Section 11-2 Homework Solutions

#17 $3-4+\frac{16}{3}-\frac{64}{9}+\cdots$ does not converge. The common ratio is $-\frac{4}{3}$ and $\left|-\frac{4}{3}\right|>1$

#18 $4+3+\frac{9}{4}+\frac{27}{16}+\cdots=\frac{\text{first term}}{1-\text{common ratio}}=\frac{4}{1-\frac{3}{4}}=16$

#19 $10-2+0.4-0.08+\cdots=10-2+\frac{2}{5}-\frac{2}{25}+\cdots=\frac{\text{first term}}{1-\text{common ratio}}=\frac{10}{1-\left(-\frac{2}{5}\right)}=\frac{50}{7}=7.1428...$

#20 $2+0.5+0.125+0.03125+\cdots=2+\frac{1}{2}+\frac{1}{8}+\frac{1}{32}\cdots=\frac{\text{first term}}{1-\text{common ratio}}=\frac{1}{1-\left(\frac{1}{4}\right)}=\frac{8}{3}=2.6666...$

#21 $\sum_{n=1}^{\infty}6(0.9)^{n-1}=\frac{\text{first term}}{1-\text{common ratio}}=\frac{6}{1-0.9}=60$

#22 $\sum_{n=1}^{\infty}\frac{10^n}{(-9)^{n-1}}=\sum_{n=1}^{\infty}\frac{10\cdot10^{n-1}}{(-9)^{n-1}}=\sum_{n=1}^{\infty}10\cdot\left(\frac{10^{n-1}}{(-9)^{n-1}}\right)=\sum_{n=1}^{\infty}10\cdot\left(-\frac{10}{9}\right)^{n-1}$

this geometric series does not converge since $\left|-\frac{10}{9}\right|>1$

#23 $\sum_{n=1}^{\infty}\frac{(-3)^{n-1}}{4^n}=\sum_{n=1}^{\infty}\frac{(-3)^{n-1}}{4\cdot4^{n-1}}=\sum_{n=1}^{\infty}\frac{1}{4}\cdot\frac{(-3)^{n-1}}{4^{n-1}}=\sum_{n=1}^{\infty}\frac{1}{4}\cdot\left(-\frac{3}{4}\right)^{n-1}=\frac{\text{first term}}{1-\text{common ratio}}=\frac{\left(\frac{1}{4}\right)}{1-\left(-\frac{3}{4}\right)}=\frac{1}{7}$

#24 $\sum_{n=0}^{\infty}\frac{1}{\left(\sqrt{2}\right)^n}=\sum_{n=0}^{\infty}\left(\frac{1}{\sqrt{2}}\right)^n=\frac{\text{first term}}{1-\text{common ratio}}=\frac{1}{1-\left(\frac{1}{\sqrt{2}}\right)}=\sqrt{2}+2$

#25 $\sum_{n=0}^{\infty}\frac{\pi^n}{3^{n+1}}=\sum_{n=0}^{\infty}\frac{\pi^n}{3\cdot3^n}=\sum_{n=0}^{\infty}\frac{1}{3}\cdot\left(\frac{\pi^n}{3^n}\right)=\sum_{n=0}^{\infty}\frac{1}{3}\cdot\left(\frac{\pi}{3}\right)^n$

this geometric series does not converge since $\left|\frac{\pi}{3}\right|>1$

#26

$$\sum_{n=1}^{\infty}\frac{e^n}{3^{n-1}}=\sum_{n=1}^{\infty}\frac{e\cdot e^{n-1}}{3^{n-1}}=\sum_{n=1}^{\infty}e\cdot\left(\frac{e^{n-1}}{3^{n-1}}\right)=\sum_{n=1}^{\infty}e\cdot\left(\frac{e}{3}\right)^{n-1}=\frac{\text{first term}}{1-\text{common ratio}}=\frac{e}{1-\left(\frac{e}{3}\right)}=-\frac{3e}{e-3}=28.9468...$$

#27 $\frac{1}{3}+\frac{1}{6}+\frac{1}{9}+\frac{1}{12}+\frac{1}{15}+\cdots=\sum_{n=1}^{\infty}\frac{1}{3n}=\sum_{n=1}^{\infty}\frac{1}{3}\cdot\frac{1}{n}=\frac{1}{3}\cdot\left[\sum_{n=1}^{\infty}\frac{1}{n}\right]$ does not converge because it is a multiple of the harmonic series.

