Integration by Parts Examples:

1. $\int x^2 \sin(x)\,dx$

<u>Repeated Iterations of Integration by Parts</u>

$$\int uv' = uv - \int u'v$$

$u = x^2 \quad v' = \sin(x)$
$u' = 2x \quad v = -\cos(x)$

$$\int x^2 \sin(x)\,dx = x^2\left(-\cos(x)\right) - \int 2x\left(-\cos(x)\right)dx$$
$$= -x^2 \cdot \cos(x) + \boxed{\int 2x\cos(x)}$$

$u = 2x \quad v' = \cos(x)$
$u' = 2 \quad v = \sin(x)$

$$\int x^2 \sin(x)\,dx = x^2\left(-\cos(x)\right) - \int 2x\left(-\cos(x)\right)dx$$
$$= -x^2 \cdot \cos(x) + \boxed{\int 2x\cos(x)}$$
$$= -x^2 \cdot \cos(x) + \boxed{2x \cdot \sin(x) - \int 2\sin(x)\,dx}$$
$$= -x^2 \cdot \cos(x) + 2x \cdot \sin(x) + 2\cos(x) + C$$

<u>Tabular Method:</u>

$u \qquad v'$

$+$
$-$
$+$
$-$
$\vdots$

u	v'
x^2	$\sin(x)$
$2x$	$-\cos(x)$
2	$-\sin(x)$
0	$\cos(x)$

$$\int x^2 \sin(x)\,dx = -x^2\cos(x) + 2x\sin(x) + 2\cos(x) + C$$

2. $\int x^4 \ln(x)\,dx$

$$\int uv' = uv - \int u'v$$

$$u = \ln(x) \quad v' = x^4$$
$$u' = \frac{1}{x} \quad v = \frac{1}{5}x^5$$

$$\begin{aligned}\int x^4 \ln(x)\,dx &= \frac{1}{5}x^5 \ln(x) - \int \frac{1}{5}x^5 \cdot \frac{1}{x}dx \\ &= \frac{1}{5}x^5 \ln(x) - \int \frac{1}{5}x^4 dx \\ &= \frac{1}{5}x^5 \ln(x) - \frac{1}{25}x^5 dx + C\end{aligned}$$

3. $\int \ln(x)\,dx = \int 1 \cdot \ln(x)\,dx$

$$\int uv' = uv - \int u'v$$

$$u = \ln(x) \quad v' = 1$$
$$u' = \frac{1}{x} \quad v = x$$

$$\begin{aligned}\int 1 \cdot \ln(x)\,dx &= x\ln(x) - \int \frac{1}{x} \cdot x dx \\ &= x\ln(x) - \int 1 dx \\ &= x\ln(x) - x + C\end{aligned}$$

4. $\int \arcsin(x)\,dx = \int 1 \cdot \arcsin(x)\,dx$

$$\int uv' = uv - \int u'v$$

$$u = \arcsin(x) \quad v' = 1$$
$$u' = \frac{1}{\sqrt{1-x^2}} \quad v = x$$

$$\begin{aligned}\int 1 \cdot \arcsin(x)\,dx &= x \cdot \arcsin(x) - \int \frac{1}{\sqrt{1-x^2}} \cdot x dx \\ &= x \cdot \arcsin(x) - \left(-\frac{1}{2}\right)\int \frac{1}{\sqrt{1-x^2}} \cdot (-2) x dx \\ &= x \cdot \arcsin(x) + \left(1-x^2\right)^{\frac{1}{2}} + C\end{aligned}$$

Do a substitution with $\begin{array}{l} w = 1 - x^2 \\ dw = -2xdx \end{array}$

5. $\int x^2 e^{2x} dx$

Repeated Iterations of Integration by Parts

$$\int uv' = uv - \int u'v$$

$u = x^2 \quad v' = e^{2x}$

$u' = 2x \quad v = \frac{1}{2}e^{2x}$

$$\begin{aligned} \int x^2 e^{2x} dx &= x^2 \cdot \frac{1}{2}e^{2x} - \int 2x \cdot \frac{1}{2}e^{2x} dx \\ &= x^2 \cdot \frac{1}{2}e^{2x} - \boxed{\int x e^{2x} dx} \end{aligned}$$

