Definite Integral in Series Notation Irregular Partition

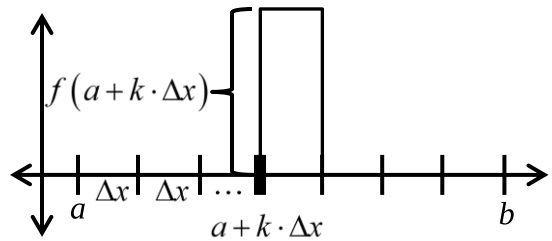
$$\int_a^b f(x)\,dx = \lim_{\|\Delta\|\to 0} \sum_{i=0}^{n} \left(f(c_i)\right)(\Delta x)_i$$

Definite Integral as a Riemann Sum – Uniform Partition

Subintervals of length $\Delta x = \dfrac{b-a}{n}$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=0}^{n} \left(f(a + k\cdot\Delta x)\right)\Delta x$$

$$= \lim_{n\to\infty} \sum_{k=0}^{n} \left(f\left(a + k\cdot\left[\frac{b-a}{n}\right]\right)\right)\left[\frac{b-a}{n}\right]$$



When approaching multiple choice questions, there are two components of the structure of the summand that will guide you in choosing the correct answer and eliminating incorrect choices.

1. $\boxed{\dfrac{b-a}{n} \leftrightarrow \Delta x}$

   a. Example: $\int_3^5 x^2 dx$ will have structure that looks like $\lim_{n\to\infty} \sum_{i=1}^{n} [\text{something}]\cdot \boxed{\dfrac{2}{n}}$

   b. Example: $\lim_{n\to\infty} \sum_{k=0}^{n} [\text{something}]\cdot \dfrac{4}{n}$ will represent a definite integral of length 4.

2. $\boxed{f\left(a + k\cdot\left[\dfrac{b-a}{n}\right]\right)}$ will inform you of the lower bound of the integral, i.e. "*a*".

   Each *x* in the integral expression will be replaced with $\left[a + k\left(\dfrac{b-a}{n}\right)\right]$

   a. Example: $\lim_{n\to\infty} \sum_{k=0}^{n} \left(2 + \underbrace{\left[3 + \frac{2k}{n}\right]^2}_{x^2} \right)\cdot \dfrac{2}{n} \longleftarrow$ length of interval is 2

   $$3 + k\left(\frac{2}{n}\right) \to a = 3 \dashrightarrow \int_3^{3+2} 2 + x^2 dx$$

   b. Example: $\int_4^7 \sqrt{\boxed{\boxed{x}} + 1}\,dx \to \lim_{n\to\infty} \sum_{k=0}^{n} \left[\sqrt{\boxed{\boxed{\left(4 + k\left(\frac{3}{n}\right)\right)}} + 1}\,\right]\cdot \dfrac{3}{n}$

1. Which of the following is equal to $\int_3^5 x^4 dx$?

(a) $\cancel{\lim_{n\to\infty}\sum_{k=1}^{n}\left(3+\frac{k}{n}\right)^4\cdot\frac{1}{n}}$ (b) $\lim_{n\to\infty}\sum_{k=1}^{n}\left(3+\frac{k}{n}\right)^4\cdot\frac{2}{n}$ (c) $\cancel{\lim_{n\to\infty}\sum_{k=1}^{n}\left(3+\frac{2k}{n}\right)^4\cdot\frac{1}{n}}$ (d) $\lim_{n\to\infty}\sum_{k=1}^{n}\left(3+\frac{2k}{n}\right)^4\cdot\frac{2}{n}$

I. $\Delta x=\frac{5-3}{n}=\frac{2}{n}$. This eliminates (a) and (c)

II. Since 3 is the lower bound, $x\to 3+k\left(\frac{2}{n}\right)$

$$\int_3^5 \boxed{\boxed{x^4}}dx \Rightarrow \quad x^4\to\left(3+k\left(\frac{2}{n}\right)\right)^4=\left(3+\frac{2k}{n}\right)^4 \qquad \Rightarrow \text{Answer is (d)}$$

2. If n is a positive integer, then $\lim_{n\to\infty}\frac{1}{n}\left[\frac{1}{1+\frac{1}{n}}+\frac{1}{1+\frac{2}{n}}+\cdots+\frac{1}{1+\frac{n}{n}}\right]$ can be expressed as

(a) $\int_0^1\frac{1}{x}dx$ (b) $\int_1^2\frac{1}{1+x}dx$ (c) $\int_1^2 x dx$ (d) $\int_1^2\frac{2}{x+1}dx$ (e) $\int_1^2\frac{1}{x}dx$

