From Pre-Calculus:

Know how to find the trigonometric values of sin, cos, tan, csc, sec, cot of any multiple of the angles $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{6}$.

You should know the double angle formulas for sin and cos, and the Pythagorean Identities

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

$$\begin{aligned}\cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta)\\ &= 2\cos^2(\theta) - 1\\ &= 1 - 2\sin^2(\theta)\end{aligned}$$

$$\sin^2(\theta) + \cos^2(\theta) = 1 \longleftarrow$$ Divide this equation by $\cos^2$ or $\sin^2$ to get the other two

$$1 + \cot^2(\theta) = \csc^2(\theta)$$

$$\tan^2(\theta) + 1 = \sec^2(\theta)$$

Properties of Logarithms

$$\log_b(xy) = \log_b(x) + \log_b(y)$$

$$\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$$

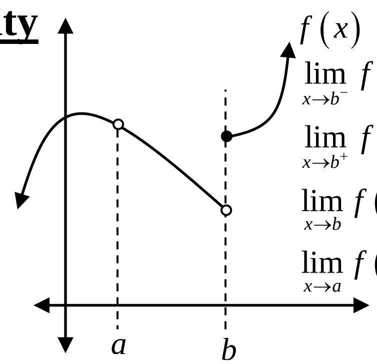
$$\log_b(x^n) = n \cdot \log_b(x)$$

$$\left.\begin{aligned}\log_b(b^m) &= m\\ b^{\log_b(m)} &= m\end{aligned}\right\} \text{“cancellation” laws}$$

Point-Slope Form

$$y - y_1 = m(x - x_1)$$

Limits and continuity



$\lim_{x\to b^-} f(x)$ exists

$\lim_{x\to b^+} f(x)$ exists

$\lim_{x\to b} f(x)$ DNE

$\lim_{x\to a} f(x)$ exists

$\lim_{x\to a} f(x)$ implies that $\underbrace{\lim_{x\to a^-} f(x)}_{\text{left hand limit}} = \underbrace{\lim_{x\to a^+} f(x)}_{\text{right hand limit}}$

To demonstrate function $f(x)$ is continuous at $x = a$, show that ALL the following are true:

(1) $f(a)$ exists

(2) $\lim_{x\to a^-} f(x) = f(a)$

(3) $\lim_{x\to a^+} f(x) = f(a)$

To demonstrate that a function $f(x)$ is not continuous at $x = a$, show that ONE of the following is true:

(1) $f(a)$ does not exist

(2) $\lim_{x\to a^-} f(x) \neq f(a)$

(3) $\lim_{x\to a^{-+}} f(x) \neq f(a)$

Average Rate of Change of a function $f(x)$ on the closed interval $[a,b]$ is given by

$$AROC = \underbrace{\frac{f(b)-f(a)}{b-a}}_{\text{slope between the points } (a,f(a)) \text{ and } (b,f(b))} \quad \text{or} \quad AROC = \underbrace{\frac{1}{b-a}\int_a^b f'(x)\,dx}_{\text{average value of the rate of change of } f \text{ on } [a,b]}$$

Instantaneous Rate of Change (a.k.a. the Derivative) of a function $f(x)$ at $x=a$ is given by

$$IROC = f'(a) = \underbrace{\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}}_{\text{limit definition of derivative (difference quotient)}} \quad \text{or} \quad \underbrace{\lim_{x\to a}\frac{f(x)-f(a)}{x-a}}_{\text{alternate form of derivative}} \quad \text{or} \quad \underbrace{\lim_{h\to 0}\frac{f(a+h)-f(a-h)}{2h}}_{\text{" symmetric difference quotient "}}$$

VISUALLY a function $f(x)$ is differentiable at $x=a$ if the function is defined at $x=a$ and the function is smooth at $x=a$

To JUSTIFY that a function $f(x)$ is differentiable at $x=a$, show that all the following are true:

(1) $f(x)$ is continuous at $x=a$

(2) $\underbrace{\lim_{h\to 0^-}\frac{f(a+h)-f(a)}{h}}_{\text{slope from the left}} = \underbrace{\lim_{h\to 0^+}\frac{f(a+h)-f(a)}{h}}_{\text{slope from the right}}$ and is *finite*

VISUALLY a function $f(x)$ is not differentiable when the function is not continuous or the slope approaching from the left is not equal to the slope approaching from the right.

To JUSTIFY that a function $f(x)$ is not differentiable at $x=a$, show that one of the following is true:

(1) $f(x)$ is not continuous at $x=a$

(2) $\lim_{h\to 0^+}\frac{f(a+h)-f(a)}{h} \to \infty/\text{DNE}$ or $\lim_{h\to 0^-}\frac{f(a+h)-f(a)}{h} \to \infty/DNE$

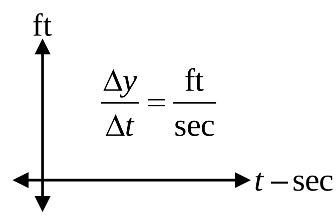
(3) $\underbrace{\lim_{h\to 0^-}\frac{f(a+h)-f(a)}{h}}_{\text{slope from the left}} \neq \underbrace{\lim_{h\to 0^+}\frac{f(a+h)-f(a)}{h}}_{\text{slope from the right}}$

The formula for the derivative is given, in general by using the following

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

If a function $f(x)$ is differentiable at $x=a$, then $f(x)$ is continuous at $x=a$.

<u>Differentiability implies Continuity!</u>



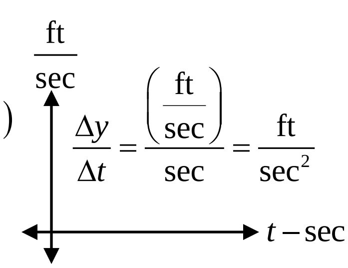
Let $p(t) =$ position of an object in feet at time t -seconds.

$\frac{d}{dt}\left[p(t)\right] = p'(t) = v(t) =$ velocity of the object at time *t*.

$\frac{d}{dt}\left[v(t)\right] = v'(t) = a(t) =$ acceleration of the object at time *t*.

speed $= |\text{velocity}|$ and distance $= |\text{displacement}|$

$p(t)$
$\downarrow$
$p'(t) = v(t)$
$\downarrow$
$p''(t) = v'(t) = a(t)$



Speed is INCREASING when $v(t)$ and $a(t)$ have the same sign.

Speed is DECREASING when $v(t)$ and $a(t)$ have opposite signs.

Properties of the derivative:

Constant multiple Rule: $\frac{d}{dx}\left[c\cdot u\right]=c\cdot u'$

Sum/Difference Rule: $\frac{d}{dx}\left[u\pm v\right]=u'\pm v'$

Product Rule: $\frac{d}{dx}\left[uv\right]=u'v+uv'$

Quotient Rule: $\frac{d}{dx}\left[\frac{u}{v}\right]=\frac{u'v-uv'}{v^2}$

Chain Rule: $\frac{d}{dx}\left[f\left(g\left(x\right)\right)\right]=f'\left(g\left(x\right)\right)\cdot g'\left(x\right)$

If u is a differentiable function of x, then : $\frac{d}{dx}\left[u^n\right]=n\cdot u^{n-1}\cdot u'$, and the following:

Basic Differentiation Rules:

$\frac{d}{dx}\left[x\right]=1$

$\frac{d}{dx}\left[|u|\right]=\frac{u}{|u|}\cdot u'$

$\frac{d}{dx}\left[\ln\left(u\right)\right]=\frac{1}{u}\cdot u'$

$\frac{d}{dx}\left[e^u\right]=e^u\cdot u'$

$\frac{d}{dx}\left[\log_a\left(u\right)\right]=\frac{1}{\ln\left(a\right)}\cdot\frac{1}{u}\cdot u'$

$\frac{d}{dx}\left[a^u\right]=\ln\left(a\right)\cdot a^u\cdot u'$

$\frac{d}{dx}\left[\sin\left(u\right)\right]=\cos\left(u\right)\cdot u'$

$\frac{d}{dx}\left[\cos\left(u\right)\right]=-\sin\left(u\right)\cdot u'$

$\frac{d}{dx}\left[\tan\left(u\right)\right]=\sec^2\left(u\right)\cdot u'$

$\frac{d}{dx}\left[\cot\left(u\right)\right]=-\csc^2\left(u\right)\cdot u'$

$\frac{d}{dx}\left[\sec\left(u\right)\right]=\sec\left(u\right)\tan\left(u\right)\cdot u'$

$\frac{d}{dx}\left[\csc\left(u\right)\right]=-\csc\left(u\right)\cot\left(u\right)\cdot u'$

$\frac{d}{dx}\left[\arcsin\left(u\right)\right]=\frac{1}{\sqrt{1-u^2}}\cdot u'$

$\frac{d}{dx}\left[\arccos\left(u\right)\right]=-\frac{1}{\sqrt{1-u^2}}\cdot u'$

$\frac{d}{dx}\left[\arctan\left(u\right)\right]=\frac{1}{1+u^2}\cdot u'$

$\frac{d}{dx}\left[\text{arccot}\left(u\right)\right]=-\frac{1}{1+u^2}\cdot u'$

$\frac{d}{dx}\left[\text{arcsec}\left(u\right)\right]=\frac{1}{|u|\sqrt{u^2-1}}\cdot u'$

$\frac{d}{dx}\left[\text{arccsc}\left(u\right)\right]=-\frac{1}{|u|\sqrt{u^2-1}}\cdot u'$

$$\left(f^{-1}\right)'\left(b\right)=\frac{1}{f'\left(\begin{matrix}\text{whatever makes}\\ f\left(x\right)=b\end{matrix}\right)}$$

Logarithmic Differentiation Examples:

Find y' given $y=x^x$

$$y=x^x$$
$$\ln\left(y\right)=\ln\left(x^x\right)$$
$$\ln\left(y\right)=x\ln\left(x\right)$$
$$\downarrow$$
$$\frac{1}{y}\cdot y'=1\cdot\ln\left(x\right)+x\cdot\frac{1}{x}$$
$$\frac{1}{y}\cdot y'=\ln\left(x\right)+1$$
$$y'=y\left[\ln\left(x\right)+1\right]$$
$$y'=x^x\left[\ln\left(x\right)+1\right]$$

$$y=\frac{x^2\sqrt{3x-2}}{\left(x-1\right)^2}$$
$$\ln\left(y\right)=\ln\left[\frac{x^2\sqrt{3x-2}}{\left(x-1\right)^2}\right]$$
$$\ln\left(y\right)=\ln\left(x^2\right)+\ln\left(\left[3x-2\right]^{\frac{1}{2}}\right)-\ln\left(\left[x-1\right]^2\right)$$
$$\ln\left(y\right)=2\ln\left(x\right)+\frac{1}{2}\ln\left(3x-2\right)-2\ln\left(x-1\right)$$
$$\downarrow$$
$$\frac{1}{y}\cdot y'=\frac{2}{x}+\frac{3}{2\left(3x-2\right)}-\frac{2}{x-1}$$
$$y'=y\left[\frac{2}{x}+\frac{3}{2\left(3x-2\right)}-\frac{2}{x-1}\right]$$
$$y'=\frac{x^2\sqrt{3x-2}}{\left(x-1\right)^2}\left[\frac{2}{x}+\frac{3}{2\left(3x-2\right)}-\frac{2}{x-1}\right]$$

Uses of the first derivative of a function $f(x)$:

The derivative of a function $f(x)$ is positive if and only if the graph of $f(x)$ is increasing.

