$$f(x)=e^x$$

$$\underset{\substack{\text{centered at}\\ x=c}}{f(x)} = f(c)+f'(c)\cdot(x-c)+\frac{f''(c)\cdot(x-c)^2}{2!}+\frac{f'''(c)\cdot(x-c)^3}{3!}+\cdots+\frac{f^{(n)}(c)\cdot(x-c)^n}{n!}+\cdots$$

$$=\sum_{n=0}^{\infty}\frac{f^{(n)}(c)\cdot(x-c)^n}{n!}$$

Consider the series centered at $x=0$

$$\begin{array}{ll} f(x)=e^x & f(0)=e^0=1 \\ f'(x)=e^x & f'(0)=e^0=1 \\ f''(x)=e^x & f''(0)=e^0=1 \\ \vdots & \vdots \\ f^{(n)}(x)=e^x & f^{(n)}(0)=e^0=1 \\ \vdots & \vdots \end{array}$$

$$\underset{\substack{\text{centered at}\\ x=0}}{e^x} = f(0)+f'(0)\cdot(x-0)+\frac{f''(0)\cdot(x-0)^2}{2!}+\frac{f'''(0)\cdot(x-0)^3}{3!}+\cdots+\frac{f^{(n)}(0)\cdot(x-0)^n}{n!}+\cdots$$

$$=1+1\cdot(x-0)+\frac{1\cdot(x-0)^2}{2!}+\frac{1\cdot(x-0)^3}{3!}+\cdots+\frac{1\cdot(x-0)^n}{n!}+\cdots$$

$$=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^n}{n!}+\cdots$$

$$=\sum_{n=0}^{\infty}\frac{x^n}{n!}$$

$$\lim_{n\to\infty}\left|\frac{\left(\frac{x^{n+1}}{(n+1)!}\right)}{\left(\frac{x^n}{n!}\right)}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}\right|$$

$$=\lim_{n\to\infty}\left|\frac{x^n\cdot x^1}{(n+1)\cdot(n!)}\cdot\frac{n!}{x^n}\right|$$

$$=\lim_{n\to\infty}\left|\frac{\cancel{x^n}\cdot x^1}{(n+1)\cdot\cancel{(n!)}}\cdot\frac{\cancel{n!}}{\cancel{x^n}}\right|$$

$$=\lim_{n\to\infty}\left|\frac{x}{(n+1)}\right|$$

$$=0$$

This series will converge for all real numbers.

Therefore

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \text{ for all real numbers.}$$

For example

$$\begin{aligned} e^2 &= 1 + (2) + \frac{(2)^2}{2!} + \frac{(2)^3}{3!} + \cdots + \frac{(2)^n}{n!} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{2^n}{n!} \end{aligned}$$

This also means that the Taylor series for $e^{\left(x^2\right)}$ centered at $c = 0$ would be given by

$$\begin{aligned} e^{\left(x^2\right)} &= 1 + \left(x^2\right) + \frac{\left(x^2\right)^2}{2!} + \frac{\left(x^2\right)^3}{3!} + \cdots + \frac{\left(x^2\right)^n}{n!} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{\left(x^2\right)^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} \end{aligned}$$

This series will also converge for all real numbers.

One should also notice that

$$1 + \pi + \frac{\pi^2}{2!} + \frac{\pi^3}{3!} + \cdots + \frac{\pi^n}{n!} + \cdots = e^{\pi}$$

$$g(x) = \sin(x)$$

$$\underset{\substack{\text{cenered at} \\ x=c}}{g(x)} = g(c) + g'(c)\cdot(x-c) + \frac{g''(c)\cdot(x-c)^2}{2!} + \frac{g'''(c)\cdot(x-c)^3}{3!} + \cdots + \frac{g^{(n)}(c)\cdot(x-c)^n}{n!} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{g^{(n)}(c)\cdot(x-c)^n}{n!}$$

