AP Calculus BC
Stewart Chapter 2 Test (Limits)
Fall 2019

Name:____________________________________

Date:_______________________ Period:________

Multiple Choice Section: 6 Questions. *No Calculator Permitted.* Suggested Time: 35 minutes.
Once you submit your Multiple Choice section, you will not be allowed to revisit it.
Free Response Section: 10 Questions. Calculator Permitted. Suggested Time: 55≈ minutes.

Multiple Choice Scoring Procedures: Each exercise is worth 5 points.	
If an response is **CIRCLED**	If no response is circled
▪ 5 points awarded if circled response is correct.	▪ 1 point awarded for each incorrect response eliminated.
▪ 0 points awarded if circled response is incorrect.	▪ 0 points awarded if the correct response is eliminated.

(5) 1. Find the limit:

$$\lim_{x\to -9}\frac{x^2+6x-27}{x+9}=\lim_{x\to -9}\frac{(x+9)(x-3)}{x+9}$$
$$=\lim_{x\to -9} x-3$$
$$=-12$$

a) -12 b) DNE c) -3

d) 0 e) None of these

(5) 2. Let $f(x)=\frac{1}{x}$ and $g(x)=x-1$. Find all values of x for which $f(g(x))$ is discontinuous.

$$f(g(x))=\frac{1}{x-1}$$

a) 0 b) 1 c) 0, 1

d) -1, 1 e) None of these

(5) 3. Let g be a continuous function in the closed interval [0,1]. Let $g(0)=1$ and $g(1)=0$. Which of the following statements is **NOT** necessarily true?

a) For all a and b in [0,1], if $a=b$, then $g(a)=g(b)$. True by def. of a function

b) There exists a number c in [0,1] such that $g(c)=\frac{1}{2}$ True by IVT

c) There exists a number c in [0,1] such that $g(c)=\frac{3}{2}$

d) For all h in the open interval (0,1), $\lim_{x\to c} g(x)=g(c)$ True by def. of continuity

(5) 4. Determine the limit: $\lim_{x\to 0}\frac{|x|}{x}$

$$\lim_{x\to 0^+}\frac{|x|}{x}=1 \text{ and } \lim_{x\to 0^-}\frac{|x|}{x}=-1$$

a) 0 b) 1 c) -1

d) DNE e) None of these f) ∞

(5) 5. The graph of $F(x)=\frac{x^2-16}{x^2-5x+4}$ has a vertical asymptote at $x=?$ and a horizontal asymptote at $y=?$

$$f(x)=\frac{x^2-16}{x^2-5x+4}=\frac{(x-4)(x+4)}{(x-4)(x-1)}$$

$$\lim_{x\to 1^+}F(x)=\infty \text{ and } \lim_{x\to 1^-}F(x)=-\infty$$

$$\lim_{x\to\pm\infty}F(x)=1$$

a) x = 0, y = 0 b) x = 1, y = 0 c) x = 1, y = 1

d) x = 1, No Horizontal Asymptote e) No Vertical Asymptote, y = 1

(5) 6. Determine the limit: $\lim_{x\to 0}\frac{\sin^5(x)}{x^4}$:

$$\begin{aligned}\lim_{x\to 0}\frac{\sin^5(x)}{x^4}&=\lim_{x\to 0}\frac{\sin(x)\cdot\sin^4(x)}{x^4}\\&=\lim_{x\to 0}\sin(x)\cdot\left(\frac{\sin(x)}{x}\right)^4\\&=\left[\lim_{x\to 0}\sin(x)\right]\cdot\left[\lim_{x\to 0}\left(\frac{\sin(x)}{x}\right)^4\right]\\&=\left[\lim_{x\to 0}\sin(x)\right]\cdot\left(\lim_{x\to 0}\frac{\sin(x)}{x}\right)^4\\&=0\cdot 1^4\\&=0\end{aligned}$$

