AP Calculus BC
Taylor Series Practice Test
Spring 2018 Solutions

Name:________________________________

Date:____________________ Period:_______

No Calculator Permitted

1. Consider the series $\sum_{n=1}^{\infty} \frac{e^n}{n!}$. If the ratio test is applied to the series, which of the following inequalities results, implying that the series converges?

$$\lim_{n\to\infty}\left|\frac{\frac{e^{n+1}}{(n+1)!}}{\frac{e^n}{n!}}\right| = \lim_{n\to\infty}\left|\frac{e^{n+1}}{(n+1)!}\cdot\frac{n!}{e^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{e}{n+1}\right|$$

(a) $\lim_{n\to\infty} \frac{e}{n!} < 1$

(b) $\lim_{n\to\infty} \frac{n!}{e} < 1$

(c) $\lim_{n\to\infty} \frac{n+1}{e} < 1$

(d) $\lim_{n\to\infty} \frac{e}{n+1} < 1$

(e) $\lim_{n\to\infty} \frac{e}{(n+1)!} < 1$

2. Which of the following series converges for all real numbers x?

(a) $\sum_{n=1}^{\infty} \frac{x^n}{n}$

(b) $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$

(c) $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$

(d) $\sum_{n=1}^{\infty} \frac{e^n x^n}{n!}$

(e) $\sum_{n=1}^{\infty} \frac{n! x^n}{e^n}$

$\sum_{n=1}^{\infty} \frac{x^n}{n}$ will not converge for $x=1$ - harmonic series

$\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ will not converge for $x=2$ - nth term test

$\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$ will not converge for $x=1$ - divergent p-series

$\sum_{n=1}^{\infty} \frac{n! x^n}{e^n}$ will not converge for $x=1$ - nth term test

$$\lim_{n\to\infty}\left|\frac{\frac{e^{n+1}x^{n+1}}{(n+1)!}}{\frac{e^n x^n}{n!}}\right| = \lim_{n\to\infty}\left|\frac{e^{n+1}x^{n+1}}{(n+1)!}\cdot\frac{n!}{e^n x^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{ex}{n+1}\right|$$

Hint: It's also an exponential divided by a factorial

$$= 0$$

3. What are all values of x for which the series $\sum_{n=1}^{\infty}\left(\frac{2}{x^2+1}\right)^n$ converges?

The interval of convergence cannot include ±1 or the series will be $\sum_{n=1}^{\infty}\left(\frac{2}{(\pm1)^2+1}\right)^n = \sum_{n=1}^{\infty}1^n$

The series will be a convergent Geometric Series if $|x|>1 \to x<-1$ or $x>1$

(a) $-1<x<1$ (b) $x>1$ only (c) $x\geq1$ only (d) $x<-1$ and $x>1$ only (e) $x\leq-1$ and $x\geq1$

4. What is the sum of the series $1+\ln(2)+\frac{(\ln(2))^2}{2!}+\cdots+\frac{(\ln(2))^n}{n!}+\cdots$

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$$

$$e^{\ln(2)}=\sum_{n=0}^{\infty}\frac{(\ln(2))^n}{n!}=1+\ln(2)+\frac{(\ln(2))^2}{2!}+\cdots+\frac{(\ln(2))^n}{n!}+\cdots$$

$$2=1+\ln(2)+\frac{(\ln(2))^2}{2!}+\cdots+\frac{(\ln(2))^n}{n!}+\cdots$$

(a) $\ln(2)$ (b) $\ln(1+\ln(2))$ (c) 2 (d) e^2 (e) The series diverges

5. $\sum_{n=1}^{\infty}a_n$ diverges and $0\leq a_n\leq b_n$ for all n, which of the following statements must be true?

