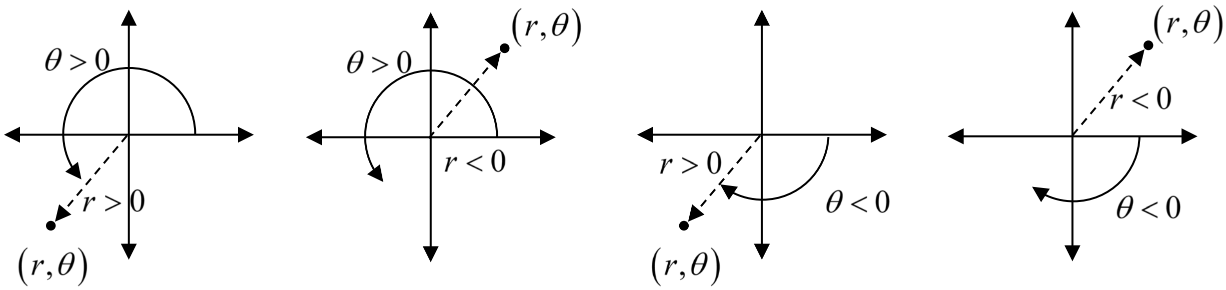
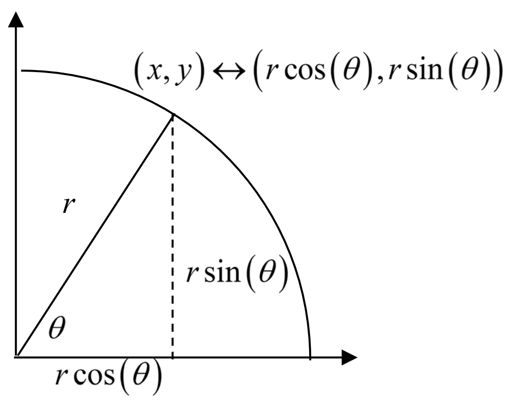
Calculus of Polar Functions

$(x,y)\leftrightarrow(r,\theta)$ Where $r=f(\theta)$	To plot a point (r,θ) in polar form (1) Rotate by θ radians in the appropriate direction ✓ Counterclockwise for $\theta>0$ ✓ Clockwise for $\theta<0$ (2) Extend from the origin the appropriate magnitude $\lvert r\rvert$ and proper direction ✓ For $r>0$, extend in the direction of the terminal side of θ by $\lvert r\rvert$ ✓ For $r<0$, extend in the opposite direction of the terminal side of θ by $\lvert r\rvert$



To convert from rectangular to polar, and vice versa

$$x^2+y^2=r^2$$
$$x=r\cos\theta \quad\leftrightarrow\quad x=r(\theta)\cos\theta$$
$$y=r\sin\theta \quad\leftrightarrow\quad y=r(\theta)\sin\theta$$

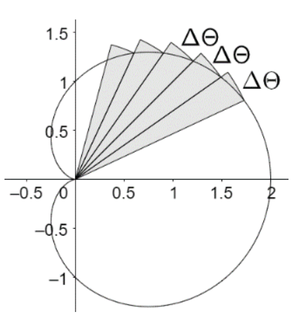


$$\frac{d}{d\theta}[y]=\frac{d}{d\theta}\left[r(\theta)\sin(\theta)\right] \qquad \frac{d}{d\theta}[x]=\frac{d}{d\theta}\left[r(\theta)\cos(\theta)\right]$$
$$\frac{dy}{d\theta}=r'(\theta)\sin(\theta)+r(\theta)\cos(\theta) \qquad \frac{dx}{d\theta}=r'(\theta)\cos(\theta)-r(\theta)\sin(\theta)$$

Using these equations, we can determine $\frac{dy}{dx}$ by the following:

$$\frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=\frac{r'(\theta)\sin(\theta)+r(\theta)\cos(\theta)}{r'(\theta)\cos(\theta)-r(\theta)\sin(\theta)}$$

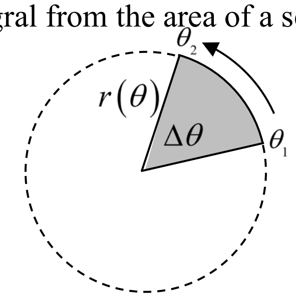
To find the area enclosed by a polar curve, you can derive the integral from the area of a sector.