a) ∞ b) 1 c) 0 d) DNE e) 4

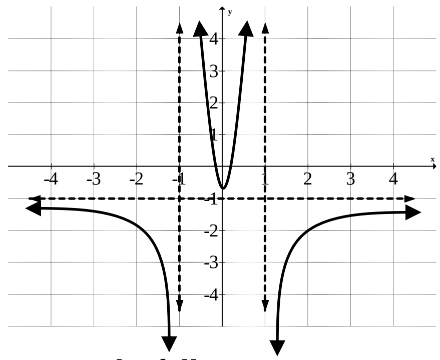
(5) 7. Determine the limit: $\lim_{x\to 0^-}\left(x-\frac{1}{x}\right)$:

$$\begin{aligned}\lim_{x\to 0^-}\left(x-\frac{1}{x}\right) &= \lim_{x\to 0^-}\left(0^- - \frac{1}{0^-}\right)\\ &= \lim_{x\to 0^-}\left(0-\frac{1}{\text{small negative}}\right)\\ &= \lim_{x\to 0^-}\left(0-[\text{big negative}]\right)\\ &\downarrow\\ &+\infty\end{aligned}$$

a) 0 b) $-\infty$ c) 1 d) -1 e) ∞

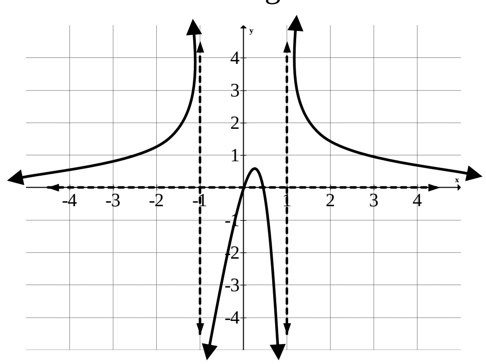
(5) 8. Sketch a graph of a function $f(x)$ that meets the following criteria:

$$\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = -1$$
$$\lim_{x\to-1^+} f(x) = \lim_{x\to 1^-} f(x) = \infty$$
$$\lim_{x\to-1^-} f(x) = \lim_{x\to 1^+} f(x) = -\infty$$



(5) 9. Sketch a graph of a function $f(x)$ that meets the following criteria:

$$\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = 0$$
$$\lim_{x\to-1^-} f(x) = \lim_{x\to 1^+} f(x) = \infty$$
$$\lim_{x\to-1^+} f(x) = \lim_{x\to 1^-} f(x) = -\infty$$



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Free Response Section: ***<u>NO Calculator Permitted.</u>***
You have the remainder of the period to complete this section.
Once you submit your Free Response Section, you will not be allowed to revisit it.

- Show all of your work. Clearly label any functions, graphs, tables, or other objects that you use. Your work will be scored on the correctness and completeness of your methods as well as your answers. Answers without supporting work will usually not receive credit. To be eligible for partial credit, methods, reasoning, and conclusions should be presented clearly.

- Justifications require that you give mathematical (non-calculator) reasons. Students should use complete sentences in responses that include explanations or justifications.

<u>ALL LIMITS MUST BE DETERMINED ANALYTICALLY!</u>
<u>No use of L'Hopital's Rule or Derivatives is permitted.</u>

(16) 1. Find the following limits.

a) $\lim_{x\to 0}(3x+2+\frac{1}{x^2})$

$$\lim_{x\to 0} 3x+2+\frac{1}{x^2} = \lim_{x\to 0}\frac{3x^3+2x^2}{x^2}+\frac{1}{x^2}$$
$$= \lim_{x\to 0}\frac{3x^3+2x^2+1}{x^2}$$
$$= \frac{\lim_{x\to 0} 3x^3+2x^2+1}{\lim_{x\to 0} x^2}$$
$$= \frac{1}{0^+} \to \infty \text{ or } DNE$$

b) $\lim_{x\to 1}\frac{x-1}{\sqrt{x}-1}$

$$\lim_{x\to 1}\frac{x-1}{\sqrt{x}-1} = \lim_{x\to 1}\frac{\left(\sqrt{x}-1\right)\left(\sqrt{x}+1\right)}{\sqrt{x}-1}$$
$$= \lim_{x\to 1}\sqrt{x}+1$$
$$= 2$$