#28

$$\begin{aligned}\frac{1}{3}+\frac{2}{9}+\frac{1}{27}+\frac{2}{81}+\frac{1}{243}+\frac{2}{729}+\cdots&=\left[\frac{1}{3}+\frac{1}{27}+\frac{1}{243}+\cdots\right]+\left[\frac{2}{9}+\frac{2}{81}+\frac{2}{729}+\cdots\right]\\&=\left[\sum_{n=0}^{\infty}\frac{1}{3}\cdot\left(\frac{1}{9}\right)^n\right]+2\cdot\left[\frac{1}{9}+\frac{1}{81}+\frac{1}{729}+\cdots\right]\\&=\frac{1}{3}\cdot\left[\sum_{n=0}^{\infty}\left(\frac{1}{9}\right)^n\right]+2\cdot\left[\sum_{n=1}^{\infty}\left(\frac{1}{9}\right)^n\right]\\&=\frac{1}{3}\cdot\frac{1}{1-\left(\frac{1}{9}\right)}+2\cdot\frac{\left(\frac{1}{9}\right)}{1-\left(\frac{1}{9}\right)}\\&=\frac{5}{8}\end{aligned}$$

#29 $\sum_{n=1}^{\infty}\frac{n-1}{3n-1}$ does not converge by the limit of the n^{th} term test: $\lim_{n\to\infty}\frac{n-1}{3n-1}\sim\lim_{n\to\infty}\frac{n}{3n}=\frac{1}{3}\neq 0$

#30 $\sum_{k=1}^{\infty}\frac{k(k+2)}{(k+3)^2}$ does not converge by the limit of the n^{th} term test:

$\lim_{k\to\infty}\frac{k(k+2)}{(k+3)^2}\sim\lim_{k\to\infty}\frac{k^2}{k^2}=1\neq 0$

#31

$$\sum_{n=1}^{\infty}\frac{1+2^n}{3^n}=\sum_{n=1}^{\infty}\left[\frac{1}{3^n}+\frac{2^n}{3^n}\right]=\left[\sum_{n=1}^{\infty}\frac{1}{3^n}\right]+\left[\sum_{n=1}^{\infty}\frac{2^n}{3^n}\right]=\left[\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n\right]+\left[\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n\right]=\frac{\left(\frac{1}{3}\right)}{1-\left(\frac{1}{3}\right)}+\frac{\left(\frac{2}{3}\right)}{1-\left(\frac{2}{3}\right)}=\frac{5}{2}$$

#32 $\sum_{n=1}^{\infty}\frac{1+3^n}{2^n}$ does not converge by the limit of the n^{th} term test:

$$\lim_{n\to\infty}\frac{1+3^n}{2^n}\sim\lim_{n\to\infty}\frac{3^n}{2^n}=\lim_{n\to\infty}\left(\frac{3}{2}\right)^n\to\infty$$

#33 $\sum_{n=1}^{\infty}\sqrt[n]{2}$ does not converge by the limit of the n^{th} term test: $\lim_{n\to\infty}\sqrt[n]{2}=1$

#34 $\sum_{n=1}^{\infty}\left[(0.8)^{n-1}-(0.3)^n\right]=\left[\sum_{n=1}^{\infty}(0.8)^{n-1}\right]-\left[\sum_{n=1}^{\infty}(0.3)^n\right]=\frac{1}{1-0.8}-\frac{0.3}{1-0.3}=\frac{32}{7}$

#35 $\sum_{n=1}^{\infty}\ln\left(\frac{n^2+1}{2n^2+1}\right)$ does not converge by the limit of the n^{th} term test:

$$\left[\ln\left(\frac{n^2+1}{2n^2+1}\right)\right]=\ln\left(\lim_{n\to\infty}\left[\frac{n^2+1}{2n^2+1}\right]\right)\sim\ln\left(\lim_{n\to\infty}\left[\frac{n^2}{2n^2}\right]\right)=\ln\left(\frac{1}{2}\right)\neq 0$$

