2012 #5 No Calculator

The rate at which a baby bird gains weight is proportional to the difference between its adult weight and its current weight. At time $t=0$, when the bird is first weighed, its weight is 20 grams. If $B(t)$ is the weight of the bird, in grams, at time *t* days after it is first weighed, then $\frac{dB}{dt}=\frac{1}{5}(100-B)$

Let $y=B(t)$ be the solution to the differential equation above with initial condition $B(0)=20$.

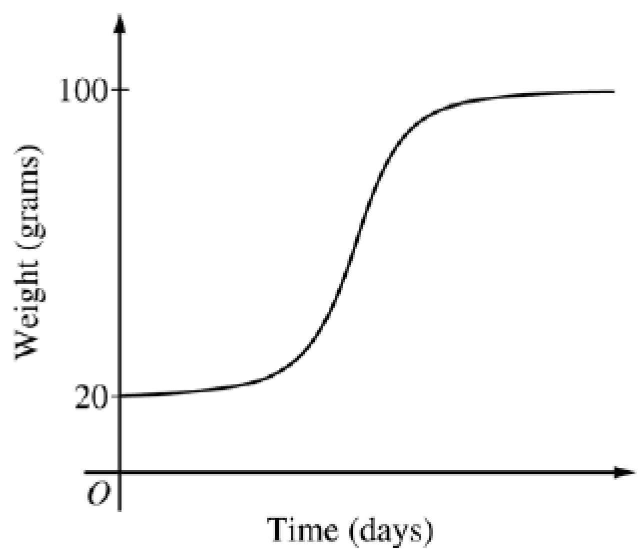
(a) Is the bird gaining weight faster when it weighs 40 grams or when it weighs 70 grams? Explain your reasoning.

$$\left.\frac{dB}{dt}\right|_{B=40}=\frac{1}{5}(100-40)$$
$$\left.\frac{dB}{dt}\right|_{B=70}=\frac{1}{5}(100-70)$$

The bird is gaining weight faster when it weighs 40 grams because $\left.\frac{dB}{dt}\right|_{B=40}>\left.\frac{dB}{dt}\right|_{B=70}$

(b) Find $\frac{d^2B}{dt^2}$ in terms of *B*. Use $\frac{d^2B}{dt^2}$ to explain why the graph of *B* cannot resemble the following graph:

$$\frac{dB}{dt}=\frac{1}{5}(100-B)=20-\frac{1}{5}B$$
$$\frac{d^2B}{dt^2}=-\frac{1}{5}B'=-\frac{1}{5}\left[\frac{1}{5}(100-B)\right]$$



The above graph cannot be a solution to the differential equation because $\frac{d^2B}{dt^2}<0$ for $B<100$, which means the graph should always be concave down if $B<100$

(c) Use separation of variables to find $y = B(t)$, the particular solution to the differential equation with initial condition $B(0) = 20$.

$$\frac{dB}{dt} = \frac{1}{5}(100 - B)$$

$$\frac{1}{(100-B)}dB = \frac{1}{5}dt$$

$$\int \frac{1}{(100-B)}dB = \int \frac{1}{5}dt$$

$$-\ln|100-B| = \frac{1}{5}t + C \text{ where } C \text{ is a constant}$$

$$\ln|100-B| = -\frac{1}{5}t + C$$

$$e^{\ln|100-B|} = e^{-\frac{1}{5}t+C}$$

$$|100-B| = Ae^{-\frac{1}{5}t} \text{ where } A = e^C$$

$$100 - B = Ae^{-\frac{1}{5}t}$$

$$B = 100 - Ae^{-\frac{1}{5}t}$$

Given that $B(0) = 20$

$$B = 100 - Ae^{-\frac{1}{5}t}$$

$$20 = 100 - Ae^{-\frac{1}{5}(0)}$$

$$-80 = -A$$

$$A = 80$$

Particular solution is $B = 100 - 80e^{-\frac{1}{5}t}$

2011 #5 No Calculator

At the beginning of 2010, a landfill contained 1400 tons of solid waste. The increasing function W models the total amount of solid waste stored at the landfill. Planners estimate that W will satisfy the differential equation $\frac{dW}{dt} = \frac{1}{25}(W - 300)$ for the next 20 years. W is measured in tons, and t is measured in years from the start of 2010.

(a) Use the line tangent to the graph of W at $t = 0$ to approximate the amount of solid waste that the landfill contains at the end of the first 3 months of 2010 (time $t = \frac{1}{4}$).

$$W(0) = 1400 \text{ and } \frac{dW}{dt} \text{ at } (0,1400) \text{ is given by } \frac{1}{25}(1400 - 300) = \frac{1100}{25} = 44$$

The equation of the tangent line is given by

$$\begin{aligned} W - 1400 &= 44(t - 0) \\ W &= 44t + 1400 \\ W\left(\frac{1}{4}\right) &= 44\left(\frac{1}{4}\right) + 1400 \\ &= 1411 \end{aligned}$$

There are approximately 1411 tons of solid waste at the end of the first three months of 2010.