$\Delta x=\frac{b-a}{n}=\frac{1}{n}$, so the length of the interval = 1. Which isn't useful

(a) $\int_0^1\frac{1}{x}dx$ Not correct $x\to 0+k\left(\frac{1}{n}\right)$ and $\frac{1}{\left[0+k\left(\frac{1}{n}\right)\right]}\not\to\frac{1}{1+\frac{k}{n}}$

(b) $\int_1^2\frac{1}{1+x}dx$ Not correct $x\to 1+k\left(\frac{1}{n}\right)$ and $\frac{1}{1+\left[1+k\left(\frac{1}{n}\right)\right]}\not\to\frac{1}{1+\frac{k}{n}}$

(c) $\int_1^2 xdx$ Not correct Can be eliminated because $x\not\to\frac{1}{1+\frac{k}{n}}$

(d) $\int_1^2\frac{2}{x+1}dx$ Not correct Can be eliminated because of the factor of 2 in the numerator

(e) $\int_1^2\frac{1}{x}dx$ $x\to 1+k\left(\frac{1}{n}\right)$ $\int_1^2\boxed{\frac{1}{x}}dx\Rightarrow\lim_{n\to\infty}\sum_{k=1}^{n}\boxed{\frac{1}{\left(1+\frac{k}{n}\right)}}\cdot\frac{1}{n}$

3. A solid has a rectangular base that lies in the first quadrant and is bounded by the x-axis, y-axis, the line $x=2$, and the line $y=1$. The height of the solid above the point (x,y), is $1+3x$. Which of the following is a Riemann sum approximation for the volume of the solid?

(a) $\sum_{i=1}^{n}\frac{1}{n}\left(1+\frac{3i}{n}\right)$ (b) $2\sum_{i=1}^{n}\frac{1}{n}\left(1+\frac{3i}{n}\right)$ (c) $2\sum_{i=1}^{n}\frac{i}{n}\left(1+\frac{3i}{n}\right)$ (d) $\sum_{i=1}^{n}\frac{2}{n}\left(1+\frac{6i}{n}\right)$ (e) $\sum_{i=1}^{n}\frac{2i}{n}\left(1+\frac{6i}{n}\right)$

4. If n is a positive integer, then $\lim_{n\to\infty}\frac{1}{n}\left[\frac{1}{1+\left(\frac{n+1}{n}\right)^2}+\frac{1}{1+\left(\frac{n+2}{n}\right)^2}+\cdots+\frac{1}{1+\left(\frac{n+n}{n}\right)^2}\right]$ could be expressed as

(a) $\int_1^2\frac{1}{x\left(x^2+1\right)}dx$ (b) $\int_1^2\frac{1}{x^2+1}dx$ (c) $\int_0^2\frac{1}{x\left(x^2+1\right)}dx$ (d) $\int_0^2\frac{1}{x^2+1}dx$ (e) $\int_0^1\frac{1}{\left(\frac{1}{x}\right)^2+1}dx$

5. If n is a positive integer, then $\lim_{n\to\infty}\frac{1}{n}\left[\cos\left(\frac{\pi}{n}\right)+\cos\left(\frac{2\pi}{n}\right)+\cdots+\cos\left(\frac{n\pi}{n}\right)\right]$ could be expressed as

(a) $\int_0^1\cos\left(\frac{\pi}{n}\right)dx$ (b) $\pi\int_0^1\cos(x)\,dx$ (c) $\int_0^1\sin(\pi x)\,dx$ (d) $\int_0^1\cos(\pi x)\,dx$ (e) $\int_0^\pi\cos(\pi x)\,dx$

6. If n is a positive integer, then $\lim_{n\to\infty}\frac{1}{n}\left[\sin\left(\frac{\pi}{n}\right)+\sin\left(\frac{2\pi}{n}\right)+\cdots+\sin\left(\frac{n\pi}{n}\right)\right]$ is

(a) 0 (b) 2 (c) $\frac{\pi}{2}$ (d) 2π (e) $\frac{2}{\pi}$

7. The expression $\frac{1}{20}\left[\left(\frac{1}{20}\right)^2+\left(\frac{2}{20}\right)^2+\left(\frac{3}{20}\right)^2+\cdots+\left(\frac{20}{20}\right)^2\right]$ is a Riemann sum approximation of

