A **<u>differential equation</u>** is an equation that involves *x*, *y*, and any order derivative of *y*.

$y^{(4)}-16y=0$	$y''+xy'-3=0$	$y^{(5)}=y$
$y'=3\cos(x)$	$y''=3\sin(3x)$	$x^2+3y=2y^{(3)}$

The *<u>solution of a differential equation</u>* is a function $y=f(x)$, such that when *y*, and its derivatives are substituted into the equation, <u>the equation is valid for all values of *x*.</u>

$y^{(4)}-16y=0$		$xy'-2y=x^3e^x$	
$y=3\cos(x)$	$y=e^{-2x}$	$y=x^2$	x^2e^x
$y'=-3\sin(x)$	$y'=-2e^{-2x}$	$y'=2x$	$y'=2xe^x+x^2e^x$
$y''=-3\cos(x)$	$y''=4e^{-2x}$	$xy'-2y\overset{?}{=}x^3e^x$ $x(2x)-2(x^2)\neq x^3e^x$ not a solution	$xy'-2y\overset{?}{=}x^3e^x$ $x\left[2xe^x+x^2e^x\right]-2(x^2e^x)=x^3e^x$ $\cancel{2x^2e^x}+x^3e^x\cancel{-2x^2e^x}=x^3e^x$ solution!
$y'''=3\sin(x)$	$y'''=-8e^{-2x}$		
$y^{(4)}=3\cos(x)$	$y^{(4)}=16e^{-2x}$		
$y^{(4)}-16y\overset{?}{=}0$ $3\cos(x)-16\cos(x)\neq 0$ not a solution	$y^{(4)}-16y\overset{?}{=}0$ $16e^{-2x}-16e^{-2x}=0$ solution!		

The solution to a differential equation is not unique, because of the constant of integration. Therefore, the solution to a differential equation is of the form $f(x)+C$, where C can be any constant.

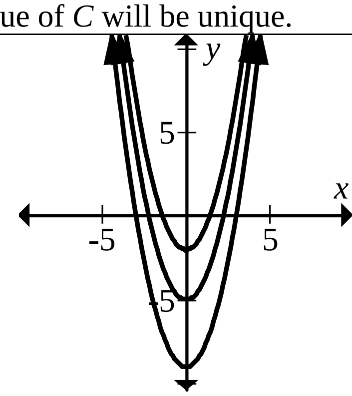
If an initial condition is given for the solution, then the value of C will be unique.

At right, these are the solutions to the differential equation $y'=2x$.

Solutions to this differential equation are of the form $y=x^2+C$.

The solution that has the initial condition of $y(0)=-2$ is

$$\left.\begin{aligned}-2&=(0)^2+C\\-2&=C\end{aligned}\right\}\rightarrow y=x^2-2$$

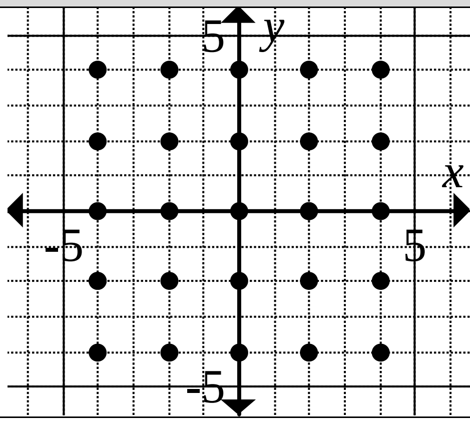
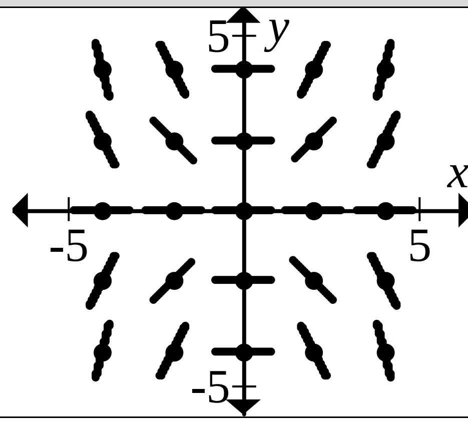