Consider the series centered at $x = 0$

$$\begin{array}{ll} g(x) = \sin(x) & g(0) = 0 \\ g'(x) = \cos(x) & g'(0) = \cos(0) = 1 \\ g''(x) = -\sin(x) & g''(0) = -\sin(0) = 0 \\ g'''(x) = -\cos(x) & g'''(0) = -\cos(0) = -1 \\ g^{(4)}(x) = \sin(x) & g^{(4)}(0) = \sin(0) = 0 \\ g^{(5)}(x) = \cos(x) & g^{(5)}(0) = \cos(0) = 1 \\ g^{(6)}(x) = -\sin(x) & g^{(6)}(0) = -\sin(0) = 0 \\ \vdots & \vdots \end{array}$$

$$\underset{\substack{\text{centered at} \\ c=0}}{\sin(x)} = g(0) + g'(0)\cdot(x-0) + \frac{g''(0)\cdot(x-0)^2}{2!} + \frac{g'''(0)\cdot(x-0)^3}{3!} + \cdots + \frac{g^{(n)}(0)\cdot(x-0)^n}{n!} + \cdots$$

$$= g(0) + g'(0)\cdot x + \frac{g''(0)\cdot x^2}{2!} + \frac{g'''(0)\cdot x^3}{3!} + \cdots + \frac{g^{(n)}(0)\cdot x^n}{n!} + \cdots$$

$$= 0 + (1)\cdot x + \frac{(0)\cdot x^2}{2!} + \frac{(-1)\cdot x^3}{3!} + \frac{(0)\cdot x^4}{4!} + \frac{(1)\cdot x^5}{5!} + \frac{(0)\cdot x^6}{6!} + \cdots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\lim_{n\to\infty}\left|\frac{\left(\frac{(-1)^{2(n+1)+1}\cdot x^{2(n+1)+1}}{(2(n+1)+1)!}\right)}{\left(\frac{(-1)^{2n+1}\cdot x^{2n+1}}{(2n+1)!}\right)}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{2(n+1)+1}\cdot x^{2(n+1)+1}}{(2(n+1)+1)!}\cdot\frac{(2n+1)!}{(-1)^{2n+1}\cdot x^{2n+1}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(-1)^{2n+3}\cdot x^{2n+3}}{(2n+3)!}\cdot\frac{(2n+1)!}{(-1)^{2n+1}\cdot x^{2n+1}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x^{2n+3}}{(2n+3)!}\cdot\frac{(2n+1)!}{x^{2n+1}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x^{2n+1}\cdot x^{2}}{(2n+3)(2n+2)\left[(2n+1)!\right]}\cdot\frac{(2n+1)!}{x^{2n+1}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{\cancel{x^{2n+1}}\cdot x^{2}}{(2n+3)(2n+2)\cancel{\left[(2n+1)!\right]}}\cdot\frac{\cancel{(2n+1)!}}{\cancel{x^{2n+1}}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x^{2}}{(2n+3)(2n+2)}\right|$$

$$= 0$$

The series will converge for all real numbers. Therefore

$$\sin(x) = \sum_{n=0}^{\infty}\frac{(-1)^n\cdot x^{2n+1}}{(2n+1)!}$$

For all real numbers.

Remember: $\sin(x)$ is an odd function. Therefore the exponents and the factorials are odd.

The expansion for $\sin(x^3)$ centered at $c=0$ will be given by

$$\sin(x^3) = \sum_{n=0}^{\infty}\frac{(-1)^n\cdot (x^3)^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n\cdot x^{6n+3}}{(2n+1)!}$$

One should also notice that

$$-\sqrt{2}+\frac{(\sqrt{2})^3}{3!}-\frac{(\sqrt{2})^5}{5!}-\frac{(\sqrt{2})^7}{7!}+\frac{(\sqrt{2})^9}{9!}+\cdots=\sin(\sqrt{2})$$

$$h(x)=\cos(x)$$

$$\underset{\substack{\text{centered at}\\ x=c}}{h(x)}=h(c)+h'(c)\cdot(x-c)+\frac{h''(c)\cdot(x-c)^2}{2!}+\frac{h'''(c)\cdot(x-c)^3}{3!}+\cdots+\frac{h^{(n)}(c)\cdot(x-c)^n}{n!}+\cdots$$

