Fundamental Theorem of Algebra states that every polynomial with real coefficients can be factored in a product of linear factors, irreducible quadratic factors, or a product of both linear and irreducible quadratic factors.

Linear factor: $(ax+b)$

Irreducible quadratic factor: (cx^2+dx+e) where $d^2-4ce<0$.

<u>That is, this quadratic polynomial cannot be factored into linear factors.</u>

Examples:

$$x^3-x^2-x+1=(x-1)^2(x+1)$$
$$x^3+4x=x(x^2+4)$$
$$x^{10}+3x^8-x^7+3x^6-3x^5+x^4-3x^3-x=x(x-1)(x^2+x+1)(x^2+1)^3$$

Linear Factors	Irreducible Quadratic Factors
$x,(x-1),(x+1),(x-1)$	$(x^2+4),(x^2+x+1),(x^2+1)$

Integration by Partial Fractions allows us to integrate rational functions of the form $\frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials and degree of $P(x)<$ degree of $Q(x)$.

Note: If degree of $P(x)\geq$ degree of $Q(x)$, you can use good old-fashioned polynomial long division to rewrite $\frac{P(x)}{Q(x)}=s(x)+\frac{r(x)}{Q(x)}$, and where $s(x)$ is a standard polynomials and degree of $r(x)<$ degree of $Q(x)$.

So if degree of $P(x)\geq$ degree of $Q(x)$, instead of $\int\frac{P(x)}{Q(x)}dx$

we will integrate $\int s(x)+\frac{r(x)}{Q(x)}dx=\int s(x)\,dx+\int\frac{r(x)}{Q(x)}dx$

We first rewrite the rational function using the following guidelines.

1. Factor $Q(x)$ into a product of linear factors, irreducible quadratic factors, or a product of both linear and irreducible quadratic factors.

2. Each linear factor $(ax+b)^n$ in the denominator will contribute the following to the decomposition:

$$(ax+b)^n \text{ contributes } \left[\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\frac{A_3}{(ax+b)^3}+\cdots+\frac{A_n}{(ax+b)^n}\right]$$

<u>When the factor is linear, the numerator is a constant</u>

3. Each irreducible quadratic $(cx^2+dx+e)^m$ in the denominator will contribute the following to the decomposition:

$$(cx^2+dx+e)^m \text{ contributes } \left[\frac{A_1\cdot x+B_1}{(cx^2+dx+e)}+\frac{A_2\cdot x+B_2}{(cx^2+dx+e)^2}+\frac{A_3\cdot x+B_3}{(cx^2+dx+e)^3}+\cdots+\frac{A_m\cdot x+B_m}{(cx^2+dx+e)^m}\right]$$

<u>When the factor is an irreducible quadratic, the numerator is a linear polynomial.</u>

Example:

$$x^4 \text{ contributes } \left[\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x^3}+\frac{D}{x^4}\right]$$

$$(x-1) \text{ contributes } \left[\frac{A}{x-1}\right]$$

$$(2x-1)^3 \text{ contributes } \left[\frac{A}{2x-1}+\frac{B}{(2x-1)^2}+\frac{C}{(2x-1)^3}\right]$$

$$(x^2+x+1) \text{ contributes } \left[\frac{A\cdot x+B}{x^2+x+1}\right]$$

$$(x^2+2)^3 \text{ contributes } \left[\frac{A\cdot x+B}{x^2+2}+\frac{C\cdot x+D}{(x^2+2)^2}+\frac{E\cdot x+F}{(x^2+2)^3}\right]$$

Remember to use a different letter constant in each of the terms of the overall decomposition:

4. Then set the original rational function to be equal to the sum of the contributions.

$$\frac{x+1}{x^2(x^2+4)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C\cdot x+D}{x^2+4}$$

$$\frac{x^3-2x+1}{(x-1)(2x+1)(x^2+7)^3}=\frac{A}{x-1}+\frac{B}{2x+1}+\frac{C\cdot x+D}{x^2+7}+\frac{E\cdot x+F}{(x^2+7)^2}+\frac{G\cdot x+H}{(x^2+7)^3}$$

5. Then rewrite the right-hand side so that each fraction has the common denominator.

$$\frac{x+1}{x^2\left(x^2+4\right)} = \frac{A}{x}\cdot\frac{x\left(x^2+4\right)}{x\left(x^2+4\right)} + \frac{B}{x^2}\cdot\frac{\left(x^2+4\right)}{\left(x^2+4\right)} + \frac{C\cdot x+D}{x^2+4}\cdot\frac{x^2}{x^2}$$

$$= \frac{Ax^3+4Ax}{x^2\left(x^2+4\right)} + \frac{Bx^2+4B}{x^2\left(x^2+4\right)} + \frac{Cx^3+Dx^2}{x^2\left(x^2+4\right)}$$

6. Collect like terms, and solve a system of linear equations to determine the values of each constant.

$$\frac{x+1}{x^2\left(x^2+4\right)} = \frac{Ax^3+4Ax}{x^2\left(x^2+4\right)} + \frac{Bx^2+4B}{x^2\left(x^2+4\right)} + \frac{Cx^3+Dx^2}{x^2\left(x^2+4\right)}$$

$$\frac{x+1}{x^2\left(x^2+4\right)} = \frac{\left(A+C\right)x^3+\left(B+D\right)x^2+4Ax+4B}{x^2\left(x^2+4\right)}$$

$$\downarrow$$

$$0x^3+0x^2+x+1 = \left(A+C\right)x^3+\left(B+D\right)x^2+4Ax+4B$$

$$\downarrow$$

$$\begin{aligned} A+C&=0 \\ B+D&=0 \\ 4A&=1 \\ 4B&=1 \end{aligned} \quad\rightarrow\quad \begin{aligned} A&=\frac{1}{4} \\ B&=\frac{1}{4} \\ C&=-\frac{1}{4} \\ D&=-\frac{1}{4} \end{aligned}$$

