
$$Q_3(x) = 7 - 4x + 2x^2 - \frac{4}{9}x^3$$

OR

$$h'(x) = f(2x),\ h''(x) = 2f'(2x),\ h'''(x) = 4f''(2x)$$
$$h'(0) = f(0) = -4,\ h''(0) = 2f'(0) = 4,\ h'''(0) = 4f''(0) = -\frac{8}{3}$$
$$Q_3(x) = 7 - 4x + 4 \cdot \frac{x^2}{2!} - \frac{8}{3} \cdot \frac{x^3}{3!} = 7 - 4x + 2x^2 - \frac{4}{9}x^3$$

$4 : \begin{cases} 2 : \text{applies } h'(x) = f(2x) \\ 1 : \text{constant term} \\ 1 : \text{remaining terms} \end{cases}$

Question 6

The Taylor series for a function f about $x = 1$ is given by $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^n}{n} (x-1)^n$ and converges to $f(x)$ for $|x-1| < R$, where R is the radius of convergence of the Taylor series.

(a) Find the value of R.

(b) Find the first three nonzero terms and the general term of the Taylor series for f', the derivative of f, about $x = 1$.

(c) The Taylor series for f' about $x = 1$, found in part (b), is a geometric series. Find the function f' to which the series converges for $|x-1| < R$. Use this function to determine f for $|x-1| < R$.

(a) Let a_n be the nth term of the Taylor series.

$$\frac{a_{n+1}}{a_n} = \frac{(-1)^{n+2} 2^{n+1} (x-1)^{n+1}}{n+1} \cdot \frac{n}{(-1)^{n+1} 2^n (x-1)^n}$$

$$= \frac{-2n(x-1)}{n+1}$$

$$\lim_{n\to\infty} \left| \frac{-2n(x-1)}{n+1} \right| = 2|x-1|$$

$$2|x-1| < 1 \Rightarrow |x-1| < \frac{1}{2}$$

The radius of convergence is $R = \frac{1}{2}$.

$3 : \begin{cases} 1 : \text{sets up ratio} \\ 1 : \text{computes limit of ratio} \\ 1 : \text{determines radius of convergence} \end{cases}$

(b) The first three nonzero terms are
$2 - 4(x-1) + 8(x-1)^2$.

The general term is $(-1)^{n+1} 2^n (x-1)^{n-1}$ for $n \geq 1$.

$3 : \begin{cases} 2 : \text{first three nonzero terms} \\ 1 : \text{general term} \end{cases}$

(c) The common ratio is $-2(x-1)$.

$$f'(x) = \frac{2}{1-(-2(x-1))} = \frac{2}{2x-1} \text{ for } |x-1| < \frac{1}{2}$$

$$f(x) = \int \frac{2}{2x-1}\, dx = \ln|2x-1| + C$$

$f(1) = 0$

$\ln|1| + C = 0 \Rightarrow C = 0$

$f(x) = \ln|2x-1|$ for $|x-1| < \frac{1}{2}$

$3 : \begin{cases} 1 : f'(x) \\ 1 : \text{antiderivative} \\ 1 : f(x) \end{cases}$

Question 6

The Maclaurin series for a function f is given by $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{n} x^n = x - \frac{3}{2}x^2 + 3x^3 - \cdots + \frac{(-3)^{n-1}}{n} x^n + \cdots$ and converges to $f(x)$ for $|x| < R$, where R is the radius of convergence of the Maclaurin series.

(a) Use the ratio test to find R.

(b) Write the first four nonzero terms of the Maclaurin series for f', the derivative of f. Express f' as a rational function for $|x| < R$.

(c) Write the first four nonzero terms of the Maclaurin series for e^x. Use the Maclaurin series for e^x to write the third-degree Taylor polynomial for $g(x) = e^x f(x)$ about $x = 0$.

(a) Let a_n be the nth term of the Maclaurin series.

$$\frac{a_{n+1}}{a_n} = \frac{(-3)^n x^{n+1}}{n+1} \cdot \frac{n}{(-3)^{n-1} x^n} = \frac{-3n}{n+1} \cdot x$$

$$\lim_{n\to\infty} \left| \frac{-3n}{n+1} \cdot x \right| = 3|x|$$

$$3|x| < 1 \Rightarrow |x| < \frac{1}{3}$$

The radius of convergence is $R = \dfrac{1}{3}$.

