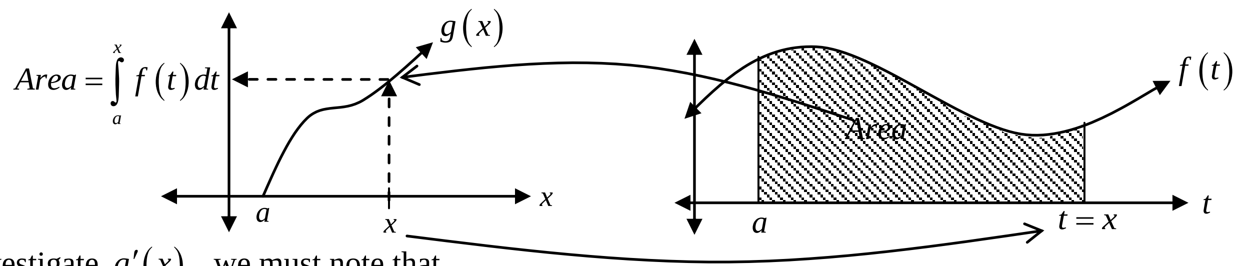
Proof of FTC Part I

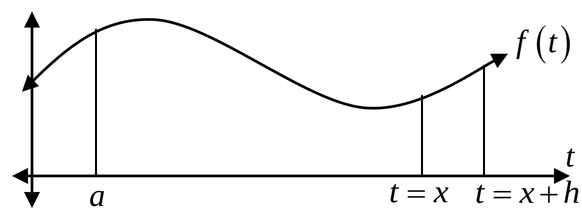
Let $g(x)=\int_a^x f(t)\,dt$ where a is any constant. A visual for $g(x)$ is provided at right. Think of the value of $g(x)$ as the area that has been accumulated under $f(t)$ from $t=a$ to $t=x$. You choose a value of x as the input of $g(x)$, and it then becomes the upper bound of the integral. You compute the value of this integral, and this value becomes the number $g(x)$.



To investigate $g'(x)$, we must note that

$$g'(x)=\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}.$$

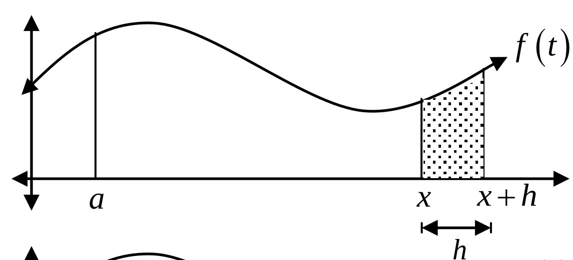
Let us first consider the location of $g(x+h)$. Consider the graph at right:



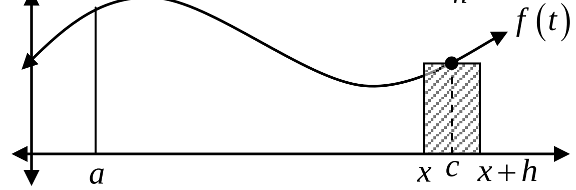
Consider the value of $g(x+h)-g(x)$.

$$\begin{aligned} g(x+h)-g(x) &= \int_a^{x+h} f(t)\,dt-\int_a^x f(t)\,dt \\ &= \int_x^{x+h} f(t)\,dt \end{aligned}$$

This corresponds to the value of the shaded area under the curve $f(t)$ from $t=x$ to $t=x+h$.



By IVT since $f(x)$ is continuous there must exist a c where $x\le c\le x+h$ and $f(c)\cdot h=\int_x^{x+h} f(t)\,dt$



Now let us rewrite this expression

$$f(c)\cdot h=\int_x^{x+h} f(t)\,dt$$

$$f(c)\cdot h=\int_a^{x+h} f(t)\,dt-\int_a^{x} f(t)\,dt$$

$$f(c)\cdot h=g(x+h)-g(x)$$

Dividing both sides by h we get $f(c)=\dfrac{g(x+h)-g(x)}{h}$.

