Types of Error Bound Exercises

In the free response portion of the AP Calculus BC exam, there are two ways to determine the bound of the error of approximating a function f at a value of x within the interval of convergence using the Taylor polynomial of degree n for f centered at $x=c$, denoted $P_n(x)$.

Let $f(x)=f(c)+f'(c)(x-c)+\dfrac{f''(c)(x-c)^2}{2!}+\cdots+\dfrac{f^{(n)}(c)(x-c)^n}{n!}+\cdots$ for all x within the interval of convergence. The error between $P_n(x)$ and $f(x)$ can be expressed as the following

$$f(x)=\underbrace{f(c)+f'(c)(x-c)+\frac{f''(c)(x-c)^2}{2!}+\cdots+\frac{f^{(n)}(c)(x-c)^n}{n!}}_{P_n(x)}+\underbrace{\frac{f^{(n+1)}(c)(x-c)^{n+1}}{(n+1)!}+\cdots}_{R_n}$$

$$f(x)=P_n(x)+R_n$$

$$f(x)-P_n(x)=R_n$$

$$\left|f(x)-P_n(x)\right|=\underbrace{\left|R_n\right|}_{\text{Error}}$$

$$\left|f(x)-P_n(x)\right|=\text{Error}$$

One way to bound the error in the approximation is to use the Alternating Series Remainder Theorem, and the other is to use the Lagrange Error Bound.

Using the Alternating Series Remainder Theorem

Let $\sum_{n=0}^{\infty}(-1)^n a_n (x-c)^n$ be the Taylor series for *f* centered at $x=c$.

Then

$$f(x)=a_0-a_1(x-c)+a_2(x-c)^2-a_3(x-c)^3+\cdots+(-1)^n a_n(x-c)^n+(-1)^{n+1}a_{n+1}(x-c)^{n+1}+\cdots$$

If *x* is in the interval of convergence, then

$$f(x)=\underbrace{a_0-a_1(x-c)+a_2(x-c)^2-a_3(x-c)^3+\cdots+(-1)^n a_n(x-c)^n}_{P_n(x)}+\underbrace{(-1)^{n+1}a_{n+1}(x-c)^{n+1}+\cdots}_{R_n}$$

$$f(x)=P_n(x)+(-1)^{n+1}a_{n+1}(x-c)^{n+1}+\cdots$$

$$f(x)-P_n(x)=(-1)^{n+1}a_{n+1}(x-c)^{n+1}+\cdots$$

$$\left|f(x)-P_n(x)\right|=\underbrace{\left|(-1)^{n+1}a_{n+1}(x-c)^{n+1}+\cdots\right|}_{\text{Error}}$$

$$\left|f(x)-P_n(x)\right|=\text{Error}$$

The Alternating Series Remainder Theorem states that

$$\text{Error}=\left|f(x)-P_n(x)\right|\le\left|(-1)^{n+1}a_{n+1}(x-c)^{n+1}\right|$$

That is, <u>the error in using the first *n* terms to approximate a convergent alternating series is bounded by the absolute value of the subsequent/next term.</u>

$$\text{Error}\le\left|\text{next term}\right|$$

AP Calculus BC 2012 Question #6

The function *g* has derivatives of all orders, and the Maclaurin series for *g* is

$$\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+3}=\frac{x}{3}-\frac{x^3}{5}+\frac{x^5}{7}-\cdots$$

(a) Using the ratio test, determine the interval of convergence for the Maclaurin series for *g*.

(b) The Maclaurin series for *g* evaluated at $x=\frac{1}{2}$ is an alternating series whose terms decrease in value to zero. The approximation for $g\left(\frac{1}{2}\right)$ using the first two nonzero terms of this series is $\frac{17}{120}$. Show that approximation differs from $g\left(\frac{1}{2}\right)$ by less than $\frac{1}{200}$.

(c) Write the first three nonzero terms and the general term of the Maclaurin series for $g'(x)$

The function g has derivatives of all orders, and the Maclaurin series for g is

$$\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+3}=\frac{x}{3}-\frac{x^3}{5}+\frac{x^5}{7}-\cdots$$

(a) Using the ratio test, determine the interval of convergence for the Maclaurin series for g.

