Homework Guidelines for Series

Students are allowed to use any test for convergence/divergence on any homework exercise, for any section of the textbook. Students can use the following to determine whether a series converges, or not:

✓ Alternating Series Test	✓ Integral Test	✓ Direct Comparison Test
✓ Root Test	✓ Ratio Test	✓ Limit Comparison Test
✓ Geometric Series Test	✓ *p*-series Test	✓ Limit of the *n*-th term
✓ Limit of the *n*-th partial sums (Telescoping Series)		

Students must quote which test(s) they are using to prove convergence/divergence, along with the necessary setup/conditions to apply the test(s).

From the AP Course Exam and Description:

"Justifications require that you give mathematical reasons, and that you verify the needed conditions under which relevant theorems, properties, definitions, or tests are applied. Your work will be scored on the correctness of your method, as well as your answers. Answers without supporting work will usually not receive credit."

Insufficient Work	Complete Work
$\sum_{n=1}^{\infty}\frac{1}{n^2+6n+13} \sim \sum_{n=1}^{\infty}\frac{1}{n^2}$ Converges	$\sum_{n=1}^{\infty}\frac{1}{n^2+6n+13} \sim \sum_{n=1}^{\infty}\frac{1}{n^2}$ $\lim_{n\to\infty}\frac{\left(\frac{1}{n^2}\right)}{\left(\frac{1}{n^2+6n+13}\right)} = \lim_{n\to\infty}\frac{n^2+6n+13}{n^2} = 1$ Since $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is a convergent p-series with $p=2$ By the Limit Comparison Test, $\sum_{n=1}^{\infty}\frac{1}{n^2+6n+13}$ also converges.
$\sum_{n=3}^{\infty}\frac{n^2}{e^n} \leftrightarrow \int_3^{\infty}\frac{x^2}{e^x}\,dx$ $\int_3^{\infty}\frac{x^2}{e^x}\,dx =$ $\left[-(\infty)^2 e^{-(\infty)} - 2(\infty)e^{-(\infty)} - 2e^{-(\infty)}\right]$ Finite. Series Converges	$\sum_{n=3}^{\infty}\frac{n^2}{e^n} \leftrightarrow \int_3^{\infty}\frac{x^2}{e^x}\,dx$ $\int_3^{\infty}\frac{x^2}{e^x}\,dx = \lim_{b\to\infty}\int_3^{b} x^2e^{-x}dx$ $= \lim_{b\to\infty}\left[-x^2e^{-x} - 2xe^{-x} - 2e^{-x}\right]_3^b$ $= \lim_{b\to\infty}\left(\left[-\frac{x^2}{e^x} - 2\frac{x}{e^x} - \frac{2}{e^x}\right] - \left[-x^2e^{-x} - 2xe^{-x} - 2e^{-x}\right]\right)$ $= 0 - \left[-(3)^2 e^{-3} - 2(3)e^{-3} - 2e^{-3}\right] = \frac{17}{e^3}$ Therefore since $\int_3^{\infty}\frac{x^2}{e^x}\,dx$ converges, $\sum_{n=3}^{\infty}\frac{n^2}{e^n}$ converges by the Integral Test

Insufficient Work	**Complete Work**
$\sum_{n=1}^{\infty} \frac{1}{n^2+6n+13} < \sum_{n=1}^{\infty} \frac{1}{n^2}$ Series Converges	$\sum_{n=1}^{\infty} \frac{1}{n^2+6n+13} < \sum_{n=1}^{\infty} \frac{1}{n^2}$ for all $n \geq 1$ Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series with $p = 2$ $\sum_{n=1}^{\infty} \frac{1}{n^2+6n+13}$ converges by the Direct Comparison Test
$\sum_{n=1}^{\infty} (-1)^{n+1} ne^{-n}$ Series Converges	$\sum_{n=1}^{\infty} (-1)^{n+1} ne^{-n} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{e^n}$ Since $\frac{n}{e^n}$ is monotonically decreasing and $\lim_{n\to\infty} \frac{n}{e^n} = 0$, $\sum_{n=1}^{\infty} (-1)^{n+1} ne^{-n}$ converges by the Alternating Series Test.
$\sum_{n=1}^{\infty} \frac{1+2^n}{3^n}$ Geometric Series, Converges	$\sum_{n=1}^{\infty} \frac{1+2^n}{3^n} = \sum_{n=1}^{\infty} \frac{1}{3^n} + \sum_{n=1}^{\infty} \frac{2^n}{3^n}$ $= \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n + \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$ $\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$ is a Geometric Series, $r = \frac{1}{3}$, therefore $\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$ converges $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$ is a Geometric Series, $r = \frac{2}{3}$, therefore $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$ converges. Since $\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$ and $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$ converge, therefore $\sum_{n=1}^{\infty} \frac{1+2^n}{3^n}$ converges. *** $\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n = \frac{\text{first term}}{1-\text{common ratio}} = \frac{\left(\frac{1}{3}\right)}{1-\left(\frac{1}{3}\right)}$ $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n = \frac{\text{first term}}{1-\text{common ratio}} = \frac{\left(\frac{2}{3}\right)}{1-\left(\frac{2}{3}\right)}$

