AP Calculus AB Solutions
Differential Equations Test   Spring 2018   Name:________________________________
Date:_____________________ Period:_____

<u>No Calculator Permitted</u>

1. A rumor spreads among a population of N people at a rate proportional to the product of the number of people who have heard the rumor and the number of people who have not heard the rumor. If p denotes the number of people who have heard the rumor, which of the following differential equations could be used to model this situation with respect to time t, where k is a positive constant?

(a) $\cancel{\dfrac{dp}{dt} = kp}$
Not proportional to the number of people who did not hear the rumor

(b) $\dfrac{dp}{dt} = kp(N-p)$
$N-p$ represents the people who have not yet heard the rumor

(c) $\dfrac{dp}{dt} = kp(p-N)$

(d) $\cancel{\dfrac{dp}{dt} = kt(N-t)}$
Rate is dependent on time

(e) $\cancel{\dfrac{dp}{dt} = kt(t-N)}$
Rate is dependent on time

2. Shown at right is a slope field for which of the following differential equations?

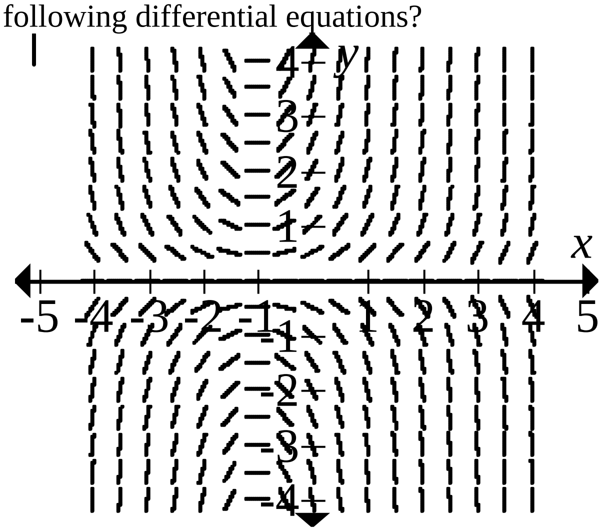
(a) $\cancel{\dfrac{dy}{dx} = xy}$ $\left.\dfrac{dy}{dx}\right|_{(-1,1)} = -1$

(b) $\cancel{\dfrac{dy}{dx} = xy - y}$ $\left.\dfrac{dy}{dx}\right|_{(-1,1)} = -1$

(c) $\dfrac{dy}{dx} = xy + y$

(d) $\cancel{\dfrac{dy}{dx} = xy + x}$ $\left.\dfrac{dy}{dx}\right|_{(-1,1)} = -2$

(e) $\cancel{\dfrac{dy}{dx} = (x+1)^3}$ Slopes would be consistent for each x value



3. Given that $y(1) = -3$ and $\frac{dy}{dx} = 2x + y$, what is the approximation for $y(2)$ if Euler's method is used with a step size of 0.5 starting at $x = 1$?

$$(1,-3) \to \left(1+0.5, -3+(0.5)\left(\left.\frac{dy}{dx}\right|_{(1,-3)}\right)\right)$$
$$\to (1.5, -3+(0.5)(-1))$$
$$\to (1.5, -3.5)$$
$$\swarrow$$
$$(1.5,-3.5) \to \left(1.5+0.5, -3.5+(0.5)\left(\left.\frac{dy}{dx}\right|_{(1.5,-3.5)}\right)\right)$$
$$\to (1.5+0.5, -3.5+(0.5)(-0.5))$$
$$\to (2, -3.75)$$

(a) –5 (b) –4.25 (c) –4 (d) –3.75 (e) –3.5

4. Which differential equation does the function $y = e^{-3t}$ satisfy?

$$y = e^{-3t}$$
$$y' = -3e^{-3t}$$
$$y'' = 9e^{-3t}$$

(a) $y'' + y' + 12y = 0$ $\quad 9e^{-3t} - 3e^{-3t} + 12e^{-3t} \neq 0$
(b) $y'' + y' - 12y = 0$ $\quad 9e^{-3t} - 3e^{-3t} - 12e^{-3t} \neq 0$
(c) $y'' - y' - 12y = 0$ $\quad 9e^{-3t} - \left(-3e^{-3t}\right) - 12e^{-3t} = 0$
(d) $y'' - y' + 12y = 0$ $\quad 9e^{-3t} - \left(-3e^{-3t}\right) + 12\left(e^{-3t}\right) \neq 0$
(e) $y'' - 3y' + 12y = 0$ $\quad 9e^{-3t} - 3\left(-3e^{-3t}\right) + 12e^{-3t} \neq 0$

