#2

$$\sum_{n=1}^{\infty}\frac{(-2)^n}{n^2}$$

Since $\lim_{n\to\infty}\left[\frac{(-2)^n}{n^2}\right]\neq 0$ the series does not converge.

#3

$$\sum_{n=1}^{\infty}\frac{n}{5^n}$$

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{\left(\frac{n+1}{5^{n+1}}\right)}{\left(\frac{n}{5^n}\right)}\right| &= \lim_{n\to\infty}\left|\frac{n+1}{5^{n+1}}\cdot\frac{5^n}{n}\right| \\ &= \lim_{n\to\infty}\left|\frac{n+1}{5^n\cdot 5}\cdot\frac{5^n}{n}\right| \\ &= \lim_{n\to\infty}\left|\frac{n+1}{n}\cdot\frac{5^n}{5^n\cdot 5}\right| \\ &= \frac{1}{5}<1\end{aligned}$$

Therefore, by the ratio test $\sum_{n=1}^{\infty}\frac{n}{5^n}$ converges absolutely.

$$\begin{aligned}\lim_{n\to\infty}\sqrt[n]{\left|\frac{n}{5^n}\right|} &= \lim_{n\to\infty}\sqrt[n]{\frac{n}{5^n}} \\ &= \lim_{n\to\infty}\frac{\sqrt[n]{n}}{\sqrt[n]{5^n}} \\ &= \frac{1}{5}\end{aligned}$$

Therefore, by the root test $\sum_{n=1}^{\infty}\frac{n}{5^n}$ converges.

#4

$$\sum_{n=1}^{\infty}(-1)^n \cdot \frac{n}{n^2+4}$$

Since the series is alternating, and $\lim_{n\to\infty}\left[\frac{n}{n^2+4}\right] \sim \lim_{n\to\infty}\left[\frac{n}{n^2}\right] = \lim_{n\to\infty}\left[\frac{1}{n}\right] = 0$, the series converges by the alternating series test.

$$\sum_{n=1}^{\infty}\left|(-1)^n \cdot \frac{n}{n^2+4}\right| = \sum_{n=1}^{\infty}\frac{n}{n^2+4}$$

$$\lim_{n\to\infty}\left[\frac{n}{n^2+4}\right] \sim \lim_{n\to\infty}\left[\frac{n}{n^2}\right] = \lim_{n\to\infty}\left[\frac{1}{n}\right]$$

$$\lim_{n\to\infty}\frac{\left[\frac{1}{n}\right]}{\left(\frac{n}{n^2+4}\right)} = \lim_{n\to\infty}\frac{\left[\frac{1}{n}\right]}{\left(\frac{1}{n}\right)} = 1$$

Since $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges, (since it is the harmonic series), $\sum_{n=1}^{\infty}\left|(-1)^n \cdot \frac{n}{n^2+4}\right| = \sum_{n=1}^{\infty}\frac{n}{n^2+4}$ does not converge by the limit comparison test.

Therefore $\sum_{n=1}^{\infty}(-1)^n \cdot \frac{n}{n^2+4}$ is conditionally convergent.

#5

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{5n+1}$$

Since the series is alternating and $\lim_{n\to\infty} \frac{1}{5n+1} = 0$, $\sum_{n=1}^{\infty} \frac{(-1)^n}{5n+1}$ converges by the alternating series test.

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{5n+1} \right| = \sum_{n=1}^{\infty} \frac{1}{5n+1}$$

$$\lim_{n\to\infty} \frac{1}{5n+1} \sim \lim_{n\to\infty} \frac{1}{5n}$$

$$\lim_{n\to\infty} \frac{\left[\frac{1}{5n}\right]}{\left(\frac{1}{5n+1}\right)} = \lim_{n\to\infty} \frac{\left[\frac{1}{5n}\right]}{\left(\frac{1}{5n}\right)} = 1$$

Since $\sum_{n=1}^{\infty} \frac{1}{5n} = \frac{1}{5} \cdot \left[\sum_{n=1}^{\infty} \frac{1}{n} \right]$ diverges (a multiple of the harmonic series), $\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{5n+1} \right| = \sum_{n=1}^{\infty} \frac{1}{5n+1}$ diverges by the limit comparison test.