(a) $\sum_{n=1}^{\infty}(-1)^n a_n$ converges

(b) $\sum_{n=1}^{\infty}(-1)^n b_n$ converges

(c) $\sum_{n=1}^{\infty}(-1)^n b_n$ diverges

(d) $\sum_{n=1}^{\infty}b_n$ converges

(e) $\sum_{n=1}^{\infty}b_n$ diverges by Direct Comparison Test

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6. Let $f(x) = \ln(1+x^3)$

(a) The Maclaurin series for $\ln(1+x)$ is $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{(-1)^{n+1} x^n}{n} + \cdots$. Use the series to write the first four nonzero terms and the general term of the Maclaurin series for *f*.

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{(-1)^{n+1} x^n}{n} + \cdots$$

$$\downarrow$$

$$\ln(1+x^3) = (x^3) - \frac{(x^3)^2}{2} + \frac{(x^3)^3}{3} - \frac{(x^3)^4}{4} + \cdots + \frac{(-1)^{n+1} (x^3)^n}{n} + \cdots$$

$$= x^3 - \frac{x^6}{2} + \frac{x^9}{3} - \frac{x^{12}}{4} + \cdots + \frac{(-1)^{n+1} x^{3n}}{n} + \cdots$$

(b) The radius of convergence of the Maclaurin series for *f* is 1. Determine the interval of convergence. Show the work that leads to your answer.

Since the power series is centered at zero, we know that the series will converge for $-1 < x < 1$. We must test for the convergence of the series when $x = -1$ and when $x = 1$.

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(1)^{3n}}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ Since this is an Alternating Series and $\lim_{n\to\infty} \frac{1}{n} = 0$, this series converges.

$$
\begin{aligned}
\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-1)^{3n}}{n} &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\left[(-1)^{3}\right]^{n}}{n} \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-1)^{n}}{n} \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{2n+1}}{n} \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{2n}(-1)}{n} \\
&= \sum_{n=1}^{\infty} \frac{\left[(-1)^{2}\right]^{n}(-1)}{n} \\
&= \sum_{n=1}^{\infty} \frac{(-1)}{n} \\
&= -\sum_{n=1}^{\infty} \frac{1}{n}
\end{aligned}
$$

This series will not converge since it is the opposite of the harmonic series.

Therefore the interval of convergence is $-1 < x \le 1$

6 Continued

Let $f\left(x\right)=\ln\left(1+x^3\right)$ and $\ln\left(1+x\right)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots+\frac{\left(-1\right)^{n+1}x^n}{n}+\cdots$

(c) Write the first four nonzero terms of the Maclaurin series for $f'\left(t^2\right)$. If $g\left(x\right)=\int_0^x f'\left(t^2\right)dt$ use the first two nonzero terms of the Maclaurin series for g to approximate $g\left(1\right)$.

$$f\left(x\right)=x^3-\frac{x^6}{2}+\frac{x^9}{3}-\frac{x^{12}}{4}+\cdots+\frac{\left(-1\right)^{n+1}x^{3n}}{n}+\cdots$$

$$\begin{aligned} f'\left(x\right)&=3x^2-\frac{6x^5}{2}+\frac{9x^8}{3}-\frac{12x^{11}}{4}+\cdots+\frac{3n\left(-1\right)^{n+1}x^{3n-1}}{n}+\cdots \\ &=3x^2-3x^5+3x^8-3x^{11}+\cdots+3\left(-1\right)^{n+1}x^{3n-1}+\cdots \end{aligned}$$

$$\begin{aligned} f'\left(t^2\right)&=3\left(t^2\right)^2-3\left(t^2\right)^5+3\left(t^2\right)^8-3\left(t^2\right)^{11}+\cdots+3\left(-1\right)^{n+1}\left(t^2\right)^{3n-1}+\cdots \\ &=3t^4-3t^{10}+3t^{16}-3t^{22}+\cdots+3\left(-1\right)^{n+1}t^{6n-2}+\cdots \end{aligned}$$