$u = x \quad v' = e^{2x}$

$u' = 1 \quad v = \frac{1}{2}e^{2x}$

$$\begin{aligned} \int x^2 e^{2x} dx &= x^2 \cdot \frac{1}{2}e^{2x} - \int 2x \cdot \frac{1}{2}e^{2x} dx \\ &= x^2 \cdot \frac{1}{2}e^{2x} - \boxed{\int x e^{2x} dx} \\ &= x^2 \cdot \frac{1}{2}e^{2x} - \left[x \cdot \frac{1}{2}e^{2x} - \int \frac{1}{2}e^{2x} dx \right] \\ &= x^2 \cdot \frac{1}{2}e^{2x} - x \cdot \frac{1}{2}e^{2x} + \int \frac{1}{2}e^{2x} dx \\ &= x^2 \cdot \frac{1}{2}e^{2x} - x \cdot \frac{1}{2}e^{2x} + \frac{1}{4}e^{2x} + C \end{aligned}$$

Tabular Method:

u	v'
x^2	e^{2x}
$2x$	$\frac{1}{2}e^{2x}$
2	$\frac{1}{4}e^{2x}$
0	$\frac{1}{8}e^{2x}$

$u \quad v'$

differentiate $\quad + \searrow \quad - \searrow \quad + \searrow \quad - \searrow \quad \vdots \quad$ antidifferentiate

$$\int x^2 e^{2x} dx = x^2 \cdot \frac{1}{2}e^{2x} - x \cdot \frac{1}{2}e^{2x} + \frac{1}{4}e^{2x} + C$$

6. $\int e^x \sin(x)\,dx$

<u>Repeated Iterations of Integration by Parts</u>

$$\int uv' = uv - \int u'v$$

$$\begin{aligned} u &= e^x & v' &= \sin(x) \\ u' &= e^x & v &= -\cos(x) \end{aligned} \qquad \begin{aligned} \int e^x \sin(x)\,dx &= -e^x\cos(x) - \int e^x\left(-\cos(x)\right)dx \\ &= -e^x\cos(x) + \int e^x\cos(x)\,dx \end{aligned}$$

$$\begin{aligned} u &= e^x & v' &= \cos(x) \\ u' &= e^x & v &= \sin(x) \end{aligned} \qquad \int e^x\cos(x)\,dx = e^x\sin(x) - \int e^x\sin(x)\,dx$$

$$\begin{aligned}
\int e^x\sin(x)\,dx &= -e^x\cos(x) - \int e^x\left(-\cos(x)\right)dx \\
\int e^x\sin(x)\,dx &= -e^x\cos(x) + \boxed{\int e^x\cos(x)\,dx} \\
\int e^x\sin(x)\,dx &= -e^x\cos(x) + e^x\sin(x) - \int e^x\sin(x)\,dx \\
\underline{+\int e^x\sin(x)\,dx} &= \underline{\qquad\qquad\qquad\qquad\qquad + \int e^x\sin(x)\,dx} \\
2\int e^x\sin(x)\,dx &= -e^x\cos x + e^x\sin(x) \\
\int e^x\sin(x)\,dx &= \frac{1}{2}\left[-e^x\cos x + e^x\sin(x)\right] + C
\end{aligned}$$

<u>Modified Tabular Method:</u>

u	v'	$\int uv'$
e^x	$\sin(x)$	$\boxed{\int e^x\sin(x)\,dx}$
e^x	$-\cos(x)$	$\int -e^x\cos(x)\,dx$
e^x	$-\sin(x)$	$\boxed{\int -e^x\sin(x)\,dx}$

$$\begin{array}{ccc}
u & v' & \int \\
\overset{+}{\searrow} & & \text{original} \\
\overset{-}{\searrow} & \overset{-}{\rightarrow} & \\
\overset{+}{\searrow} & \overset{+}{\rightarrow} & \\
\overset{-}{\searrow} & \overset{-}{\rightarrow} & \\
\vdots & \vdots &
\end{array}$$

$$\begin{aligned}
\int e^x\sin(x)\,dx &= -e^x\cos x + e^x\sin(x) + \int -e^x\sin(x)\,dx \\
\int e^x\sin(x)\,dx &= -e^x\cos x + e^x\sin(x) - \int e^x\sin(x)\,dx \\
\underline{+\int e^x\sin(x)\,dx} &= \underline{\qquad\qquad\qquad\qquad\qquad + \int e^x\sin(x)\,dx} \\
2\int e^x\sin(x)\,dx &= -e^x\cos x + e^x\sin(x) \\
\int e^x\sin(x)\,dx &= \frac{1}{2}\left[-e^x\cos x + e^x\sin(x)\right] + C
\end{aligned}$$