Strategies for finding limits:

1. **Direct Substitution** – When substituting will not result in any problems (i.e. No DNE, Division by Zero, ∞, etc.)

2. **Dividing Out/Factoring** - Used often with rational functions that are not factored. Cancel out common factors and then use Direct Substitution

$$\begin{aligned}\lim_{x\to1}\frac{x^3-1}{x-1} &= \lim_{x\to1}\frac{(x-1)(x^2+x+1)}{x-1}\\ &= \lim_{x\to1}\frac{\cancel{(x-1)}(x^2+x+1)}{\cancel{x-1}}\\ &= \lim_{x\to1}x^2+x+1\\ &= 3\end{aligned}$$

3. **Rationalizing (Multiplying by the conjugate)** – Used often with expressions that involve square roots, or trig expression where the Pythagorean Identity can be used.

$$\begin{aligned}\lim_{x\to0}\frac{\sqrt{x+1}-1}{x} &= \lim_{x\to0}\frac{\left(\sqrt{x+1}-1\right)}{x}\cdot\frac{\left(\sqrt{x+1}+1\right)}{\left(\sqrt{x+1}+1\right)}\\ &= \lim_{x\to0}\frac{x+1-1}{x\left(\sqrt{x+1}+1\right)}\\ &= \lim_{x\to0}\frac{x}{x\left(\sqrt{x+1}+1\right)}\\ &= \lim_{x\to0}\frac{\cancel{x}}{\cancel{x}\left(\sqrt{x+1}+1\right)}\\ &= \lim_{x\to0}\frac{1}{\sqrt{x+1}+1}\\ &= \frac{1}{2}\end{aligned}$$

$$\begin{aligned}\lim_{x\to0}\frac{1-\cos(x)}{x} &= \lim_{x\to0}\frac{(1-\cos(x))}{x}\cdot\frac{(1+\cos(x))}{(1+\cos(x))}\\ &= \lim_{x\to0}\frac{1-\cos^2(x)}{x(1+\cos(x))}\\ &= \lim_{x\to0}\frac{\sin^2(x)}{x(1+\cos(x))}\\ &= \lim_{x\to0}\left[\frac{\sin(x)}{x}\cdot\frac{\sin(x)}{1+\cos(x)}\right]\\ &= \lim_{x\to0}\left[\frac{\sin(x)}{x}\right]\cdot\lim_{x\to0}\left[\frac{\sin(x)}{1+\cos(x)}\right]\\ &= 1\cdot0\\ &= 0\end{aligned}$$

4. **Squeezing** – Used often with functions that exhibit damped oscillation

$\lim_{x\to0}x\sin\left(\frac{1}{x}\right)$

$$-x\le x\sin\left(\frac{1}{x}\right)\le x \text{ since } \left|\sin\left(\frac{1}{x}\right)\right|\le1$$

$$\lim_{x\to0}-x\le\lim_{x\to0}x\sin\left(\frac{1}{x}\right)\le\lim_{x\to0}x$$

$$0\le\lim_{x\to0}x\sin\left(\frac{1}{x}\right)\le0$$

$$\downarrow$$

$$\lim_{x\to0}x\sin\left(\frac{1}{x}\right)=0$$

5. **Multiplying Out** – Used often with expression that are of the form of the Difference Quotient $\dfrac{f(x+h)-f(x)}{h}$

$$\begin{aligned}\lim_{h\to 0}\frac{(x+h)^2-x^2}{h} &= \lim_{h\to 0}\frac{x^2+2xh+h^2-x^2}{h}\\ &= \lim_{h\to 0}\frac{\cancel{x^2}+2xh+h^2\cancel{-x^2}}{h}\\ &= \lim_{h\to 0}\frac{2xh+h^2}{h}\\ &= \lim_{h\to 0}\frac{2xh}{h}+\lim_{h\to 0}\frac{h^2}{h}\\ &= \lim_{h\to 0}2x+\lim_{h\to 0}h\\ &= 2x\end{aligned}$$

