<u>Sequences and Series</u>

$a_1, a_2, a_3, \ldots, a_n, \ldots$ is an infinite sequence. An infinite sequence may be denoted as $\{a_n\}_1^{\infty}$

We say that a sequence $\{a_n\}_1^{\infty}$ converges if $\lim\limits_{n\to\infty} a_n = C$ for some finite value C.

$$\text{Let } \lim_{n\to\infty} a_n = L \text{ and } \lim_{n\to\infty} b_n = K$$

$$\lim_{n\to\infty}(a_n \pm b_n) = \left[\lim_{n\to\infty} a_n\right] \pm \left[\lim_{n\to\infty} b_n\right] = L \pm K \qquad \lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n} = \frac{L}{K} \text{ provided } b_n \neq 0 \text{ and } K \neq 0$$

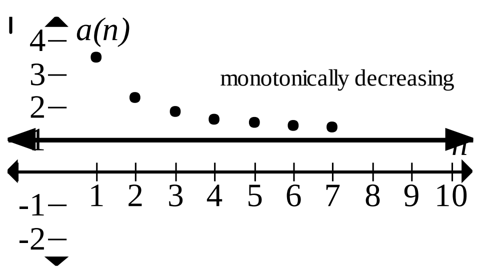
$$\lim_{n\to\infty}(a_n \cdot b_n) = \left[\lim_{n\to\infty} a_n\right] \cdot \left[\lim_{n\to\infty} b_n\right] = L \cdot K \qquad \lim_{n\to\infty}[a_n]^p = \left[\lim_{n\to\infty} a_n\right]^p = L^p \; ; \; p > 0 \text{ and } L > 0$$

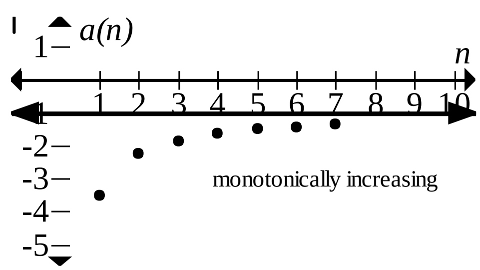
$$\lim_{n\to\infty}(c \cdot a_n) = c \cdot \left[\lim_{n\to\infty} a_n\right] = c \cdot L \text{ where } c \in \mathbb{R} \qquad \lim_{n\to\infty} c = c$$

$$\lim_{n\to\infty} c^{a_n} = c^{\lim\limits_{n\to\infty} a_n} = c^L \qquad \lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right)$$

$\left\{(-1)^n\right\}_1^{\infty} = \underset{a_1}{-1}, \underset{a_2}{1}, \underset{a_3}{-1}, \underset{a_4}{1}, \ldots$ is a sequence that does not converge.

This series does not *<u>diverge.</u>* A sequence that diverges is a sequence whose terms tend to either $+\infty$ or $-\infty$. Therefore, it is best to say that a sequence does not converge, just like it is best to say that a limit DNEs.





A sequence is monotonically decreasing if $a_1 \geq a_2 \geq \cdots \geq a_n \geq a_{n+1} \geq \cdots$

A sequence is monotonically increasing if $a_1 \leq a_2 \leq \cdots \leq a_n \leq a_{n+1} \leq \cdots$

We say that a sequence $\{a_n\}$ is bounded above if there exists a real number U such that $a_n \leq U$ for all *n*. We say that U is an upper bound of $\{a_n\}$.

We say that a sequence $\{a_n\}$ is bounded below if there exists a real number L such that $a_n \geq L$ for all *n*. We say that L is a lower bound of $\{a_n\}$.

If $\{a_n\}$ is both bounded above AND bounded below, then we say that $\{a_n\}$ is bounded.

If $\{a_n\}$ is monotonically increasing and bounded above, then $\{a_n\}$ is convergent.

If $\{a_n\}$ is monotonically decreasing and bounded below, then $\{a_n\}$ is convergent.