The derivative of a function $f(x)$ is negative if and only if the graph of $f(x)$ is decreasing.

(1) Determine when $f(x)$ is increasing.

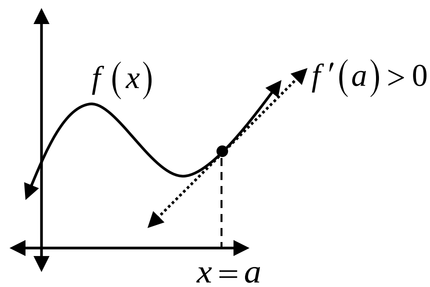
a. To identify that $f(x)$ is increasing at $x = a$

i. USING CALCULUS - To justify that $f(x)$ is increasing at $x = a$, then demonstrate that $f'(a) > 0$.

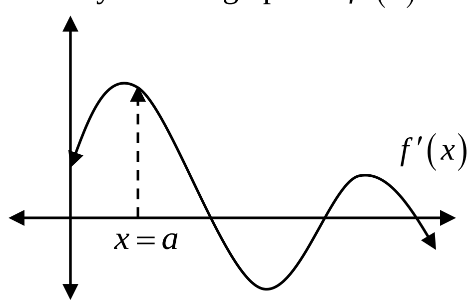
1. Determine the equation of $f'(x)$ and evaluate it at $x = a$.
2. State that "$f(x)$ is increasing at $x = a$ because $f'(a)$ is positive."

ii. VISUALLY – To identify that $f(x)$ is increasing at $x = a$

1. Identify that the slope of the tangent to the graph of $f(x)$ at $x = a$ has positive slope.

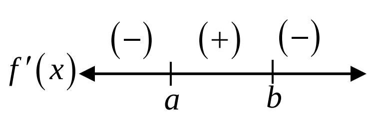


2. Identify that the graph of $f'(x)$ at $x = a$ is above the *x*-axis.



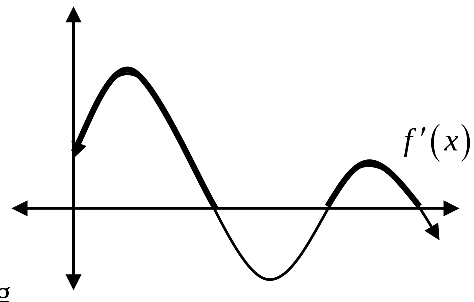
b. To identify that $f(x)$ is increasing on an interval (a, b), then demonstrate that $f'(x) > 0$ for all values *c* in the interval (a, b)

i. USING CALCULUS – demonstrate the equation of the derivative and make a labeled sign chart indicating the intervals on which the derivative is positive, negative, zero, and DNE.



1. State that "$f(x)$ is increasing on (a, b) because $f'(x)$ is positive."

ii. GIVEN THE GRAPH OF $f'(x)$ - identify the open intervals where the graph of $f'(x)$ lies above the *x*-axis



(2) Determine when $f(x)$ is decreasing

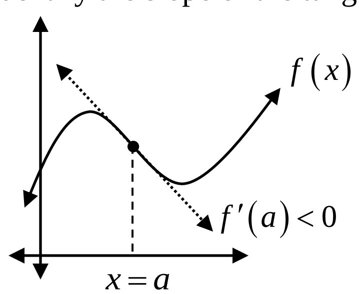
a. To identify that $f(x)$ is decreasing at $x=a$

i. USING CALCULUS- To justify that $f(x)$ is decreasing $x=a$, then demonstrate that $f'(a)<0$

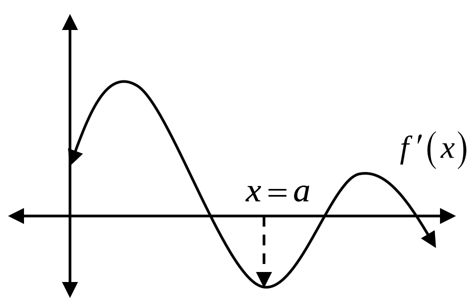
1. Demonstrate the equation of $f'(x)$ and evaluate it at $x=a$.
2. State "$f(x)$ is decreasing at $x=a$ because $f'(a)$ is negative."

ii. VISUALLY – To identify that $f(x)$ is decreasing at $x=a$

1. Identify the slope of the tangent to the graph of $f(x)$ at $x=a$ has negative slope.

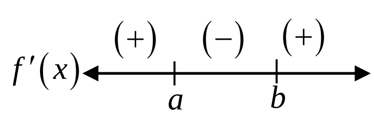


2. Identify that the graph of $f'(x)$ at $x=a$ lies below the *x*-axis



b. To identify that $f(x)$ is decreasing on an interval (a,b), then demonstrate that $f'(x)<0$ for all values *c* in the interval (a,b)

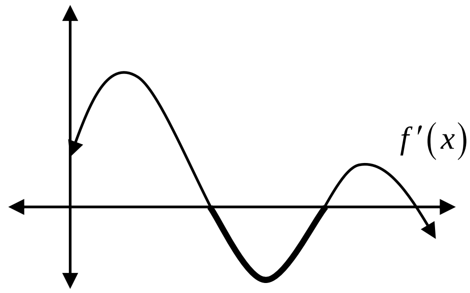
i. USING CALCULUS – demonstrate the equation of the derivative and make a labeled sign chart indicating the intervals on which the derivative is positive, negative, zero, and DNE.



1. State that "$f(x)$ is decreasing on (a,b) because $f'(x)$ is negative."

ii. GIVEN THE GRAPH OF $f'(x)$

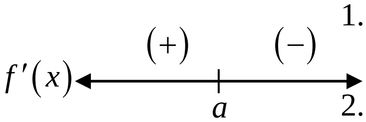
1. Identify the open intervals where $f'(x)$ lies below the *x*-axis.



(3) Identify candidates for local relative maximums or minimums of $f(x)$

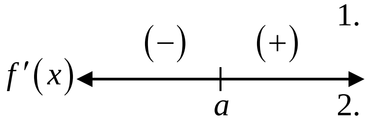
a. $f(x)$ has a candidate for a relative maximum or minimum at $x = a$ when $f'(a) = 0$ or $f'(a)$ DNE. The values of *f* for which $f'(x) = 0$ or $f'(x)$ DNE are called **critical values** *of* $f(x)$.

i. To JUSTIFY that $f(x)$ attains a relative maximum at $x = a$.



1. Demonstrate that $f'(x)$ changes sign from positive to negative at $x = a$. This can be done with the use of a labeled sign chart.
2. State " $f(x)$ has a relative maximum at $x = a$ because $f'(x)$ changes sign from positive to negative."

ii. To JUSTIFY that $f(x)$ attains a relative minimum at $x = a$.



1. Demonstrate that $f'(x)$ changes sign from negative to positive at $x = a$. This can be done with the use of a labeled sign chart.
2. State" $f(x)$ has a relative minimum at $x = a$ because $f'(x)$ changes sign from negative to positive."

*** NOTE ***

To demonstrate that a function $f(x)$ has an ABSOLUTE maximum or minimum on a CLOSED interval $[a, b]$ using the Extreme Value Theorem

(1) You must demonstrate the function is continuous on the closed interval
  a. This can be concluded if $f(x)$ is differentiable on $[a, b]$ since *differentiability implies continuity.*

(2) Demonstrate the following values
  a. $f(b)$ and $f(a)$
  b. $f(c_i)$ where c_i is a critical value of f in the interval $[a, b]$

(3) Conclude that least value of the set $\{f(a), \ldots, f(c_i), \ldots, f(b)\}$ is the absolute minimum, and the greatest value of the set is the absolute maximum of *f on the closed interval* $[a, b]$

Use of the second derivative of a function $f(x)$

The second derivative of a function $f(x)$ is positive if and only if the graph of $f(x)$ is concave up.

The second derivative of a function $f(x)$ is negative if and only if the graph of $f(x)$ is concave down.

(1) Determine when the graph of $f(x)$ is concave up or concave down.

a. To demonstrate that the graph of $f(x)$ is concave up, you must demonstrate that $f''(x) > 0$.

   i. At $x = a$

      1. Demonstrate the equation of $f''(x)$ and evaluate it at $x = a$.
      2. State " $f(x)$ is concave up at $x = a$ because $f''(a)$ is positive."

   ii. On an open interval (a,b)

      1. Demonstrate the equation of $f''(x)$ and make a labeled sign chart.
      2. State " $f(x)$ is concave up on (a,b) because $f''(x)$ is positive."

b. To demonstrate that the graph of $f(x)$ is concave down, you must demonstrate that $f''(x) < 0$.

   i. At $x = a$

      1. Demonstrate the equation of $f''(x)$, and evaluate it at $x = a$.
      2. State " $f(x)$ is concave down at $x = a$ because $f''(a)$ is negative."

   ii. On an open interval (a,b)

      1. Demonstrate the equation of $f''(x)$ and make a labeled sign chart.
      2. State " $f(x)$ is concave down on (a,b) because $f''(x)$ is negative.

(2) Justify whether a critical value of $f(x)$ is a minimum or a maximum

a. To justify that $f(x)$ has a relative maximum at $x = c$

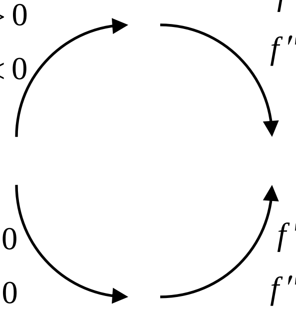
   i. Demonstrate that $f'(c) = 0$ AND $f''(c) < 0$
   ii. State " By the Second Derivative Test $f(x)$ has a relative maximum at $x = c$ because $f'(c) = 0$ and $f''(c) < 0$."

b. To justify that $f(x)$ has a relative minimum at $x = c$

   i. Demonstrate that $f'(c) = 0$ AND $f''(c) > 0$
   ii. State "By the Second Derivative Test $f(x)$ has a relative minimum at $x = c$ because $f'(c) = 0$ and $f''(c) > 0$."

Increasing at a decreasing rate
Slopes are positive, and becoming less positive
(f is increasing by a smaller amount as x increases)

$f'(x) > 0$
$f''(x) < 0$



$f'(x) < 0$
$f''(x) < 0$

Decreasing at a decreasing rate
Slopes are negative, and becoming more negative
(f is decreasing by a larger amount as x increases)

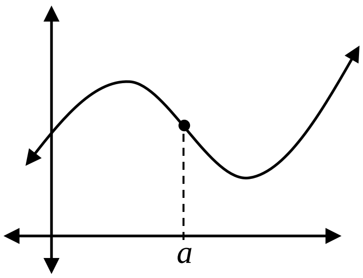
Decreasing at an increasing rate
Slopes are negative, and becoming less negative
(f is decreasing by a smaller amount as x increases)

$f'(x) < 0$
$f''(x) > 0$

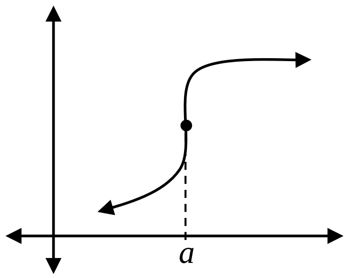
$f'(x) > 0$
$f''(x) > 0$

Increasing at an increasing rate
Slopes are positive, and becoming more positive
(f is increasing by a larger amount as x increases)

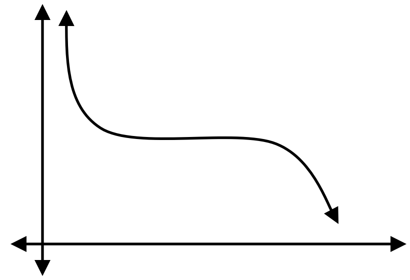
Point of Inflection:
A point of inflection on the graph of $f(x)$ is a coordinate (a,b) where a tangent to the graph of $f(x)$ exists and the graph of $f(x)$ changes concavity. In calculus terms, *the sign of the* <u>*second derivative*</u> *of f immediately changes from positive to negative, or negative to positive at* $x=a$.