Therefore $\sum_{n=1}^{\infty} \frac{(-1)^n}{5n+1}$ is conditionally convergent.

#6 $\sum_{n=1}^{\infty}\frac{(-3)^n}{(2n+1)!}$

$$\lim_{n\to\infty}\left|\frac{\frac{(-3)^{n+1}}{(2(n+1)+1)!}}{\frac{(-3)^n}{(2n+1)!}}\right|=\lim_{n\to\infty}\left|\frac{(-3)^{n+1}}{(2(n+1)+1)!}\cdot\frac{(2n+1)!}{(-3)^n}\right|$$

$$=\lim_{n\to\infty}\left|\frac{(-3)^n\cdot(-3)}{(2n+3)!}\cdot\frac{(2n+1)!}{(-3)^n}\right|$$

$$=\lim_{n\to\infty}\left|\frac{(2n+1)!}{(2n+3)\cdot(2n+2)\cdot\left[(2n+1)!\right]}\cdot\frac{(-3)^n\cdot(-3)}{(-3)^n}\right|$$

$$=\lim_{n\to\infty}\left|\frac{-3}{(2n+3)(2n+2)}\right|$$

$$=0$$

$\sum_{n=1}^{\infty}\frac{(-3)^n}{(2n+1)!}$ converges absolutely by the ratio test.

$$\lim_{n\to\infty}\sqrt[n]{\left|\frac{(-3)^n}{(2n+1)!}\right|}=\lim_{n\to\infty}\frac{\sqrt[n]{\left|(-3)^n\right|}}{\sqrt[n]{(2n+1)!}}$$

$$=\lim_{n\to\infty}\frac{\sqrt[n]{\left|(-3)\right|^n}}{\sqrt[n]{(2n+1)!}}$$

$$=\lim_{n\to\infty}\frac{\sqrt[n]{3^n}}{\sqrt[n]{(2n+1)!}}$$

$$\sim\frac{3}{\infty}\to 0$$

Therefore, by the root test, $\sum_{n=1}^{\infty}\frac{(-3)^n}{(2n+1)!}$ converges absolutely.

#7

$$\sum_{k=1}^{\infty} k\left(\frac{2}{3}\right)^k$$

$$\begin{aligned}
\lim_{k\to\infty}\left|\frac{(k+1)\left(\frac{2}{3}\right)^{k+1}}{k\left(\frac{2}{3}\right)^k}\right| &= \lim_{k\to\infty}\left|\frac{(k+1)\left(\frac{2}{3}\right)^k\cdot\left(\frac{2}{3}\right)}{k\left(\frac{2}{3}\right)^k}\right| \\
&= \lim_{k\to\infty}\left|\frac{k+1}{k}\cdot\frac{\left(\frac{2}{3}\right)^k}{\left(\frac{2}{3}\right)^k}\cdot\left(\frac{2}{3}\right)\right| \\
&= \frac{2}{3}
\end{aligned}$$

By the ratio test, $\sum_{k=1}^{\infty} k\left(\frac{2}{3}\right)^k$ converges absolutely.

$$\begin{aligned}
\lim_{k\to\infty}\sqrt[k]{\left|k\left(\frac{2}{3}\right)^k\right|} &= \lim_{k\to\infty}\sqrt[k]{k\left(\frac{2}{3}\right)^k} \\
&= \lim_{k\to\infty}\sqrt[k]{k}\cdot\sqrt[k]{\left(\frac{2}{3}\right)^k} \\
&= 1\cdot\frac{2}{3} \\
&= \frac{2}{3}
\end{aligned}$$

Therefore, by the root test, $\sum_{k=1}^{\infty} k\left(\frac{2}{3}\right)^k$ converges absolutely.

#8

$$\sum_{n=1}^{\infty} \frac{n!}{100^n}$$

$\lim_{n\to\infty} \frac{n!}{100} = \frac{\text{factorial growth}}{\text{exponential growth}} \to \infty$. Therefore $\sum_{n=1}^{\infty} \frac{n!}{100^n}$ does not converge by the limit of the n^{th} term test.

