2002 #6

The MacLaurin series for the function f is given by

$$f(x)=\sum_{n=0}^{\infty}\frac{(2x)^{n+1}}{n+1}=2x+\frac{4x^2}{2}+\frac{8x^3}{3}+\frac{16x^4}{4}+\cdots+\frac{(2x)^{n+1}}{n+1}+\cdots$$

on its interval of convergence.

(a) Find the interval of convergence of the MacLaurin series for f. Justify your answer.

Using the ratio test:	Using the Root Test
$\lim_{n\to\infty}\left\|\frac{\frac{(2x)^{(n+1)+1}}{(n+1)+1}}{\frac{(2x)^{n+1}}{n+1}}\right\|=\lim_{n\to\infty}\left\|\frac{(2x)^{n+2}}{n+2}\cdot\frac{n+1}{(2x)^{n+1}}\right\|$ $=\lim_{n\to\infty}\left\|\frac{(2x)^{n+2}}{(2x)^{n+1}}\cdot\frac{n+1}{n+2}\right\|$ $=\lim_{n\to\infty}\left\|\frac{\cancel{(2x)^{n+1}}\cdot(2x)^1}{\cancel{(2x)^{n+1}}}\right\|$ $=\|2x\|$ $\downarrow$ $\|2x\|<1$ $-1<2x<1$ $-\frac{1}{2}<x<\frac{1}{2}$	$\lim_{n\to\infty}\sqrt[n]{\left\|\frac{(2x)^{n+1}}{n+1}\right\|}=\lim_{n\to\infty}\sqrt[n]{\frac{(2x)^n\cdot(2x)}{n+1}}$ $=\lim_{n\to\infty}\frac{\sqrt[n]{\|(2x)\|^n}\cdot\sqrt[n]{\|(2x)\|}}{\sqrt[n]{n+1}}$ $=\|2x\|$ $\downarrow$ $-1<2x<1$ $-\frac{1}{2}<x<\frac{1}{2}$

when $x=-\frac{1}{2}$	when $x=\frac{1}{2}$
$\sum_{n=0}^{\infty}\frac{\left(2\left(-\frac{1}{2}\right)\right)^{n+1}}{n+1}=\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n+1}$ Since $\lim_{n\to\infty}\left[\frac{1}{n+1}\right]=0$ This series converges by the Alternating Series test	$\sum_{n=0}^{\infty}\frac{\left(2\left(\frac{1}{2}\right)\right)^{n+1}}{n+1}=\sum_{n=0}^{\infty}\frac{(1)^{n+1}}{n+1}$ $=\sum_{n=0}^{\infty}\frac{1}{n+1}$ This series does not converge, because it is a form of the Harmonic series.

The interval of convergence is $-\frac{1}{2}\le x<\frac{1}{2}$.

(b) Find the first four terms, and the general term for the MacLaurin series for $f'(x)$

$$f(x)=\sum_{n=0}^{\infty}\frac{(2x)^{n+1}}{n+1}=2x+\frac{4x^2}{2}+\frac{8x^3}{3}+\frac{16x^4}{4}+\cdots+\frac{(2x)^{n+1}}{n+1}+\cdots$$

$$\downarrow$$

$$f'(x)=\sum_{n=0}^{\infty}(2x)^n=2+4x+8x^2+16x^3+\cdots+(2x)^n\cdot 2+\cdots$$

$$=2\cdot(2x)^0+2(2x)^1+2(2x)^2+2(2x)^3+\cdots+2\cdot(2x)^n+\cdots$$

(c) Use the MacLaurin series found in part (b) to find the value of $f'\left(-\frac{1}{3}\right)$

The series found in part (b) is a geometric series with common ratio of $(2x)$.