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}\dfrac{x^{2(n+1)+1}}{2(n+1)+3}}{(-1)^n\dfrac{x^{2n+1}}{2n+3}}\right|=\lim_{n\to\infty}\left|\frac{x^{2n+3}(2n+3)}{[2n+5]\left(x^{2n+1}\right)}\right|$$

$$=\lim_{n\to\infty}\left|\frac{x^{2n+3}(2n+3)}{[2n+5]\left(x^{2n+1}\right)}\right|$$

$$=\left|x^2\right|$$

$$\downarrow$$

$$\left|x^2\right|<1$$

$$|x|<1$$

When $x=-1$	When $x=1$
$\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{(-1)^{2n+1}}{2n+3}=\sum_{n=0}^{\infty}\frac{(-1)^n\cdot(-1)^{2n+1}}{2n+3}$ $\displaystyle=\sum_{n=0}^{\infty}\frac{(-1)^{3n+1}}{2n+3}$ $\displaystyle=\sum_{n=0}^{\infty}\frac{(-1)^{3n}(-1)}{2n+3}$ $\displaystyle=\sum_{n=0}^{\infty}\frac{\left[(-1)^3\right]^n(-1)}{2n+3}$ $\displaystyle=\sum_{n=0}^{\infty}\frac{\left[(-1)\right]^n(-1)}{2n+3}$ $\displaystyle=\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{2n+3}$ This is a convergent Alternating Series.	$\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{(1)^{2n+1}}{2n+3}=\sum_{n=0}^{\infty}(-1)^n\frac{1}{2n+3}$ This is a convergent Alternating Series.

The interval of convergence is $-1\le x\le 1$.

(b) The Maclaurin series for g evaluated at $x = \frac{1}{2}$ is an alternating series whose terms decrease in value to zero. The approximation for $g\left(\frac{1}{2}\right)$ using the first two nonzero terms of this series is $\frac{17}{120}$. Show that approximation differs from $g\left(\frac{1}{2}\right)$ by less than $\frac{1}{200}$.

$$g\left(\frac{1}{2}\right) = \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{1}{2}\right)^{2n+1}}{2n+3} = \underbrace{\frac{\left(\frac{1}{2}\right)}{3} - \frac{\left(\frac{1}{2}\right)^3}{5}}_{\text{approximation}} + \underbrace{\frac{\left(\frac{1}{2}\right)^5}{7}}_{\text{next term}} - \cdots$$

By the Alternating Series Remainder Theorem, the difference between $g\left(\frac{1}{2}\right)$ and the approximation using the first two nonzero terms is bounded by the absolute value of the next term:

$$\text{Error} = \left| f\left(\frac{1}{2}\right) - P_3\left(\frac{1}{2}\right) \right| = \left| \frac{\left(\frac{1}{2}\right)^5}{7} \right| = \frac{1}{2^5 \cdot 7} = \frac{1}{224} < \frac{1}{200}$$

(c) Write the first three nonzero terms and the general term of the Maclaurin series for $g'(x)$

$$g(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+3} = \frac{x}{3} - \frac{x^3}{5} + \frac{x^5}{7} - \cdots + \frac{(-1)^n x^{2n+1}}{2n+3} + \cdots$$

$$\downarrow$$

$$g'(x) = \sum_{n=0}^{\infty} (-1)^n \frac{(2n+1)x^{2n}}{2n+3} = \frac{1}{3} - \frac{3x^2}{5} + \frac{5x^4}{7} - \cdots + \frac{(-1)^n (2n+1)x^{2n}}{2n+3} + \cdots$$

Using the Lagrange Error Bound

Let $f(x)=f(c)+f'(c)(x-c)+\dfrac{f''(c)(x-c)^2}{2!}+\cdots+\dfrac{f^{(n)}(c)(x-c)^n}{n!}+\cdots$ for all x within the radius of convergence. The error between $P_n(x)$ and $f(x)$ can be expressed as the following

$$f(x)=\underbrace{f(c)+f'(c)(x-c)+\frac{f''(c)(x-c)^2}{2!}+\cdots+\frac{f^{(n)}(c)(x-c)^n}{n!}}_{P_n(x)}+\underbrace{\frac{f^{(n+1)}(c)(x-c)^{n+1}}{(n+1)!}+\cdots}_{R_n}$$