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{\frac{(n+1)!}{100^{n+1}}}{\frac{n!}{100^n}} \right| &= \lim_{n\to\infty} \left| \frac{(n+1)!}{100^{n+1}} \cdot \frac{100^n}{n!} \right| \\
&= \lim_{n\to\infty} \left| \frac{(n+1)\cdot[n!]}{100^n \cdot 100} \cdot \frac{100^n}{n!} \right| \\
&= \lim_{n\to\infty} \left| \frac{100^n}{100^n \cdot 100} \cdot \frac{(n+1)\cdot[n!]}{n!} \right| \\
&= \lim_{n\to\infty} \left| \frac{n+1}{100} \right| \\
&\downarrow \\
&\infty
\end{aligned}$$

Therefore $\sum_{n=1}^{\infty} \frac{n!}{100^n}$ diverges by the ratio test.

$$\begin{aligned}
\lim_{n\to\infty} \sqrt[n]{\left| \frac{n!}{100^n} \right|} &= \lim_{n\to\infty} \frac{\sqrt[n]{n!}}{\sqrt[n]{100^n}} \\
&\sim \frac{\infty}{100} \\
&\to \infty
\end{aligned}$$

Therefore, by the root test, $\sum_{n=1}^{\infty} \frac{n!}{100^n}$ does not converge.

#9

$$\sum_{n=1}^{\infty}(-1)^n\cdot\frac{(1.1)^n}{n^4}$$

$\lim\limits_{n\to\infty}(-1)^n\cdot\frac{(1.1)^n}{n^4}\to(-1)^n\cdot\frac{\text{exponential growth}}{\text{power growth}}\to$ alternatning $\pm\infty\neq 0$, therefore $\sum\limits_{n=1}^{\infty}(-1)^n\cdot\frac{(1.1)^n}{n^4}$ does not converge.

$$\begin{aligned}
\lim_{n\to\infty}\sqrt[n]{\left|(-1)^n\cdot\frac{(1.1)^n}{n^4}\right|}&=\lim_{n\to\infty}\sqrt[n]{\left|(-1)^n\right|\cdot\left|\frac{(1.1)^n}{n^4}\right|}\\
&=\lim_{n\to\infty}\sqrt[n]{\left|\frac{(1.1)^n}{n^4}\right|}\\
&=\lim_{n\to\infty}\frac{\sqrt[n]{\left|(1.1)^n\right|}}{\sqrt[n]{n^4}}\\
&=\frac{1.1}{1}
\end{aligned}$$

$\sum\limits_{n=1}^{\infty}(-1)^n\cdot\frac{(1.1)^n}{n^4}$ does not converge by the root test.

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{(-1)^{n+1}\cdot\frac{(1.1)^{n+1}}{(n+1)^4}}{(-1)^n\cdot\frac{(1.1)^n}{n^4}}\right|&=\lim_{n\to\infty}\left|\frac{\frac{(1.1)^{n+1}}{(n+1)^4}}{\frac{(1.1)^n}{n^4}}\right|\\
&=\lim_{n\to\infty}\left|\frac{(1.1)^{n+1}}{(n+1)^4}\cdot\frac{n^4}{(1.1)^n}\right|\\
&=\lim_{n\to\infty}\left|\frac{n^4}{(n+1)^4}\cdot\frac{(1.1)^{n+1}}{(1.1)^n}\right|\\
&=\lim_{n\to\infty}\left|\frac{n^4}{(n+1)^4}\cdot\frac{(1.1)^n\cdot(1.1)}{(1.1)^n}\right|\\
&=1.1
\end{aligned}$$

Therefore $\sum\limits_{n=1}^{\infty}(-1)^n\cdot\frac{(1.1)^n}{n^4}$ does not converge by the ratio test.

#10

$$\sum_{n=1}^{\infty}(-1)^n \cdot \frac{n}{\sqrt{n^3+2}}$$

Since the series is alternating and $\lim_{n\to\infty}\frac{n}{\sqrt{n^3+2}} = 0$, the series converges by the alternating series test.

$$\lim_{n\to\infty}\left|(-1)^n \cdot \frac{n}{\sqrt{n^3+2}}\right| = \lim_{n\to\infty}\left|\frac{n}{\sqrt{n^3+2}}\right| \sim \lim_{n\to\infty}\left[\frac{n}{\sqrt{n^3}}\right] = \lim_{n\to\infty}\left[\frac{n}{n^{\frac{3}{2}}}\right] = \lim_{n\to\infty}\left[\frac{1}{n^{\frac{1}{2}}}\right]$$