$3 : \begin{cases} 1 : \text{sets up ratio} \\ 1 : \text{computes limit of ratio} \\ 1 : \text{determines radius of convergence} \end{cases}$

(b) The first four nonzero terms of the Maclaurin series for f' are $1 - 3x + 9x^2 - 27x^3$.

$$f'(x) = \frac{1}{1 - (-3x)} = \frac{1}{1 + 3x}$$

$3 : \begin{cases} 2 : \text{first four nonzero terms} \\ 1 : \text{rational function} \end{cases}$

(c) The first four nonzero terms of the Maclaurin series for e^x are $1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!}$.

The product of the Maclaurin series for e^x and the Maclaurin series for f is

$$\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(x - \frac{3}{2}x^2 + 3x^3 - \cdots\right)$$
$$= x - \frac{1}{2}x^2 + 2x^3 + \cdots$$

The third-degree Taylor polynomial for $g(x) = e^x f(x)$ about $x = 0$ is $T_3(x) = x - \dfrac{1}{2}x^2 + 2x^3$.

$3 : \begin{cases} 1 : \text{first four nonzero terms of the Maclaurin series for } e^x \\ 2 : \text{Taylor polynomial} \end{cases}$

Question 6

The function f has a Taylor series about $x = 1$ that converges to $f(x)$ for all x in the interval of convergence. It is known that $f(1) = 1$, $f'(1) = -\frac{1}{2}$, and the nth derivative of f at $x = 1$ is given by $f^{(n)}(1) = (-1)^n \frac{(n-1)!}{2^n}$ for $n \ge 2$.

(a) Write the first four nonzero terms and the general term of the Taylor series for f about $x = 1$.

(b) The Taylor series for f about $x = 1$ has a radius of convergence of 2. Find the interval of convergence. Show the work that leads to your answer.

(c) The Taylor series for f about $x = 1$ can be used to represent $f(1.2)$ as an alternating series. Use the first three nonzero terms of the alternating series to approximate $f(1.2)$.

(d) Show that the approximation found in part (c) is within 0.001 of the exact value of $f(1.2)$.

(a) $f(1) = 1,\ f'(1) = -\frac{1}{2},\ f''(1) = \frac{1}{2^2},\ f'''(1) = -\frac{2}{2^3}$

$$f(x) = 1 - \frac{1}{2}(x-1) + \frac{1}{2^2 \cdot 2}(x-1)^2 - \frac{1}{2^3 \cdot 3}(x-1)^3 + \cdots + \frac{(-1)^n}{2^n \cdot n}(x-1)^n + \cdots$$

$4 : \begin{cases} 1 : \text{first two terms} \\ 1 : \text{third term} \\ 1 : \text{fourth term} \\ 1 : \text{general term} \end{cases}$

(b) $R = 2$. The series converges on the interval $(-1, 3)$.

When $x = -1$, the series is $1 + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$.
Since the harmonic series diverges, this series diverges.

When $x = 3$, the series is $1 - 1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \cdots$.
Since the alternating harmonic series converges, this series converges.

Therefore, the interval of convergence is $-1 < x \le 3$.

$2 : \begin{cases} 1 : \text{identifies both endpoints} \\ 1 : \text{analysis and interval of convergence} \end{cases}$

(c) $f(1.2) \approx 1 - \frac{1}{2}(0.2) + \frac{1}{8}(0.2)^2 = 1 - 0.1 + 0.005 = 0.905$

1 : approximation

(d) The series for $f(1.2)$ alternates with terms that decrease in magnitude to 0.

$$|f(1.2) - T_2(1.2)| \le \left| \frac{-1}{2^3 \cdot 3}(0.2)^3 \right| = \frac{1}{3000} \le 0.001$$

$2 : \begin{cases} 1 : \text{error form} \\ 1 : \text{analysis} \end{cases}$

Question 5

(a) $f'(x) = \dfrac{-3(4x-7)}{\left(2x^2 - 7x + 5\right)^2}$

$f'(3) = \dfrac{(-3)(5)}{(18-21+5)^2} = -\dfrac{15}{4}$

2 : $f'(3)$

(b) $f'(x) = \dfrac{-3(4x-7)}{\left(2x^2 - 7x + 5\right)^2} = 0 \Rightarrow x = \dfrac{7}{4}$

The only critical point in the interval $1 < x < 2.5$ has x-coordinate $\dfrac{7}{4}$.

f' changes sign from positive to negative at $x = \dfrac{7}{4}$.

