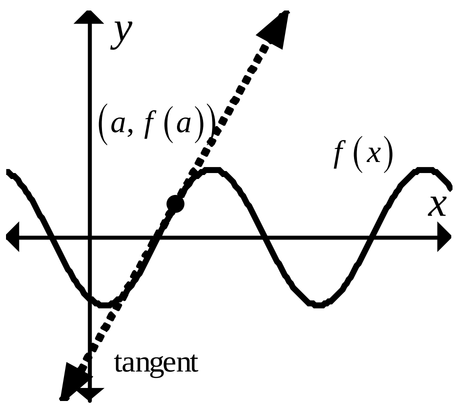
Tangent Line Approximations:

The tangent to a graph $y = f(x)$ at the point $(a, f(a))$ is a reasonable model for $f(x)$ around $x = a$. That is, the *y*-value of the tangent line is a reasonable approximation for $f(x)$ close to $x = a$.

<table>
<tr>
<td>
</td>
<td>To find the tangent line approximation for $f(x)$ at $x = a$, you need two pieces of information:
I. The value of $f(a)$
II. The value of $f'(a)$, the slope of the tangent line.
$$y - y_1 = m(x - x_1)$$
$$\downarrow$$
$$y - f(a) = f'(a)(x - a)$$
The slope of the tangent line is the instantaneous rate of change of $f(x)$ at $x = a$.</td>
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The graph above is the graph of $f(x) = 2\sin(x-2)$, and the point of tangency is at $x = 2.5$. Estimate the value of $f(3)$ using the equation of the tangent line to *f* at $x = 2.5$. What is the difference between $f(3)$ and the tangent line approximation for $f(3)$?

<table>
<tr>
<td>$$\begin{aligned} f(2.5) &= 2\sin(2.5-2) \\ &= 2\sin\frac{1}{2} \\ &= 0.9588... \end{aligned}$$</td>
<td>$$f(x) = 2\sin(x-2)$$
$$\downarrow$$
$$f'(x) = 2\cos(x-2)$$</td>
<td>$$\begin{aligned} f'(x) &= 2\cos(x-2) \\ f'(2.5) &= 2\cos(2.5-2) \\ &= 2\cos\left(\frac{1}{2}\right) \\ &= 1.7551... \end{aligned}$$</td>
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<tr>
<td colspan="2">$$y - y_1 = m(x - x_1)$$
$$\downarrow$$
$$y - f(a) = f'(a)(x - a)$$
$$y - f(2.5) = f'(2.5)(x - 2.5)$$
$$y - 2\sin\left(\frac{1}{2}\right) = 2\cos\left(\frac{1}{2}\right)(x - 2.5)$$
$$y - 0.9588... = 1.7551...(x - 2.5)$$</td>
<td>$$y(3) - 2\sin\left(\frac{1}{2}\right) = 2\cos\left(\frac{1}{2}\right)(3 - 2.5)$$
$$\begin{aligned} y(3) &= 2\sin\left(\frac{1}{2}\right) + 2\cos\left(\frac{1}{2}\right)(3 - 2.5) \\ &= 3.5103... \end{aligned}$$
$$y(3) - 0.9588... = 1.7551...(3 - 2.5)$$
$$\begin{aligned} y(3) &= 0.9588... + 1.7551...(3 - 2.5) \\ &= 3.5103... \end{aligned}$$</td>
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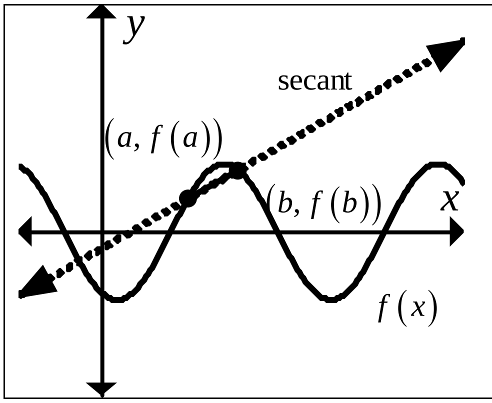	A secant line is a line that passes through two points on the graph of $y = f(x)$. The slope of the secant line is the average rate of change of $f(x)$ on the interval $[a,b]$. $m = \frac{f(b)-f(a)}{b-a}$

The graph above is the graph of $f(x) = 2\sin(x-2)$. In the secant diagram, the interval for which the secant is constructed is $[2.5, 4]$. Estimate the value of $f(3)$ using the equation of the secant line on $[2.5, 4]$. What is the difference between the secant approximation for $f(3)$ and $f(3)$?

