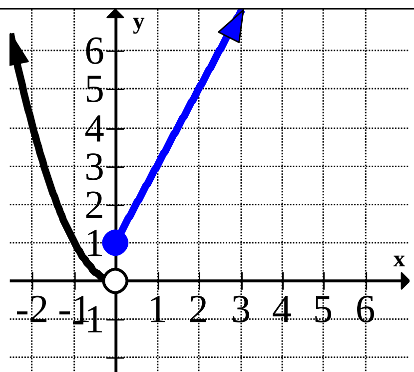
How to determine if a piecewise function of the form

$$f(x)=\begin{cases} h(x) & x<c \\ k(x) & x\geq c \end{cases}$$

is

Continuous at $x=c$	Differentiable at $x=c$
$h(c)=k(c)$	$h(c)=k(c)$ AND $h'(c)=k'(c)$



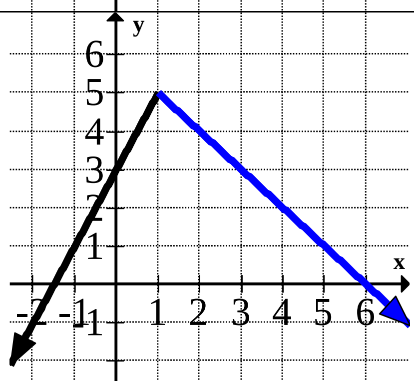
Example #1: $f(x)=\begin{cases} x^2 & x<0 \\ 2x+1 & x\geq 0 \end{cases}$

$$x^2\Big|_{x=0}=0$$

$$2x+1\Big|_{x=0}=1$$

$f(x)$ is not continuous at $x=0$

$f(x)$ is not differentiable at $x=0$ because it is not continuous at $x=0$

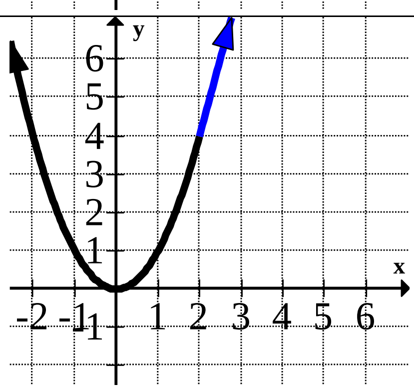


Example #2: $f(x)=\begin{cases} 2x+3 & x\leq 1 \\ -x+6 & x>1 \end{cases}$

$2x+3\Big|_{x=1}=5$ and $\frac{d}{dx}(2x+3)\Big|_{x=1}=2\Big|_{x=1}=2$

$-x+6\Big|_{x=1}=5$ and $\frac{d}{dx}(-x+6)\Big|_{x=1}=-1\Big|_{x=1}=-1$

$f(x)$ is continuous at $x=1$, but is not differentiable at $x=1$.



Example #3: $f(x)=\begin{cases} x^2 & x\leq 2 \\ 4x-4 & x>2 \end{cases}$

$x^2\Big|_{x=2}=4$ and $\frac{d}{dx}(x^2)\Big|_{x=2}=2x\Big|_{x=2}=4$

$4x-4\Big|_{x=2}=4$ and $\frac{d}{dx}(4x-4)\Big|_{x=2}=4\Big|_{x=2}=4$

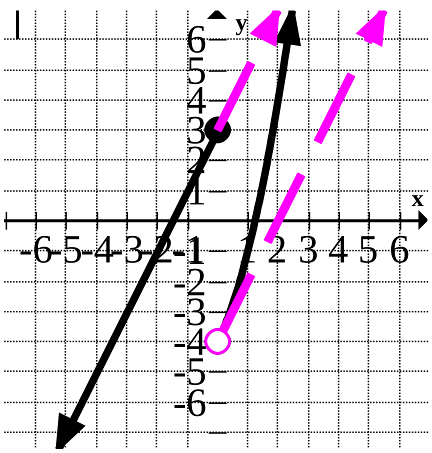
$f(x)$ is continuous and differentiable at $x=2$

SPECIAL CASE!

If you are checking for differentiability at $x=c$, and only verify that $h'(c)=k'(c)$, you could make an incorrect conclusion.

Example:

$$f(x)=\begin{cases} 2x+3 & x\le 0 \\ x^2+2x-4 & x>0 \end{cases}$$



$$\frac{d}{dx}(2x+3)\Big|_{x=0} = 2\big|_{x=0} = 2$$

$$\frac{d}{dx}\left(x^2+2x-4\right)\Big|_{x=0} = (2x+2)\big|_{x=0} = 2$$

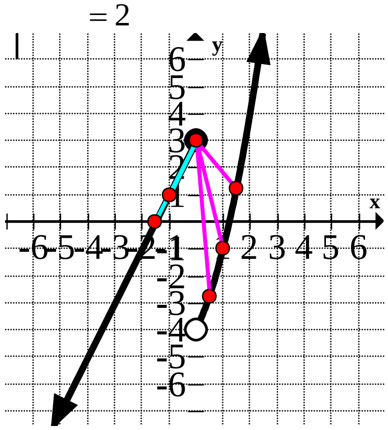
However, $f(x)$ is not differentiable at $x=0$, because $f(x)$ is not continuous at $x=0$.

$$\text{differentiable} \rightarrow \text{continuous} \equiv \neg\text{continuous} \rightarrow \neg\text{differentiable}$$

Formally, $\lim\limits_{h\to 0^-}\frac{f(0+h)-f(0)}{h} \ne \lim\limits_{h\to 0^+}\frac{f(0+h)-f(0)}{h}$. That is, the slope using $(0, f(0))$ and points from the left, do not match the slope using $(0, f(0))$ and points from the right.

$h=0^-$

$$\begin{aligned}\lim_{h\to 0^-}\frac{f(0+h)-f(0)}{h} &= \lim_{h\to 0^-}\frac{f(0+h)-f(3)}{h} \\ &= \lim_{h\to 0^-}\frac{(2h+3)-(3)}{h} \\ &= \lim_{h\to 0^-}\frac{2h}{h} \\ &= 2\end{aligned}$$



$h=0^+$

$$\begin{aligned}\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h} &= \lim_{h\to 0^+}\frac{f(0+h)-f(3)}{h} \\ &= \lim_{h\to 0^+}\frac{\left(h^2+2h-4\right)-(3)}{h} \\ &= \lim_{h\to 0^+}\frac{h^2+2h-7}{h} \\ &= \lim_{h\to 0^+}\frac{h^2}{h}+\lim_{h\to 0^+}\frac{2h}{h}+\lim_{h\to 0^+}\frac{-7}{h} \\ &= \lim_{h\to 0^+}h+\lim_{h\to 0^+}2+\lim_{h\to 0^+}\frac{-7}{h} \\ &= 0+2+(-\infty)\end{aligned}$$

$$\downarrow$$

$$DNE \;/\; -\infty$$