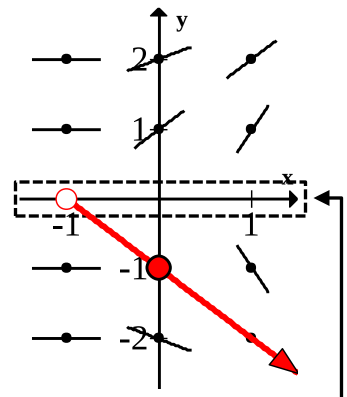
AP Calculus AB 2010 Form B #5 No Calculator



#5 Consider the differential equation $\frac{dy}{dx}=\frac{x+1}{y}$.

(a) On the axes provided, sketch a slope field for the given differential equation at the twelve points indicated, and for $-1<x<1$, sketch the solution curve that passes through the point $(0,-1)$.

The solution curve cannot extend above the x-axis. This is due to the location of the initial condition in the *xy*-plane at $(0,-1)$ and the fact that $\frac{dy}{dx}$ *DNE* for $y=0$.

Slopes must be correctly positive, negative, or zero.
Relative steepness of slopes must be consistent in rows and in columns.

(x,y)	$(-1,2)$	$(-1,1)$	$(0,2)$	$(0,1)$	$(1,2)$	$(1,1)$
$\frac{dy}{dx}$	$\frac{(-1)+1}{2}=0$	$\frac{(-1)+1}{1}=0$	$\frac{0+1}{2}=\frac{1}{2}$	$\frac{0+1}{1}=1$	$\frac{1+1}{2}=1$	$\frac{1+1}{1}=2$
(x,y)	$(-1,-2)$	$(-1,-1)$	$(0,-2)$	$(0,-1)$	$(1,-2)$	$(1,-1)$
$\frac{dy}{dx}$	$\frac{(-1)+1}{-2}=0$ 0	$\frac{(-1)+1}{-1}=0$	$\frac{0+1}{-2}=-\frac{1}{2}$	$\frac{0+1}{-1}=-1$	$\frac{1+1}{-2}=-1$	$\frac{1+1}{-1}=-2$

(b) While the slope field in part (a) is drawn only at twelve points, it is defined at every point in the *xy*-plane for which $y\neq 0$. Describe all points in the *xy*-plane, $y\neq 0$, for which $\frac{dy}{dx}=-1$.

$$\frac{dy}{dx}=\frac{x+1}{y}$$
$$-1=\frac{x+1}{y}$$
$$-y=x+1$$
$$y=-x-1$$

The points on the plane for which $\frac{dy}{dx}=-1$ are all the points on the line $y=-x-1$ and $y\neq 0$.

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(0) = -2$.

$$\frac{dy}{dx} = \frac{x+1}{y}$$

$$ydy = x+1dx$$

$$\int ydy = \int x+1dx$$

$$\frac{1}{2}y^2 = \frac{1}{2}x^2 + x + C$$

$$y^2 = x^2 + 2x + C$$

$$y = \pm\sqrt{x^2 + 2x + C}$$

Since the particular solution passes through $(0,-2)$, we must use the negative square root.

$$y = -\sqrt{x^2 + 2x + C}$$

$$\downarrow$$

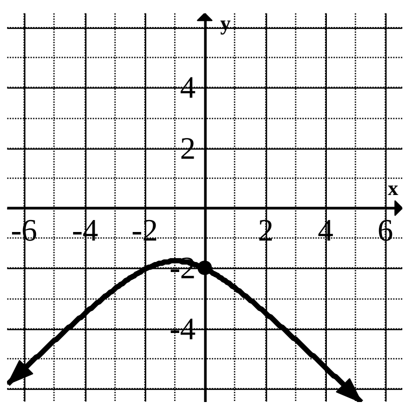
$$-2 = -\sqrt{(0)^2 + 2(0) + C}$$

$$2 = \sqrt{C}$$

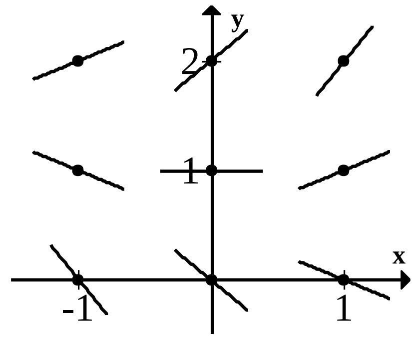
$$C = 4$$

$$\downarrow$$

$$y = -\sqrt{x^2 + 2x + 4}$$



AP Calculus AB 2007 Form B #5 No Calculator



#5 Consider the differential equation $\frac{dy}{dx} = \frac{1}{2}x + y - 1$

(a) On the axes provided, sketch a slope field for the given differential equation at the nine points indicated.

