Area and Sums:

The sum of the first n terms of a sequence $a_1, a_2, \ldots, a_n$ is given by

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

Where

$i =$ index of the summation

$a_i =$ i^{th} term / summand

$1 =$ starting index of the summation

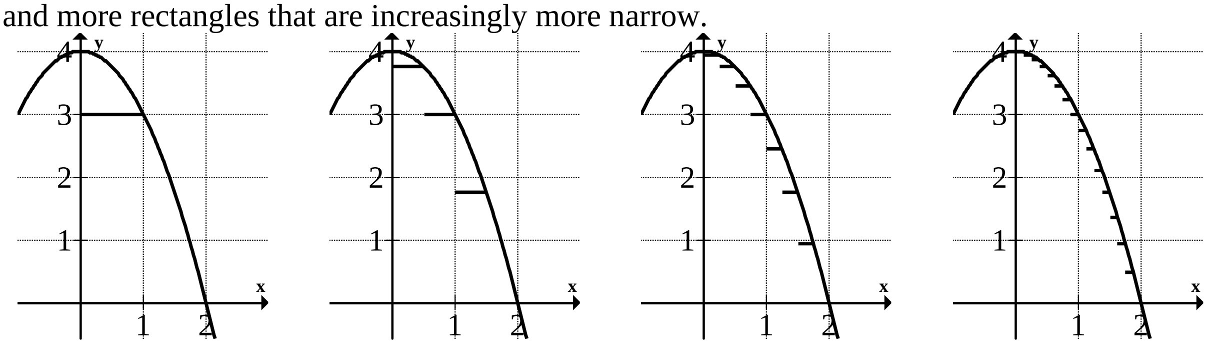
$n =$ ending index of the summation

Properties of Summations:

$$\sum_{i=1}^{n} k \cdot a_i = k \cdot \sum_{i=1}^{n} a_i$$

$$\sum_{i=1}^{n} \left(a_i \pm b_i\right) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$$

To estimate the area underneath a positive function and between the *x*-axis from $x = a$ to $x = b$ can be done by estimating the area with rectangles, and improving the estimates by making more and more rectangles that are increasingly more narrow.



$\Delta x = 1$ $\Delta x = 0.5$ $\Delta x = 0.25$ $\Delta x = 0.125$

The sums of the areas of the rectangles are called Riemann Sums.

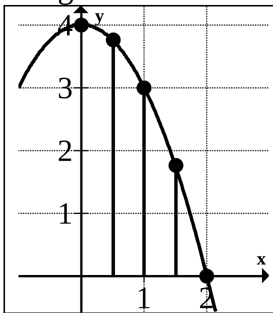
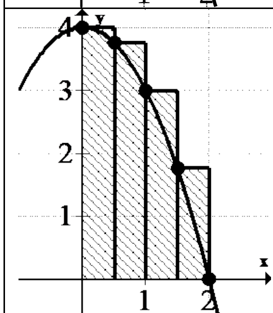
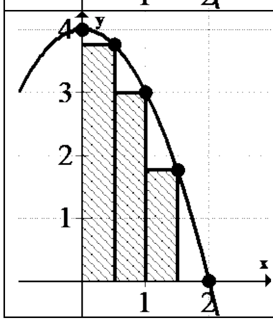
In order to start this process, you must first decide on the interval on which you want to find the area. In the above case, we were estimating the area under the curve of $f\left(x\right) = -x^2 + 4$ from $x = 0$ to $x = 2$.

To find the area under the curve, we must first divide the interval $\left[0, 2\right]$ into smaller subintervals. This is called <u>*partitioning the interval.*</u> The following are the partitions for each graph:

$\left\{0,1,2\right\}$	$\left\{0,\frac{1}{2},1,\frac{3}{2},2\right\}$	$\left\{0,\frac{1}{4},\frac{1}{2},\frac{3}{4},1,\frac{5}{4},\frac{3}{2},\frac{7}{4},2\right\}$	$\left\{0,\frac{1}{8},\frac{1}{4},\frac{3}{8},\frac{1}{2},\frac{5}{8},\frac{3}{4},\frac{7}{8},1,\frac{9}{8},\frac{5}{4},\frac{11}{8},\frac{3}{2},\frac{13}{8},\frac{7}{4},\frac{15}{8},2\right\}$

These partitions are considered *regular* because all subintervals are of equal length.

Consider the partition $\left\{0, \frac{1}{2}, 1, \frac{3}{2}, 2\right\}$. This divides up the interval into four subintervals of equal length of 0.5 each.

