ERROR AND UNCERTAINTY

Most experiments are designed to test a hypothesis or theoretical prediction. However, it is very rare that the experimental result will be an exact match for the theoretical prediction. How, then, can you tell whether the difference between experiment and theory reflects a real physical effect or just a measurement error? In order to make this determination, every experimental result must have an accompanying uncertainty estimate: that is, experiments must estimate the amount of possible error due to experimental methods. This uncertainty can then be compared to the difference between the experimental result and the theoretical prediction to draw conclusions about how well the data fits the theory.

A good example of the importance of considering error and uncertainty comes from a 2011 experiment run the OPERA collaboration in Europe. Established physical theory predicts that nothing can go faster than the speed of light, and many experiments have been done to test this theory. For example, in 2011, OPERA measured a particle called a neutrino making a 454 mile trip 0.000000060 seconds (60 nanoseconds) faster than light. Is this difference big enough that we should reject the theory that nothing can go faster than light? We need to know the uncertainty of the measurements involved to find out!

DIFFERENCES VS. UNCERTAINTY

Students often confuse differences and uncertainty. Understanding these terms is critical to understanding how to interpret data,

- **Differences: Error/ discrepancy/ difference** refer to the **difference between** an experimental result and a known value, a theoretical prediction, or another experimental result. Error, difference, and discrepancy have slightly different meanings, as described below, but they all quantify how close the experimental result was to the theoretical value. The important thing to remember about these quantities is that you get these numbers **after** your experiment is done by comparing your result to some prediction or other result. These numbers should never alter your experiment or the uncertainty estimates.
    - **Error** refers to a difference between an experimental result and the true value of physical quantity. In situations where the theoretical value of a physical quantity is broadly accepted and therefore assumed to be true, the error can be calculated by subtracting the accepted value from the experimental value. In most experiments, the error is unknowable, since there is no way to know the true value of most physical quantities.
    - **Discrepancy** refers to a difference between an experimental result and a theory that is being tested. This is the most commonly calculated quantity, as most experiments aim to test a theory. Discrepancy is calculated by subtracting the theoretical value from the experimental value.
    - **Difference** usually refers to a difference between two experimental results. It is sometimes used interchangeably with the terms “discrepancy” and “error”, however.
- **Uncertainty: Uncertainty/possible error** refer to the amount that the measurements **could be off** from the true value due to **bias or limitations in the experimental apparatus**. The terms “uncertainty” and “possible error” mean the same thing, so you

can use them interchangeably. You get these numbers by estimating the uncertainty of each measurement you, and combining these uncertainties to find a total uncertainty. The most important mistake to avoid is that your uncertainty should NEVER depend on the error/discrepancy/difference: if you base your uncertainty on what you think the result "should" be, (rather than an assessment of the accuracy of your experimental methods) you are not really conducting science: you are just setting your uncertainty so that your results "agree" with a given theory.

Experimental conclusions are drawn by **comparing the error/difference/discrepancy to the uncertainty/possible error**. For example, the OPERA scientists discussed above estimated their uncertainty to be 10 nanoseconds. This means that they thought their data could be off by no more than 10 nanoseconds from the actual value. Since the neutrinos were measured to be travelling 60 nanoseconds faster than light, the difference was bigger than the uncertainty, which led the scientists to claim they had found a particle that moved faster than light. **In our class, when the difference is larger than the uncertainty, that difference will be considered a significant difference**. In more rigorous approaches, scientists account for features of the data such as the number of data points to determine statistical significance, so more complex statistical tests (t-tests, z-tests, etc.) are used.

In the end, the OPERA scientists discovered that a faulty piece of equipment had biased their results. That is, the uncertainty in their results had actually been much higher than the 10 nanosecond uncertainty they had claimed. With the new, higher uncertainty, they could no longer claim to have measured a significant difference between the speed of the neutrino and the speed of light.

