2000 #3

The Taylor series about $x = 5$ for a certain function *f* converges to $f(x)$ for all *x* in the interval of convergence.The n^{th} derivative of *f* at $z = 5$ is given by $f^{(n)}(5) = \dfrac{(-1)^n \cdot n!}{2^n \cdot (n+2)}$ and $f(5) = \dfrac{1}{2}$.

(a) Write the third-degree Taylor polynomial for *f* about $x = 5$.

$$f'(5) = \frac{(-1)^1 \cdot 1!}{2^1 \cdot (1+2)} = -\frac{1}{2 \cdot 3}$$

$$f''(5) = \frac{(-1)^2 \cdot 2!}{2^2 \cdot (2+2)} = \frac{2!}{2^2 \cdot 4}$$

$$f'''(5) = \frac{(-1)^3 \cdot 3!}{2^3 \cdot (3+2)} = -\frac{3!}{2^3 \cdot 5}$$

$$\begin{aligned}
T_3(x) &= f(c) + f'(c)\cdot(x-c) + \frac{f''(c)\cdot(x-c)^2}{2!} + \frac{f'''(c)\cdot(x-c)^3}{3!} \\
&= f(5) + f'(5)\cdot(x-5) + \frac{f''(5)\cdot(x-5)^2}{2!} + \frac{f'''(5)\cdot(x-5)^3}{3!} \\
&= \frac{1}{2} + \left[-\frac{1}{2\cdot 3}\right]\cdot(x-5) + \frac{\left[\frac{2!}{2^2 \cdot 4}\right]\cdot(x-5)^2}{2!} + \frac{\left[-\frac{3!}{2^3 \cdot 5}\right]\cdot(x-5)^3}{3!}
\end{aligned}$$

(b) Find the radius of convergence of the Taylor series for *f* about $x = 5$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\dfrac{\left[\dfrac{(-1)^{n+1}\cdot(n+1)!}{2^{n+1}\cdot((n+1)+2)}\right](x-5)^{n+1}}{(n+1)!}}{\dfrac{\left[\dfrac{(-1)^{n}\cdot n!}{2^{n}\cdot(n+2)}\right](x-5)^{n}}{n!}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{\dfrac{\left[\dfrac{(n+1)!}{2^{n+1}\cdot(n+3)}\right](x-5)^{n+1}}{(n+1)!}}{\dfrac{\left[\dfrac{n!}{2^{n}\cdot(n+2)}\right](x-5)^{n}}{n!}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{\left[\dfrac{(n+1)!}{2^{n+1}\cdot(n+3)}\right](x-5)^{n+1}}{(n+1)!}\cdot\frac{n!}{\left[\dfrac{n!}{2^{n}\cdot(n+2)}\right](x-5)^{n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+1)!(x-5)^{n+1}}{2^{n+1}\cdot(n+3)\cdot[(n+1)!]}\cdot\frac{[n!]\cdot 2^{n}\cdot(n+2)}{[n!]\cdot(x-5)^{n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{\cancel{(n+1)!}(x-5)^{n+1}}{2^{n+1}\cdot(n+3)\cdot\cancel{[(n+1)!]}}\cdot\frac{\cancel{[n!]}\cdot 2^{n}\cdot(n+2)}{\cancel{[n!]}\cdot(x-5)^{n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(x-5)^{n+1}}{2^{n+1}\cdot(n+3)}\cdot\frac{2^{n}\cdot(n+2)}{(x-5)^{n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(x-5)^{n}\cdot(x-5)^{1}}{2^{n}\cdot 2^{1}\cdot(n+3)}\cdot\frac{2^{n}\cdot(n+2)}{(x-5)^{n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{\cancel{(x-5)^{n}}\cdot(x-5)^{1}}{\cancel{2^{n}}\cdot 2^{1}\cdot(n+3)}\cdot\frac{\cancel{2^{n}}\cdot(n+2)}{\cancel{(x-5)^{n}}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(x-5)^{1}}{2^{1}\cdot(n+3)}\cdot\frac{(n+2)}{1}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(x-5)}{2}\cdot\frac{(n+2)}{(n+3)}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(x-5)}{2}\right|$$

$$= \left|\frac{x-5}{2}\right|$$

Radius of convergence is 2

$$\left|\frac{x-5}{2}\right| < 1$$

$$\downarrow$$

$$-1 < \frac{x-5}{2} < 1$$

$$-2 < x-5 < 2$$

$$3 < x < 7$$

The series is centered at $\dfrac{7+3}{2} = 5$ and has radius of convergence $\dfrac{7-3}{2} = 2$

(c) Show that the sixth-degree Taylor polynomial for f about $x=5$ approximates $f(6)$ with error less than $\frac{1}{1000}$.

$$\begin{aligned}
f(6) &\approx f(5)+f'(5)\cdot(6-5)+\frac{f''(5)\cdot(6-5)^2}{2!}+\frac{f'''(5)\cdot(6-5)^3}{3!}+\cdots+\frac{f^{(6)}(5)(6-5)^6}{6!} \\
&= f(5)+f'(5)\cdot(1)+\frac{f''(5)\cdot(1)^2}{2!}+\frac{f'''(5)\cdot(1)^3}{3!}+\cdots+\frac{f^{(6)}(5)(1)^6}{6!} \\
&= f(5)+f'(5)+\frac{f''(5)}{2!}+\frac{f'''(5)}{3!}+\cdots+\frac{f^{(6)}(5)}{6!} \\
&= \frac{1}{2}+\left[-\frac{1}{2\cdot 3}\right]+\frac{\left[\frac{2!}{2^2\cdot 4}\right]}{2!}+\frac{\left[\frac{3!}{2^3\cdot 5}\right]}{3!}+\cdots+\frac{\left[\frac{6!}{2^6\cdot 8}\right]}{6!} \\
&= \frac{1}{2}-\frac{1}{2\cdot 3}+\frac{2!}{2^2\cdot 4\cdot 2!}-\frac{3!}{2^3\cdot 5\cdot 3!}+\cdots+\frac{6!}{2^6\cdot 8\cdot 6!} \\
&= \frac{1}{2}-\frac{1}{6}+\frac{1}{2^2\cdot 4}-\frac{1}{2^3\cdot 5}+\cdots+\frac{1}{2^6\cdot 8}
\end{aligned}$$

The series for $f(6)$ is an alternating series whose terms decrease in absolute value to zero. By the Alternating Series Remainder Theorem

$$\begin{aligned}
\text{error} &\le \left|\text{next term}\right| \\
&\le \left|\frac{1}{2^7\cdot 9}\right| < \frac{1}{1000}
\end{aligned}$$

2001 #6

A function f is defined by

$$f\left(x\right)=\frac{1}{3}+\frac{2}{3^2}\cdot x+\frac{3}{3^3}\cdot x^2+\cdots+\frac{n+1}{3^{n+1}}\cdot x^n+\cdots$$

For all x in the interval of convergence of the given power series.

