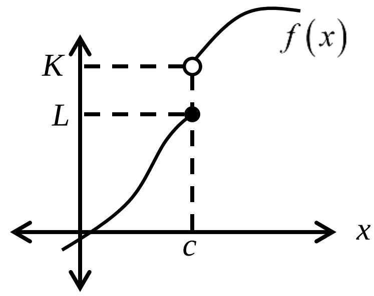
Properties of Limits

One-Sided Limits:



Right Hand Limit

$\boxed{\lim_{x\to c^+} f(x) = K}$ is read

"The limit, as *x* approaches *c* from the right of $f(x)$, is *K*."

Left Hand Limit

$\boxed{\lim_{x\to c^-} f(x) = L}$ is read

"The limit, as *x* approaches *c* from the left of $f(x)$, is *L*."

We say that $\lim_{x\to c^\pm} f(x)$ DNE (does not exist) if any of the following situations arise:

I. If $f(x)$ goes to $\pm\infty$ as $x\to c^\pm$.

II. If $f(x)$ does not *converge* to a single *y*-value as $x\to c^\pm$.

Two-Sided Limits:

$\lim_{x\to c} f(x)$ is a two-sided limit. A two-sided limit is identified by the absence of a $\pm$ near the *c*.

$$\boxed{\lim_{x\to c} f(x) = L} \text{ if and only if } \boxed{\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x)}$$

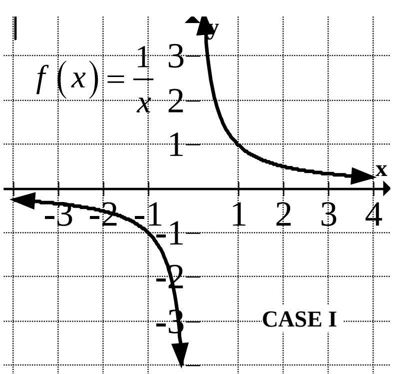
A two sided limit exists only if the left hand limit and the right hand limit both exist, and both are equal.

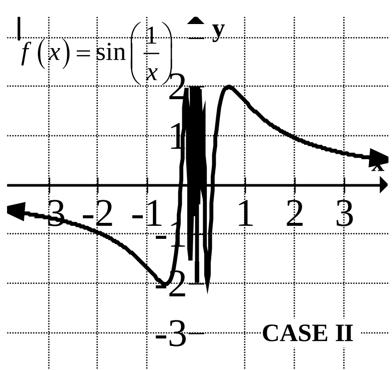
We say that $\lim_{x\to c} f(x)$ DNE (does not exist) if any of the following situations arise:

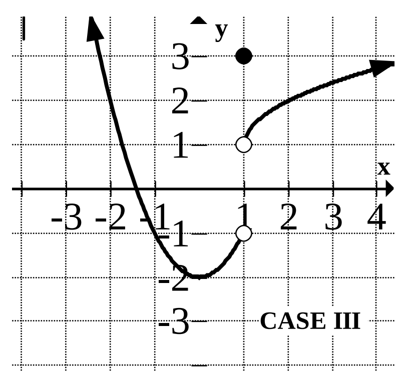
I. If $f(x)$ goes to $\pm\infty$ as $x\to c$.

II. If $f(x)$ does not *converge* to a single *y*-value as $x\to c$.

III. If $\lim_{x\to c^-} f(x) \neq \lim_{x\to c^+} f(x)$.
That is, if the limit approaching from the left does not match the limit approaching from the right.







A function $f(x)$ has a **vertical asymptote** at $x = c$ if $\lim_{x\to c^\pm} f(x) \to \pm\infty$

A function $f(x)$ has a **horizontal asymptote** at $y = k$ if $\lim_{x\to \pm\infty} f(x) = k$

Properties of Limits

Let b and c be real numbers, and n a positive integer. Let f and g be functions such that

$$\lim_{x\to c} f(x) = L \text{ and } \lim_{x\to c} g(x) = K$$

Then the following limit properties hold.

Scalar Multiple Property of Limits	$\lim_{x\to c}\left[b\cdot f(x)\right] = b\cdot\left[\lim_{x\to c} f(x)\right] = b\cdot L$
Sum/Difference Property of Limits	$\lim_{x\to c}\left[f(x)\pm g(x)\right] = \left[\lim_{x\to c} f(x)\right]\pm\left[\lim_{x\to c} g(x)\right] = L\pm K$
Product Property of Limits	$\lim_{x\to c}\left[f(x)\cdot g(x)\right] = \left[\lim_{x\to c} f(x)\right]\cdot\left[\lim_{x\to c} g(x)\right] = L\cdot K$
Quotient Property of Limits	$\lim_{x\to c}\left[\frac{f(x)}{g(x)}\right] = \frac{\left[\lim_{x\to c} f(x)\right]}{\left[\lim_{x\to c} g(x)\right]} = \frac{L}{K}$ provided $K\neq 0$
Power Property of Limits	$\lim_{x\to c}\left(\left[f(x)\right]^n\right) = \left(\lim_{x\to c} f(x)\right)^n = L^n$

These properties hold for one-sided limits as well.

If $\lim_{x\to c} f(x)$ DNE or $\lim_{x\to c} g(x)$ DNE, then the limit properties may not hold.

I. Investigate $\lim_{x\to c^+} f(x)$ and $\lim_{x\to c^+} g(x)$ to see if $\lim_{x\to c^+}[\cdots]$ exists.

II. Investigate $\lim_{x\to c^-} f(x)$ and $\lim_{x\to c^-} g(x)$ to see if $\lim_{x\to c^-}[\cdots]$ exists.

III. If $\lim_{x\to c^-}[\cdots]$ or $\lim_{x\to c^+}[\cdots]$ DNE, then one can conclude that $\lim_{x\to c}[\cdots]$ DNE

IV. If are both finite and $\lim_{x\to c^-}[\cdots] = \lim_{x\to c^+}[\cdots]$, then $\lim_{x\to c}[\cdots]$ exists.

Some Basic Limit Properties:

Let b and c be any real numbers, and n be any positive integer.

$$\lim_{x\to c} b = b \qquad \lim_{x\to c} x = c \qquad \lim_{x\to c} x^n = c^n$$

If $p(x)$ is a polynomial function and c is any real number, then $\lim_{x\to c} p(x) = p(c)$.	If $r(x)$ is a rational function given by $\frac{p(x)}{q(x)}$ and c is any real number, then $\lim_{x\to c} r(x) = r(c) = \frac{p(c)}{q(c)}$ provided $q(c)\neq 0$

If n is a positive odd integer, then $\lim_{x\to c}\sqrt[n]{x} = \sqrt[n]{c}$ for all real numbers c.

If n is a positive even integer, then $\lim_{x\to c}\sqrt[n]{x} = \sqrt[n]{c}$ provided that $c>0$.

Limits of Trigonometric Functions Let c be a real number in the domain of the given trigonometric function		
$\lim_{x\to c}\left[\sin(x)\right] = \sin(c)$	$\lim_{x\to c}\left[\cos(x)\right] = \cos(c)$	$\lim_{x\to c}\left[\tan(x)\right] = \tan(c)$
$\lim_{x\to c}\left[\csc(x)\right] = \csc(c)$	$\lim_{x\to c}\left[\sec(x)\right] = \sec(c)$	$\lim_{x\to c}\left[\cot(x)\right] = \cot(c)$
$\lim_{x\to 0}\frac{\sin(x)}{x} = 1$ or $\lim_{x\to 0}\frac{x}{\sin(x)} = 1$		$\lim_{x\to 0}\frac{1-\cos(x)}{x} = 0$