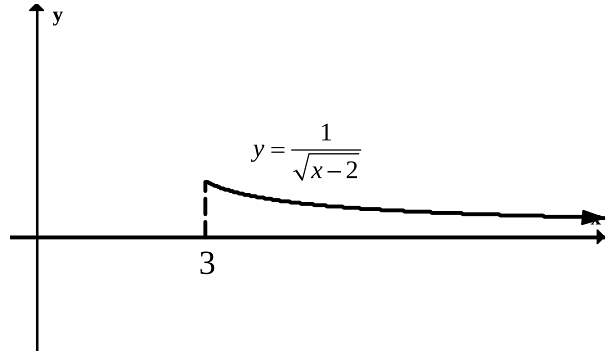
“Tail-End” Improper Integrals

I. If $\int_a^t f(x)\,dx$ exists for all $t \ge a$, then

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$$
$$= \lim_{t\to\infty}\left[F(t) - F(a)\right]$$

Provided the limit exists.

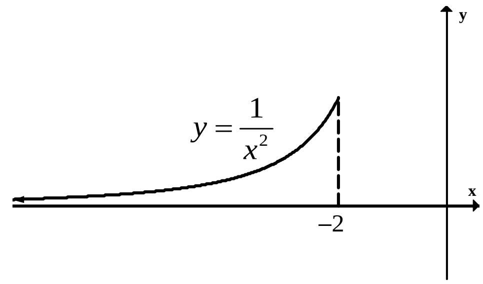


$$\int_3^\infty \frac{1}{\sqrt{x-2}}\,dx = \lim_{t\to\infty}\int_3^t \frac{1}{\sqrt{x-2}}\,dx$$
$$= \lim_{t\to\infty}\int_3^t (x-2)^{-\frac{1}{2}}\,dx$$
$$= \lim_{t\to\infty}\left[2(x-2)^{\frac{1}{2}}\right]_3^t$$
$$= \lim_{t\to\infty}\left[2(t-2)^{\frac{1}{2}} - 2(3-2)^{\frac{1}{2}}\right]$$
$$\downarrow$$
∞ / DNE

II. If $\int_t^b f(x)\,dx$ exists for all $t \le b$, then

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx$$
$$= \lim_{t\to-\infty}\left[F(b) - F(t)\right]$$

Provided the limit exists.

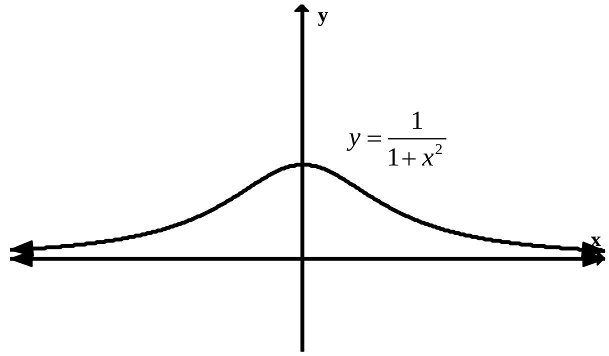


$$\int_{-\infty}^{-2} \frac{1}{x^2}\,dx = \lim_{t\to-\infty}\int_t^{-2} \frac{1}{x^2}\,dx$$
$$= \lim_{t\to-\infty}\int_t^{-2} x^{-2}\,dx$$
$$= \lim_{t\to-\infty}\left[-x^{-1}\right]_t^{-2}$$
$$= \lim_{t\to-\infty}\left[\left[-(-2)^{-1}\right] - \left[-(t)^{-1}\right]\right]$$
$$= \frac{1}{2} - 0$$
$$= \frac{1}{2}$$

III. If both $\int_{-\infty}^{a} f(x)\,dx$ and $\int_{a}^{\infty} f(x)\,dx$ exist, then we can define

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx$$

The choice of *a* can be any real number.



$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx$$

$$= 2 \cdot \int_{0}^{\infty} \frac{1}{1+x^2}\,dx$$

$$= 2 \cdot \lim_{t\to\infty} \int_{0}^{t} \frac{1}{1+x^2}\,dx$$

$$= 2 \cdot \lim_{t\to\infty} \Big[\arctan(x)\Big]_0^t$$

$$= 2 \cdot \lim_{t\to\infty} \Big[\arctan(t) - \arctan(0)\Big]$$

$$= 2\left[\frac{\pi}{2} - 0\right]$$

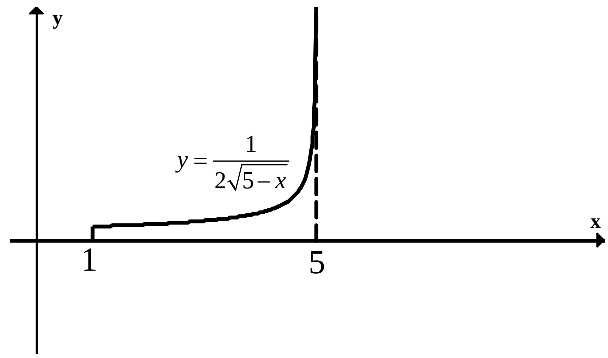
$$= \pi$$

Asymptotic Improper Integrals

I. If $f(x)$ is continuous on $[a,b)$, and $f(x)$ has an infinite discontinuity at $x=b$, then

$$\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx$$
$$= \lim_{t\to b^-}\left[F(t)-F(a)\right]$$

Provided the limit exists.