$\sum_{n=1}^{\infty}\frac{1}{n^{\frac{1}{2}}}$ is a divergent *p*-series.

$$\lim_{n\to\infty}\frac{\left[\frac{1}{n^{\frac{1}{2}}}\right]}{\left(\frac{n}{\sqrt{n^3+2}}\right)} = \lim_{n\to\infty}\frac{\left[\frac{1}{n^{\frac{1}{2}}}\right]}{\left(\frac{1}{n^{\frac{1}{2}}}\right)} = 1$$

By the limit comparison test, since $\sum_{n=1}^{\infty}\frac{1}{n^{\frac{1}{2}}}$, so does $\sum_{n=1}^{\infty}\frac{n}{\sqrt{n^3+2}}$

Therefore, $\sum_{n=1}^{\infty}(-1)^n \cdot \frac{n}{\sqrt{n^3+2}}$ is conditionally convergent.

#11

$$\sum_{n=1}^{\infty}\frac{(-1)^n\cdot e^{\frac{1}{n}}}{n^3}$$

Since the series is alternating and $\lim_{n\to\infty}\left[\frac{e^{\frac{1}{n}}}{n^3}\right]=\lim_{n\to\infty}\left[\frac{1}{n^3}\right]=0$, the series converges by the alternating series test.

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^n\cdot e^{\frac{1}{n}}}{n^3}\right|=\sum_{n=1}^{\infty}\frac{e^{\frac{1}{n}}}{n^3}$$

$$\frac{e^{\frac{1}{n}}}{n^3}<\frac{5}{n^3}$$

Therefore $\sum_{n=1}^{\infty}\frac{e^{\frac{1}{n}}}{n^3}<\sum_{n=1}^{\infty}\frac{5}{n^3}$

Since $\sum_{n=1}^{\infty}\frac{5}{n^3}$ is a convergent *p*-series, $\sum_{n=1}^{\infty}\left|\frac{(-1)^n\cdot e^{\frac{1}{n}}}{n^3}\right|=\sum_{n=1}^{\infty}\frac{e^{\frac{1}{n}}}{n^3}$ converges by the direct comparison test.

Therefore $\sum_{n=1}^{\infty}\frac{(-1)^n\cdot e^{\frac{1}{n}}}{n^3}$ converges absolutely.

#12

$$\sum_{n=1}^{\infty}\frac{\sin(4n)}{4^n}$$

$$\frac{\sin(4n)}{4^n}<\frac{1}{4^n}=\left(\frac{1}{4}\right)^n$$

$$\sum_{n=1}^{\infty}\left|\frac{\sin(4n)}{4^n}\right|<\sum_{n=1}^{\infty}\frac{1}{4^n}=\sum_{n=1}^{\infty}\left(\frac{1}{4}\right)^n$$

Since $\sum_{n=1}^{\infty}\left(\frac{1}{4}\right)^n$ is a convergent geometric series, $\sum_{n=1}^{\infty}\left|\frac{\sin(4n)}{4^n}\right|$ converges.

Therefore $\sum_{n=1}^{\infty}\frac{\sin(4n)}{4^n}$ converges absolutely.

#13

$$\sum_{n=1}^{\infty}\frac{10^n}{(n+1)4^{2n+1}}$$

$$\frac{10^n}{(n+1)4^{2n+1}}=\frac{1}{n+1}\cdot\frac{10^n}{4^{2n+1}}<\frac{10^n}{4^{2n+1}}=\frac{10^n}{4\cdot 4^{2n}}=\frac{10^n}{4\cdot 16^n}=\frac{1}{4}\cdot\left(\frac{10}{16}\right)^n=\frac{1}{4}\cdot\left(\frac{5}{8}\right)^n$$

$$\sum_{n=1}^{\infty}\left|\frac{10^n}{(n+1)4^{2n+1}}\right|=\sum_{n=1}^{\infty}\frac{10^n}{(n+1)4^{2n+1}}<\sum_{n=1}^{\infty}\frac{1}{4}\cdot\left(\frac{5}{8}\right)^n$$

Since $\sum_{n=1}^{\infty}\frac{1}{4}\cdot\left(\frac{5}{8}\right)^n$ is a convergent geometric series, $\sum_{n=1}^{\infty}\frac{10^n}{(n+1)4^{2n+1}}$ converges by the direct comparison test.