Series

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$$

i is called the index of the series.
a_i is called the summand of the series.
n is called the upper bound of the series.
1 is the starting index of the series.

$$\sum_{i=1}^{\infty} a_i = a_1 + a_2 + a_3 + \cdots + a_i + \cdots$$

Is an infinite series. Infinite series are identified by

(1) The upper bound of the series is ∞
(2) The expansion of the series ends with an ellipses $"\cdots"$

Let $S_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$ be defined as the "*n*th partial sum of $\sum_{i=1}^{\infty} a_i$

$$S_1 = \sum_{i=1}^{1} a_i = a_1$$

$$S_2 = \sum_{i=1}^{2} a_i = a_1 + a_2$$

$$S_3 = \sum_{i=1}^{3} a_i = a_1 + a_2 + a_3$$

$$\vdots$$

$$S_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$$

If the sequence of partial sums converges, then $\sum_{i=1}^{\infty} a_i$ converges.

If the sequence of partial sums does not converge, then $\sum_{i=1}^{\infty} a_i$ does not converge.

<u>Convergence is an end-behavior quality</u>. That is, it is the end-behavior of the series that determines whether or not a series converges. This can be useful to know, since you can disregard a finite number of terms at the beginning of the series, and still not affect whether or not the series will converge or not. Therefore, if the first few terms of a series are giving you trouble in determining whether the series converges or not, disregard them, and deal with the sum that does not include those terms.

$$\sum_{i=1}^{\infty} a_i = \underbrace{a_1 + a_2 + \cdots + a_i}_{\substack{\text{can be disregarded for} \\ \text{convergence}}} + \underbrace{a_{i+1} + a_{i+2} \cdots}_{\substack{\text{"tail end" is what} \\ \text{determines convergence}}}$$

Properties of Series:

Let $\sum_{i=1}^{\infty} a_i = A$ and $\sum_{i=1}^{\infty} b_i = B$, then

1.	Sum/Difference Rule $\sum_{i=1}^{\infty}(a_i \pm b_i) = \left(\sum_{i=1}^{\infty} a_i\right) \pm \left(\sum_{i=1}^{\infty} b_i\right)$	$\sum_{i=1}^{\infty}(a_i \pm b_i) = (a_1 \pm b_1) + (a_2 \pm b_2) + \cdots + (a_k \pm b_k) + \cdots$ $= (a_1 + a_2 + \cdots + a_k + \cdots) \pm (b_1 + b_2 + \cdots + b_k + \cdots)$ $= \sum_{i=1}^{\infty} a_i \pm \sum_{i=1}^{\infty} b_i$ $= A \pm B$
2.	Constant Multiple Rule $\sum_{i=1}^{\infty} c \cdot a_i = c \cdot \left(\sum_{i=1}^{\infty} a_i\right)$	$\sum_{i=1}^{\infty} c \cdot a_i = c \cdot a_1 + c \cdot a_2 + \cdots + c \cdot a_k + \cdots$ $= c(a_1 + a_2 + \cdots + a_k + \cdots)$ $= c \cdot \sum_{i=1}^{\infty} a_i$ $= c \cdot A$
3.	If $\sum_{i=1}^{\infty} a_i$ converges, then $\lim_{n\to\infty} a_n = 0$.	
4.	If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{i=1}^{\infty} a_i$ does not converge.	

Harmonic Series: $\sum_{n=1}^{\infty} \frac{1}{n}$ is called the Harmonic Series, and does not converge.

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{>\frac{1}{2}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{>\frac{1}{2}} + \underbrace{\frac{1}{9} + \cdots + \frac{1}{16}}_{>\frac{1}{2}} + \cdots > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \cdots = 1 + \sum_{n=1}^{\infty} \frac{1}{2} \leftrightarrow \infty$$

$$\underbrace{\frac{1}{2^n+1} + \frac{1}{2^n+2} + \cdots + \frac{1}{2^n+2^{n-1}} + \frac{1}{2^{n+1}}}_{2^n \text{ terms}}$$

Since $\frac{1}{2^{n+1}}$ is the smallest term in the sequence, we have that

$$\underbrace{\frac{1}{2^n+1} + \frac{1}{2^n+2} + \cdots + \frac{1}{2^n+2^{n-1}} + \frac{1}{2^{n+1}}}_{2^n \text{ terms}} > \underbrace{\frac{1}{2^{n+1}} + \frac{1}{2^{n+1}} + \cdots + \frac{1}{2^{n+1}} + \frac{1}{2^{n+1}}}_{2^n \text{ terms}} = 2^n\left(\frac{1}{2^{n+1}}\right) = \cancel{2^n}\,\frac{1}{\cancel{2^n}} \cdot \frac{1}{2} = \frac{1}{2}$$

Because we have an infinite series, we can continue to group an infinite amount of subsequences of increasing length that will add up to something greater than $\frac{1}{2}$.

