Taylor Series

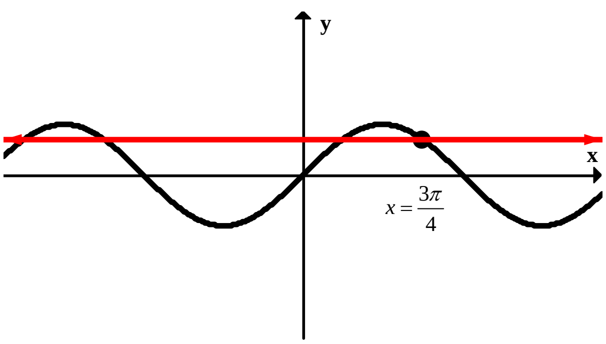
Earlier in the year, you learned to construct lines tangent to the graph of a function at a given coordinate. This tangent line $l(x)$ can be considered a degree-one polynomial that meets the following two conditions simultaneously:

(1) $l(c) = f(c)$ function values match

(2) $l'(c) = f'(c)$ slopes match (i.e. first derivatives match)

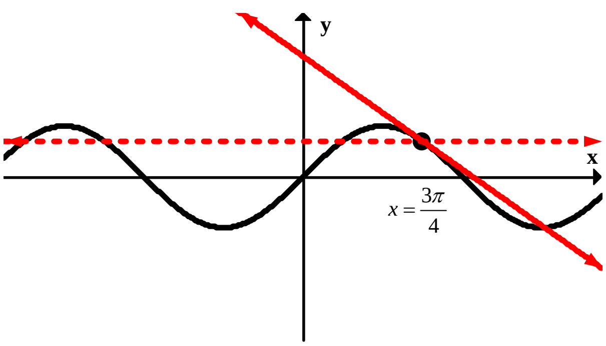
If you wanted to create a degree zero polynomial that approximated $f(x)$ at $x = c$, one would just make the constant function $f(c)$.

Consider the following example of $f(x) = \sin(x)$ at $x = \frac{3\pi}{4}$.



Notice that this approximation is good at $x = \frac{3\pi}{4}$, and not very good anywhere near $\frac{3\pi}{4}$. It's a zero-degree polynomial, what can you expect?

Making a degree-one polynomial that matches the function value and the slope of the function at the given location is even better.



Notice that this tangent line (i.e. degree-one polynomial approximation) is a good estimate for $\sin(x)$ for a small interval around $x = \frac{3\pi}{4}$.

Making a degree-two polynomial $p(x)$ such that

(1) $p(c) = f(c)$

(2) $p'(c) = f'(c)$

(3) $p''(c) = f''(c)$

Would be the next step. It takes a little finesse to get this to happen.

Let's look at the structure of the polynomials we are familiar with so far. Let $p_n(x)$ be the polynomial of degree *n* used to approximate $f(x)$ at $x=c$.

Degree-zero	$p_0(x)=f(c)$
Degree-one (a.k.a. the tangent line)	$y-y_1=m(x-x_1)$ $f(x)-f(c)=f'(c)(x-c)$ $f(x)=f(c)+f'(c)(x-c)$ $p_1(x)=f(c)+f'(c)(x-c)$

You may notice the start of a pattern in the column on the right that might look like this:

Degree-zero	$p_0(x)=f(c)$
Degree-one	$p_1(x)=f(c)+f'(c)(x-c)$
Degree-two	$p_2(x)=f(c)+f'(c)(x-c)+f''(c)(x-c)^2$

This hypothesis should be tested by seeing if the degree-two polynomial meets the three criteria that we wish it to possess:

(1) $p_2(c)=f(c)$	(2) $p_2'(c)=f'(c)$	(3) $p_2''(c)=f''(c)$
Function values match	First derivatives match	Second derivatives match

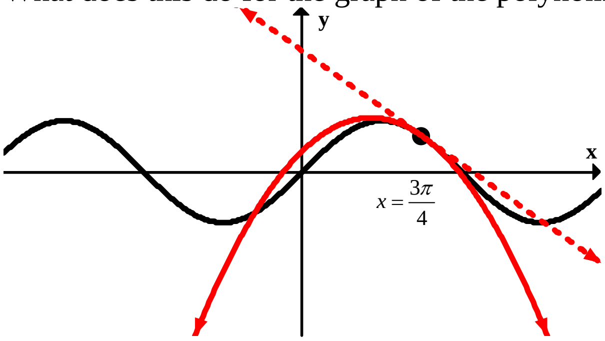
$$p_2(x)=f(c)+f'(c)(x-c)+f''(c)(x-c)^2$$
$$p_2'(x)=f'(c)+2f''(c)(x-c)^1$$
$$p_2''(x)=2f''(c)$$

$p_2(c)=f(c)+f'(c)(c-c)+f''(c)(c-c)^2$	$p_2(c)=f(c)+0+0$	function values match ✓
$p_2'(c)=f'(c)+2f''(c)(c-c)^1$	$p_2'(c)=f'(c)+0$	first derivtives match ✓
$p_2''(c)=2f''(c)$	$p_2''(c)=2f''(c)$	second derivatives do not match ✗

The polynomial we guessed was close, but we must adjust for the unwanted factor of 2, by dividing the second degree term by 2. The table is now corrected:

Degree-zero	$p_0(x)=f(c)$
Degree-one	$p_1(x)=f(c)+f'(c)(x-c)$
Degree-two	$p_2(x)=f(c)+f'(c)(x-c)+\dfrac{f''(c)(x-c)^2}{2}$

What does this do for the graph of the polynomial in comparison to the original graph?