(a) Find the interval of convergence for this power series. Show the work that leads to your answer.

$$\lim_{n\to\infty}\left|\frac{\frac{\left(n+1\right)+1}{3^{(n+1)+1}}\cdot x^{n+1}}{\frac{n+1}{3^n}\cdot x^n}\right|=\lim_{n\to\infty}\left|\frac{\left[\left(n+1\right)+1\right]\cdot x^{n+1}}{3^{(n+1)+1}}\cdot\frac{3^{n+1}}{\left(n+1\right)\cdot x^n}\right|$$

$$=\lim_{n\to\infty}\left|\frac{\left(n+2\right)\cdot x^{n+1}}{3^{n+2}}\cdot\frac{3^{n+1}}{\left(n+1\right)\cdot x^n}\right|$$

$$=\lim_{n\to\infty}\left|\frac{\left(n+2\right)\cdot x^{n+1}}{\left(n+1\right)\cdot x^n}\cdot\frac{3^{n+1}}{3^{n+2}}\right|$$

$$=\lim_{n\to\infty}\left|\frac{\left(n+2\right)\cdot x^n\cdot x^1}{\left(n+1\right)\cdot x^n}\cdot\frac{3^{n+1}}{3^{n+1}\cdot 3^1}\right|$$

$$=\lim_{n\to\infty}\left|\frac{\left(n+2\right)\cdot \cancel{x^n}\cdot x^1}{\left(n+1\right)\cdot \cancel{x^n}}\cdot\frac{\cancel{3^{n+1}}}{\cancel{3^{n+1}}\cdot 3^2}\right|$$

$$=\lim_{n\to\infty}\left|\frac{\left(n+2\right)\cdot x^1}{\left(n+1\right)}\cdot\frac{1}{3}\right|$$

$$=\lim_{n\to\infty}\left|\frac{\left(n+2\right)}{\left(n+1\right)}\cdot\frac{x}{3}\right|$$

$$=\left|\frac{x}{3}\right|$$

The series will converge so long as

$$|x| < 3$$
$$\downarrow$$
$$-3 < x < 3$$

$x = -3$	$x = 3$
$\sum_{n=0}^{\infty} \frac{n+1}{3^{n+1}} \cdot (-)^n = \sum_{n=0}^{\infty} \frac{n+1}{3^{n+1}} \cdot (-1)^n \cdot 3^n$ $= \sum_{n=0}^{\infty} \frac{3^n}{3^{n+1}} \cdot (-1)^n \cdot (n+1)$ $= \sum_{n=0}^{\infty} \frac{1}{3} \cdot (-1)^n \cdot (n+1)$ This does not converge because the terms to not go to zero as $n \to \infty$.	$\sum_{n=0}^{\infty} \frac{n+1}{3^{n+1}} \cdot 3^n = \sum_{n=0}^{\infty} \frac{3^n}{3^{n+1}} \cdot (n+1)$ $= \sum_{n=0}^{\infty} \frac{1}{3} \cdot (n+1)$ This does not converge because the terms to not go to zero as $n \to \infty$.

Therefore the interval of convergence is $-3 < x < 3$.

(b) Find $\lim_{x \to 0} \frac{f(x) - \frac{1}{3}}{x}$

$$\lim_{x \to 0} \frac{f(x) - \frac{1}{3}}{x} = \lim_{n \to 0} \frac{\left[\frac{1}{3} + \frac{2}{3^2} \cdot x + \frac{3}{3^3} \cdot x^2 + \cdots + \frac{n+1}{3^{n+1}} \cdot x^n + \cdots\right] - \frac{1}{3}}{x}$$
$$= \lim_{n \to 0} \frac{\left[\frac{2}{3^2} \cdot x + \frac{3}{3^3} \cdot x^2 + \cdots + \frac{n+1}{3^{n+1}} \cdot x^n + \cdots\right]}{x}$$
$$= \lim_{n \to 0} \frac{1}{x} \cdot \left[\frac{2}{3^2} \cdot x + \frac{3}{3^3} \cdot x^2 + \cdots + \frac{n+1}{3^{n+1}} \cdot x^n + \cdots\right]$$
$$= \lim_{n \to 0} \left[\frac{2}{3^2} + \frac{3}{3^3} \cdot x + \cdots + \frac{n+1}{3^{n+1}} \cdot x^{n-1} + \cdots\right]$$
$$= \frac{2}{3^2}$$

(c) Write the first three nonzero terms and the general term for an infinite series that represents $\int_0^1 f(x)\,dx$

$$\int_0^1 f(x)\,dx = \int_0^1 \left[\frac{1}{3}+\frac{2}{3^2}\cdot x+\frac{3}{3^3}\cdot x^2+\cdots+\frac{n+1}{3^{n+1}}\cdot x^n+\cdots\right]dx$$

$$=\left[\frac{1}{3}x+\frac{1}{2}\cdot\frac{2}{3^2}\cdot x+\frac{1}{3}\cdot\frac{3}{3^3}\cdot x^2+\cdots+\frac{1}{n+1}\cdot\frac{n+1}{3^{n+1}}\cdot x^{n+1}+\cdots\right]_0^1$$

$$=\left[\frac{1}{3}x+\frac{1}{3^2}\cdot x+\frac{1}{3^3}\cdot x^2+\cdots+\frac{1}{3^{n+1}}\cdot x^{n+1}+\cdots\right]_0^1$$

$$=\left[\frac{1}{3}(1)+\frac{1}{3^2}\cdot(1)+\frac{1}{3^3}\cdot(1)^2+\cdots+\frac{1}{3^{n+1}}\cdot(1)^{n+1}+\cdots\right]-\left[\frac{1}{3}(0)+\frac{1}{3^2}\cdot(0)+\frac{1}{3^3}\cdot(0)^2+\cdots+\frac{1}{3^{n+1}}\cdot(0)^{n+1}+\cdots\right]$$

$$=\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\cdots+\frac{1}{3^{n+1}}+\cdots$$