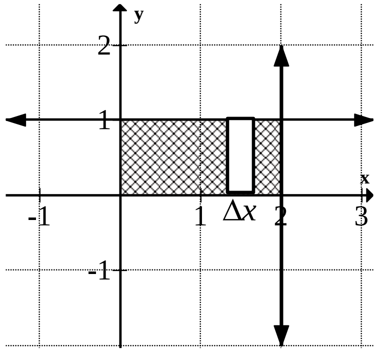
(a) $\int_0^1 x^2dx$ (b) $\frac{1}{20}\int_0^1 x^2dx$ (c) $\frac{1}{20}\int_0^1\left(\frac{x}{20}\right)^2dx$ (d) $\int_0^1\left(\frac{x}{20}\right)^2dx$ (e) $\frac{1}{20}\int_1^{20}\left(\frac{x}{20}\right)^2dx$

8. The expression $\frac{1}{75}\left[\ln\left(\frac{76}{75}\right)+\ln\left(\frac{77}{75}\right)+\ln\left(\frac{78}{75}\right)+\cdots+\ln(2)\right]$ is a Riemann sum approximation for

(a) $\int_1^2\ln\left(\frac{x}{75}\right)dx$ (b) $\int_{76}^{150}\ln\left(\frac{x}{75}\right)dx$ (c) $\frac{1}{75}\int_{76}^{100}\ln(x)\,dx$ (d) $\int_1^2\ln(x)\,dx$ (e) $\frac{1}{75}\int_1^2\ln(x)\,dx$

Solutions

3. A solid has a rectangular base that lies in the first quadrant and is bounded by the x-axis, y-axis, the line $x = 2$, and the line $y = 1$. The height of the solid above the point (x, y), is $1+3x$. Which of the following is a Riemann sum approximation for the volume of the solid?



$$V_{\text{slice}} = (1+3x)\cdot(1)\cdot\Delta x$$

$$V_{\text{solid}} = \int_0^2 (1+3x)\,dx$$

$\Delta x = \dfrac{2-0}{n} = \dfrac{2}{n}$ This eliminates answer choices A, C and E.

$x \to 0 + k\left(\dfrac{2}{n}\right)$ therefore the integrand should be of the form $1+3\left[0+k\left(\dfrac{2}{n}\right)\right] = 1+\dfrac{6k}{n}$, making the correct answer choice (d)

(a) $\displaystyle\sum_{i=1}^{n} \frac{1}{n}\left(1+\frac{3i}{n}\right)$

(b) $\displaystyle 2\sum_{i=1}^{n} \frac{1}{n}\left(1+\frac{3i}{n}\right)$

(c) $\displaystyle 2\sum_{i=1}^{n} \frac{i}{n}\left(1+\frac{3i}{n}\right)$

(d) $\displaystyle\sum_{i=1}^{n} \frac{2}{n}\left(1+\frac{6i}{n}\right)$

(e) $\displaystyle\sum_{i=1}^{n} \frac{2i}{n}\left(1+\frac{6i}{n}\right)$

4. If n is a positive integer, then $\lim\limits_{n\to\infty}\frac{1}{n}\left[\frac{1}{1+\left(\frac{n+1}{n}\right)^2}+\frac{1}{1+\left(\frac{n+2}{n}\right)^2}+\cdots+\frac{1}{1+\left(\frac{n+n}{n}\right)^2}\right]$ could be expressed as

$$\lim_{n\to\infty}\frac{1}{n}\left[\frac{1}{1+\left(1+\frac{1}{n}\right)^2}+\frac{1}{1+\left(1+\frac{2}{n}\right)^2}+\cdots+\frac{1}{1+\left(1+\frac{n}{n}\right)^2}\right]$$

The factor of $\frac{1}{n}$ indicates that the length of the definite integral is 1. This eliminates answer choices C and D.

The term $\left(1+k\left(\frac{1}{n}\right)\right)\leftrightarrow "x"$, therefore the lower bound must be 1, eliminating E.

$\frac{1}{1+\left(1+\frac{1}{n}\right)^2}\leftrightarrow\frac{1}{1+x^2}$, indicating that the integrand must be $\frac{1}{1+x^2}$. The answer is therefore B.

(a) $\int_1^2\frac{1}{x\left(x^2+1\right)}dx$ (b) $\int_1^2\frac{1}{x^2+1}dx$ (c) $\cancel{\int_0^2\frac{1}{x\left(x^2+1\right)}dx}$ (d) $\cancel{\int_0^2\frac{1}{x^2+1}dx}$ (e) $\cancel{\int_0^1\frac{1}{\left(\frac{1}{x}\right)^2+1}dx}$

5. If n is a positive integer, then $\lim\limits_{n\to\infty}\frac{1}{n}\left[\cos\left(\frac{\pi}{n}\right)+\cos\left(\frac{2\pi}{n}\right)+\cdots+\cos\left(\frac{n\pi}{n}\right)\right]$ could be expressed as

The factor of $\frac{1}{n}$ indicates that the definite integral is of length 1. Eliminating E.
Since each term involves cos, we can eliminate C.
"n" should not be part of the definite integral, eliminating A.
Since the term π cannot be factored out of the sum, B is eliminated.

