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On the Role of Sign Charts in AP® Calculus Exams for Justifying Local or Absolute Extrema

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Sign charts can provide a useful tool to investigate and summarize the behavior of a function. We commend their use as an investigative tool. However, the Development Committee has recommended, and the Chief Reader concurs, that sign charts, by themselves, should not be accepted as a sufficient response when a problem asks for a justification for the existence of either a local or an absolute extremum of a function at a particular point in its domain. This is a policy that took effect with the 2005 AP Calculus Exams and Reading.

1. Local Extrema, the First Derivative Test

One way to justify that a critical value is, in fact, the location of a local maximum or a local minimum is to use the First Derivative Test. If the first derivative changes from positive immediately to the left of the critical point to negative immediately to the right of the critical point, then there is a local maximum at the critical point. Similarly, a change in the sign of the first derivative from negative to positive guarantees that there is a local minimum at the critical point. A sign chart may contain all of the necessary information to make the conclusion that there is a local maximum or minimum, but the Development Committee and Chief Reader want to see that the student knows what it is about this information that enables the appropriate conclusion. As an example, see **1987 AB4 (a)** in the appendix.

The labeled sign chart, even with the indication that f is decreasing between -3 and 1, increasing between 1 and 3, and decreasing between 3 and 5, is not, by itself, sufficient justification. We want to see the student demonstrate a knowledge of the First Derivative Test by recognizing that there is a relative minimum at $x = 1$ because f' changes from negative to positive. The word "because," while not required, is a useful indication that the student has given a reason rather than simply assembled information. Note that it would not be sufficient justification for a relative minimum at $x = 1$ if the student said "because f changes from decreasing to increasing." This is a statement of what can be meant by a local minimum rather than an appeal to an argument based on calculus. It would be acceptable to give as justification that f is decreasing to the left of $x = 1$ because f' is negative and it is increasing to the right because f' is positive.

2. Local Extrema, the Second Derivative Test

Another way to justify that a critical value is the location of a local maximum or minimum is to use the Second Derivative Test. Again, a sign chart for the second derivative is not enough. As an example, see **2002 Form B AB5/BC5 (a)** in the appendix. After showing that the first derivative is 0 at $x = 3$ and the second derivative is $\frac{1}{2}$, the student needs to state that f has a local minimum at $x = 3$ because the first derivative is 0 and the second derivative is positive.

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3. Absolute Extrema

On a closed interval, the justification of an absolute maximum or minimum can be accomplished by identifying all critical values as well as the endpoints, evaluating the function at each of these values, and then identifying which value of x corresponds to the absolute maximum or minimum of the function. The student can also use arguments based on where the function is increasing or decreasing or the amount of change in the function to explain why certain critical or end points can be eliminated as candidates for the location of an absolute maximum or minimum. For example, see **2001 AB3/BC3 (c)** in the appendix.

On an open interval, the only points that need to be considered are critical values, but students must indicate that they have considered what is happening over the entire interval. For example, in **1998 AB2 (b)** in the appendix, the justification for an absolute minimum at $x = -\frac{1}{2}$ requires the observation that f' is negative for all $x < -\frac{1}{2}$ and f' is positive for all $x > -\frac{1}{2}$. It would also be a correct justification to find a value of x to the left of $-\frac{1}{2}$ for which f' is negative, a value to the right of $-\frac{1}{2}$ at which f' is positive, and then to observe that $x = -\frac{1}{2}$ is the only critical point for the function.

Appendix

1987 AB4 (a)

Let f be the function given by $f(x) = 2\ln(x^2+3) - x$ with domain $-3 \le x \le 5$. Find the x-coordinate of each relative maximum point and each relative minimum point of f. Justify your answer.

Solution

$$f'(x) = 2 \cdot \frac{2x}{x^2+3} - 1 = -\frac{(x-3)(x-1)}{x^2+3}.$$

f'	$-$	$+$	$-$
f	dec	inc	dec

$-3 \qquad 1 \qquad 3 \qquad 5$

There is a relative minimum at $x = 1$ because f' changes from negative to positive.
There is a relative maximum at $x = 3$ because f' changes from positive to negative.

Comment

The sign chart, by itself, is not sufficient justification. We need to see that the student knows what it is about the sign chart that implies a relative minimum at $x = 1$ and a relative maximum at $x = 3$.

2002 Form B AB5/BC5 (a)

Consider the differential equation $\frac{dy}{dx} = \frac{3-x}{y}$. Let $y = f(x)$ be the particular solution to the given differential equation for $1 < x < 5$ such that the line $y = -2$ is tangent to the graph of f. Find the x-coordinate of the point of tangency, and determine whether f has a local maximum, local minimum, or neither at this point. Justify your answer.

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Solution

Since $\dfrac{dy}{dx} = 0$ when $x = 3$, the graph of $y = f(x)$ is tangent to the line $y = -2$ at the point $(3, -2)$. The second derivative is equal to

$$\frac{d^2y}{dx^2} = \frac{-y - y'(3-x)}{y^2}, \quad \text{and therefore } f''(3) = \frac{-(-2) - 0\cdot(3-3)}{(-2)^2} = \frac{1}{2} > 0.$$