Point of inflection



Point of inflection



No point(s) of inflection
Since the second derivative does not *immediately change sign* for any value of *x*.

To JUSTIFY that $f(x)$ has a point of inflection at $x=a$:

1. Make a labeled sign chart for $f''(x)$
2. State " $f(x)$ has a point of inflection at $x=a$ because $f''(x)$ <u>*changes sign*</u> at $x=a$."

<u>EXTREMA DECISION TREE</u>

Find the extrema of $f(x)$

Determine the Critical Values of $f(x)$

$f'(x)=0$ or DNE $\rightarrow x=c_1,c_2,\cdots,c_n$

Bounded Interval / Absolute Min or Max

<u>USE EVT</u>

Test Endpoints and all Critical Values in the bounded interval

$f(a)=$

$\vdots$

$f(c_i)=$

$\vdots$

$f(b)=$

Claim the greatest value as the Absolute Maximum
Claim the least value as the Absolute Minimum

Not closed interval / Relative Min or Max

You CAN make a sign chart for $f'(x)$

You CANNOT make a sign chart for $f'(x)$ [Differential Equations]

2nd Derivative Test:
Demonstrate the values of $f'(c_i)$ and $f''(c_i)$

- $f(x)$ has a relative max at $x=c_i$ because $f'(c_i)=0$ and $f''(c_i)<0$
- $f(x)$ has a relative min at $x=c_i$ because $f'(c_i)=0$ and $f''(c_i)>0$
- If $f''(c_i)=0$, First Derivative Test must be used .

1st Derivative Test:
Make a <u>labeled</u>
Sign Chart for $f'(x)$

- $f(x)$ has a relative max at $x=c_i$ because $f'(x)$ goes from $+\rightarrow-$
- $f(x)$ has a relative min at $x=c_i$ because $f'(x)$ goes from $-\rightarrow+$

<u>Integration</u>

The antiderivative of a function $f(x)$ is a function $F(x)$ such that $\frac{d}{dx}\left[F(x)\right] = f(x)$.
The notation for the antiderivative of $f(x)$ with respect to *x* is $\int f(x)\,dx$.
The *dx* indicates that you are antidifferentiating with respect to *x*.

Since the derivative of a constant is equal to zero, the antiderivative of ANY function must include the *constant of integration*. That is,

$$\int f'(x)\,dx = f(x) + C \text{, where } C \text{ is a constant}$$

Integrals without constants or $\pm\infty$ at the top and bottom of the integral sign are called *indefinite integrals.*

$$\int f(x)\,dx$$

Integrals with constants at the top and bottom of the integral sign are called *definite integrals.* The value below the integral symbol is called the <u>*lower bound*</u> and the value above the integral symbol is called the <u>*upper bound.*</u>

$$\int_a^b f(x)\,dx$$

Integrals with $\pm\infty$ in the upper bound, lower bound, or both lower and upper bounds are called *improper integrals.*

$$\int_a^{\infty} f(x)\,dx,\ \int_{-\infty}^{b} f(x)\,dx,\ \int_{-\infty}^{\infty} f(x)\,dx$$

<u>Basic Integration Formulas:</u>

$\int k \cdot f(u)\,du = k\int f(u)\,du$ $\quad$ $\int f(u) \pm g(u)\,du = \int f(u)\,du \pm \int g(u)\,du$ $\quad$ $\int du = u + C$

$\int a^u\,du = \frac{1}{\ln(a)}a^u + C$ $\quad$ $\int e^u\,du = e^u + C$ $\quad$ $\int \frac{1}{u}\,du = \ln|u| + C$

$\int \cos(u)\,du = \sin(u) + C$ $\quad$ $\int \sin(u)\,du = -\cos(u) + C$ $\quad$ $\int \sec^2(u)\,du = \tan(u) + C$

$\int \csc^2(u)\,du = -\cot(u) + C$ $\quad$ $\int \sec(u)\tan(u)\,du = \sec(u) + C$ $\quad$ $\int \csc(u)\cot(u)\,du = -\csc(u) + C$

$\int \frac{1}{\sqrt{a^2 - u^2}}\,du = \arcsin\left(\frac{u}{a}\right) + C$ $\quad$ $\int \frac{1}{a^2 + u^2}\,du = \frac{1}{a}\arctan\left(\frac{u}{a}\right) + C$ $\quad$ $\int \frac{1}{u\sqrt{u^2 - a^2}}\,du = \frac{1}{a}\operatorname{arcsec}\left(\frac{|u|}{a}\right) + C$

<u>The Definite Integral</u>

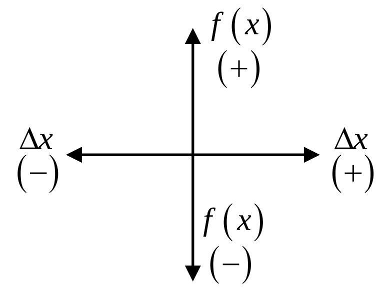
The definite integral of a function $f(x)$ from $x=a$ to $x=b$ is denoted $\int_a^b f(x)\,dx$.

a and *b* are called the lower and upper bounds, respectively. They indicate two important characteristics of the integral.

(1) The interval on the *x*-axis over which to integrate

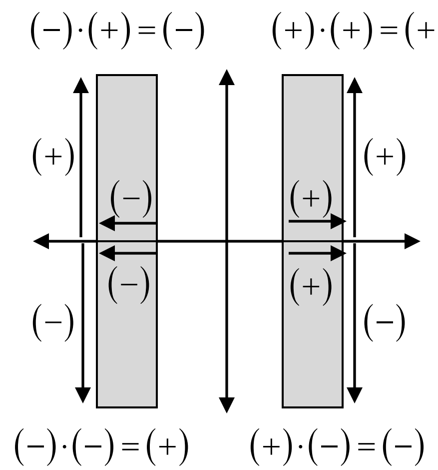
(2) The *direction* of integration

- a. If $a<b$, then integration is directed in the positive *x* direction
    - i. Δx or dx is considered to be a positive value.
- b. If $b<a$, then integration is directed in the negative *x* direction
    - i. Δx or dx is considered to be a negative value.



$f(x)$ is called the integrand.

dx indicates that you are integrating with respect to *x*.



To calculate $\int_a^b f'(x)\,dx$, we do the following: $\int_a^b f'(x)\,dx = f(b)-f(a)$.

The result of the definite integral is a value/number. The value of the definite integral $\int_a^b f'(x)\,dx$ has two very important interpretations in calculus:

(1) Area under the curve of $f'(x)$ from $x=a$ to $x=b$.

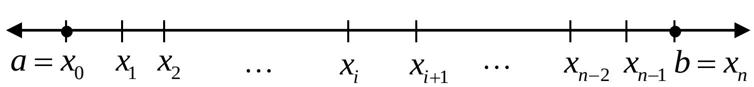
(2) Net Change in $f(x)$ from $x=a$ to $x=b$. [noted $f(b)-f(a)$]

To calculate the exact value of the definite integral, we must use the formula

$$\int_a^b f'(x)\,dx = f(b)-f(a)$$

To *estimate* the value of the definite integral, we use Riemann Sums (Left-Sum, Right-Sum, and Midpoint-Sum).

First we must partition the interval $[a,b]$ into small subintervals such that



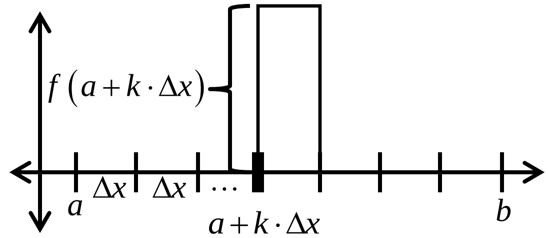
The subintervals need not be of equal length to approximate the integral. However, in order to refine the approximation in such a way to have the successive estimates converge to the value of the integral, we must create a more refined partition [create subintervals of the existing subintervals] such that the width of the longest subinterval approaches zero.

Definite Integral as a Riemann Sum – Uniform Partition

Subintervals of length $\Delta x = \dfrac{b-a}{n}$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{k=0}^{n}\big(f(a+k\cdot\Delta x)\big)\Delta x$$

$$= \lim_{n\to\infty}\sum_{k=0}^{n}\left(f\left(a+k\cdot\left[\frac{b-a}{n}\right]\right)\right)\left[\frac{b-a}{n}\right]$$



When approaching multiple choice questions, there are two components of the structure of the summand that will guide you in choosing the correct answer and eliminating incorrect choices.

1. $\boxed{\dfrac{b-a}{n} \leftrightarrow \Delta x}$

   a. Example: $\int_3^5 x^2\,dx$ will have structure that looks like $\lim_{n\to\infty}\sum_{i=1}^{n}[\text{something}]\cdot\boxed{\dfrac{2}{n}}$

   b. Example: $\lim_{n\to\infty}\sum_{k=0}^{n}[\text{something}]\cdot\dfrac{4}{n}$ will represent a definite integral of length 4.

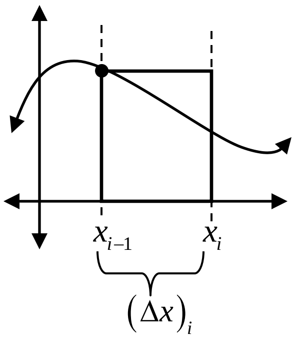
2. $\boxed{f\left(a+k\cdot\left[\dfrac{b-a}{n}\right]\right)}$ will inform you of the lower bound of the integral, i.e. "*a*".

