Improper Integral that has everything:

$$\int_0^\infty \frac{x\ln\left(x^2+1\right)}{\left(x^2+2\right)^2}dx$$

Solution:

$$\int_0^\infty \frac{x\ln\left(x^2+1\right)}{\left(x^2+2\right)^2}dx \quad = \lim_{b\to\infty}\int_0^b \frac{x\ln\left(x^2+1\right)}{\left(x^2+2\right)^2}dx$$

$$= \lim_{b\to\infty}\int_0^b \frac{x}{\left(x^2+2\right)^2}\cdot\ln\left(x^2+1\right)dx$$

u	v'
$\ln\left(x^2+1\right)$	$x\left(x^2+2\right)^{-2}$
$\frac{1}{x^2+1}\cdot 2x$	$-\frac{1}{2}\left(x^2+2\right)^{-1}$

$$= \lim_{b\to\infty}\left[\ln\left(x^2+1\right)\frac{1}{2}\left(x^2+2\right)^{-1}\right]_0^b - \lim_{b\to\infty}\int_0^b \frac{2x}{x^2+1}\cdot\left[-\frac{1}{2}\left(x^2+2\right)^{-1}\right]dx$$

$$= \lim_{b\to\infty}\left[\frac{\ln\left(x^2+1\right)}{2\left(x^2+2\right)}\right]_0^b + \lim_{b\to\infty}\int_0^b \frac{x}{x^2+1}\cdot\frac{1}{x^2+2}dx$$

$$= \lim_{b\to\infty}\left[\frac{\ln\left(b^2+1\right)}{2\left(b^2+2\right)} + \frac{\ln\left(0^2+1\right)}{2\left(0^2+2\right)}\right] + \lim_{b\to\infty}\int_0^b \frac{x}{x^2+1}\cdot\frac{1}{x^2+2}dx$$

$$= \lim_{b\to\infty}\frac{\ln\left(b^2+1\right)}{2\left(b^2+2\right)} + \lim_{b\to\infty}\int_0^b \frac{x}{x^2+1}\cdot\frac{1}{x^2+2}dx$$

$$\frac{x}{x^2+1}\cdot\frac{1}{x^2+2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+2}$$

$$= \lim_{b\to\infty}\frac{\frac{1}{b^2+1}\cdot 2b}{2b} + \lim_{b\to\infty}\int_0^b \frac{x}{x^2+1} + \frac{-x}{x^2+2}dx$$

$$\frac{x}{x^2+1}\cdot\frac{1}{x^2+2} = \frac{Ax+B}{x^2+1}\cdot\frac{x^2+2}{x^2+2} + \frac{Cx+D}{x^2+2}\cdot\frac{x^2+1}{x^2+1}$$

$$= \lim_{b\to\infty}\frac{1}{b^2+1} + \lim_{b\to\infty}\int_0^b \frac{x}{x^2+1} - \frac{x}{x^2+2}dx$$

$$\frac{x}{x^2+1}\cdot\frac{1}{x^2+2} = \frac{(a+c)x^3+(b+d)x^2+(2a+c)x+(2b+d)}{\left(x^2+1\right)\left(x^2+2\right)}$$

$$=$$

$$0x^3+0x^2+x+0 = (a+c)x^3+(b+d)x^2+(2a+c)x+(2b+d)$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$= \lim_{b\to\infty}\int_0^b \frac{x}{x^2+1} - \frac{x}{x^2+2}dx$$

$$= \lim_{b\to\infty}\left[\frac{1}{2}\ln\left|x^2+1\right| - \frac{1}{2}\ln\left|x^2+2\right|\right]_0^b$$

$$= \lim_{b\to\infty}\left[\frac{1}{2}\ln\left|\frac{x^2+1}{x^2+2}\right|\right]_0^b$$

$$= \lim_{b\to\infty}\left[\frac{1}{2}\ln\left|\frac{b^2+1}{b^2+2}\right| - \frac{1}{2}\ln\left|\frac{x^2+1}{x^2+2}\right|\right]$$

$$= 0 - \frac{1}{2}\ln\left(\frac{1}{2}\right)$$