(d) Find the sum of the series determined in part (c).

$$\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\cdots+\frac{1}{3^{n+1}}+\cdots=\sum_{n=1}^{\infty}\frac{1}{3^n}$$

$$=\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n$$

This is a geometric series with common ratio $r=\frac{1}{3}$ and first term $\frac{1}{3}$

Therefore, the sum of the series is $\dfrac{\left(\frac{1}{3}\right)}{1-\left(\frac{1}{3}\right)}=\dfrac{1}{2}$

2002 Form B #6

The Maclaurin series for $\ln\left(\frac{1}{1-x}\right)$ is $\sum_{n=1}^{\infty}\frac{x^n}{n}$ with interval of convergence $-1 \le x < 1$.

(a) Find the Maclaurin series for $\ln\left(\frac{1}{1+3x}\right)$ and determine the interval of convergence.

$$\ln\left(\frac{1}{1-x}\right)=\sum_{n=1}^{\infty}\frac{x^n}{n} \text{ with interval of convergence } -1 \le x < 1$$

$$\downarrow$$

$$\ln\left(\frac{1}{1+3x}\right)=\ln\left(\frac{1}{1-(-3x)}\right)$$

$$=\sum_{n=1}^{\infty}\frac{(-3x)^n}{n}$$

Taking advantage of the fact that the interval of convergence of $\ln\left(\frac{1}{1-x}\right)=\sum_{n=1}^{\infty}\frac{x^n}{n}$ is $-1 \le x < 1$, this new series will converge so long as

$$-1 \le -3x < 1$$

$$\downarrow$$

$$\frac{1}{3} \ge x > -\frac{1}{3}$$

The interval of convergence for $\ln\left(\frac{1}{1+3x}\right)=\sum_{n=1}^{\infty}\frac{(-3x)^n}{n}$ is $-\frac{1}{3} < x \le \frac{1}{3}$

(b) Find the value of $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n}=\sum_{n=1}^{\infty}\frac{\left(-3\left[\frac{1}{3}\right]\right)^n}{n}=\ln\left(\frac{1}{1+3\left[\frac{1}{3}\right]}\right)=\ln\left(\frac{1}{2}\right)$$

(c) Give a value of p such that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ converges, but $\sum_{n=1}^{\infty} \frac{1}{n^{2p}}$ diverges. Give reasons why your value of p is correct.

A possible value of p: $p = \frac{1}{2}$

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{\left(\frac{1}{2}\right)}}$ is an alternating series, and $\lim_{n \to \infty} \left[\frac{1}{n^{\frac{1}{2}}} \right] = 0$. Therefore, the series converges by the Alternating Series Test.

$\sum_{n=1}^{\infty} \frac{1}{n^{2\left(\frac{1}{2}\right)}} = \sum_{n=1}^{\infty} \frac{1}{n}$ is the harmonic series, which diverges.

(d) Give a value of p such that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges, but $\sum_{n=1}^{\infty} \frac{1}{n^{2p}}$ converges. Give reasons why your value of p is correct.

A possible value of p: $p = 1$

$\sum_{n=1}^{\infty} \frac{1}{n^1} = \sum_{n=1}^{\infty} \frac{1}{n}$ is the harmonic series, which diverges.

$\sum_{n=1}^{\infty} \frac{1}{n^{2(1)}} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ which is a convergent p-series.

2005 #6

Let *f* be a function with derivatives of all orders and for which $f(2)=7$. When *n* is odd, the n^{th} derivative of *f* at $x=2$ is 0. When *n* is even and $n \ge 2$, the n^{th} derivative of *f* at $x=2$ is given by $f^{(n)}(2)=\dfrac{(n-1)!}{3^n}$

.

(a) Write the sixth-degree Taylor polynomial for *f* about $x=2$.

$$f(x)=f(c)+f'(c)\cdot(x-c)+\frac{f''(c)\cdot(x-c)^2}{2!}+\frac{f'''(c)\cdot(x-c)^3}{3!}+\cdots+\frac{f^{(6)}(c)\cdot(x-c)^6}{6!}$$

$$=f(2)+f'(2)\cdot(x-2)+\frac{f''(2)\cdot(x-2)^2}{2!}+\frac{f'''(2)\cdot(x-2)^3}{3!}+\cdots+\frac{f^{(6)}(2)\cdot(x-2)^6}{6!}$$

$$=7+(0)\cdot(x-2)+\frac{\left[\frac{(2-1)!}{3^2}\right]\cdot(x-2)^2}{2!}+\frac{(0)\cdot(x-2)^3}{3!}+\frac{\left[\frac{(4-1)!}{3^4}\right]\cdot(x-2)^4}{4!}+\frac{(0)\cdot(x-2)^5}{5!}+\frac{\left[\frac{(6-1)!}{3^6}\right]\cdot(x-2)^6}{6!}$$

$$=7+\frac{\left[\frac{1}{3^2}\right]\cdot(x-2)^2}{2!}+\frac{\left[\frac{3!}{3^4}\right]\cdot(x-2)^4}{4!}+\frac{\left[\frac{5!}{3^6}\right]\cdot(x-2)^6}{6!}$$

$$=7+\left[\frac{1}{3^2\cdot 2!}\right]\cdot(x-2)^2+\left[\frac{3!}{3^4\cdot 4!}\right]\cdot(x-2)^4+\left[\frac{5!}{3^6\cdot 6!}\right](x-2)^6$$

$$=7+\left[\frac{1}{3^2\cdot 2}\right]\cdot(x-2)^2+\left[\frac{1}{3^4\cdot 4}\right]\cdot(x-2)^4+\left[\frac{1}{3^6\cdot 6}\right](x-2)^6$$

(b) In the Taylor series for *f* about $x=2$, what is the coefficient of $(x-2)^{2n}$ for $n \ge 1$.

$$\frac{f^{(2n)}(2)\cdot(x-2)^{2n}}{(2n)!}=\frac{\left[\frac{(2n-1)!}{3^{2n}}\right]\cdot(x-2)^{2n}}{(2n)!}$$

$$=\left[\frac{(2n-1)!}{3^{2n}\cdot(2n)!}\right]\cdot(x-2)^{2n}$$

$$=\left[\frac{(2n-1)!}{3^{2n}\cdot(2n)\cdot\left[(2n-1)!\right]}\right]\cdot(x-2)^{2n}$$

$$=\left[\frac{\cancel{(2n-1)!}}{3^{2n}\cdot(2n)\cdot\cancel{\left[(2n-1)!\right]}}\right]\cdot(x-2)^{2n}$$

$$=\left[\frac{1}{3^{2n}\cdot(2n)}\right]\cdot(x-2)^{2n}$$

The coefficient of $(x-2)^{2n}$ is $\left[\dfrac{1}{3^n\cdot(2n)}\right]$ for $n \ge 1$.

