Integration by Substitution

In order to understand Integration by Substitution, you must master 3 things:

I. **Understanding of the Chain Rule**
II. **Recognize patterns of the Chain Rule**
III. **Perform *Change of Variables***

I. Understanding Chain Rule

Let

$$y = F(u)$$
$$u = g(x)$$

Then

$$\frac{d}{dx}\left[F(u)\right] = \frac{d}{dx}\left[F(g(x))\right]$$
$$= F'(g(x)) \cdot g'(x)$$

Applying antidifferentiation knowledge we get

$$\int F'(g(x)) \cdot g'(x)dx = F(g(x)) + C$$

Theorem:
Let g be a function whose range is an interval I
Let f be a function that is continuous on I
If g is a differentiable on its domain and F is an antiderivative of f on I, then

$$\int f(g(x)) \cdot g'(x)\,dx = F(g(x)) + C$$

Making the substitution $u = g(x)$ we get that

On homework, you can go from the first line to last line.

$$u = g(x)$$
$$\frac{d}{dx}[u] = \frac{d}{dx}\left[g(x)\right]$$
$$\frac{du}{dx} = g'(x)$$
$$dx \cdot \left[\frac{du}{dx}\right] = dx \cdot \left[g'(x)\right]$$
$$du = g'(x)\,dx$$

We can then rewrite the integral as follows:

$$\int f\big(\underbrace{g(x)}_{u}\big) \cdot \underbrace{g'(x)\,dx}_{du} = \int f(u)\,du$$
$$= F(u) + C$$

The point, is that we have made the more complicated looking integral on the left look much less complicated on the right!

II. Pattern Recognition

Recognize the form of the previous theorem in these examples:

$$\int \left(x^2+1\right)2x\,dx \qquad \int e^{2x^3}\cdot 6x^2\,dx \qquad \int 5\cos\left(5x\right)dx$$

Note about the patterns: sometimes you have the essential part, but are missing constant multiple for $g'\left(x\right)$. You can insert a factor of "1" into the integrand, and take the part of "1" you don't need out.

$$\begin{array}{ccc}
\int \left(x^2+1\right)x\,dx & \int \cos\left(5x\right)dx & \int e^{2x^3}x^2\,dx \\
\downarrow & \downarrow & \downarrow \\
\int \left(x^2+1\right)\frac{2}{2}x\,dx & \int \cos\left(5x\right)\cdot\frac{5}{5}dx & \int e^{2x^3}\cdot\frac{6}{6}\cdot x^2\,dx \\
\frac{1}{2}\int \left(x^2+1\right)2x\,dx & \frac{1}{5}\int \cos\left(5x\right)5\,dx & \frac{1}{6}\int e^{2x^3}6x^2\,dx
\end{array}$$

You can introduce a factor of *x* into the integrand so long as (1) it is a form of "1" and (2) all *x*'s stay inside the integral.

III. Change of Variables (Indefinite Integrals)

Choose $u=g\left(x\right)$ as the inner part of a function composition.

Given $\int \sqrt{2x+1}\,dx$. Let

$$u=2x+1$$
$$du=2dx$$

Since the integral is missing a factor of 2, we can introduce a factor of $1=\frac{2}{2}$ into the integrand.

$$\begin{array}{c}
\int \sqrt{2x+1}\,dx \\
\int \sqrt{2x+1}\cdot\frac{2}{2}dx \\
\frac{1}{2}\int \sqrt{2x+1}\cdot 2dx \\
\frac{1}{2}\int \sqrt{u}\,du \\
\frac{1}{2}\int u^{\frac{1}{2}}du \\
\frac{1}{2}\left[\frac{2}{3}u^{\frac{3}{2}}+C\right] \\
\frac{1}{3}u^{\frac{3}{2}}+C \\
\frac{1}{3}\left(2x+1\right)^{\frac{3}{2}}+C
\end{array}$$

Guidelines for Change of Variables:

1. Choose $u=g\left(x\right)$. Usually best to choose the inner part of a composite function.
2. Compute $du=g'\left(x\right)dx$.

   [Keep an eye out for a constant multiple of $g'\left(x\right)$] Multiply by a form of "1" if necessary to get $g'\left(x\right)dx$.
3. Rewrite the original integral in terms of *u*.
4. Find the resulting integral with respect to *u*.
5. Switch the expression to be a function of *x* again.
6. Check by differentiation.

For homework, the choice of $\begin{array}{c} u=g\left(x\right) \\ du=g'\left(x\right)dx \end{array}$ must be demonstrated for each exercise.