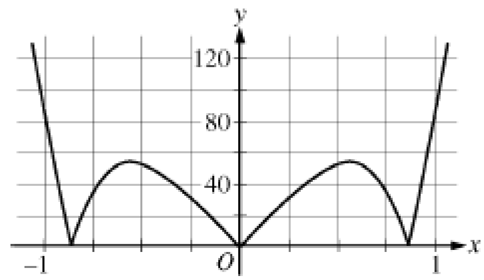
$$\begin{aligned} g\left(x\right)&=\int_0^x f'\left(t^2\right)dt \\ &=\int_0^x 3t^4-3t^{10}+3t^{16}-3t^{22}+\cdots+3\left(-1\right)^{n+1}t^{6n-2}+\cdots dt \\ &=\left[\frac{3}{5}t^5-\frac{3}{11}t^{11}+\frac{3}{17}t^{17}-\frac{3}{23}t^{23}+\cdots+\frac{3\left(-1\right)^{n+1}}{6n-1}t^{6n-1}+\cdots\right]_0^x \\ &=\frac{3}{5}x^5-\frac{3}{11}x^{11}+\frac{3}{17}x^{17}-\frac{3}{23}x^{23}+\cdots+\frac{3\left(-1\right)^{n+1}}{6n-1}x^{6n-1}+\cdots \end{aligned}$$

$$g\left(1\right)\approx\frac{3}{5}\left(1\right)^5-\frac{3}{11}\left(1\right)^{11}$$

(d) The Maclaurin series for g, evaluated at $x = 1$, is a convergent alternating series with individual terms that decrease in absolute value to 0. Show that your approximation in part (c) must differ from $g(1)$ by less than $\frac{1}{5}$.

By the Remainder Theorem for Alternating Series, the difference between the actual value and the approximate value differ by the absolute value of the next term. Therefore the error is bounded by $\left|\frac{3}{17}(1)^{17}\right| = \frac{3}{17} < \frac{1}{5}$

7. Let $f(x)=\sin\left(x^2\right)+\cos(x)$. The graph of $y=\left|f^{(5)}(x)\right|$ is shown at right.



Graph of $y=\left|f^{(5)}(x)\right|$

(a) Write the first four nonzero terms of the Taylor series for $\sin(x)$ about $x=0$, and write the first four nonzero terms of the Taylor series for $\sin\left(x^2\right)$ about $x=0$.

$$\sin(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\approx x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}$$

$$\sin\left(x^2\right)=\sum_{n=0}^{\infty}\frac{(-1)^n \left(x^2\right)^{2n+1}}{(2n+1)!}$$

$$\approx \left(x^2\right)-\frac{\left(x^2\right)^3}{3!}+\frac{\left(x^2\right)^5}{5!}-\frac{\left(x^2\right)^7}{7!}+\cdots+\frac{(-1)^n\left(x^2\right)^{2n+1}}{(2n+1)!}+\cdots$$

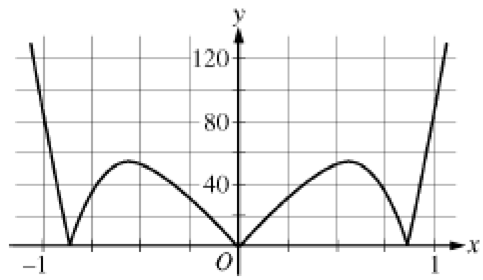
$$\approx x^2-\frac{x^6}{3!}+\frac{x^{10}}{5!}-\frac{x^{14}}{7!}$$

(b) Write the first four nonzero terms of the Taylor series for $\cos(x)$ about $x=0$. Use this series and the series for
$\sin(x^2)$, found in part (a), to write the first four nonzero
terms of the Taylor series for *f* about $x=0$.

