A **power series** is a series of the form $\sum_{n=0}^{\infty} a_n (x-c)^n$.

a_n are the coefficients of the terms of the power series.

$(x-c)^n \rightarrow$ the power series is centered at $x = c$. The value of the constant *c* must be given.

We can now define a function $f(x)$ by a power series, so long as the power series converges for the value of *x* we input to $f(x)$.

That is $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$, provided $\sum_{n=0}^{\infty} a_n (x-c)^n$ converges.

Consider $f(x) = \sum_{n=0}^{\infty} \frac{(x-2)^n}{3^n}$

$$f(x) = \sum_{n=0}^{\infty} \frac{(x-2)^n}{3^n} = \sum_{n=0}^{\infty} \left(\frac{(x-2)}{3} \right)^n \text{ is a Geometric Series.}$$

This series will converge so long as

$$\left| \frac{x-2}{3} \right| < 1$$

$$\frac{|x-2|}{3} < 1 \rightarrow \begin{array}{c} -3 < x-2 < 3 \\ -1 < x < 5 \end{array}$$

$$|x-2| < 3$$

Therefore the domain of $f(x) = \sum_{n=0}^{\infty} \frac{(x-2)^n}{3^n}$ is $-1 < x < 5$. Since this is a Geometric Series, we can also determine the value of $f(x)$ in this interval. That is

$$f(1) = \sum_{n=0}^{\infty} \frac{(1-2)^n}{3^n} = \sum_{n=0}^{\infty} \left(-\frac{1}{3} \right)^n = \frac{1}{1 - \left(-\frac{1}{3} \right)} = \frac{3}{4}$$

For any given power series $\sum_{n=0}^{\infty} a_n (x-c)^n$, there are three scenarios for which the power series will converge:

I. The series will converge only at $x = c$.

II. The series will converge for all real numbers *x*.

III. The series will converge for a certain interval around $x = c$.

   a. The series will converge for $|x-c| < R$

   b. The series will not converge for $|x-c| > R$

The number ***R*** above is called the **radius of convergence**. In the example above, the radius of convergence is 3. **The series is centered at $x = 2$.**

To find the radius of convergence of any power series you must use either:

I. Ratio Test
II. Root Test

Note:

x ***is considered a constant when testing if the limit converges or not.***

c ***is a given constant.***

n ***is the only term that is changing when taking the limit.***

Consider $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$. Using the ratio test we get:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(x-3)^{n+1}}{n+1}}{\frac{(x-3)^n}{n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(x-3)^{n+1}}{n+1}\cdot\frac{n}{(x-3)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{n}{n+1}\cdot\frac{\cancel{(x-3)^n}(x-3)}{\cancel{(x-3)^n}}\right|$$

$$= |x-3|$$

This power series is centered at $x=3$.

So long as $|x-3|<1$ the series will converge.

Therefore, the radius of convergence of the power series is $R=1$.

Therefore we can conclude that the power series will converge for all x, where $2<x<4$.

To find the ***interval of convergence*** we must test for convergence when $x=2$ and $x=4$.

$x=2$	$x=4$
$\sum_{n=0}^{\infty}\frac{(2-3)^n}{n} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n}$ By the Alternating Series Test $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$	$\sum_{n=1}^{\infty}\frac{(4-3)^n}{n} = \sum_{n=1}^{\infty}\frac{(1)^n}{n}$ $= \sum_{n=1}^{\infty}\frac{1}{n}$ This is the Harmonic Series, which diverges.

Therefore, ***interval of convergence*** is $2 \le x < 4$.

To find the interval of convergence:

1) Determine the center of the power series, $x=c$.
2) Use the Ratio Test or Root test to determine the radius of convergence, *R*. The power series will converge for $|x-c|<R$.
3) Test each of the endpoints of the open interval above to determine the interval of convergence.

$$|x-c|<R$$
$$-R<x-c<R$$
$$-R+c<x<R+c$$

   a. Test the series for convergence with *x* replaced by the value of $R+c$
   b. Test the series for convergence with *x* replaced by the value of $-R+c$

Example: $\sum_{n=0}^{\infty} n!x^n$

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(n+1)!x^{n+1}}{n!x^n}\right| \\ &= \lim_{n\to\infty}\left|\frac{(n+1)\cdot \cancel{n!}\,\cancel{x^n}\cdot x}{\cancel{n!}\,\cancel{x^n}}\right| \\ &= \lim_{n\to\infty}\left|(n+1)x\right| \\ &\to \infty\end{aligned}$$

$\sum_{n=0}^{\infty} n!x^n$ will not converge for any x other than $x = 0$ (the center of the power series).

Example: $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{\dfrac{(-1)^{n+1}x^{2(n+1)}}{2^{2(n+1)}\left([n+1]!\right)^2}}{\dfrac{(-1)^n x^{2n}}{2^{2n}(n!)^2}}\right| \\ &= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{2(n+1)}}{2^{2(n+1)}\left([n+1]!\right)^2}\cdot\frac{2^{2n}(n!)^2}{(-1)^n x^{2n}}\right| \\ &= \lim_{n\to\infty}\left|\frac{x^{2n+2}}{2^{2n+2}\left([n+1]\cdot n!\right)^2}\cdot\frac{2^{2n}(n!)^2}{x^{2n}}\right| \\ &= \lim_{n\to\infty}\left|\frac{\cancel{x^{2n}}\cdot x^2}{\cancel{2^{2n}}\cdot 2^2\cdot(n+1)^2\cdot\cancel{(n!)^2}}\cdot\frac{\cancel{2^{2n}}\,\cancel{(n!)^2}}{\cancel{x^{2n}}}\right| \\ &= \lim_{n\to\infty}\left|\frac{x^2}{2^2\cdot(n+1)^2}\right|\end{aligned}$$

Since x is a fixed value, and $n\to\infty$ $\lim_{n\to\infty}\left|\frac{x^2}{2^2\cdot(n+1)^2}\right| = 0$ for all x.

Therefore, the power series converges for all real numbers x.

Tips for Interval and Radius of Convergence

Let the result of the ratio/root test be $\lim_{n\to\infty}\left|\cdots\right|$

Case 1: If $\lim_{n\to\infty}\left|\cdots\right| = 0$, the radius of convergence will be ∞, interval of convergence will be all real numbers

Case 2: If $\lim_{n\to\infty}\left|\cdots\right| \to \infty$, the radius of convergence will be zero, and the series will converge only at $x = c$.

Case 3: The radius of convergence will be some finite interval if $\lim_{n\to\infty}\left|\cdots\right| = k$, where $k > 0$.

Solve $\left|\cdots\right| < k$ to be of the form $a < x < b$

The radius of convergence will be $\frac{b-a}{2}$

The series will be centered at $x = \frac{a+b}{2}$