$$f(x)=P_n(x)+\frac{f^{(n+1)}(c)(x-c)^{n+1}}{(n+1)!}+\cdots$$

$$f(x)-P_n(x)=\frac{f^{(n+1)}(c)(x-c)^{n+1}}{(n+1)!}+\cdots$$

$$\left|f(x)-P_n(x)\right|=\underbrace{\left|\frac{f^{(n+1)}(c)(x-c)^{n+1}}{(n+1)!}+\cdots\right|}_{\text{Error}}$$

$$\left|f(x)-P_n(x)\right|=\text{Error}$$

The Lagrange error bound states that $\text{Error}=\left|f(x)-P_n(x)\right|\le\dfrac{\max\limits_{z\text{ between }x\text{ and }c}\left|f^{(n+1)}(z)\right|\cdot\left|x-c\right|^{n+1}}{(n+1)!}$. The value of *n*, *x*, and *c* must be given in the context of the problem. The only value that must be determined is $\max\limits_{z\text{ between }x\text{ and }c}\left|f^{(n+1)}(z)\right|$.

$$\max_{z\text{ between }x\text{ and }c}\left|f^{(n+1)}(z)\right|$$

is **the maximum of the $(n+1)^{\text{th}}$ derivative of *f* on the interval between *x* and *c*.** Where *x* is the value/location at which you are making an approximation, and *c* is the center of the series.

There are four ways to find $\max\limits_{z\text{ between }x\text{ and }c}\left|f^{(n+1)}(z)\right|$

I. $\max\limits_{z\text{ between }x\text{ and }c}\left|f^{(n+1)}(z)\right|$ is explicitly given

II. $\max\limits_{z\text{ between }x\text{ and }c}\left|f^{(n+1)}(z)\right|$ can be determined by the bounded nature of the function $f^{(n+1)}(z)$

III. $\max\limits_{z\text{ between }x\text{ and }c}\left|f^{(n+1)}(z)\right|$ can be determined visually from the graph of $\left|f^{(n+1)}(z)\right|$

IV. $\max\limits_{z\text{ between }x\text{ and }c}\left|f^{(n+1)}(z)\right|$ can be determined from a table of values of $f^{(n+1)}(z)$, along with information about whether $f^{(n+1)}(z)$ is increasing or decreasing

I. AP Calculus BC 2004 Form B # 2 (Calculator)

Let f be the function having derivatives of all orders for all real numbers. The third-degree Taylor polynomial for f about $x=2$ is given by $T_3(x)=7-9(x-2)^2-3(x-2)^3$.

(a) Find $f(2)$ and $f''(2)$.

(b) Is there enough information given to determine whether f has a critical point at $x=2$? If not, explain why not. If so, determine whether $f(2)$ is a relative maximum, a relative minimum, or neither, and justify your answer.

(c) Use $T_3(x)$ to find an approximation for $f(0)$. Is there enough information given to determine whether f has a critical point at $x=0$? If not, explain why not. If so, determine whether $f(0)$ is a relative maximum, a relative minimum, or neither, and justify your answer.

(d) The fourth derivative of f satisfies the inequality $\left|f^{(4)}(x)\right|\le 6$ for all x in the closed interval $[0,2]$. Use the Lagrange error bound on the approximation to $f(0)$ found in part (c) to explain why $f(0)$ is negative.

II. AP Calculus BC 2004 #6 (No Calculator)

Let f be the function given by $f(x) = \sin\left(5x + \frac{\pi}{4}\right)$ and let $P(x)$ be the third degree Taylor polynomial for f about $x = 0$.

(a) Find $P(x)$

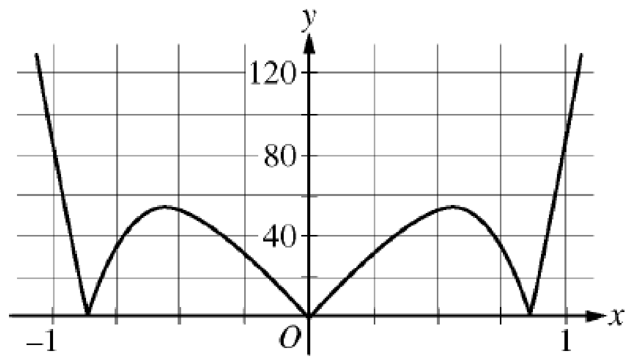
(b) Find the coefficient of x^{22} in the Taylor series for $x = 0$.