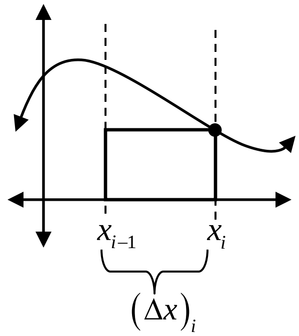
Each *x* in the integral expression will be replaced with $\left[a+k\left(\dfrac{b-a}{n}\right)\right]$

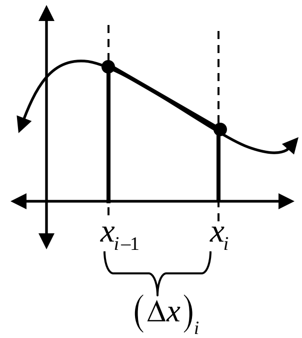
Example: $\lim_{n\to\infty}\sum_{k=0}^{n}\left(2+\underbrace{\left[3+\frac{2k}{n}\right]^2}_{x^2}\right)\cdot\dfrac{2}{n} \leftarrow$ length of interval is 2

$$3+k\left(\frac{2}{n}\right) \rightarrow a=3 \qquad \int_3^{3+2} 2+x^2\,dx$$

Example: $\int_4^7 \sqrt{\boxed{x}+1}\,dx \rightarrow \lim_{n\to\infty}\sum_{k=0}^{n}\left[\sqrt{\boxed{\left(4+k\left(\frac{3}{n}\right)\right)}+1}\right]\cdot\dfrac{3}{n}$



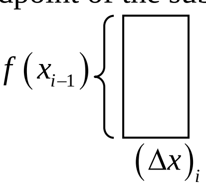




<u>Left Sum</u>

$f(x_{i-1})\cdot(\Delta x)_i$

Create rectangles with width Δx, and height using the function value at the left endpoint of the subinterval

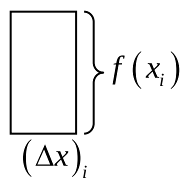


$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(x_{i-1})(\Delta x)_i$$

<u>Right Sum</u>

$f(x_i)\cdot(\Delta x)_i$

Create rectangles with width Δx, and height using the function value at the right endpoint of the subinterval

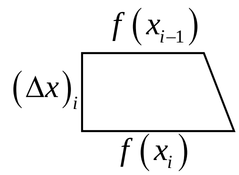


$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(x_i)(\Delta x)_i$$

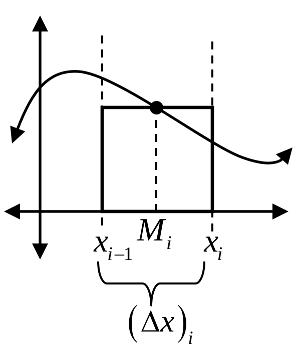
<u>Trapezoidal Sum</u>

$\frac{1}{2}\left(f(x_{i-1})+f(x_i)\right)\cdot(\Delta x)_i$

Create trapezoids with height Δx, and bases using the function values of at the endpoints of the subinterval



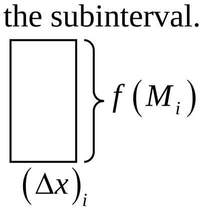
$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} \frac{1}{2}\left[f(x_{i-1})+f(x_i)\right](\Delta x)_i$$



<u>Midpoint Sum:</u>

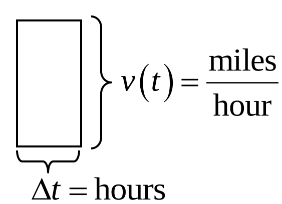
$f(M_i)\cdot(\Delta x)_i$

Create rectangles with width Δx, and height using the function value at the midpoint of the subinterval.



$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(M_i)(\Delta x)_i$$

<u>Integral as Net Change in</u> $f(x)$



If the integrand of a definite integral is a derivative/rate, then the value of the integral gives the net change in the antiderivative of the integrand.

For example, Let $v(t)$ be the velocity of a car given in miles per hour, and t be measured in hours.

$$\text{Area} = \left(\frac{\text{miles}}{\text{hour}}\right)\cdot \text{hours} = \text{ miles}$$

Then $\int_a^b v(t)\,dt$ is approximated by rectangles like those at right:

$$\int_a^b v(t)\,dt$$

This area represents a change in miles over the time interval Δt.

$$\int_a^b \left(\frac{\text{miles}}{\text{hour}}\right)(\text{hour}) \rightarrow \text{miles}$$

This area represents a change in miles over the time interval Δt. Therefore, if we know the value of $p(a)$, then we can find $p(b)$ by using the following:

$$\underset{\substack{\text{position at}\\ \text{time } t=b}}{p(b)} = \underset{\substack{\text{position at}\\ \text{time } t=a}}{p(a)} + \underbrace{\int_a^b v(t)\,dt}_{\substack{\text{net change in}\\ \text{position from}\\ t=a \text{ to } t=b}}$$

$\int_a^b v(t)\,dt =$ net change in position from $t=a$ to $t=b$

$\int_a^b |v(t)|\,dt =$ distance traveled from $t=a$ to $t=b$

Rearranging the terms of the equation we derive Part I of the Fundamental Theorem of Calculus:

$$\underbrace{\int_a^b f'(x)\,dx}_{\substack{\text{net change in } f(x)\\ \text{from } t=a \text{ to } t=b}} = \underbrace{f(b) - f(a)}_{\substack{\text{net change in } f(x)\\ \text{from } t=a \text{ to } t=b}}$$

It is important to note that given
(1) $f'(x)$ and (2) $f(a)$, it is possible to find $f(b)$ by using

$$\underset{\substack{\text{value of } f\\ \text{at } x=b}}{f(b)} = \underset{\substack{\text{value of } f\\ \text{at } x=a}}{f(a)} + \underbrace{\int_a^b f'(x)\,dx}_{\substack{\text{net change in } f\\ \text{from } x=a \text{ to } x=b}}$$

Properties of definite integrals:

$$\int_a^a f(x)\,dx = 0 \qquad \int_b^a f(x)\,dx = -\int_a^b f(x)\,dx \qquad \int_a^b k \cdot f(x)\,dx = k\int_a^b f(x)\,dx$$

$$\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

Integration by Parts:

$$\int uv' = uv - \int u'v$$

$$\begin{array}{ccc} u & & v' \\ \text{differentiate} & \overset{+}{\searrow} & \text{antidifferentiate} \\ & \overset{-}{\searrow} & \\ & \overset{+}{\searrow} & \\ & \overset{-}{\searrow} & \\ & \vdots & \end{array}$$

Indefinite Integals:
Let *a* be any arbitrary value,

$$\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx \qquad \int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx$$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_a^{\infty} f(x)\,dx$$

Part II of the Fundamental Theorem of Calculus

Let $g(x)=\int_a^x f(t)\,dt$. Note that the upper bound is a different variable than the integral.

$g(x)$ is considered the accumulated area of $f(t)$ from $t=a$ to $t=x$.

$$\frac{d}{dx}\left[g(x)\right]=\frac{d}{dx}\left[\int_a^x f(t)\,dt\right]$$
$$g'(x)=f(x)$$

Furthermore, with chain rule,

$$g(x)=\int_a^{h(x)} f(t)\,dt$$
$$\downarrow$$
$$g'(x)=f\left(h(x)\right)\cdot h'(x)$$

It follows that $g''(x)=f'(x)$.

Example: $\frac{d}{dx}\left[\int_2^{x^3}\sin(t)\,dt\right]=\sin\left(x^3\right)\cdot 3x^2$

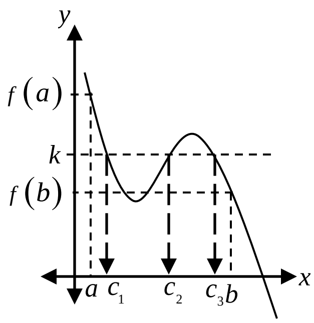
$$\begin{aligned}\frac{d}{dx}\left[\int_{m(x)}^{h(x)} f(t)\,dt\right] &= \frac{d}{dx}\left[\int_a^{h(x)} f(t)\,dt-\int_a^{m(x)} f(t)\,dt\right]\\ &=\frac{d}{dx}\left[\int_a^{h(x)} f(t)\,dt\right]-\frac{d}{dx}\left[\int_a^{m(x)} f(t)\,dt\right]\\ &= f\left(h(x)\right)\cdot h'(x)-f\left(m(x)\right)\cdot m'(x)\end{aligned}$$

Example: Let *a* be any constant you choose.

$$\begin{aligned}\frac{d}{dx}\left[\int_{\cos(x)}^{x^3} t^2+1dt\right] &= \frac{d}{dx}\left[\int_a^{x^3} t^2+1dt-\int_a^{\cos(x)} t^2+1dt\right]\\ &=\frac{d}{dx}\left[\int_a^{x^3} t^2+1dt\right]-\frac{d}{dx}\left[\int_a^{\cos(x)} t^2+1dt\right]\\ &=\left[\left(x^3\right)^2+1\right]\left(3x^2\right)-\left[\cos^2(x)+1\right]\left(-\sin(x)\right)\end{aligned}$$

Existence Theorems

Intermediate Value Theorem: If f is continuous on the closed interval $[a,b]$ and k is any number between $f(a)$ and $f(b)$, then there is *at least* one number c in $[a,b]$ such that $f(c)=k$.



In order to use the Intermediate Value Theorem you must

(1) Demonstrate f is continuous on $[a,b]$

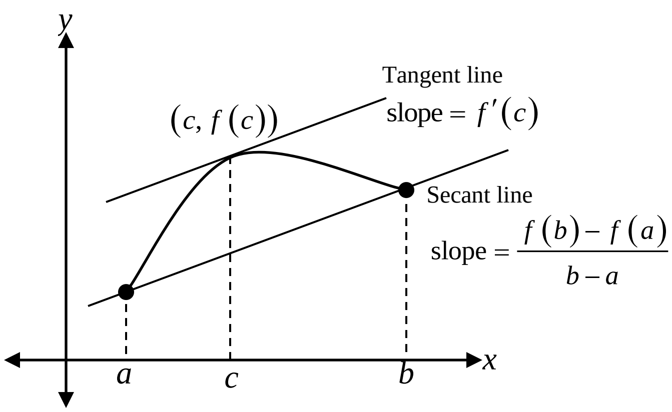
(2) Demonstrate the values $f(b)$ and $f(a)$.

Then you must state the conclusion of the theorem

(3) By IVT, there exists a c in $[a,b]$, such that $f(c)=k$.

Mean Value Theorem for Derivatives: If f is continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b), then there exists a number c in (a,b) such that

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$



$$\underset{\text{slope of tangent}}{f'(c)} = \underbrace{\frac{f(b)-f(a)}{b-a}}_{\text{slope of secant}}$$

$$\underset{\text{instantaneous rate of change of } f \text{ at } x=c}{f'(c)} = \underbrace{\frac{f(b)-f(a)}{b-a}}_{\text{average rate of change of } f \text{ from } x=a \text{ to } x=b}$$

In order to use the Mean Value Theorem you must:

(1) Demonstrate that f is continuous on the closed interval $[a,b]$

(2) Demonstrate the f is differentiable on the open interval (a,b)

(3) Demonstrate the value of $\frac{f(b)-f(a)}{b-a}$

Then you may state the conclusion of the theorem.

(4) By MVT, there exists a c in the interval (a,b) such that $f'(c)=\frac{f(b)-f(a)}{b-a}$.

Rolle's Theorem: If f is a continuous function on the closed interval $[a,b]$, differentiable on the open interval (a,b), and $f(a)=f(b)$, then there exists a c in (a,b) such that $f'(c)=0$.

In order to use Rolle's Theorem you must:

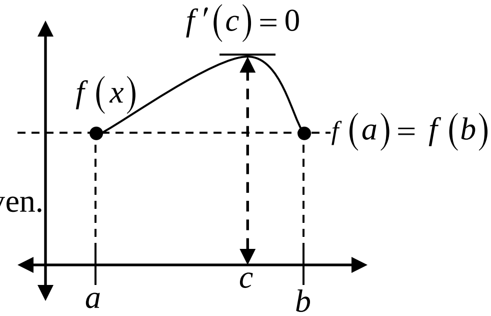
(1) Demonstrate that f is continuous on the closed interval $[a,b]$.

