Integration by Parts – The "Anti-Product Rule"

$$\frac{d}{dx}[u\cdot v]=uv'+u'v$$

$$(u\cdot v)'=uv'+u'v$$

$$\int (u\cdot v)'\,dx=\int uv'dx+\int u'vdx$$

$$u\cdot v=\int u\frac{dv}{dx}dx+\int v\frac{du}{dx}dx$$

$$u\cdot v=\int udv+\int vdu$$

$$u\cdot v=\int u\cdot v'+\int u'v$$

$$\downarrow$$

$$\boxed{\int u\cdot v'=u\cdot v-\int u'v}$$

Tips for integration by parts:

I. Let v' be the more complicated function.

II. Let u be the function whose derivative is more simple than u. Unless there is $\ln(x)$.

$$\int xe^x dx\rightarrow \begin{array}{cc} u & v' \\ \hline x & e^x \\ & \overset{+}{\searrow} \\ 1 & e^x \end{array}$$

$$\int xe^x=xe^x-1\cdot e^x+C$$

Tabular Method: Make a column for u and a column for u and one column for v'. Differentiate down the u column and antidifferentiate down the v' column. You will stop when you get a zero in the u column. Your terms will be the products of the diagonals, alternating in sign starting with $(+),(-),(+),(-),\cdots$. This method will work for indefinite integrals where integration by parts must be used over and over again.

$$\begin{array}{ccc} u & & v' \\ & \overset{+}{\searrow} & \\ \text{Differentiate} & \overset{-}{\searrow} & \text{Antidifferentiate} \\ & \overset{+}{\searrow} & \\ & \overset{-}{\searrow} & \\ & \vdots & \end{array}$$

$$\int x\cos(x)dx\rightarrow \begin{array}{cc} u & v' \\ \hline x & \cos(x) \\ & \overset{+}{\searrow} \\ 1 & \sin(x) \end{array}$$

$$\int x\cos(x)dx=x\sin(x)-\int\sin(x)dx$$

$$\int x^2\ln(x)dx\rightarrow \begin{array}{cc} u & v' \\ \hline \ln(x) & x^2 \\ & \overset{+}{\searrow} \\ \frac{1}{x} & \frac{1}{3}x^3 \end{array}$$

$$\int x^2\ln(x)dx=\frac{1}{3}x^3\ln(x)-\int\frac{1}{3}x^2dx$$

Let $f(x) = xe^{-x^2}$ and $\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$. Determine the value of $\int_0^\infty x \cdot f(x)\,dx$.

First we will determine $\int f(x)\,dx = \int xe^{-x^2}dx$

Let $\begin{aligned} u &= -x^2 \\ du &= -2x\,dx \end{aligned}$	$\begin{aligned} \int f(x)\,dx &= \int xe^{-x^2}dx \\ &= -\frac{1}{2}\int e^{-x^2}\cdot(-2x)\,dx \\ &= -\frac{1}{2}\int e^{u}du \\ &= -\frac{1}{2}e^{u} + C \end{aligned}$ Let $F(x) = -\frac{1}{2}e^{-x^2} + C$	Let $\begin{aligned} u &= x & v' &= f(x) \\ u' &= 1 & v &= F(x) \end{aligned}$ $\begin{array}{cc} u & v' \\ \hline x & xe^{-x^2} \\ 1 & -\frac{1}{2}e^{-x^2} \end{array}$ $\int x \cdot f(x)\,dx = -\frac{1}{2}xe^{-x^2} - \int e^{-x^2}dx$

Now to determine the original integral

$$\begin{aligned}
\int_0^\infty x \cdot f(x)\,dx &= \lim_{b\to\infty}\int_0^b x \cdot f(x)\,dx \\
&= \lim_{b\to\infty}\left[x\cdot\left(-\frac{1}{2}e^{-x^2}\right) - \int 1\cdot\left(-\frac{1}{2}e^{-x^2}\right)dx\right]_0^b \\
&= \lim_{b\to\infty}\left[-\frac{1}{2}xe^{-x^2}\right]_0^b - \lim_{b\to\infty}\int_0^b 1\cdot\left(-\frac{1}{2}e^{-x^2}\right)dx \\
&= \lim_{b\to\infty}\left[\left(-\frac{1}{2}\cdot b\cdot e^{-b^2}\right) - \left(-\frac{1}{2}(0)e^{-(0)^2}\right)\right] - \int_0^\infty -\frac{1}{2}e^{-x^2}dx \\
&= \lim_{b\to\infty}\left(-\frac{b}{2e^{b^2}}\right) - (0) - \int_0^\infty -\frac{1}{2}e^{-x^2}dx \\
&= 0 - 0 + \frac{1}{2}\int_0^\infty e^{-x^2}dx \\
&= \frac{1}{2}\left(\frac{\sqrt{\pi}}{2}\right) \\
&= \frac{\sqrt{\pi}}{4}
\end{aligned}$$

$$\int uv' = uv - \int u'v$$

$$\downarrow$$

$$\begin{aligned}
\int_a^b uv' &= \left[uv - \int u'v\right]_a^b \\
&= \left[uv\right]_a^b - \left[\int u'v\right]_a^b \\
&= \left[uv\right]_a^b - \int_a^b u'v
\end{aligned}$$

Has been differentiated

To be determined