Sometimes a differential equation is not solvable by algebraic methods. In such a case, sometimes just having an idea of what the graph of the solution looks like is helpful. In order to construct the general shape of the solution, two things are needed:

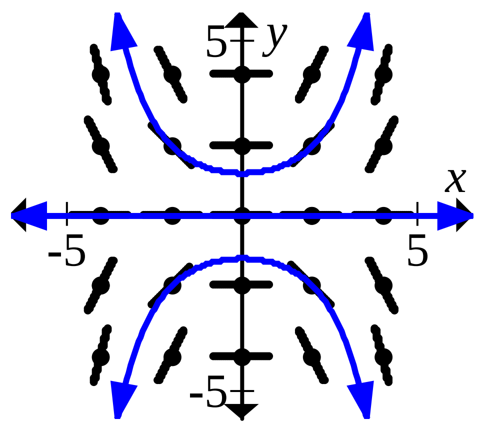
- ✓ A slope field needs to be produced
- ✓ A starting point/initial value of the solution must be given.

To create a slope field, you need to pick the points in the plane at which you want to create the slope segments. These are little segments of the tangent line to the solution of the differential equation at the particular location.

At each location, you use the differential equation to determine the slope value at that point. This calculation may involve <u>both ***x*** **and** ***y*****.**</u> Then, at each point, construct a segment of the line that (1) passes through that point (2) has the determined slope.

$\frac{dy}{dx}=\frac{1}{4}xy$	
	

Looking at the slope field, you can get a general sense of what shape and behavior a solution will have based on the structure of the slope field.



Notice that the solution that passes through $(2,2)$ is tangent to the segment in the slope field at $(2,2)$.

The solution that passes through $(1,0)$ is a straight line.

You can notice that the range of a solution with an initial condition in quadrants I or II will not include negative numbers since the solution will not cross the line $y=0$.

Differential equations are an entire field in the study of mathematics. There are courses devoted to many levels of the topic, and one can major in the study of this field. In AP Calculus, we will only study first-order differential equations that can be solved by separation of variables.

A <u>first-order differential equation</u> is a differential equation that involves *x*, *y*, and y' only – no other higher order derivative of *y* can appear in the equation.

<u>Separation of variables</u> is a method to solve differential equations algebraically. This method does not work on all differential equations out there, but it will work on a subset.

To start separation of variables, you rewrite y' as $\frac{dy}{dx}$

Then multiply both side of the equation by dx, and get all the *x*'s to the *dx* side, and all the *y*'s to the *dy* side to get an equation of the form $f(y)dy = f(x)dx$.

Then integrate both sides respectively, and solve the new equation for *y*. An example of such work is provided at right.