7. Then integrate:

$$\int \frac{x+1}{x^2\left(x^2+4\right)}dx = \int \frac{\left(\frac{1}{4}\right)}{x} + \frac{\left(\frac{1}{4}\right)}{x^2} + \frac{\left(-\frac{1}{4}\right)\cdot x+\left(-\frac{1}{4}\right)}{x^2+4}dx$$

$$= \int \frac{1}{4}\cdot\frac{1}{x} + \frac{1}{4}\cdot\frac{1}{x^2} - \frac{1}{4}\cdot\frac{x+1}{x^2+4}dx$$

$$= \frac{1}{4}\int\frac{1}{x}dx + \frac{1}{4}\int\frac{1}{x^2}dx - \frac{1}{4}\int\frac{x+1}{x^2+4}dx$$

$$= \frac{1}{4}\int\frac{1}{x}dx + \frac{1}{4}\int\frac{1}{x^2}dx - \frac{1}{4}\left[\int\frac{x}{x^2+4}dx + \int\frac{1}{x^2+4}dx\right]$$

Use a $u-$substiution

$$u = x^2+4$$
$$du = 2xdx$$

Use $\int\frac{1}{a^2+x^2}dx = \frac{1}{a}\cdot\arctan\left(\frac{x}{a}\right)$

$$\int \frac{5x^2+20x+6}{x^3+2x^2+x}dx \qquad x^3+2x^2+x = x\left(x^2+2x+1\right) = x\left(x+1\right)^2$$

$$\frac{5x^2+20x+6}{x\left(x+1\right)^2} = \frac{A}{x}+\frac{B}{\left(x+1\right)}+\frac{C}{\left(x+1\right)^2}$$

$$\frac{5x^2+20x+6}{x\left(x+1\right)^2} = \frac{A}{x}\cdot\frac{\left(x+1\right)^2}{\left(x+1\right)^2}+\frac{B}{\left(x+1\right)}\cdot\frac{x\left(x+1\right)}{x\left(x+1\right)}+\frac{C}{\left(x+1\right)^2}\cdot\frac{x}{x}$$

$$\frac{5x^2+20x+6}{x\left(x+1\right)^2} = \frac{Ax^2+2Ax+A}{x\left(x+1\right)^2}+\frac{Bx^2+Bx}{x\left(x+1\right)^2}+\frac{Cx}{x\left(x+1\right)^2}$$

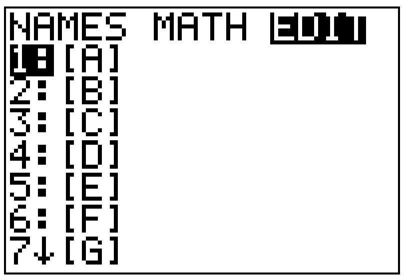
$$5x^2+20x+6 = Ax^2+2Ax+A+Bx^2+Bx+Cx$$

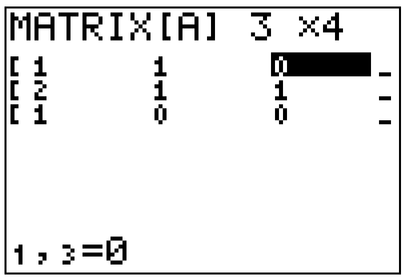
$$5x^2+20x+6 = Ax^2+Bx^2+2Ax+Bx+Cx+A$$

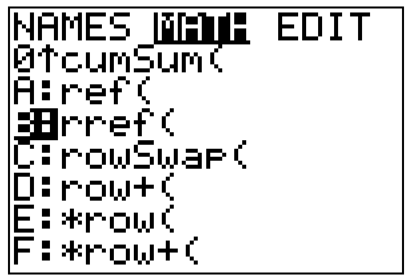
$$\underline{\underline{5}}x^2+\boxed{20}x+\underline{6} = \underline{\underline{\left(A+B\right)}}x^2+\boxed{\left(2A+B+C\right)}x+\underline{A}$$

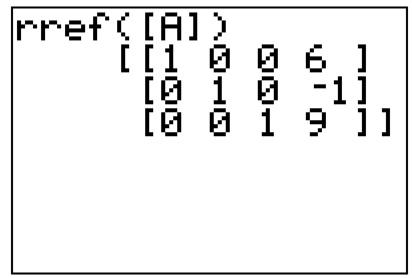
$$\begin{aligned} 5 &= A+B \\ 20 &= 2A+B+C \rightarrow \\ 6 &= A \end{aligned}$$

A	B	C	#
1	1	0	5
2	1	1	20
1	0	0	6









$A=6$; $B=-1$; $C=9$

$$\int \frac{5x^2+20x+6}{x^3+2x^2+x}dx = \int \frac{6}{x}-\frac{1}{\left(x+1\right)}+\frac{9}{\left(x+1\right)^2}dx$$

$$= 6\ln\left|x\right|-\ln\left|x+1\right|-9\left(x+1\right)^{-1}+C$$

FRQ: AP Calc BC 2015 #5

$$\int \frac{7x}{\left(2x-3\right)\left(x+2\right)}dx =$$

(a) $\frac{3}{2}\ln\left|2x-3\right|+2\ln\left|x+2\right|+C$
(b) $3\ln\left|2x-3\right|+2\ln\left|x+2\right|+C$
(c) $3\ln\left|2x-3\right|-2\ln\left|x+2\right|+C$
(d) $-\frac{6}{\left(2x-3\right)^2}-\frac{2}{\left(x+2\right)^2}+C$
(e) $-\frac{3}{\left(2x-3\right)^2}-\frac{2}{\left(x+2\right)^2}+C$