Slopes must be correctly positive, negative, or zero.
Relative steepness of slopes must be consistent in rows and in columns.

(x, y)	$\frac{dy}{dx} =$	(x, y)	$\frac{dy}{dx} =$	(x, y)	$\frac{dy}{dx} =$
$(-1, 2)$	$\frac{1}{2}(-1)+(2)-1=\frac{1}{2}$	$(0, 2)$	$\frac{1}{2}(0)+(2)-1=1$	$(1, 2)$	$\frac{1}{2}(1)+(2)-1=\frac{3}{2}$
$(-1, 1)$	$\frac{1}{2}(-1)+(1)-1=-\frac{1}{2}$	$(0, 1)$	$\frac{1}{2}(0)+(1)-1=0$	$(1, 1)$	$\frac{1}{2}(1)+(1)-1=\frac{1}{2}$
$(-1, 0)$	$\frac{1}{2}(-1)+(0)-1=-\frac{3}{2}$	$(0, 0)$	$\frac{1}{2}(0)+(0)-1=-1$	$(1, 0)$	$\frac{1}{2}(1)+(0)-1=-\frac{1}{2}$

(b) Find $\frac{d^2y}{dx^2}$ in terms of *x* and *y*. Describe the region in the *xy*-plane in which all solution curves to the differential equation are concave up.

$$\frac{dy}{dx} = \frac{1}{2}x + y - 1$$

$$y' = \frac{1}{2}x + y - 1$$

$$\downarrow$$

$$y'' = \frac{1}{2} + y'$$

$$\frac{d^2y}{dx^2} = \frac{1}{2} + \frac{1}{2}x + y - 1$$

$$= \frac{1}{2}x + y - \frac{1}{2}$$

Concave up $\leftrightarrow$ $\frac{d^2y}{dx^2} > 0$

$$\frac{1}{2}x + y - \frac{1}{2} > 0$$

$$y > -\frac{1}{2}x + \frac{1}{2}$$

The region of the plane for which the solution curves to the differential equation are concave up are all the points in the plane satisfying the inequality $y > -\frac{1}{2}x + \frac{1}{2}$, i.e. all the points in the plane that lie above the line $y = -\frac{1}{2}x + \frac{1}{2}$.

(c) Let $y = f(x)$ be a solution to the differential equation with the initial condition $f(0) = 1$. Does *f* have a relative minimum, relative maximum, or neither at $x = 0$? Justify your answer.

$$\left.\frac{dy}{dx}\right|_{(0,1)} = \frac{1}{2}(0) + (1) - 1 = 0$$

and

$$\left.\frac{d^2y}{dx^2}\right|_{(0,1)} = \frac{1}{2}(0) + (1) - \frac{1}{2} = \frac{1}{2}$$

Since $\left.\frac{dy}{dx}\right|_{(0,1)} = 0$ and $\left.\frac{d^2y}{dx^2}\right|_{(0,1)} = \frac{1}{2} > 0$, by the Second Derivative Test $f(x)$ has a relative minmum at $(0,1)$.

(d) Find the values of m and b for which $y = mx + b$ is a solution to the differential equation.

Option #1

If $y = mx + b$ is a solution to the differential equation, then

$$\frac{d^2y}{dx^2} = 0 \text{ for all } (x, y) \text{ in the plane}$$

$$\downarrow$$

$$0 = \frac{1}{2}x + y - \frac{1}{2}$$

$$y = -\frac{1}{2}x + \frac{1}{2}$$

Option #2

$$y = mx + b \rightarrow \frac{dy}{dx} = m$$

$$\frac{1}{2}x + y - 1 = m$$

$$\frac{1}{2}x + (mx + b) - 1 = m$$

$$\underline{\left(\frac{1}{2} + m\right)}x + \underline{\underline{(b-1)}} = \underline{0}x + \underline{\underline{m}}$$

Matching coefficients of the polynomial on the left with the coefficients of the polynomial on the right

$$\frac{1}{2} + m = 0 \qquad b - 1 = m$$

$$m = -\frac{1}{2} \qquad b = m + 1 \rightarrow b = \frac{1}{2}$$

$$y = -\frac{1}{2}x + \frac{1}{2}$$

AP Calculus AB 2010 #6 No Calculator

Solutions to the differential equation $\frac{dy}{dx} = xy^3$ also satisfy $\frac{d^2y}{dx^2} = y^3\left(1+3x^2y^2\right)$. Let $y = f\left(x\right)$ be a particular solution to the differential equation $\frac{dy}{dx} = xy^3$ with $f\left(1\right) = 2$.