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$$\begin{aligned} m &= \frac{f(b)-f(a)}{b-a} \\ &= \frac{f(4)-f(2.5)}{4-2.5} \\ &= \frac{2\sin(2)-2\sin(0.5)}{1.5} \end{aligned}$$

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$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - f(2.5) &= m(x-2.5) \\ y - 2\sin\left(\frac{1}{2}\right) &= \frac{2\sin(2)-2\sin(0.5)}{1.5}(x-2.5) \end{aligned}$$

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$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - f(4) &= m(x-4) \\ y - 2\sin(2) &= \frac{2\sin(2)-2\sin(0.5)}{1.5}(x-4) \end{aligned}$$

</td>
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</table>

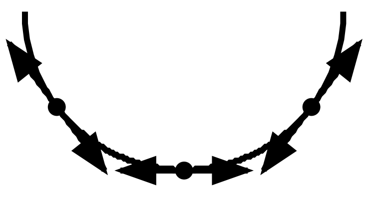
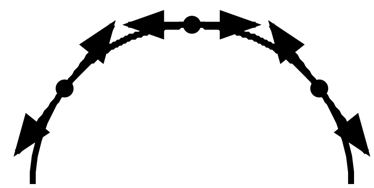
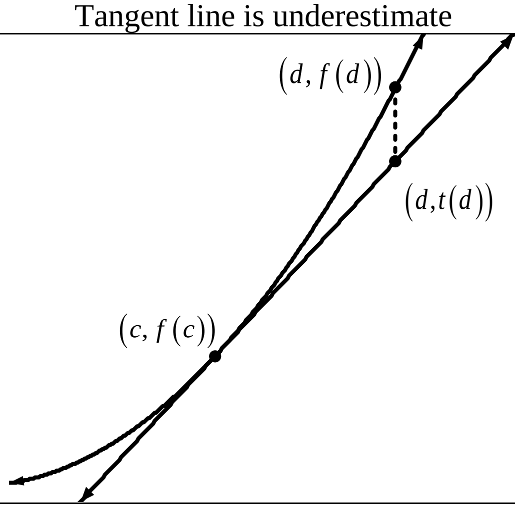
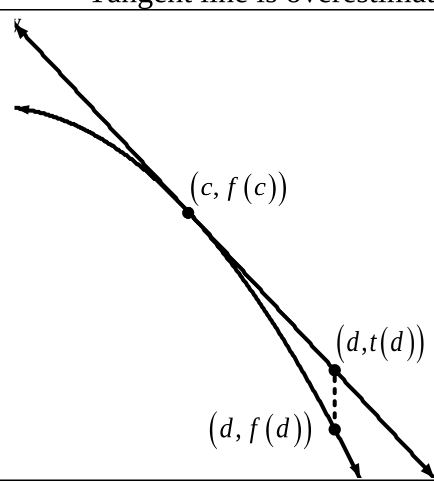
$$\begin{aligned} y(3) - 2\sin\left(\frac{1}{2}\right) &= \frac{2\sin(2)-2\sin(0.5)}{1.5}(3-2.5) \\ y(3) &= 2\sin\left(\frac{1}{2}\right) + \frac{2\sin(2)-2\sin(0.5)}{1.5}(3-2.5) \\ &= 1.2454.... \end{aligned}$$

$$\begin{aligned} y(3) - 2\sin(2) &= \frac{2\sin(2)-2\sin(0.5)}{1.5}((3)-4) \\ y(3) &= 2\sin(2) + \frac{2\sin(2)-2\sin(0.5)}{1.5}((3)-4) \\ &= 1.2454.... \end{aligned}$$

Tangent Line Underestimates/Overestimates

Let $f(d)$ be approximated by the line tangent to the graph of $f(x)$ at $x=c$. Let $t(d)$ be the approximate value of $f(d)$ approximated by using the line tangent to the graph of $f(x)$ at $x=c$.

