Arc Length

Standard Functions

$$\sum d = \sum \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

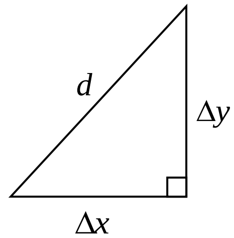
$$= \sum \sqrt{\left[\frac{(\Delta x)^2}{(\Delta x)^2} + \frac{(\Delta y)^2}{(\Delta x)^2}\right] \cdot (\Delta x)^2}$$

$$= \sum \sqrt{1 + \frac{(\Delta y)^2}{(\Delta x)^2}} \cdot \sqrt{(\Delta x)^2}$$

$$= \sum \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2} \cdot \Delta x$$

$$\downarrow$$

$$= \int_a^b \sqrt{1 + [f'(x)]^2}\,dx$$



$$\sum d = \sum \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

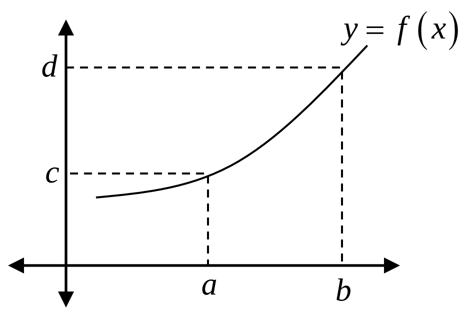
$$= \sum \sqrt{\left[\frac{(\Delta x)^2}{(\Delta y)^2} + \frac{(\Delta y)^2}{(\Delta y)^2}\right] \cdot (\Delta y)^2}$$

$$= \sum \sqrt{1 + \frac{(\Delta x)^2}{(\Delta y)^2}} \cdot \sqrt{(\Delta y)^2}$$

$$= \sum \sqrt{1 + \left(\frac{\Delta x}{\Delta y}\right)^2} \cdot \Delta y$$

$$\downarrow$$

$$= \int_c^d \sqrt{1 + [f'(y)]^2}\,dy$$



Parametric Functions

$$\int_a^b f(x)\,dx = \sqrt{1+\left[f'(x)\right]^2}\,dx$$

$$\downarrow$$

$$= \sqrt{1+\left[\frac{dy}{dx}\right]^2}\,dx$$

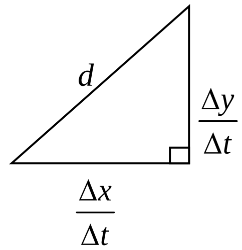
$$= \sqrt{1+\left[\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right]^2}\,dx$$

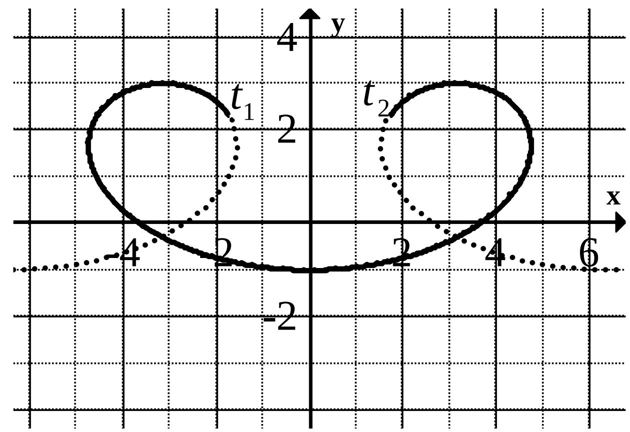
$$= \sqrt{\frac{\left[\frac{dx}{dt}\right]^2+\left[\frac{dy}{dt}\right]^2}{\left[\frac{dx}{dt}\right]^2}}\,dx$$

$$= \sqrt{\left[\frac{dx}{dt}\right]^2+\left[\frac{dy}{dt}\right]^2}\sqrt{\frac{1}{\left[\frac{dx}{dt}\right]^2}}\,dx$$

$$= \sqrt{\left[\frac{dx}{dt}\right]^2+\left[\frac{dy}{dt}\right]^2}\cdot\frac{dt}{dx}\cdot dx$$

