<u>Differentiability of Piecewise Functions:</u>

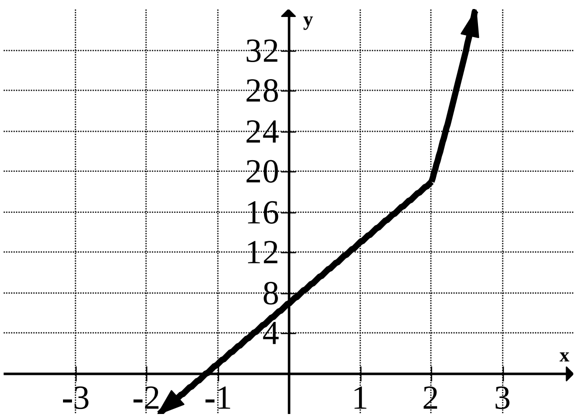
1. For what value of c is the function f continuous on $(-\infty,\infty)$? Justify your answer.

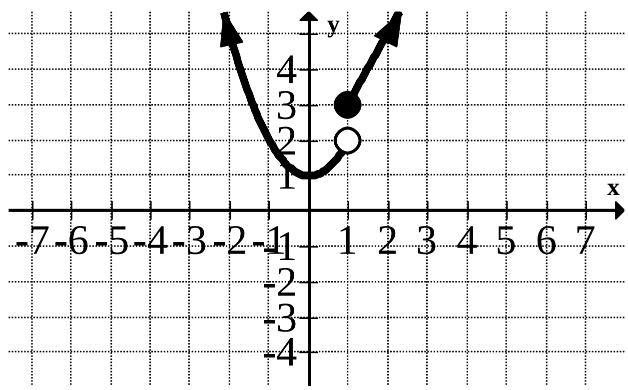
$$f(x)=\begin{cases} cx+7 & x\le 2 \\ cx^2-5 & x>2 \end{cases}$$

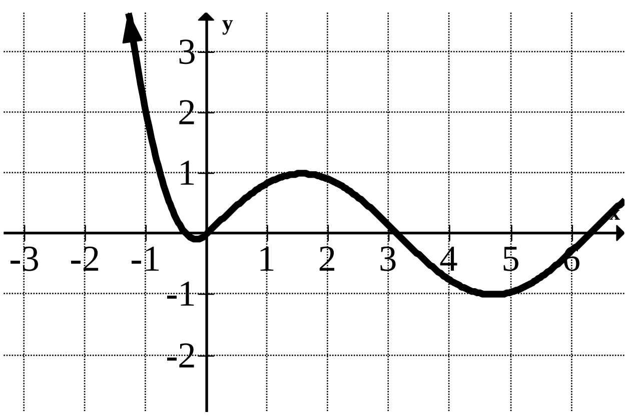
Is the function differentiable with this value of c? Justify your answer.

2. Let $f(x)=\begin{cases} x^2+1 & x<1 \\ 2x+1 & x\ge 1 \end{cases}$. Is f differentiable at $x=1$? Justify your answer.

3. Let $f(x)=\begin{cases} 3x^2+x & x\le 0 \\ \sin(x) & x>0 \end{cases}$. Is $f(x)$ differentiable at $x=0$? Justify your answer.







$$f(x)=\begin{cases} h(x) & x\ge c \\ k(x) & x<c \end{cases}$$

$$h(c)\overset{?}{=}k(c)$$

YES
$f(x)$ is continuous at $x=c$

NO
$f(x)$ not continuous at $x=c$

$$h'(c)\overset{?}{=}k'(c)$$

YES
$f(x)$ is differentiable at $x=c$

NO
$f(x)$ not differentiable at $x=c$

For what value of c is the function f continuous on $(-\infty,\infty)$? Justify your answer.

$$f(x)=\begin{cases} cx+7 & x\le 2 \\ cx^2-5 & x>2 \end{cases}$$

Is the function differentiable at this value? Justify your answer.

Informally	
$2c+7=4c-5\to c=6$	$f(x)$ is continuous when $c=6$
$\left[6x+7\right]'\Big\vert_{x=2}=6$ $\left[6x^2-5\right]'\Big\vert_{x=2}=24$	$f(x)$ is not differentiable at $x=2$

FORMALLY: In order for $f(x)$ to be continuous at $x=2$, we must have

$$\lim_{x\to 2^-} f(x)=f(2)=\lim_{x\to 2^+} f(x)$$

$$\begin{aligned} f(2)&=c(2)+7 \\ &=2c+7 \\ \lim_{x\to 2^-} f(x)&=\lim_{x\to 2^-} cx+7 \\ &=2c+7 \\ \lim_{x\to 2^+} f(x)&=\lim_{x\to 2^+} cx^2-5 \\ &\lim_{x\to 2^+} c(2)^2-5 \\ &=4c-5 \end{aligned}$$

