<u>AP Calculus Homework Guidelines</u>

Relating $f(x)$, $f'(x)$, and $f''(x)$

A continuous function $f(x)$ on a closed interval is guaranteed an Absolute Minimum value, and an Absolute Maximum value by the Extreme Value Theorem.

To determine the Absolute Minimum/Maximum of $f(x)$ on a closed interval $[a,b]$, use the Extreme Value Theorem:

Step 1: Establish that $f(x)$ is continuous on $[a,b]$. This can be done by stating that the function is a polynomial or one of the twelve basic functions from Pre-Calculus, or that the function is differentiable on $[a,b]$.

Step 2: Determine the critical values of $f(x)$ in the interval (a,b). Suppose this list is $\{c_1, c_2, ..., c_n\}$.

Step 3: Evaluate $f(x)$ at all the critical values, and the endpoints.

x	$f(x)$
a	$f(a)$
c_1	$f(c_1)$
c_2	$f(c_2)$
$\vdots$	$\vdots$
c_n	$f(c_n)$
b	$f(b)$

Step4: Claim that the Absolute Minimum is the least of all the $f(x)$ values, and that the Absolute Maximum is the greatest of all the $f(x)$ values. The Absolute Minimum and Absolute Maximum *occur at* the corresponding *x* values. When claiming the absolute min/max in a sentence, it is suggested to write that "$f(x)$ has an absolute min/max on $[a,b]$ at $(c_i, f(c_i))$." This is due to the fact that the coordinate $(c_i, f(c_i))$ communicates the *location* and *absolute min/max value* <u>*simultaneously.*</u> $(c_i, f(c_i)) \leftrightarrow (\text{location, value})$. You will never be penalized for communicating both pieces of information at once, regardless of whether the exercise asks for one or the other.

Example: Section 3-1 # 36 Locate the absolute extrema of the function on the closed interval.
$y = x^2 - 2 - \cos(x) \quad [-1,3]$.

$y' = 2x + \sin(x)$
$y' = 0$ or DNE when $x = 0$

$$y(-1) = -1 - \cos(-1) \approx -1.540$$
$$y(0) = -3$$
$$y(3) = 7 - \cos(3) \approx 7.989 \text{ or } 7.990$$

The absolute min of *y* on $[-1,3]$ is at $(0,-3)$.

The absolute max of *y* on $[-1,3]$ is at $(3, 7-\cos(3))$.

A continuous function $f(x)$ on an open interval is *NOT* guaranteed an Absolute Minimum value, or Absolute Maximum value. <u>You must test the limits at the open endpoints!</u>

To determine the Absolute Minimum/Maximum of a continuous function on an interval (a,b)

Step 1: Determine the critical values of $f(x)$ in the interval (a,b).

State: $f'(x)=0$ or DNE when $x=c_1,c_2,...,c_n$.

Step 2: Evaluate $f(x)$ at all the critical values, and evaluate $\lim\limits_{x\to a^+} f(x)$ and $\lim\limits_{x\to b^-} f(x)$

x	$f(x)$
c_1	$f(c_1)$
c_2	$f(c_2)$
$\vdots$	$\vdots$
c_n	$f(c_n)$

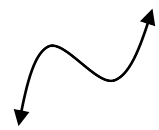
and

$\lim\limits_{x\to a^+} f(x)$

$\lim\limits_{x\to b^-} f(x)$

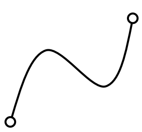
Step 2:

<u>No Maximum or Minimum</u>



If either $\lim\limits_{x\to a^+} f(x)$ or $\lim\limits_{x\to b^-} f(x)$ is ∞, then there is no Absolute Maximum

If either $\lim\limits_{x\to a^+} f(x)$ or $\lim\limits_{x\to b^-} f(x)$ is $-\infty$, then there is no Absolute Minimum



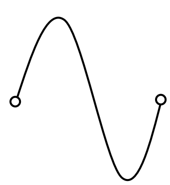
If all the above values are finite, and $\lim\limits_{x\to a^+} f(x)$ or $\lim\limits_{x\to b^-} f(x)$ is the largest value, then there is no Absolute Maximum.