#36 $\sum_{n=1}^{\infty}\frac{1}{1+\left(\frac{2}{3}\right)^n}$ does not converge by the limit of the n^{th} term test: $\lim_{n\to\infty}\left[\frac{1}{1+\left(\frac{2}{3}\right)^n}\right]=1$

#37 $\sum_{k=0}^{\infty}\left(\frac{\pi}{3}\right)^k$ is a geometric series that does not converge since $\left|\frac{\pi}{3}\right|>1$

#38 $\sum_{k=1}^{\infty}\left(\cos(1)\right)^k$ is a geometric series with $\left|\cos(1)\right|<1$ so $\sum_{k=1}^{\infty}\left(\cos(1)\right)^k=\frac{\cos(1)}{1-\cos(1)}$

#39 $\sum_{n=1}^{\infty}\arctan(n)$ does not converge by the limit of the n^{th} term test: $\lim_{n\to\infty}\left[\arctan(n)\right]=\frac{\pi}{2}\neq 0$ t

#40 $\sum_{n=1}^{\infty}\left(\frac{3}{5^n}+\frac{2}{n}\right)=\left[\sum_{n=1}^{\infty}\left(\frac{3}{5^n}\right)\right]+\left[\sum_{n=1}^{\infty}\frac{2}{n}\right]=\left[3\cdot\sum_{n=1}^{\infty}\left(\frac{1}{5}\right)^n\right]+\left[2\cdot\sum_{n=1}^{\infty}\frac{1}{n}\right]$ will not converge because $3\cdot\sum_{n=1}^{\infty}\left(\frac{1}{5}\right)^n$ is a geometric series that will converge since $\left|\frac{1}{5}\right|<1$ but $2\cdot\sum_{n=1}^{\infty}\frac{1}{n}$ is a multiple of the harmonic series which goes to ∞.

#42 $\sum_{n=1}^{\infty}\frac{e^n}{n^2}$ does not converge by the limit of the n^{th} term test: $\lim_{n\to\infty}\frac{e^n}{n^2}\to\infty$.

#53

$$
\begin{aligned}
2.\overline{516} &= 2 + 0.516 + 000516 + 000000516 + \cdots \\
&= 2 + 516 \cdot 10^{-3} + 516 \cdot 10^{-6} + 516 \cdot 10^{-9} + \cdots \\
&= 2 + 516 \cdot 10^{-3} + 516 \cdot \left(10^{-3}\right)^2 + 516 \cdot \left(10^{-3}\right)^3 + \cdots \\
&= 2 + 516 \cdot \left[10^{-3} + \left(10^{-3}\right)^2 + \left(10^{-3}\right)^3 + \cdots\right] \\
&= 2 + 516\left[\sum_{n=1}^{\infty} (0.001)^n\right] \\
&= 2 + 516\left[\frac{0.001}{1-0.001}\right] \\
&= \frac{838}{333}
\end{aligned}
$$

#54

$$
\begin{aligned}
10.1\overline{35} &= 10.1 + 0.035 + 0.00035 + 0.0000035 + \cdots \\
&= 10.1 + 35 \cdot 10^{-3} + 35 \cdot 10^{-5} + 35 \cdot 10^{-7} + \cdots \\
&= 10.1 + 3.5 \cdot 10^{-2} + 3.5 \cdot 10^{-4} + 3.5 \cdot 10^{-6} + \cdots \\
&= 10.1 + 3.5 \cdot 10^{-2} + 3.5 \cdot \left(10^{-2}\right)^2 + 3.5 \cdot \left(10^{-2}\right)^3 + \cdots \\
&= 10.1 + \sum_{n=1}^{\infty} 3.5\left(10^{-2}\right)^n \\
&= 10.1 + \frac{0.035}{1-10^{-2}} \\
&= \frac{5017}{495}
\end{aligned}
$$

For the following, you need to use the fact that a geometric series will converge if $|\text{common ratio}| < 1$