c) $\lim_{x\to\infty}\frac{x^2-2x+3}{4-2x^2}$

$$\lim_{x\to\infty}\frac{x^2-2x+3}{4-2x^2} = \lim_{x\to\infty}\frac{x^2-2x+3}{-2x^2+4}$$
$$\downarrow$$
$$\lim_{x\to\infty}\frac{x^2}{-2x^2} \to -\frac{1}{2}$$

d) $\lim_{x\to 0}\frac{\sin(\frac{1}{2}x)}{\sin(3x)}$

$$\lim_{x\to 0}\frac{\sin\left(\frac{1}{2}x\right)}{\sin(3x)}=\lim_{x\to 0}\frac{\sin\left(\frac{1}{2}x\right)}{1}\cdot\frac{1}{\sin(3x)}$$

$$=\lim_{x\to 0}\frac{\frac{1}{2}}{\frac{1}{2}}\cdot\frac{\sin\left(\frac{1}{2}x\right)}{x}\cdot\frac{3}{3}\cdot\frac{x}{\sin(3x)}$$

$$=\frac{1}{3}\cdot\frac{1}{2}\lim_{x\to 0}\frac{\sin\left(\frac{1}{2}x\right)}{\frac{1}{2}x}\cdot\frac{3x}{\sin(3x)}$$

$$=\frac{1}{6}\lim_{x\to 0}\frac{\sin\left(\frac{1}{2}x\right)}{\frac{1}{2}x}\cdot\lim_{x\to 0}\frac{3x}{\sin(3x)}$$

$$=\frac{1}{6}\lim_{x\to 0}\frac{\sin\left(\frac{1}{2}x\right)}{\frac{1}{2}x}\cdot\lim_{x\to 0}\frac{1}{\frac{\sin(3x)}{3x}}$$

$$=\frac{1}{6}\lim_{x\to 0}\frac{\sin\left(\frac{1}{2}x\right)}{\frac{1}{2}x}\cdot\frac{1}{\lim_{x\to 0}\frac{\sin(3x)}{3x}}$$

$$=\frac{1}{6}\cdot 1\cdot 1$$

$$=\frac{1}{6}$$

$$\approx 0.166 \text{ or } 0.167$$

(10) 2. Determine the value of c so that $f(x)$ is continuous for all x values.

$$f(x)=\begin{cases} x^2 & \text{for } x \le 3 \\ \dfrac{c}{x} & \text{for } x > 3 \end{cases}$$

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} x^2 \quad \text{and} \quad f(3)=9$$
$$= 9$$

Therefore, in order for the function to be continuous at $x=3$ we need $\lim_{x\to 3^+} f(x) = 9$.

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \frac{c}{x} = 9$$
$$\downarrow$$
$$\lim_{x\to 3^+} \frac{c}{x} = 9$$
$$\frac{\lim\limits_{x\to 3^+} c}{\lim\limits_{x\to 3^+} x} = 9$$
$$\frac{\lim\limits_{x\to 3^+} c}{3} = 9$$
$$\lim_{x\to 3^+} c = 27$$
$$c = 27$$

Therefore in order for the function to be continuous, the value of c must be 27.

(8) 3. Use the Intermediate Value Theorem to show that the function $f(x) = x^4 - 2x^2 + 3x$ has a zero in the interval [-2, -1].

First, $f(x)$ is continuous on the interval $[-2,-1]$ since it is a polynomial. Polynomials are continuous on the entire real line. $f(-2) = 2$ and $f(-1) = -4$. By the Intermediate Value theorem, there exists a c in the interval $[-2,-1]$ such that $f(c) = 0$. Therefore $f(x)$ has a root in the interval $[-2,-1]$.

