Concavity and the Second Derivative Test

Concavity:

Let *f* be a differentiable function on an open interval *I*. The graph of *f* is

Convave up – if f' is increasing on the interval.

Concave down – if f' is decreasing on the interval.

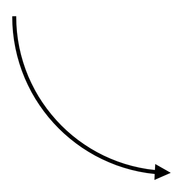
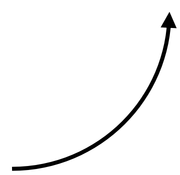
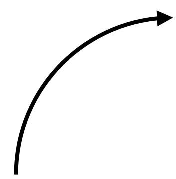
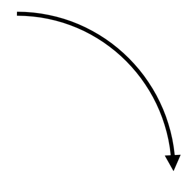
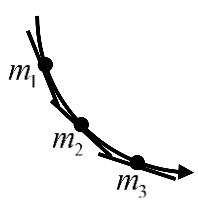
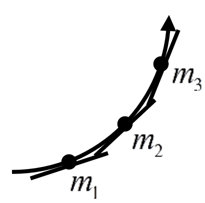
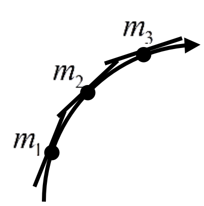
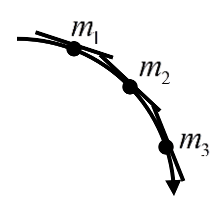
Note that a line has no concavity. Therefore, if $f''=0$ *on the entire interval* then we say that the graph of *f* is neither concave down nor concave up.

Let *f* be a function whose second derivative exists on an open interval *I*:

I. If $f''>0$ for all x in *I*, then *f* is concave up on *I*.

II. If $f''<0$ for all x in *I*, then *f* is concave down on *I*.

Visually:

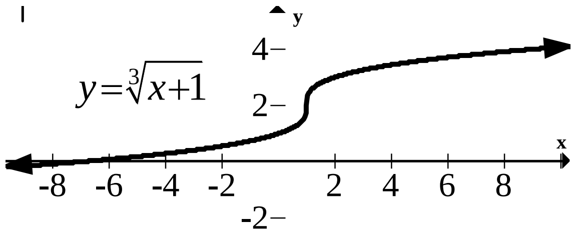
Concave Up		**Concave Down**	
			
 $m_1<m_2<m_3$	 $m_1<m_2<m_3$	 $m_1>m_2>m_3$	 $m_1>m_2>m_3$
$f'<0$ and f' is increasing	$f'>0$ and f' is increasing	$f'>0$ and f' is decreasing	$f'<0$ and f' is decreasing
"*f* is decreasing at an increasing rate."	"*f* is increasing at an increasing rate."	"*f* is increasing at a decreasing rate."	"*f* is decreasing at a decreasing rate."

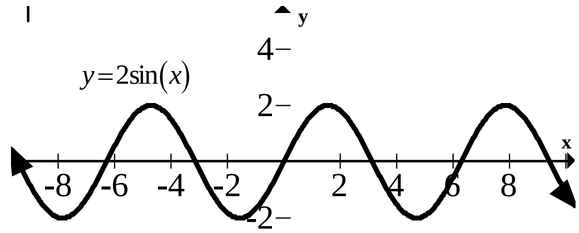
If both of the following hold simultaneously:

I. f has a “tangent” to its graph at $(c, f(c))$. (possible vertical tangent)

II. f changes concavity at $(c, f(c))$.

Then we say that f has an **inflection point** at $(c, f(c))$.





If $(c, f(c))$ is an inflection point of f, then either $f''(c)=0$ or $f''(c)$ DNE.
That is, if f has an inflection point at $x=c$ then either $f''(c)=0$ or DNE.

If $f(x)$ has a tangent line (possibly a vertical tangent line) at $(c, f(c))$ and $f''(x)$ changes sign at $x=c$, then $f(x)$ has an inflection point at $(c, f(c))$.

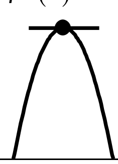
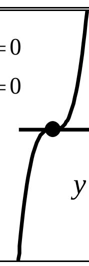
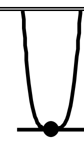
Second Derivative Test:

Let f be a function such $f''(x)$ exists on an open interval containing c.

I. If $f'(c)=0$ and $f''(c)>0$, then f has a relative minimum at $(c, f(c))$.

II. If $f'(c)=0$ and $f''(c)<0$, then f has a relative maximum at $(c, f(c))$.

III. If $f'(c)=0$ and $f''(c)=0$, the test is *inconclusive.* The First Derivative Test must be used to determine if f has a relative min, max, or neither at $(c, f(c))$.

 $f'(c)=0$ $f''(c)>0$	$f'(c)=0$ $f''(c)<0$ 	$f'(0)=0$ $f''(0)=0$  $y=x^3$	 $f'(0)=0$ $f''(0)=0$ $y=x^4$
Case I	Case II	The three different possible situations for Case III	$f'(0)=0$ $f''(0)=0$ $y=-x^4$