(c) Find the interval of convergence of the Taylor series for *f* about $x = 2$. Show the work that leads to your answer.

$$\lim_{n\to\infty}\left|\frac{\left[\frac{1}{3^{2(n+1)}\cdot(2(n+1))}\right]\cdot(x-2)^{2(n+1)}}{\left[\frac{1}{3^{2n}\cdot(2n)}\right]\cdot(x-2)^{2n}}\right| = \lim_{n\to\infty}\left|\frac{\left[\frac{(x-2)^{2n+2}}{3^{2n+2}\cdot(2n+2)}\right]}{\left[\frac{(x-2)^{2n}}{3^{2n}\cdot(2n)}\right]}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(x-2)^{2n+2}}{3^{2n+2}\cdot(2n+2)}\cdot\frac{3^{2n}\cdot(2n)}{(x-2)^{2n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(x-2)^{2n}\cdot(x-2)^{2}}{3^{2n}\cdot 3^{2}\cdot(2n+2)}\cdot\frac{3^{2n}\cdot(2n)}{(x-2)^{2n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{\cancel{3^{2n}}\cdot\cancel{(x-2)^{2n}}\cdot(x-2)^{2}}{\cancel{3^{2n}}\cdot 3^{2}\cdot\cancel{(x-2)^{2n}}}\cdot\frac{(2n)}{(2n+2)}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(x-2)^{2}}{9}\cdot\frac{(2n)}{(2n+2)}\right|$$

$$= \left|\frac{(x-2)^{2}}{9}\right|$$

The series will converge so long as

$$\left|\frac{(x-2)^{2}}{9}\right| < 1$$

$$\sqrt{\left|\frac{(x-2)^{2}}{9}\right|} < \sqrt{1}$$

$$\left|\frac{x-2}{3}\right| < 1$$

$$\downarrow$$

$$-1 < \frac{x-2}{3} < 1$$

$$-3 < x-2 < 3$$

$$-1 < x < 5$$

$x=-1$
$$7+\sum_{n=1}^{\infty}\left[\frac{1}{3^{2n}\cdot(2n)}\right]\cdot(-1-2)^{2n}=7+\sum_{n=1}^{\infty}\left[\frac{1}{3^{2n}\cdot(2n)}\right]\cdot(-3)^{2n}$$ $$=7+\sum_{n=1}^{\infty}\left[\frac{(-3)^{2n}}{3^{2n}\cdot(2n)}\right]$$ $$=7+\sum_{n=1}^{\infty}\left[\frac{(-1)^{2n}\cdot 3^{2n}}{3^{2n}\cdot(2n)}\right]$$ $$=7+\sum_{n=1}^{\infty}\frac{1}{2n}$$ This diverges because of the harmonic series component.

$x=5$
$$7+\sum_{n=1}^{\infty}\left[\frac{1}{3^{2n}\cdot(2n)}\right]\cdot(5-2)^{2n}=7+\sum_{n=1}^{\infty}\left[\frac{1}{3^{2n}\cdot(2n)}\right]\cdot 3^{2n}$$ $$=7+\sum_{n=1}^{\infty}\left[\frac{3^{2n}}{3^{2n}\cdot(2n)}\right]$$ $$=7+\sum_{n=1}^{\infty}\frac{1}{2n}$$ This diverges because of the harmonic series component.

2007 Form B #6

Let f be the function given by $f(x)=6e^{-\frac{x}{3}}$ for all x.

(a) Find the first four nonzero terms and the general term for the Taylor series for f about $x=0$.

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^n}{n!}+\cdots$$

$$6e^{\left(-\frac{x}{3}\right)}=6\cdot\sum_{n=0}^{\infty}\frac{\left(-\frac{x}{3}\right)^n}{n!}=6\cdot\left[1+\left(-\frac{x}{3}\right)+\frac{\left(-\frac{x}{3}\right)^2}{2!}+\frac{\left(-\frac{x}{3}\right)^3}{3!}+\cdots+\frac{\left(-\frac{x}{3}\right)^n}{n!}+\cdots\right]$$

$$=6\cdot\sum_{n=0}^{\infty}\frac{(-1)^n\cdot\frac{x^n}{3^n}}{n!}=6\cdot\left[1-\frac{x}{3}+\frac{\left(\frac{x^2}{3^2}\right)}{2!}-\frac{\left(\frac{x^3}{3^3}\right)}{3!}+\cdots+\frac{(-1)^n\cdot\left(\frac{x^n}{3^n}\right)}{n!}+\cdots\right]$$

$$=\sum_{n=0}^{\infty}\frac{2\cdot(-1)^n\cdot x^n}{3^{n-1}\cdot(n!)}=\left[6-2x+\frac{2\cdot x^2}{3^1\cdot(2!)}-\frac{2\cdot x^3}{3^2\cdot(3!)}+\cdots+\frac{(-1)^n\cdot 2\cdot x^n}{3^{n-1}\cdot(n!)}+\cdots\right]$$

(b) Let g be the function given by $g(x)=\int_0^x f(t)\,dt$. Find the first four nonzero terms and the general term for the Taylor series for g about $x=0$.

