Section 11-7 Complete Solutions:

#1

$$\sum_{n=1}^{\infty}\frac{1}{n+3^n}$$

$$\frac{1}{n+3^n}<\frac{1}{3^n}=\left(\frac{1}{3}\right)^n$$

$$\sum_{n=1}^{\infty}\frac{1}{n+3^n}<\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n$$

Since $\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n$ is a convergent geometric series, $\sum_{n=1}^{\infty}\frac{1}{n+3^n}$ converges by the direct comparison test.

#2

$$\sum_{n=1}^{\infty}\frac{(2n+1)^n}{n^{2n}}$$

$$\lim_{n\to\infty}\sqrt[n]{\frac{(2n+1)^n}{n^{2n}}}=\lim_{n\to\infty}\frac{\sqrt[n]{(2n+1)^n}}{\sqrt[n]{n^{2n}}}=\lim_{n\to\infty}\frac{2n+1}{n^2}=0$$

By the root test, the series converges absolutely.

#3

$$\sum_{n=1}^{\infty}(-1)^n\cdot\frac{n}{n+2}$$

Since $\lim_{n\to\infty}\left[(-1)^n\cdot\frac{n}{n+2}\right]\neq 0$, the series does not converge by the limit of the n^{th} term

#4

$$\sum_{n=1}^{\infty}(-1)^n\cdot\frac{n}{n^2+2}$$

The series is alternating and $\lim_{n\to\infty}\frac{n}{n^2+2}=0$, the series converges by the alternating series test.

#5

$$\sum_{n=1}^{\infty}\frac{n^2 2^{n-1}}{(-5)^n}$$

$$\lim_{n\to\infty}\sqrt[n]{\left|\frac{n^2 2^{n-1}}{(-5)^n}\right|}=\lim_{n\to\infty}\sqrt[n]{\frac{n^2 2^{n-1}}{5^n}\cdot\frac{2}{2}}=\lim_{n\to\infty}\sqrt[n]{\frac{n^2 2^n}{2\cdot 5^n}}=\lim_{n\to\infty}\frac{\sqrt[n]{n^2}\cdot\sqrt[n]{2^n}}{\sqrt[n]{2}\cdot\sqrt[n]{5^n}}=\frac{2}{5}$$

By the root test, the series converges absolutely.

#6

$$\sum_{n=1}^{\infty}\frac{1}{2n+1}$$

$$\lim_{n\to\infty}\frac{1}{2n+1}\sim\lim_{n\to\infty}\frac{1}{2n}$$

$$\lim_{n\to\infty}\frac{\left(\frac{1}{2n+1}\right)}{\left(\frac{1}{2n}\right)}=\lim_{n\to\infty}\frac{\left(\frac{1}{2n}\right)}{\left(\frac{1}{2n}\right)}=1$$

Since $\sum_{n=1}^{\infty}\frac{1}{2n}=\frac{1}{2}\cdot\left[\sum_{n=1}^{\infty}\frac{1}{n}\right]$ is the divergent harmonic series, by the limit comparison test $\sum_{n=1}^{\infty}\frac{1}{2n+1}$ diverges.

#7

$$\sum_{n=1}^{\infty}\frac{1}{n\sqrt{\ln(n)}}$$

$$\int_1^{\infty}\frac{1}{x\sqrt{\ln(x)}}dx=\lim_{t\to\infty}\int_2^{t}\frac{1}{x\sqrt{\ln(x)}}dx$$

$$\text{let}\quad \begin{array}{ll} u=\ln(x) & u(2)=\ln(2) \\ du=\frac{1}{x}dx & u(\infty)=\infty \end{array}$$

$$=\lim_{t\to\infty}\int_2^{t}\frac{1}{\sqrt{\ln(x)}}\cdot\frac{1}{x}dx$$

$$=\lim_{t\to\infty}\int_{\ln(2)}^{t}\frac{1}{\sqrt{u}}du$$

$$=\lim_{t\to\infty}\int_{\ln(2)}^{t}u^{-\frac{1}{2}}du$$

$$=\lim_{t\to\infty}\left[2u^{\frac{1}{2}}\right]_{\ln(2)}^{t}$$

$$=\lim_{t\to\infty}\left(\left[2t^{\frac{1}{2}}\right]-\left[2(\ln(2))^{\frac{1}{2}}\right]\right)$$

$$\to\infty$$

Since $\int_1^{\infty}\frac{1}{x\sqrt{\ln(x)}}dx$ does not converge, $\sum_{n=1}^{\infty}\frac{1}{n\sqrt{\ln(n)}}$ does not converge by the integral test.