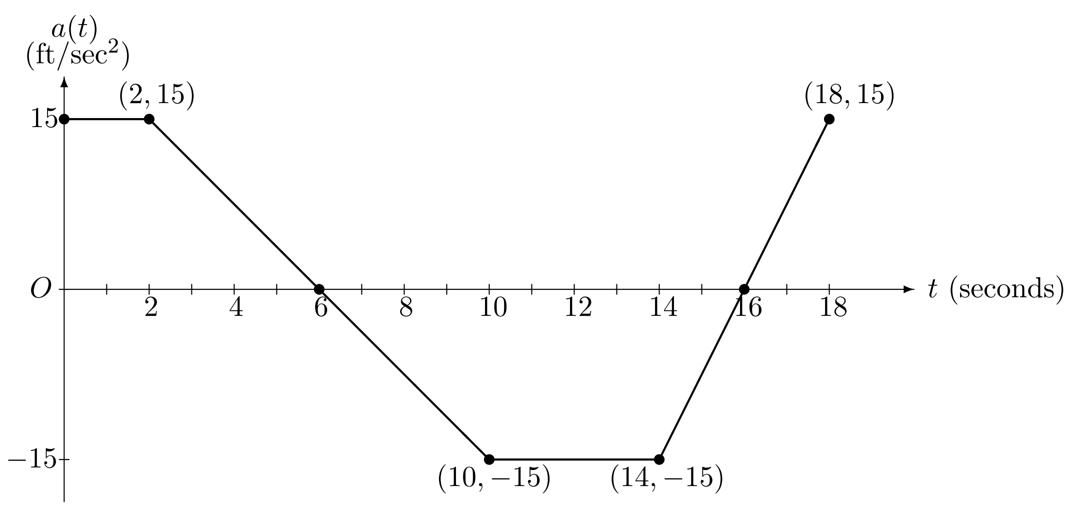
Since the first derivative is 0 and the second derivative is positive, there must be a local minimum at $x = 3$.

Comment

In most cases, a student can use the First Derivative Test to justify a local maximum or minimum, and the sign of the first derivative can be found either by inspecting the formula for the derivative, by inspecting the graph of the derivative that has been provided, or by evaluating the derivative at values on either side of the critical value. The situation in this problem is more difficult because there is no explicit representation of either the function or its derivative.

The student who tries to use the First Derivative Test to justify that there is a local minimum at $x = 3$ needs to explain why the derivative must be negative to the left of $x = 3$ and positive to the right of this value. The first step in a justification by the First Derivative Test is to observe that $y = f(x)$ is a solution of a first order differential equation for $1 < x < 5$, and so it must be continuous on that interval. The next step is to state that since $y = -2$ when $x = 3$, there must be an open interval containing 3 on which $y < 0$. On this open interval and to the left of $x = 3$ we have $x < 3$ and $y < 0$, so $\dfrac{dy}{dx} = \dfrac{3-x}{y} < 0$. On this open interval and to the right of $x = 3$ we have $x > 3$ and $y < 0$, so $\dfrac{dy}{dx} > 0$. We can now conclude that f has a local minimum at $x = 3$ because f' changes sign from negative to positive. This is a problem for which it is much easier to justify the answer using the Second Derivative Test.

2001 AB3/BC3 (c)



A car is traveling on a straight road with velocity 55 ft/sec at time $t = 0$. For $0 \le t \le 18$ seconds, the car's acceleration $a(t)$, in ft/sec^2, is the piecewise linear function defined by the graph above. On the time interval $0 \le t \le 18$, what is the car's absolute maximum velocity, in ft/sec, and at what time does it occur? Justify your answer.

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Solution

Since $v'(t) = a(t)$, the derivative of v is zero only at $t = 6$ and $t = 16$. The four values that need to be checked are $t = 0$, 6, 16, and 18.

$$\begin{aligned}
v(0) &= 55 \text{ ft/sec}, \\
v(6) &= 55 + \int_0^6 a(t)\,dt \;=\; 55 + 30 + 30 \;=\; 115 \text{ ft/sec}, \\
v(16) &= v(6) + \int_6^{16} a(t)\,dt \;=\; 115 - 30 - 60 - 15 = 10 \text{ ft/sec}, \\
v(18) &= v(16) + \int_{16}^{18} a(t)\,dt \;=\; 10 + 15 = 25 \text{ ft/sec}.
\end{aligned}$$

The car's absolute maximum velocity is 115 ft/sec, occurring at $t = 6$.

Comment

The student can also argue from the sign of $v'(t)$ that the velocity is increasing on the intervals $[0, 6]$ and $[16, 18]$ and decreasing on the interval $[6, 16]$, and therefore the only candidates for the location of the absolute maximum are at $t = 6$ and $t = 18$. Furthermore, the student can argue that since the area between the graph of $a(t)$ and the t-axis for $6 \le t \le 16$ is greater than the area between the graph of $a(t)$ and the t-axis for $16 \le t \le 18$, the velocity at $t = 6$ must be greater than the velocity at $t = 18$, and so the absolute maximum velocity occurs at $t = 6$. For this particular problem, the student still needs to find the velocity at $t = 6$.

1998 AB2 (b)

Let f be the function given by $f(x) = 2xe^{2x}$. Find the absolute minimum value of f. Justify that your answer is an absolute minimum.

Solution

$$\begin{aligned}
f'(x) &= 2e^{2x} + 2x \cdot 2e^{2x} \;=\; 2e^{2x}(1 + 2x), \\
f'(x) &= 0 \quad \text{at} \quad x = -\frac{1}{2}.
\end{aligned}$$

f'	$-$	$+$
f	dec	inc
	$-\frac{1}{2}$	

There is an absolute minimum at $x = -\frac{1}{2}$ because $f'(x) < 0$ for all $x < -\frac{1}{2}$ and $f'(x) > 0$ for all $x > -\frac{1}{2}$.

Comment

The key to justifying that we have an absolute minimum at $x = -\frac{1}{2}$ is that the derivative is negative **for all** $x < -\frac{1}{2}$ and positive **for all** $x > -\frac{1}{2}$. It is not enough to establish that the derivative changes sign from negative to positive at $x = -\frac{1}{2}$.

An equally valid justification would be that the derivative changes sign from negative to positive at $x = -\frac{1}{2}$ **and** $x = -\frac{1}{2}$ is the only critical value.

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