	Raw Partition $\Delta x = \frac{2-0}{4} = \frac{1}{2}$
	Using Left-Endpoints for height of rectangle: $\Delta x = \frac{2-0}{4} = \frac{1}{2}$ $\sum_{i=0}^{3} f\left(0 + i \cdot \Delta x\right) \cdot \Delta x = f\left(0\right) \cdot \frac{1}{2} + f\left(\frac{1}{2}\right) \cdot \frac{1}{2} + f\left(1\right) \cdot \frac{1}{2} + f\left(\frac{3}{2}\right) \cdot \frac{1}{2}$
	Using Right-Endpoints for height of rectangles: $\Delta x = \frac{2-0}{4} = \frac{1}{2}$ $\sum_{i=1}^{4} f\left(0 + i \cdot \Delta x\right) \cdot \Delta x = f\left(\frac{1}{2}\right) \cdot \frac{1}{2} + f\left(1\right) \cdot \frac{1}{2} + f\left(\frac{3}{2}\right) \cdot \frac{1}{2} + f\left(2\right) \cdot \frac{1}{2}$

<u>Theorem:</u>

Let *f* be continuous and non-negative function on $\left[a, b\right]$. Then the limit as $n \to \infty$ of both the lower and upper sums exist, and are equal to each other.

$$\lim_{n \to \infty} \sum_{i=1}^{n} f\left(m_i\right) \Delta x = \lim_{n \to \infty} \sum_{i=1}^{n} f\left(M_i\right) \Delta x$$

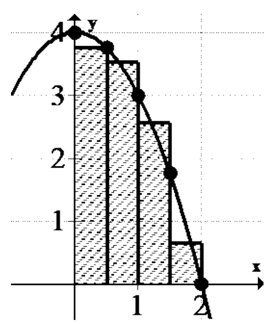
where

$$\Delta x = \frac{b-a}{n}$$

$f\left(m_i\right) =$ is the minimum value of $f\left(x\right)$ on $\left[x_{i-1}, x_i\right]$

$f\left(M_i\right) =$ is the maximum value of $f\left(x\right)$ on $\left[x_{i-1}, x_i\right]$

Overestimates and underestimates will converge to the true value as more and more rectangles of narrower width are used.



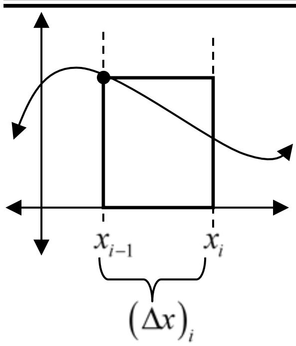
Because of the Squeeze Theorem and the two limits being equal, you can choose any function value in the subinterval as the height of the rectangle and still achieve the same result.

Definition of the Area of a Region in the Plane:

Let *f* be continuous a non-negative function on $[a,b]$. The area of the region bounded vertically by $f(x)$ and the *x*-axis, and horizontally by $x=a$ and $x=b$ is

$$A=\lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x=\int_a^b f(x)\,dx$$

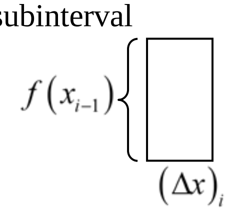
where $x_{i-1}\le c_i\le x_i$ and $\Delta x=\dfrac{b-a}{n}$.



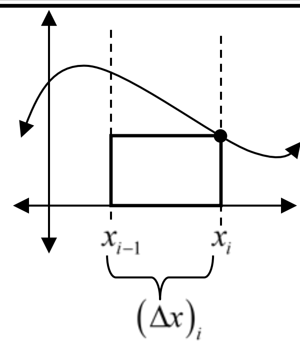
Left Sum

$f(x_{i-1})\cdot(\Delta x)_i$

Create rectangles with width Δx, and height using the function value at the left endpoint of the subinterval



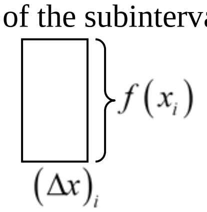
$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(x_{i-1})(\Delta x)_i$$



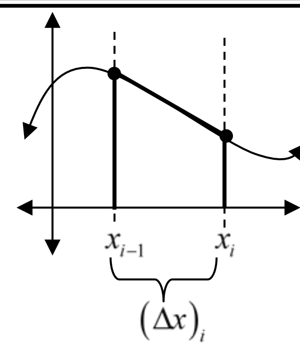
Right Sum

$f(x_i)\cdot(\Delta x)_i$

Create rectangles with width Δx, and height using the function value at the right endpoint of the subinterval



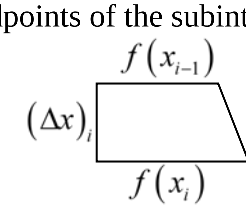
$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(x_i)(\Delta x)_i$$



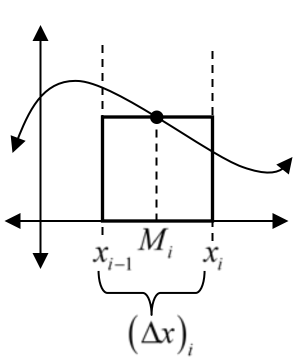
Trapezoidal Sum

$\frac{1}{2}\left(f(x_{i-1})+f(x_i)\right)\cdot(\Delta x)_i$

Create trapezoids with height Δx, and bases using the function values of at the endpoints of the subinterval



$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n}\frac{1}{2}\left[f(x_{i-1})+f(x_i)\right](\Delta x)_i$$



Midpoint Sum:

$f(M_i)\cdot(\Delta x)_i$

Create rectangles with width Δx, and height using the function value at the midpoint of the subinterval.

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(M_i)(\Delta x)_i$$