AP Calculus BC
Stewart Chapter 2 Test (Limits)
Fall 2020 Solutions

Name:________________________________

Date:______________________ Period:________

Free Response Section: *NO CAS Calculator Permitted.*
You have the remainder of the period to complete this section.
Once you submit your Free Response Section, you will not be allowed to revisit it.

- Show all of your work. Clearly label any functions, graphs, tables, or other objects that you use. Your work will be scored on the correctness and completeness of your methods as well as your answers. Answers without supporting work will usually not receive credit. To be eligible for partial credit, methods, reasoning, and conclusions should be presented clearly.

- Justifications require that you give mathematical (non-calculator) reasons. Students should use complete sentences in responses that include explanations or justifications.

ALL LIMITS MUST BE DETERMINED ANALYTICALLY!
No use of L'Hopital's Rule or Differentiation Rules are permitted.

1. $\lim\limits_{x\to 0}\dfrac{\sqrt{4+x}-2}{x}$	**2.** $\lim\limits_{x\to 0}\dfrac{x\cos(x)}{\lvert x\rvert}$
$\lim\limits_{x\to 0}\dfrac{\sqrt{4+x}-2}{x}=\lim\limits_{x\to 0}\dfrac{\sqrt{4+x}-2}{x}\cdot\dfrac{\left(\sqrt{4+x}+2\right)}{\left(\sqrt{4+x}+2\right)}$ $=\lim\limits_{x\to 0}\dfrac{(4+x)-4}{x\left(\sqrt{4+x}+2\right)}$ $=\lim\limits_{x\to 0}\dfrac{\cancel{x}}{\cancel{x}\left(\sqrt{4+x}+2\right)}$ $=\lim\limits_{x\to 0}\dfrac{1}{\sqrt{4+x}+2}$ $=\dfrac{1}{4}$	$\lim\limits_{x\to 0}\dfrac{x\cos(x)}{\lvert x\rvert}=\lim\limits_{x\to 0}\left[\dfrac{x}{\lvert x\rvert}\right]\cdot\lim\limits_{x\to 0}\left[\cos(x)\right]$ $=\underbrace{\lim\limits_{x\to 0}\left[\dfrac{x}{\lvert x\rvert}\right]}_{\text{DNE}}\cdot 1$ DNE Formally: $\lim\limits_{x\to 0^+}\dfrac{x}{\lvert x\rvert}\cos(x)=\left[\lim\limits_{x\to 0^+}\dfrac{x}{\lvert x\rvert}\right]\cdot\left[\lim\limits_{x\to 0^+}\cos(x)\right]$ $=1\cdot 1$ $=1$ $\lim\limits_{x\to 0^-}\dfrac{x}{\lvert x\rvert}\cos(x)=\left[\lim\limits_{x\to 0^-}\dfrac{x}{\lvert x\rvert}\right]\cdot\left[\lim\limits_{x\to 0^-}\cos(x)\right]$ $=(-1)\cdot 1$ $=-1$ $\lim\limits_{x\to 0}\dfrac{x}{\lvert x\rvert}\cos(x)$ DNE because $\lim\limits_{x\to 0^+}\dfrac{x}{\lvert x\rvert}\cos(x)\neq\lim\limits_{x\to 0^-}\dfrac{x}{\lvert x\rvert}\cos(x)$

3. $\lim\limits_{x\to\frac{\pi}{4}} \dfrac{1-\tan(x)}{1-\cos(x)}$	**4.** $\lim\limits_{x\to 0} \dfrac{3x(x+1)}{\sqrt{x^4+2x^2+2}}$
$\lim\limits_{x\to\frac{\pi}{4}} \dfrac{1-\tan(x)}{1-\cos(x)} = \lim\limits_{x\to\frac{\pi}{4}} \dfrac{1-\tan\left(\frac{\pi}{4}\right)}{1-\cos\left(\frac{\pi}{4}\right)}$ $= \dfrac{1-1}{1-\frac{1}{\sqrt{2}}}$ $=0$	$\lim\limits_{x\to 0} \dfrac{3x(x+1)}{\sqrt{x^4+2x^2+2}} = \dfrac{0}{2}$ $=0$