If all the above values are finite, and $\lim\limits_{x\to a^+} f(x)$ or $\lim\limits_{x\to b^-} f(x)$ is is the least value, then there is no Absolute Maximum.

<u>Maximum or Minimum Exists</u>

If all the above values are finite, and $f(c_i)$ is the largest value, then you may claim $f(c_i)$ as the Absolute Maximum.

If all the above values are finite, and $f(c_i)$ is the least value, then you may claim $f(c_i)$ as the Absolute Minimum.



How to demonstrate Analytically that a function is increasing/decreasing.

YOU MUST DEMONSTRATE $f'(x)$

To claim that $f(x)$ is increasing/decreasing at a specific x value, $x=c$:

Claim $f(x)$ is increasing at $x=c$ by demonstrating $f'(c)>0$.

Claim $f(x)$ is decreasing at $x=c$ by demonstrating $f'(c)<0$.

To claim that $f(x)$ is increasing/decreasing on an open interval (a,b)

YOU MUST MAKE A LABELED SIGN CHART FOR $f'(x)$

To claim that $f(x)$ is increasing on (a,b), demonstrate through making the sign chart that $f'(x)>0$ on (a,b)

To claim that $f(x)$ is decreasing on (a,b), demonstrate through making the sign chart that $f'(x)<0$ on (a,b)

How to demonstrate Analytically that a function is concave up/concave down.

YOU MUST DEMONSTRATE $f''(x)$

To claim that $f(x)$ is concave up/concave down at a specific x value, $x=c$:

Claim $f(x)$ is concave up at $x=c$ by demonstrating $f''(c)>0$.

Claim $f(x)$ is concave down at $x=c$ by demonstrating $f''(c)<0$.

To claim that $f(x)$ is concave up/concave on an open interval (a,b)

YOU MUST MAKE A LABELED SIGN CHART FOR $f''(x)$

To claim that $f(x)$ is concave up on (a,b), demonstrate through making the sign chart of $f''(x)$ that $f''(x)>0$ on (a,b)

To claim that $f(x)$ is concave down on (a,b), demonstrate through making the sign chart of $f''(x)$ that $f''(x)<0$ on (a,b)

How to demonstrate analytically that $f(x)$ has a relative maximum or minimum at $x = c$

Note: Before making any claim at $x = c$, makes sure that $f(c)$ exists. If not, there is no maximum or minimum occurring at $x = c$.

By using the First Derivative Test:

Demonstrate $f'(x)$

Make a labeled sign chart for $f'(x)$.

To claim that $f(x)$ has a relative minimum at $x = c$, state that $f'(x)$ changes sign from negative to positive at $x = c$.
To claim that $f(x)$ has a relative maximum at $x = c$, state that $f'(x)$ changes sign from positive to negative at $x = c$.

By using the Second Derivative Test:

Demonstrate that $f'(c) = 0$

Demonstrate $f''(x)$

To claim that $f(x)$ has a relative minimum at $x = c$, demonstrate that $f''(x) > 0$ at $x = c$.

To claim that $f(x)$ has a relative maximum at $x = c$, demonstrate that $f''(x) < 0$ at $x = c$.

Local Minimum/Maximum of a Continuous Function at the Endpoints of a Closed Interval

Left Endpoint of a closed interval $[a,b]$	**Right Endpoint of a closed interval $[a,b]$**
To demonstrate that there is a local minimum at the left endpoint, you must demonstrate that $f'(x) > 0$ to the right of *a*.	To demonstrate that there is a local maximum at the right endpoint, you must demonstrate that $f'(x) > 0$ to the left of *b*.
To demonstrate that there is a local maximum at the left endpoint, you must demonstrate that $f'(x) < 0$ to the right of *a*.	To demonstrate that there is a local minimum at the right endpoint, you must demonstrate that $f'(x) < 0$ to the left of *b*.

How to demonstrate analytically that $f(x)$ has a point of inflection at $x = c$

Note: Before making any claim at $x = c$, makes sure that $f(x)$ has a tangent line at $x = c$. It acceptable for the tangent to be vertical at $x = c$. If no tangent exists, there is no inflection point occurring at $x = c$.