$$g(x)=\int_0^x f(t)\,dt$$

$$=\int_0^x\left[6-2t+\frac{2\cdot t^2}{3^1\cdot(2!)}-\frac{2\cdot t^3}{3^2\cdot(3!)}+\cdots+\frac{(-1)^n\cdot 2\cdot t^n}{3^{n-1}\cdot(n!)}+\cdots\right]dt$$

$$=\left[6t-\frac{1}{2}\cdot 2t^2+\frac{2\cdot t^3}{3^1\cdot(2!)}-\frac{1}{4}\cdot\frac{2\cdot t^4}{3^2\cdot(3!)}+\cdots+\frac{1}{n+1}\cdot\frac{(-1)^n\cdot 2\cdot t^{n+1}}{3^{n-1}\cdot(n!)}+\cdots\right]_0^x$$

$$=\left[6x-\frac{1}{2}\cdot 2x^2+\frac{2\cdot x^3}{3^1\cdot(2!)}-\frac{1}{4}\cdot\frac{2\cdot x^4}{3^2\cdot(3!)}+\cdots+\frac{1}{n+1}\cdot\frac{(-1)^n\cdot 2\cdot x^{n+1}}{3^{n-1}\cdot(n!)}+\cdots\right]$$

$$-\left[6(0)-\frac{1}{2}\cdot 2(0)^2+\frac{2\cdot(0)^3}{3^1\cdot(2!)}-\frac{1}{4}\cdot\frac{2\cdot(0)^4}{3^2\cdot(3!)}+\cdots+\frac{1}{n+1}\cdot\frac{(-1)^n\cdot 2\cdot(0)^{n+1}}{3^{n-1}\cdot(n!)}+\cdots\right]$$

$$=6x-\frac{1}{2}\cdot 2x^2+\frac{2\cdot x^3}{3^1\cdot(2!)}-\frac{1}{4}\cdot\frac{2\cdot x^4}{3^2\cdot(3!)}+\cdots+\frac{1}{n+1}\cdot\frac{(-1)^n\cdot 2\cdot x^{n+1}}{3^{n-1}\cdot(n!)}+\cdots$$

(c) The function h satisfies $h(x) = k \cdot f'(ax)$ for all x, where a and k are constants. The Taylor series for h about $x = 0$ is given by

$$h(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

Find the values of a and k.

$$\begin{aligned} h(x) &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots \\ &= e^x \end{aligned}$$

Method #1

$f(x) = 6 - \frac{6x}{3} + \frac{6x^2}{2! \cdot 3^2} - \frac{6x^3}{3! \cdot 3^3} + \cdots + \frac{6 \cdot (-1)^n x^n}{n! \cdot 3^n} + \cdots$ $f'(x) = -2 + \frac{2 \cdot 6x}{2! \cdot 3^2} - \frac{3 \cdot 6x^2}{3! \cdot 3^3} + \cdots + \frac{6(-1)^n x^{n-1}}{(n-1)! \cdot 3^n} + \cdots$ $= -2 + \frac{2}{3}x - \frac{x^2}{3^2} + \cdots$ $\downarrow$ $f'(ax) = -2 + \frac{2}{3}(ax) - \frac{(ax)^2}{3^2} + \cdots$ $k \cdot f'(ax) = -2k + \frac{2k}{3}(ax) - \frac{k(ax)^2}{3^2} + \cdots$	$-2k = 1$ $k = -\frac{1}{2}$ $\frac{2}{3}ak = 1$ $\frac{2}{3}\left(-\frac{1}{2}\right)a = 1$ $a = -3$

Method #2

$h(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$ $= e^x$ $f(x) = 6e^{-\frac{x}{3}}$ $\downarrow$ $f'(x) = -2e^{-\frac{x}{3}}$ $f'(ax) = -2e^{-\frac{ax}{3}}$	$k \cdot f'(ax) = h(x)$ $k\left[-2e^{-\frac{ax}{3}}\right] = e^x$ $-2ke^{-\frac{ax}{3}} = e^x$ $\downarrow$ $-2k = 1$ $k = -\frac{1}{2}$ $-\frac{ax}{3} = x$ $-\frac{a}{3} = 1$ $a = -3$

2009 #6

The Maclaurin series for $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^n}{n!} + \cdots$ The continuous function f is defined by $f(x) = \frac{e^{(x-1)^2} - 1}{(x-1)^2}$ for $x \neq 1$ and $f(1) = 1$. The function *f* has derivatives of all orders at $x = 1$.

(a) Write the first four nonzero terms and the general term of the Taylor series for $e^{(x-1)^2}$ about $x = 1$.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

$$\downarrow$$

$$e^{(x-1)^2} = \sum_{n=0}^{\infty} \frac{\left[(x-1)^2\right]^n}{n!}$$

$$= 1 + \left[(x-1)^2\right] + \frac{\left[(x-1)^2\right]^2}{2!} + \frac{\left[(x-1)^2\right]^3}{3!} + \cdots + \frac{\left[(x-1)^2\right]^n}{n!} + \cdots$$

$$= 1 + (x-1)^2 + \frac{(x-1)^4}{2!} + \frac{(x-1)^6}{3!} + \cdots + \frac{(x-1)^{2n}}{n!} + \cdots$$

(b) Use the Taylor series found in part (a) to write the first four nonzero terms and the general term of the Taylor series for f about $x = 1$.

$$f(x) = \frac{e^{(x-1)^2} - 1}{(x-1)^2}$$

$$= \frac{\left[1 + (x-1)^2 + \frac{(x-1)^4}{2!} + \frac{(x-1)^6}{3!} + \cdots + \frac{(x-1)^{2n}}{n!} + \cdots\right] - 1}{(x-1)^2}$$

$$= \frac{(x-1)^2 + \frac{(x-1)^4}{2!} + \frac{(x-1)^6}{3!} + \cdots + \frac{(x-1)^{2n}}{n!} + \cdots}{(x-1)^2}$$

$$= \frac{1}{(x-1)^2} \cdot \left[(x-1)^2 + \frac{(x-1)^4}{2!} + \frac{(x-1)^6}{3!} + \cdots + \frac{(x-1)^{2n}}{n!} + \cdots\right]$$

$$= 1 + \frac{(x-1)^2}{2!} + \frac{(x-1)^4}{3!} + \frac{(x-1)^6}{4!} + \cdots + \frac{(x-1)^{2n-2}}{n!} + \cdots$$

(c) Use the ratio test to find the interval of convergence for the Taylor series found in part (b)

$$\lim_{n\to\infty}\left|\frac{\frac{(x-1)^{2(n+1)-2}}{(n+1)!}}{\frac{(x-1)^{2n-2}}{n!}}\right| = \lim_{n\to\infty}\left|\frac{\frac{(x-1)^{2n}}{(n+1)!}}{\frac{(x-1)^{2n-2}}{n!}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(x-1)^{2n}}{(n+1)!}\cdot\frac{n!}{(x-1)^{2n-2}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{n!}{(n+1)!}\cdot\frac{(x-1)^{2n}}{(x-1)^{2n-2}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{n!}{(n+1)\cdot[n!]}\cdot\frac{(x-1)^{2n-2}\cdot(x-1)^{2}}{(x-1)^{2n-2}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{\cancel{n!}}{(n+1)\cdot\cancel{[n!]}}\cdot\frac{\cancel{(x-1)^{2n-2}}\cdot(x-1)^{2}}{\cancel{(x-1)^{2n-2}}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(x-1)^{2}}{(n+1)}\right|$$

$$= 0$$

Therefore, the interval of convergence is all real numbers.

