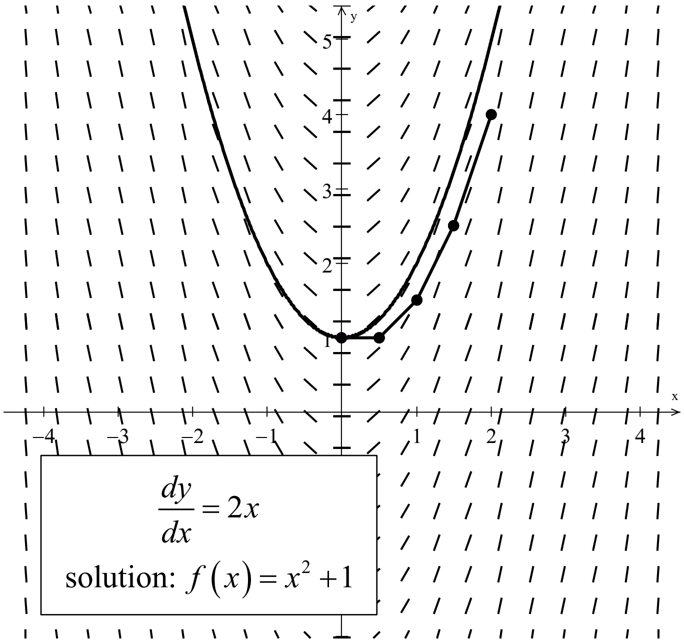

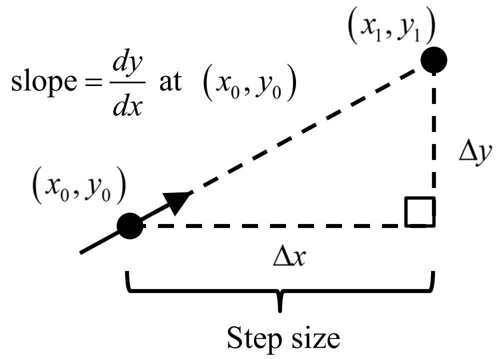

Euler's Method is a process of estimating what the general shape of a solution to a given differential equation looks like.

When sketching the solution to a given differential equation in its slope field, you

1) Choose a starting point
2) Go in the direction that the slope field indicates at the starting point for a short distance ending up at a new location.
3) At this new location, go in the direction that the slope field indicates for a short distance, ending up at a new location
4) Repeat this iterative process.

Euler's Method *__is a numerical method__* that describes what you do by hand sketching a solution passing through the starting point in the slope field. In order to start Euler's Method, you need three things

I. A **starting coordinate** (x_0, y_0)

II. The **equation of the differential equation** $\frac{dy}{dx} = \cdots$

III. A **step size** – how far in the *x*-direction you will move each time between points.

$$(x_0, y_0) \to (x + \Delta x, y_0 + \Delta y)$$

$$\to \left(x + \Delta x, y_0 + \frac{\Delta y}{\cancel{\Delta x}} \cdot \cancel{\Delta x} \right)$$

$$\to \left(x + \Delta x, y_0 + \frac{dy}{dx} \cdot \Delta x \right)$$

$$\boxed{(x_0, y_0) \to \left(x_0 + (\text{step size}), y_0 + (\text{derivative})(\text{step size}) \right)}$$

$$\to \left(x + \Delta x, y_0 + \left[\frac{dy}{dx} \text{ at } (x_0, y_0) \right] \cdot \Delta x \right)$$

$$\to (x_1, y_1)$$

No calculator: Let $y = f(x)$ be the solution to the differential equation $\frac{dy}{dx} = x + y$ with the initial condition $f(1) = 2$. What is the approximation for $f(2)$ if Euler's method is used, starting at $x = 1$ with a step size of 0.5?

(a) 3 (b) 5 (c) 6 (d) 10 (e) 12

$$
\begin{aligned}
(1,2) &\to \left(1+(\text{step size}), 2+(\text{derivative})(\text{step size})\right) \\
&\to \left(1+0.5, 2+\left(\frac{dy}{dx} \text{ at } (1,2)\right)(0.5)\right) \\
&\to \left(1.5, 2+(1+2)(0.5)\right) \\
&\to (1.5, 3.5) \\
&\swarrow \\
(1.5,3.5) &\to \left(1.5+(\text{step size}), 3.5+(\text{derivative})(\text{step size})\right) \\
&\to \left(1.5+0.5, 3.5+\left(\frac{dy}{dx} \text{ at } (1.5,3.5)\right)(0.5)\right) \\
&\to \left(2, 3.5+(1.5+3.5)(0.5)\right) \\
&\to (2,6)
\end{aligned}
$$

Use Euler's Method with step size 0.1 to estimate $y(0.2)$, where $y(x)$ is the solution of the initial value problem $\frac{dy}{dx} = y + xy$ where $y(0) = 1$.

$$
\begin{aligned}
(0,1) &\to \left(0+\text{step size}\ , y(0)+\left[\frac{dy}{dx} \text{ at } (0,1)\right]\cdot(\text{step size})\right) \\
&\approx \left(0.1, 1+[1+0\cdot 1]\cdot 0.1\right) \\
&\approx (0.1, 1.1) \\
&\swarrow \\
(0.1,1.1) &\to \left(0.1+\text{step size}\ , y(0.1)+\left[\frac{dy}{dx} \text{ at } (0.1,1.1)\right]\cdot(\text{step size})\right) \\
&\approx \left(0.1+0.1, 1.1+[1.1+(0.1)(1.1)]\cdot 0.1\right) \\
&\approx \left(0.2, 1.1+[1.1+(0.1)(1.1)]\cdot 0.1\right) \\
&\approx (0.2, 1.221)
\end{aligned}
$$

Use Euler's Method with step size 0.2 to estimate $y(0.4)$ where $y(x)$ is the solution of the initial-value problem $\frac{dy}{dx} = xy - x^2$ where $y(0) = 1$.

$$
\begin{aligned}
(0,1) &\rightarrow \left(x(0) + \text{step size}\ ,\ y(0) + \left[\frac{dy}{dx} \text{ at } (0,1) \right] \cdot (\text{step size}) \right) \\
&\approx \left(0 + 0.2, 1 + \left[(0)(2) - (0)^2 \right] \cdot 0.2 \right) \\
&\approx (0.2, 1) \\
&\swarrow \\
(0.2,1) &\rightarrow \left(0.2 + \text{step size}\ ,\ y(0.2) + \left[\frac{dy}{dx} \text{ at } (0.2,1) \right] \cdot (\text{step size}) \right) \\
&\approx \left(0.2 + 0.2, 1 + \left[(0.2)(1) - (0.2)^2 \right] \cdot 0.2 \right) \\
&\approx (0.4, 1.032)
\end{aligned}
$$

$y(0.4) \approx 1.032$