6. **Rewrite in terms of $\sin(x)$ and $\cos(x)$** - Used when expression involve trig functions that involve $\tan(x), \cot(x), \csc(x),$ and $\sec(x)$.

$$\begin{aligned}\lim_{x\to 0}\frac{\tan(x)}{x} &= \lim_{x\to 0}\frac{\left[\dfrac{\sin(x)}{\cos(x)}\right]}{x}\\ &= \lim_{x\to 0}\frac{\sin(x)}{x\cos(x)}\\ &= \lim_{x\to 0}\frac{\sin(x)}{x}\cdot\lim_{x\to 0}\frac{1}{\cos(x)}\\ &= 1\cdot 1\\ &= 1\end{aligned}$$

7. **Rewrite to involve factors of $\lim\limits_{x\to 0}\dfrac{\sin(x)}{x}=1$ or $\lim\limits_{x\to 0}\dfrac{1-\cos(x)}{x}=0$**

8. **Substitution**

$$\lim_{x\to\infty} x\cdot\sin\left(\frac{1}{x}\right)$$

Let $u=\dfrac{1}{x}$. If $x\to\infty$, then $u\to 0$. If $u=\dfrac{1}{x}$, then $x=\dfrac{1}{u}$.

$$u=\frac{1}{x}$$
$$ux=1$$
$$x=\frac{1}{u}$$

$$\begin{aligned}\lim_{x\to\infty} x\cdot\sin\left(\frac{1}{x}\right) &= \lim_{u\to 0}\frac{1}{u}\sin(u)\\ &= \lim_{u\to 0}\frac{\sin(u)}{u}\\ &= 1\end{aligned}$$

9. **Use a calculator to test your conclusion** – Use $\mathrm{Y}_1(\ \ )$ to test values around the given value of x to see if your conclusion is correct.

Properties of Limits

Let b and c be real numbers. Let n be a positive integer, and let f and g be functions with the following limits:

$$\lim_{x\to c} f(x) = L \text{ and } \lim_{x\to c} g(x) = K$$

1. Scalar Multiple:

$$\begin{aligned}\lim_{x\to c}\left[bf(x)\right] &= b\cdot\lim_{x\to c}\left[f(x)\right]\\ &= bL\end{aligned}$$

2. Sum or Difference:

$$\begin{aligned}\lim_{x\to c}\left[f(x)\pm g(x)\right] &= \lim_{x\to c} f(x)\pm\lim_{x\to c} g(x)\\ &= L+K\end{aligned}$$

3. Product:

$$\begin{aligned}\lim_{x\to c}\left[f(x)\cdot g(x)\right] &= \left[\lim_{x\to c} f(x)\right]\cdot\left[\lim_{x\to c} g(x)\right]\\ &= L\cdot K\end{aligned}$$

4. Quotient:

$$\begin{aligned}\lim_{x\to c}\left[\frac{f(x)}{g(x)}\right] &= \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}\\ &= \frac{L}{K} \qquad \text{provided } K\neq 0\end{aligned}$$

5. Power:

$$\begin{aligned}\lim_{x\to c}\left[f(x)\right]^n &= \left[\lim_{x\to c} f(x)\right]^n\\ &= L^n\end{aligned}$$

If $p(x)$ is a polynomial function, then $\lim_{x\to c} p(x) = p(c)$

If $r(x)=\dfrac{p(x)}{q(x)}$ is a rational function, and $q(c)\neq 0$, then $\lim_{x\to c} r(x) = r(c)$

Let n be a positive integer, then

if n is odd $\qquad \lim_{x\to c}\sqrt[n]{x}=\sqrt[n]{c}$

if n is even $\qquad \lim_{x\to c}\sqrt[n]{x}=\sqrt[n]{c}$ so long as $c>0$