AP Calculus BC
Stewart Chapter 2 Test (Limits)
Fall 2020

Name:______________________________________

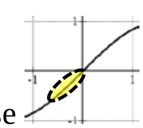
Date:________________________ Period:________

Free Response Section: *NO CAS Calculator Permitted.*
You have the remainder of the period to complete this section.
Once you submit your Free Response Section, you will not be allowed to revisit it.

1. [4 points] $\lim\limits_{t\to 0}\dfrac{\sqrt{1-\cos(t)}}{t}$

$$\lim_{t\to 0}\frac{\sqrt{1-\cos(t)}}{t}=\lim_{t\to 0}\frac{\sqrt{1-\cos(t)}}{t}\cdot\frac{\sqrt{1+\cos(t)}}{\sqrt{1+\cos(t)}}$$
$$=\lim_{t\to 0}\frac{\sqrt{1-\cos^2(t)}}{t\sqrt{1+\cos(t)}}$$
$$=\lim_{t\to 0}\frac{\sqrt{\sin^2(t)}}{t\sqrt{1+\cos(t)}}$$
$$=\lim_{t\to 0}\frac{|\sin(t)|}{t}\cdot\frac{1}{\sqrt{1+\cos(t)}}$$
$$=\left[\lim_{t\to 0}\frac{|\sin(t)|}{t}\right]\cdot\left[\lim_{t\to 0}\frac{1}{\sqrt{1+\cos(t)}}\right]$$
$$=\text{DNE}\cdot\frac{1}{\sqrt{2}}$$
$$\downarrow$$
$$\text{DNE}$$

$$\left.\begin{aligned}\lim_{t\to 0^+}\frac{|\sin(t)|}{t}&=\lim_{t\to 0^+}\frac{\sin(t)}{t}=1\\ \lim_{t\to 0^-}\frac{|\sin(t)|}{t}&=\lim_{t\to 0^-}\frac{-\sin(t)}{t}=-1\end{aligned}\right\}\to\lim_{t\to 0}\frac{|\sin(t)|}{t}\ \text{DNE}$$

$|\sin(t)|=-\sin(t)$ when t is just to the left of zero because 

2. [4 points] $\lim\limits_{x\to 0}\dfrac{\sin(x)}{2x^2-x}$

$$\lim_{x\to 0}\frac{\sin(x)}{2x^2-x}=\lim_{x\to 0}\frac{\sin(x)}{x(2x-1)}$$
$$=\lim_{x\to 0}\left[\frac{\sin(x)}{x}\cdot\frac{1}{(2x-1)}\right]$$
$$=\left[\lim_{x\to 0}\frac{\sin(x)}{x}\right]\cdot\left[\lim_{x\to 0}\frac{1}{(2x-1)}\right]$$
$$=1\cdot(-1)$$
$$=-1$$

3. [4 points] $\lim_{\theta\to 0}\frac{\csc(\theta)-\cot(\theta)}{\theta\csc(\theta)}$

$$\lim_{\theta\to 0}\frac{\csc(\theta)-\cot(\theta)}{\theta\csc(\theta)}=\lim_{\theta\to 0}\frac{\frac{1}{\sin(\theta)}-\frac{\cos(\theta)}{\sin(\theta)}}{\theta\left(\frac{1}{\sin(\theta)}\right)}$$

$$=\lim_{\theta\to 0}\frac{\frac{1}{\sin(\theta)}-\frac{\cos(\theta)}{\sin(\theta)}}{\theta\left(\frac{1}{\sin(\theta)}\right)}$$

$$=\lim_{\theta\to 0}\frac{\frac{1}{\sin(\theta)}-\frac{\cos(\theta)}{\sin(\theta)}}{\left(\frac{\theta}{\sin(\theta)}\right)}$$

$$=\lim_{\theta\to 0}\left(\frac{1}{\sin(\theta)}-\frac{\cos(\theta)}{\sin(\theta)}\right)\frac{\sin(\theta)}{\theta}$$

$$=\lim_{\theta\to 0}\left(\frac{1-\cos(\theta)}{\theta}\right)\cdot\frac{\sin(\theta)}{\sin(\theta)}$$

$$=\left[\lim_{\theta\to 0}\left(\frac{1-\cos(\theta)}{\theta}\right)\right]\cdot\left[\lim_{\theta\to 0}\frac{\sin(\theta)}{\sin(\theta)}\right]$$

$$=0\cdot 1$$

$$=0$$

4. [2 points] $\lim_{x\to 1}\frac{3x^2+1}{\sqrt{x^3-3x^2+x+1}}$

$$\lim_{x\to 1}\frac{3x^2+1}{\sqrt{x^3-3x^2+x+1}}\to\frac{3(1)^2+1}{\sqrt{(1)^3-3(1)^2+(1)+1}}$$

$$=\frac{4}{0}$$

$$\downarrow$$

DNE

5. [4 points] $\lim_{h\to 0}\frac{\sqrt{5+h}-\sqrt{5}}{h}$

$$\lim_{h\to 0}\frac{\sqrt{5+h}-\sqrt{5}}{h}=\lim_{h\to 0}\frac{\sqrt{5+h}-\sqrt{5}}{h}\cdot\frac{\sqrt{5+h}+\sqrt{5}}{\sqrt{5+h}+\sqrt{5}}$$