$$=\sum_{n=0}^{\infty}\frac{f^{(n)}(c)\cdot(x-c)^n}{n!}$$

Consider the series centered at $x=0$

$$\begin{array}{ll}
h(x)=\cos(x) & h(0)=\cos(0)=1\\
h'(x)=-\sin(x) & h'(0)=-\sin(0)=0\\
h''(x)=-\cos(x) & h''(0)=-\cos(0)=-1\\
h'''(x)=\sin(x) & h'''(0)=\sin(0)=0\\
h^{(4)}(x)=\cos(x) & h^{(4)}(0)=\cos(0)=1\\
h^{(5)}(x)=-\sin(x) & h^{(6)}(0)=-\sin(0)=0\\
h^{(6)}(x)=-\cos(0) & h^{(6)}(0)=-\cos(0)=-1\\
\vdots & \vdots
\end{array}$$

$$\underset{\substack{\text{centered at}\\ c=0}}{h(x)}=h(0)+h'(0)\cdot(x-0)+\frac{h''(0)\cdot(x-0)^2}{2!}+\frac{h'''(0)\cdot(x-0)^3}{3!}+\cdots+\frac{h^{(n)}(0)\cdot(x-0)^n}{n!}+\cdots$$

$$=h(0)+h'(0)\cdot x+\frac{h''(0)\cdot x^2}{2!}+\frac{h'''(0)\cdot x^3}{3!}+\cdots+\frac{h^{(n)}(0)\cdot x^n}{n!}+\cdots$$

$$=1+(0)\cdot x+\frac{(-1)\cdot x^2}{2!}+\frac{(0)\cdot x^3}{3!}+\frac{(1)\cdot x^4}{4!}+\frac{(0)\cdot x^5}{5!}+\frac{(-1)\cdot x^6}{6!}+\cdots$$

$$=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}-\frac{x^{10}}{10!}+\cdots$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n\cdot x^{2n}}{(2n)!}$$

$$\lim_{n\to\infty}\left|\frac{\left(\frac{(-1)^{n+1}\cdot x^{2(n+1)}}{(2(n+1))!}\right)}{\frac{(-1)^n\cdot x^{2n}}{(2n)!}}\right|=\lim_{n\to\infty}\left|\frac{(-1)^{n+1}\cdot x^{2(n+1)}}{(2(n+1))!}\cdot\frac{(2n)!}{(-1)^n\cdot x^{2n}}\right|$$

$$=\lim_{n\to\infty}\left|\frac{x^{2(n+1)}}{(2(n+1))!}\cdot\frac{(2n)!}{\cdot x^{2n}}\right|$$

$$=\lim_{n\to\infty}\left|\frac{x^{2n+2}}{(2n+2)!}\cdot\frac{(2n)!}{\cdot x^{2n}}\right|$$

$$=\lim_{n\to\infty}\left|\frac{x^{2n}\cdot x^2}{(2n+2)\cdot(2n+1)\cdot[(2n)!]}\cdot\frac{(2n)!}{x^{2n}}\right|$$

$$=\lim_{n\to\infty}\left|\frac{\cancel{x^{2n}}\cdot x^2}{(2n+2)\cdot(2n+1)\cdot\cancel{[(2n)!]}}\cdot\frac{\cancel{(2n)!}}{\cancel{x^{2n}}}\right|$$

$$=\lim_{n\to\infty}\left|\frac{x^2}{(2n+2)\cdot(2n+1)}\right|$$

$$=0$$

This series will converge for all real numbers. Therefore

$$\cos(x)=\sum_{n=1}^{\infty}\frac{(-1)^n\cdot x^{2n}}{(2n)!}$$

Remember that $\cos(x)$ is an even function, therefore the exponents and the factorials are even.

Notice the following:

$$1-\frac{3^2}{2!}+\frac{3^4}{4!}-\frac{3^6}{6!}+\frac{3^8}{8!}-\frac{3^{10}}{10!}+\cdots=\cos(3)$$

The Taylor series for $\cos(\sqrt{x})$ will be given by:

$$\cos(\sqrt{x})=\sum_{n=1}^{\infty}\frac{(-1)^n\cdot(\sqrt{x})^{2n}}{(2n)!}$$

$$=\sum_{n=1}^{\infty}\frac{(-1)^n\cdot\left(x^{\frac{1}{2}}\right)^{2n}}{(2n)!}$$

$$=\sum_{n=1}^{\infty}\frac{(-1)^n\cdot x^n}{(2n)!}$$