(d) Use the Taylor series for *f* about $x=1$ to determine whether the graph of *f* has any points of inflection.

$$f(x) = 1+\frac{(x-1)^2}{2!}+\frac{(x-1)^4}{3!}+\frac{(x-1)^6}{4!}+\cdots+\frac{(x-1)^{2n-2}}{n!}+\cdots$$

$$\downarrow$$

$$f'(x) = \frac{2\cdot(x-1)^1}{2!}+\frac{4\cdot(x-1)^3}{3!}+\frac{6\cdot(x-1)^5}{4!}+\cdots+\frac{(2n-2)\cdot(x-1)^{2n-3}}{n!}+\cdots$$

$$\downarrow$$

$$f''(x) = 1+\frac{4\cdot3\cdot(x-1)^2}{3!}+\frac{6\cdot5\cdot(x-1)^4}{4!}+\cdots+\frac{(2n-2)\cdot(2n-3)\cdot(x-1)^{2n-4}}{n!}+\cdots$$

$$= 1+\frac{4\cdot3\cdot(x-1)^2}{3!}+\frac{6\cdot5\cdot(x-1)^4}{4!}+\cdots+\frac{(2n-2)\cdot(2n-3)\cdot\left[(x-1)^{n-2}\right]^2}{n!}+\cdots$$

Every term of the series is nonnegative. Therefore, the graph of $f''(x)$ will not change sign, Therefore the graph of *f* has no points of inflection.

2010 Form B #6

The Maclaurin series for the function *f* is given by $f(x)=\sum_{n=2}^{\infty}\frac{(-1)^n\cdot(2x)^n}{n-1}$ on its interval of convergence.

(a) Find the interval of convergence for the Maclaurin series of *f*. Justify your answer.

$$\lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}\cdot(2x)^{n+1}}{(n+1)-1}}{\frac{(-1)^n\cdot(2x)^n}{n-1}}\right|=\lim_{n\to\infty}\left|\frac{\frac{2^{n+1}\cdot x^{n+1}}{n}}{\frac{2^n\cdot x^n}{n-1}}\right|$$

$$=\lim_{n\to\infty}\left|\frac{2^{n+1}\cdot x^{n+1}}{n}\cdot\frac{n-1}{2^n\cdot x^n}\right|$$

$$=\lim_{n\to\infty}\left|\frac{n-1}{n}\cdot\frac{2^{n+1}\cdot x^{n+1}}{2^n\cdot x^n}\right|$$

$$=\lim_{n\to\infty}\left|\frac{n-1}{n}\cdot\frac{2^n\cdot 2^1\cdot x^n\cdot x^1}{2^n\cdot x^n}\right|$$

$$=\lim_{n\to\infty}\left|\frac{n-1}{n}\cdot\frac{\cancel{2^n}\cdot 2^1\cdot \cancel{x^n}\cdot x^1}{\cancel{2^n}\cdot \cancel{x^n}}\right|$$

$$=\lim_{n\to\infty}\left|\frac{n-1}{n}\cdot\frac{2^1\cdot x^1}{1}\right|$$

$$=|2x|$$

The series will converge for

$$|2x|<1$$

$$\downarrow$$

$$-1<2x<1$$

$$-\frac{1}{2}<x<\frac{1}{2}$$

$x=-\frac{1}{2}$	$x=\frac{1}{2}$
$\sum_{n=2}^{\infty}\frac{(-1)^n\cdot\left(2\left(-\frac{1}{2}\right)\right)^n}{n-1}=\sum_{n=2}^{\infty}\frac{(-1)^n\cdot(-1)^n}{n-1}$ $=\sum_{n=2}^{\infty}\frac{(-1)^{2n}}{n-1}$ $=\sum_{n=2}^{\infty}\frac{\left[(-1)^2\right]^n}{n-1}$ $=\sum_{n=2}^{\infty}\frac{[1]^n}{n-1}$ $=\sum_{n=2}^{\infty}\frac{1}{n-1}$ This series will not converge, since it is the harmonic series.	$\sum_{n=2}^{\infty}\frac{(-1)^n\cdot\left(2\left(\frac{1}{2}\right)\right)^n}{n-1}=\sum_{n=2}^{\infty}\frac{(-1)^n\cdot 1^n}{n-1}$ $=\sum_{n=2}^{\infty}\frac{(-1)^n}{n-1}$ Since $\lim_{n\to\infty}\left[\frac{1}{n-1}\right]=0$, this series converges by the Alternating Series Test

Therefore, the interval of convergence is $-\frac{1}{2}<x\le\frac{1}{2}$.

(b) Show that $y = f(x)$ is a solution to the differential eqution $xy' - y = \frac{4x^2}{1+2x}$ for $|x| < R$, where R is the radius of convergence from part (a).

$$
\begin{aligned}
f(x) &= \sum_{n=2}^{\infty} \frac{(-1)^n \cdot (2x)^n}{n-1} \\
&= \frac{(-1)^2 \cdot (2x)^2}{2-1} + \frac{(-1)^3 \cdot (2x)^3}{3-1} + \frac{(-1)^4 \cdot (2x)^4}{4-1} + \cdots + \frac{(-1)^n \cdot (2x)^n}{n-1} \cdots \\
&= (2x)^2 - \frac{(2x)^3}{2} + \frac{(2x)^4}{3} + \cdots + \frac{(-1)^n \cdot (2x)^n}{n-1} \cdots
\end{aligned}
$$