Geometric Series:

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \cdots + ar^k + \cdots \qquad (a \neq 0)$$

A geometric series with common ratio r diverges if $|r| \geq 1$, and converges for $|r| < 1$.

Proof:

$$\begin{array}{rl} S_n = & a + ar + ar^2 + \cdots + ar^n \\ -(r \cdot S_n = & \phantom{a +{}} ar + ar^2 + \cdots + ar^n + ar^{n+1}) \\ \hline S_n - rS_n = & a - ar^{n+1} \\ S_n(1-r) = & a - ar^{n+1} \\ S_n = & \dfrac{a - ar^{n+1}}{1-r} \end{array}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a - ar^{n+1}}{1-r} = \frac{a}{1-r} \text{ if } |r| < 1$$

A more general way to remember the result is

$$\boxed{\frac{\text{first term}}{1 - \text{common ratio}}}$$

This is one of the few types of series that one can actually figure out the exact value of the sum converges to. Other series we only know that the series converges, and do not have definitive means to determine the exact value the series converges to.

You can use this formula to figure out what fraction repeating decimals are equal to. That is, if you want figure out what fraction $0.\overline{123} = 0.123123123\ldots$ is equivalent to.

$$\begin{array}{lcllll} 0.\overline{123} & = & 0.123123123\ldots & & \\ & = & 0.123 & 123 \cdot 10^{-3} & 123 \cdot \left(10^{-3}\right)^1 \\ & + & 0.000123 & 123 \cdot 10^{-6} & 123 \cdot \left(10^{-3}\right)^2 \\ & + & 0.000000123 & 123 \cdot 10^{-9} & 123 \cdot \left(10^{-3}\right)^3 \\ & \vdots & \vdots & \vdots & \vdots \\ \hline & & & & \displaystyle\sum_{n=1}^{\infty} 123(0.001)^n \\ & & & & = \dfrac{0.123}{1-(0.001)} \\ & & & & = \dfrac{41}{333} \end{array}$$

Telescoping Series:

A telescoping series is a series in which part of the n^{th} term will cancel out part of the $(n+1)^{\text{th}}$ term.

$$\sum_{n=1}^{\infty}\frac{1}{n}-\frac{1}{n+1}=\left(1-\cancel{\frac{1}{2}}\right)+\left(\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}\right)+\left(\cancel{\frac{1}{3}}-\frac{1}{4}\right)+\cdots$$

$$S_1=\left(1-\frac{1}{2}\right)$$

$$S_2=\left(1-\cancel{\frac{1}{2}}\right)+\left(\cancel{\frac{1}{2}}-\frac{1}{3}\right)=1-\frac{1}{3}$$

$$S_3=\left(1-\cancel{\frac{1}{2}}\right)+\left(\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}\right)+\left(\cancel{\frac{1}{3}}-\frac{1}{4}\right)=1-\frac{1}{4}$$

$$\vdots$$

$$S_n=1-\frac{1}{n+1}$$

$$\lim_{n\to\infty}1-\frac{1}{n+1}=1$$

$$\downarrow$$

$$\sum_{n=1}^{\infty}\frac{1}{n}-\frac{1}{n+1}=1$$

You will rarely see telescoping series on the exam. To see if a telescoping series converges, you <u>must</u> take the limit of the n^{th} partial sums. This might require using partial fractions to see the series as telescoping, and writing out the first few terms to determine what the structure of S_n looks like.

