Why the Harmonic Series Diverges

Consider

$$\sum_{n=1}^{\infty}\frac{1}{n}=\frac{1}{2}+\underbrace{\frac{1}{3}+\frac{1}{4}}_{>\frac{1}{2}}+\underbrace{\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}}_{>\frac{1}{2}}+\underbrace{\frac{1}{9}+\cdots+\frac{1}{16}}_{>\frac{1}{2}}+\cdots$$

It will be demonstrated that this grouping of subsequences to form sums that are greater than $\frac{1}{2}$ can be done indefinitely.

Consider the following sum:

$$\underbrace{\frac{1}{2^n+1}+\frac{1}{2^n+2}+\cdots+\frac{1}{2^n+2^{n-1}}+\frac{1}{2^n+2^n}}_{2^n \text{ terms}}$$

Note that the last term in the sequence is $\frac{1}{2^n+2^n}=\frac{1}{2\left(2^n\right)}=\frac{1}{2^{n+1}}$

$$\underbrace{\frac{1}{2^n+1}+\frac{1}{2^n+2}+\cdots+\frac{1}{2^n+2^{n-1}}+\frac{1}{2^{n+1}}}_{2^n \text{ terms}}$$

Since $\frac{1}{2^{n+1}}$ is the smallest term in the sequence, we have that

$$\underbrace{\frac{1}{2^n+1}+\frac{1}{2^n+2}+\cdots+\frac{1}{2^n+2^{n-1}}+\frac{1}{2^{n+1}}}_{2^n \text{ terms}}>\underbrace{\frac{1}{2^{n+1}}+\frac{1}{2^{n+1}}+\cdots+\frac{1}{2^{n+1}}+\frac{1}{2^{n+1}}}_{2^n \text{ terms}}=2^n\left(\frac{1}{2^{n+1}}\right)=\frac{1}{2}$$

Because we have an infinite series, we can continue to group an infinite amount of subsequences of increasing length that will add up to something greater than $\frac{1}{2}$.

Therefore

$$\sum_{n=1}^{\infty}\frac{1}{n}=\frac{1}{2}+\underbrace{\frac{1}{3}+\frac{1}{4}}_{>\frac{1}{2}}+\underbrace{\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}}_{>\frac{1}{2}}+\underbrace{\frac{1}{9}+\cdots+\frac{1}{16}}_{>\frac{1}{2}}+\cdots>1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\cdots=1+\sum_{n=1}^{\infty}\frac{1}{2}\leftrightarrow\infty$$

Hence the Harmonic Series diverges, because it is greater than an infinite number of $\frac{1}{2}'s$ added together.