(10) 4. Let $f(x) = \begin{cases} x^2 + 1 & \text{for } x < 1 \\ -2x + 4 & \text{for } x \geq 1 \end{cases}$

a) State the Domain and Range of $f(x)$

The domain of f is all real numbers.
The range of f is all real numbers.

b) Find $\lim\limits_{x \to 1} f(x)$

$$\begin{aligned} \lim_{x \to 1^+} f(x) &= \lim_{x \to 1^+} -2x + 4 \\ &= 2 \end{aligned} \quad \text{and} \quad \begin{aligned} \lim_{x \to 1^-} f(x) &= \lim_{x \to 1^-} x^2 + 1 \\ &= 2 \end{aligned}$$

Therefore $\lim\limits_{x \to 1} f(x) = 2$

c) Prove analytically/algebraically whether or not $f(x)$ is a continuous function.

$$\begin{aligned} f(1) &= 2 \\ \lim_{x \to 1^+} f(x) &= \lim_{x \to 1^+} -2x + 4 \\ &= 2 \\ \lim_{x \to 1^-} f(x) &= \lim_{x \to 1^-} x^2 + 1 \\ &= 2 \end{aligned}$$

The three conditions for continuity are met.

Therefore the function is continuous at $x = 1$.

(10) 5. Find $\lim\limits_{\Delta x \to 0} \dfrac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x}$

$$\begin{aligned} \lim_{\Delta x \to 0} \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} &= \lim_{\Delta x \to 0} \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} \cdot \frac{\sqrt{x + \Delta x} + \sqrt{x}}{\sqrt{x + \Delta x} + \sqrt{x}} \\ &= \lim_{\Delta x \to 0} \frac{x + \Delta x - x}{\Delta x\left(\sqrt{x + \Delta x} + \sqrt{x}\right)} \\ &= \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x\left(\sqrt{x + \Delta x} + \sqrt{x}\right)} \\ &= \lim_{\Delta x \to 0} \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}} \\ &= \frac{1}{2\sqrt{x}} \end{aligned}$$

(6) 6. Find $\lim_{x\to\infty} \frac{2x}{\sqrt{3x^2+1}}$. Prove the limit analytically.

$$\begin{aligned}
\lim_{x\to\infty} \frac{2x}{\sqrt{3x^2+1}} &\sim \lim_{x\to\infty} \frac{2x}{\sqrt{3x^2}} \\
&= \lim_{x\to\infty} \frac{2x}{\sqrt{3}\sqrt{x^2}} \\
&= \lim_{x\to\infty} \frac{2x}{\sqrt{3}|x|} \\
&= \lim_{x\to\infty} \frac{2x}{\sqrt{3}x} \\
&= \lim_{x\to\infty} \frac{2}{\sqrt{3}} \\
&= \frac{2}{\sqrt{3}}
\end{aligned}$$

(6) 7. Find $\lim_{x\to-\infty} \left(3x+\sqrt{9x^2-x}\right)$ analytically.

Hint: Treat the expression as a fraction whose denominator is 1 and rationalize the numerator.

$$\begin{aligned}
\lim_{x\to-\infty} \left(3x+\sqrt{9x^2-x}\right) &= \lim_{x\to-\infty} \frac{3x+\sqrt{9x^2-x}}{1} \cdot \frac{3x-\sqrt{9x^2-x}}{3x-\sqrt{9x^2-x}} \\
&= \lim_{x\to-\infty} \frac{9x^2-\left(9x^2-x\right)}{3x-\sqrt{9x^2-x}} \\
&= \lim_{x\to-\infty} \frac{x}{3x-\sqrt{9x^2-x}} \\
&\sim \lim_{x\to-\infty} \frac{x}{3x-\sqrt{9x^2}} \\
&= \lim_{x\to-\infty} \frac{x}{3x-\sqrt{9}\sqrt{x^2}} \\
&= \lim_{x\to-\infty} \frac{x}{3x-3|x|} \\
&= \lim_{x\to-\infty} \frac{x}{3x-3(-x)} \quad \text{since } |x| \leftrightarrow (-x) \text{ when } x<0 \\
&= \lim_{x\to-\infty} \frac{x}{3x+3x} \\
&= \lim_{x\to-\infty} \frac{x}{6x} \\
&= \lim_{x\to-\infty} \frac{1}{6} \\
&= \frac{1}{6}
\end{aligned}$$