Integration by Partial Fractions Examples:

$$\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx = \int \frac{x^2+2x-1}{x(x+2)(2x-1)}\,dx$$

$$\frac{x^2+2x-1}{x(x+2)(2x-1)} = \frac{A}{x}+\frac{B}{x+2}+\frac{C}{2x-1}$$

$$\frac{x^2+2x-1}{x(x+2)(2x-1)} = \frac{A}{x}\cdot\frac{(x+2)(2x-1)}{(x+2)(2x-1)}+\frac{B}{x+2}\cdot\frac{x(2x-1)}{x(2x-1)}+\frac{C}{2x-1}\cdot\frac{x(x+2)}{x(x+2)}$$

$$\frac{x^2+2x-1}{x(x+2)(2x-1)} = \frac{A(x+2)(2x-1)}{x(x+2)(2x-1)}+\frac{Bx(2x-1)}{x(2x-1)(x+2)}+\frac{Cx(x+2)}{x(x+2)(2x-1)}$$

$$\frac{x^2+2x-1}{x(x+2)(2x-1)} = \frac{A(2x^2+3x-2)}{x(x+2)(2x-1)}+\frac{B(2x^2-x)}{x(2x-1)(x+2)}+\frac{C(x^2+2x)}{x(x+2)(2x-1)}$$

$$\frac{x^2+2x-1}{x(x+2)(2x-1)} = \frac{2Ax^2+3Ax-2A}{x(x+2)(2x-1)}+\frac{2Bx^2-Bx}{x(2x-1)(x+2)}+\frac{Cx^2+2Cx}{x(x+2)(2x-1)}$$

$$\frac{x^2+2x-1}{x(x+2)(2x-1)} = \frac{(2A+2B+C)x^2+(3A-B+2C)x-(2a)}{x(x+2)(2x-1)}$$

$$x^2+2x-1 = (2A+2B+C)x^2+(3A-B+2C)x-2A$$

$$\downarrow$$

$$1 = 2A+2B+C$$

$$2 = 3A-B+2C$$

$$-1 = -2A$$

$$\downarrow$$

$$A = \frac{1}{2}$$

$$B = -\frac{1}{10}$$

$$C = \frac{1}{5}$$

$$\int \frac{x^2+2x-1}{x(x+2)(2x-1)}\,dx = \int \frac{A}{x}+\frac{B}{x+2}+\frac{C}{2x-1}\,dx$$

$$= \int \frac{\left(\frac{1}{2}\right)}{x}+\frac{\left(-\frac{1}{10}\right)}{x+2}+\frac{\left(\frac{1}{5}\right)}{2x-1}\,dx$$

$$= \int \frac{1}{2}\cdot\frac{1}{x}-\frac{1}{10}\cdot\frac{1}{x+2}+\frac{1}{5}\cdot\frac{1}{2x-1}\,dx$$

$$= \int \frac{1}{2}\cdot\frac{1}{x}\,dx-\int \frac{1}{10}\cdot\frac{1}{x+2}\,dx+\int \frac{1}{5}\cdot\frac{1}{2x-1}\,dx$$

$$= \frac{1}{2}\cdot\int \frac{1}{x}\,dx-\frac{1}{10}\cdot\int \frac{1}{x+2}\,dx+\frac{1}{5}\cdot\int \frac{1}{2x-1}\,dx$$

$$= \frac{1}{2}\ln|x|-\frac{1}{10}\ln|x+2|+\frac{1}{10}\ln|2x+1|+C$$

$$\int \frac{2x^2-x+4}{x^3+4x}\,dx = \int \frac{2x^2-x+4}{x\left(x^2+4\right)}\,dx$$

$$\frac{2x^2-x+4}{x\left(x^2+4\right)} = \frac{A}{x}+\frac{Bx+C}{x^2+4}$$

$$\frac{2x^2-x+4}{x\left(x^2+4\right)} = \frac{A}{x}\cdot\frac{x^2+4}{x^2+4}+\frac{Bx+C}{x^2+4}\cdot\frac{x}{x}$$

$$\frac{2x^2-x+4}{x\left(x^2+4\right)} = \frac{A}{x}\cdot\frac{x^2+4}{x^2+4}+\frac{Bx+C}{x^2+4}\cdot\frac{x}{x}$$

$$\frac{2x^2-x+4}{x\left(x^2+4\right)} = \frac{A\left(x^2+4\right)}{x\left(x^2+4\right)}+\frac{x\left(Bx+C\right)}{x\left(x^2+4\right)}$$

$$\frac{2x^2-x+4}{x\left(x^2+4\right)} = \frac{Ax^2+4A}{x\left(x^2+4\right)}+\frac{Bx^2+Cx}{x\left(x^2+4\right)}$$

$$2x^2-x+4 = Ax^2+4A+Bx^2+Cx$$

$$2x^2-x+4 = \left(A+B\right)x^2+Cx+4A$$

$$\downarrow$$

$$A+B=2$$

$$C=-1$$

$$4A=4$$

$$\downarrow$$

$$A=1$$

$$B=1$$

$$C=-1$$

$$\int \frac{2x^2-x+4}{x\left(x^2+4\right)}\,dx = \int \frac{1}{x}+\frac{x-1}{x^2+4}\,dx$$