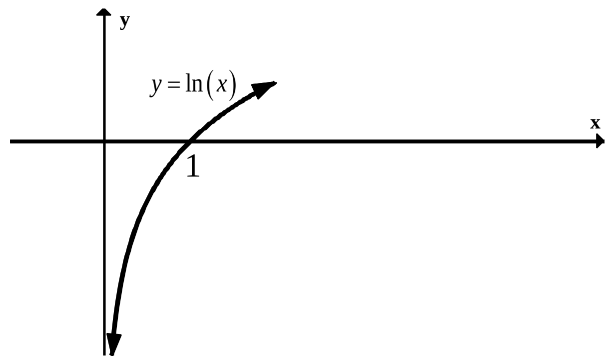


$$\int_1^5 \frac{1}{2\sqrt{5-x}}\,dx = \lim_{t\to 5^-}\int_1^t \frac{1}{2\sqrt{5-x}}\,dx$$
$$= \lim_{t\to 5^-}\int_1^t \frac{1}{2}(5-x)^{-\frac{1}{2}}\,dx$$
$$= \lim_{t\to 5^-}\left[-(5-x)^{\frac{1}{2}}\right]_1^t$$
$$= \lim_{t\to 5^-}\left[\left(-(5-t)^{\frac{1}{2}}\right)-\left(-(5-1)^{\frac{1}{2}}\right)\right]$$
$$= 2$$

II. If $f(x)$ is continuous on $(a,b]$, and $f(x)$ has an infinite discontinuity at $x=a$, then

$$\int_a^b f(x)\,dx = \lim_{t\to a^+}\int_t^b f(x)\,dx$$
$$= \lim_{t\to a^+}\left[F(b)-F(t)\right]$$

Provided the limit exists.

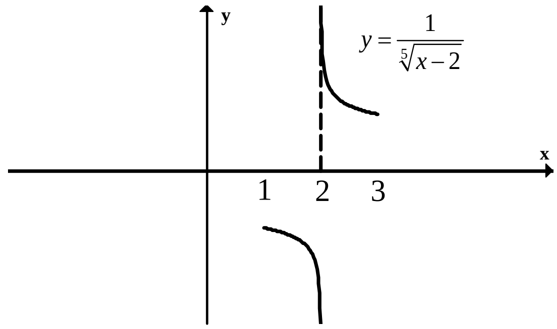


$$\int_0^1 \ln(x)\,dx = \lim_{t\to 0^+}\int_t^1 \ln(x)\,dx$$
$$= \lim_{t\to 0^+}\left[x\ln(x)-x\right]_t^1$$
$$= \lim_{t\to 0^+}\left[\left(1\cdot\ln(1)-1\right)-\left(t\ln(t)-t\right)\right]$$
$$= \lim_{t\to 0^+}\left[-1-t\ln(t)\right]$$
$$= -1-\lim_{t\to 0^+}\left[t\ln(t)\right]$$
$$= -1-\lim_{t\to 0^+}\left[\frac{\ln(t)}{t^{-1}}\right]$$
$$= -1-\lim_{t\to 0^+}\left[\frac{\left(\frac{1}{t}\right)}{-t^{-2}}\right]$$
$$= -1-\lim_{t\to 0^+}\left[-t\right]$$
$$= -1$$

Use L'Hopitals's Rule:

$$0\cdot\infty \to \begin{cases} \dfrac{0}{\left(\frac{1}{\infty}\right)} \\[2ex] \dfrac{\infty}{\left(\frac{1}{0}\right)} \end{cases}$$

III. If $f(x)$ has an infinite discontinuity at $x = c$, where $a < c < b$ and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ exist, then we define



$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$
$$= \lim_{m \to c^-} \int_a^m f(x)\,dx + \lim_{k \to c^+} \int_k^b f(x)\,dx$$
$$= \lim_{m \to c^-} \left[F(m) - F(a) \right] + \lim_{k \to c^+} \left[F(b) - F(k) \right]$$

Note: If $\int_a^c f(x)\,dx$ DNE or $\int_c^b f(x)\,dx$ DNE, then $\int_a^b f(x)\,dx$ DNE.

$$\int_1^3 \frac{1}{\sqrt[5]{x-2}}\,dx = \lim_{w \to 2^-} \int_1^w \frac{1}{\sqrt[5]{x-2}}\,dx + \lim_{t \to 2^+} \int_t^3 \frac{1}{\sqrt[5]{x-2}}\,dx$$
$$= \lim_{w \to 2^-} \int_1^w (x-2)^{-\frac{1}{5}}\,dx + \lim_{t \to 2^+} \int_t^3 (x-2)^{-\frac{1}{5}}\,dx$$
$$= \lim_{w \to 2^-} \left[\frac{5}{4}(x-2)^{\frac{4}{5}} \right]_1^w + \lim_{t \to 2^+} \left[\frac{5}{4}(x-2)^{\frac{4}{5}} \right]_t^3$$
$$= \lim_{w \to 2^-} \left[\frac{5}{4}(w-2)^{\frac{4}{5}} - \frac{5}{4}(1-2)^{\frac{4}{5}} \right] + \lim_{t \to 2^+} \left[\frac{5}{4}(3-2)^{\frac{4}{5}} - \frac{5}{4}(t-2)^{\frac{4}{5}} \right]$$
$$= -\frac{5}{4} + \frac{5}{4}$$
$$= 0$$

Limits to be familiar with:

$$\lim_{x\to\infty} \sqrt[n]{x} = \infty \text{ for all } n$$

$$\lim_{x\to\infty}\left[\ln(x)\right] = \infty$$

$$\lim_{x\to 0^+}\left[\ln(x)\right] = -\infty$$

$$\lim_{x\to\infty}\left[\arctan(x)\right] = \frac{\pi}{2}$$

$$\lim_{x\to-\infty}\left[\arctan(x)\right] = -\frac{\pi}{2}$$

$$\lim_{x\to-\infty}\left[a^x\right] = 0$$