The degree-two polynomial is a better match for a larger interval around the given location, and the polynomial "bends" to match the graph of $\sin(x)$ more closely.

What if we could continue this pattern? More importantly, since the degree-two term has changed, what's the new pattern?

One might guess that the pattern is as follows:

Degree-zero	$p_0(x) = f(c)$
Degree-one	$p_1(x) = f(c) + f'(c)(x-c)$
Degree-two	$p_2(x) = f(c) + f'(c)(x-c) + \frac{f''(c)(x-c)^2}{2}$
Degree-three	$p_3(x) = f(c) + f'(c)(x-c) + \frac{f''(c)(x-c)^2}{2} + \frac{f'''(c)(x-c)^3}{3}$

Which again, needs to be tested:

$$p_3(x) = f(c) + f'(c)(x-c) + \frac{f''(c)(x-c)^2}{2} + \frac{f'''(c)(x-c)^3}{3}$$
$$p_3'(x) = f'(c) + f''(c)(x-c)^1 + f''(c)(x-c)^2$$
$$p_3''(x) = f''(c) + 2f'''(c)(x-c)$$
$$p_3'''(x) = 2f'''(c)$$

$p_2(c) = f(c) + f'(c)(c-c) + f''(c)(c-c)^2$	$p_2(c) = f(c)$	function values match ✓
$p_2'(x) = f'(c) + 2f''(c)(c-c)^1$	$p_2'(c) = f'(c) + 0$	first derivtives match ✓
$p_2''(x) = 2f''(c)$	$p_2''(c) = 2f''(c)$	second derivatives do not match ✗

Again, we must adjust the new term to get rid of the unwanted factor. The table will be changed as follows:

Degree-zero	$p_0(x) = f(c)$
Degree-one	$p_1(x) = f(c) + f'(c)(x-c)$
Degree-two	$p_2(x) = f(c) + f'(c)(x-c) + \frac{f''(c)(x-c)^2}{2}$
Degree-three	$p_3(x) = f(c) + f'(c)(x-c) + \frac{f''(c)(x-c)^2}{2} + \frac{f'''(c)(x-c)^3}{3 \cdot 2}$ $= f(c) + f'(c)(x-c) + \frac{f''(c)(x-c)^2}{2 \cdot 1} + \frac{f'''(c)(x-c)^3}{3 \cdot 2 \cdot 1}$ $= f(c) + f'(c)(x-c) + \frac{f''(c)(x-c)^2}{2!} + \frac{f'''(c)(x-c)^3}{3!}$

Exciting $\boxed{!}$ isn't it! Turns out that the pattern involves a factorial in the denominator because repeated differentiation using the power rule will create a factorial!

$$\begin{aligned}
f(x) &= x^3 & g(x) &= k \cdot (x-c)^3 \\
f'(x) &= 3x^2 & g'(x) &= 3 \cdot k \cdot (x-c)^2 \\
f''(x) &= 3 \cdot 2 \cdot x & g''(x) &= 3 \cdot 2 \cdot k \cdot (x-c)^2 \\
f'''(x) &= \underbrace{3 \cdot 2 \cdot 1}_{3!} & g'''(x) &= \underbrace{3 \cdot 2 \cdot 1} \cdot k \\
&= 3! & &= (3!) \cdot k
\end{aligned}$$

$$\frac{d}{dx}\left[f^{(n)}(c)(x-c)^n \right] = n \cdot f^{(n)}(c)(x-c)^{n-1}$$

$$\frac{d^2}{dx}\left[f^{(n)}(c)(x-c)^n \right] = n \cdot (n-1)\, f^{(n)}(c)(x-c)^{n-2}$$

$$\vdots$$

$$\frac{d^{n-1}}{dx}\left[f^{(n)}(c)(x-c)^n \right] = (n)(n-1)(n-2)\cdots(3)(2) \cdot f^{(n)}(c)(x-c)^1$$

$$\frac{d^n}{dx}\left[f^{(n)}(c)(x-c)^n \right] = \underbrace{(n)(n-1)(n-2)\cdots(3)(2)(1)}_{n!} \cdot f^{(n)}(c)(x-c)^1$$

$$= n! \cdot f^{(n)}(c)$$

Degree-four	$p_4(x) = f(c) + f'(c)(x-c) + \frac{f''(c)(x-c)^2}{2!} + \frac{f'''(c)(x-c)^3}{3!} + \frac{f^{(4)}(c)(x-c)^4}{4!}$
$\vdots$	$\vdots$

The pattern in the table is finalized as follows:

Degree-zero	$p_0(x) = f(c)$
Degree-one	$p_1(x) = f(c) + f'(c)(x-c)$
Degree-two	$p_2(x) = f(c) + f'(c)(x-c) + \dfrac{f''(c)(x-c)^2}{2}$
Degree-three	$p_3(x) = f(c) + f'(c)(x-c) + \dfrac{f''(c)(x-c)^2}{2!} + \dfrac{f'''(c)(x-c)^3}{3!}$
$\vdots$	$\vdots$
Degree-*n*	$p_n(x) = f(c) + f'(c)(x-c) + \dfrac{f''(c)(x-c)^2}{2!} + \dfrac{f'''(c)(x-c)^3}{3!} + \cdots + \dfrac{f^{(n)}(c)(x-c)^n}{n!}$