$$\underbrace{\pi\left[r(\theta)\right]^2}_{\text{area of the circle}}\cdot\underbrace{\frac{d\theta}{2\pi}}_{\text{fraction of the circle}}$$

$$\downarrow$$

$$\frac{1}{2}\int_{\theta_1}^{\theta_2}\left[r(\theta)\right]^2\,d\theta$$

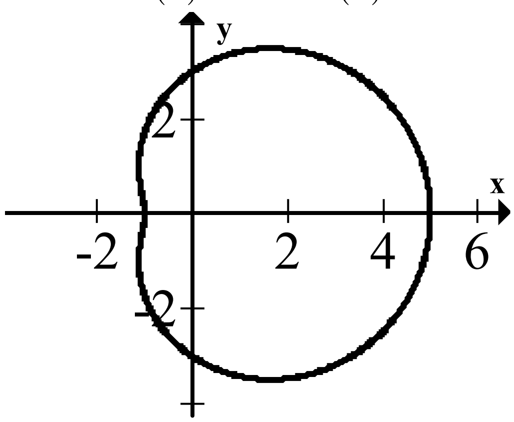


To find the area enclosed by a polar curve, or between two polar curves, *<u>the most important/challenging task is to determine the interval of</u>* θ *<u>that correspond to the lower bound and upper bound of the integral that sweep out the region in question.</u>*

Make sure to use clues like:

The structure of $r(\theta)$ — The graph of the curve

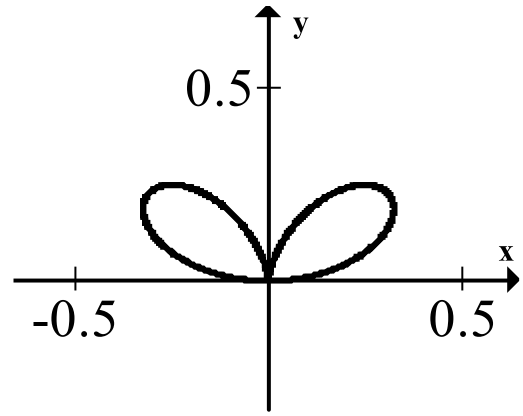
$r(\theta) = 3 + 2\cos(\theta)$



Notice that the radius will be the shortest when $3 + 2\cos(\theta)$ attains its smallest value. This will occur when $\theta = \pi$.
Then use the fact that the graph is symmetric about the *x*-axis, and set up the following integral:

$$2 \cdot \left[\frac{1}{2} \int_0^{\pi} \left[r(\theta) \right]^2 d\theta \right]$$

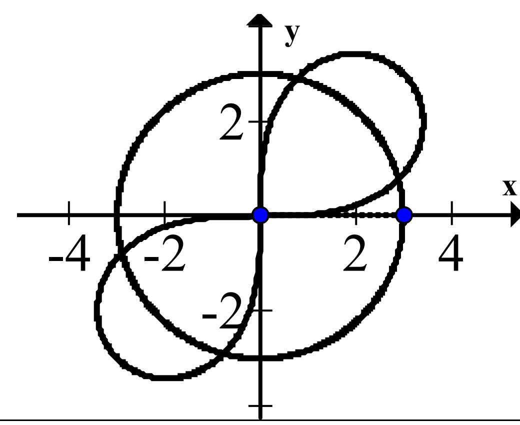
$r(\theta) = \sin(\theta)\cos^2(\theta)$



Notice that the curve will have radius zero when $\sin(\theta)\cos^2(\theta) = 0$.
The first values for which this will occur will be $\theta = \frac{\pi}{2}$ and π.
Use the fact that the graph is symmetric about the *y*-axis, and set up the following integral:

$$2 \cdot \left[\frac{1}{2} \int_0^{\frac{\pi}{2}} \left[r(\theta) \right]^2 d\theta \right]$$

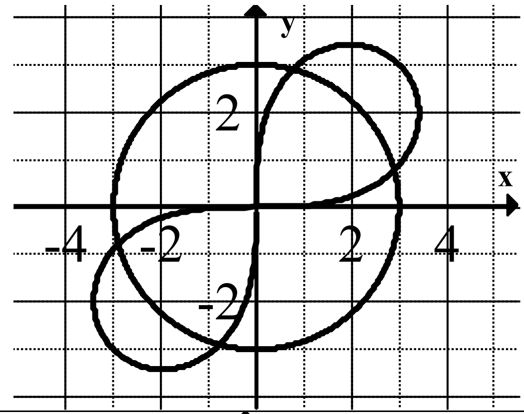
Area in the first quadrant outside the polar curve and inside the polar curve $r^2 = 18\sin(2\theta)$.