(a) Write an equation of the line tangent to the graph of $y = f\left(x\right)$ at $x = 1$.

$$\left.\frac{dy}{dx}\right|_{(1,2)} = \left(1\right)\left(2\right)^3 = 8$$

$$y - y_1 = m\left(x - x_1\right)$$

$$y - 2 = 8\left(x-1\right)$$

(b) Use the tangent line from part (a) to approximate $f\left(1.1\right)$. Given that $f\left(x\right) > 0$ for $1 < x < 1.1$, is the approximation for $f\left(1.1\right)$ greater or less than $f\left(1.1\right)$? Explain your reasoning.

$$y - 2 = 8\left(x-1\right)$$

$$y = 8\left(x-1\right)+2$$

$$f\left(x\right) \approx 8\left(x-1\right)+2$$

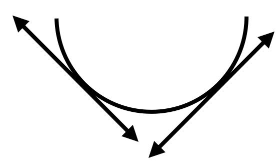
$$f\left(1.1\right) \approx 8\left(1.1-1\right)+2$$

$$\approx 8\left(0.1\right)+2$$

$$\approx 2.8$$

Since $f\left(x\right) > 0 \rightarrow y > 0$ for $1 < x < 1.1$.

$$\frac{d^2y}{dx^2} = \underset{(+)}{y^3}\underbrace{\left(1+3x^2y^2\right)}_{(+)} \rightarrow \frac{d^2y}{dx^2} > 0 \text{ for } 1 < x < 1.1$$



All tangents to a graph of a function that is concave up *__from the point of tangency to the estimate__* will be an underestimate. Therefore the approximation for $f\left(1.1\right)$ is an underestimate because $\frac{d^2y}{dx^2} > 0$ on $1 < x < 1.1$.

(c) Find the particular solution $y = f\left(x\right)$ with the initial condition $f\left(1\right) = 2$.

$$\frac{dy}{dx} = xy^3$$
$$dy = xy^3 dx$$
$$\frac{1}{y^3} dy = x dx$$
$$\int \frac{1}{y^3} dy = \int x dx$$
$$\int y^{-3} dy = \int x dx$$
$$-\frac{1}{2} y^{-2} = \frac{1}{2} x^2 + C$$
$$\frac{1}{y^2} = -x^2 + C$$
$$y^2 = \frac{1}{-x^2 + C}$$
$$y = \pm\sqrt{\frac{1}{-x^2 + C}}$$
$$y = \pm\frac{1}{\sqrt{-x^2 + C}}$$

Since the particular solution passes through $\left(1, 2\right)$, we must use the positive square root.

$$y = \frac{1}{\sqrt{-x^2 + C}}$$
$$2 = \frac{1}{\sqrt{-\left(1\right)^2 + C}}$$
$$4 = \frac{1}{-1 + C}$$
$$-4 + 4C = 1$$
$$4C = 5$$
$$C = \frac{5}{4}$$
$$y = \frac{1}{\sqrt{-x^2 + \frac{5}{4}}}$$

$$-\sqrt{\frac{5}{4}} < x < \sqrt{\frac{5}{4}}$$
$$-\frac{\sqrt{5}}{2} < x < \frac{\sqrt{5}}{2}$$

AP Calculus AB 2009 #5 No Calculator

x	2	3	5	8	13
$f(x)$	1	4	–2	3	6

Let f be a function that is twice differentiable for all real numbers. The table above gives values of f for selected points in the closed interval $2 \le x \le 13$.

