AP Calculus Integral Homework Guidelines:

From College Board

Students are expected to show all of their work. They may also be asked to use complete sentences to explain their methods or the reasonableness of their answers, or to interpret their results.

For results obtained using one of the four required calculator capabilities listed below, students are required to write the setup (e.g., the equation being solved, or the derivative or definite integral being evaluated) that leads to the solution, along with the result produced by the calculator. For example, if the student is asked to find the area of a region, the student is expected to show a definite integral (i.e., the setup) and the answer. The student need not compute the antiderivative; the calculator may be used to calculate the value of the definite integral without further explanation. For solutions obtained using a calculator capability other than one of the four required ones, students must also show the mathematical steps that lead to the answer; a calculator result is not sufficient. For example, if the student is asked to find a relative minimum value of a function, the student is expected to use calculus and show the mathematical steps that lead to the answer. It is not sufficient to graph the function or use a built-in minimum folder.

When a student is asked to justify an answer, the justification must include mathematical reasons, not merely calculator results. Functions, graphs, tables, or other objects that are used in a justification should be clearly identified.

Required calculator skills/functions

- plot the graph of a function within an arbitrary viewing window
- find the zeros of functions (solve equations numerically)
- numerically calculate the derivative of a function
- numerically calculate the value of a definite integral

Indefinite Integrals:

When evaluating indefinite integrals, students must copy the original indefinite integral. Leaving out the "*dx*" from any integral expression will be severely penalized. Students are expected to show a reasonable amount of written work, which includes rewriting the algebraic expression of the integrand to make executing the antiderivative easier. Students must write out the general antiderivative with the constant of integration as their final solution. If the exercise asks for a particular solution, students must demonstrate the work used to solve for the particular solution. If students do not wish to write out "… where *C* is a constant" for each exercise, students can write at the top of their paper before starting any work "For the following assignment, all *C*'s are constants, unless stated otherwise." If you use substitution/change of variables you must state the substitution/change of variable and write your final answer in terms of the variable the exercise is written with respect to.

Ex:

Find the indefinite integral with the given condition:

$$f(x)=\int x\left(x^2+1\right)^3 dx \text{ with } f(0)=6$$

Solution:

$$\text{Let } \begin{aligned} u &= x^2+1 \\ du &= 2x\,dx \end{aligned}$$

$$\begin{aligned}
\int x\left(x^2+1\right)^3 dx &= \frac{1}{2}\int \left(x^2+1\right)^3 2x\,dx \\
&= \frac{1}{2}\int u^3 du \\
&= \frac{1}{2}\left(\frac{1}{4}u^4+C\right) \\
&= \frac{1}{8}u^4+C \\
&= \frac{1}{8}\left(x^2+1\right)^4+C
\end{aligned}$$

$$\to \frac{1}{8}\left((0)^2+1\right)+C=6 \qquad C=\frac{47}{8}$$

$$\to \text{Solution: } f(x)=\frac{1}{8}\left(x^2+1\right)^4+\frac{47}{8}$$

Definite Integrals:

When evaluating definite integrals, student solutions must include the antiderivative/indefinite integral with the evaluation bar noted with the upper and lower bounds of the integral. The algebra of evaluating the indefinite integrals at the bounds is not necessary. On the AP Exam Free Response, students should write out an expression for the solution, but not simplify.

Ex: Evaluate the definite integral of the algebraic function. $\int_1^4 x^3 dx$

Solution:

$$\int_1^4 x^3 dx = \left[\frac{1}{4}x^4\right]_1^4 \leftarrow \text{required for AP Exam and homework}$$

$$= \left[\frac{1}{4}(4)^4\right] - \left[\frac{1}{4}(1)^4\right] \leftarrow \text{recommended for AP Free Response}$$

$$= 63.75 \text{ or } \frac{255}{4} \text{ or } \left[\frac{1}{4}(4)^4\right] - \left[\frac{1}{4}(1)^4\right] \leftarrow \text{required for homework}$$

When evaluating definite integrals that may be determined using the method of u-substitution, students have two options to solve for the value of the integral – evaluate in terms of x or in terms of u. If one chooses to evaluate the integral in terms of u, one must demonstrate (1) the choice of u, (2) du, and (3) change the bounds according to u.