(b) Find $\dfrac{d^2W}{dt^2}$ in terms of *W*. [*Hint*: Use implicit differentiation and substitute the expression for $\dfrac{dW}{dt}$ for W']. Use $\dfrac{d^2W}{dt^2}$ to determine whether your answer in part (a) is an underestimate or an overestimate of the amount of solid waste that the landfill contains at time $t=\dfrac{1}{4}$.

$$\frac{dW}{dt}=\frac{1}{25}(W-300)$$

$$\frac{dW}{dt}=\frac{1}{25}W-12 \qquad \text{since } \frac{dW}{dt}=\frac{1}{25}(W-300)$$

$$\frac{d^2W}{dt^2}=\frac{1}{25}W'$$

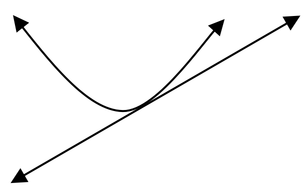
$$\frac{d^2W}{dt^2}=\frac{1}{25}\left[\frac{1}{25}(W-300)\right]$$

$$\left.\frac{d^2W}{dt^2}\right|_{t=0}=\frac{1}{25}\left[\frac{1}{25}(W-300)\right]$$

$$\frac{d^2W}{dt^2} \text{ at } t=0 \text{ is given by} \qquad =\frac{1}{25}\left[\frac{1}{25}(1400-300)\right]$$

$$>0$$

Therefore the graph of the tangent line to the graph of *W* at $t=0$ is an underestimate because the graph is concave up for $t\ge 0$ since $W(t)\ge 1400$ [because $W(t)$ is an increasing function] making the second derivative is positive for $t\ge 0$.



(c) Find the particular solution to the differential equation $\frac{dW}{dt} = \frac{1}{25}(W-300)$ with the initial condition $W(0)=1400$.

$$\frac{dW}{dt} = \frac{1}{25}(W-300)$$

$$\frac{1}{W-300}dW = \frac{1}{25}dt$$

$$\int \frac{1}{W-300}dW = \int \frac{1}{25}dt$$

$$\ln|W-300| = \frac{1}{25}t + C \text{ where } C \text{ is a constant}$$

$$e^{\ln|W-300|} = e^{\frac{1}{25}t+C}$$

$$|W-300| = e^{\frac{1}{25}t} \cdot e^{C} \text{ Let } e^{C} = A$$

$$|W-300| = Ae^{\frac{1}{25}t}$$

$$W-300 = Ae^{\frac{1}{25}t}$$

$$W = 300 + Ae^{\frac{1}{25}t}$$

$$1400 = 300 + Ae^{\frac{1}{25}(0)}$$

$$1400 = 300 + A$$

$$A = 1100$$

$$\downarrow$$

$$W = 300 + 1100e^{\frac{1}{25}t}$$

At time $t = 0$, a boiled potato is taken from a pot on a stove and left to cool in a kitchen. The internal temperature of the potato is 91 degrees Celsius $(°\text{C})$ at time $t = 0$, and the internal temperature of the potato is greater than $27°\text{C}$ for all times $t > 0$. The internal temperature of the potato at time *t* minutes can be modeled by the function *H* that satisfies the differential equation $\frac{dH}{dt} = -\frac{1}{4}(H - 27)$, where $H(t)$ is measured in degrees Celsius and $H(0) = 91$.

(a) Write an equation for the line tangent to the graph of *H* at $t = 0$. Use this equation to approximate the internal temperature of the potato at time $t = 3$.

$$\left.\frac{dH}{dt}\right|_{(0,91)} = -\frac{1}{4}(91 - 27) = -16$$

$$y - y_1 = m(x - x_1)$$
$$y - 91 = -16(x - 0)$$
$$y = -16x + 91$$

$$y(3) \approx -16(3) + 91 = 43$$

(b) Use $\frac{d^2H}{dt^2}$ to determine whether your answer in part (a) is an underestimate or an overestimate of the internal temperature of the potato at time $t = 3$.

$$\frac{dH}{dt} = -\frac{1}{4}(H - 27)$$
$$\frac{d^2H}{dt^2} = -\frac{1}{4}H'$$
$$= -\frac{1}{4}\left[-\frac{1}{4}(H - 27)\right]$$
$$= \frac{1}{16}(H - 27)$$

The answer in part (a) is an underestimate of the internal temperature of the potato at time $t = 3$ because $\frac{d^2H}{dt^2} > 0$ for $H > 27$.

(c) For $t < 10$, an alternate model for the internal temperature of the potato at time t minutes is the function G that satisfies the differential equation $\frac{dG}{dt} = -(G-27)^{\frac{2}{3}}$, where $G(t)$ is measured in degrees Celsius and $G(0) = 91$. Find an expression for $G(t)$. Based on this model, what is the internal temperature of the potato at time $t = 3$?

$$\frac{dG}{dt} = -(G-27)^{\frac{2}{3}}$$

$$-(G-27)^{-\frac{2}{3}}\,dG = dt$$

$$\int -(G-27)^{-\frac{2}{3}}\,dG = \int dt$$

$$-3(G-27)^{\frac{1}{3}} = t + C$$

$$(G-27)^{\frac{1}{3}} = -\frac{1}{3}t + C$$

$$G - 27 = \left(-\frac{1}{3}t + C\right)^3$$

$$G(t) = \left(-\frac{1}{3}t + C\right)^3 + 27$$

$$G(0) = 91$$

$$\downarrow$$

$$91 = \left(-\frac{1}{3}(0) + C\right)^3 + 27$$

$$64 = C^3$$

$$C = 4$$

$$\downarrow$$

$$G(t) = \left(-\frac{1}{3}t + 4\right)^3 + 27$$

$$\downarrow$$

$$G(3) = \left(-\frac{1}{3}(3) + 4\right)^3 + 27$$

$$= 54$$