Quick Quiz 1 *(quick quiz answers are at the end of the introduction)*
In lab you measure the value of the acceleration due to gravity to be 9.7 m/s^2. The accepted value is 9.8 m/s^2. The difference between these two values (9.7 - 9.8 = -0.1 m/s^2) could be described as:

a) Error
b) Uncertainty
c) Both of the above

ESTIMATING UNCERTAINTY

Uncertainty (i.e. possible error) is due limitations in the accuracy and/or precision of the experimental apparatus and design. Each measurement you make should have an uncertainty associated with it. Some instruments will state the uncertainty (described as the accuracy of the instrument) in their documentation. For others, you must consider the limitations in the instrument's accuracy and/or precision. When an instrument accurate but not precise (that is, the results are noisy but not biased in a particular direction), you can estimate the uncertainty by doing multiple measurements and seeing how much they vary. (The standard deviation and standard error are used for this purpose). In other cases, if you suspect an instrument may have a consistent bias in one direction, you must estimate the possible size of the bias by considering the equipment and procedure. For example, the accuracy of a handheld stopwatch is limited by the reaction time of the user. A typical human reaction time of 0.1s would be a reasonable uncertainty estimate. If you time a runner running a sprint to take 12.3 s, you could write that the race time was 12.3 $\pm$ 0.1 s.

Below is a table of some of the common ways to estimate the uncertainty of a measurement. Only by careful consideration of your apparatus, experimental setup, and the likely sources of error can you choose the correct method to estimate the uncertainty of a measurement.

Method	**Pros**	**Cons**
Standard error of multiple measurements of the same quantity.	• Does not require estimation	• Time consuming to do multiple trials • Does not capture systematic errors (biases) • Assumes a normal distribution of errors.
Measurement rule (half the smallest marking on the device).	• Easy to implement	• Assumes error is only due to lack of precision of the instrument. Often, other sources of error are more important.
Estimation based on knowledge of the device and methods	• Gets you to think about all sources of uncertainty	• Subjective. • Sources of uncertainty may not be obvious.
Accuracies reported in device manual	• Objective source	• Assumes device is the main source of error. • May not always be available.
Calibration	• Best way to get accurate error estimates.	• Often experimentally impractical

Note that if many measurements of the same type are taken in an experiment, it is often useful to express all of their uncertainties as having the same absolute uncertainty, although some times, the uncertainty in a measurement will vary with the magnitude of the measurement.

Quick Quiz 2
A standard ruler has millimeter markings. You use the ruler to measure the length of a rectangular block of wood, and get a result of 12.35 cm. What is a reasonable estimate of the uncertainty?

a) ± 1 cm
b) ± 0.5 cm
c) ± 0.05 cm
d) ± 0.01 cm

Relative and Absolute Quantities
It's often useful to express differences and uncertainties as percentages: these are called **relative** uncertainties/differences. Regular uncertainties and differences (with units) are called **absolute**. For example, the absolute uncertainty of the measurement of the time of the sprinter discussed above is ± 0.1 s.

- The **relative uncertainty/relative possible error/percent uncertainty** is the ratio of the absolute uncertainty to the measurement value. In the sprinter example, it's ± 0.1 s / 12.3 s = 0.008 = 0.8%. You can express the measurement with uncertainty as 12.3 s ± 0.8%
- The **relative discrepancy/percent discrepancy/relative error/percent error** is the absolute discrepancy/error divided by the theoretical or reference value. For example, if the record time for the race above was 12.4 seconds, the absolute discrepancy would be -0.1 seconds, so the relative discrepancy would be -0.1 s / 12.4 s = -0.008 = -0.8%. That is: $Percent\ error = (absolute\ error/thoery) * 100$.
- The **relative difference/percent difference** is the absolute difference divided by the average of the two values being compared. For example, if a second stopwatch recorded the time for the runner above as 12.1 s, the absolute difference would be 0.2 seconds, and the average value of the measurements being compared is Average(12.1 s, 12.3 s) = 12.2 s, the relative difference would be 0.2s / 12.2 = 0.02 = 2%. That is: $Percent\ difference = (absolute\ error/average\ measurement) * 100$.

Quick Quiz 3
The length of the track the sprinter ran the race on is 100.0 ± 0.1 m. What is the relative uncertainty of the value?