5. The acceleration *a* of a body moving in a straight line is given in terms of time *t* by $a = 8 - 6t$. If the velocity of the body is 25 at $t = 1$ and if $s(t)$ is the distance of the body from the origin at time *t*, what is $s(4) - s(2)$?

$$a(t) = 8 - 6t \qquad v(1) = 25$$

$$\downarrow \qquad\qquad \downarrow$$

$$v(t) = -3t^2 + 8t + C \qquad 25 = -3(1)^2 + 8(1) + C$$

$$C = 20$$

$$\begin{aligned} s(4) - s(2) &= \int_2^4 v(t)\,dt \\ &= \int_2^4 -3t^2 + 8t + 20dt \\ &= \left[-t^3 + 4t^2 + 20t\right]_2^4 \\ &= \left[-(4)^3 + 4(4)^2 + 20(4)\right] - \left[-(2)^3 + 4(2)^2 + 20(2)\right] \\ &= 32 \end{aligned}$$

(a) 20 (b) 24 (c) 28 (d) 32 (e) 42

6. $\lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h} =$

$$\lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h} = \left[\sin(x)\right]' = \cos(x)$$

(a) ~~0~~ Must be a function

(b) ~~1~~ Must be a function

(c) $\sin(x)$

(d) $\cos(x)$

(e) DNE

7. The average value of $y = \frac{1}{x}$ on the interval closed interval $[1,3]$ is

$$\begin{aligned} A.V. &= \frac{1}{3-1}\int_1^3 \frac{1}{x}\,dx \\ &= \frac{1}{2}\int_1^3 \frac{1}{x}\,dx \\ &= \frac{1}{2}\left(\left[\ln|x|\right]_1^3\right) \\ &= \frac{1}{2}\left(\left[\ln|3|\right]-\left[\ln|1|\right]\right) \\ &= \frac{\ln(3)}{2} \end{aligned}$$

(a) $\frac{1}{2}$ (b) $\frac{2}{3}$ (c) $\frac{\ln(2)}{2}$ (d) $\frac{\ln(3)}{2}$ (e) $\ln(3)$

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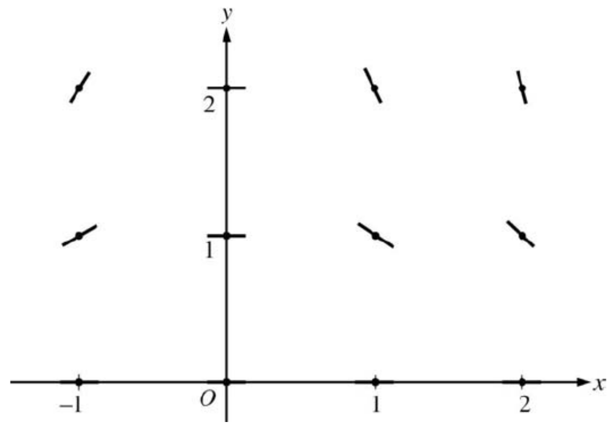
<u>Calculator Required</u>

8. Consider the differential equation $\frac{dy}{dx} = \frac{-xy^2}{2}$.

   Let $y = f(x)$ be the particular solution to this differential equation with the initial condition $f(-1) = 2$.

   a. On the axis provided, sketch a slope field for the given differential equation at the twelve points indicated.



   b. Write an equation for the line tangent to the graph of *f* at $x = -1$.

$$\left.\frac{dy}{dx}\right|_{(-1,2)} = \frac{-(-1)(2)^2}{2} = 2$$

$$y - y_1 = m(x - x_1)$$
$$y - 2 = 2(x+1)$$

   c. Find the solution $y = f(x)$ to the given differential equation with the initial condition $f(-1) = 2$.

$$\frac{dy}{dx} = \frac{-xy^2}{2}$$
$$\frac{1}{y^2}dy = -\frac{1}{2}xdx$$
$$\int \frac{1}{y^2}dy = \int -\frac{1}{2}xdx$$
$$-y^{-1} = -\frac{1}{4}x^2 + C$$
$$-\frac{1}{y} = -\frac{1}{4}x^2 + C$$
$$\frac{1}{y} = \frac{1}{4}x^2 + C$$
$$y = \frac{1}{\frac{1}{4}x^2 + C}$$

$$f(-1) = 2$$
$$\downarrow$$
$$2 = \frac{1}{\frac{1}{4}(-1)^2 + C}$$
$$2 = \frac{1}{\frac{1}{4} + C}$$
$$\frac{1}{2} + 2C = 1$$
$$2C = \frac{1}{2}$$
$$C = \frac{1}{4}$$