(2) Demonstrate that f is differentiable on the open interval (a,b).

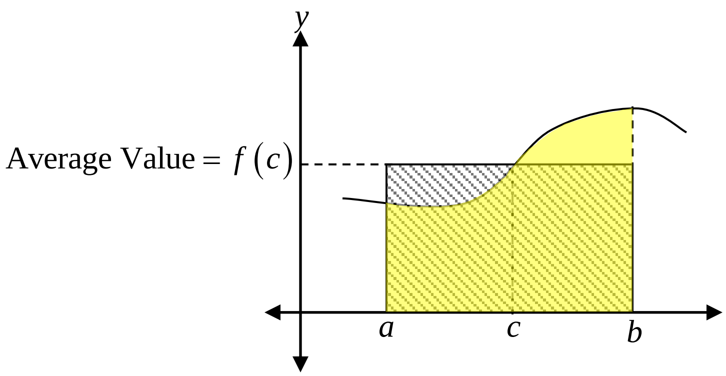
(3) Demonstrate the values of $f(a)$ and $f(b)$, if they are not already given.

Then you must state the conclusion of the theorem:

(4) By Rolle's Theorem, there exists a c in (a,b) such that $f'(c)=0$.



Mean Value Theorem for Integrals: If f is continuous on the closed interval $[a,b]$, then there exists a number c in the closed interval $[a,b]$ such that $\int_a^b f(x)\,dx = f(c)\cdot(b-a)$.



$$\underbrace{\int_a^b f(x)\,dx}_{\text{area under curve}} = \underbrace{f(c)(b-a)}_{\substack{\text{area of rectangle}\\ \text{width } b-a\\ \text{height } f(c)}}$$

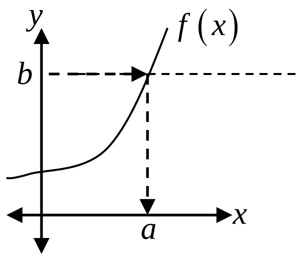
There exists a rectangle whose base has dimension $b-a$, and an appropriate height $f(c)$ such that the ***area of the rectangle has the same area as the area under the curve***. In the figure above, the function achieves its average value only once in the interval $[a,b]$.

<u>The value $f(c)$, which is the height of the rectangle, is the average value of the function $f(x)$ on the closed interval $[a,b]$.</u>

Average Value of $f(x)$ on the interval $[a,b]$: If f is integrable on the closed interval $[a,b]$, then the average value of f on the interval $[a,b]$ is given by:

$$\frac{1}{b-a}\int_a^b f(x)\,dx = \underbrace{f(c)}_{\substack{\text{average value}\\ \text{of } f(x) \text{ from}\\ x=a \text{ to } x=b}}$$

Derivative of the inverse function



$$f^{-1}\left(f(x)\right)=x$$

$$\downarrow$$

$$\left(f^{-1}\right)'\left(f(x)\right)\cdot f'(x)=1$$

$$\left(f^{-1}\right)'\left(\underbrace{f(x)}_{b}\right)=\frac{1}{f'\left(\underbrace{x}_{a}\right)}$$

Given a function $f(x)$, $\left(f^{-1}\right)'(b)=\dfrac{1}{f'(a)}$, where (a,b) is a point on the graph of $f(x)$.

To solve for such a value, first solve the equation $f(x)=b$. The solution will be your *a* value. Then take the derivative of $f(x)$ at $x=a$, and take the reciprocal of this value.

Differential Equations

Implicit differentiation

Not all functions can be written explicitly in terms of *x*. Some functions are implied by an equation. So when you want to take the derivative of an equation that involves both *x* and *y*, you must <u>treat *y* as a function of *x* and apply the chain rule.</u>

$$x^2-2y^3+4y=2\leftrightarrow x^2-2\left[f(x)\right]^3+4f(x)=2$$

Differentiating both sides with respect to *x* we have

$$\frac{d}{dx}\left[x^2-2y^3+4y\right]=\frac{d}{dx}[2]\leftrightarrow\frac{d}{dx}\left[x^2-2\left[f(x)\right]^3+4f(x)\right]=\frac{d}{dx}[2]$$

$$2x-6y^2y'+4y'=0\qquad 2x-6\left[f(x)\right]^2\cdot f'(x)+4f'(x)=0$$

To solve for y' or $\dfrac{dy}{dx}$

(1) Implicitly differentiate the equation.
(2) Move all terms not involving y' to one side, and all the ones that do to the other.
(3) Factor out y', then solve.

To Solve for y''.

(1) Solve for y'
(2) Differentiate y' implicitly.
(3) Get all the terms that involve y'' to one side, everything else to the other.
(4) Solve for y''
(5) Replace all y' using the equation for y' in step one.

A **differential equation** is an equation involving a function and any of its derivatives ($f, f', f'', f''', \ldots$). For example: $3y' + 2yx + 7 - 3y'' = 4$.

<u>The solution to a differential equation is a function</u> $f(x)$ such that when you replace y with $f(x)$, y' with $f'(x)$, y'' with $f''(x)$, etc. the equation is true.

The general solution to a differential equation is a function involving a constant C where any value of C will satisfy the differential equation. This creates a family of functions that are solutions to the given differential equation.

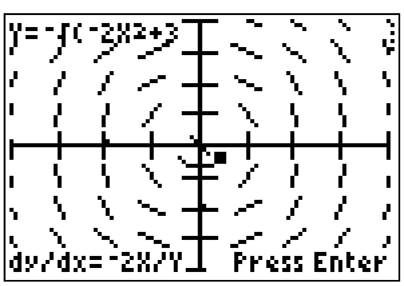
A particular solution to a differential equation is a function satisfying an initial condition, such that the value of the constant C must be fixed based on the initial condition. The initial condition is oftentimes a coordinate that the solution $f(x)$ must pass through.

Separation of variables

To solve a differential equation involving both x and y, use the technique of **<u>separation of variables.</u>**

(1) Isolate y' on one side of the equation, and rewrite it as $\frac{dy}{dx}$.
(2) Factor all the terms involving y out of the side.
(3) Divide both sides by the factor involving y, and multiply both sides by dx to get an equation of the form $(\text{terms involving only } y)\,dy = (\text{terms only involving } x)\,dx$
(4) Antidifferentiate both sides, and solve for y.

$$\frac{dy}{dx} = -\frac{2x}{y}$$
$$ydy = -2xdx$$
$$\int ydy = \int -2xdx$$
$$\frac{1}{2}y^2 = -x^2 + C$$
$$y^2 = -2x^2 + C$$
$$y = \pm\sqrt{-2x^2 + C}$$



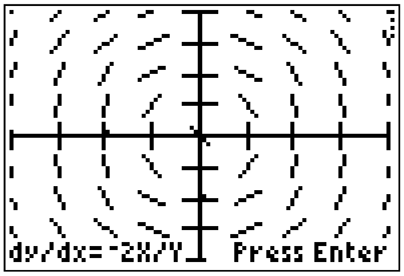
The graph of the particular solution passing through $(1,-1)$ will be restricted to the third and fourth quadrants, because the slope field is undefined along the line $y = 0$.

The particular solution in this case is $y = -\sqrt{-2x^2 + 3}$. It would be incorrect to write $y^2 = -2x^2 + 6$, since this would imply a graph that is both above and below the x-axis.

Slope Fields

For differential equations that are a function of both *x* and *y*, *the value of the derivative will depend on a coordinate* (x, y). That is you need both an *x* and *y* value to determine the overall value of $\frac{dy}{dx}$. For example:

$$\frac{dy}{dx} = 3xy + 2y^2 \text{ evaluated at the point } (2,1) \text{ is } \left.\frac{dy}{dx}\right|_{(2,1)} = 3(2)(1) + 2(1)^2 = 8$$



If you choose selected points on the plane, and draw short line segments passing through the points with the slope equal to the value of the derivative at that point, you will create a slope field.

Be sure to note the values for which $\frac{dy}{dx}$ does not exist. In the case to the right, there is a restriction that $y \neq 0$. Therefore these is a "barrier" that the solution graph cannot pass, the line $y = 0$.

Euler's Method

Some functions are impossible to antidifferentiate, such as e^{-x^2}. In order to figure out what the shape of the solution to the differential equation $\frac{dy}{dx} = e^{-x^2}$ looks like, we use Euler's Method.

First you start off with a coordinate that the solution must pass through (x_0, y_0). You then proceed in the direction indicated by the slope from this point to another point. Decide on a step size *h*, and use this step size to find the next point in the direction of the slope from the given point by:

$$x_1 = x_0 + h \quad \text{and} \quad y_1 = y_0 + h\left(\frac{dy}{dx} \text{ at } (x_0, y_0)\right)$$

Alternately, if you think of $h = \Delta x$, then

$$x_1 = x_0 + \Delta x \quad \text{and} \quad y_1 = y_0 + \Delta x\left(\frac{dy}{dx} \text{ at } (x_0, y_0)\right)$$

In general,

$$x_n = x_{n-1} + \Delta x \quad \text{and} \quad y_n = y_{n-1} + \Delta x\left(\frac{dy}{dx} \text{ at } (x_{n-1}, y_{n-1})\right)$$

You should be able to calculate two iterations of Euler's Method without a calculator!

Arc Length Formulas

Standard	Parametric	Polar
$\int_a^b \sqrt{1+\left[\frac{dy}{dx}\right]^2}\,dx$	$\int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$	$\int_{\theta=a}^{\theta=b} \sqrt{\left[f'(\theta)\right]^2+\left[f(\theta)\right]^2}\,d\theta$

Surface Areas of Revolution

General setup: $\int_a^b \text{radius}\cdot(\text{arc length})\,dx$

Standard

Horizontal Axis of Revolution	Vertical Axis of Revolution
$\int_{x=a}^{x=b} 2\pi r(x)\sqrt{1+\left[\frac{dy}{dx}\right]^2}\,dx$	$\int_{x=c}^{x=d} 2\pi x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$ $\int_{y=a}^{y=b} 2\pi r(y)\sqrt{1+\left[\frac{dx}{dy}\right]^2}\,dy$

Parametric

$x = x(t)$

$y = y(t)$

Axis of Revolution is *x*-axis	Axis of Revolution is *y*-axis
$\int_{t=a}^{t=b} 2\pi y(t)\sqrt{\left[\frac{dx}{dt}\right]^2+\left[\frac{dy}{dt}\right]^2}\,dt$	$\int_{t=a}^{t=b} 2\pi x(t)\sqrt{\left[\frac{dx}{dt}\right]^2+\left[\frac{dy}{dt}\right]^2}\,dy$

Polar

$x = r(\theta)\cdot\cos(\theta)$

$y = r(\theta)\cdot\sin(\theta)$

Axis of Revolution is Polar Axis (*x*-axis)	Axis of Revolution is $\theta=\frac{\pi}{2}$ (*y*-axis)
$\int_{\theta=a}^{\theta=b} 2\pi r(\theta)\sin\theta\sqrt{\left[r(\theta)\right]^2+\left[r'(\theta)\right]^2}\,d\theta$	$\int_{\theta=a}^{\theta=b} 2\pi r(\theta)\cos\theta\sqrt{\left[r(\theta)\right]^2+\left[r'(\theta)\right]^2}\,d\theta$

<u>**Calculus of Parametric Curves:**</u>

Let C be a smooth curve defined by the parametric equations $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$.

