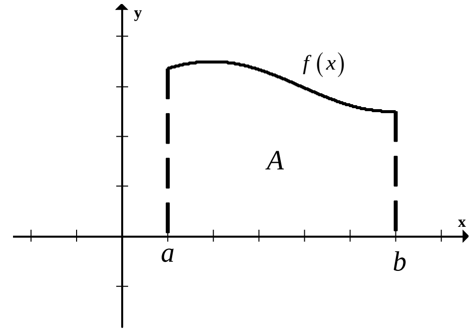
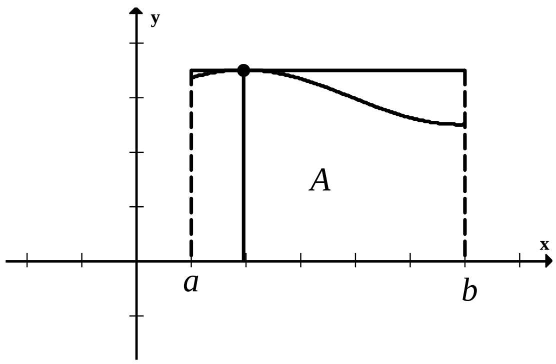
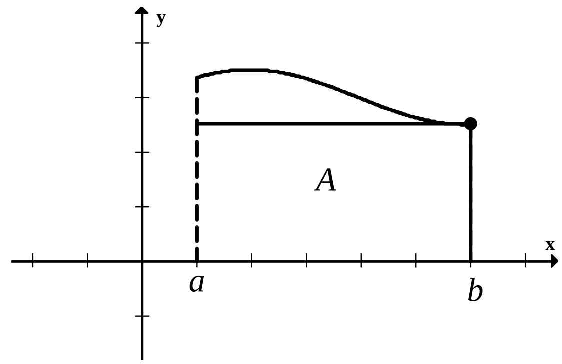
To estimate the area under a positive function $f(x)$ from $x = a$ to $x = b$ (A), we can use a single rectangle to do so. The height of the rectangle we use must be a representative height of the function somewhere within the interval $[a,b]$.



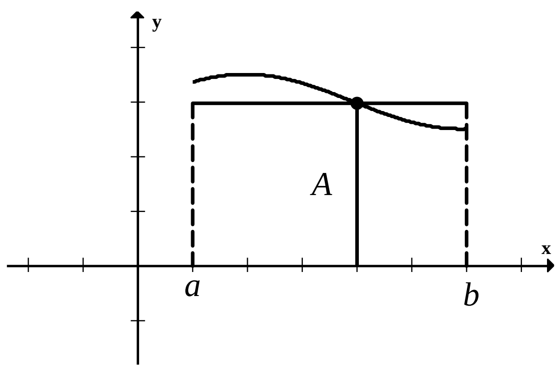
Rectangle whose height is the largest value of $f(x)$ on the interval $[a,b]$



Rectangle whose height is the least value of $f(x)$ on the interval $[a,b]$



Rectangle using a height that is somewhere between the largest value of $f(x)$ and the least value of $f(x)$ on the interval $[a,b]$.



To improve the estimate of the area under the curve, we will need to use more rectangles to get a more accurate approximation.

Using more rectangles improves our estimate (figure #1). However, as we increase the number of rectangles, we must also make sure that each rectangle becomes narrower, or else we will not approach the true value of the area under the curve, as in figure #2.

