AP Calculus BC
Taylor Series Exam
Fall 2021

Name:__________________________________

Date:____________________ Period:________

Calculator Permitted

Once you submit your Free Response Section, you will not be allowed to revisit it.
You have the remainder of the period to complete this portion of the exam.

- Show all of your work, even though a question may not explicitly remind you to do so. Clearly label any functions, graphs, tables, or other objects that you use. Justifications require that you give mathematical reasons, and that you verify the needed conditions under which relevant theorems, properties, definitions, or tests are applied. Your work will be scored on the correctness of your method, as well as your answers. Answers without supporting work will usually not receive credit.

- Your work must be expressed in standard mathematical notation rather than calculator syntax: $\int_1^5 x^2 dx$ may not be written as $\text{fnInt}\left(x^2, x, 1, 5\right)$. $f'(2) = 6$ may not be written as $\left.\frac{d}{dx}(Y_1)\right|_{x=2} = 6$

- Unless otherwise specified, answers (numeric or algebraic) need not be simplified. If you use decimal approximations in calculations, your work will be scored on accuracy. Unless otherwise specified, your final answers should be accurate to three places after the decimal point.

  A graphing calculator appropriate for use on the exam is expected to have the built-in capability to:
  (1) plot the graph of a function within an arbitrary viewing window,
  (2) find the zeros of functions (solve equations numerically),
  (3) numerically calculate the derivative of a function, and
  (4) numerically calculate the value of a definite integral.

- For results obtained using one of the four required calculator capabilities, students are required to write the mathematical setup that leads to the solution along with the result produced by the calculator. These setups include the equation being solved, the derivative being evaluated, or the definite integral being evaluated. In general, in a calculator-active problem that requires the value of a definite integral, students may use a calculator to determine the value; they do not need to compute an antiderivative as an intermediate step. Similarly, if a calculator-active problem requires the value of a derivative of a given function at a specific point, students may use a calculator to determine the value; they do not need to state the symbolic derivative expression. For solutions obtained using a calculator capability other than one of the four required, students must show the mathematical steps necessary to produce their results; a calculator result alone is not sufficient. For example, if students are asked to find a relative minimum value of a function, they are expected to use calculus and show the mathematical steps that lead to the answer. It is not sufficient to graph the function or use a calculator application that finds minimum values.

- Students may bring to the exam one or two (but no more than two) graphing calculators from the approved list. Calculator memories will not be cleared. Students are allowed to bring calculators containing whatever programs they want. They are expected to bring calculators that are set to radian mode. **NO CALCULATOR WITH A CAS SYSTEM ALLOWED ON THIS EXAM.**

- $(a)(-b)$ written as $a \cdot -b$ will be interpreted at $a - b$.

1. The Maclaurin series for $\ln(1+x)$ is

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{(-1)^{n+1} \cdot x^n}{n} + \cdots$$

The radius of convergence is 1. The continuous function *f* is defined by $f(x) = x \cdot \ln(1+x)$ for for $-1 < x \le 1$.

(a) Write the first three nonzero terms and the general term for the Maclaurin series for $f'(x)$.

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{(-1)^{n+1} \cdot x^n}{n} + \cdots$$

$$\downarrow$$

$$\begin{aligned} f(x) &= x \cdot \ln(1+x) \\ &= x \cdot \left[x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{(-1)^{n+1} \cdot x^n}{n} + \cdots \right] \\ &= x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \frac{x^5}{4} + \cdots + \frac{(-1)^{n+1} \cdot x^{n+1}}{n} + \cdots \end{aligned}$$

$$\downarrow$$

$$f'(x) = 2x - \frac{3 \cdot x^2}{2} + \frac{4x^3}{3} - \frac{5x^4}{4} + \cdots + \frac{(-1)^{n+1} \cdot (n+1) x^n}{n} + \cdots$$

(b) Use the result in part (a) to find the sum of the following infinite series.

$$1-\frac{3}{2\cdot 2^2}+\frac{4}{3\cdot 2^3}-\frac{5}{4\cdot 2^4}+\cdots+\frac{(-1)^{n+1}\cdot(n+1)}{n\cdot 2^n}\cdots$$

Your result may not be an infinite sum. Your answer must be the exact value or accurate to 6 (six) decimal places.