Taking the limit as $h\to 0$ of both sides we get that

$$\lim_{h\to 0} f(c)=\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

LEFT HAND SIDE	*RIGHT HAND SIDE*
As $h\to 0$, the width of the interval will get smaller and smaller. As this happens, the value of c will approach the value of *x*. $$\lim_{h\to 0} x\le \lim_{h\to 0} c\le \lim_{h\to 0} x+h$$ $$x\le \lim_{h\to 0} c\le x$$ $$x\le c\le x$$ Hence $c=x$ and it follows that $$\lim_{h\to 0} f(c)=f\left(\lim_{h\to 0} c\right)=f(x)$$ Therefore the Left Hand Side is equivalent to $f(x)$	$$\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$ $$\downarrow$$ $$g'(x)$$

Putting it all together, we get that

$$\lim_{h\to 0} f(c)=\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$$\downarrow$$

$$f(x)=g'(x)$$

Continued on the next page.

FTC Explanation & Proof

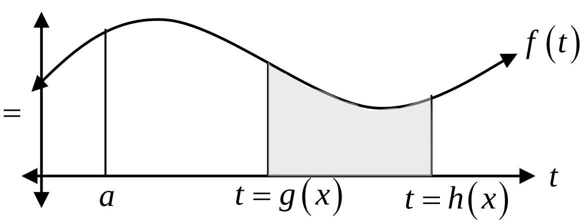
The first part of FTC is proved. We have $\underbrace{\frac{d}{dx}\left[\int_a^x f(t)\,dt\right]}_{\substack{\text{The rate at which area}\\ \text{is accumulated under } f(t)\\ \text{from } t=a \text{ to } t=x \text{ at the}\\ \text{moment t}=x}} = \underbrace{f(x)}_{\substack{\text{The height of } f(t)\\ \text{at } t=x}}$.

Combined with Chain Rule, we get that $\frac{d}{dx}\left[\int_a^{g(x)} f(t)\,dt\right] = f(g(x))\cdot g'(x)$.

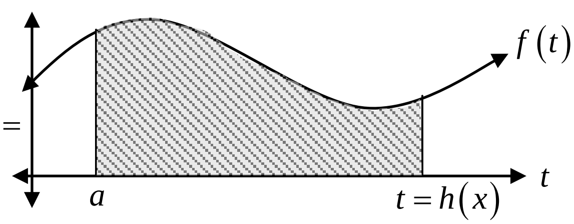
For any two functions $g(x)$ and $h(x)$ we have that

$$\begin{aligned}\frac{d}{dx}\left[\int_{g(x)}^{h(x)} f(t)\,dt\right] &= \frac{d}{dx}\left[\int_a^{h(x)} f(t)\,dt - \int_a^{g(x)} f(t)\,dt\right]\\ &= f(h(x))\cdot h'(x) - f(g(x))\cdot g'(x)\end{aligned}$$

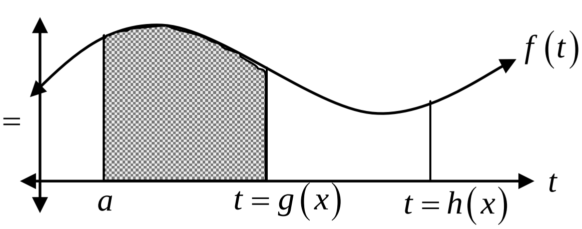
$\int_{g(x)}^{h(x)} f(t)\,dt =$



$\int_a^{h(x)} f(t)\,dt =$



$\int_a^{h(x)} f(t)\,dt =$



Proof of FTC Part II.

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Let $g(x) = \int_a^x f(t)\,dt$. Then $g'(x) = f(x)$. Therefore $g(x)$ is an antiderivative of $f(x)$.

Let $F(x)$ be any antiderivative of $f(x)$.