(c) Use the Lagrange error bound to show that $\left| f\left(\frac{1}{10}\right) - P\left(\frac{1}{10}\right) \right| < \frac{1}{100}$

(d) Let G be the function given by $G(x) = \int_0^x f(t)\,dt$. Write the third degree Taylor Polynomial for G about $x = 0$.

III. AP Calculus BC 2011 #6 (No Calculator)

Let $f(x) = \sin(x^2) + \cos(x)$. The graph of $y = \left| f^{(5)}(x) \right|$ is shown at right.



Graph of $y = \left| f^{(5)}(x) \right|$

(a) Write the first four nonzero terms of the Taylor series for $\sin(x)$ about $x = 0$, and write the first four nonzero terms of the Taylor series for $\sin(x^2)$ about $x = 0$.

(b) Write the first four nonzero terms for $\cos(x)$ about $x = 0$. Use this series and the series for $\sin(x^2)$, found in part (a), to write the first four nonzero terms of the Taylor series for *f* about $x = 0$.

(c) Find the value of $f^{(6)}(0)$.

(d) Let $P_4(x)$ be the fourth-degree Taylor polynomial for *f* about $x = 0$. Using information from the graph of $y = \left| f^{(5)}(x) \right|$ shown above, show that $\left| P_4\left(\frac{1}{4}\right) - f\left(\frac{1}{4}\right) \right| < \frac{1}{3000}$.

IV. AP Calculus BC 2008 #3 (Calculator)

x	$h(x)$	$h'(x)$	$h''(x)$	$h'''(x)$	$h^{(4)}(x)$
1	11	30	42	99	18
2	80	128	$\frac{488}{3}$	$\frac{448}{3}$	$\frac{584}{9}$
3	317	$\frac{753}{2}$	$\frac{1383}{4}$	$\frac{3483}{16}$	$\frac{1125}{16}$

Let h be a function having derivatives of all orders for $x > 0$. Selected values of h and its first four derivatives are indicated in the table above. The function h and these four derivatives are increasing on the interval $1 \le x \le 3$.

(a) Write the first-degree Taylor polynomial for h about $x = 2$ and use it to approximate $h(1.9)$. Is this approximation greater or less than $h(1.9)$? Explain your reasoning.

(b) Write the third-degree Taylor polynomial for h about $x = 2$ and use it to approximate $h(1.9)$.

(c) Use the Lagrange error bound to show that the third-degree Taylor polynomial for h about $x = 2$ approximates $h(1.9)$ with error less than 3×10^{-4}.

I. AP Calculus BC 2004 Form B # 2 Solutions

Let f be the function having derivatives of all orders for all real numbers. The third-degree Taylor polynomial for f about $x=2$ is given by $T(x)=7-9(x-2)^2-3(x-2)^3$.

(a) Find $f'(2)$ and $f''(2)$.

$$T_3(x)=7-9(x-2)^2-3(x-2)^3$$
$$= f(2)+f'(2)(x-2)+\frac{f''(2)(x-2)^2}{2!}+\frac{f'''(2)(x-2)^3}{3!}$$

$f(2)$ is the constant term of the Taylor Series expansion, which means $f(2)=7$

The coefficient of $(x-2)^2$ is -9. Therefore

$$-9=\frac{f''(2)}{2!}$$
$$-18=f''(2)$$

(b) Is there enough information given to determine whether f has a critical point at $x=2$?
If not, explain why not. If so, determine whether $f(2)$ is a relative maximum, a relative minimum, or neither, and justify your answer.

$f'(2)$ is the coefficient of $(x-2)$. Since there is no $(x-2)$ term, we can conclude that $\frac{f'(2)(x-2)^1}{1}=0 \rightarrow f'(2)=0$.

Yes. Since $f'(2)=0$ and $f''(2)<0$, by the Second Derivative Test $f(x)$ has a relative maximum at $x=2$.