$$\sum_{n=1}^{\infty}\frac{2}{4n^2-1}\to\sum_{n=1}^{\infty}\frac{1}{2n-1}-\frac{1}{2n+1}$$

$$S_1=\frac{1}{1}-\frac{1}{3}$$

$$S_2=\left(\frac{1}{1}-\cancel{\frac{1}{3}}\right)+\left(\cancel{\frac{1}{3}}-\frac{1}{5}\right)$$

$$S_2=\left(\frac{1}{1}\cancel{-\frac{1}{3}}\right)+\left(\cancel{\frac{1}{3}}\cancel{-\frac{1}{5}}\right)+\left(\cancel{\frac{1}{5}}-\frac{1}{7}\right)$$

$$\vdots$$

$$S_n=\left(\frac{1}{1}\cancel{-\frac{1}{3}}\right)+\left(\cancel{\frac{1}{3}}\cancel{-\frac{1}{5}}\right)+\cdots+\left(\cancel{\frac{1}{2n-1}}-\frac{1}{2n+1}\right)$$

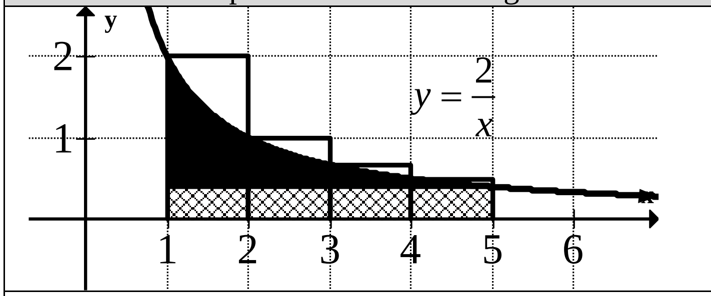
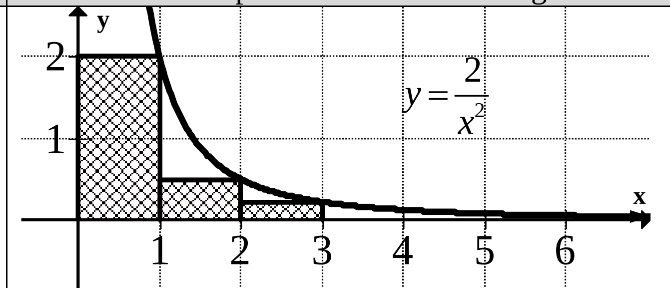
$$\begin{aligned}\sum_{n=1}^{\infty}\frac{2}{4n^2-1}&=\lim_{n\to\infty}S_n\\&=\lim_{n\to\infty}\left[1-\frac{1}{2n+1}\right]\\&=1\end{aligned}$$

Integral Test:

If $f(x) > 0$ is continuous and decreasing for $x \geq 1$, and $a_n = f(n)$, then

$$\sum_{n=1}^{\infty} a_n \text{ and } \int_1^{\infty} f(x)\,dx$$

either both converge, or both diverge.

<table>
<tr><th>To prove the series diverges</th><th>To prove the series converges</th></tr>
<tr><td>
</td><td>
</td></tr>
<tr><td>The series $\sum_{n=1}^{\infty} \frac{2}{n}$ will diverge since each rectangle has area $\left(\frac{2}{n}\right)(1)$. The sum of the areas of the rectangles will be more than the area under the curve. Since the area under the curve $\to \infty$, the $\sum_{n=1}^{\infty} \frac{2}{n} \to \infty$.

$$\sum_{n=1}^{\infty} \frac{2}{n} \leftrightarrow \int_1^{\infty} \frac{2}{x}\,dx$$</td><td>The series $\sum_{n=1}^{\infty} \frac{2}{n^2}$ will converge since each rectangle has area $\left(\frac{2}{n^2}\right)(1)$, which is less than the area under the curve from $x = 1$ to ∞. The first rectangle is disregarded because leaving out one rectangle will not affect whether the series/sum of the rectangles converges or diverges, and this way the value of the integral $\int_1^{\infty} \frac{2}{x^2}\,dx$ is finite. Since the area under the curve from $x = 1$ to ∞ converges, so does $\sum_{n=1}^{\infty} \frac{2}{n^2}$ (because the sequence of the n^{th} partial sums is a monotonically increasing sequence that is bounded above [by the value of the integral] must converge). Note that the series may not converge to the area under the curve, we can only conclude series converges – stating nothing about what it converges to.