#8

$$\sum_{k=1}^{\infty}\frac{2^k\cdot k!}{(k+2)!}$$

$$\lim_{k\to\infty}\left|\frac{\dfrac{2^{k+1}\cdot(k+1)!}{((k+1)+2)!}}{\dfrac{2^k\cdot k!}{(k+2)!}}\right|=\lim_{k\to\infty}\left|\frac{\dfrac{2^{k+1}\cdot(k+1)!}{(k+3)!}}{\dfrac{2^k\cdot k!}{(k+2)!}}\right|$$

$$=\lim_{k\to\infty}\left|\frac{2^{k+1}\cdot(k+1)!}{(k+3)!}\cdot\frac{(k+2)!}{2^k\cdot k!}\right|$$

$$=\lim_{k\to\infty}\left|\frac{(k+2)!}{(k+3)!}\cdot\frac{(k+1)!}{k!}\cdot\frac{2^{k+2}}{2^k}\right|$$

$$=\lim_{k\to\infty}\left|\frac{(k+2)!}{(k+3)\cdot[(k+2)!]}\cdot\frac{(k+1)\cdot[k!]}{k!}\cdot\frac{2^k\cdot 2^2}{2^k}\right|$$

$$=\lim_{k\to\infty}\left|\frac{1}{k+3}\cdot\frac{k+1}{1}\cdot\frac{2}{1}\right|$$

$$=2$$

By the ratio test, $\sum_{k=1}^{\infty}\frac{2^k\cdot k!}{(k+2)!}$ does not converge.

#9

$$\sum_{k=1}^{\infty} k^2 e^{-k} = \sum_{k=1}^{\infty} \frac{k^2}{e^k}$$

$$\lim_{k\to\infty} \sqrt[k]{\frac{k^2}{e^k}} = \lim_{k\to\infty} \frac{\sqrt[k]{k^2}}{\sqrt[k]{e^k}} = \frac{1}{e}$$

By the root test, $\sum_{k=1}^{\infty} k^2 e^{-k}$ converges absolutely.

$\int_1^{\infty} k^2 e^{-k} dk = \lim_{t\to\infty} \int_1^t k^2 e^{-k} dk$ $= \lim_{t\to\infty} \left[-k^2 e^{-k} - 2ke^{-k} - 2e^{-k}\right]_1^t$ $= \lim_{t\to\infty} \left(\left[-(t)^2 e^{-(t)} - 2(t)e^{-(t)} - 2e^{-(t)}\right] - \left[-(1)^2 e^{-1} - 2(1)e^{-1} - 2e^{-1}\right]\right)$ $= \lim_{t\to\infty} \left(\left[-\frac{t^2}{e^t} - \frac{2t}{e^t} - \frac{2}{e^t}\right] - \left[-(1)^2 e^{-1} - 2(1)e^{-1} - 2e^{-1}\right]\right)$ $= -\left[-(1)^2 e^{-1} - 2(1)e^{-1} - 2e^{-1}\right]$	u v' k^2 e^{-k} $2k$ $-e^{-k}$ 2 e^{-k} 0 $-e^{-k}$

Since $\int_1^{\infty} k^2 e^{-k} dk$ converges, $\sum_{k=1}^{\infty} k^2 e^{-k}$ converges.

#10

$$\sum_{n=1}^{\infty} n^2 e^{-n^3}$$

$$\begin{aligned}
\lim_{n\to\infty}\left[\sqrt[n]{n^2 e^{-n^3}}\right] &= \lim_{n\to\infty}\left[\sqrt[n]{n^2}\cdot\sqrt[n]{e^{-n^3}}\right] \\
&= \lim_{n\to\infty}\left[1\cdot\left(e^{-n^3}\right)^{\frac{1}{n}}\right] \\
&= \lim_{n\to\infty}\left[1\cdot e^{-n^2}\right] \\
&= 0
\end{aligned}$$

Therefore, by the root test, $\sum_{n=1}^{\infty} n^2 e^{-n^3}$ converges absolutely.