Evaluate in terms of u

$$\int_{-\sqrt{3}}^{0} \frac{x}{1+x^2} dx$$

$$\text{Let } \left.\begin{aligned} u &= 1+x^2 \\ du &= 2xdx \\ u(0) &= 1 \\ u(2) &= 4 \end{aligned}\right\} \text{must be demonstrated}$$

$$\int_{-\sqrt{3}}^{0} \frac{x}{1+x^2} dx = \int_{-\sqrt{3}}^{0} \frac{1}{1+x^2} \cdot xdx$$

$$= \frac{1}{2}\int_{-\sqrt{3}}^{0} \frac{1}{1+x^2} \cdot 2xdx$$

$$\downarrow$$

$$= \frac{1}{2}\int_{u(-\sqrt{3})}^{u(0)} \frac{1}{u} du$$

$$= \frac{1}{2}\int_{4}^{1} \frac{1}{u} du$$

$$= \frac{1}{2}\left[\ln|u|\right]_4^1$$

$$= \frac{1}{2}\left(\ln|1| - \ln|4|\right)$$

Evaluate in terms of x

$$\int_{-\sqrt{3}}^{0} \frac{x}{1+x^2} dx = \frac{1}{2}\int_{-\sqrt{3}}^{0} \frac{1}{1+x^2} \cdot 2xdx$$

$$= \frac{1}{2}\left[\ln\left|1+x^2\right|\right]_{-\sqrt{3}}^{0}$$

$$= \frac{1}{2}\left[\ln\left|1+(0)^2\right| - \ln\left|1+\left(-\sqrt{3}\right)^2\right|\right]$$

NOTE: If at any intermediate step in your solution your bounds are not correct for the variable you are integrating with respect to, your work will be penalized. For example, if your solution on the left had the following expression

$$\frac{1}{2}\int_{\boxed{-\sqrt{3}}}^{\boxed{0}} \frac{1}{u} d\boxed{u}$$

your work would be penalized.

Improper Integrals

Improper integrals are integrals of the form $\int_c^{\infty} f(x)\,dx$, $\int_{-\infty}^{c} f(x)\,dx$, $\int_{-\infty}^{\infty} f(x)\,dx$, or $\int_a^b f(x)\,dx$ where $f(x)$ has an infinite discontinuity at some x in the interval $[a,b]$. All of these integrals *must* be evaluated using limits.

$$\int_c^{\infty} f(x)\,dx \to \lim_{b\to\infty}\int_c^b f(x)\,dx$$

$$\int_{-\infty}^{c} f(x)\,dx \to \lim_{b\to-\infty}\int_c^b f(x)\,dx$$

$$\int_{-\infty}^{\infty} f(x)\,dx \to \lim_{a\to-\infty}\int_a^k f(x)\,dx+\lim_{b\to\infty}\int_k^b f(x)\,dx$$

$$\int_a^b f(x)\,dx \to \lim_{w\to c^-}\int_a^w f(x)\,dx+\lim_{w\to c^+}\int_w^b f(x)\,dx \text{ where } a<c<b, \text{ and } f(x) \text{ has an infinite discontinuity at } x=c$$

$$\int_1^{\infty}\frac{1}{x^3}\,dx=\lim_{b\to\infty}\int_1^b\frac{1}{x^3}\,dx$$

$$=\lim_{b\to\infty}\int_1^b x^{-3}dx$$

$$=-\lim_{b\to\infty}\left[-\frac{1}{2}x^{-2}\right]_1^b$$

$$=\lim_{b\to\infty}\left[\left(-\frac{1}{2}(b)^{-2}\right)-\left(-\frac{1}{2}(1)^{-2}\right)\right]$$

$$=\lim_{b\to\infty}\left[\left(-\frac{1}{2}\cdot\frac{1}{b^2}\right)-\left(-\frac{1}{2}\cdot\frac{1}{1^2}\right)\right]$$