#57

$$\sum_{n=1}^{\infty}(-5)^n x^n = \sum_{n=1}^{\infty}(-5x)^n$$

This will converge if

$$\begin{aligned}
|-5x| &< 1 \\
|-5| \cdot |x| &< 1 \\
5|x| &< 1 \\
|x| &< \frac{1}{5} \\
-\frac{1}{5} < x &< \frac{1}{5}
\end{aligned}$$

The sum of the series for $-\frac{1}{5} < x < \frac{1}{5}$ is $\frac{\text{first term}}{1-\text{common ratio}} = \frac{(-5x)}{1-(-5x)} = \frac{-5x}{1+5x}$

#58

$\sum_{n=1}^{\infty}(x+2)^n$

This will converge if

$$\begin{gathered}
|x+2| < 1 \\
\downarrow \\
-1 < x+2 < 1 \\
-3 < x < -1
\end{gathered}$$

The sum of the series for $-3 < x < 1$ is $\frac{\text{first term}}{1-\text{common ratio}} = \frac{(x+2)}{1-(x+2)} = \frac{x+2}{-1-x}$

#59

$$\sum_{n=0}^{\infty}\frac{(x-2)^n}{3^n}=\sum_{n=0}^{\infty}\left(\frac{x-2}{3}\right)^n$$

This will converge if

$$\left|\frac{x-2}{3}\right|<1$$

$$\downarrow$$

$$-1<\frac{x-2}{3}<1$$

$$-3<x-2<3$$

$$-1<x<5$$

The sum of the series for $-1<x<5$ is $\dfrac{\text{first term}}{1-\text{common ratio}}=\dfrac{1}{1-\left(\dfrac{x-2}{3}\right)}=\dfrac{3}{5-x}$

#60

$$\sum_{n=0}^{\infty}(-4)^n(x-5)^n=\sum_{n=0}^{\infty}\left[-4(x-5)\right]^n$$

This will converge if

$$|-4(x-5)|<1$$

$$4\cdot|x-5|<1$$

$$|x-5|<\frac{1}{4}$$

$$\downarrow$$

$$-\frac{1}{4}<x-5<\frac{1}{4}$$

$$\frac{19}{4}<x<\frac{21}{4}$$

The sum of the series for $\dfrac{19}{4}<x<\dfrac{21}{4}$ is given by $\dfrac{\text{first term}}{1-\text{common ratio}}=\dfrac{1}{1-(-4(x-5))}=\dfrac{1}{4x-19}$

#61

$$\sum_{n=0}^{\infty}\frac{2^n}{x^n}=\sum_{n=0}^{\infty}\left(\frac{2}{x}\right)^n$$

This will converge if

$$\left|\frac{2}{x}\right|<1$$

$$\frac{2}{|x|}<1$$

$$2<|x|$$

$$\downarrow$$

$$x<-2 \text{ or } x>2$$

The sum of the series for $x<-2$ or $x>2$ is given by $\dfrac{\text{first term}}{1-\text{common ratio}}=\dfrac{1}{1-\left(\dfrac{2}{x}\right)}=\dfrac{x}{x-2}$

#62

$$\sum_{n=0}^{\infty}\frac{\sin^n(x)}{3^n}=\sum_{n=0}^{\infty}\left(\frac{\sin(x)}{3}\right)^n$$

This will converge if

$$\left|\frac{\sin(x)}{3}\right|<1$$

$$\frac{|\sin(x)|}{3}<1$$

$$|\sin(x)|<3$$

This inequality is true for all real numbers.

The sum of the series will be $\dfrac{\text{first term}}{1-\text{common ratio}}=\dfrac{1}{1-\dfrac{\sin(x)}{3}}=\dfrac{3}{3-\sin(x)}$

#63

$\sum_{n=0}^{\infty} e^{nx} = \sum_{n=0}^{\infty} \left(e^{x}\right)^{n}$

This will converge if

$$\begin{gathered} \left|e^{x}\right| < 1 \\ e^{x} < 1 \\ \ln\left(e^{x}\right) < \ln\left(1\right) \\ x < 0 \end{gathered}$$

The sum of the series, for $x < 0$ will be given by $\dfrac{\text{first term}}{1 - \text{common ratio}} = \dfrac{1}{1 - e^{x}}$