(a) $\cancel{\int_0^1\cos\left(\frac{\pi}{n}\right)dx}$ (b) $\cancel{\pi\int_0^1\cos(x)\,dx}$ (c) $\cancel{\int_0^1\sin(\pi x)\,dx}$ (d) $\int_0^1\cos(\pi x)\,dx$ (e) $\cancel{\int_0^\pi\cos(\pi x)\,dx}$

6. If n is a positive integer, then $\lim_{n\to\infty}\frac{1}{n}\left[\sin\left(\frac{\pi}{n}\right)+\sin\left(\frac{2\pi}{n}\right)+\cdots+\sin\left(\frac{n\pi}{n}\right)\right]$ is

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\left[\sin\left(\frac{\pi}{n}\right)+\sin\left(\frac{2\pi}{n}\right)+\cdots+\sin\left(\frac{n\pi}{n}\right)\right]&=\lim_{n\to\infty}\sum_{k=1}^{n}\sin\left(\pi\left[0+k\left(\frac{1}{n}\right)\right]\right)\cdot\left(\frac{1}{n}\right)\\&=\int_0^1\sin(\pi x)\,dx\\&=\frac{1}{\pi}\int_0^1\sin(\pi x)\pi\,dx\\&=\frac{1}{\pi}\left[-\cos(\pi x)\right]_0^1\\&=\frac{1}{\pi}\left[\left(-\cos(\pi\cdot1)\right)-\left(-\cos(\pi\cdot0)\right)\right]\\&=\frac{1}{\pi}[1+1]\\&=\frac{2}{\pi}\end{aligned}$$

(a) 0 (b) 2 (c) $\frac{\pi}{2}$ (d) 2π (e) $\frac{2}{\pi}$

7. The expression $\frac{1}{20}\left[\left(\frac{1}{20}\right)^2+\left(\frac{2}{20}\right)^2+\left(\frac{3}{20}\right)^2+\cdots+\left(\frac{20}{20}\right)^2\right]$ is a Riemann sum approximation of

$$\Delta x=\frac{b-a}{n}=\frac{1}{20}$$

$$\frac{1}{20}\left[\left(0+\frac{1}{20}\right)^2+\left(0+2\left(\frac{1}{20}\right)\right)^2+\left(0+3\left(\frac{1}{20}\right)\right)^2+\cdots+\left(0+20\left(\frac{1}{20}\right)\right)^2\right]$$

(a) $\int_0^1 x^2dx$ (b) $\frac{1}{20}\int_0^1 x^2dx$ (c) $\frac{1}{20}\int_0^1\left(\frac{x}{20}\right)^2dx$ (d) $\int_0^1\left(\frac{x}{20}\right)^2dx$ (e) $\frac{1}{20}\int_1^{20}\left(\frac{x}{20}\right)^2dx$

8. The expression $\frac{1}{75}\left[\ln\left(\frac{76}{75}\right)+\ln\left(\frac{77}{75}\right)+\ln\left(\frac{78}{75}\right)+\cdots+\ln(2)\right]$ is a Riemann sum approximation for

$$\frac{1}{75}\left[\ln\left(\frac{76}{75}\right)+\ln\left(\frac{77}{75}\right)+\ln\left(\frac{78}{75}\right)+\cdots+\ln(2)\right]$$

$$\frac{1}{75}\left[\ln\left(\frac{75+1}{75}\right)+\ln\left(\frac{75+2}{75}\right)+\ln\left(\frac{75+3}{75}\right)+\cdots+\ln\left(\frac{75+75}{75}\right)\right]$$

$$\frac{1}{75}\left[\ln\left(1+\frac{1}{75}\right)+\ln\left(1+\frac{2}{75}\right)+\ln\left(1+\frac{3}{75}\right)+\cdots+\ln\left(1+\frac{75}{75}\right)\right]$$

$\frac{b-a}{n}=\frac{1}{75}\rightarrow b-a=1$ and $n=75$

$1+\frac{k}{75}\sim 1+k\left(\frac{1}{n}\right)\sim x$ where $a=1$, making the answer choice (d)

(a) $\int_1^2 \ln\left(\frac{x}{75}\right)dx$ (b) $\int_{76}^{150} \ln\left(\frac{x}{75}\right)dx$ (c) $\frac{1}{75}\int_{76}^{100} \ln(x)\,dx$ (d) $\int_1^2 \ln(x)\,dx$ (e) $\frac{1}{75}\int_1^2 \ln(x)\,dx$