Method 1:

$$1-\frac{3}{2\cdot 2^2}+\frac{4}{3\cdot 2^3}-\frac{5}{4\cdot 2^4}+\cdots+\frac{(-1)^{n+1}\cdot(n+1)}{n\cdot 2^n}\cdots=f'\left(\frac{1}{2}\right)$$

$$f(x)=x\cdot\ln(1+x)$$

$$\downarrow$$

$$f'(x)=\ln(1+x)+x\cdot\frac{1}{1+x}$$

$$f'\left(\frac{1}{2}\right)=\ln\left(\frac{1}{2}\right)+\frac{1}{2}\cdot\frac{1}{1+\frac{1}{2}}$$

$$=0.738798...$$

Method 2:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cdot(n+1)\left(\frac{1}{2}\right)^{n}}{n}=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cdot\left(\frac{1}{2}\right)^{n}+(-1)^{n+1}\cdot n\cdot\left(\frac{1}{2}\right)^{n}}{n}$$

$$=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cdot\left(\frac{1}{2}\right)^{n}}{n}+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cdot n\cdot\left(\frac{1}{2}\right)^{n}}{n}$$

$$=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cdot\left(\frac{1}{2}\right)^{n}}{n}+\sum_{n=1}^{\infty}(-1)^{n+1}\cdot\left(\frac{1}{2}\right)^{n}$$

$$=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cdot\left(\frac{1}{2}\right)^{n}}{n}+\sum_{n=1}^{\infty}(-1)^{n}\cdot(-1)\cdot\left(\frac{1}{2}\right)^{n}$$

$$=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cdot\left(\frac{1}{2}\right)^{n}}{n}-\sum_{n=1}^{\infty}(-1)^{n}\left(\frac{1}{2}\right)^{n}$$

$$=\underbrace{\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cdot\left(\frac{1}{2}\right)^{n}}{n}}_{\ln\left(1+\frac{1}{2}\right)}+\underbrace{\sum_{n=1}^{\infty}\left(-\frac{1}{2}\right)^{n}}_{\text{geometric}}$$

$$=\ln\left(\frac{3}{2}\right)-\frac{\left(-\frac{1}{2}\right)}{1-\left(-\frac{1}{2}\right)}$$

$$=\ln\left(\frac{3}{2}\right)+\frac{1}{3}$$

$$=0.738798\ldots$$

(c) Let g be the function given by $g(x)=\int_0^x f(t)\,dt$. Find the first four nonzero terms and the general term for the Maclaurin series representing $g(x)$.

$$\begin{aligned}
g(x)&=\int_0^x f(t)\,dt\\
&=\int_0^x\left[t^2-\frac{t^3}{2}+\frac{t^4}{3}-\frac{t^5}{4}+\cdots+\frac{(-1)^{n+1}\cdot t^{n+1}}{n}+\cdots\right]dt\\
&=\left[\frac{t^3}{3}-\frac{t^4}{4\cdot 2}+\frac{t^5}{5\cdot 3}-\frac{t^6}{6\cdot 4}+\cdots+\frac{(-1)^{n+1}\cdot t^{n+2}}{(n+2)\cdot n}+\cdots\right]_0^x\\
&=\frac{x^3}{3}-\frac{x^4}{4\cdot 2}+\frac{x^5}{5\cdot 3}-\frac{x^6}{6\cdot 4}+\cdots+\frac{(-1)^{n+1}\cdot x^{n+2}}{(n+2)\cdot n}+\cdots
\end{aligned}$$

(d) Show that the first four nonzero terms found in part (c) approximates $g(1)$ with an error less than $\frac{1}{25}$.

$$g(1)\approx\frac{1^3}{3}-\frac{1^4}{4\cdot 2}+\frac{1^5}{5\cdot 3}-\frac{1^6}{6\cdot 4}$$

The series for $g(1)$ is an alternating series whose terms decrease in absolute value to zero. By the Alternating Series Remainder Theorem

$$\begin{aligned}
\left|g(1)-\left[\frac{1^3}{3}-\frac{1^4}{4\cdot 2}+\frac{1^5}{5\cdot 3}-\frac{1^6}{6\cdot 4}\right]\right|&\le|\text{next term}|\\
&\le\frac{1}{7\cdot 5}\\
&<\frac{1}{25}
\end{aligned}$$

2. Let $P(x)=\ln(2)+\frac{\sqrt{2}}{5}\cdot(x-1)-\frac{\sqrt{2}\cdot(x-1)^2}{2\cdot 5^2}-\frac{\sqrt{2}\cdot(x-1)^4}{4\cdot 5^4}$ be the fourth-degree Taylor polynomial for the function *f* about $x=1$. Assume that *f* has derivatives of all orders for all real numbers.

