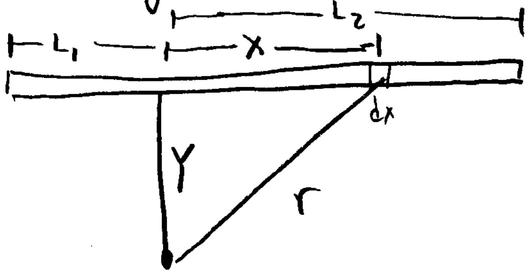
Introductory Problem

a)



Chop the rod into differential length w/ length dx

Define x as shown.

then $-L_1 \le x \le L_2$

notice x will be negative in this region.

dm will be $\lambda\,dx$. for our segment.

$r^2 = x^2 + Y^2$ from the geometry.

so $dI = dm\, r^2 = (\lambda\, dx)(x^2+Y^2)$

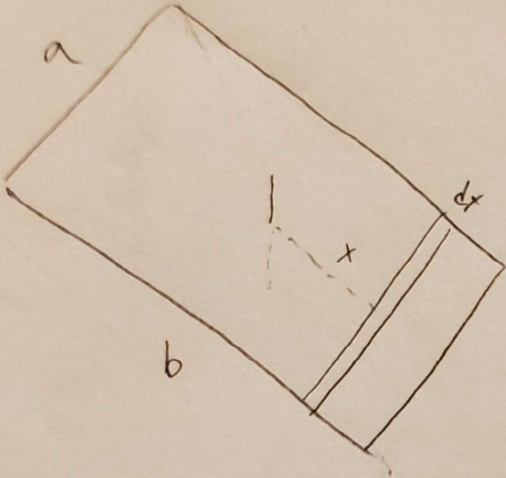
so $I = \int_{-L_1}^{L_2} \lambda (x^2+Y^2)\,dx = \lambda\left(\frac{x^3}{3} + Y^2 x\right)\Big|_{-L_1}^{L_2}$

$= \lambda\left(\frac{L_2^3 - (-L_1^3)}{3} + Y^2(L_2 - -L_1)\right)$

$= \frac{\lambda(L_2^3+L_1^3)}{3} + \lambda Y^2(L_2+L_1)$ and $M = (L_1+L_2)\lambda$ so $\lambda = \frac{M}{L_1+L_2}$

$I = M\left(Y^2 + \frac{L_2^3+L_1^3}{3(L_1+L_2)}\right)$

P+P



ab) Chop up as shown

c) parameterize ~~as~~ with x, so

d) $X_{min} = -\frac{b}{2}$ $\quad X_{max} = \frac{b}{2}$

e) $dA = a\,dx$ (since it's a rectangle)

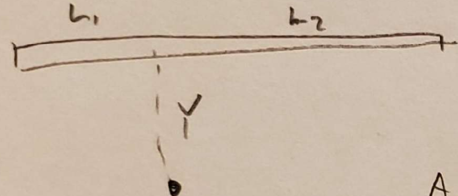
f) $dm = \sigma\, dA$, and $\sigma = \frac{M}{ab}$

so $dm = \frac{M}{ab} \cdot a\,dx = \frac{M}{b}\,dx$

Solve

We found earlier that in the diagram below

a)



$$I = M\left(Y^2 + \frac{L_2^3 + L_1^3}{3(L_1+L_2)}\right)$$

Applying that to our situation,

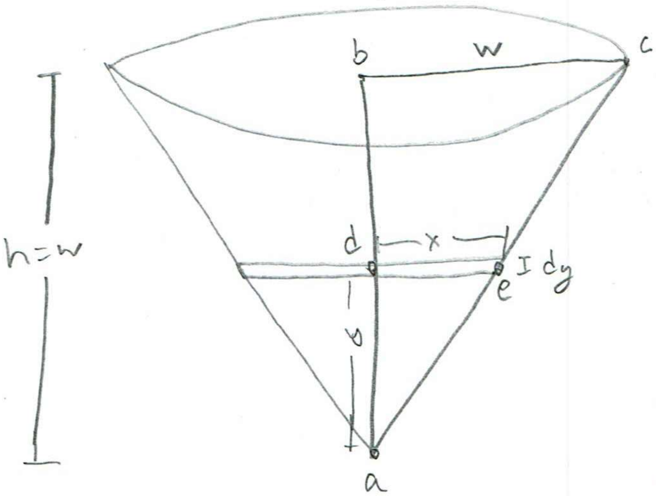
Y becomes x, $L_1 = L_2 = \frac{a}{2}$. So