Then

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{y'(t)}{x'(t)} \text{ provided that } \frac{dx}{dt} \neq 0$$

The second derivative is found easiest by the fact that you've already determined $\frac{dy}{dx}$ and $\frac{dx}{dt}$:

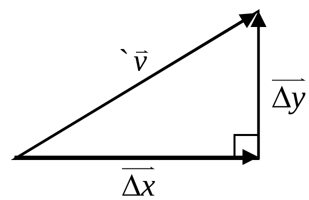
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{\frac{d}{dt}}{\frac{dx}{dt}} \cdot \left[\frac{dy}{dx}\right] = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}}$$

<u>How to Determine the Second Derivative of a Parametric Curve</u>

Step 1: Find $\frac{dy}{dx}$

Step 2: Differentiate $\frac{dy}{dx}$ with respect to t.

Step 3: Divide the result in Step #2 by the expression for $\frac{dx}{dt}$



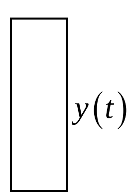
If the position of a particle at time t is given by $\left(x(t), y(t)\right)$, then

$$\text{Velocity vector } \vec{v} = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle, \text{ where speed is } |\vec{v}| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}.$$

$$\text{Acceleration Vector } \vec{a} = \left\langle \frac{d^2x}{dt^2}, \frac{d^2y}{dx^2} \right\rangle$$

Area under a parametric curve:

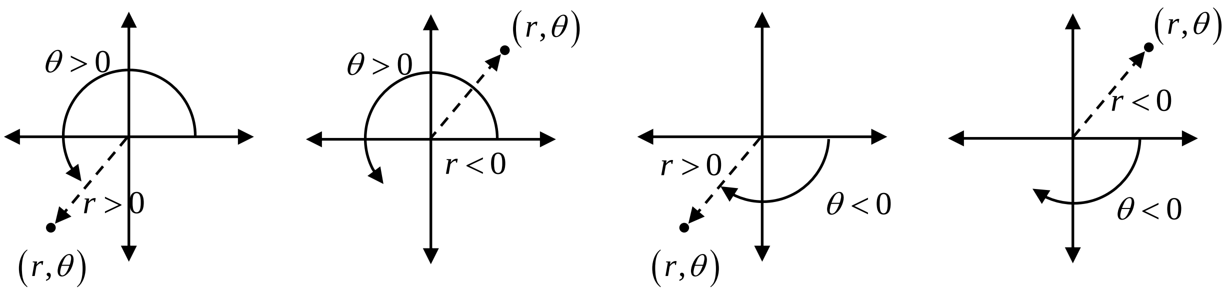
$$\int_{t=a}^{t=b} y(t)\,x'(t)\,dt$$



$$\Delta x(t) = \frac{dx}{\cancel{dt}} \cdot \cancel{dt}$$

Calculus of Polar Functions

$(x,y)\leftrightarrow(r,\theta)$ Where $r=f(\theta)$	To plot a point (r,θ) in polar form (1) Rotate by θ radians in the appropriate direction ✓ Counterclockwise for $\theta>0$ ✓ Clockwise for $\theta<0$ (2) Extend from the origin the appropriate magnitude $\lvert r\rvert$ and proper direction ✓ For $r>0$, extend in the direction of the terminal side of θ by $\lvert r\rvert$ ✓ For $r<0$, extend in the opposite direction of the terminal side of θ by $\lvert r\rvert$



To convert from rectangular to polar, and vice versa

$$x^2+y^2=r^2$$

$$x=r\cos\theta \quad\leftrightarrow\quad x=r(\theta)\cos\theta$$

$$y=r\sin\theta \quad\leftrightarrow\quad y=r(\theta)\sin\theta$$

$$\frac{d}{d\theta}[y]=\frac{d}{d\theta}\left[r(\theta)\sin(\theta)\right] \qquad \frac{d}{d\theta}[x]=\frac{d}{d\theta}\left[r(\theta)\cos(\theta)\right]$$

$$\frac{dy}{d\theta}=r'(\theta)\sin(\theta)+r(\theta)\cos(\theta) \qquad \frac{dx}{d\theta}=r'(\theta)\cos(\theta)-r(\theta)\sin(\theta)$$

Using these equations, we can determine $\frac{dy}{dx}$ by the following:

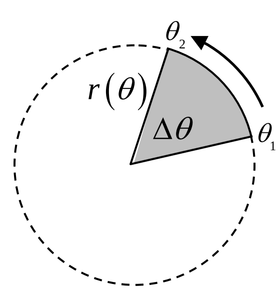
$$\frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=\frac{r(\theta)\cos\theta+r'(\theta)\sin\theta}{-r(\theta)\sin\theta+r'(\theta)\cos\theta}$$

To find the area enclosed by a polar curve, you can derive the integral from the area of a sector.

$$\underset{\text{area of circle}}{\pi\left[r(\theta)\right]^2}\cdot\underset{\text{fraction of circle}}{\frac{d\theta}{2\pi}}$$

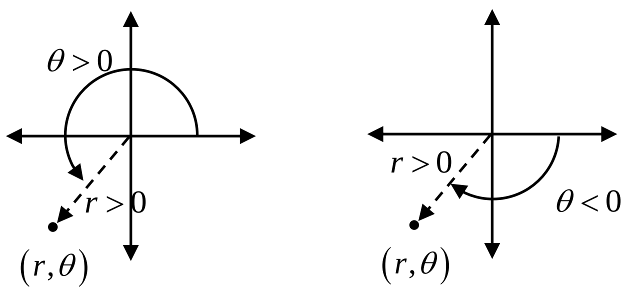
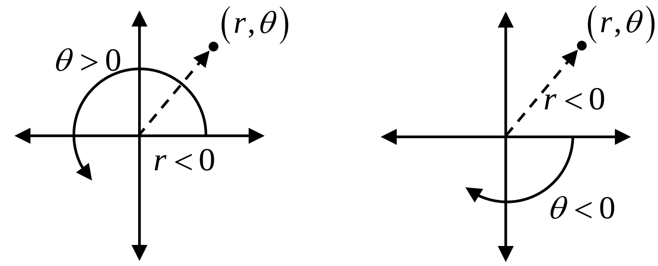
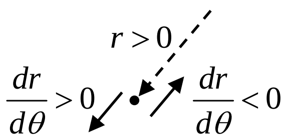
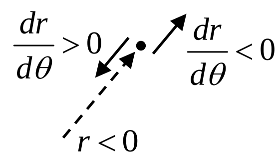
$$\downarrow$$

$$\frac{1}{2}\int_{\theta_1}^{\theta_2}\left[r(\theta)\right]^2\,d\theta$$



How to determine whether a particle is moving towards or away from the origin:

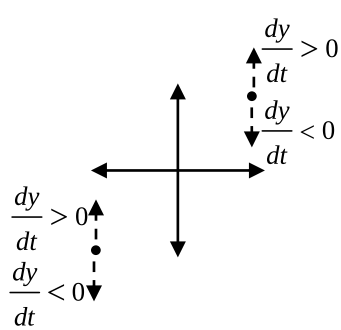
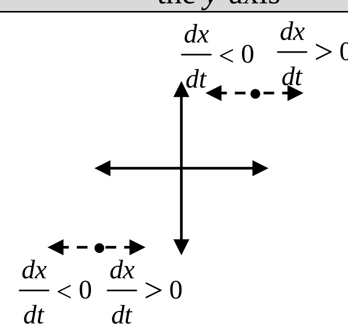
Determine whether r is positive or negative, AND whether $\frac{dr}{d\theta}$ is positive or negative.

$r > 0$	$r < 0$
	
	

If r and $\frac{dr}{d\theta}$ have the same sign, then the particle is moving away from the origin.

If r and $\frac{dr}{d\theta}$ have opposite signs, then the particle is moving towards the origin.

How to determine whether a point is moving towards or away from the *x*-axis or *y*-axis:

the *x*-axis				the *y*-axis			
							
$y > 0$		$y < 0$		$x < 0$		$x > 0$	
$\frac{dy}{dt} < 0$	$\frac{dy}{dt} > 0$	$\frac{dy}{dt} < 0$	$\frac{dy}{dt} > 0$	$\frac{dx}{dt} < 0$	$\frac{dx}{dt} > 0$	$\frac{dx}{dt} < 0$	$\frac{dx}{dt} > 0$
towards *x*-axis	away from *x*-axis	away from *x*-axis	towards *x*-axis	away from *y*-axis	towards *y*-axis	towards *y*-axis	away from *y*-axis

L'Hopital's Rule:

If $\lim_{x\to c}\frac{f(x)}{g(x)}=\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$, then $\lim_{x\to c}\frac{f(x)}{g(x)}=\lim_{x\to c}\frac{f'(x)}{g'(x)}$

Indeterminate forms other than $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$ that can be handled by L'Hopital's Rule:

$0\cdot\infty$	0^0	1^∞	∞^0
$0\cdot\infty=\frac{0}{\frac{1}{\infty}}$ $\downarrow$ $=\frac{0}{0}$	$y=0^0$ $\downarrow$ $\ln(y)=\ln(0^0)$ $\ln(y)=0\cdot\ln(0)$ $\ln(y)=0\cdot-\infty$ $\ln(y)=\frac{-\infty}{\frac{1}{0}}$ $\downarrow$ $\ln(y)=\frac{-\infty}{\infty}$	$y=1^\infty$ $\downarrow$ $\ln(y)=\ln(1^\infty)$ $\ln(y)=\infty\cdot\ln(1)$ $\ln(y)=\infty\cdot 0$ $\downarrow$ $\ln(y)=\frac{0}{\frac{1}{\infty}}$ $\ln(y)=\frac{0}{0}$	$y=\infty^0$ $\downarrow$ $\ln(y)=\ln(\infty^0)$ $\ln(y)=0\cdot\ln(\infty)$ $\ln(y)=\frac{\ln(\infty)}{\frac{1}{0}}$ $\downarrow$ $\ln(y)=\frac{\infty}{\infty}$

You should recognize the limit definition of the number *e*:

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e$$

Furthermore, if *a* is a constant where $a\neq 0$, then

$$\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n=e^a$$

Geometric Series

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

(1) $n=0$ or the formula will not work.

(2) $|r|<1$ or the series diverges.