PROPAGATION OF UNCERTAINTY

Often, you will use one or more measurement results to calculate a quantity. For example, say that the sprinter discussed above was running a race of length 100.0 m $\pm$ 0.1%. You can calculate her average speed from the formula $\bar{v} = \frac{distance}{\Delta t} = \frac{100.0m}{12.3s} = 8.13m/s$. But what is the uncertainty in this value? The average speed was calculated using two experimental values: the distance of 100.0 m $\pm$ 0.1%, and the time of 12.3 s $\pm$ 0.8%. How do we combine those two uncertainties? The answer depends on the mathematical form of the equation used to calculate the new quantity (in this case, the average speed). The basic rules (derived in the next section) are as follows:

- If two experimental quantities are added or subtracted, add their **absolute** uncertainties together to get the combined uncertainty.
- If an experimental quantity is multiplied/divided by a number with no uncertainty x, the absolute uncertainty is also multiplied/divided by x. The relative uncertainty remains the same.
- If two experimental quantities are multiplied or divided, add their **relative** uncertainties together to get the combined relative uncertainty.

These rules come from determining how far off the calculated result could possibly be given the uncertainties. In the example above, since the distance is divided by the time, we add the **relative** uncertainties to get an uncertainty in the average speed of 0.1% + 0.8% = 0.9%.

Quick Quiz 4
A runner in a longer race records lap times of 70.3, 72.5, 67.5, and 62.7 seconds, all with absolute uncertainty of $\pm$ 0.1 s. Each lap has a length of 400.0 $\pm$ 0.1 m. What is the average speed of the runner, with uncertainty?

Correlated Errors
It is important to note that the above rules only work if the errors in multiple quantities are assumed to be independent- that is, that the errors in one quantity are not related to the errors in the other quantities. For example, if you are timing the time a runner starts and ends a race, the individual measurements are likely to both be a bit late due to human reaction time. However, since both measurements are likely to be off in the same direction these errors would be correlated, and it would be inappropriate to use the rules above to combine the uncertainties in the start and end time of the race. Such scenarios can be quite complex and are generally beyond the scope of this course, but several special cases are worth discussion:

Differences of Correlated Values
Whenever you subtract two values, any systematic errors shared by both values vanish. The race scenario described above is an example: even if you start and stop your stopwatch 3 seconds late, as long as the start and the stop are both late by the same amount of time, this has no effect on your measurement of the amount of time it takes the runner to run the race. Since it's often hard to determine how consistent such errors are, a god rule of thumb in such cases is that if you believe the errors in two measurements are correlated and you subtract the two measurements, take the uncertainty in the result to be

the equal to the absolute uncertainty of the original measurements. For example, if a runner running a race starts the second lap at time 60.0 ± 0.1 s and ends at 120.3 ± 0.1 s, then the time for the lap is more like 60.3 ± 0.1 s, not 60.3 ± 0.2 s as would be predicted by the rule in the previous section.

Averages of Multiple Measurements of a Quantity

If you have made multiple measurements of the same physical quantity and are averaging them, how to propagate the uncertainty depends on the type of error you suspect you have. For example, say three separate timers record the race time of the same race as 10.2 ± 0.1 s, 10.1 ± 0.1 s, and 10.0 ± 0.1 s respectively. The average of these measurements, 10.1 s, is our best estimate of the runner's actual time, but what is the uncertainty? In this case, since correlated errors have already been removed by the process discussed above, we expect the errors to be mostly random in nature. In such cases, a statistical metric called the "standard error of the mean" gives a reasonable approximation of the uncertainty of the mean. (There are a number of assumptions implicit in using the standard error of the mean, and it's not really statistically valid to use with smaller data sets, but it still gives us a reasonable ballpark approximation of the uncertainty. If you have taken only a few measurements, finding the range (max-min) of the measurements and dividing by two is a good quick-and-dirty way to get a ballpark figure). The standard error of the mean is the standard deviation of the data divided by the square root of the number of measurements. In this case, the standard deviation (which can be calculated by any statistical program) is 0.1 s, so the standard error is 0.1 s / Sqrt (3) = 0.06 s. This makes sense since the uncertainty in the average is less than that of the individual measurements. On the other hand, this method should be used with caution, since systematic errors can sneak in in unexpected ways. Even if you had 1000 people timing a runner with stopwatches, it would be absurd to report her time with an uncertainty of less than 0.01 s, since the stopwatches themselves only display data to the nearest 0.01 second.