(c) Use $T_3(x)$ to find an approximation for $f(0)$. Is there enough information given to determine whether f has a critical point at $x=0$? If not, explain why not. If so, determine whether $f(0)$ is a relative maximum, a relative minimum, or neither, and justify your answer.

$$\begin{aligned} f(0) &\approx T_3(0) \\ &\approx 7-9(-2)^2-3(-2)^3 \\ &\approx -5 \end{aligned}$$

There is not enough information to determine whether $f(x)$ has a critical point at $x=0$ because the Taylor Polynomial for $f(x)$ centered at $x=2$ does not give us information about the derivative of $f(x)$ at $x=0$.

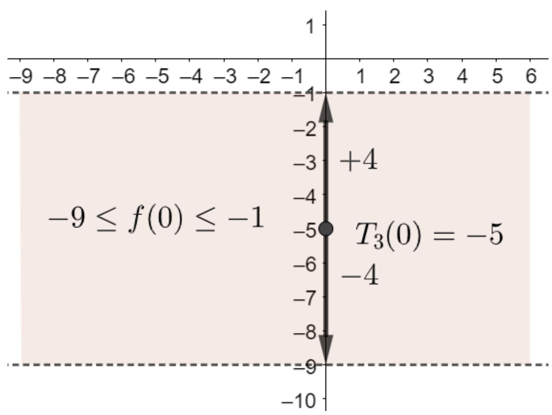
(d) The fourth derivative of f satisfies the inequality $\left|f^{(4)}(x)\right| \le 6$ for all x in the closed interval $[0,2]$. Use the Lagrange error bound on the approximation to $f(0)$ found in part (c) to explain why $f(0)$ is negative.

We know that $f(0)=T_3(0)+\text{Remainder}$

By the Lagrange Error Bound, we know that

$$\begin{aligned} \text{Remainder} &\le \frac{\underbrace{f^{(4)}(z)}_{0\le z\le 2}|0-2|^4}{4!} \\ &\le \frac{6|-2|^4}{4!} \\ &\le 4 \end{aligned}$$

$$\left|f(0)-T_3(0)\right| \le 4$$
$$\left|f(0)-(-5)\right| \le 4$$
$$\downarrow$$
$$-4 \le f(0)+5 \le 4$$
$$-9 \le f(0) \le -1$$



Therefore $f(0) \le -1$, which means that $f(0)$ must be negative.

II. AP Calculus BC 2004 #6 Solutions

Let f be the function given by $f(x)=\sin\left(5x+\frac{\pi}{4}\right)$ and let $P(x)$ be the third degree Taylor polynomial for f about $x=0$.

$$f(x)=\sin\left(5x+\frac{\pi}{4}\right) \qquad f(0)=\frac{\sqrt{2}}{2}$$

$$f'(x)=5\cos\left(5x+\frac{\pi}{4}\right) \qquad f'(0)=\frac{5\sqrt{2}}{2}$$

$$f''(x)=-5^2\sin\left(5x+\frac{\pi}{4}\right) \qquad f''(0)=\frac{-5^2\sqrt{2}}{2}$$

$$f'''(x)=-5^3\cos\left(5x+\frac{\pi}{4}\right) \qquad f'''(0)=\frac{-5^3\sqrt{2}}{2}$$

$$f^{(4)}(x)=5^4\sin\left(5x+\frac{\pi}{4}\right) \qquad f^{(4)}(0)=\frac{5^4\sqrt{2}}{2}$$

$$f^{(n)}(x)=\begin{cases}(-1)^{\frac{n}{2}}\sin\left(5x+\frac{\pi}{4}\right) & \text{if } n \text{ is even}\\ (-1)^{\frac{n-1}{2}}\cos\left(5x+\frac{\pi}{4}\right) & \text{if } n \text{ is odd}\end{cases} \qquad f^{(n)}(0)=\begin{cases}\dfrac{(-1)^{\frac{n}{2}}5^n\sqrt{2}}{2} & \text{if } n \text{ is even}\\ \dfrac{(-1)^{\frac{n-1}{2}}5^n\sqrt{2}}{2} & \text{if } n \text{ is odd}\end{cases}$$