Arc Length: $\alpha \le \theta \le \beta$ length of $r(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ is given by

$$s = \int_{\alpha}^{\beta} \sqrt{\left(r(\theta)\right)^2 + \left(r'(\theta)\right)^2}\, d\theta$$

Area in the first quadrant outside the polar curve $r=3$ and inside the polar curve $r^2=18\sin(2\theta)$



$$3=\pm\sqrt{18\sin(2\theta)}$$

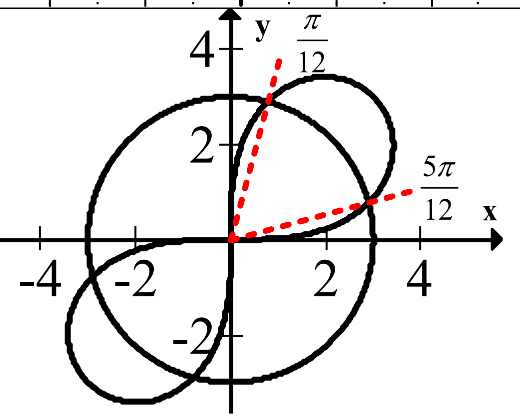
$$9=18\sin(2\theta)$$

$$\frac{1}{2}=\sin(2\theta)$$

$$\downarrow$$

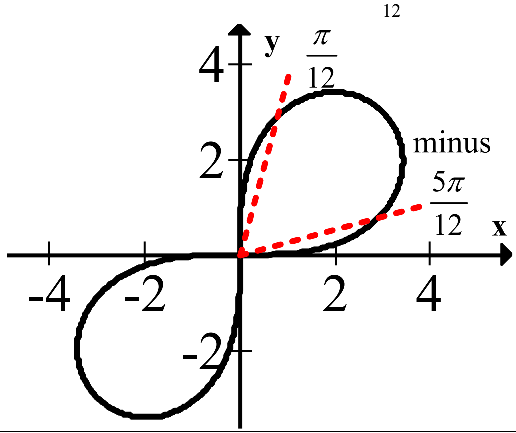
$$2\theta=\left(\frac{\pi}{6}\text{ or }\frac{5\pi}{6}\right)+2\pi k$$

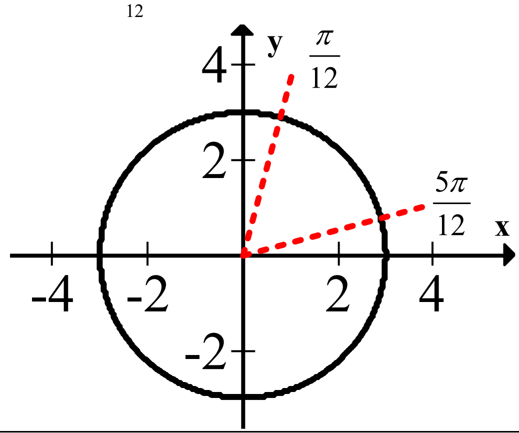
$$\theta=\left(\frac{\pi}{12}\text{ or }\frac{5\pi}{12}\right)+\pi k$$



$$\text{Area}=\frac{1}{2}\int_{\theta_1}^{\theta_2}\left[r_1(\theta)\right]^2 d\theta-\frac{1}{2}\int_{\theta_1}^{\theta_2}\left[r_2(\theta)\right]^2 d\theta$$

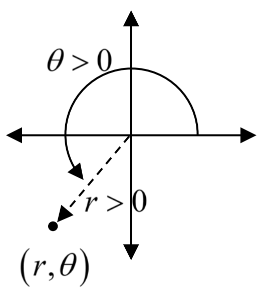
$$=\frac{1}{2}\int_{\frac{\pi}{12}}^{\frac{5\pi}{12}}18\sin(2\theta)\,d\theta-\frac{1}{2}\int_{\frac{\pi}{12}}^{\frac{5\pi}{12}}9\,d\theta$$

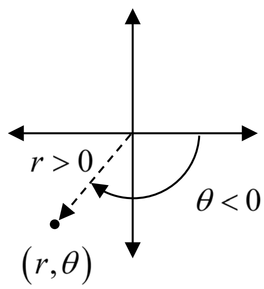
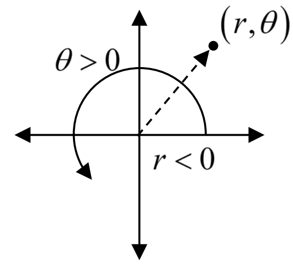


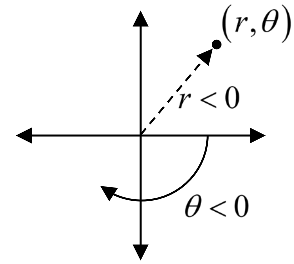


How to determine whether a particle is moving towards or away from the origin:

Determine whether r is positive or negative, AND whether $\frac{dr}{d\theta}$ is positive or negative.