$$\frac{dy}{dx} = \frac{1}{4}xy$$

$$\frac{1}{y}dy = \frac{1}{4}xdx$$

$$\int \frac{1}{y}dy = \int \frac{1}{4}xdx$$

$$\ln|y| = \frac{1}{8}x^2 + C$$

$$e^{\ln|y|} = e^{\frac{1}{8}x^2+C}$$

$$|y| = e^{\frac{1}{8}x^2+C}$$

$$|y| = e^{\frac{1}{8}x^2} \cdot e^C$$

$$|y| = Ae^{\frac{1}{8}x^2}$$

$$y = \pm Ae^{\frac{1}{8}x^2}$$

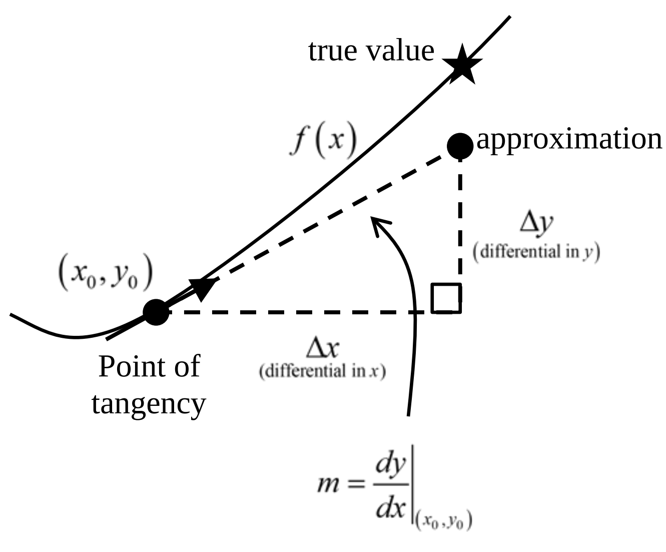
$$y = Ae^{\frac{1}{8}x^2}$$

When a differential equation cannot be solved by algebraic methods, then it is possible to construct a visual approximation to the solution of the differential equation by using Euler's Method.

Euler's (pronounced "oil"-er's) Method is based off of the idea of using the tangent line as an approximation of given a function for points close to the point of tangency.

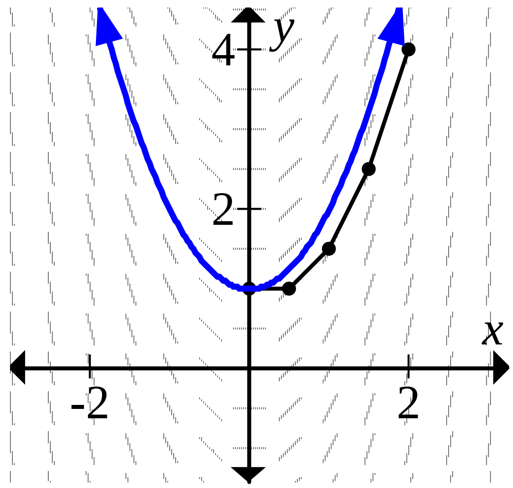
Given a point (x_0, y_0) and the slope of the tangent at that point $\left.\frac{dy}{dx}\right|_{(x_0,y_0)}$ you can find an approximation to the function by deciding on a Δx value and using the following formula:

$$\boxed{\begin{aligned}\text{approximation} &= (x_0 + \Delta x\ ,\ y_0 + \Delta y) \\ &= \left(x_0 + \Delta x\ ,\ y_0 + \left[\frac{dy}{dx}\right]\Delta x\right) \\ &= \left(x_0 + \Delta x, y_0 + \left[\left.\frac{dy}{dx}\right|_{(x_0,y_0)}\right] \cdot \Delta x\right)\end{aligned}}$$



If you connect the point of tangency to the approximation with a segment, this is a linear approximation of the function on that interval.

If you repeat re-evaluating the derivative and constructing a line segment (this time with the location of the approximation) over and over again, you will get a reasonable method for approximating the general shape of the solution to the differential equation that has the initial condition of (x_0, y_0), oftentimes referred to as the "seed."



At right, $\frac{dy}{dx} = 2x$, $(0,1)$ is the seed value, and the step-size $\Delta x = 0.5$.

The solution passing through $(0,1)$ is $y = x^2 + 1$

Step #	**Coordinate**	**Approximation**
0	$(0,1)$	$(0+0.5, 1+[2(0)]\cdot(0.5))$ $\downarrow$ $(0.5,1)$
1	$(0.5,1)$	$(0.5+05, 1+[2(0.5)](0.5))$ $\downarrow$ $(1,1.5)$
2	$(1,1.5)$	$(1+0.5, 1.5+[2(1)](0.5))$ $\downarrow$ $(1.5, 2.5)$
3	$(1.5, 2.5)$	$(1.5+0.5, 2.5+[2(1.5)](0.5))$ $\downarrow$ $(2,4)$

As you can see, the segmented graph produced by Euler's Method conveys the same general behavior as the true solution, even though is it off by a bit. So if all you need is an understanding of general shape and behavior, Euler's Method will inform you about these qualities.

In addition, this recursive method is easily programmed for computers/calculators to do, which is nice. However, be prepared to execute two steps of Euler's Method by hand on the AP Calculus AB/BC exams.