$$\sum_{n=1}^{\infty} \frac{2}{n^2} \leftrightarrow \int_1^{\infty} \frac{2}{x^2}\,dx$$</td></tr>
</table>

$$\sum_{n=1}^{\infty} \frac{1}{n^2+4} \leftrightarrow \int_1^{\infty} \frac{1}{x^2+4}\,dx \qquad \sum_{n=1}^{\infty} n^2 e^{-n^3} \leftrightarrow \int_1^{\infty} x^2 e^{-x^3}\,dx$$

P-Series Test:

The series $\sum_{n=1}^{\infty}\frac{1}{n^p}=\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\cdots$

(1) Converges if $p>1$

(2) Diverges if $p\le 1$

This is given without proof. Memorize this. (proven via the Integral Test).

Series Comparison Tests:

There are two types of comparison tests when it comes to determining whether series converge or not. *Both require that you are familiar with series that converge and series that diverge, or you will not have anything to compare to.*

Direct Comparison Test	**Limit Comparison Test**
"Hit the brakes or step on the gas"	"Behaves like …"

Direct Comparison Test:

If $0\le a_n\le b_n$ for all *n*, then $\sum_{n=1}^{\infty}a_n\le\sum_{n=1}^{\infty}b_n$ and…

(1) If $\sum_{n=1}^{\infty}a_n$ diverges, then $\sum_{n=1}^{\infty}b_n$ diverges. "Step on the gas." $\sum_{n=2}^{\infty}\frac{1}{n}\le\sum_{n=2}^{\infty}\frac{\ln(n)}{n}$

(2) If $\sum_{n=1}^{\infty}b_n$ converges, then $\sum_{n=1}^{\infty}a_n$ converges. "Hit the brakes." $\sum_{n=1}^{\infty}\frac{1}{n^2+3n+1}\le\sum_{n=1}^{\infty}\frac{1}{n^2}$

Imagine that a_n and b_n are two cars on a one lane road, where a_n is behind b_n. If a_n steps on the gas, b_n will have no choice but to hit the gas as well or be rear-ended.
If b_n hits the brakes, then a_n will have to hit the brakes as well.

$\sum_{n=1}^{\infty}a_n\le\boxed{\sum_{n=1}^{\infty}b_n}$ converges $\to\sum_{n=1}^{\infty}a_n$ converges	diverges $\boxed{\sum_{n=1}^{\infty}a_n}\le\sum_{n=1}^{\infty}b_n\to\sum_{n=1}^{\infty}b_n$ diverges

Limit Comparison Test:

Suppose that $a_n>0$ and $b_n>0$, and $\lim_{n\to\infty}\frac{a_n}{b_n}=L$. If *L* is finite and positive, then either $\sum_{n=0}^{\infty}a_n$ and $\sum_{n=0}^{\infty}b_n$ both converge, or $\sum_{n=1}^{\infty}a_n$ and $\sum_{n=1}^{\infty}b_n$ both diverge. (Note: does not matter if it is $\frac{a_n}{b_n}$ or $\frac{b_n}{a_n}$).

$$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n^2+1}}\sim\sum_{n=1}^{\infty}\frac{1}{n}\qquad\sum_{n=1}^{\infty}\frac{n-1}{n^2\sqrt{n}}\sim\sum_{n=1}^{\infty}\frac{1}{n^{1.5}}$$

Alternating Series:

An alternating series is a series in which the terms alternate signs.

If $a_n > 0$ for all *n*, then $\sum_{n=1}^{\infty}(-1)^n a_n$ and $\sum_{n=1}^{\infty}(-1)^{n+1} a_n$ converge if both of the following hold:

(1) $\lim_{n\to\infty} a_n = 0$

(2) a_n is monotonically decreasing

The language used by the College Board to communicate (1) and (2) simultaneously is:

<u>"The terms of the series decrease in absolute value to zero."</u> $\leftrightarrow \lim_{n\to\infty}\left|(-1)^{\cdots} a_n\right| = 0$