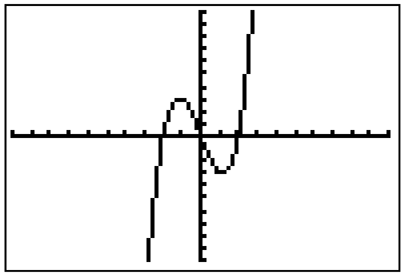
Demonstrate $f''(x)$

Make a labeled sign chart for $f''(x)$

To claim that $f(x)$ has a point of inflection at $x = c$, state that $f''(x)$ changes sign at $x = c$.

Given

$$f(x) = x(x-2)(x+2)$$
$$= x^3 - 4x$$



$$f(x) = x^3 - 4x$$
$$f'(x) = 3x^2 - 4$$
$$f''(x) = 6x$$

$$f'(x) = 0$$
$$\downarrow$$
$$x = -\sqrt{\frac{4}{3}}, \sqrt{\frac{4}{3}}$$

$f'(x)$	$(+)$	0	$(-)$	0	$(+)$
x		$-\sqrt{\frac{4}{3}}$		$\sqrt{\frac{4}{3}}$	

Labeled Sign chart for $f'(x)$

$$f''(x) = 0$$
$$\downarrow$$
$$x = 0$$

$f''(x)$	$(-)$	0	$(+)$
x		0	

Labeled Sign chart for $f''(x)$

Increasing/Decreasing

Decreasing:

$f(x)$ is decreasing on $\left(-\sqrt{\frac{4}{3}}, \sqrt{\frac{4}{3}}\right)$ because $f'(x) < 0$ on this interval.

Increasing:

$f(x)$ is increasing on $\left(-\infty, -\sqrt{\frac{4}{3}}\right) \cup \left(\sqrt{\frac{4}{3}}, \infty\right)$ $f'(x) > 0$ on these intervals.

Concave Up/Down

$f(x)$ is concave up on $(0, \infty)$ because $f''(x) > 0$ on this interval.
$f(x)$ is concave down on $(-\infty, 0)$ because $f''(x) < 0$ on this interval.

Justifying Local Minimum and Maximums

$$f'(x)=3x^2-4$$

$f'(x)$	$(+)$	0	$(-)$	0	$(+)$
x		$-\sqrt{\frac{4}{3}}$		$\sqrt{\frac{4}{3}}$	

Labeled Sign chart for $f'(x)$

First Derivative Test:

$f(x)$ has a local minimum at $x=\sqrt{\frac{4}{3}}$ because $f'(x)$ changes from negative to positive at $x=\sqrt{\frac{4}{3}}$.

$f(x)$ has a local maximum at $x=-\sqrt{\frac{4}{3}}$ because $f'(x)$ changes from positive to negative at $x=-\sqrt{\frac{4}{3}}$.

Second Derivative Test:

$f'\left(\sqrt{\frac{4}{3}}\right)=0$ AND $f''\left(\sqrt{\frac{4}{3}}\right)=6\sqrt{\frac{4}{3}}>0$, therefore by the Second Derivative Test $\left(\sqrt{\frac{4}{3}}, f\left(\sqrt{\frac{4}{3}}\right)\right)$ is a local minimum.

$f'\left(-\sqrt{\frac{4}{3}}\right)=0$ AND $f''\left(-\sqrt{\frac{4}{3}}\right)=6\left(-\sqrt{\frac{4}{3}}\right)<0$, therefore by the Second Derivative Test $\left(-\sqrt{\frac{4}{3}}, f\left(-\sqrt{\frac{4}{3}}\right)\right)$ is a local maximum.

Justifying Inflection Points:

$$f''(x)=6x$$

$$f''(x)=0$$

$$\downarrow$$

$$x=0$$

$f''(x)$	$(-)$	0	$(+)$
x		0	

Labeled Sign chart for $f''(x)$

$f(x)$ has an inflection point at $x=0$ because $f''(x)$ changes sign at $x=0$

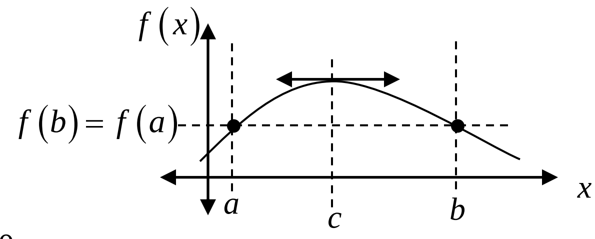
Rolle's Theorem

Rolle's Theorem states that given a closed interval $[a,b]$ and function $f(x)$ that meets all the following criteria:

(1) $f(x)$ is continuous on the closed interval $[a,b]$

(2) $f(x)$ is differentiable on the open interval (a,b)

(3) $f(b)=f(a)$



Then there is at least one value c such that $a<c<b$ and $f'(c)=0$.