$\downarrow$

$$
f'(x) = 2(2x)^1 \cdot 2 - \frac{3(2x)^2 \cdot 2}{2} + \frac{4(2x)^3 \cdot 2}{3} + \cdots + \frac{(-1)^n \cdot n \cdot (2x)^{n-1} \cdot 2}{n-1} \cdots
$$

$\downarrow$

$$
\begin{aligned}
x \cdot f'(x) &= x \cdot \left[2(2x)^1 \cdot 2 - \frac{3(2x)^2 \cdot 2}{2} + \frac{4(2x)^3 \cdot 2}{3} + \cdots + \frac{(-1)^n \cdot n \cdot (2x)^{n-1} \cdot 2}{n-1} \cdots \right] \\
&= \left[2(2x)^1 \cdot 2x - \frac{3(2x)^2 \cdot 2x}{2} + \frac{4(2x)^3 \cdot 2x}{3} + \cdots + \frac{(-1)^n \cdot n \cdot (2x)^{n-1} \cdot 2x}{n-1} \cdots \right] \\
&= 2(2x)^2 - \frac{3(2x)^3}{2} + \frac{4(2x)^4}{3} + \cdots + \frac{(-1)^n \cdot n \cdot (2x)^n}{n-1} \cdots
\end{aligned}
$$

$\downarrow$

$$
\begin{aligned}
x \cdot f'(x) - f(x) &= \left[2(2x)^2 - \frac{3(2x)^3}{2} + \frac{4(2x)^4}{3} + \cdots + \frac{(-1)^n \cdot n \cdot (2x)^n}{n-1} \cdots \right] - \left[(2x)^2 - \frac{(2x)^3}{2} + \frac{(2x)^4}{3} + \cdots + \frac{(-1)^n \cdot (2x)^n}{n-1} \cdots \right] \\
&= (2-1) \cdot (2x)^2 + \left(-\frac{3}{2} + \frac{1}{2} \right) \cdot (2x)^3 + \left(\frac{4}{3} - \frac{1}{3} \right) \cdot (2x)^4 + \cdots + \left(\frac{(-1)^n \cdot n}{n-1} - \frac{(-1)^n}{n-1} \right) \cdot (2x)^n + \cdots \\
&= (1) \cdot (2x)^2 + \left(-\frac{2}{2} \right) \cdot (2x)^3 + \left(\frac{3}{3} \right) \cdot (2x)^4 + \cdots + \left(\frac{(-1)^n \cdot (n-1)}{n-1} \right) \cdot (2x)^n + \cdots \\
&= (2x)^2 - (2x)^3 + (2x)^4 + \cdots + (-1)^n \cdot (2x)^n + \cdots \\
&= (-2x)^2 - (2x) + (-2x)^2 - \cdots + (-2x)^{n-2} + \cdots
\end{aligned}
$$

The series $\sum_{n=0}^{\infty} (-2x)^n$ is a geometric series with common ratio $-2x$, therefore

$$
\text{sum} = \frac{\text{first term}}{1 - \text{common ratio}} = \frac{4x^2}{1-(-2x)} = \frac{4x^2}{1+2x}
$$

$$4x^2\left[\sum_{n=0}^{\infty}(-2x)^n\right]=4x^2\left[\frac{1}{1+2x}\right]=\frac{4x^2}{1+2x}$$

2013 #6

A function *f* has derivatives of all orders at $x=0$. Let $P_n(x)$ denote the n^{th} degree Taylor polynomial for *f* about $x=0$.

(a) It is known that $f(0)=-4$ and that $P_1\left(\frac{1}{2}\right)=-3$. Show that $f'(0)=2$.

$$P_1(x)=f(0)+f'(0)\cdot(x-0)$$

$$\downarrow$$

$$-3=-4+f'(0)\cdot\left(\frac{1}{2}\right)$$

$$1=f'(0)\cdot\left(\frac{1}{2}\right)$$

$$f'(0)=2$$

(b) It is known that $f''(0)=-\frac{2}{3}$ and $f'''(0)=\frac{1}{3}$. Find $P_3(x)$.

$$\begin{aligned}P_3(x)&=f(0)+f'(0)\cdot(x-0)+\frac{f''(0)\cdot(x-0)^2}{2!}+\frac{f'''(0)\cdot(x-0)^3}{3!}\\&=-4+2\cdot(x-0)+\frac{\left(-\frac{2}{3}\right)\cdot(x-0)^2}{2!}+\frac{\left(\frac{1}{3}\right)\cdot(x-0)^3}{3!}\\&=-4+2x-\frac{1}{3}x^2+\frac{1}{18}x^3\end{aligned}$$

(c) The function h has first derivative given by $h'(x) = f(2x)$. It is known that $h(0) = 7$. Find the third-degree Taylor polynomial for h about $x = 0$.

$$
\begin{aligned}
h'(x) &= f(2x) \\
&= -4 + 2(2x) - \frac{1}{3}(2x)^2 + \frac{1}{18}(2x)^3 + \cdots \\
&= -4 + 4x - \frac{4}{3}x^2 + \frac{4}{9}x^3 + \cdots \\
h(x) &= \int \left[-4 + 4x - \frac{4}{3}x^2 + \frac{4}{9}x^3 + \cdots \right] dx \\
&= C - 4x + 2x^2 - \frac{4}{9}x^3 + \frac{4}{27}x^4 + \cdots
\end{aligned}
$$

Since $h(0) = 7$,

$$
h(x) = 7 - 4x + 2x^2 - \frac{4}{9}x^3 + \frac{4}{27}x^4 + \cdots
$$

$$\downarrow$$

$$
T_3(x) = 7 - 4x + 2x^2 - \frac{4}{9}x^3
$$

2016 #6

The function f has a Taylor Series about $x=1$ that converges to $f(x)$ for all x in the interval of convergence. It is known that $f(1)=1$, $f'(1)=-\frac{1}{2}$, and the n^{th} derivative of f at $x=1$ is given by $f^{(n)}(1)=(-1)^n\cdot\frac{(n-1)!}{2^n}$ for $n\geq 2$.