Therefore, f has a relative maximum at $x = \dfrac{7}{4}$.

$2 : \begin{cases} 1 : x\text{-coordinate} \\ 1 : \text{relative maximum} \\ \quad\ \text{with justification} \end{cases}$

(c) $\displaystyle\int_5^\infty f(x)\,dx = \lim_{b\to\infty}\int_5^b \frac{3}{2x^2 - 7x + 5}\,dx = \lim_{b\to\infty}\int_5^b \left(\frac{2}{2x-5} - \frac{1}{x-1}\right)dx$

$\displaystyle = \lim_{b\to\infty}\Big[\ln(2x-5) - \ln(x-1)\Big]_5^b = \lim_{b\to\infty}\left[\ln\left(\frac{2x-5}{x-1}\right)\right]_5^b$

$\displaystyle = \lim_{b\to\infty}\left[\ln\left(\frac{2b-5}{b-1}\right) - \ln\left(\frac{5}{4}\right)\right] = \ln 2 - \ln\left(\frac{5}{4}\right) = \ln\left(\frac{8}{5}\right)$

$3 : \begin{cases} 1 : \text{antiderivative} \\ 1 : \text{limit expression} \\ 1 : \text{answer} \end{cases}$

(d) f is continuous, positive, and decreasing on $[5, \infty)$.

2 : answer with conditions

The series converges by the integral test since $\displaystyle\int_5^\infty \frac{3}{2x^2 - 7x + 5}\,dx$ converges.

— OR —

$\dfrac{3}{2n^2 - 7n + 5} > 0$ and $\dfrac{1}{n^2} > 0$ for $n \ge 5$.

Since $\displaystyle\lim_{n\to\infty}\frac{\frac{3}{2n^2 - 7n + 5}}{\frac{1}{n^2}} = \frac{3}{2}$ and the series $\displaystyle\sum_{n=5}^\infty \frac{1}{n^2}$ converges,

the series $\displaystyle\sum_{n=5}^\infty \frac{3}{2n^2 - 7n + 5}$ converges by the limit comparison test.

Question 6

(a) $f(0) = 0$

$f'(0) = 1$

$f''(0) = -1(1) = -1$

$f'''(0) = -2(-1) = 2$

$f^{(4)}(0) = -3(2) = -6$

The first four nonzero terms are

$$0 + 1x + \frac{-1}{2!}x^2 + \frac{2}{3!}x^3 + \frac{-6}{4!}x^4 = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}.$$

The general term is $\dfrac{(-1)^{n+1}x^n}{n}$.

$3 : \begin{cases} 1 : f''(0),\ f'''(0), \text{ and } f^{(4)}(0) \\ 1 : \text{verify terms} \\ 1 : \text{general term} \end{cases}$

(b) For $x = 1$, the Maclaurin series becomes $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$.

The series does not converge absolutely because the harmonic series diverges.

The series alternates with terms that decrease in magnitude to 0, and therefore the series converges conditionally.

2 : converges conditionally with reason

(c)
$$\int_0^x f(t)\,dt = \int_0^x \left(t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots + \frac{(-1)^{n+1}t^n}{n} + \cdots \right) dt$$
$$= \left[\frac{t^2}{2} - \frac{t^3}{3\cdot 2} + \frac{t^4}{4\cdot 3} - \frac{t^5}{5\cdot 4} + \cdots + \frac{(-1)^{n+1}t^{n+1}}{(n+1)n} + \cdots \right]_{t=0}^{t=x}$$
$$= \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^5}{20} + \cdots + \frac{(-1)^{n+1}x^{n+1}}{(n+1)n} + \cdots$$

$3 : \begin{cases} 1 : \text{two terms} \\ 1 : \text{remaining terms} \\ 1 : \text{general term} \end{cases}$

(d) The terms alternate in sign and decrease in magnitude to 0. By the alternating series error bound, the error $\left| P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right) \right|$ is bounded by the magnitude of the first unused term, $\left| -\dfrac{(1/2)^5}{20} \right|$.

Thus, $\left| P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right) \right| \le \left| -\dfrac{(1/2)^5}{20} \right| = \dfrac{1}{32\cdot 20} < \dfrac{1}{500}$.

1 : error bound