$$\cos(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$$
$$\approx 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}$$

$$f(x)=\sin(x^2)+\cos(x)$$
$$=\left[x^2-\frac{x^6}{3!}+\frac{x^{10}}{5!}-\frac{x^{14}}{7!}+\cdots+\frac{(-1)^n x^{4n+2}}{(2n+1)!}+\cdots\right]+\left[1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots+\frac{(-1)^n x^{2n}}{(2n)!}+\cdots\right]$$
$$\downarrow$$
$$f(x)\approx 1+\left(x^2-\frac{x^2}{2!}\right)+\frac{x^4}{4!}+\left(-\frac{x^6}{3!}-\frac{x^6}{6!}\right)$$
$$\approx 1+\frac{x^2}{2}+\frac{x^4}{4!}-\left(\frac{6\cdot 5\cdot 4+1}{6!}\right)x^6$$
$$\approx 1+\frac{x^2}{2}+\frac{x^4}{4!}-\frac{121x^6}{6!}$$

7 Continued



Graph of $y = \left|f^{(5)}(x)\right|$

(c) Find the value of $f^{(6)}(0)$.

We know that $\dfrac{f^{(6)}(0)\cdot x^6}{6!}$ is the degree six term of the Taylor polynomial around $x=0$. Therefore we can look at the factor of x^6 in the Talyor Series Polynomial to determine the value of $f^{(6)}(0)$

$$\frac{f^{(6)}(0)\cdot x^6}{6!} = -\frac{121x^6}{6!}$$
$$\downarrow$$
$$f^{(6)}(0) = -121$$

(d) Let $P_4(x)$ be the fourth-degree Taylor polynomial for f about $x=0$. Using information from the graph of $y=\left|f^{(5)}(x)\right|$ shown at right, show that $\left|P_4\left(\frac{1}{4}\right) - f\left(\frac{1}{4}\right)\right| \le \frac{1}{3000}$.

The Lagrange Error Bound is given by

$$\text{error} \le \frac{\displaystyle\max_{\substack{z \text{ between} \\ 0 \text{ and } \frac{1}{4}}} f^{(5)}(z)\cdot\left(\frac{1}{4}\right)^5}{5!}$$
$$\le \frac{40\left(\frac{1}{4}\right)^5}{5!}$$
$$\le \frac{1}{3072}$$
$$\le \frac{1}{3000}$$

** Solution for 6c if $f'\left(t^2\right)$ was mistakenly identified at $\left[f\left(t^2\right)\right]'$

$$f\left(t^2\right)=\left(t^2\right)^3-\frac{\left(t^2\right)^6}{2}+\frac{\left(t^2\right)^9}{3}-\frac{\left(t^2\right)^{12}}{4}+\cdots+\frac{\left(-1\right)^{n+1}\left(t^2\right)^{3n}}{n}+\cdots$$

$$=t^6-\frac{t^{12}}{2}+\frac{t^{18}}{3}-\frac{t^{24}}{4}+\cdots+\frac{\left(-1\right)^{n+1}t^{6n}}{n}+\cdots$$

$$\downarrow$$

$$\left[f\left(t^2\right)\right]'=6t^5-6t^{11}+6t^{17}-6t^{23}+\cdots+\left(-1\right)^{n+1}6t^{6n-1}+\cdots$$

$$\downarrow$$

$$\int_0^x\left[f\left(t^2\right)\right]'dt=\left[t^6-\frac{1}{2}t^{12}+\frac{1}{3}t^{18}-\frac{1}{4}t^{24}+\cdots+\frac{\left(-1\right)^{n+1}t^{6n}}{n}+\cdots\right]_0^x$$

$$=x^6-\frac{1}{2}x^{12}+\frac{1}{3}x^{18}-\frac{1}{4}x^{24}+\cdots+\frac{\left(-1\right)^{n+1}x^{6n}}{n}+\cdots$$

$$\downarrow$$

$$g\left(1\right)=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$$

$$g\left(1\right)\approx 1-\frac{1}{2}$$

The error is bounded by $\text{Error}\le\left|\frac{1}{3}\right|$