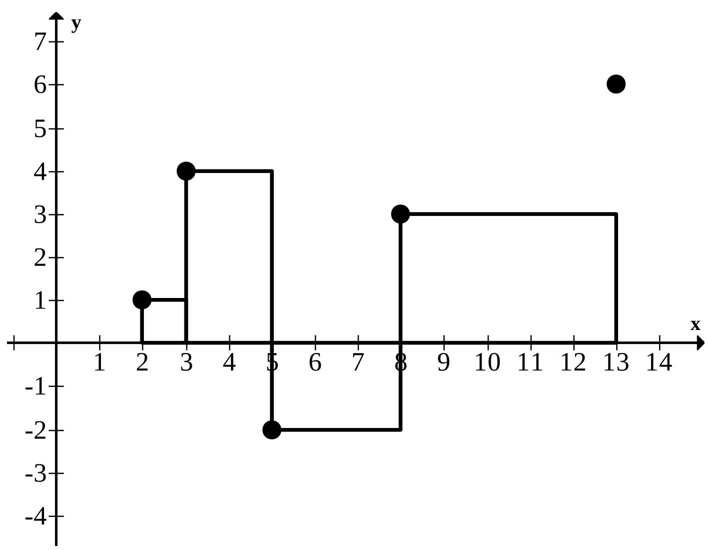
(a) Estimate $f'(4)$. Show the work that leads to your answer.

$$f'(4) \approx \frac{f(5)-f(3)}{5-3} = \frac{(-2)-(4)}{2} = -3$$

(b) Evaluate $\int_2^{13} \left(3-5f'(x)\right)dx$. Show the work that leads to your answer.

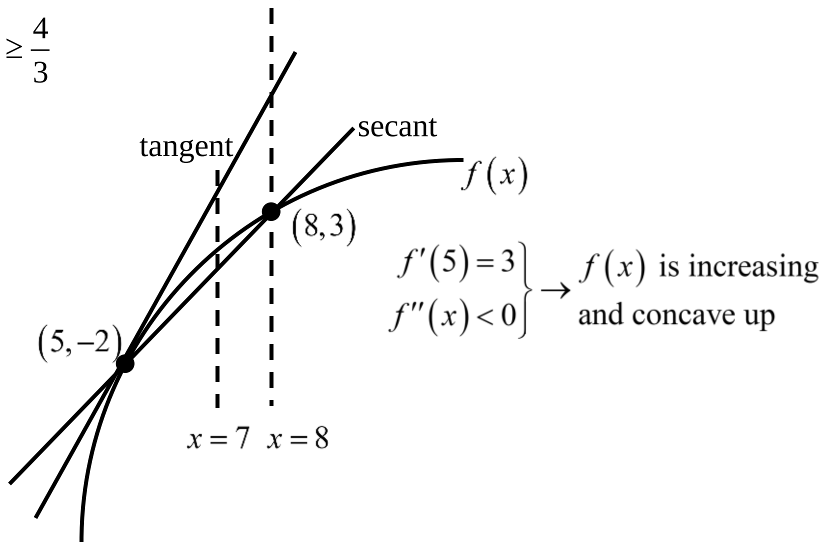
$$\begin{aligned}\int_2^{13}\left(3-f'(x)\right)dx &= \int_2^{13} 3dx - \int_2^{13} 5f'(x)\,dx \\ &= \int_2^{13} 3dx - 5\int_2^{13} f'(x)\,dx \\ &= 3(13-2)-5\left[f(13)-f(2)\right] \\ &= 3(11)-5[6-1] \\ &= 33-25 \\ &= 8\end{aligned}$$

(c) Use a left Riemann sum with subintervals indicated by the data in the table to approximate $\int_2^{13} f(x)\,dx$. Show the work that leads to your answer.



$$\int_2^{13} f(x)\,dx \approx (1)(3-2)+(4)(5-3)+(-2)(8-5)+(3)(13-8) = 18$$

(d) Suppose that $f'(5)=3$ and $f''(x)<0$ for all x in the closed interval $5\le x\le 8$. Use the line tangent to the graph of f at $x=5$ to show that $f(7)\le 4$. Use the secant line for the graph of f on $5\le x\le 8$ to show that $f(7)\ge \frac{4}{3}$