Therefore $\sum_{n=1}^{\infty}\frac{10^n}{(n+1)4^{2n+1}}$ converges absolutely.

#14

$$\sum_{n=1}^{\infty}\frac{n^{10}}{(-10)^{n+1}}$$

$$\lim_{n\to\infty}\left|\frac{\frac{(n+1)^{10}}{(-10)^{(n+1)+1}}}{\frac{n^{10}}{(-10)^{n+1}}}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n+1)^{10}}{(-1)^{n+2}(10)^{(n+1)+1}}}{\frac{n^{10}}{(-1)^{n+1}(10)^{n+1}}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{\frac{(n+1)^{10}}{10^{n+2}}}{\frac{n^{10}}{10^{n+1}}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+1)^{10}}{10^{n+2}}\cdot\frac{10^{n+1}}{n^{10}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{10^{n+1}}{10^{n+2}}\cdot\frac{(n+1)^{10}}{n^{10}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{10^{n+1}}{10^{n+1}\cdot 10}\cdot\frac{(n+1)^{10}}{n^{10}}\right|$$

$$= \frac{1}{10}$$

Therefore converges absolutely by the ratio test.

$$\lim_{n\to\infty}\sqrt[n]{\left|\frac{n^{10}}{(-10)^{n+1}}\right|} = \lim_{n\to\infty}\sqrt[n]{\frac{n^{10}}{10^{n+1}}} = \lim_{n\to\infty}\sqrt[n]{\frac{n^{10}}{10\cdot 10^{n}}} = \lim_{n\to\infty}\frac{\sqrt[n]{n^{10}}}{\sqrt[n]{10}\cdot\sqrt[n]{10^{n}}} = \frac{1}{10}$$

Therefore by the root test, $\sum_{n=1}^{\infty}\frac{n^{10}}{(-10)^{n+1}}$ converges absolutely.

#15

$$\sum_{n=1}^{\infty}\frac{(-1)^n\cdot\arctan(n)}{n^2}$$

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^n\cdot\arctan(n)}{n^2}\right|=\sum_{n=1}^{\infty}\frac{\arctan(n)}{n^2}$$

$$\frac{\arctan(n)}{n^2}<\frac{\left(\frac{\pi}{2}\right)}{n^2}=\frac{\pi}{2}\cdot\frac{1}{n^2}$$

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^n\cdot\arctan(n)}{n^2}\right|=\sum_{n=1}^{\infty}\frac{\arctan(n)}{n^2}<\frac{\pi}{2}\cdot\left[\sum_{n=1}^{\infty}\frac{1}{n^2}\right]$$

Since $\frac{\pi}{2}\cdot\left[\sum_{n=1}^{\infty}\frac{1}{n^2}\right]$ is a convergent *p*-series, $\sum_{n=1}^{\infty}\left|\frac{(-1)^n\cdot\arctan(n)}{n^2}\right|=\sum_{n=1}^{\infty}\frac{\arctan(n)}{n^2}$ converges.

Therefore, $\sum_{n=1}^{\infty}\frac{(-1)^n\cdot\arctan(n)}{n^2}$ converges absolutely.

#16

$$\sum_{n=1}^{\infty}\frac{3-\cos(n)}{n^{\frac{2}{3}}-2}$$

$$\frac{2}{n^{\frac{2}{3}}}<\frac{2}{n^{\frac{2}{3}}-2}<\frac{3-\cos(n)}{n^{\frac{2}{3}}-2}$$

$$\sum_{n=1}^{\infty}\frac{2}{n^{\frac{2}{3}}}<\sum_{n=1}^{\infty}\frac{3-\cos(n)}{n^{\frac{2}{3}}-2}$$

Since $\sum_{n=1}^{\infty}\frac{2}{n^{\frac{2}{3}}}=2\cdot\left[\sum_{n=1}^{\infty}\frac{1}{n^{\frac{2}{3}}}\right]$ is a divergent *p*-series, $\sum_{n=1}^{\infty}\frac{3-\cos(n)}{n^{\frac{2}{3}}-2}$ diverges by the direct comparison test.