(a) Find $P(x)$

$$P(x)=f(0)+\frac{f'(0)(x-0)}{1!}+\frac{f''(0)(x-0)^2}{2!}+\frac{f'''(0)(x-0)^3}{3!}$$

$$=\frac{\sqrt{2}}{2}+\frac{\left(\frac{\sqrt{2}}{2}\right)x}{1!}+\frac{\left(\frac{5\sqrt{2}}{2}\right)x^2}{2!}+\frac{\left(\frac{-5^3\sqrt{2}}{2}\right)x^3}{3!}$$

(b) Find the coefficient of x^{22} in the Taylor series for $x=0$.

$$\frac{f^{(n)}(0)x^n}{n!}\rightarrow\frac{f^{(22)}(0)x^{22}}{22!}=\frac{(-1)^{\frac{22}{2}}\frac{5^{22}\sqrt{2}}{2}x^{22}}{22!}$$

The coefficient of x^{22} is given by $\dfrac{(-1)^{\frac{22}{2}}\frac{5^{22}\sqrt{2}}{2}}{22!}=\dfrac{(-1)^{11}\frac{5^{22}\sqrt{2}}{2}}{22!}=-\dfrac{\left(\frac{5^{22}\sqrt{2}}{2}\right)}{22!}=-\dfrac{5^{22}\sqrt{2}}{2\cdot(22!)}$

(c) Use the Lagrange error bound to show that $\left| f\left(\frac{1}{10}\right) - P\left(\frac{1}{10}\right) \right| < 100$

$$\text{Error} \le \frac{\max\limits_{z \text{ between 0 and } \frac{1}{10}} f^{(4)}(z) \cdot \left| \frac{1}{10} - 0 \right|^4}{4!}$$

Since $f^{(4)}(x) = 5^4 \sin\left(5x + \frac{\pi}{4}\right)$, we know that $\left| f^{(4)}(x) \right| \le 5^4$ for all real numbers x.

$$\begin{aligned}
\text{Error} &\le \frac{\max\limits_{z \text{ between 0 and } \frac{1}{10}} f^{(4)}(z) \cdot \left| \frac{1}{10} - 0 \right|^4}{4!} \\
&\le \frac{5^4 \cdot \left| \frac{1}{10} - 0 \right|^4}{4!} \\
&\le \frac{5^4}{4! \cdot 10^4} \\
&\le \frac{5^4}{4! \cdot (2 \cdot 5)^4} \\
&\le \frac{\cancel{5^4}}{4! \cdot 2^4 \cdot \cancel{5^4}} \\
&\le \frac{1}{24 \cdot 32} \\
&< \frac{1}{100}
\end{aligned}$$

(d) Let G be the function given by $G(x)=\int_0^x f(t)\,dt$. Write the third-degree Taylor Polynomial for G about $x=0$.

$$G(x)=\int_0^x f(t)\,dt$$

$$=\int_0^x \frac{\sqrt{2}}{2}+\frac{\left(\frac{\sqrt{2}}{2}\right)t}{1!}+\frac{\left(\frac{5\sqrt{2}}{2}\right)t^2}{2!}+\frac{\left(\frac{-5^3\sqrt{2}}{2}\right)t^3}{3!}+\cdots dt$$

$$=\left[\frac{\sqrt{2}}{2}t+\frac{\left(\frac{\sqrt{2}}{2}\right)t^2}{2\cdot(1!)}+\frac{\left(\frac{5\sqrt{2}}{2}\right)t^3}{3\cdot(2!)}+\frac{\left(\frac{-5^3\sqrt{2}}{2}\right)t^4}{4\cdot(3!)}+\cdots\right]_0^x$$

$$=\left[\frac{\sqrt{2}}{2}x+\frac{\left(\frac{\sqrt{2}}{2}\right)x^2}{2\cdot(1!)}+\frac{\left(\frac{5\sqrt{2}}{2}\right)x^3}{3\cdot(2!)}+\frac{\left(\frac{-5^3\sqrt{2}}{2}\right)x^4}{4\cdot(3!)}+\cdots\right]-\left[\frac{\sqrt{2}}{2}0+\frac{\left(\frac{\sqrt{2}}{2}\right)0^2}{2\cdot(1!)}+\frac{\left(\frac{5\sqrt{2}}{2}\right)0^3}{3\cdot(2!)}+\frac{\left(\frac{-5^3\sqrt{2}}{2}\right)0^4}{4\cdot(3!)}+\cdots\right]$$

$$=\left[\frac{\sqrt{2}}{2}x+\frac{\left(\frac{\sqrt{2}}{2}\right)x^2}{2\cdot(1!)}+\frac{\left(\frac{5\sqrt{2}}{2}\right)x^3}{3\cdot(2!)}+\frac{\left(\frac{-5^3\sqrt{2}}{2}\right)x^4}{4\cdot(3!)}+\cdots\right]$$