(a) Write the first four nonzero terms and the general term of the Taylor series for f about $x=1$.

$$f(1)+f'(1)(x-1)+\frac{f''(1)(x-1)^2}{2!}+\frac{f'''(1)(x-1)^3}{3!}$$

$$1+\left(-\frac{1}{2}\right)(x-1)+\frac{\left[(-1)^2\frac{(2-1)!}{2^2}\right](x-1)^2}{2!}+\frac{\left[(-1)^3\frac{(3-1)!}{2^3}\right](x-1)^3}{3!}$$

$$1+\left(-\frac{1}{2}\right)(x-1)+\left(\frac{1}{8}\right)(x-1)^2+\left(-\frac{1}{24}\right)(x-1)^3$$

(b) The Taylor series for f about $x=1$ has a radius of convergence of 2. Find the interval of convergence. Show the work that leads to your answer.

$x=-1$
$$\sum_{n=0}^{\infty}\frac{f^{(n)}(1)(-2)^n}{n!}=\sum_{n=0}^{\infty}\frac{(-1)^n\frac{(n-1)!}{2^n}(-2)^n}{n!}$$ $$=\sum_{n=0}^{\infty}\frac{\frac{(n-1)!}{\cancel{2^n}}\cancel{(2)^n}}{n!}$$ $$=\sum_{n=0}^{\infty}\frac{(n-1)!}{n!}$$ $$=\sum_{n=0}^{\infty}\frac{1}{n}$$ This is the harmonic series, which diverges. Therefore -1 is not included in the interval of convergence.

$x=3$
$$\sum_{n=0}^{\infty}\frac{f^{(n)}(1)(2)^n}{n!}=\sum_{n=0}^{\infty}\frac{(-1)^n\frac{(n-1)!}{2^n}(2)^n}{n!}$$ $$=\sum_{n=0}^{\infty}\frac{(-1)^n\frac{(n-1)!}{\cancel{2^n}}\cancel{(2)^n}}{n!}$$ $$=\sum_{n=0}^{\infty}\frac{(-1)^n(n-1)!}{n!}$$ $$=\sum_{n=0}^{\infty}(-1)^n\cdot\frac{1}{n}$$ This is a convergent alternating series. Therefore $x=3$ is included in the interval of convergence.

The interval of convergence for the Taylor series is $(-1,3]$ or $-1<x\le 3$.

(c) The Taylor series for *f* about $x = 1$ can be used to represent $f(1.2)$ as an alternating series. Use the first three nonzero terms of the alternating series to approximate $f(1.2)$.

$$f(1.2) \approx 1 + \left(-\frac{1}{2}\right)(1.2-1) + \left(\frac{1}{8}\right)(1.2-1)^2 = \frac{181}{200} = 0.905$$

(d) Show that the approximation found in part (c) is within 0.001 of the exact value of $f(1.2)$.

The series for $f(1.2)$ is an alternating series, whose terms decrease in absolute value to zero.

By the Alternating Series Remainder Theorem, the error in part (c) bounded by

$$\left|\frac{1}{24}(1.2-1)^3\right| = \frac{1}{24} \cdot \left(\frac{1}{5}\right)^3 = \frac{1}{24} \cdot \frac{1}{125} = \frac{1}{3000} = 0.0003$$

2017 #6

$$f(0)=0$$
$$f'(0)=1$$
$$f^{(n+1)}(0)=(-n)\cdot f^{(n)}(0) \text{ for all } n\geq 1$$

A function *f* has derivatives of all order for $-1<x<1$. The derivatives of *f* satisfy the conditions above. The Maclaurin series for *f* converges to $f(x)$ for $|x|<1$.

(a) Show that the first four nonzero terms of the Maclaurin series for *f* are $x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}$, and write the general term of the Maclaurin series for *f*.

$$f''(0)=-1f'(0)=-1\cdot 1=-1$$
$$f^{(3)}(0)=-2f''(0)=-2(-1)=2$$
$$f^{(4)}(0)=-3f^{(3)}(0)=-6$$
$$f(0)+f'(0)(x-0)+\frac{f''(0)(x-0)^2}{2!}+\frac{f^{(3)}(0)(x-0)^3}{3!}+\frac{f^{(4)}(0)(x-0)^4}{4!}$$
$$0+1\cdot x+\frac{(-1)x^2}{2!}+\frac{2x^3}{3!}+\frac{(-6)x^4}{4!}$$
$$\boxed{\underbrace{x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}}_{\text{first four nonzero terms}}+\cdots+\underbrace{\frac{(-1)^{n+1}x^n}{n}}_{\text{general term}}+\cdots}$$

(b) Determine whether the Maclaurin series described in part (a) converges absolutely, converges conditionally, or diverges at $x=1$. Explain your reasoning.

$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(1)^n}{n}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$ converges by the Alternating Series Test since $\lim_{n\to\infty}\frac{1}{n}=0$.

$\sum_{n=1}^{\infty}\left|\frac{(-1)^{n+1}(1)^n}{n}\right|=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$ diverges because it is the harmonic series.

Therefore, the Macluarin series described in part (a) converges conditionally

(c) Write the first four nonzero terms and the general term of the Maclaurin series for $g(x)=\int_0^x f(t)\,dt$.

$$g(x)=\int_0^x f(t)\,dt$$
$$=\int_0^x \sum_{n=1}^{\infty}\frac{(-1)^{n+1}t^n}{n}dt$$
$$=\int_0^x t-\frac{t^2}{2}+\frac{t^3}{3}-\frac{t^4}{4}+\cdots+\frac{(-1)^{n+1}t^n}{n}+\cdots dt$$
$$=\left[\frac{1}{2}t^2-\frac{t^3}{6}+\frac{t^4}{12}-\frac{t^5}{20}+\cdots+\frac{(-1)^{n+1}t^{n+1}}{n(n+1)}\right]_0^x$$
$$=\frac{x^2}{2}-\frac{x^3}{6}+\frac{x^4}{12}-\frac{x^5}{20}+\cdots+\frac{(-1)^{n+1}x^{n+1}}{n(n+1)}$$

(d) Let $P_n\left(\frac{1}{2}\right)$ represent the n^{th} degree Taylor polynomial for g about $x=0$ evaluated at $x=\frac{1}{2}$, where g is the function defined in part (c). Use the alternating series error bound to show that $\left|P_4\left(\frac{1}{2}\right)-g\left(\frac{1}{2}\right)\right|<\frac{1}{500}$.

$$\left|P_4\left(\frac{1}{2}\right)-g\left(\frac{1}{2}\right)\right|<\left|\frac{\left(\frac{1}{2}\right)^5}{20}\right|=\frac{1}{32\cdot 20}=\frac{1}{640}<\frac{1}{500}$$