“Even Factorial” and “Odd Factorial” Explained

Define $(\text{odd})! = 1\cdot 3\cdot 5\cdot 7\cdots\cdots(2k+1)$ where k is a positive integer

$(2k+2)! = 1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdots\cdots(2k)(2k+1)(2k+2)$

$$1\cdot 3\cdot 5\cdot 7\cdots\cdots(2k+1) = \frac{1\cdot \cancel{2}\cdot 3\cdot \cancel{4}\cdot 5\cdot \cancel{6}\cdot 7\cdots\cdots \cancel{(2k)}\,(2k+1)\,\cancel{(2k+2)}}{\cancel{2}\quad\cdot \cancel{4}\quad \cdot\cancel{6}\cdot\quad\cdots\cdots\cancel{(2k)}\qquad\qquad \cancel{(2k+2)}}$$

$$\downarrow$$

$$= \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdots\cdots(2k)(2k+1)(2k+2)}{2\quad\cdot 4\quad\cdot 6\quad\cdots\cdots(2k)\qquad\qquad(2k+2)}$$

$$= \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdots\cdots(2k)(2k+1)(2k+2)}{\underbrace{(1\cdot 2)\cdot(2\cdot 2)\cdot(3\cdot 2)\cdots\cdots(2\cdot k)\big(2\cdot(k+1)\big)}_{(k+1)-\text{terms, each with a factor of 2}}}$$

$$= \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdots\cdots(2k)(2k+1)(2k+2)}{2^{k+1}\cdot(1)\cdot(2)\cdot(3)\cdots\cdots(k)(k+1)}$$

$$= \frac{(2k+2)!}{2^{k+1}(k+1)!}$$

Define $(\text{even})! = 2\cdot 4\cdot 6\cdots\cdots(2k-2)(2k)$ where k is a positive integer.

$$2\cdot 4\cdot 6\cdots\cdots(2k-2)(2k) = \underbrace{(2\cdot 1)\cdot(2\cdot 2)\cdot(2\cdot 3)\cdots\cdots\big(2(k-1)\big)(2k)}_{k-\text{factors each with with a factor of 2}}$$

$$= 2^k\big(1\cdot 2\cdot 3\cdots\cdots(k-1)(k)\big)$$

$$= 2^k\cdot k!$$

Note: These concepts are more appropriately written/expressed in *direct product* notation.

$$2\cdot 4\cdot 6\cdot 8\cdots\cdots(2k-2)(2k) = \prod_{n=1}^{k} 2n$$

$$1\cdot 3\cdot 5\cdot 7\cdots\cdots(2k-1)(2k+1) = \prod_{n=0}^{k} 2n+1$$

We use $\sum$, the capital Greek letter **S**igma for Sums.

We use $\prod$, the capital Greek letter **P**i for Products.