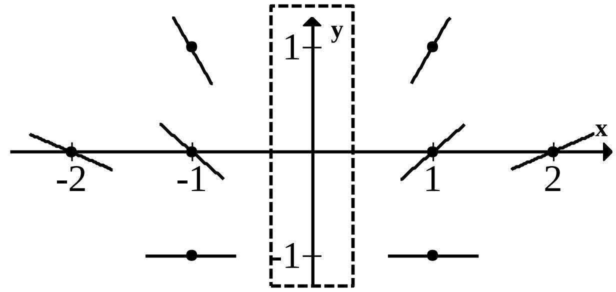


Equation of Tangent Line	Equation of Secant Line
$y-y_1=m(x-x_1)$ $y-f(5)=f'(5)(x-5)$ $y-(-2)=3(x-5)$ $y=3x-17$	$f(8)=3$ from the given table $y-y_1=m(x-x_1)$ $y-f(5)=\left[\frac{f(8)-f(5)}{8-5}\right]\cdot(x-5)$ $y-(-2)=\left[\frac{(3)-(-2)}{3}\right](x-5)$ $y+2=\frac{5}{3}(x-5)$ $y=\frac{5}{3}(x-5)-2$
$f(7)\approx 3(7)-17$ ≈ 4	$f(7)\approx -2+\frac{5}{3}(7-5)$ $\approx -2+\frac{10}{3}$ $\approx -\frac{6}{3}+\frac{10}{3}$ $\approx \frac{4}{3}$

Since $\boldsymbol{f''(x)<0}$ / $\boldsymbol{f(x)}$ **is concave down** on $5\le x\le 8$, the of $f(x)$, and the **<u>secant line is an under-approximation</u>** of $f(x)$. Therefore we can conclude that $\frac{4}{3}\le f(7)\le 4$.

AP Calculus AB 2006 #5 No Calculator

$\frac{dy}{dx}$ DNE if $x = 0$



Consider the differential equation $\frac{dy}{dx} = \frac{1+y}{x}$, where $x \neq 0$.
(a) On the axes provided, sketch a slope field for the given differential equation at the eight points indicated.

Slopes must be correctly positive, negative, or zero.
Relative steepness of slopes must be consistent in rows and in columns.

(x, y)	$\frac{dy}{dx} =$	(x, y)	$\frac{dy}{dx} =$
$(-2, 0)$	$\frac{0+1}{-2} = -\frac{1}{2}$	$(2, 0)$	$\frac{0+1}{2} = \frac{1}{2}$
$(-1, 1)$	$\frac{1+1}{-1} = -2$	$(1, 1)$	$\frac{1+1}{1} = 2$
$(-1, 0)$	$\frac{0+1}{-1} = -1$	$(1, 0)$	$\frac{0+1}{1} = 1$
$(-1, -1)$	$\frac{-1+1}{-1} = 0$	$(1, -1)$	$\frac{-1+1}{1} = 0$

Note that the slope field is undefined for $x = 0$ (i.e. the y-axis).

(b) Find the particular solution $y = f\left(x\right)$ to the differential equation with the initial condition $f\left(-1\right)=1$ and state its domain.

$$\frac{dy}{dx} = \frac{1+y}{x}$$

$$dy = \frac{1+y}{x}dx$$

$$\frac{1}{y+1}dy = \frac{1}{x}dx$$

$$\int\frac{1}{y+1}dy = \int\frac{1}{x}dx$$

$$\ln\left|y+1\right| = \ln\left|x\right| + C$$

$$e^{\ln|y+1|} = e^{\ln|x|+C}$$

$$\left|y+1\right| = e^{\ln|x|}e^{C} \quad \text{Let } A = e^{C}$$

$$\left|y+1\right| = Ae^{\ln|x|}$$

$$\left|y+1\right| = A\left|x\right|$$

$$y+1 = \pm A\left|x\right|$$

$$y+1 = A\left|x\right|$$

$$y = A\left|x\right| - 1$$

Rules with Exponents

$$e^{m+n} = e^{m}\cdot e^{n}$$

$$e^{\ln(m)} = m$$

Solving absolute value

$$\left|x\right| = k$$

$$\downarrow$$

$$x = \pm k$$

Since $f\left(-1\right)=1$

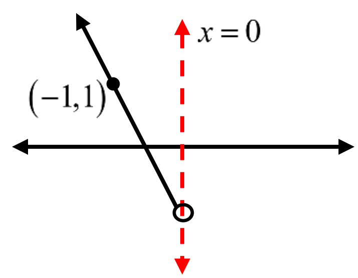
$$1 = A\left|-1\right| - 1$$

$$1 = A - 1$$

$$A = 2$$

$$\downarrow$$

$$y = 2\left|x\right| - 1$$



This function is valid for $x<0$ because $\frac{dy}{dx}=\frac{y+1}{x}$ is not defined for $x=0$. Since the initial condition is located at $\left(-1,1\right)$, the function cannot cross $x=0$. This limits the domain of the function with this particular condition to $x<0$.