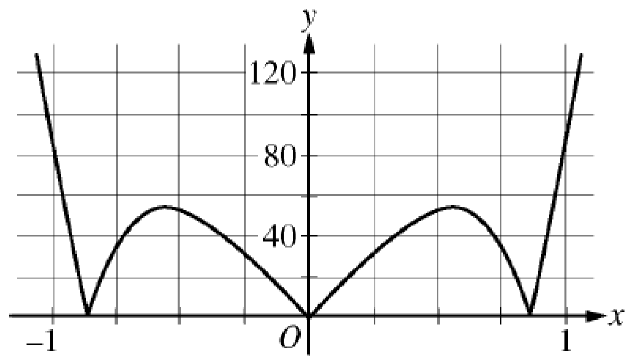
$$=\frac{\sqrt{2}}{2}x+\frac{\left(\frac{\sqrt{2}}{2}\right)x^2}{2\cdot(1!)}+\frac{\left(\frac{5\sqrt{2}}{2}\right)x^3}{3\cdot(2!)}+\frac{\left(\frac{-5^3\sqrt{2}}{2}\right)x^4}{4\cdot(3!)}+\cdots$$

The third degree Taylor polynomial for $G(x)=\int_0^x f(t)\,dt$ about $x=0$ is

$$\frac{\sqrt{2}}{2}x+\frac{\left(\frac{\sqrt{2}}{2}\right)x^2}{2\cdot(1!)}+\frac{\left(\frac{5\sqrt{2}}{2}\right)x^3}{3\cdot(2!)}$$

Let $f(x)=\sin\left(x^2\right)+\cos(x)$. The graph of $y=\left|f^{(5)}(x)\right|$ is shown at right.

(a) Write the first four nonzero terms of the Taylor series for $\sin(x)$ about $x=0$, and write the first four nonzero terms of the Taylor series for $\sin\left(x^2\right)$ about $x=0$.



Graph of $y=\left|f^{(5)}(x)\right|$

$$\sin(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{n!}$$

$$=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots+\frac{(-1)^n x^{2n+1}}{n!}+\cdots$$

$$\downarrow$$

$$\sin\left(x^2\right)=\sum_{n=0}^{\infty}\frac{(-1)^n \left(x^2\right)^{2n+1}}{n!}$$

$$=\left(x^2\right)-\frac{\left(x^2\right)^3}{3!}+\frac{\left(x^2\right)^5}{5!}-\frac{\left(x^2\right)^7}{7!}+\cdots+\frac{(-1)^n \left(x^2\right)^{2n+1}}{n!}+\cdots$$

$$=x^2-\frac{x^6}{3!}+\frac{x^{10}}{5!}-\frac{x^{14}}{7!}+\cdots+\frac{(-1)^n x^{4n+2}}{n!}+\cdots$$

(b) Write the first four nonzero terms for $\cos(x)$ about $x = 0$. Use this series and the series for $\sin(x^2)$, found in part (a), to write the first four nonzero terms of the Taylor series for f about $x = 0$.

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$
$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots$$

$$f(x) = \sin(x^2) + \cos(x)$$
$$= \left[x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots + \frac{(-1)^n x^{4n+2}}{n!} + \cdots \right] + \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots \right]$$
$$\downarrow$$
$$f(x) \approx 1 + \left(x^2 - \frac{x^2}{2!} \right) + \frac{x^4}{4!} + \left(-\frac{x^6}{3!} - \frac{x^6}{6!} \right)$$
$$\approx 1 + \frac{x^2}{2} + \frac{x^4}{4!} - \left(\frac{6 \cdot 5 \cdot 4 + 1}{6!} \right) x^6$$
$$\approx 1 + \frac{x^2}{2} + \frac{x^4}{4!} - \frac{121x^6}{6!}$$

(c) Find the value of $f^{(6)}(0)$.