<table>
<tr><th>$r > 0$</th><th>$r < 0$</th></tr>
<tr><td>


</td><td>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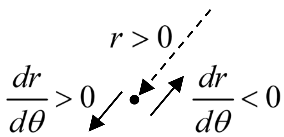
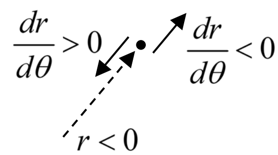
<tr><td>
</td><td>
</td></tr>
<tr><td colspan="2">If r and $\frac{dr}{d\theta}$ have the same sign, then the particle is moving away from the origin.
If r and $\frac{dr}{d\theta}$ have opposite signs, then the particle is moving towards the origin.</td></tr>
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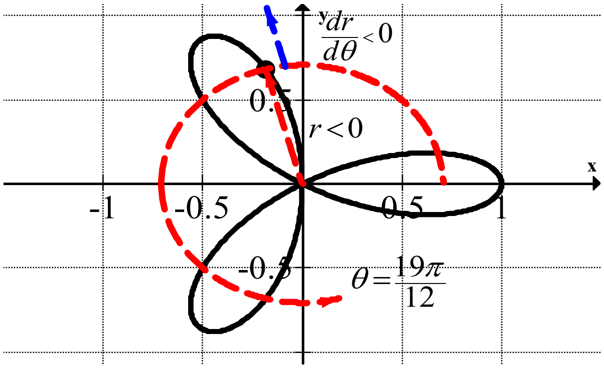
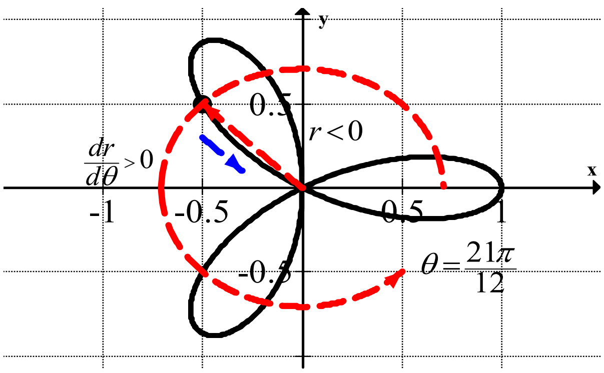
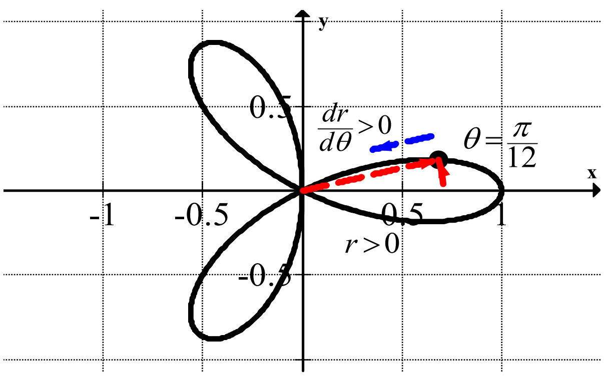
Examples:



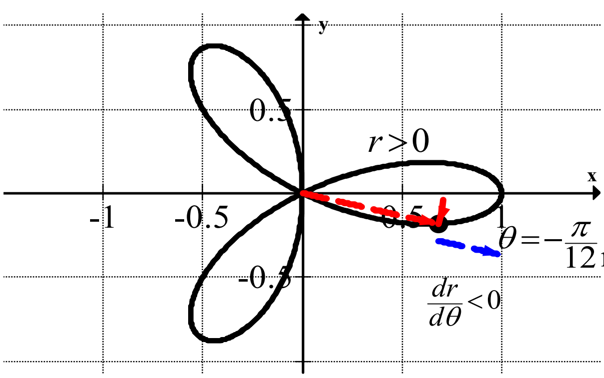
$r=\cos(3\theta)$ when $\theta=\frac{19\pi}{12}$. $r<0$ and $\frac{dr}{d\theta}<0$ and the point is moving away from the origin



$r=\cos(3\theta)$ when $\theta=\frac{21\pi}{12}$. $r<0$ and $\frac{dr}{d\theta}>0$ and the point is moving towards the origin.



$r=\cos(3\theta)$ when $\theta=\frac{\pi}{12}$. $r>0$ and $\frac{dr}{d\theta}<0$ and the point is moving towards the origin.



$r=\cos(3\theta)$ when $\theta=-\frac{\pi}{12}$. $r>0$ and $\frac{dr}{d\theta}>0$ and the point is moving away from the origin.