(a) Find $f'(1)$ and $f^{(3)}(1)$.

$$f'(1)\cdot(x-1)=\frac{\sqrt{2}}{5}\cdot(x-1)$$

$$f'(1)=\frac{\sqrt{2}}{5}$$

$$\frac{f^{(3)}(1)\cdot(x-1)}{3!}=0$$

$$\downarrow$$

$$f^{(3)}(1)=0$$

(b) Write the third-degree Taylor polynomial for f' about $x=1$, and use it to approximate $f'(1.1)$.

$$P(x)=\ln(2)+\frac{\sqrt{2}}{5}\cdot(x-1)-\frac{\sqrt{2}\cdot(x-1)^2}{2\cdot 5^2}-\frac{\sqrt{2}\cdot(x-1)^4}{4\cdot 5^4}$$

$$\downarrow$$

$$P'(x)=\frac{\sqrt{2}}{5}-\frac{\sqrt{2}\cdot(x-1)}{5^2}-\frac{\sqrt{2}\cdot(x-1)^3}{5^4}$$

$$f'(x)\approx\frac{\sqrt{2}}{5}-\frac{\sqrt{2}\cdot(x-1)}{5^2}-\frac{\sqrt{2}\cdot(x-1)^3}{5^4}$$

$$f'(1.1)\approx\frac{\sqrt{2}}{5}-\frac{\sqrt{2}\cdot(1.1-1)}{5^2}-\frac{\sqrt{2}\cdot(1.1-1)^3}{5^4}$$

(c) Write the third-degree Taylor polynomial for $g(x)=\int_1^x f(t)\,dt$ about $x=1$.

$$g(x)\approx\int_1^x \ln(2)+\frac{\sqrt{2}}{5}\cdot(t-1)-\frac{\sqrt{2}\cdot(t-1)^2}{2\cdot 5^2}-\frac{\sqrt{2}\cdot(t-1)^4}{4\cdot 5^4}\,dt$$

$$\approx\left[\ln(2)\cdot t+\frac{\sqrt{2}}{2\cdot 5}\cdot(t-1)^2-\frac{\sqrt{2}\cdot(t-1)^3}{3\cdot 2\cdot 5^2}-\frac{\sqrt{2}\cdot(t-1)^5}{5\cdot 4\cdot 5^4}\right]_1^x$$

$$\approx\left[\ln(2)\cdot x+\frac{\sqrt{2}}{2\cdot 5}\cdot(x-1)^2-\frac{\sqrt{2}\cdot(x-1)^3}{3\cdot 2\cdot 5^2}-\frac{\sqrt{2}\cdot(x-1)^5}{5\cdot 4\cdot 5^4}\right]-\left[\ln(2)\cdot 1+\frac{\sqrt{2}}{2\cdot 5}\cdot(1-1)^2-\frac{\sqrt{2}\cdot(1-1)^3}{3\cdot 2\cdot 5^2}-\frac{\sqrt{2}\cdot(1-1)^5}{5\cdot 4\cdot 5^4}\right]$$

$$\approx-\ln(2)+\ln(2)\cdot x+\frac{\sqrt{2}}{2\cdot 5}\cdot(x-1)^2-\frac{\sqrt{2}\cdot(x-1)^3}{3\cdot 2\cdot 5^2}-\frac{\sqrt{2}\cdot(x-1)^5}{5\cdot 4\cdot 5^4}$$

$$\downarrow$$

$$\approx-\ln(2)+\ln(2)\cdot x+\frac{\sqrt{2}}{2\cdot 5}\cdot(x-1)^2-\frac{\sqrt{2}\cdot(x-1)^3}{3\cdot 2\cdot 5^2}$$

$$\approx\ln(2)\cdot(x-1)+\frac{\sqrt{2}}{2\cdot 5}\cdot(x-1)^2-\frac{\sqrt{2}\cdot(x-1)^3}{3\cdot 2\cdot 5^2}$$