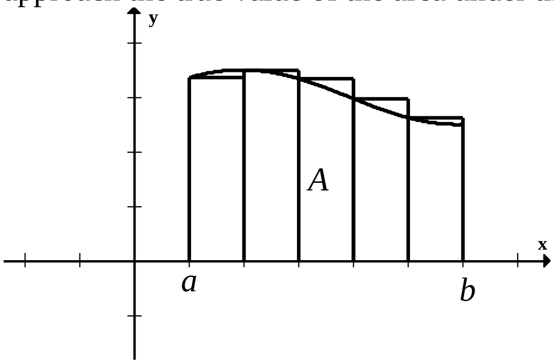


Figure #1

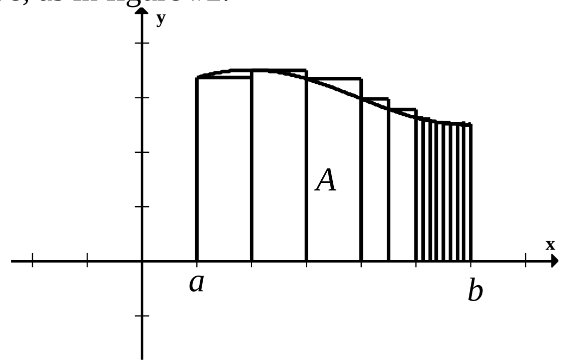
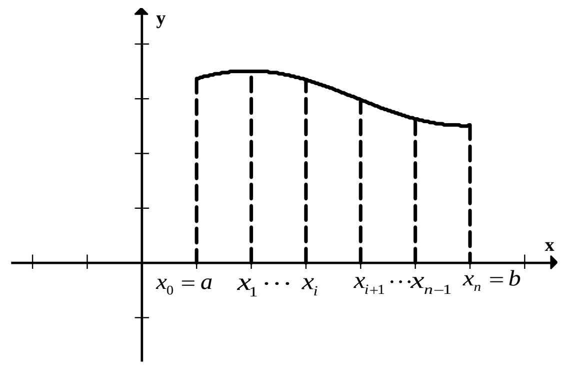
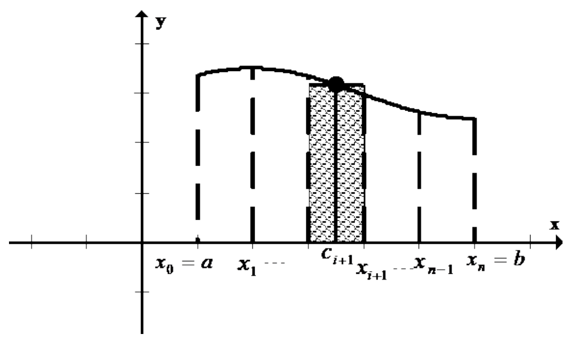


Figure #2

In order to improve our estimate we must first *partition the interval* $[a,b]$ into smaller subintervals. That is we select a set of *x*-values $\{x_{0,}x_1, x_2, \ldots, x_n\}$ such that $[a,b]=[x_0,x_1]\cup[x_{1,}x_2]\cup\cdots\cup[x_{n-1},x_n]$. The partition of $[a,b]$ is $\Delta=\{x_{0,}x_1,x_2,\ldots,x_n\}$. Within each subinterval $[x_i,x_{i+1}]$ we choose a c_{i+1} such that $x_i \le c_{i+1} \le x_{i+1}$



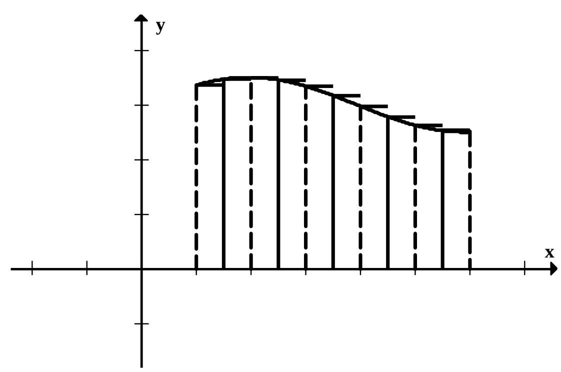
The area of the $(i+1)^{th}$ rectangle is $f(c_{i+1})(x_{i+1}-x_i)$.



The estimate for the area under the function from $x=a$ to $x=b$ is given by $f(c_1)\underbrace{(x_1-x_0)}_{(\Delta x)_1}+f(c_2)\underbrace{(x_2-x_1)}_{(\Delta x)_2}+\cdots+f(c_n)\underbrace{(x_n-x_{n-1})}_{(\Delta x)_n}$

$$=\sum_{i=1}^{n} f(c_i)(\Delta x)_i \,.$$

In order to improve the estimate, we must create (1) more rectangles, where (2) each rectangle becomes narrower and narrower.



We define the "norm of delta", denoted $\|\Delta\|$ to be the width of the largest subinterval of the partition Δ. To create more and more rectangles of narrower and narrower width, we let $\|\Delta\| \to 0$.

If $f(x)$ is defined on the closed interval $[a,b]$ and $\lim\limits_{\|\Delta\|\to 0} \sum\limits_{i=1}^{n} f(c_i)(\Delta x)_i$ exists, then we say that $f(x)$ is integrable on $[a,b]$ and

$$\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(c_i)(\Delta x)_i = \int_a^b f(x)\,dx$$

Where $\int_a^b f(x)\,dx$ is read "The integral from $x = a$ to $x = b$, of $f(x)$, with respect to *x*."