#17

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{\ln(n)}$$

Is an alternating series with $\lim_{n\to\infty}\frac{1}{\ln(n)}=0$, therefore $\sum_{n=1}^{\infty}\frac{(-1)^n}{\ln(n)}$ converges by the alternating series test.

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{\ln(n)}\right|=\sum_{n=1}^{\infty}\frac{1}{\ln(n)}$$

Since $\frac{1}{n}<\frac{1}{\ln(n)}$

$$\sum_{n=1}^{\infty}\frac{1}{n}<\sum_{n=1}^{\infty}\frac{1}{\ln(n)}$$

Since $\sum_{n=1}^{\infty}\frac{1}{n}$ is the divergent harmonic series, by the direct comparison test, $\sum_{n=1}^{\infty}\frac{1}{\ln(n)}$ diverges.

Therefore $\sum_{n=1}^{\infty}\frac{(-1)^n}{\ln(n)}$ is conditionally convergent

#18

$$\sum_{n=1}^{\infty}\frac{n!}{n^n}$$

$$\lim_{n\to\infty}\left|\frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!}{(n+1)^{n+1}}\frac{n^n}{n!}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{n!}\right|$$

$$= \lim_{n\to\infty}\left|\frac{n^n}{(n+1)^{n+1}}\cdot\frac{(n+1)!}{n!}\right|$$

$$= \lim_{n\to\infty}\left|\frac{n^n}{(n+1)^n\cdot(n+1)}\cdot\frac{(n+1)\cdot[n!]}{n!}\right|$$

$$= \lim_{n\to\infty}\left|\frac{n^n}{(n+1)^n}\right|$$

$$= \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n$$

$$= \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^{-n}$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{-n}$$

$$= \frac{1}{e}$$

Therefore, by the ratio test, $\sum_{n=1}^{\infty}\frac{n!}{n^n}$ converges absolutely.

$$\lim_{n\to\infty}\sqrt[n]{\frac{n!}{n^n}} = \lim_{n\to\infty}\frac{\sqrt[n]{n!}}{\sqrt[n]{n^n}} \to \lim_{n\to\infty}\frac{\infty}{n} \sim \frac{\infty}{\infty}$$

Since we cannot use L'Hopital's Rule and differentiate $n!$, the Root test will not apply.

#19

$$\sum_{n=1}^{\infty} \frac{\cos\left(\frac{n\cdot\pi}{3}\right)}{n!}$$

$$\sum_{n=1}^{\infty} \left| \frac{\cos\left(\frac{n\cdot\pi}{3}\right)}{n!} \right|$$

$\left| \frac{\cos\left(\frac{n\cdot\pi}{3}\right)}{n!} \right| < \frac{1}{n!} < \frac{1}{n^2}$ for sufficiently large *n*.

$$\sum_{n=1}^{\infty} \left| \frac{\cos\left(\frac{n\cdot\pi}{3}\right)}{n!} \right| < \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent *p*-series, $\sum_{n=1}^{\infty} \frac{\cos\left(\frac{n\cdot\pi}{3}\right)}{n!}$ converges absolutely.

#20

$$\sum_{n=1}^{\infty}\frac{(-2)^n}{n^n}$$

$$\lim_{n\to\infty}\sqrt[n]{\left|\frac{(-2)^n}{n^n}\right|}=\lim_{n\to\infty}\sqrt[n]{\frac{2^n}{n^n}}=\lim_{n\to\infty}\frac{\sqrt[n]{2^n}}{\sqrt[n]{n^n}}=\lim_{n\to\infty}\frac{2}{n}=0$$

Therefore, by the root test, $\sum_{n=1}^{\infty}\frac{(-2)^n}{n^n}$ converges absolutely.