$$y = \frac{1}{\frac{1}{4}x^2 + \frac{1}{4}}$$
$$= \frac{4}{x^2 + 1}$$

9. Let f be the function satisfying $f'(x) = x\sqrt{f(x)}$ for all real numbers x, where $f(3) = 25$.

a. Find $f''(3)$

$$\begin{aligned}
f'(x) &= x\sqrt{f(x)} \\
y' &= x\sqrt{y} \\
&= xy^{\frac{1}{2}} \\
&\downarrow \\
y'' &= y^{\frac{1}{2}} + x\left[\frac{1}{2}y^{-\frac{1}{2}} \cdot y'\right] \\
&= y^{\frac{1}{2}} + x\left[\frac{1}{2}y^{-\frac{1}{2}} \cdot \left(xy^{\frac{1}{2}}\right)\right] \\
&= y^{\frac{1}{2}} + \frac{1}{2}x^2 \\
y''\big|_{(3,25)} &= (25)^{\frac{1}{2}} + \frac{1}{2}(3)^2 = \frac{19}{2}
\end{aligned}$$

b. Write an expression for $y = f(x)$ by solving the differential equation $\frac{dy}{dx} = x\sqrt{y}$ with the initial condition $f(3) = 25$.

$$\frac{dy}{dx} = x\sqrt{y}$$

$$\frac{dy}{dx} = xy^{\frac{1}{2}}$$

$$y^{-\frac{1}{2}}dy = xdx$$

$$\int y^{-\frac{1}{2}}dy = \int xdx$$

$$2y^{\frac{1}{2}} = \frac{1}{2}x^2 + C$$

$$\sqrt{y} = \frac{1}{4}x^2 + C$$

$$y = \left(\frac{1}{4}x^2 + C\right)^2$$

$$f(3) = 25$$

$$\downarrow$$

$$25 = \left(\frac{1}{4}[3]^2 + C\right)^2$$

$$25 = \left(\frac{9}{4} + C\right)^2$$

$$\sqrt{25} = \sqrt{\left(\frac{9}{4} + C\right)^2}$$

$$5 = \left|\frac{9}{4} + C\right|$$

$$\pm 5 = \frac{9}{4} + C$$

$$-\frac{9}{4} \pm 5 = C$$

$$C = -\frac{29}{4} \text{ or } \frac{11}{4}$$

Since $y=\left(\frac{1}{4}x^2+\frac{11}{4}\right)^2$ and $y=\left(\frac{1}{4}x^2-\frac{29}{4}\right)^2$ both satisfy the differential equation, either function is acceptable.

$\begin{aligned} y&=\left(\frac{1}{4}x^2+\frac{11}{4}\right)^2 \\ &\downarrow \\ y'&=2\left(\frac{1}{4}x^2+\frac{11}{4}\right)\cdot\left(2\cdot\frac{1}{4}x\right) \\ &=x\left(\frac{1}{4}x^2+\frac{11}{4}\right) \\ &=x\sqrt{y} \end{aligned}$ $\begin{aligned} y&=\left(\frac{1}{4}(3)^2+\frac{11}{4}\right)^2 \\ &=\left(\frac{9}{4}+\frac{11}{4}\right) \\ &=\left(\frac{20}{4}\right)^2 \\ &=5^2 \\ &=25 \end{aligned}$	$\begin{aligned} y&=\left(\frac{1}{4}x^2-\frac{29}{4}\right)^2 \\ &\downarrow \\ y'&=2\left(\frac{1}{4}x^2-\frac{29}{4}\right)\left(\frac{1}{2}x\right) \\ &=x\left(\frac{1}{4}x^2-\frac{29}{4}\right) \\ &=x\sqrt{y} \end{aligned}$ $\begin{aligned} y&=\left(\frac{1}{4}(3)^2-\frac{29}{4}\right)^2 \\ &=\left(\frac{9}{4}-\frac{29}{4}\right) \\ &=\left(-\frac{20}{4}\right) \\ &=(-5)^2 \\ &=25 \end{aligned}$

c. Use Euler's Method starting at $x=3$ with two steps of equal size to approximate the value of $f(5)$. Show the work that leads to your answer.

$$
\begin{aligned}
(3,25) &\to \left(3+1, 25+(1)\left(\left.\frac{dy}{dx}\right|_{(3,25)}\right)\right) \\
&\to \left(4, 25+(1)\left(3\sqrt{25}\right)\right) \\
&\to (4,40) \\
&\swarrow \\
(4,40) &\to \left(4+1, 40+(1)\left(\left.\frac{dy}{dx}\right|_{(4,70)}\right)\right) \\
&\to \left(4+1, 40+(1)\left(4\sqrt{40}\right)\right) \\
&\to \left(5, 40+4\sqrt{40}\right) \\
&\to \left(5, 40+8\sqrt{10}\right)
\end{aligned}
$$

$$
\begin{aligned}
f(5) &\approx 40+4\sqrt{40} \\
&\approx 65.298
\end{aligned}
$$