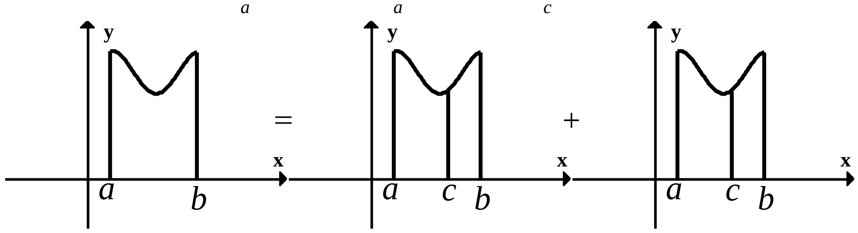
a is called the **lower bound** of the integral
b is called the **upper bound** of the integral
$f(x)$ is called the **integrand**
dx indicates that integration is done with respect to *x*.

Properties of Definite Integrals

I. $\int_a^a f(x)\,dx = 0$

II. $\int_b^a f(x)\,dx = -\left[\int_a^b f(x)\,dx\right]$ "switch the direction, switch the sign"

III. If $a \le c \le b$, then $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

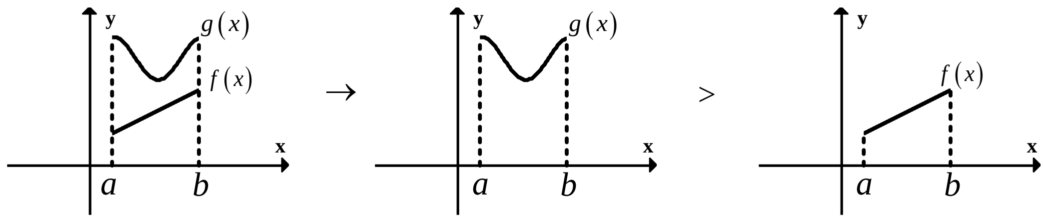


IV. If *k* is a constant then $\int_a^b k \cdot f(x)\,dx = k \cdot \int_a^b f(x)\,dx$

V. $\int_a^b f(x) \pm g(x)\,dx = \left[\int_a^b f(x)\,dx\right] \pm \left[\int_a^b (x)\,dx\right]$

VI. If *f* is integrable and non-negative on $[a,b]$, then $0 \le \int_a^b f(x)\,dx$

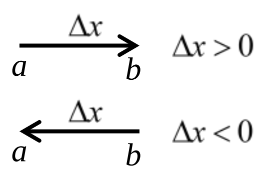
VII. If $f(x) \le g(x)$ for all x in $[a,b]$, then $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$

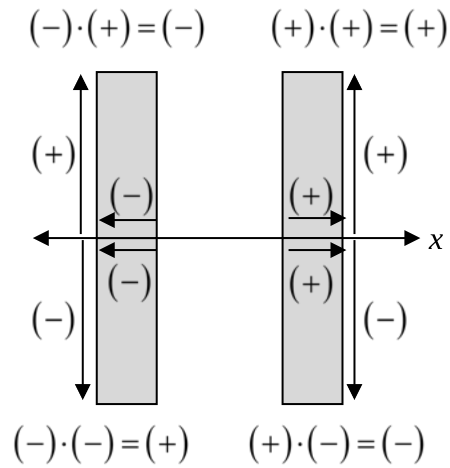


VIII. The definite integral can take on negative, zero, and positive values depending on

a. The direction of integration

b. The sign of $f(x)$





Series

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$$

i is called the **index** of the series.
a_i is called the **summand** of the series.
n is called the **ending index** of the series.
1 is the **starting index** of the series.

Let $\sum_{i=1}^{n} a_i = A$ and $\sum_{i=1}^{n} b_i = B$, then

$\sum_{i=1}^{n}(a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$	$\sum_{i=1}^{n} c \cdot a_i = c \cdot \sum_{i=1}^{n} a_i$
$\sum_{i=1}^{n}(a_i \pm b_i) = (a_1 \pm b_1) + (a_2 \pm b_2) + \cdots + (a_n \pm b_n)$ $= (a_1 + a_2 + \cdots + a_n) \pm (b_1 + b_2 + \cdots + b_n)$ $= \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$	$\sum_{i=1}^{n} c \cdot a_i = c \cdot a_1 + c \cdot a_2 + \cdots + c \cdot a_n$ $= c(a_1 + a_2 + \cdots + a_n)$ $= c \cdot \sum_{i=1}^{n} a_i$