Powers

Say you measure the side of a square to be 10.0 cm ± 2%. What is the uncertainty in the square's area? $A = L^2 = L*L$, so according to our simple propagation rules, the uncertainty in the area is 2 + 2 = 4%. But the errors are certainly correlated here, since we are using the same length measurement twice- should we modify our rule? In this case, no: any error in L will in fact be compounded when L is multiplied by itself. So 4% is the correct uncertainty for the area.

Propagating Uncertainty through Complex Expressions

Sometimes, you need to apply more complex functions to measured quantities. For example, say you want to compute the sine of the measured angle θ=1.21 ± 0.05 rad. What is the uncertainty in the sine of the angle? In this class we will use one of two methods.

Worst-case Scenario Analysis

This method assesses the "worst case scenario" that could occur with the given uncertainty. In the example above, sin(1.21) = 0.936, but it could be as high sin(1.21 + 0.05) = sin(1.26) = 0.952, given the uncertainty in the angle. So the value of θ could be off by as much as 0.952 - 0.936 = 0.016 in the positive direction. Sin(θ) could also be as low as sin(1.21-0.05) = sin(1.16)=0.917, which is 0.936-0.917 = 0.019 lower than the original value. To be conservative, we'll take the larger difference, and report the result as sin(θ) = 0.936 ± 0.019. Note that with small uncertainties and functions that are relatively well-behaved, the differences in the uncertainties from the highest and lowest worst cases are usually small, so using just one or the other may be fine.

This method can be tricky with more complex functions. For example, consider the function

$$\frac{1.00 - x}{1.00 - y}$$

Let's say you have measured experimental values of x = 0.30 ±0.02 and y = 0.40 ±0.02. What is the uncertainty? The experimental value is (1.00-0.30)/(1.00-0.40) = 1.17. The maximum value, given the uncertainties is

$$\frac{1.00 - (0.30 - 0.02)}{1.00 - (0.40 + 0.02)} = \frac{1.00 - 0.28}{1.00 - 0.42} = 1.24$$

and the minimum value is

$$\frac{1.00 - (0.30 + 0.02)}{1.00 - (0.40 - 0.02)} = \frac{1.00 - 0.32}{1.00 - 0.38} = 1.10$$

In this case, the difference in both cases is 0.07, so the result is 1.17 ± 0.07. Note that the choices of signs to maximize and minimize the function are not obvious: you have to think carefully about whether adding of subtracting the uncertainty to a value will make the result smaller or bigger.

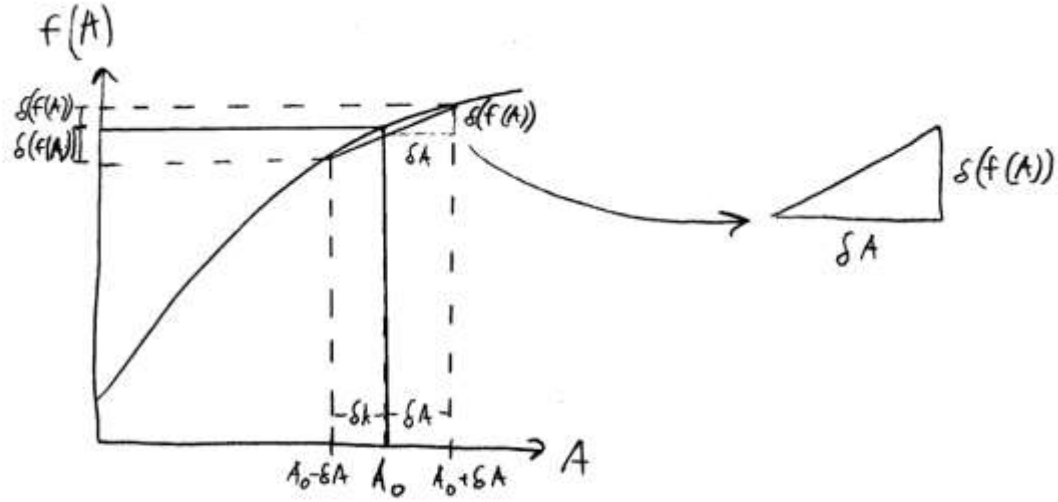
Quick Quiz 5

A runner is measured to run a race with a length of 1320 ± 10 m in a time of 323 ± 1 s.

