Improper Integrals Examples Stewart Section 7-8

#11

$$\int_0^\infty \frac{x^2}{\sqrt{1+x^3}}\,dx = \lim_{t\to\infty}\int_0^t \frac{x^2}{\sqrt{1+x^3}}\,dx$$

$$= \lim_{t\to\infty}\frac{1}{3}\cdot\int_0^t \frac{3x^2}{\sqrt{1+x^3}}\,dx$$

$$= \frac{1}{3}\lim_{t\to\infty}\int_0^t \frac{1}{\sqrt{1+x^3}}\cdot 3x^2dx$$

Let $\begin{array}{ll} u = 1+x^3 & u(t) = 1+t^3 \\ du = 3x^2dx & u(0) = 1 \end{array}$

$$= \frac{1}{3}\lim_{t\to\infty}\int_1^{1+t^3} \frac{1}{\sqrt{u}}\cdot du$$

$$= \frac{1}{3}\lim_{t\to\infty}\int_1^{1+t^3} u^{-\frac{1}{2}}\cdot du$$

$$= \frac{1}{3}\lim_{t\to\infty}\left[2u^{\frac{1}{2}}\right]_1^{1+t^3}$$

$$= \frac{1}{3}\lim_{t\to\infty}\left[2\left(1+t^3\right)^{\frac{1}{2}} - 2\left(1\right)^{\frac{1}{2}}\right]$$

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#14

$$\int_1^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx = \lim_{t\to\infty}\int_1^t \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx$$

$$= \lim_{t\to\infty}\int_1^t e^{-\sqrt{x}}\cdot\frac{1}{\sqrt{x}}\,dx$$

Let $\begin{array}{ll} u = \sqrt{x} & u(1) = 1 \\ du = \frac{1}{2}\cdot\frac{1}{\sqrt{x}}dx & u(t) = \sqrt{t} \end{array}$

$$= 2\lim_{t\to\infty}\int_1^t e^{-\sqrt{x}}\cdot\frac{1}{2\sqrt{x}}\,dx$$

$$= 2\lim_{t\to\infty}\int_1^{\sqrt{t}} e^{-u}du$$

$$= 2\lim_{t\to\infty}\left[-e^{-u}\right]_1^{\sqrt{t}}$$

$$= 2\lim_{t\to\infty}\left[\left(-e^{-\sqrt{t}}\right)-\left(-e^{-1}\right)\right]$$

$$= 2\left[0+\frac{1}{e}\right]$$

$$= \frac{2}{e}$$

#21

$$\int_1^\infty \frac{\ln(x)}{x}\,dx = \lim_{t\to\infty}\int_1^t \frac{\ln(x)}{x}\,dx$$

Let $\begin{array}{ll} u = \ln(x) & u(1) = 0 \\ du = \frac{1}{x}dx & u(t) = \ln(t) \end{array}$

$$= \lim_{t\to\infty}\int_1^t \ln(x)\cdot\frac{1}{x}\,dx$$

$$= \lim_{t\to\infty}\int_0^{\ln(t)} u\,du$$

$$= \lim_{t\to\infty}\left[\frac{1}{2}u^2\right]_0^{\ln(t)}$$

$$= \lim_{t\to\infty}\left[\frac{1}{2}(\ln(t))^2 - \frac{1}{2}(0)^2\right]$$

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#23

$$\int_{-\infty}^{\infty} \frac{x^2}{9+x^6}\,dx = \int_{-\infty}^{\infty} \frac{x^2}{3^2+\left(x^3\right)^2}\,dx$$

$$= \int_{-\infty}^{0} \frac{x^2}{3^2+\left(x^3\right)^2}\,dx + \int_{0}^{\infty} \frac{x^2}{3^2+\left(x^3\right)^2}\,dx$$

$$= \frac{1}{3}\int_{-\infty}^{0} \frac{1}{3^2+\left(x^3\right)^2}\cdot 3x^2dx + \frac{1}{3}\int_{0}^{\infty} \frac{1}{3^2+\left(x^3\right)^2}\cdot 3x^2dx$$

$\begin{array}{ll} u = x^3 & u(0) = 0 \\ du = 3x^2dx & u(d) = d^3 \\ & u(w) = w^3 \end{array}$

$$= \frac{1}{3}\lim_{d\to-\infty}\int_{d}^{0} \frac{1}{3^2+\left(x^3\right)^2}\cdot 3x^2dx + \frac{1}{3}\lim_{w\to\infty}\int_{0}^{w} \frac{1}{3^2+\left(x^3\right)^2}\cdot 3x^2dx$$

$$= \frac{1}{3}\lim_{d\to-\infty}\int_{d^3}^{0} \frac{1}{3^2+u^2}\,du + \frac{1}{3}\lim_{w\to\infty}\int_{0}^{w^3} \frac{1}{3^2+u^2}\,du$$

$$= \frac{1}{3}\lim_{d\to-\infty}\left[\frac{1}{3}\arctan\left(\frac{u}{3}\right)\right]_{d^3}^{0} + \frac{1}{3}\lim_{w\to\infty}\left[\frac{1}{3}\arctan\left(\frac{u}{3}\right)\right]_{0}^{w^3}$$

$$= \frac{1}{3}\lim_{d\to-\infty}\left[\frac{1}{3}\arctan\left(\frac{0}{3}\right) - \frac{1}{3}\arctan\left(\frac{d^3}{3}\right)\right] + \frac{1}{3}\lim_{w\to\infty}\left[\frac{1}{3}\arctan\left(\frac{w^3}{3}\right) - \frac{1}{3}\arctan\left(\frac{0}{3}\right)\right]$$

$$= \frac{1}{3}\left[-\frac{1}{3}\left(-\frac{\pi}{2}\right)\right] + \frac{1}{3}\left[\frac{1}{3}\left(\frac{\pi}{2}\right) - 0\right]$$

$$= \frac{\pi}{9}$$

#34

$$\int_0^5 \frac{w}{w-2}\,dw = \int_0^5 1+\frac{2}{w-2}\,dw$$

$$\begin{array}{r|l} & \quad 1 \\ \hline w-2 & \quad w+0 \\ & -(w-2) \\ \hline & \quad 2 \end{array}$$

$$\int_0^5 \frac{w}{w-2}\,dw = \int_0^5 1+\frac{2}{w-2}\,dw$$

$$= \lim_{k\to 2^-}\int_0^k 1+\frac{2}{w-2}\,dw + \lim_{a\to 2^+}\int_a^5 1+\frac{2}{w-2}\,dw$$

$$= \lim_{k\to 2^-}\Big[w+2\ln|w-2|\Big]_0^k + \lim_{a\to 2^+}\int_a^5 1+\frac{2}{w-2}\,dw$$

$$= \lim_{k\to 2^-}\Big[\big(k+2\ln|k-2|\big)-\big(0+2\ln|0-2|\big)\Big] + \lim_{a\to 2^+}\int_a^5 1+\frac{2}{w-2}\,dw$$

$$= \lim_{k\to 2^-}\left[\left(\lim_{k\to 2^-} k + \underbrace{\lim_{k\to 2^-} 2\ln|k-2|}_{\text{DNE}/-\infty}\right) - \lim_{k\to 2^-}\big(0+2\ln|0-2|\big)\right] + \lim_{a\to 2^+}\int_a^5 1+\frac{2}{w-2}\,dw$$

$\downarrow$

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