Suppose $\sum_{n=1}^{\infty}(-1)^n a_n$ is a convergent alternating series, and

$$\sum_{k=1}^{\infty}(-1)^k a_k = -a_1 + a_2 - a_3 + \cdots + (-1)^N a_N + (-1)^{N+1} a_{N+1} + (-1)^{N+2} a_{N+2} \cdots$$

If the series is ended at $k = N$, we get the N^{th} partial sum:

$$\sum_{k=1}^{N}(-1)^k a_k = -a_1 + a_2 - a_3 + \cdots (-1)^N a_N .$$

This finite series is off from the infinite sum by $\sum_{k=N+1}^{\infty}(-1)^k a_k = (-1)^{N+1} a_{N+1} + (-1)^{N+2} a_{N+2} + \cdots$

$$\underbrace{\sum_{k=1}^{\infty}(-1)^k a_k}_{\substack{\text{true value}\\ \text{"}S\text{"}}} = \underbrace{-a_1 + a_2 - a_3 + \cdots + (-1)^N a_N}_{\substack{N^{\text{th}}\text{partial sum}\\ \text{"}S_N\text{"}}} + \underbrace{(-1)^{N+1} a_{N+1} + (-1)^{N+2} a_{N+2} \cdots}_{\substack{\text{error / remainder}\\ \text{"}R_N\text{"}}}$$

$$\sum_{k=1}^{\infty}(-1)^k a_k = \sum_{k=1}^{N}(-1)^k a_k + \sum_{k=N+1}^{\infty}(-1)^k a_k$$

$$\sum_{k=1}^{\infty}(-1)^k a_k - \sum_{k=1}^{N}(-1)^k a_k = \sum_{k=N+1}^{\infty}(-1)^k a_k$$

$$\left|\sum_{k=1}^{\infty}(-1)^k a_k - \sum_{k=1}^{N}(-1)^k a_k\right| \geq \sum_{k=N+1}^{\infty}(-1)^k a_k$$

$$\left|S - S_N\right| \geq \text{remainder}$$

Alternating Series Remainder Theorem:

If $\sum_{k=1}^{\infty}(-1)^k a_k$ is terminated at $k = N$, then the remainder/error is bounded by error $\leq \left|(-1)^{N+1} a_{N+1}\right|$. That is, *<u>the error is bounded by the absolute value of the subsequent term.</u>*

If $\sum |a_n|$ converges, then $\sum a_n$ must also converge.

If $\sum |a_n|$ converges, then we say that $\sum a_n$ is **absolutely convergent.**

If $\sum |a_n|$ does not converge, but $\sum a_n$ converges, then we say that $\sum a_n$ is **conditionally convergent.**

Ratio Test:

$\sum a_n$ is absolutely convergent if $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| < 1$.

$\sum a_n$ diverges if $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| > 1$ or ∞.

If $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = 1$ more investigation is necessary to make any claim about divergence or convergence.

The Ratio Test is often used when the summand has

- a factorial, or factorial-ish part.
- terms to the power of n

$$\sum_{n=1}^{\infty} \frac{n!}{e^n} \qquad\qquad \sum_{n=1}^{\infty} \frac{n^2+1}{5^n}$$

Root Test:

$\sum a_n$ is absolutely convergent if $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} < 1$.

$\sum a_n$ diverges if $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} > 1$ or ∞.

If $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = 1$ more investigation is necessary to make any claim about divergence or convergence.

Special Limits to remember when using the Root Test:

If $k \geq 1$ is a constant, then: $\lim\limits_{n\to\infty} \sqrt[n]{n^k} = 1 \qquad \lim\limits_{n\to\infty} \sqrt[n]{k} = 1 \qquad \lim\limits_{n\to\infty} \sqrt[n]{n!} = \infty$

The Root Test is often used when the summand can be expressed as a power of n.

$$\sum_{n=1}^{\infty} \frac{(-5)^{2n}}{n^2 9^n} = \sum_{n=1}^{\infty} \frac{25^n}{n^2 9^n} \qquad\qquad \sum_{n=1}^{\infty} \frac{n^{2n}}{\left(1+2n^2\right)^n} = \sum_{n=1}^{\infty} \left(\frac{n^2}{\left(1+2n^2\right)}\right)^n$$