Test	What to show for convergence	What to show if the series does not converge
n^{th} term test	This test is not used to show convergence.	$\lim_{n\to\infty}(a_n) \neq 0$
Geometric Series	Rewrite series to be of the form $\sum ar^n$ Identify the value of r State "$\sum ar^n$ converges because this is a Geometry series with $\lvert r\rvert < 1$"	Rewrite series to be of the form $\sum ar^n$ Identify the value of r State "$\sum ar^n$ does not converge because this is a Geometry series with $\lvert r\rvert \geq 1$"
Telescoping Series	Write the general expression for the n^{th} partial sum Demonstrate $\lim_{n\to\infty} S_n = k$ for some finite value k. State "The series converges because $\lim_{n\to\infty} S_n = k$"	Write the general expression for the n^{th} partial sum Demonstrate $\lim_{n\to\infty} S_n$ DNE State" The series does not converge because $\lim_{n\to\infty} S_n$ does not converge."
***p*-series**	Identify that the series is of the form $\sum \frac{1}{n^p}$ State: "The series converges since this is a *p*-series with $p > 1$"	Identify that the series is of the form $\sum \frac{1}{n^p}$ State: "The series does not converge since this is a *p*-series with $p > 1$"
Limit Comparison Test	Given $\sum a_n$, define a series $\sum b_n$ Show that $\lim_{n\to\infty}\left[\frac{a_n}{b_n} \text{ or } \frac{b_n}{a_n}\right]$ is finite and positive. Justify that $\sum b_n$ converges State "Since $\sum b_n$ converges and $\lim_{n\to\infty}\left[\frac{a_n}{b_n} \text{ or } \frac{a_n}{b_n}\right]$ is finite, $\sum a_n$ converges by the Limit Comparison Test."	Given $\sum a_n$, define a series $\sum b_n$ Show that $\lim_{n\to\infty}\left[\frac{a_n}{b_n} \text{ or } \frac{b_n}{a_n}\right]$ is finite and positive. Justify that $\sum b_n$ does not converge State "Since $\sum b_n$ does not converge and $\lim_{n\to\infty}\left[\frac{a_n}{b_n} \text{ or } \frac{a_n}{b_n}\right]$ is finite, $\sum a_n$ does not converge by the Limit Comparison Test."
Direct Comparison Test	Given $\sum a_n$, define a series $\sum b_n$ Show that $a_n \leq b_n$ for sufficiently large n. Justify that $\sum b_n$ converges State "Since $\sum a_n \leq \sum b_n$, by the Direct Comparison Test, $\sum a_n$ converges."	Given $\sum a_n$, define a series $\sum b_n$ Show that $b_n \leq a_n$ for sufficiently large n. Justify that $\sum b_n$ does not converge State "Since $\sum b_n \leq \sum a_n$, by the Direct Comparison Test, $\sum a_n$ does not converge."

Test	What to show for convergence	What to show if the series does not converge
Integral Test	Given $\sum_{n=k}^{\infty} a_n$, rewrite the sum as the integral $\int_k^{\infty} f(x)dx$ and demonstrate that $\int_k^{\infty} f(x)dx$ converges. State "Since $\int_k^{\infty} f(x)dx$ converges, $\sum_{n=k}^{\infty} a_n$ converges by the Integral Test."	Given $\sum_{n=k}^{\infty} a_n$, rewrite the sum as the integral $\int_k^{\infty} f(x)dx$ and demonstrate that $\int_k^{\infty} f(x)dx$ does not converge. State "Since $\int_k^{\infty} f(x)dx$ does not converge, $\sum_{n=k}^{\infty} a_n$ does not converge by the Integral Test."
Alternating Series Test	Given $\sum(-1)^n a_n$ or $\sum(-1)^{n+1} a_n$, show that $\lim_{n\to\infty} a_n = 0$ State "Since the sum is an Alternating Series whose terms decrease in absolute value to zero, the series converges."	Given $\sum(-1)^n a_n$ or $\sum(-1)^{n+1} a_n$, show that $\lim_{n\to\infty} a_n \neq 0$ State "Since the sum is an Alternating Series and $\lim_{n\to\infty} a_n \neq 0$, the series does not converge."
Ratio Test	Given $\sum a_n$ demonstrate that $\lim_{n\to\infty}\left\lvert\frac{a_{n+1}}{a_n}\right\rvert < 1$ State "Since $\lim_{n\to\infty}\left\lvert\frac{a_{n+1}}{a_n}\right\rvert < 1$, the series converges by the Ratio Test."	Given $\sum a_n$ demonstrate that $\lim_{n\to\infty}\left\lvert\frac{a_{n+1}}{a_n}\right\rvert > 1$ State "Since $\lim_{n\to\infty}\left\lvert\frac{a_{n+1}}{a_n}\right\rvert < 1$, the series does not converge by the Ratio Test."
Root Test	Given $\sum a_n$ demonstrate that $\lim_{n\to\infty}\sqrt[n]{\lvert a_n\rvert} < 1$ State "Since $\lim_{n\to\infty}\sqrt[n]{\lvert a_n\rvert} < 1$, the series converges by the Root Test."	Given $\sum a_n$ demonstrate that $\lim_{n\to\infty}\sqrt[n]{\lvert a_n\rvert} > 1$ State "Since $\lim_{n\to\infty}\sqrt[n]{\lvert a_n\rvert} > 1$, the series does not converges by the Root Test."