a) Use the worst-case scenario method to determine the (absolute) uncertainty in the runner's average speed (the distance divided by the time).
b) Use the basic rules at the beginning of the section to determine the (absolute) uncertainty in the runner's average speed. Note that this will require changing back in forth between relative and absolute uncertainties.

Propagation Using Calculus

A more subtle approach than the worst-case scenario analysis if offered by calculus. Say we have a measurement A_0, with uncertainty δA, and we want to apply a function f(A) on that measurement value. What is the uncertainty, $\delta f(A)$ in the value $f(A_0)$? Consider the graph of f(A) vs. A below. The graph shows us the values of $f(A_0)$, as well as the upper and lower bounds on the possible values of $f(A_0)$, $f(A_0) \pm \delta f(A))$. From the graph, we see that if δA is relatively small and the function is well-behaved, the uncertainty $\delta f(A))$ can be



approximated by looking at the magnified triangle. The slope of the triangle is close to the slope of the tangent line, that is, the derivative of f(A) evaluated at A_0. So, looking at the triangle,

$$\frac{df(A)}{dA}|_{A_o} = \frac{rise}{run} = \frac{\delta(f(A))}{\delta A}$$

So

$$\delta(f(A)) = \left(\frac{df(A)}{dA}|_{A_o}\right)\delta A$$

This rule can be used to approximate uncertainty in f(A), as long as the uncertainty in A is not too large and f is a "well-behaved" function. For example, say you measure an angle $\theta = 1.21 \pm 0.05$ rad, and you want to find the uncertainty in $\sin(\theta)$. Then

$$\delta(\sin(\theta)) = \left(\frac{d\sin(\theta)}{d\theta}|_{1.21}\right) * 0.05 = \cos(1.21) * 0.05 = 0.018$$

This result is very close to that given by the worst case scenario analysis above.

Quick Quiz 6

You measure length L to be 1.29 ±0.05 m.

a. Use propagation of uncertainty with calculus to determine the uncertainty in ln(L) (with L in meters).

b. Use worst-case scenario analysis to determine the uncertainty in ln(L) (with L in meters), and compare to the calculus method.

Propagating Uncertainty when Graphing

Often, two sets of data will be plotted against each other and the slope and intercept of the best-fit line will have physical interpretations. A method is therefore needed to use the uncertainty estimates of the original data to estimate the uncertainty in the slope and intercept of the best fit line. This task is non-trivial, but one way to approach it is to use a computer-based technique called a Monte Carlo simulation. Your instructor will provide you with a Microsoft Excel file entitled "Plot and Monte Carlo.xlsm"; this file has the capability of running the Monte Carlo simulation to estimate the uncertainty in a best fit line's slope and intercept. To run the simulation, simply enter your data and the associated uncertainty values in columns A through D, then click the "Find Uncert." button. (Note that the best fit slope and intercept are displayed in cells G2 and H2, and a tab at the bottom brings up a graph of your data with error bars.)

The Monte Carlo simulation works by having the computer randomly generate a new set of data based on the original data and associated uncertainty. For example, if a data value is 11 ± 1, the simulation might generate a new value of 11.5 or 10.3. In this particular simulation, the new values are distributed normally about the original data point, with a standard deviation equal to the uncertainty. The computer does this for each data value, generating a new data set. The computer then fits a line through this new data set, and finds the slope and intercept of the line. Since computers don't get bored, we then have the computer repeat the process 500 times, generating 500 new data sets (all based on the original data and uncertainties) and associated slopes and intercepts. Finally, the computer looks at how much the values of the slopes (or intercepts) vary from each other by taking the standard deviation of all the slopes (or intecepts). This variation is an estimate of the uncertainty in the slope (or intercept) of the original best fit line.