$$=\frac{1}{2}$$

$$\int_0^1\frac{1}{\sqrt[3]{x}}\,dx=\lim_{b\to0^+}\int_b^1 x^{-\frac{1}{3}}dx$$

$$=\lim_{b\to0^+}\left[\frac{3}{2}x^{\frac{2}{3}}\right]_b^1$$

$$=\lim_{b\to0^+}\left[\left(\frac{3}{2}[1]^{\frac{2}{3}}\right)-\left(\frac{3}{2}[b]^{\frac{2}{3}}\right)\right]$$

$$=\frac{3}{2}$$

$$\int_{-\infty}^{\infty}\frac{2}{4+x^2}\,dx$$

$$2\int_{-\infty}^{\infty}\frac{1}{4+x^2}\,dx$$

$$2\left[\lim_{a\to-\infty}\int_a^0\frac{1}{4+x^2}\,dx+\lim_{b\to\infty}\int_0^b\frac{1}{4+x^2}\,dx\right]$$

$$2\left[\lim_{a\to-\infty}\left[\frac{1}{2}\arctan\left(\frac{x}{2}\right)\right]_a^0+\lim_{b\to\infty}\left[\frac{1}{2}\arctan\left(\frac{x}{2}\right)\right]_0^b\right]$$

$$2\left[\lim_{a\to-\infty}\left[\left(\frac{1}{2}\arctan\left(\frac{0}{2}\right)\right)-\left(\frac{1}{2}\arctan\left(\frac{a}{2}\right)\right)\right]+\lim_{b\to\infty}\left[\left(\frac{1}{2}\arctan\left(\frac{b}{2}\right)\right)-\left(\frac{1}{2}\arctan\left(\frac{0}{2}\right)\right)\right]\right]$$

$$2\left[\frac{1}{2}\left[0-\left(-\frac{\pi}{2}\right)\right]+\frac{1}{2}\left[\left(\frac{\pi}{2}\right)-0\right]\right]=\pi$$

Describing the Meaning of Definite Integrals:

To describe the meaning of a definite integral in a word problem, one must remember that the definite integral of a rate of change describes the net change in the original function.

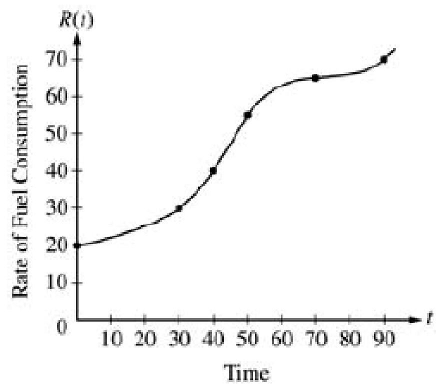
$$\underbrace{\int_a^b f'(x)\,dx}_{\substack{\text{definite integral} \\ \text{of the rate of change} \\ \text{of } f \text{ from } x=a \text{ to } x=b}} = \underbrace{f(b)-f(a)}_{\substack{\text{net change in the} \\ \text{function } f \text{ from} \\ x=a \text{ to } x=b}}$$

When describing the meaning of the integral, one must always make reference to the bounds of the definite integral.

Example: 2003 AB FRQ #3

The rate of fuel consumption, in gallons per minute, recorded during an airplane flight is given by a twice differentiable and strictly increasing function R of time t. The graph of R and a table of selected values of $R(t)$, for the time interval $0 \le t \le 90$ minutes are shown.

(e) For $0 < b \le 90$ minutes, explain the meaning of $\int_0^b R(t)\,dt$ in terms of fuel consumption for the plane. Explain the meaning of $\frac{1}{b}\int_0^b R(t)\,dt$ in terms of fuel consumption for the plane. Indicate units of measure.



t (minutes)	$R(t)$ (gallons per minute)
0	20
30	30
40	40
50	55
70	65
90	70

Response:

$\int_0^b R(t)\,dt$ represents the total fuel, in gallons, consumed from time $t = 0$ to time $t = b$.

$\frac{1}{b}\int_0^b R(t)\,dt$ represents the average rate of fuel consumption, in gallons per minute, from time $t = 0$ to time $t = b$.