Unit Summary for Derivatives Solutions

1. The slope of a function at a given coordinate is the slope of the tangent to the function at the given coordinate.

(a) Explain in words how the slope of a function $f(x)$ at $x = c$ is **estimated**. You may include diagrams as well.

The slope of a function $f(x)$ at $x = c$ is estimated by taking a point close to the point $(c, f(c))$, call it $(k, f(k))$, and finding the slope between those two points. The slope between those two points is the estimate for the slope of $f(x)$ at $x = c$.

(b) Explain the process of how these estimates are improved.

The estimates are improved by moving the point $(k, f(k))$ closer and closer to the point $(c, f(c))$. That is the number k must get closer and closer to the number c.

2. To formally estimate the slope of a function $f(x)$ at $x = c$, the following expressions may used:

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \qquad \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Choose one expression above and explain how the notation in the chosen expression is connected to the way the slope of the tangent to $f(x)$ at $x = c$ is estimated in your answer to 1.

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

$\frac{f(c+h) - f(c)}{h}$ represents the slope between the coordinates $(c, f(c))$ and $(c+h, f(c+h))$. The estimate of the slope of the line tangent to $f(x)$ at $(c, f(x))$ is improved by moving the coordinate $(c+h, f(c+h))$ closer to $(c, f(c))$ - which results in $h \to 0$. This is the reason $\lim\limits_{h \to 0}$ is in the expression.

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$\frac{f(x) - f(a)}{x - a}$ represents the slope between the coordinates $(a, f(a))$ and $(x, f(x))$. This is the approximation for the slope of the line tangent to $f(x)$ at $x = a$. The approximation is improved by moving $(x, f(x))$ closed to $(a, f(a))$, which results in $x \to a$. This is the reason $\lim\limits_{x \to a}$ is in the expression.

3. Explain in words what each expression represents visually:

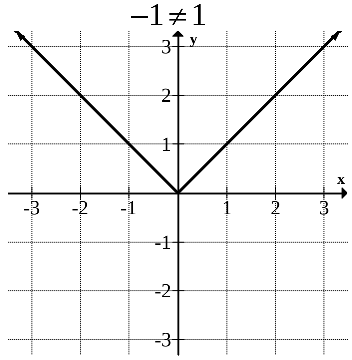
$\lim\limits_{h\to0^+}\dfrac{f(c+h)-f(c)}{h}$	$\lim\limits_{h\to0^-}\dfrac{f(c+h)-f(c)}{h}$
This limit represents the estimate of the slope of $f(x)$ at $x=c$ using the limit of the difference quotient with points to the right of *c*.	This limit represents the estimate of the slope of $f(x)$ at $x=c$ using the limit of the difference quotient with points top to the left of *c*.

THE RESULTS OF THESE LIMITS DO NOT NEED TO BE THE SAME!

$$f(x)=|x|$$

$$\lim_{h\to0^-}\frac{f(0+h)-f(0)}{h}\neq\lim_{h\to0^+}\frac{f(0+h)-f(0)}{h}$$

$$-1\neq1$$



4. The derivative of $f(x)$ at $x=c$, denoted $f'(c)$, is defined as $f'(c)=\lim\limits_{h\to0}\dfrac{f(c+h)-f(c)}{h}$.

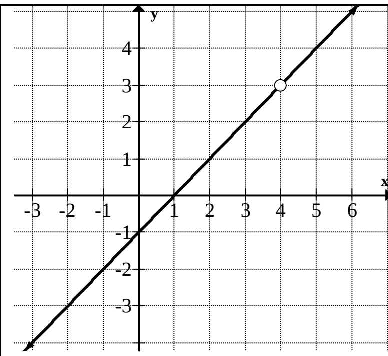
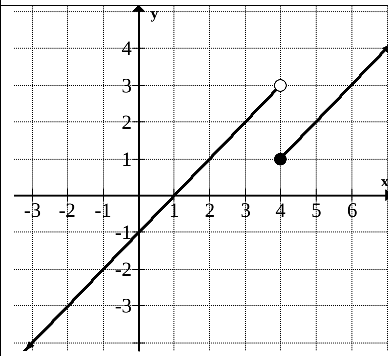
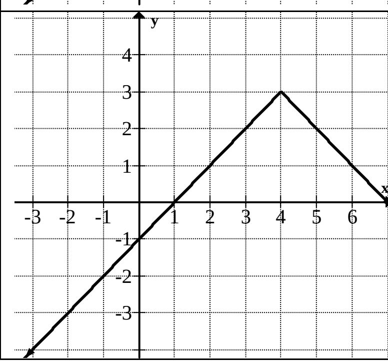
State the similarities and differences between the results/meaning of the following two expressions:

$$\lim_{h\to0}\frac{f(c+h)-f(c)}{h}\qquad\lim_{h\to0}\frac{f(x+h)-f(x)}{h}$$

Similarity: They are both limits of the difference quotient

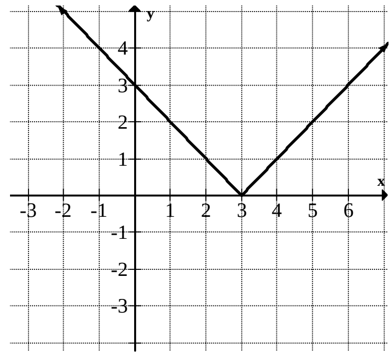
Difference: The result of $\lim\limits_{h\to0}\dfrac{f(c+h)-f(c)}{h}$ is a <u>value/number</u>, whereas the result of $\lim\limits_{h\to0}\dfrac{f(x+h)-f(x)}{h}$ is a <u>function</u>.