Reporting Your Results: Significant Figures

In quick quiz five, we considered a runner that ran a distance of 1320 ± 10 m in a time of 323 ± 1 s. If we calculate the average speed by dividing the distance by the time, we get 4.0866873065. However, when reporting that result at the end of the analysis, we should not report all of the above decimal places, because that would imply a higher level of precision than we actually have. Significant figures are a reporting convention that gives a reader a quick indication of the precision of a measurement. To find the number of significant figures in final result, you first need to know the number of significant figures in your initial values. The number of significant figures in a value is the number of digits that contain actual measured values, as opposed to digits that may be placeholders. Another useful metric is the number of decimal places in a number- that's the number of digits after the decimal point. The following table shows the number of decimal places and the number of significant figures in five numbers.

Number	Number of Decimal Places	Number of Significant Figures
15.73	2	4
0.0072	4	2
200.6	1	4
1.2700	4	5
4300	0	2, 3 or 4

The ambiguity in the number of significant figures in the last example is easily removed by using scientific notation. 4.30×10^3 is three significant figures.

The number of significant figures you should report at the end of your analysis is can be found by using looking at the mathematical operations you used to get to your final value. When values are added or subtracted, keep as many decimal places in the result as the smallest number of decimal places found in any of the numbers being added or subtracted.

Examples: $20.5 + 1.483 = 22.0$

$19.03 - 18.96 = 0.07$

$10.512 - 9.8 = 0.7$

$4.93 + 6.26 = 11.19$

In multiplication, division, or more complex operations keep as many significant figures as the smallest number of significant figures found in any of the numbers being multiplied or divided.

Examples: $38.75(49.186)/1.48 = 1.29\times10^3$ or 1290, but NOT 1287.81.

$(1.237)(43.9)^2 = (1.237)\ (43.9)\ (43.9) = 2380$ or 2.38×10^3

$\mathrm{Ln}(1.436) = 0.3619$

So, according to the above rule, in quick quiz 5, the result should be reported as 4.09 m/s, since both of the input values had three significant figures. The uncertainty should be reported to the last digit of the result: so we report our result with uncertainty as 4.09 ± 0.05 m/s.

Significant Figures and Round-off Errors

A lot of science courses use significant figure rules like those above as an easy way to do uncertainty propagation. While this method can be useful to give a rough sense of uncertainty, it is dangerous to do this for several reasons. Firstly, significant figures should indicate precision, not accuracy. For example, a typical stopwatch displays digits to the nearest 100th of a second, say 5.35 seconds. But a human operating a stopwatch is likely only accurate to at best 0.1 seconds. If you do an experiment with a stopwatch, you should record all of the displayed digits, because the precision of the device is to the nearest 100th of a second. But just because you have recorded 5.35 seconds does not mean your result is accurate to the nearest 100th of a second- it is not. What you should do instead is record your result with uncertainty: 5.35 ± 0.1 s. This tells the reader that the device has a precision of a 100th of a second, but was only accurate to the nearest 10th of a second.

An even worse approach above would be to round your stopwatch result immediately: i.e. record 5.4 s instead of 5.35 s. This too-early rounding can lead to significant errors in your analysis. Say these are times in a race over a distance of 40 meters. Then the true average speed should be reported as 40 m / 4.35 s = 7.48 m/s = 7.5 m/s (with the last rounding done because 40 m has only two significant figures). However, if you had rounded your original stopwatch reaching to 5.4 s, you would get an average speed of 40 m/ 5.4 s = 7.4 m/s. By rounding early, you would have reported the incorrect result.

Thus the most important thing to remember about significant figures is to **only round off to the proper number of significant figures at the <u>very end</u> of your analysis**. Significant figures should be used for reporting purposes only. They should not be used for uncertainty propagation in any setting where uncertainty is important. **As you do your data collection and analysis, always record enough digits so you can be sure round-off errors will not be a factor.** One helpful trick is to use the memory function on your calculator to store many digits for each intermediate quantity: this will help you avoid calculator errors and round-off errors.