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{\frac{(-2)^{n+1}}{(n+1)^{n+1}}}{\frac{(-2)^n}{n^n}}\right| &= \lim_{n\to\infty}\left|\frac{\frac{2^{n+1}}{(n+1)^{n+1}}}{\frac{2^n}{n^n}}\right| \\
&= \lim_{n\to\infty}\left|\frac{2^{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{2^n}\right| \\
&= \lim_{n\to\infty}\left|\frac{n^n}{(n+1)^{n+1}}\cdot\frac{2^{n+1}}{2^n}\right| \\
&= \lim_{n\to\infty}\left|\frac{n^n}{(n+1)^n\cdot(n+1)}\cdot\frac{2^n\cdot 2}{2^n}\right| \\
&= \lim_{n\to\infty}\left|\frac{n^n}{(n+1)^n}\cdot\frac{2}{n+1}\right| \\
&= \lim_{n\to\infty}\left|\left(\frac{n}{n+1}\right)^n\cdot\frac{2}{n+1}\right| \\
&= \lim_{n\to\infty}\left|\left(\frac{n+1}{n}\right)^{-n}\cdot\frac{2}{n+1}\right| \\
&= \lim_{n\to\infty}\left|\left(1+\frac{1}{n}\right)^{-n}\cdot\frac{2}{n+1}\right| \\
&= \frac{1}{e}\cdot 0 \\
&= 0
\end{aligned}$$

Therefore, by the ratio test, $\sum_{n=1}^{\infty}\frac{(-2)^n}{n^n}$ converges absolutely.

#21

$$\sum_{n=1}^{\infty}\left(\frac{n^2+1}{2n^2+1}\right)^n$$

$$\lim_{n\to\infty}\sqrt[n]{\left|\left(\frac{n^2+1}{2n^2+1}\right)^n\right|}=\lim_{n\to\infty}\sqrt[n]{\left(\frac{n^2+1}{2n^2+1}\right)^n}=\lim_{n\to\infty}\frac{n^2+1}{2n^2+1}=\frac{1}{2}$$

Therefore $\sum_{n=1}^{\infty}\left(\frac{n^2+1}{2n^2+1}\right)^n$ converges absolutely by the root test.

$$\lim_{n\to\infty}\left|\frac{\left(\frac{(n+1)^2+1}{2(n+1)^2+1}\right)^{n+1}}{\left(\frac{n^2+1}{2n^2+1}\right)^n}\right|=\lim_{n\to\infty}\left|\left(\frac{(n+1)^2+1}{2(n+1)^2+1}\right)^{n+1}\cdot\left(\frac{2n^2+1}{n^2+1}\right)^n\right|$$

$$=\lim_{n\to\infty}\left|\frac{\text{degree } 4\cdot n\cdot(n+1) \text{ polynomial with leading coefficient } 2^n}{\text{degree } 4\cdot n\cdot(n+1) \text{ polynomial with leading coefficient } 2^{n+1}}\right|$$

$$=\frac{1}{2}$$

Therefore, $\sum_{n=1}^{\infty}\left(\frac{n^2+1}{2n^2+1}\right)^n$ converges absolutely by the ratio test.

#22

$$\sum_{n=1}^{\infty}\left(\frac{-2n}{n+1}\right)^{5n}$$

$$\lim_{n\to\infty}\sqrt[n]{\left|\left(\frac{-2n}{n+1}\right)^{5n}\right|}=\lim_{n\to\infty}\sqrt[n]{\left(\frac{2n}{n+1}\right)^{5n}}=\lim_{n\to\infty}\left(\frac{2n}{n+1}\right)^{5}=2^{5}$$

Therefore $\sum_{n=1}^{\infty}\left(\frac{-2n}{n+1}\right)^{5n}$ diverges by the root test.

$$\lim_{n\to\infty}\left|\frac{\left(\frac{-2(n+1)}{(n+1)+1}\right)^{5(n+1)}}{\left(\frac{-2n}{n+1}\right)^{5n}}\right|=\lim_{n\to\infty}\left|\frac{\left(\frac{2(n+1)}{n+2}\right)^{5n+5}}{\left(\frac{2n}{n+1}\right)^{5n}}\right|$$

$$=\lim_{n\to\infty}\left|\left(\frac{2(n+1)}{n+2}\right)^{5n+5}\cdot\left(\frac{n+1}{2n}\right)^{5n}\right|$$

$$=\lim_{n\to\infty}\left|\frac{\text{polynomial of degree } 2\cdot 5n\cdot(5n+5) \text{ with leading coefficient } 2^{5n+5}}{\text{polynomial of degree } 2\cdot 5n\cdot(5n+5) \text{ with leading coefficient } 2^{5n}}\right|$$

$$=2^{5}$$

Therefore, $\sum_{n=1}^{\infty}\left(\frac{-2n}{n+1}\right)^{5n}$ diverges by the ratio test.