$$dI = dm\left(x^2 + \frac{\left(\frac{a}{2}\right)^3 + \left(\frac{a}{2}\right)^3}{3\left(\frac{a}{2} + \frac{a}{2}\right)}\right) = dm\left(x^2 + \frac{a^2}{12}\right) = \frac{M}{b}\left(x^2 + \frac{a^2}{12}\right)dx$$

b) Finally,

$$I = \int_{-\frac{b}{2}}^{b/2} \frac{M}{b}\left(x^2 + \frac{a^2}{12}\right)dx = \frac{M}{b}\left(\frac{x^3}{3} + \frac{a^2}{12}x\right)\Bigg|_{-\frac{b}{2}}^{b/2}$$

$$= \frac{M}{b}\left(\frac{b^3}{12} + \frac{a^2 b}{12}\right) = \frac{M}{12}(a^2 + b^2)$$



We could chop it up into Cylinders or solid disks. I'll choose disks here ~~since in the next problem we'll use cylinders~~. I know

$$I_{\text{disk, axis of symmetry}} = \frac{1}{2} m r^2.$$

I'll parameterize with y ranging from 0 to h (or 0 to w).

Each disk has radius x and volume $dV = \pi x^2 dy$. But x varies with y. Noting that triangle abc is similar to ade, I know the ratios of sides are

$$\frac{h}{w} (=1) = \frac{y}{x},$$ so $y = x$, so $dV = \pi y^2 dy$.

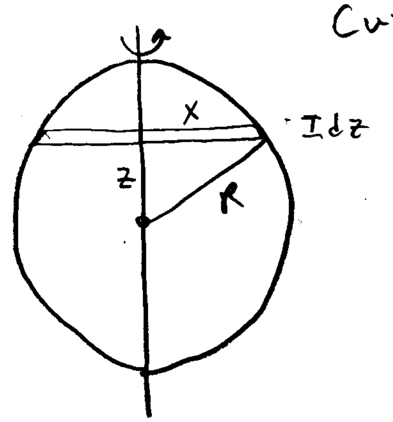
$$dm = \rho\, dV = \pi \rho y^2 dy.$$

Finally $dI = \frac{1}{2} dm\, x^2$ since x is the radius of the disk

and so $dI = \frac{1}{2}\rho\pi y^2 \cdot y^2 dy = \frac{1}{2}\rho\pi y^4 dy.$

$$\text{so } I = \int_0^w \frac{1}{2}\pi\rho y^4 dy = \rho\frac{\pi}{10} w^5$$

Additional problem 1



Cut the sphere into ~~cylinders~~ disks as shown.
They are paramaterized by z, as shown in the diagram.
$-R \leq z \leq R$ is the range of z.

Looking at the triangle in the diagram, we see $z^2 + x^2 = R^2$.
so $x^2 = R^2 - z^2$.

so $dV = (\pi x^2)\,dz = \pi(R^2 - z^2)\,dz$

so $dm = \rho\, dV = \rho\pi (R^2 - z^2)\,dz$

Since the moment of of inertia of a ~~cylinder~~ disk about this axis is $\frac{1}{2} m R^2$, $dI = \frac{dm\, x^2}{2} = \frac{\rho\pi(R^2 - z^2)x^2}{2} dz$

$$= \frac{\rho\pi(R^2 - z^2)^2}{2} dz = \frac{\rho\pi}{2}(R^4 - 2R^2z^2 + z^4)\,dz$$

so $I = \int dI = \int_{-R}^{R} \frac{\rho\pi}{2}(R^2 - 2R^2 z^2 + z^4)\,dz$

$$= 2\int_0^R \frac{\rho\pi}{2}(R^4 - 2R^2 z^2 + z^4)\,dz \quad \text{(since the function is even)}$$

$$= \rho\pi\left(R^5 - \frac{2}{3}R^5 + \frac{1}{5}R^5\right) = \rho\pi\left(\frac{15 - 10 + 3}{15} R^5\right)$$

$I = \frac{8}{15}\rho\pi R^5$

And $M = \rho \frac{4}{3}\pi R^3$, so $\rho = \frac{3m}{4\pi R^3}$

so $I = \frac{8}{15}\left(\frac{3m}{4\pi R^3}\right)\pi R^5 = \frac{2}{5} M R^2$