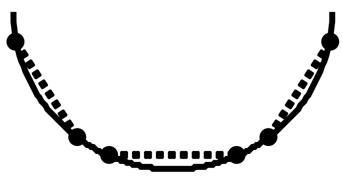
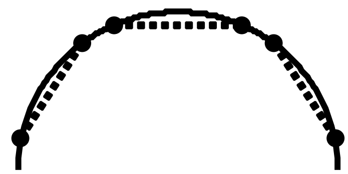
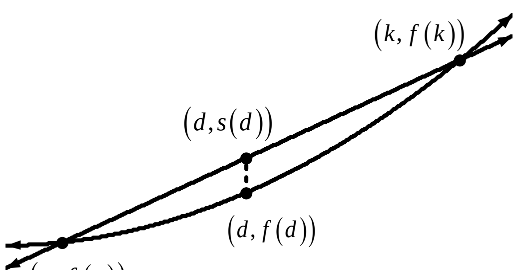
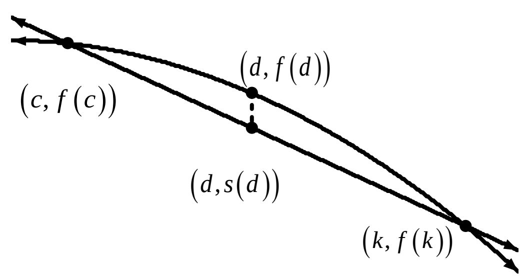
- $t(d)$ is an underapproximation of $f(d)$ if $f''(x)>0$ for all x between $x=c$ and $x=d$
- $t(d)$ is an overapproximation of $f(d)$ if $f''(x)<0$ for all x between $x=c$ and $x=d$

If $f(x)$ is concave up: $f''(x)>0$	If $f(x)$ is concave down: $f''(x)<0$
 Tangent line is underestimate	 Tangent line is overestimate
	

<u>Secant Line Underestimates/Overestimates</u>

Let $f(d)$ be approximated by the secant line through the points $(c, f(c))$ and $(k, f(k))$ where d is between c and k. Let $s(d)$ be the approximate value of $f(d)$ approximated by using the secant line through the points $(c, f(c))$ and $(k, f(k))$.

- $s(d)$ is an overapproximation of $f(d)$ if $f''(x) > 0$ for all x between $x = c$ and $x = k$.
- $s(d)$ is an underapproximation of $f(d)$ if $f''(x) < 0$ for all x between $x = c$ and $x = k$

If $f(x)$ is concave up: $f''(x) > 0$	If $f(x)$ is concave down: $f''(x) < 0$
 Secant line is overestimate	 Secant line is underestimate
	

[AP Calculus AB 2009 Question #5(d)](#)

Suppose $f'(5) = 3$ and $f''(x) < 0$ for all x in the closed interval $5 \le x \le 8$. Use the line tangent to the graph of f at $x = 5$ to show that $f(7) \le 4$. Use the secant line for the graph of f on $5 \le x \le 8$ to show that $f(7) \ge \frac{4}{3}$.

AP Calculus AB 2001 #4(d)   AP Calculus AB 2010 Form B 2(d)   AP Calculus BC 2005 #4(d)

AP Calculus AB 2009 Question #5(d)

x	2	3	5	8	13
$f(x)$	1	4	–2	3	6

Suppose $f'(5)=3$ and $f''(x)<0$ for all *x* in the closed interval $5\le x\le 8$. Use the line tangent to the graph of *f* at $x=5$ to show that $f(7)\le 4$. Use the secant line for the graph of *f* on $5\le x\le 8$ to show that $f(7)\ge \frac{4}{3}$.

Tangent line approximation for $f(7)$:

$$y-y_1=m(x-x_1)$$
$$y-(-2)=3(x-5)$$

$$f(7)-(-2)\approx 3(7-5)$$
$$f(7)\approx 4$$

Since $f''(x)<0$/concave down for all *x* in the closed interval $5\le x\le 8$, we know that the tangent line lies above the graph of *f* on the interval $5\le x\le 8$, so $f(7)\le 4$.

Secant line approximation through $(5, f(5))$ and $(8, f(8))$

slope between $(5, f(5))$ and $(8, f(8))=\dfrac{f(8)-f(5)}{8-5}$

$$=\frac{3-(-2)}{8-5}$$
$$=\frac{5}{3}$$

Secant line equation:

$$y-y_1=m(x-x_1)$$
$$y-3=\frac{5}{3}(x-8)$$

$$f(7)-3\approx\frac{5}{3}(7-8)$$
$$f(7)\approx\frac{4}{3}$$

Since $f''(x)<0$/concave down for all *x* in the closed interval $5\le x\le 8$, we know that the secant line lies below the graph of *f* on the interval $5\le x\le 8$, so $f(7)\ge\frac{4}{3}$.