Since $g(x)$ and $F(x)$ are both antiderivatives of the same function, they must differ only by a constant. Hence

$$g(x) = F(x) + C$$

Now we investigate $g(b) - g(a)$

$$\begin{aligned} g(b) - g(a) &= \left[F(b) + C\right] - \left[F(a) + C\right] \\ &= F(b) - F(a) \end{aligned}$$

Since $g(x) = \int_a^x f(t)\,dt$, we know that $g(a) = \int_a^a f(t)\,dt = 0$. Substituting this in to the above equation we get:

$$\begin{aligned} g(b) - g(a) &= F(b) - F(a) \\ g(b) - 0 &= F(b) - F(a) \\ g(b) &= F(b) - F(a) \end{aligned}$$

Since $g(x) = \int_a^x f(t)\,dt$, we know that $g(b) = \int_a^b f(t)\,dt$. One more substitution and we get:

$$\int_a^b f(t)\,dt = F(b) - F(a)$$

Alternate Proof of FTC Part II

Let $f(x)$ be continuous on $[a,b]$, and let $F(x)$ be an antiderivative of $f(x)$.

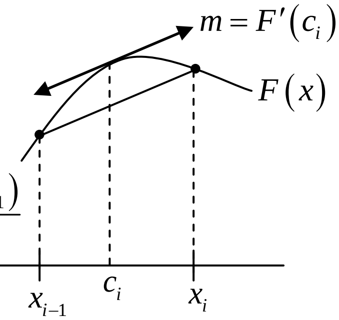
Let Δ be the following partition of the interval $[a,b]$: $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$

We can express $F(b)-F(a)$ in the following way

$$F(b)-F(a) = F(x_n)\underbrace{-F(x_{n-1})+F(x_{n-1})}_{=0}\underbrace{-F(x_{n-2})+F(x_{n-3})}_{=0}-\cdots\underbrace{-F(x_1)+F(x_1)}_{=0}-F(x_0)$$

$$= \sum_{i=1}^{n}\left[F(x_i)-F(x_{i-1})\right]$$

Let us consider the general subinterval of $[a,b]$,
$[x_{i-1}, x_i]$



By the Mean Value Theorem for derivatives, there must exist a c_i such that $x_{i-1} < c_i < x_i$ and

$$\underbrace{F'(c_i)}_{\text{slope of tangent}} = \underbrace{\frac{F(x_i)-F(x_{i-1})}{x_i - x_{i-1}}}_{\text{slope of secant}}$$

Which can be expressed alternately as

$$F'(c_i)\cdot(x_i - x_{i-1}) = F(x_i)-F(x_{i-1})$$

Since $F(x)$ is an antiderivative of $f(x)$, $F'(x) = f(x)$ and therefore $F'(c_i) = f(c_i)$.

$$F'(c_i)\cdot(x_i - x_{i-1}) = F(x_i)-F(x_{i-1})$$

$$\downarrow$$

$$f(c_i)\cdot(x_i - x_{i-1}) = F(x_i)-F(x_{i-1})$$

If we let $(\Delta x)_i = x_i - x_{i-1}$, we can now rewrite $\sum_{i=1}^{n}\left[F(x_i)-F(x_{i-1})\right]$ as follows:

$$F(b)-F(a) = \sum_{i=1}^{n}\left[F(x_i)-F(x_{i-1})\right] = \sum_{i=1}^{n} f(c_i)(\Delta x)_i$$

$$\downarrow$$

$$F(b)-F(a) = \sum_{i=1}^{n} f(c_i)(\Delta x)_i$$

Taking $\lim\limits_{\|\Delta\|\to 0}$ of both sides we get

$$\lim_{\|\Delta\|\to 0} F(b)-F(a) = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n} f(c_i)(\Delta x)_i$$

$$F(b)-F(a) = \int_a^b f(x)\,dx$$

FTC Explanation & Proof

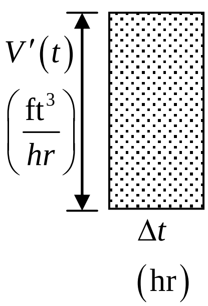
Integral County needs to know about how much water is flowing into its reservoirs. One of Norm of Delta's many jobs is to check the gauging station on the River Calc once each day. The station tells the current rate of flow in thousands of cubic feet per hour. His weekly report shows the following.