$$f'\left(-\frac{1}{3}\right)=\sum_{n=0}^{\infty}\left(2\left(-\frac{1}{3}\right)\right)^n=2+4\left(-\frac{1}{3}\right)+8\left(-\frac{1}{3}\right)^2+16\left(-\frac{1}{3}\right)^3+\cdots+\left(2\left(-\frac{1}{3}\right)\right)^n\cdot 2+\cdots$$

$$=2\cdot\left(2\left(-\frac{1}{3}\right)\right)^0+2\left(2\left(-\frac{1}{3}\right)\right)^1+2\left(2\left(-\frac{1}{3}\right)\right)^2+2\left(2\left(-\frac{1}{3}\right)\right)^3+\cdots+2\cdot(2)^n+\cdots$$

$$=2\cdot\left(-\frac{2}{3}\right)^0+2\left(-\frac{2}{3}\right)^1+2\left(-\frac{2}{3}\right)^2+2\left(-\frac{2}{3}\right)^3+\cdots+2\cdot\left(-\frac{2}{3}\right)^n+\cdots$$

$$=\frac{\text{first term}}{1-\text{common ratio}}$$

$$=\frac{2}{1-\left(-\frac{2}{3}\right)}=\frac{6}{5}$$

2005 Form B #3

The Taylor Series about $x=0$ for a certain function *f* converges to $f(x)$ for all *x* in the interval of convergence. The n^{th} derivative of *f* at $x=0$ is given by

$$f^{(n)}(0)=\frac{(-1)^{n+1}\cdot(n+1)!}{5^n\cdot(n-1)^2} \text{ for } n\ge 2.$$

The graph of *f* has a horizontal tangent line at $x=0$ and $f(0)=6$.

(a) Determine whether *f* has a relative maximum, relative minimum, or neither at $x=0$. Justify your answer.

Since *f* has a horizontal tangent line at $x=0$, $f'(0)=0$.

$$f^{(2)}(0)=\frac{(-1)^{2+1}\cdot(2+1)!}{5^2\cdot(2-1)^2}<0$$

Since $f''(0)<0$ and $f'(0)=0$, by the Second Derivative Test, *f* has a relative maximum at $x=0$.

(b) Write the third-degree Taylor polynomial for *f* about $x=0$.

$$T_3(x)=f(0)+f'(0)\cdot(x-0)+\frac{f''(0)\cdot(x-0)^2}{2!}+\frac{f'''(0)\cdot(x-0)^3}{3!}$$

$$=6+0+\frac{\frac{(-1)^{2+1}\cdot(2+1)!}{5^2\cdot(2-1)^2}\cdot x^2}{2!}+\frac{\frac{(-1)^{3+1}\cdot(3+1)!}{5^3\cdot(3-1)^2}\cdot x^3}{3!}$$

$$=6-\frac{3}{25}x^2+\frac{1}{125}x^3$$

(c) Find the radius of convergence of the Taylor series for *f* about $x=0$. Show the work that leads to your answer.

Since the derivative terms involve factorials, the ratio test must be used, and the root test cannot be used.

$$\lim_{n\to\infty}\left|\frac{\dfrac{\dfrac{(-1)^{(n+1)+1}\cdot((n+1)+1)!}{5^{(n+1)}\cdot((n+1)-1)^2}\cdot x^{n+1}}{(n+1)!}}{\dfrac{\dfrac{(-1)^{n+1}\cdot(n+1)!}{5^n\cdot(n-1)^2}\cdot x^n}{n!}}\right|=\lim_{n\to\infty}\left|\frac{\dfrac{(n+2)!}{5^{n+1}\cdot n^2}\cdot x^{n+1}}{(n+1)!}\cdot\frac{n!}{\dfrac{(n+1)!}{5^n\cdot(n-1)^2}\cdot x^n}\right|$$

$$=\lim_{n\to\infty}\left|\frac{(n+2)!\cdot x^{n+1}}{5^{n+1}\cdot n^2\cdot(n+1)!}\cdot\frac{n!\cdot 5^n\cdot(n-1)^2}{(n+1)!\cdot x^n}\right|$$

$$=\lim_{n\to\infty}\left|\frac{(n+2)!}{(n+1)!}\cdot\frac{x^{n+1}}{x^n}\cdot\frac{n!}{(n+1)!}\cdot\frac{5^n}{5^{n+1}}\cdot\frac{(n-1)^2}{n^2}\right|$$

$$=\lim_{n\to\infty}\left|\frac{(n+2)}{1}\cdot\frac{x}{1}\cdot\frac{1}{(n+1)}\cdot\frac{1}{5}\cdot\frac{(n-1)^2}{n^2}\right|$$

$$=\lim_{n\to\infty}\left|\frac{(n+2)}{(n+1)}\cdot\frac{x}{5}\cdot\frac{(n-1)^2}{n^2}\right|$$

$$=\left|\frac{x}{5}\right|$$

$$\downarrow$$

$$-1<\frac{x}{5}<1$$

$$-5<x<5$$

The radius of convergence is 5.