Alternately, if ar^k is the first term of the geometric series, then

$$ar^k + ar^{k+1} + ar^{k+2} + \cdots = \sum_{n=k}^{\infty} ar^n = \frac{ar^k}{1-r} = \frac{\text{first term}}{1-\text{common ratio}}$$

Remainder Theorem for Alternating Series

The remainder of an alternating series is bounded by the absolute value of the first neglected term. Given $\sum_{n=1}^{\infty} (-1)^n a_n = S$, then

$$\left| S - \sum_{n=1}^{k} (-1)^n a_n \right| \le |a_{k+1}|$$

$$|\text{Remainder}| \le |\text{next term}|$$

Taylor Series:

The n^{th} degree Taylor Series Polynomial for a function $f(x)$ about the point $(c, f(c))$ is a polynomial of degree n that approximates the function around $x=c$ such that $\rightarrow$

$$p(c) = f(c)$$
$$p'(c) = f'(c)$$
$$p''(c) = f''(c)$$
$$\downarrow$$
$$p^{(n)}(c) = f^{(n)}(c)$$

The general form for the Taylor Series of $f(x)$ centered at $x=c$ is given by

$$f(x) \approx f(c) + f'(c)(x-c) + \frac{f''(c)(x-c)^2}{2!} + \frac{f'''(c)(x-c)^3}{3!} + \cdots + \frac{f^{(n)}(c)(x-c)^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)(x-c)^n}{n!}$$

The **Lagrange Error Bound** for the n^{th} degree Taylor Series Polynomial centered at $x=c$ is given by

$$R_n(x) \le \frac{|x-c|^{n+1}}{(n+1)!} \cdot \max_{\text{where } z \text{ is between } x \text{ and } c} \left| f^{(n+1)}(z) \right|$$

Series you must memorize

Series	Radius of Convergence
$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$	$-\infty < x < \infty$
$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$	$-\infty < x < \infty$
$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$	$-\infty < x < \infty$

$$\sum_{i=1}^{\infty}\frac{1}{(n+1)(n+2)}$$

$$\frac{1}{(n+1)(n+2)}=\frac{a}{n+1}+\frac{b}{n+2}$$

$$=\frac{a(n+2)}{(n+1)(n+2)}+\frac{b(n+1)}{(n+2)(n+1)}$$

$$=\frac{an+2a+bn+b}{(n+1)(n+2)}$$

$$\downarrow$$

$$0n+1=(a+b)n+(2a+b)$$

$$\downarrow$$

$$\begin{cases}a+b=0\\2a+b=1\end{cases}$$

$$\downarrow$$

$$a=1,\ b=-1$$

$$\downarrow$$

$$\frac{1}{(n+1)(n+2)}=\frac{1}{n+1}+\frac{1}{n+2}$$

Partial Fractions!

Sequence of Partial Sums

$$S_1=\sum_{i=1}^{1}\frac{1}{n+1}-\frac{1}{n+2}=\frac{1}{2}-\frac{1}{3}$$

$$S_2=\sum_{i=1}^{2}\frac{1}{n+1}-\frac{1}{n+2}=\left(\frac{1}{2}-\cancel{\frac{1}{3}}\right)+\left(\cancel{\frac{1}{3}}-\frac{1}{4}\right)$$

$$S_3=\sum_{i=1}^{3}\frac{1}{n+1}-\frac{1}{n+2}=\left(\frac{1}{2}-\cancel{\frac{1}{3}}\right)+\left(\cancel{\frac{1}{3}}-\cancel{\frac{1}{4}}\right)+\left(\cancel{\frac{1}{4}}-\frac{1}{5}\right)$$

$$\vdots$$

$$S_k=\sum_{i=1}^{k}\frac{1}{n+1}-\frac{1}{n+2}=\left(\frac{1}{2}-\cancel{\frac{1}{3}}\right)+\left(\cancel{\frac{1}{3}}-\cancel{\frac{1}{4}}\right)+\left(\cancel{\frac{1}{4}}-\cancel{\frac{1}{5}}\right)+\cdots+\left(\cancel{\frac{1}{k+1}}-\frac{1}{k+2}\right)$$ ← Telescoping Series

$$\boxed{\begin{aligned}\lim_{k\to\infty}S_k&=\lim_{k\to\infty}\frac{1}{2}+\frac{1}{k+2}\\&=\frac{1}{2}+0\\&=\frac{1}{2}\end{aligned}}$$

Remember that the most basic way to show that a series converges is to show that the *sequence* of partial sums Converges.

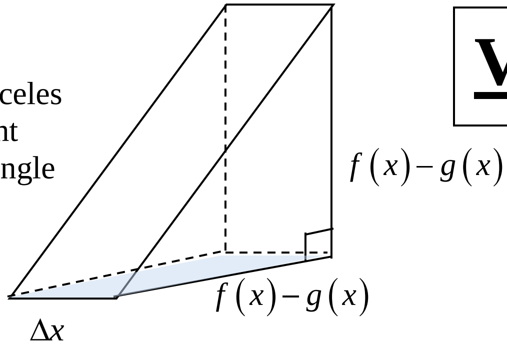
$$\lim_{n\to\infty}S_n=k$$

$$\text{Where }S_n=a_1+a_2+\cdots+a_n=\sum_{i=0}^{n}a_i$$

$$\Rightarrow \int_a^b (\text{cross sectional area})\,dx \Leftarrow$$

Volumes on the Base Cross Sections

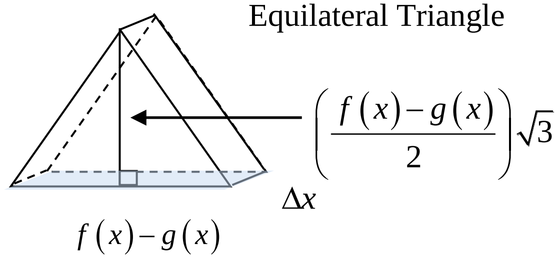
Isosceles Right Triangle



$$V_{\text{slice}} = \frac{1}{2}\left(f(x)-g(x)\right)^2 \Delta x$$

$$V_{\text{solid}} = \int_a^b \frac{1}{2}\left(f(x)-g(x)\right)^2 dx$$

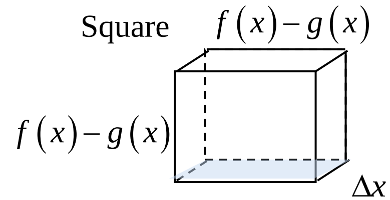
Equilateral Triangle



$$V_{\text{slice}} = \frac{1}{2}\left(f(x)-g(x)\right)\left(\frac{f(x)-g(x)}{2}\right)\sqrt{3}\cdot\Delta x$$

$$V_{\text{solid}} = \int_a^b \frac{1}{2}\left(f(x)-g(x)\right)\left(\frac{f(x)-g(x)}{2}\right)\sqrt{3}dx$$

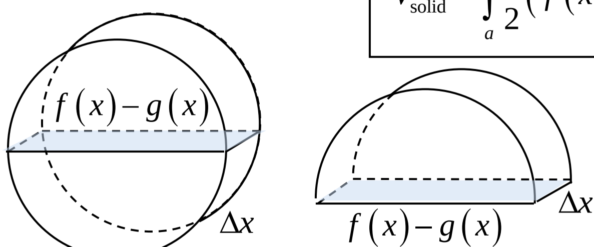
Square



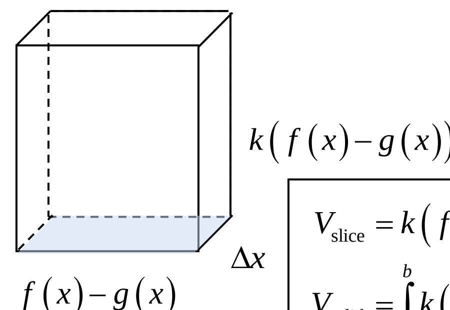
$$V_{\text{slice}} = \left(f(x)-g(x)\right)^2 \Delta x$$

$$V_{\text{solid}} = \int_a^b \left(f(x)-g(x)\right)^2 dx$$

Disk or Semicircle



Rectangle



$$V_{\text{slice}} = k\left(f(x)-g(x)\right)^2 \Delta x$$

$$V_{\text{solid}} = \int_a^b k\left(f(x)-g(x)\right)^2 dx$$

Disc

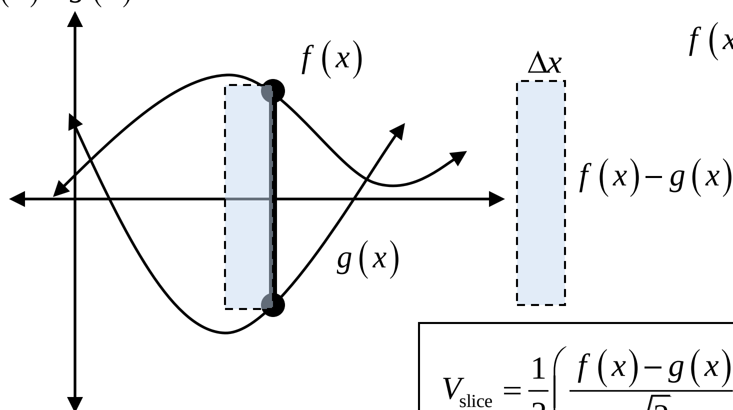
$$V_{\text{slice}} = \pi\left(\frac{f(x)-g(x)}{2}\right)^2 \Delta x$$

$$V_{\text{solid}} = \int_a^b \pi\left(\frac{f(x)-g(x)}{2}\right)^2 dx$$

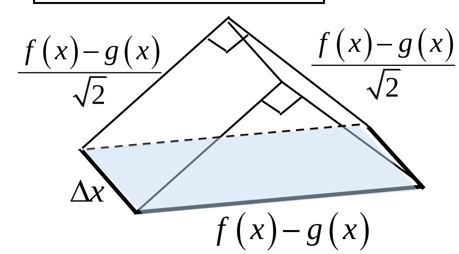
Semicircle

$$V_{\text{slice}} = \frac{1}{2}\pi\left(\frac{f(x)-g(x)}{2}\right)^2 \Delta x$$

$$V_{\text{solid}} = \int_a^b \frac{1}{2}\pi\left(\frac{f(x)-g(x)}{2}\right)^2 dx$$

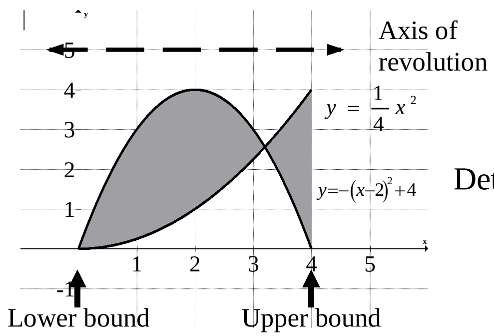


Isosceles Triangle Hypotenuse in the *x*-*y* plane.



$$V_{\text{slice}} = \frac{1}{2}\left(\frac{f(x)-g(x)}{\sqrt{2}}\right)^2 \Delta x$$

$$V_{\text{solid}} = \int_a^b \frac{1}{2}\left(\frac{f(x)-g(x)}{\sqrt{2}}\right)^2 dx$$



Horizontal Axis of Revolution

↓

Integrate with respect to $x \rightarrow \int \underline{\underline{dx}}$

↓

Determine the greatest and least x-coordinates of your region

Greatest→ Upper Bound

Least →Lower Bound

↓

Determine each subinterval on the x-axis for which the greater and lesser functions change.

