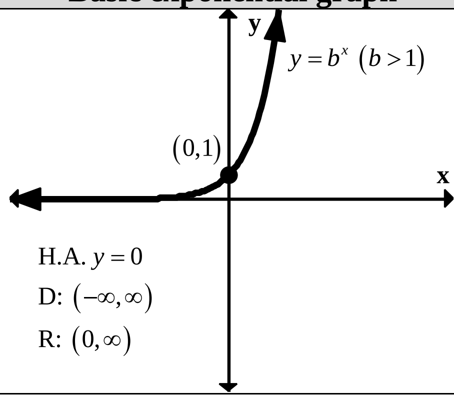
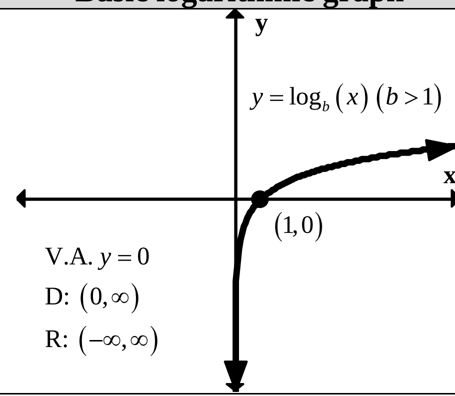
Logarithmic Differentiation Notes

Properties of Logs that one need to know in AP Calculus:

<table>
<tr><td colspan="2">$y = \log_b(x) \leftrightarrow b^y = x$</td></tr>
<tr><td>$\log_b(x \cdot y) = \log_b(x) + \log_b(y)$</td><td>$\log_b\left(b^k\right) = k$</td></tr>
<tr><td>$\log_b\left(\dfrac{x}{y}\right) = \log_b(x) - \log_b(y)$</td><td>$b^{\log_b(k)} = k$</td></tr>
<tr><td></td><td>$\log_b\left(x^m\right) = m \cdot \log_b(x)$</td></tr>
<tr><th>Basic exponential graph</th><th>Basic logarithmic graph</th></tr>
<tr><td>
</td><td>
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Differentiation Rules with Log

$\dfrac{d}{dx}\left[\log_b(x)\right] = \dfrac{1}{\ln(b)} \cdot \dfrac{1}{x}$	$\dfrac{d}{dx}\left[\ln(x)\right] = \dfrac{1}{x}$
$\dfrac{d}{dx}\left[\log_b(u)\right] = \dfrac{1}{\ln(b)} \cdot \dfrac{1}{u} \cdot u'$	$\dfrac{d}{dx}\left[\ln(u)\right] = \dfrac{1}{u} \cdot u'$

Logarithmic Differentiation Example:

Find y'

$$y = \frac{x^2\sqrt{3x-2}}{(x-1)^2}$$

$$\ln(y) = \ln\left[\frac{x^2\sqrt{3x-2}}{(x-1)^2}\right]$$

$$\ln(y) = \ln\left(x^2\right) + \ln\left([3x-2]^{\frac{1}{2}}\right) - \ln\left([x-1]^2\right)$$

$$\ln(y) = 2\ln(x) + \frac{1}{2}\ln(3x-2) - 2\ln(x-1)$$

$$\downarrow$$

$$\frac{1}{y} \cdot y' = \frac{2}{x} + \frac{3}{2(3x-2)} - \frac{2}{x-1}$$

$$y' = y\left[\frac{2}{x} + \frac{3}{2(3x-2)} - \frac{2}{x-1}\right]$$

$$y' = \frac{x^2\sqrt{3x-2}}{(x-1)^2}\left[\frac{2}{x} + \frac{3}{2(3x-2)} - \frac{2}{x-1}\right]$$

<u>Derivative of Exponential Functions:</u>

$\frac{d}{dx}\left[e^x\right]=e^x$	$\frac{d}{dx}\left[a^x\right]=\ln\left(a\right)\cdot a^x$
$\frac{d}{dx}\left[e^u\right]=e^u\cdot u'$	$\frac{d}{dx}\left[a^u\right]=\ln\left(a\right)\cdot a^u\cdot u'$

Functions similar to $\left[\cos\left(x\right)\right]^x$, x^{2x-1}, etc. cannot be handled by

using the power rule	using the exponential rule
$\frac{d}{dx}\left[u^n\right]=n\cdot u^{n-1}\cdot u'$	$\frac{d}{dx}\left[a^u\right]=\ln\left(a\right)\cdot a^u\cdot u'$
the exponent is not a constant	*the base is not a constant*

To differentiate functions of this form, <u>***you need to use logarithmic differentiation combined with implicit differentiation***</u>:

I. Take the natural logarithm of both sides of the equation.
II. Use the exponent properties of logarithms to bring the exponent down
III. Use implicit differentiation to differentiate the new equation
IV. Isolate y'
V. Replace any *y* in the derivative with the expression that involves *x* only.
   a. If *y* is defined explicitly in terms of *x*, then y' must be defined explicitly in terms of *x*.

Find y' given $y=x^x$	Find y' given $y=\left[\cos\left(x\right)\right]^x$
$y=x^x$ $\ln\left(y\right)=\ln\left(x^x\right)$ $\ln\left(y\right)=x\ln\left(x\right)$ $\downarrow$ $\frac{1}{y}\cdot y'=1\cdot\ln\left(x\right)+x\cdot\frac{1}{x}$ $\frac{1}{y}\cdot y'=\ln\left(x\right)+1$ $y'=y\left[\ln\left(x\right)+1\right]$ $y'=x^x\left[\ln\left(x\right)+1\right]$	$y=\left[\cos\left(x\right)\right]^x$ $\ln\left(y\right)=\ln\left(\left[\cos\left(x\right)\right]^x\right)$ $\ln\left(y\right)=x\ln\left(\cos\left(x\right)\right)$ $\frac{d}{dx}\left[\ln\left(y\right)\right]=\frac{d}{dx}\left[x\ln\left(\cos\left(x\right)\right)\right]$ $\frac{1}{y}\cdot y'=1\cdot\ln\left(\cos\left(x\right)\right)+x\cdot\frac{1}{\cos\left(x\right)}\cdot\left(-\sin\left(x\right)\right)$ $y'=y\left(\ln\left(\cos\left(x\right)\right)+x\cdot\frac{1}{\cos\left(x\right)}\cdot\left(-\sin\left(x\right)\right)\right)$ $y'=\left[\cos\left(x\right)\right]^x\left(\ln\left(\cos\left(x\right)\right)+x\cdot\frac{1}{\cos\left(x\right)}\cdot\left(-\sin\left(x\right)\right)\right)$