5. There are three different types of reasons why $f'(c)$ will not exist. Sketch three different functions for which $f'(4)$ does not exist where the reason for $f'(4)$ not existing is different from the other two graphs.

The function is not defined at $x=4$	
The function is defined and discontinuous at $x=4$	
The function is continuous at $x=4$, but $\lim\limits_{h\to 0^-}\dfrac{f(4+h)-f(4)}{h} \neq \lim\limits_{h\to 0^+}\dfrac{f(4+h)-f(4)}{h}$	

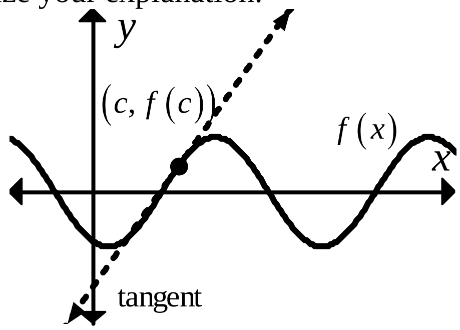
6. Give an equation of a continuous function that is not differentiable at $x = 3$ and sketch its graph.

$$y = |x - 3|$$



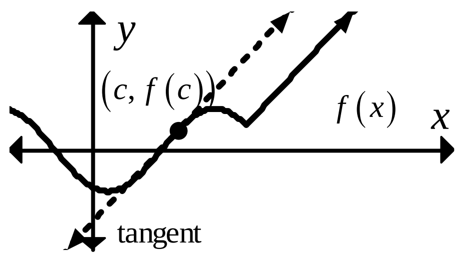
7.

(a) Explain in words why the line tangent to the function $f(x)$ at $x=c$ is a good linear approximation to $f(x)$ for *x*-values close to *c*, and not a good approximation for *x*-values far away from *c*. Include a sketched diagram to visualize your explanation.



The tangent to the function $f(x)$ at $x=c$ is a good linear approximation to $f(x)$ for *x*-values close to *c* because a the graph of the tangent and the graph of the function are close to each other for a very small interval near the point of tangency. The farther away *x* is from the point of tangency, the function and the line tangent to the graph of $f(x)$ at $x=c$ may be further away from teach other - making the tangent line approximation not a good estimate.

The reason for the difference in y-values has nothing to do with the value of $f'(x)$ being different from the value of the slope of the tangent. In fact, a function can be created so that $f'(x)$ is equal to the slope of the tangent for all values of *x* away from the point of tangency and still have the *y*-values be different.



(b) Explain how to estimate the value of $f(2)$ using the tangent to the graph of a function $f(x)$ at $x=1$.

(i) Find the value of

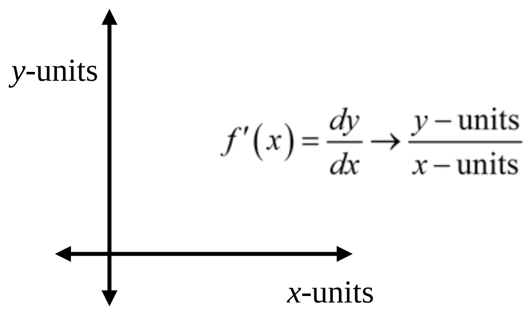
(ii) Find the value of $f'(1)$

(iii) Use the point-slope equation $y-y_1=m(x-x_1)$ to create an equation for the line tangent to the graph of $f(x)$ at $x=1$ by substituting $1\leftrightarrow x_1$, $y_1\leftrightarrow f(1)$, and $f'(1)\leftrightarrow m$ to get $y-f(1)=f'(1)(x-1)$.

(iv) Replace $x\leftrightarrow 2$ and then solve for *y*: $y-f(1)=f'(1)(2-1)$. The value of *y* will be the estimate for $f(2)$ using the line tangent to the graph of $f(x)$ at $x=1$.

8. Explain how to determine the units of the derivative of a function $f(x)$ given the units on the x-axis and the units on the y-axis. Sketch a made-up function $f(x)$ on a set of axes. Label the units on each axis with whatever units you like, and demonstrate what the units of $f'(x)$ will be.

To determine the units of the derivative of a function $f(x)$ given the units on the x-axis and the units on the y-axis, take the units of the y-axis and divide them by the units of the x-axis. The resulting unit will be the unit of $f'(x)$.



$$f'(x) = \frac{dy}{dx} \to \frac{y-\text{units}}{x-\text{units}}$$