	Mon	**Tues**	**Wed**	**Thurs**	**Fri**	**Sat**	**Sun**
Time t_i	9:25 am	1:30 pm	8:00 am	11:30 am	8:30 am	10:20 am	11:30 am
Flow Rate $V'(t_i)$ **1000 ft^3 / hr**	13	17	12	16	10	14	18

Norm of Delta can estimate the volume of water that flowed into the reservoir over the course of the week by doing the following calculation:

$$V = (13{,}000)(24) + (17{,}000)(24) + \cdots + (18{,}000)(24)$$

Each term of the sum can be viewed as the area of a Riemann Rectangle:



$$A = V'(t)\Delta t$$
$$= \left(\frac{\text{ft}^3}{\cancel{\text{hr}}}\right)(\cancel{\text{hr}}) = \text{ft}^3$$

Since we are adding up all the areas of each of the rectangles to get and *estimate* of the volume of water that flowed into the reservoir, we can write it our as the following sum

$$\sum_{i=1}^{n} V'(t_i)(\Delta t)_i$$

To make the estimate better, we must take more readings of the rate of flow over smaller and smaller time intervals. That is we need to take the limit as our $(\Delta x)_i \to 0$

If represents the width of the longest subinterval of time, then to get the true value of the amount of water that flowed in to the reservoir we are making $\left\|(\Delta x)_i\right\| \to 0$.

$$\lim_{\left\|(\Delta x)_i\right\| \to 0} \sum_{i=1}^{n} V'(t_i)(\Delta t)_i = \int_0^{168} V'(t)\,dt$$

Note that there are 24 hours in a day, over the course of 7 days, means that 168 hours have passed over the course of the week.

However, $\int_0^{168} V'(t)\,dt$ <u>DOES NOT REPRESENT THE TOTAL VOLUME OF THE RESERVOIR!</u>

In order to determine the volume of water that is in the reservoir at the end of the week we need to know how much water was in the reservoir at the beginning of the week, i.e. $V(0)$.

Since $V(168)$ represents the volume of water that was in the reservoir at the end of the week, we can state that

FTC Explanation & Proof

$$\underbrace{V(0)}_{\substack{\text{Volume of water}\\ \text{at } t=0}} + \underbrace{\int_0^{168} V'(t)\,dt}_{\substack{\text{Net change in volume}\\ \text{from } t=0 \text{ to } t=168}} = \underbrace{V(168)}_{\substack{\text{Volume of water}\\ \text{at } t=168}}$$

Subtracting $V(0)$ from both sides we get

$$\int_0^{168} V'(t)\,dt = V(168) - V(0)$$

This is the same as the statement of the FTC Part II

$$\boxed{\underbrace{\int_a^b V'(t)\,dt}_{\substack{\text{Definite Integral of}\\ \text{rate of change of volume}\\ \text{from } t=a \text{ to } t=b}} = \underbrace{V(b) - V(a)}_{\substack{\text{Net change in volume}\\ \text{from } t=a \text{ to } t=b}}}$$

Hence this is an alternate explanation of the proof/discovery of FTC Part II.

It is important to note that you can find the value of any function so long as you know two things:

(1) The value of the function at a specific time
(2) The rate of change/derivative of that function.

That is, if you know $f(a)$ and $f'(x)$, you can find the value of $f(b)$ *for any b* by using the following formula:

$$\boxed{f(a) + \int_a^b f'(x)\,dx = f(b)}$$

$$f(a) + \int_a^b f'(x)\,dx = f(b)$$

$$f(a) + \int_a^b \left(\frac{dy}{\cancel{dx}}\right)\cancel{dx} = f(b)$$

$$f(a) + \underset{\substack{\text{from } x=a\\ \text{to } x=b}}{\Delta y} = f(b)$$