To use Rolle's Theorem you *must:*

(A) Demonstrate conditions (1) and (2) are met.

(B) Demonstrate algebraically/numerically that $f(b)=f(a)$.

(C) State in words "By Rolle's Thereom there exists at least one value c in (a,b) such that $f'(c)=0$."

To demonstrate that Rolle's Theorem cannot be applied, you must demonstrate only one of the following:

(1) $f(x)$ is not continuous on the closed interval $[a,b]$

(2) $f(x)$ is not differentiable on the open interval (a,b)

(3) $f(b)\neq f(a)$

Common Questions

Question: How does one prove continuity on $[a,b]$?

Answer:

There are two ways to prove the function is continuous. One way is to state the function you are dealing with is a polynomial, or any of the basic functions you learned from Pre-Calculus, you can assume continuity. The second way to prove continuity on a closed interval is to use the fact that *differentiability implies continuity.* So if it is given that the function is differentiable on the interval, then one can claim that the function is continuous on the interval.

Question: How does one prove differentiability on (a,b)?

Answer:

There are a few ways to claim differentiability. One way is to claim that a function is differentiable on an open interval is by stating function is one of the basic functions from Pre-Calculus.

A second way to assert differentiability is if it is given that the function is twice differentiable, or if the second derivative exists for every value in the interval, then the function is differentiable on the interval.

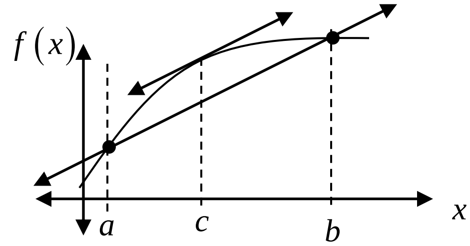
The third way to prove that a function is differentiable is to take the derivative of the function, and see if there are any values of x in the interval (a,b) for which the derivative will not exist. If there is no such x, then you can claim that the function is differentiable. If there exists at least one value where the derivative does not exist, then the function is not differentiable on (a,b).

Mean Value Theorem

Mean Value Theorem states that given a closed interval $[a,b]$ and function $f(x)$ that meets all the following criteria:

(1) $f(x)$ is continuous on the closed interval $[a,b]$

(2) $f(x)$ is differentiable on the open interval (a,b)



Then there is at least one value c such that $a<c<b$ and $f'(c)=\dfrac{f(b)-f(a)}{b-a}$.

To use Mean Value Theorem you *must:*

(A) Demonstrate conditions (1) and (2) are met.

(B) Demonstrate algebraically/numerically the value $\dfrac{f(b)-f(a)}{b-a}$.

(C) State in words "By Mean Value Thereom there exists at least one value c in (a,b) such that $f'(c)=\dfrac{f(b)-f(a)}{b-a}$."

To demonstrate that Mean Value Theorem cannot be applied, you must demonstrate only one of the following:

(1) $f(x)$ is not continuous on the closed interval $[a,b]$

(2) $f(x)$ is not differentiable on the open interval (a,b)

Some notes about the Mean Value Theorem:

$$\underset{\substack{\text{instanatneous rate of change}\\ \text{of } f(x) \text{ at } x=c}}{f'(c)} = \underbrace{\frac{f(b)-f(a)}{b-a}}_{\substack{\text{average rate of change}\\ \text{of } f(x) \text{ from } x=a \text{ to } x=b}}$$

$$\text{instaneous rate of change} = \text{average rate of change}$$

$$\text{slope of tangent} = \text{slope of secant}$$

Some notes about Rolle's Theore and Mean Value Theorem:

They are most often used on the exam when you are given at TABLE of information about a function. Both theorems allow you to make assertions about the derivative of the function when you only know information about specific coordinates of the function.