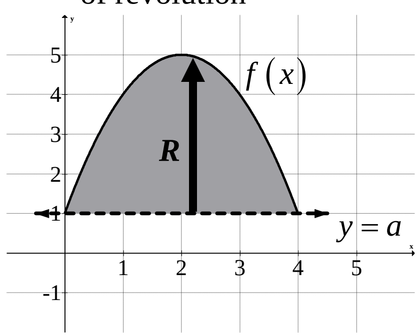
FOR EACH SUBINTERVAL

↓

Determine whether your region lies above or below the axis of revolution

Above the axis of revolution

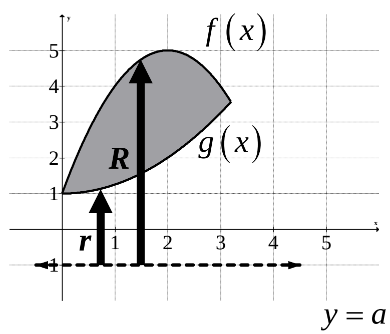
Adjacent to axis of revolution



$$\pi\int_c^d \left[R(x)\right]^2 dx$$

$$R(x)=f(x)-a$$

Not adjacent to axis of revolution



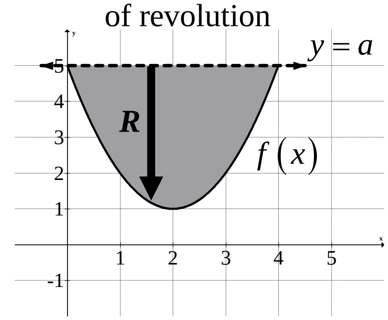
$$\pi\int_c^d \left[R(x)\right]^2-\left[r(x)\right]^2 dx$$

$$R(x)=f(x)-a$$

$$r(x)=g(x)-a$$

Below the axis of revolution

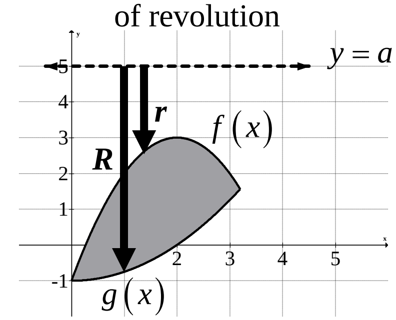
Adjacent to axis of revolution



$$\pi\int_c^d \left[R(x)\right]^2 dx$$

$$R(x)=a-f(x)$$

Not adjacent to axis of revolution



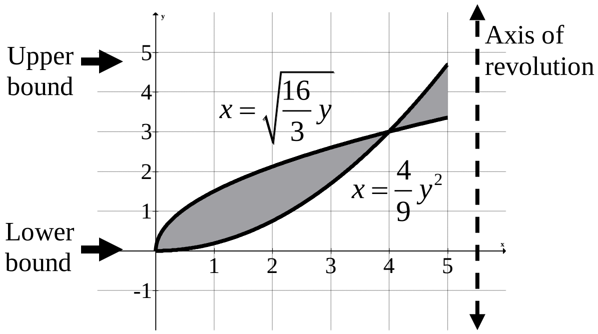
$$\pi\int_c^d \left[R(x)\right]^2-\left[r(x)\right]^2 dx$$

$$R(x)=a-g(x)$$

$$r(x)=a-f(x)$$

$R(x)$ is always measured from the axis of revolution to the function farthest from axis of revolution (i,e. outside edge of the region.)

$r(x)$ is always measured from the axis of revolution to the function closest to the axis of revolution (i.e. inside edge of the region.)



Vertical Axis of Revolution

↓

Integrate with respect to $y \rightarrow \int \underline{\underline{dy}}$

Rewrite your functions as functions of y: $x = f(y)$

↓

Determine the greatest and least y-coordinates of your region

Greatest→ Upper Bound

Least →Lower Bound

↓

Determine each subinterval on the y-axis for which the greater and lesser functions change.

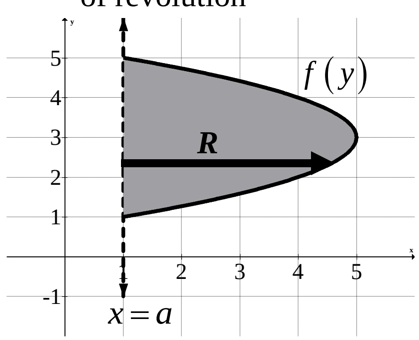
FOR EACH SUBINTERVAL

↓

Determine whether your region lies to the right or to the left of the axis of revolution

To the right of the axis of revolution

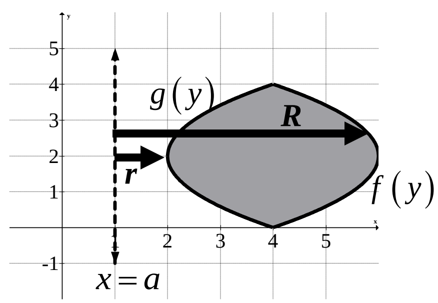
Adjacent to axis of revolution



$$\pi\int_c^d \left[R(y)\right]^2 dy$$

$$R(y) = f(y) - a$$

Not adjacent to axis of revolution



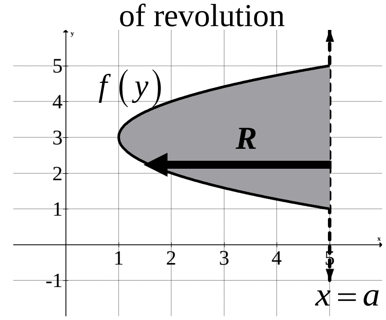
$$\pi\int_c^d \left[R(y)\right]^2 - \left[r(y)\right]^2 dy$$

$$R(y) = f(y) - a$$

$$r(y) = g(y) - a$$

To left of the axis of revolution

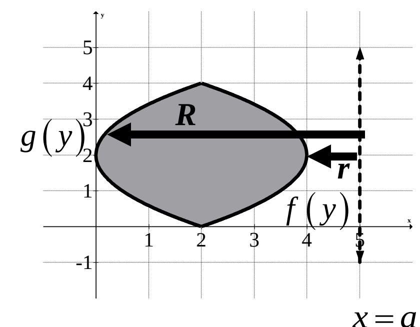
Adjacent to axis of revolution



$$\pi\int_c^d \left[R(y)\right]^2 dy$$

$$R(y) = a - f(y)$$

Not adjacent to axis of revolution



$$\pi\int_c^d \left[R(y)\right]^2 - \left[r(y)\right]^2 dy$$

$$R(y) = a - g(y)$$

$$r(y) = a - f(y)$$

$R(x)$ is always measured from the axis of revolution to the function farthest from axis of revolution (i,e. outside edge of the region.)

$r(x)$ is always measured from the axis of revolution to the function closest to the axis of revolution (i.e. inside edge of the region.)

You suspect that $\sum_{n=c}^{\infty} a_n$ **CONVERGES**

Identify the type of series

Geometric Series $\sum_{n=c}^{\infty} ar^n$	***p*-series** $\sum_{n=c}^{\infty} \frac{1}{n^p}$	**Telescoping Series** $\sum_{n=c}^{\infty} (a_n - a_{n+1})$	**Alternating Series** $\sum_{n=c}^{\infty} (-1)^n a_n$	**None of these**
Show that $\lvert r\rvert < 1$ $\text{Sum} = \frac{\text{first term}}{1-\text{common ratio}}$	Show that $p > 1$	Show that the partial sums converge $\lim_{n\to\infty} S_n$ exists	Show the Alternating Series Test holds Verify: "Alternating series whose terms decrease in absolute value to zero." $\text{Remainder} \le \lvert\text{next term}\rvert$	

"Behaves like" Test

Integral Test	****Ratio Test****	****Root Test****	**Direct Comparison Test**	**Limit Comparison Test**
Show that all are true: $f(n) = a_n$ is 1. Positive 2. Continuous 3. Decreasing 4. $\int_0^{\infty} f(x)$ converges *	Show that $\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert < 1$ If the limit is one, the test is inconclusive.	Show that $\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} < 1$ If the limit is one, the test is inconclusive	Write down a series $\sum_{n=k}^{\infty} b_n$ and demonstrate the following 1. $a_n, b_n > 0$ for all $n \ge k \ge c$ 2. $\sum_{n=k}^{\infty} b_n$ converges* 3. $a_n \le b_n$ for all $n \ge k$ *	Write down a series $\sum_{n=k}^{\infty} b_n$ and demonstrate: 1. $a_n, b_n > 0$ for all $n \ge k \ge c$ 2. $\sum_{n=k}^{\infty} b_n$ converges* 3. $\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$ * L must be finite and positive

*Must be demonstrated ** Use these tests to determine the interval of convergence of a given power series.

$\lim_{n\to\infty} \sqrt[n]{c} = 1 \ (c > 0)$; $\lim_{n\to\infty} \left(1 + \frac{c}{n}\right)^n = e^c$; $\lim_{n\to\infty} \sqrt[n]{n} = 1$; $\lim_{n\to\infty} \sqrt[n]{n!} = \infty$; $\lim_{n\to\infty} \sqrt[n]{n^c} = 1$; **Growth Order:** $c < \ln(\ln(n)) < \ln(n) < [\ln(n)]^c < n^c < c^n < n! < n^n$

You suspect that $\sum_{n=c}^{\infty} a_n$ **DOES NOT CONVERGE**

Test if $\lim_{n\to\infty} a_n \neq 0$

$\neq 0$: $\sum_{n=c}^{\infty} a_n$ Does not converge. Done.

(Diverges if $\lim_{n\to\infty} a_n = c \neq 0$)

$= 0$: Identify the type of series

Geometric Series $\sum_{n=c}^{\infty} ar^n$	***p*-series** $\sum_{n=c}^{\infty} \frac{1}{n^p}$	**Telescoping Series** $\sum_{n=c}^{\infty} (a_n - a_{n+1})$	**Alternating Series** $\sum_{n=c}^{\infty} (-1)^n a_n$	**None of these**
Show that $\lvert r \rvert \geq 1$	Show that $p < 1$	Show that the partial sums do not converge $\lim_{n\to\infty} S_n = \pm\infty$ or DNE	Show that $\lim_{n\to\infty} a_n \neq 0$ or $\lim_{n\to\infty} a_n \to \infty$ or DNE	

"Behaves like" Test

Integral Test	**Ratio Test**	**Root Test**	**Direct Comparison Test**	**Limit Comparison Test**
Show that $f(n) = a_n$ is 1. Positive 2. Continuous 3. Decreasing 4. $\int_0^{\infty} f(x)$ diverges *	Show that $\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert > 1$ If the limit is one, the test is inconclusive.	Show that $\lim_{n\to\infty} \sqrt[n]{\lvert a_n \rvert} > 1$ If the limit is one, the test is inconclusive	Write down a series $\sum_{n=k}^{\infty} b_n$ and demonstrate the following: 1. $a_n, b_n > 0$ for all $n \geq k \geq c$ 2. $\sum_{n=k}^{\infty} b_n$ divereges* 3. $b_n \leq a_n$ for all $n \geq k$ *	Write down a series $\sum_{n=k}^{\infty} b_n$ and demonstrate: 1. $a_n, b_n > 0$ for all $n \geq k \geq c$ 2. $\sum_{n=k}^{\infty} b_n$ Diverges* 3. $\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$ * L is finite and positive

* Must be demonstrated* $\lim_{n\to\infty} \sqrt[n]{c} = 1 \ (c > 0)$; $\lim_{n\to\infty} \left(1 + \frac{c}{n}\right)^n = e^c$; $\lim_{n\to\infty} \sqrt[n]{n} = 1$; $\lim_{n\to\infty} \sqrt[n]{n!} = \infty$; $\lim_{n\to\infty} \sqrt[n]{n^c} = 1$; **Growth**

Order: $c < \ln(\ln(n)) < \ln(n) < [\ln(n)]^c < n^c < c^n < n! < n^n$