Alternating Series Test:

Let $a_n > 0$ for all n. The alternating series

$$\sum_{n=1}^{\infty} (-1)^n a_n \quad \text{and} \quad \sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

converge if the following two conditions are met simultaneously:

I. $\lim_{n\to\infty} a_n = 0$

II. $a_{n+1} \le a_n$ for all n (monotonically decreasing)

The way to demonstrate the following is to demonstrate that:

"the terms of the alternating series decrease in absolute value to zero." $\leftrightarrow \lim_{n\to\infty} \left|(-1)^{\cdots} a_n\right| = 0$

Alternating Series Remainder Theorem:

If an alternating series $\sum_{n=1}^{\infty} (-1)^n a_n$ satisfies the condition that $a_{n+1} \le a_n$, then the absolute value of the remainder of R_N involved in approximating the sum S with the N^{th} partial sum S_N is ***less than or equal to the absolute value of the first neglected term*** [Since the sum starts at $n=1$, this would be $(N+1)^{\text{th}}$ term].

$$|S - S_N| = |R_N| \le |a_{N+1}|$$

Note that another way to express the bound of the difference is $|a_{N+1}| = |S_{N+1} - S_N|$

Proof: The series obtained by deleting the first N terms of the given series satisfies the conditions of the Alternating Series Test and has remainder of R_N. S_N represents the first N terms of the series and S the actual infinite sum.

$$\begin{aligned}
S_N + R_N &= S \\
R_N &= S - S_N \\
&= \sum_{n=1}^{\infty} (-1)^{n+1} a_n - \sum_{n=1}^{N} (-1)^{n+1} a_n \\
&= (-1)^{N+1} a_{N+1} + (-1)^{N+2} a_{N+2} + (-1)^{N+3} a_{N+3} + \cdots \\
&= (-1)^{N+1} \left[a_{N+1} - a_{N+2} + a_{N+3} - \cdots\right] \\
|R_N| &= \left|(-1)^{N+1} \left[a_{N+1} - a_{N+2} + a_{N+3} - \cdots\right]\right| \\
&= \left|(-1)^{N+1}\right| \left|\left[a_{N+1} - a_{N+2} + a_{N+3} - \cdots\right]\right|
\end{aligned}$$

since a_n are positive for all n...

$$\begin{aligned}
&= a_{N+1} - a_{N+2} + a_{N+3} - a_{N+4} + a_{N+5} \cdots \\
&= a_{N+1} \underbrace{-\left(a_{N+2} - a_{N+3}\right)}_{\substack{\text{negative value} \\ a_i \ge a_{i+1}}} \underbrace{-\left(a_{N+4} - a_{N+5}\right)}_{\substack{\text{negative value} \\ a_i \ge a_{i+1}}} \cdots \\
&\le a_{N+1}
\end{aligned}$$

Example #1:

$$S=\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}=-1+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\cdots$$

$$S_4=\sum_{n=1}^{4}\frac{(-1)^n}{n!}=-1+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}$$

$|S-S_4|=|R_4|\le\left|-\frac{1}{5!}\right|$

Example # 2:

$$S=\sum_{n=0}^{\infty}\left(-\frac{2}{3}\right)^n=1-\frac{2}{3}+\left(\frac{2}{3}\right)^2-\left(\frac{2}{3}\right)^3+\left(\frac{2}{3}\right)^4-\cdots$$

$$S_4=\sum_{n=0}^{3}\left(-\frac{2}{3}\right)^n=1-\frac{2}{3}+\left(\frac{2}{3}\right)^2-\left(\frac{2}{3}\right)^3$$

$|S-S_4|=|R_4|\le\left|\left(\frac{2}{3}\right)^4\right|$

A series is ***<u>conditionally convergent</u>*** if : $\sum a_n$ converges and $\sum \lvert a_n\rvert$ diverges	A series ***<u>converges absolutely</u>*** if $\sum \lvert a_n\rvert$ converges
$\sum_{n=1}^{\infty}\frac{(-1)^n}{\ln(n+1)}$ Converges by Alternating Series Test $\sum_{n=1}^{\infty}\left\lvert\frac{(-1)^n}{\ln(n+1)}\right\rvert$ does not converge by using the Direct Comparison Test with the Harmonic Series $\frac{1}{n}\le\left\lvert\frac{(-1)^n}{\ln(n+1)}\right\rvert=\frac{1}{\ln(n+1)}$	$\sum_{n=1}^{\infty}\frac{(-1)^n}{(n-1)!}$ $\sum_{n=1}^{\infty}\left\lvert\frac{(-1)^n}{(n-1)!}\right\rvert=\sum_{n=1}^{\infty}\frac{1}{(n-1)!}$ converges by direct comparison test with $\frac{1}{n^2}$ $\frac{1}{(n-1)!}\le\frac{1}{n^2}$ for sufficiently large *n*.