Therefore

$$\begin{aligned} 2c+7&=4c-5 \\ 2c&=12 \\ c&=6 \end{aligned}$$

In order for $f(x)$ to be continuous at $x=2$, $f(x)$ must be

$$f(x)=\begin{cases} 6x+7 & x\le 2 \\ 6x^2-5 & x>2 \end{cases}$$

Now

$$\underbrace{\lim_{h\to 0^-}\frac{f(2+h)-f(2)}{h}}_{\text{derivative using points to the left of 2}}=\left[6x+7\right]'\Big\vert_{x=2} \qquad \underbrace{\lim_{h\to 0^+}\frac{f(2+h)-f(2)}{h}}_{\text{derivative using points to the right of 2}}=\left[6x^2-5\right]'\Big\vert_{x=2}$$

$$=6\big\vert_{x=2} \qquad\qquad =12x\big\vert_{x=2}$$

$$=6 \qquad\qquad =24$$

$f(x)$ is not differentiable at $x=2$ because $\underbrace{\lim_{h\to 0^-}\frac{f(2+h)-f(2)}{h}}_{\text{derivative using points to the left of 2}}\neq\underbrace{\lim_{h\to 0^+}\frac{f(2+h)-f(2)}{h}}_{\text{derivative using points to the right of 2}}$

Let $f(x)=\begin{cases} x^2+1 & x<1 \\ 2x+1 & x\geq 1 \end{cases}$. Is f differentiable at $x=1$? Justify your answer.

INFORMALLY: $\begin{matrix} [x+1]\big|_{x=1}=2 \\ [2x+1]\big|_{x=1}=3 \end{matrix}$. $f(x)$ is not differentiable at $x=1$ since $f(x)$ is not continuous at $x=1$.

FORMALLY: In order for $f(x)$ to be differentiable at $x=1$, $f(x)$ must be continuous at $x=1$. In order for $f(x)$ to be continuous at $x=1$,

$$\lim_{x\to 1^-} f(x)=f(1)=\lim_{x\to 1^+} f(x)$$

$$\begin{aligned} \lim_{x\to 1^-} f(x) &= \lim_{x\to 1^-} x^2+1 \\ &= 2 \\ f(1) &= 1+1 \\ &= 2 \\ \lim_{x\to 1^+} f(x) &= \lim_{x\to 1^+} 2x+1 \\ &= 3 \end{aligned}$$

Since $\lim_{x\to 1^-} f(x)=f(1)\neq \lim_{x\to 1^+} f(x)$, $f(x)$ is not continuous at $x=1$, and therefore $f(x)$ is not differentiable at $x=1$.

Let $f(x)=\begin{cases}3x^2+x & x\le 0\\ \sin(x) & x>0\end{cases}$. Is $f(x)$ differentiable at $x=0$? Justify your answer.

$$\left.\begin{aligned}\left[3x^2+x\right]\Big|_{x=0}&=0\\ \left[\sin(x)\right]\Big|_{x=0}&=0\end{aligned}\right\}\rightarrow f(x) \text{ is continous at } x=0$$

INFORMALLY:
$$\left.\begin{aligned}\left[3x^2+x\right]'\Big|_{x=0}&=\left[6x+1\right]\Big|_{x=0}=1\\ \left[\sin(x)\right]'\Big|_{x=0}&=\left[\cos(x)\right]\Big|_{x=0}=1\end{aligned}\right\}\rightarrow f(x) \text{ is differentiable at } x=0$$

FORMALLY: In order for $f(x)$ to be continuous at $x=0$, $f(x)$ must be continuous at $x=0$. In order for $f(x)$ to be continuous at $x=0$,

$$\lim_{x\to 0^-} f(x)=f(0)=\lim_{x\to 0^+} f(x)$$

$$\begin{aligned}\lim_{x\to 0^-} f(x)&=\lim_{x\to 0^-} 3x^2+x\\ &=0\\ f(0)&=3(0)^2+(0)\\ &=0\\ \lim_{x\to 0^+} f(x)&=\lim_{x\to 0^+}\sin(x)\\ &=0\end{aligned}$$

Hence $f(x)$ is continuous at $x=0$.

In order for $f(x)$ to be differentiable at $x=0$, $\lim_{h\to 0^-}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h}$

$$\begin{aligned}\underbrace{\lim_{h\to 0^-}\frac{f(0+h)-f(0)}{h}}_{\text{derivative using poinst to the left of zero}}&=\left[3x^2+x\right]'\Big|_{x=0}\\ &=6x+1\big|_{x=0}\\ &=1\end{aligned}$$

$$\begin{aligned}\underbrace{\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h}}_{\text{derivative using points to the right of zero}}&=\left[\sin(x)\right]'\Big|_{x=0}\\ &=\cos(x)\big|_{x=0}\\ &=1\end{aligned}$$

Since $\lim_{h\to 0^-}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h}$, we can conclude that $f(x)$ is differentiable at $x=0$.