We know that $\dfrac{f^{(6)}(0) \cdot x^6}{6!} = -\dfrac{121x^6}{6!}$ is the degree six term of the Taylor polynomial around $x = 0$. Therefore $f^{(6)}(0) = -121$.

(d) Let $P_4(x)$ be the fourth-degree Taylor polynomial for f about $x=0$. Using information from the graph of $y=\left|f^{(5)}(x)\right|$ shown above, show that $\left|P_4\left(\frac{1}{4}\right)-f\left(\frac{1}{4}\right)\right|<\frac{1}{3000}$.

The Lagrange Error Bound is given by

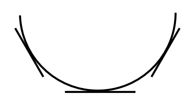
$$
\begin{aligned}
\text{Error} &\le \frac{\left|\max\limits_{\substack{z \text{ between} \\ 0 \text{ and } \frac{1}{4}}} f^{(5)}(z)\right| \cdot \left(\frac{1}{4}-0\right)^5}{5!} \\
&\le \frac{40\left(\frac{1}{4}\right)^5}{5!} \\
&\le \frac{\cancel{5}\cdot\cancel{4}\cdot\cancel{2}}{\cancel{5}\cdot\cancel{4}\cdot 3\cdot\cancel{2}\cdot 1\cdot 4^5} \\
&\le \frac{1}{3\cdot 1024} \\
&\le \frac{1}{3072} \\
&< \frac{1}{3000}
\end{aligned}
$$

IV. AP Calculus BC 2008 #3 Solutions

x	$h(x)$	$h'(x)$	$h''(x)$	$h'''(x)$	$h^{(4)}(x)$
1	11	30	42	99	18
2	80	128	$\frac{488}{3}$	$\frac{448}{3}$	$\frac{584}{9}$
3	317	$\frac{753}{2}$	$\frac{1383}{4}$	$\frac{3483}{16}$	$\frac{1125}{16}$

Let h be a function having derivatives of all orders for $x > 0$. Selected values of h and its first four derivatives are indicated in the table above. The function h and these four derivatives are increasing on the interval $1 \le x \le 3$.

(a) Write the first-degree Taylor polynomial for h about $x = 2$ and use it to approximate $h(1.9)$. Is this approximation greater or less than $h(1.9)$? Explain your reasoning.



$$T_1 = 80 + 128(x-2)$$
$$h(1.9) \approx 80 + 128(1.9-2)$$

This approximation is an underestimate because $f''(x)$ is positive on $1 \le x \le 3$.

(b) Write the third-degree Taylor polynomial for h about $x = 2$ and use it to approximate $h(1.9)$.

$$T_3 = 80 + 128(x-2) + \frac{\left(\frac{488}{3}\right)(x-2)^2}{2!} + \frac{\left(\frac{448}{3}\right)(x-2)^3}{3!}$$
$$h(1.9) \approx 80 + 128(1.9-2) + \frac{\left(\frac{488}{3}\right)(1.9-2)^2}{2!} + \frac{\left(\frac{448}{3}\right)(1.9-2)^3}{3!}$$

(c) Use the Lagrange error bound to show that the third-degree Taylor polynomial for h about $x=2$ approximates $h(1.9)$ with error less than 3×10^{-4}.

3×10^{-4}.

x	$h(x)$	$h'(x)$	$h''(x)$	$h'''(x)$	$h^{(4)}(x)$
1	11	30	42	99	18
2	80	128	$\frac{488}{3}$	$\frac{448}{3}$	$\frac{584}{9}$
3	317	$\frac{753}{2}$	$\frac{1383}{4}$	$\frac{3483}{16}$	$\frac{1125}{16}$

Since $f^{(4)}(x)$ is increasing on the closed interval $[1,2]$, $\left|f^{(4)}(x)\right|\le\frac{584}{9}$ on the closed interval $[1,2]$.

$$\begin{aligned}
|\text{Error}| &\le \frac{\max\limits_{1.9\le z\le 2} f^{(4)}(z)\cdot|1.9-2|^4}{4!}\\
&\le \frac{\left(\frac{584}{9}\right)|1.9-2|^4}{4!}\\
&< 3\times10^{-4}
\end{aligned}$$