5. $\lim\limits_{h\to 2} \dfrac{2(h-3)^2-h}{h-2}$	**6.** $\lim\limits_{h\to 0} \dfrac{\frac{1}{h-4}+\frac{1}{4}}{h}$
$\lim\limits_{h\to 2} \dfrac{2(h-3)^2-h}{h-2} = \lim\limits_{h\to 2} \dfrac{2h^2-12h+18-h}{h-2}$ $= \lim\limits_{h\to 2} \dfrac{2h^2-13h+18}{h-2}$ $= \lim\limits_{h\to 2} \dfrac{\cancel{(h-2)}(2h-9)}{\cancel{h-2}}$ $= \lim\limits_{h\to 2}(2h-9)$ $=-5$	$\lim\limits_{h\to 0} \dfrac{\frac{1}{h-4}+\frac{1}{4}}{h} = \lim\limits_{h\to 0} \dfrac{\frac{1}{h-4}\cdot\frac{(4)}{(4)}+\frac{1}{4}\cdot\frac{(h-4)}{(h-4)}}{h}$ $= \lim\limits_{h\to 0} \dfrac{\frac{4}{4(h-4)}+\frac{h-4}{4(h-4)}}{h}$ $= \lim\limits_{h\to 0} \dfrac{\left[\frac{h}{4(h-4)}\right]}{h}$ $= \lim\limits_{h\to 0} \dfrac{\cancel{h}}{4\cancel{h}(h-4)}$ $=-\dfrac{1}{16}$

7. $\lim_{x\to 0}\dfrac{\sin(x)-\cos(x)}{\cot(x)-1}$	8. $\lim_{x\to 0}\dfrac{3^x-3^{3x}}{1-3^x}$
$\begin{aligned}\lim_{x\to 0}\frac{\sin(x)-\cos(x)}{\cot(x)-1} &= \lim_{x\to 0}\frac{\sin(x)-\cos(x)}{\frac{\cos(x)}{\sin(x)}-1}\\ &= \lim_{x\to 0}\frac{\sin(x)-\cos(x)}{\frac{\cos(x)}{\sin(x)}-\frac{\sin(x)}{\sin(x)}}\\ &= \lim_{x\to 0}\frac{\sin(x)-\cos(x)}{\left[\frac{\cos(x)-\sin(x)}{\sin(x)}\right]}\\ &= \lim_{x\to 0}\frac{\sin(x)-\cos(x)}{1}\cdot\frac{\sin(x)}{\cos(x)-\sin(x)}\\ &= \lim_{x\to 0}\frac{\sin(x)\left[\sin(x)-\cos(x)\right]}{\cos(x)-\sin(x)}\\ &= \lim_{x\to 0}\frac{\sin(x)\left[\sin(x)-\cos(x)\right]}{-\left[\sin(x)-\cos(x)\right]}\\ &= \lim_{x\to 0}\frac{\sin(x)\cancel{\left[\sin(x)-\cos(x)\right]}}{-\cancel{\left[\sin(x)-\cos(x)\right]}}\\ &= \lim_{x\to 0}\left[-\sin(x)\right]\\ &= 0\end{aligned}$	$\lim_{x\to 0}\dfrac{7\cdot 3^x-3^{2x}}{1-3^x}=\dfrac{7\cdot 1-1}{0}$ $\downarrow$ DNE

9. Explain why the function is not continuous using limits.

$$f(x)=\begin{cases}3-x & x<2\\ \dfrac{x}{2}+1 & x\geq 2\end{cases}$$

$\lim_{x\to 2^-} f(x)=\lim_{x\to 2^-} 3-x$ $=1$	$f(2)=2$	$\lim_{x\to 2^+} f(x)=\lim_{x\to 2^+}\left[\dfrac{x}{2}+1\right]$ $=2$

Any of the following statements are acceptable so long as the values of the one-sided limits are demonstrated.

$f(x)$ is not continuous at $x=2$ because $\lim_{x\to 2} f(x)$ DNE.

$f(x)$ is not continuous at $x=2$ because $\lim_{x\to 2^-} f(x)\neq f(2)$

10. Use the Intermediate Value Theorem to prove that the function $g(x) = x - 4\cos(x)$ has a zero in the interval $\left[\frac{\pi}{3}, \frac{2\pi}{3}\right]$. Your response must demonstrate that the conditions of the hypothesis are met, state "…by the Intermediate Value Theorem…", and have the conclusion of the theorem tailored to the conditions of the exercise.

$$\begin{aligned} g\left(\frac{\pi}{3}\right) &= \left(\frac{\pi}{3}\right) - 4\cos\left(\frac{\pi}{3}\right) \\ &= \frac{\pi}{3} - 4\left(\frac{1}{2}\right) \\ &= \frac{\pi}{3} - 2 \\ &= -0.9528... \end{aligned}$$

$$\begin{aligned} g\left(\frac{2\pi}{3}\right) &= \left(\frac{2\pi}{3}\right) - 4\cos\left(\frac{2\pi}{3}\right) \\ &= \frac{2\pi}{3} - 4\left(-\frac{1}{2}\right) \\ &= \frac{2\pi}{3} + 2 \\ &= 4.0943... \end{aligned}$$

Since $g(x)$ is continuous on $\left[\frac{\pi}{3}, \frac{2\pi}{3}\right]$ and $g\left(\frac{\pi}{3}\right) < 0$ and $g\left(\frac{2\pi}{3}\right) > 0$ by the Intermediate Value Theorem, there is a c where $\frac{\pi}{3} < c < \frac{2\pi}{3}$ and $f(c) = 0$.