Quick Quiz Answers

1. **a. Error.** -0.1 m/s^2 is the difference between the accepted value and the experimental value. (The term discrepancy would also be appropriate here, as the theoretical value does vary somewhat from place to place, and so we don't always expect experimental results to be 9.8 m/s^2 exactly). This quantity is NOT the uncertainty of the measurement, which must be determined by considering the experimental apparatus and procedure.

2. **c. $\pm$ 0.05 cm.** If the ruler is marked to the nearest millimeter, you can "read between the lines" and estimate to an accuracy of about half a millimeter. It would also be reasonable to use the more conservative estimate of $\pm$ 0.1 cm. Note that if you were measuring the length of an object that was harder to measure like an oddly shaped rock, then the uncertainty might be larger.

3. **0.1%.** The relative uncertainty is the absolute uncertainty divided by the value of the measurement. Here, that is $\pm$ 0.1 m/100 m = 0.001 = 0.1%.

4. We will use the formula $\bar{v} = \frac{distance}{\Delta t}$. The total distance is 400.0 m + 400.0 m +400.0 m +400.0 m = 1600.0 m. Since these quantities are added together, we add the absolute uncertainties to get the uncertainty in the distance: 0.1 m + 0.1 m + 0.1 m + 0.1 m = 0.4 m. So the distance is 1600 $\pm$ 0.4 m. We do the same thing with the time: Δt_{total} = 70.3 + 72.5 + 67.5 + 62.7 =273.0 s and uncertainty in time is 0.1 s + 0.1 s + 0.1 s + 0.1 s = 0.4 s. So Δt_{total} = 273 $\pm$ 0.4 s. The average speed is 1600.0 m / 273.0 s = 5.86 m / s.

   Since the distance is **divided** by the time we must add the **relative** uncertainties in the distance and time to find the uncertainty in the average speed. The relative uncertainty in the distance is $\pm$ 0.4 m/1600 m = 0.00025 = $\pm$ 0.025%. The relative uncertainty in the time is $\pm$ 0.4 s/273 s = 0.0015 = $\pm$ 0.15%. So the uncertainty in the average speed is 0.025% + 0.15% = 0.18%. The average speed can be expressed as 5.86 m / s $\pm$ 0.18%. Or, this can be converted to an absolute uncertainty: 0.0018 * 5.86 m/s = 0.01 m/s, so the average speed is 5.86 $\pm$ 0.01 m/s.

5. a. The runner's average speed is 1320 m/ 323 s = 4.09 m/s. Given the uncertainties, the highest this could be is (1320+10 m)/(323 – 1 s) = 1330 m/ 322 s = 4.13 m/s, 0.04 m/s faster than the original value. The lowest the speed could be is (1320 – 10 m)/(323 + 1 s) = 1310 m/ 324 s = 4.04 m/s, 0.05 m/s slower than the original value. We'll take the higher difference, and report the speed as 4.09 ± 0.05 m/s.

   b. The third basic rule says that when two quantities are divided, we must add their relative uncertainties. The relative uncertainty in the distance is 10/1320 = 0.76 %. The relative uncertainty in the time is 1/323 = 0.31 %. So the relative uncertainty in the average speed is 0.76 + 0.31 = 1.07 %. If we multiply this by the reported value for the average speed, 4.09 m/s, we find an absolute uncertainty of ± 0.04 m/s, close to that given by the worst-case scenario analysis.'

6. a. The formula $\delta\big(f(A)\big) = \left(\frac{df(A)}{dA}|_{A_o}\right)\delta A$, as applied to this situation becomes

$$\delta(\ln(L)) = \left(\frac{dln(L)}{dL}|_{L_o}\right)\delta L = \frac{1}{L_o}\delta L = \frac{0.05}{1.29} = 0.039$$

b. Ln(1.29) = 0.255. The maximum value is ln(1.29+0.05) = ln(1.34)=0.293, a difference of 0.038. The minimum value is ln(1.29-0.05) = ln (1.24) = 0.215, a difference of 0.040. Taking the larger difference, that gives an uncertainty of 0.040, very close to that given by the calculus method.