$$=\lim_{h\to 0}\frac{(5+h)-5}{h\left(\sqrt{5+h}+\sqrt{5}\right)}$$

$$=\lim_{h\to 0}\frac{\cancel{h}}{\cancel{h}\left(\sqrt{5+h}+\sqrt{5}\right)}$$

$$=\lim_{h\to 0}\frac{1}{\sqrt{5+h}+\sqrt{5}}$$

$$=\frac{1}{2\sqrt{5}}$$

6. [4 points] Given that a is a constant, find the following limit:

$$\lim_{h\to 0}\frac{\frac{1}{a+h}-\frac{1}{a}}{h}$$

$$\lim_{h\to 0}\frac{\frac{1}{a+h}-\frac{1}{a}}{h}=\lim_{h\to 0}\frac{\frac{1}{a+h}\cdot\frac{a}{a}-\frac{1}{a}\cdot\frac{a+h}{a+h}}{h}$$

$$=\lim_{h\to 0}\frac{\frac{a}{a(a+h)}-\frac{a+h}{a(a+h)}}{h}$$

$$=\lim_{h\to 0}\frac{\left[\frac{a-(a+h)}{a(a+h)}\right]}{h}$$

$$=\lim_{h\to 0}\frac{\left[\frac{-h}{a(a+h)}\right]}{h}$$

$$=\lim_{h\to 0}\frac{-\cancel{h}}{a\cdot\cancel{h}\cdot(a+h)}$$

$$=\lim_{h\to 0}\frac{-1}{a\cdot(a+h)}$$

$$=-\frac{1}{a^2}$$

7. [4 points] $\lim_{x\to 0}\frac{x\csc(x)+1}{x\csc(x)}$

$$\begin{aligned}
\lim_{x\to 0}\frac{x\csc(x)+1}{x\csc(x)} &= \lim_{x\to 0}\frac{x\csc(x)+1}{x\csc(x)} \\
&= \lim_{x\to 0}\left[\frac{x\csc(x)}{x\csc(x)}+\frac{1}{x\csc(x)}\right] \\
&= \left[\lim_{x\to 0}\frac{x\csc(x)}{x\csc(x)}\right]+\left[\lim_{x\to 0}\frac{1}{x\csc(x)}\right] \\
&= 1+\lim_{x\to 0}\frac{1}{x\left(\frac{1}{\sin(x)}\right)} \\
&= 1+\lim_{x\to 0}\frac{1}{\left[\frac{x}{\sin(x)}\right]} \\
&= 1+\lim_{x\to 0}\frac{\sin(x)}{x} \\
&= 1+1 \\
&= 2
\end{aligned}$$

8. [4 points] $\lim_{x\to\infty}\frac{x\sin(x)+2\sin(x)}{x^2}$

$$\begin{aligned}
\lim_{x\to\infty}\frac{x\sin(x)+2\sin(x)}{x^2} &= \lim_{x\to\infty}\left[\frac{x\sin(x)}{x^2}+\frac{2\sin(x)}{x^2}\right] \\
&= \lim_{x\to\infty}\left[\frac{x\sin(x)}{x^2}\right]+\lim_{x\to\infty}\left[\frac{2\sin(x)}{x^2}\right] \\
&= \lim_{x\to\infty}\left[\frac{\sin(x)}{x}\right]+\lim_{x\to\infty}\left[\frac{2\sin(x)}{x^2}\right] \\
&= 0+0 \\
&= 0
\end{aligned}$$

9. [4 points] Find the value of k that makes the function continuous at $x = 2$. Justify your reasoning using calculus.

$$f(x) = \begin{cases} kx & 0 \le x < 2 \\ 3x^2 & 2 \le x \end{cases}$$

$\lim_{x \to 2^-} f(x)$	$f(2)$	$\lim_{x \to 2^+} f(x)$
$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} kx$ $= 2k$	$f(2) = 3(2)^2$ $= 12$	$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} 3x^2$ $= 3(2)^2$ $= 12$

In order for the function to be continuous at $x = 2$, $\lim_{x \to 2} f(x) = f(2)$. This means that

$$2k = 12$$
$$k = 6$$

10. [4 points] Use the Intermediate Value Theorem to prove that the function $f(x) = 2e^{\cos(x)} + 1$ is equal to 5 at some point in the interval $\left[\frac{\pi}{2}, 2\pi\right]$

$f(x)$ is continuous on $\left[\frac{\pi}{2}, 2\pi\right]$.

$$f\left(\frac{\pi}{2}\right) = 2e^{\cos\left(\frac{\pi}{2}\right)} + 1 = 3$$

$$\begin{aligned} f(2\pi) &= 2e^{\cos(2\pi)} + 1 \\ &= 2e + 1 \\ &= 6.4365... \\ &> 5 \end{aligned}$$

By IVT, there exists a c in $\left[\frac{\pi}{2}, 2\pi\right]$ such that $f(c) = 5$. Therefore $f(x)$ must equal 5 at least once in the interval $\left[\frac{\pi}{2}, 2\pi\right]$.