2009 Form B #6

The function f is defined by the power series

$$f(x) = 1 + (x+1) + (x+1)^2 + \cdots + (x+1)^n + \cdots = \sum_{n=0}^{\infty} (x+1)^n$$

for all real numbers x for which the series converges.

(a) Find the interval of convergence of the power series for f. Justify your answer.

Using the Ratio Test	**Using the Root Test**
$\lim_{n\to\infty}\left\|\frac{(x+1)^{n+1}}{(x+1)^n}\right\| = \lim_{n\to\infty}\|x+1\|$ $= \|x+1\|$ $\downarrow$ $\|x+1\| < 1$ $-1 < x+1 < 1$ $-2 < x < 0$	$\lim_{n\to\infty}\sqrt[n]{\left\|(x+1)^n\right\|} = \lim_{n\to\infty}\|x+1\|$ $= \|x+1\|$ $\downarrow$ $\|x+1\| < 1$ $-1 < x+1 < 1$ $-2 < x < 0$

when $x = -2$	when $x = 0$
$\sum_{n=0}^{\infty}(-2+1)^n = \sum_{n=0}^{\infty}(-1)^n$ Since $\lim_{n\to\infty}[1] \neq 0$, the series does not converge by the Alternating Series test.	$\sum_{n=0}^{\infty}(0+1)^n = \sum_{n=0}^{\infty}1$ This series does not converge because the terms do not go to zero as $n \to \infty$.

The interval of convergence is $-2 < x < 0$.

(b) The power series above is the Tayor series for f about $x=-1$. Find the sum of the series for f.

$f(x)=1+(x+1)+(x+1)^2+\cdots+(x+1)^n+\cdots=\sum_{n=0}^{\infty}(x+1)^n$ is a geometric series with common ratio of $(x+1)$. Therefore

$$\begin{aligned} f(x)&=1+(x+1)+(x+1)^2+\cdots+(x+1)^n+\cdots \\ &=\sum_{n=0}^{\infty}(x+1)^n \\ &=\frac{\text{first term}}{1-\text{common ratio}} \\ &=\frac{1}{1-(x+1)} \\ &=-\frac{1}{x} \end{aligned}$$

So long as $-2<x<0$.

(c) Let g be the function defined by $g(x)=\int_{-1}^{x} f(t)\,dt$. Find the value of $g\left(-\frac{1}{2}\right)$, if it exists, or explain why $g\left(-\frac{1}{2}\right)$ does not exist.

$$\begin{aligned}
g\left(-\frac{1}{2}\right) &= \int_{-1}^{-\frac{1}{2}} f(t)\,dt \\
&= \int_{-1}^{-\frac{1}{2}} -\frac{1}{t}\,dt \\
&= -\int_{-1}^{-\frac{1}{2}} \frac{1}{t}\,dt \\
&= -\Big[\ln|t|\Big]_{-1}^{-\frac{1}{2}} \\
&= -\left(\ln\left|-\frac{1}{2}\right| - \ln|-1|\right) \\
&= -\left(\ln\left|\frac{1}{2}\right| - \ln|1|\right) \\
&= -\ln\left|\frac{1}{2}\right| \\
&= -\ln\left|2^{-1}\right| \\
&= \ln|2|
\end{aligned}$$

(d) Let h be the function defined by $h(x) = f\left(x^2 - 1\right)$. Find the first three nonzero terms and the general term for the Taylor series for f about $x = 0$, and find the value of $h\left(\frac{1}{2}\right)$.

$$\begin{aligned} h(x) &= f\left(x^2 - 1\right) \\ &= 1 + \left(\left(x^2 - 1\right) + 1\right) + \left(\left(x^2 - 1\right) + 1\right)^2 + \cdots + \left(\left(x^2 - 1\right) + 1\right)^n + \cdots \\ &= 1 + x^2 + x^4 + \cdots + x^{2n} + \cdots \end{aligned}$$

$$\begin{aligned} h\left(\frac{1}{2}\right) &= f\left(\left(\frac{1}{2}\right)^2 - 1\right) \\ &= f\left(-\frac{3}{4}\right) \\ &= -\frac{1}{\left(-\frac{3}{4}\right)} \\ &= \frac{4}{3} \end{aligned}$$