$$= \int \frac{1}{x}+\frac{x}{x^2+4}-\frac{1}{x^2+4}\,dx$$

$$= \int \frac{1}{x}\,dx+\int \frac{x}{x^2+4}\,dx-\int \frac{1}{x^2+4}\,dx$$

$$= \int \frac{1}{x}\,dx+\frac{1}{2}\int \frac{1}{x^2+4}\cdot 2x\,dx-\int \frac{1}{x^2+\left(2\right)^2}\,dx$$

$$= \ln\left|x\right|+\frac{1}{2}\ln\left|x^2+4\right|-\frac{1}{2}\arctan\left(\frac{x}{2}\right)+C$$

$$\int \frac{-x^3+2x^2-x+1}{x\left(x^2+1\right)^2}\,dx$$

$$\frac{-x^3+2x^2-x+1}{x\left(x^2+1\right)^2}=\frac{A}{x}+\frac{Bx+C}{x^2+1}+\frac{Dx+E}{\left(x^2+1\right)^2}$$

$$\frac{-x^3+2x^2-x+1}{x\left(x^2+1\right)^2}=\frac{A}{x}\cdot\frac{\left(x^2+1\right)^2}{\left(x^2+1\right)^2}+\frac{Bx+C}{x^2+1}\cdot\frac{x^2+1}{x^2+1}+\frac{Dx+E}{\left(x^2+1\right)^2}\cdot\frac{x}{x}$$

$$\frac{-x^3+2x^2-x+1}{x\left(x^2+1\right)^2}=\frac{A\left(x^2+1\right)^2}{x\left(x^2+1\right)^2}+\frac{\left(Bx+C\right)\left(x^2+1\right)}{x\left(x^2+1\right)^2}+\frac{x\left(Dx+E\right)}{x\left(x^2+1\right)^2}$$

$$\frac{-x^3+2x^2-x+1}{x\left(x^2+1\right)^2}=\frac{A\left(x^4+2x^2+1\right)}{x\left(x^2+1\right)^2}+\frac{Bx^3+Cx^2+Bx+C}{x\left(x^2+1\right)^2}+\frac{Dx^2+Ex}{x\left(x^2+1\right)^2}$$

$$\frac{-x^3+2x^2-x+1}{x\left(x^2+1\right)^2}=\frac{Ax^4+2Ax^2+A}{x\left(x^2+1\right)^2}+\frac{Bx^3+Cx^2+Bx+C}{x\left(x^2+1\right)^2}+\frac{Dx^2+Ex}{x\left(x^2+1\right)^2}$$

$$-x^3+2x^2-x+1=Ax^4+2Ax^2+A+Bx^3+Cx^2+Bx+C+Dx^2+Ex$$

$$-x^3+2x^2-x+1=Ax^4+Bx^3+\left(2A+C+D\right)x^2+\left(B+E\right)x+\left(A+C\right)$$

$$\downarrow$$

$$0=A$$

$$-1=B$$

$$2=2A+C+D$$

$$-1=B+E$$

$$1=A+C$$

$$\downarrow$$

$$A=0$$

$$B=-1$$

$$C=1$$

$$D=1$$

$$E=0$$

$$\frac{-x^3+2x^2-x+1}{x\left(x^2+1\right)^2}=\frac{1}{x}-\frac{x+1}{x^2+1}+\frac{x}{\left(x^2+1\right)^2}$$

$$\int\frac{-x^3+2x^2-x+1}{x\left(x^2+1\right)^2}dx=\int\frac{1}{x}-\frac{x+1}{x^2+1}+\frac{x}{\left(x^2+1\right)^2}dx$$

$$\int\frac{-x^3+2x^2-x+1}{x\left(x^2+1\right)^2}dx=\int\frac{1}{x}dx-\int\frac{x+1}{x^2+1}dx+\int\frac{x}{\left(x^2+1\right)^2}dx$$

$$\int\frac{-x^3+2x^2-x+1}{x\left(x^2+1\right)^2}dx=\int\frac{1}{x}dx-\int\frac{x}{x^2+1}dx-\int\frac{1}{x^2+1}dx+\int\frac{x}{\left(x^2+1\right)^2}dx$$

$$\int\frac{-x^3+2x^2-x+1}{x\left(x^2+1\right)^2}dx=\int\frac{1}{x}dx-\frac{1}{2}\int\frac{1}{x^2+1}\cdot 2xdx-\int\frac{1}{x^2+1}dx+\frac{1}{2}\int\frac{1}{\left(x^2+1\right)^2}\cdot 2xdx$$

$$\int\frac{-x^3+2x^2-x+1}{x\left(x^2+1\right)^2}dx=\ln\left|x\right|-\frac{1}{2}\ln\left|x^2+1\right|-\arctan\left(x\right)-\frac{1}{2}\left(x^2+1\right)^{-1}+C$$