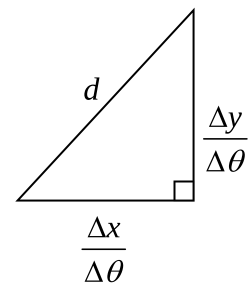
$$= \int_{t_1}^{t_2}\sqrt{\left[\frac{dx}{dt}\right]^2+\left[\frac{dy}{dt}\right]^2}\,dt$$





Polar Functions

First one must note that $r = r(\theta)$ and that

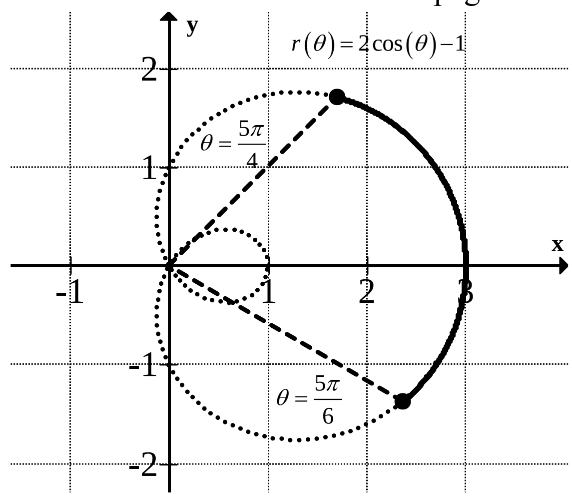


$$x = r(\theta)\cos\theta$$
$$y = r(\theta)\sin\theta$$

Therefore

$$\frac{dx}{d\theta} = r'(\theta)\cos\theta + r(\theta)(-\sin\theta) \quad \text{and} \quad \frac{dy}{d\theta} = r'(\theta)\sin\theta + r(\theta)\cos\theta$$
$$= r'(\theta)\cos\theta - r(\theta)\sin\theta$$

Derivation continued on next page…



$$\int_a^b \sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2}\, dt \to \int_a^b \sqrt{\left[\frac{dx}{d\theta}\right]^2 + \left[\frac{dy}{d\theta}\right]^2}\, d\theta$$

$$= \int_a^b \sqrt{\left[r'(\theta)\cos\theta - r(\theta)\sin\theta\right]^2 + \left[r'(\theta)\sin\theta + r(\theta)\cos\theta\right]^2}\, d\theta$$

$$= \int_a^b \sqrt{\left[r'(\theta)\cos\theta\right]^2 - 2r'(\theta)\cos\theta \cdot r(\theta)\cdot\sin\theta + \left[r(\theta)\sin\theta\right]^2 + \left[r'(\theta)\sin\theta\right]^2 + 2r'(\theta)\sin\theta \cdot r(\theta)\cdot\cos\theta + \left[r(\theta)\cos\theta\right]^2}\, d\theta$$

$$= \int_a^b \sqrt{\left[r'(\theta)\right]^2\cos^2\theta - 2r'(\theta)r(\theta)\sin\theta\cos\theta + \left[r(\theta)\right]^2\sin^2\theta + \left[r'(\theta)\right]^2\sin^2\theta + 2r'(\theta)r(\theta)\sin\theta\cos\theta + \left[r(\theta)\right]^2\cos^2\theta}\, d\theta$$

$$= \int_a^b \sqrt{\left[r'(\theta)\right]^2\cos^2\theta + \left[r(\theta)\right]^2\sin^2\theta + \left[r'(\theta)\right]^2\sin^2\theta + \left[r(\theta)\right]^2\cos^2\theta}\, d\theta$$

$$= \int_a^b \sqrt{\left[r'(\theta)\right]^2\cos^2\theta + \left[r'(\theta)\right]^2\sin^2\theta + \left[r(\theta)\right]^2\sin^2\theta + \left[r(\theta)\right]^2\cos^2\theta}\, d\theta$$

$$= \int_a^b \sqrt{\left[r'(\theta)\right]^2\left(\cos^2\theta + \sin^2\theta\right) + \left[r(\theta)\right]^2\left(\sin^2\theta + \cos^2\theta\right)}\, d\theta$$

$$= \int_a^b \sqrt{\left[r'(\theta)\right]^2 + \left[r(\theta)\right]^2}\, d\theta$$