$$\int_1^{\infty} x^2 e^{-x^3}\,dx = \lim_{t\to\infty}\int_1^t x^2 e^{-x^3}\,dx$$

$$\text{let}\quad \begin{array}{ll} u = x^3 & u(1) = -1 \\ du = 3x^2 dx & u(\infty) = \infty \end{array}$$

$$\begin{aligned}
&= \left(\frac{1}{3}\right)\lim_{t\to\infty}\int_1^t e^{-x^3}\cdot\left(3x^2\right)dx \\
&= \lim_{t\to\infty}\int_1^t e^{-u}\,du \\
&= \lim_{t\to\infty}\left[-e^{-u}\right]_1^t \\
&= \lim_{t\to\infty}\left(\left[-e^{-t}\right]-\left[-e^{-1}\right]\right) \\
&= -\left[-e^{-1}\right]
\end{aligned}$$

Since $\int_1^{\infty} x^2 e^{-x^3}\,dx$, by the integral test $\sum_{n=1}^{\infty} n^2 e^{-n^3}$ converges.

#11

$$\sum_{n=1}^{\infty}\left(\frac{1}{n^3}+\frac{1}{3^n}\right)=\left[\sum_{n=1}^{\infty}\frac{1}{n^3}\right]+\left[\sum_{n=1}^{\infty}\frac{1}{3^n}\right]$$

$\sum_{n=1}^{\infty}\frac{1}{n^3}$ is a convergent *p*-series

$\sum_{n=1}^{\infty}\frac{1}{3^n}=\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n$ is a convergent geometric series.

Therefore, since the sum of two convergent series is convergent, $\sum_{n=1}^{\infty}\left(\frac{1}{n^3}+\frac{1}{3^n}\right)$ is convergent.

#12

$$\sum_{k=1}^{\infty}\frac{1}{k\sqrt{k^2+1}}$$

$$\lim_{k\to\infty}\frac{1}{k\sqrt{k^2+1}}\sim\lim_{k\to\infty}\frac{1}{k\sqrt{k^2}}=\lim_{k\to\infty}\frac{1}{k^2}$$

$$\lim_{k\to\infty}\frac{\left[\frac{1}{k^2}\right]}{\left(\frac{1}{k\sqrt{k^2+1}}\right)}=\lim_{k\to\infty}\frac{\left[\frac{1}{k^2}\right]}{\left(\frac{1}{k\sqrt{k^2}}\right)}=\lim_{k\to\infty}\frac{\left[\frac{1}{k^2}\right]}{\left(\frac{1}{k^2}\right)}=1$$

Since $\sum_{k=1}^{\infty}\frac{1}{k^2}$ is a convergent *p*-series, by the limit comparison test $\sum_{k=1}^{\infty}\frac{1}{k\sqrt{k^2+1}}$ converges.

#13

$$\sum_{n=1}^{\infty}\frac{3^n\cdot n^2}{n!}$$

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{\frac{3^{n+1}\cdot(n+1)^2}{(n+1)!}}{\frac{3^n\cdot n^2}{n!}}\right|&=\lim_{n\to\infty}\left|\frac{3^{n+1}\cdot(n+1)^2}{(n+1)!}\cdot\frac{n!}{3^n\cdot n^2}\right|\\&=\lim_{n\to\infty}\left|\frac{3^{n+1}}{3^n}\cdot\frac{(n+1)^2}{n^2}\cdot\frac{n!}{(n+1)!}\right|\\&=\lim_{n\to\infty}\left|\frac{3^n\cdot 3}{3^n}\cdot\frac{(n+1)^2}{n^2}\cdot\frac{n!}{(n+1)\cdot(n!)}\right|\\&=\lim_{n\to\infty}\left|\frac{3}{n+1}\right|\\&=0\end{aligned}$$

Therefore by the ratio test $\sum_{n=1}^{\infty}\frac{3^n\cdot n^2}{n!}$ converges absolutely.

#14

$$\sum_{n=1}^{\infty}\frac{\sin(2n)}{1+2^n}$$

$$\frac{\sin(2n)}{1+2^n}<\frac{1}{1+2^n}<\frac{1}{2^n}$$

So

$$\sum_{n=1}^{\infty}\frac{\sin(2n)}{1+2^n}<\sum_{n=1}^{\infty}\frac{1}{2^n}=\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n$$

Since $\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n$ is a convergent geometric series, $\sum_{n=1}^{\infty}\frac{\sin(2n)}{1+2^n}$ converges by the direct comparison test.

#15

$$\sum_{k=1}^{\infty}\frac{2^{k-1}\cdot 3^{k-1}}{k^k}$$

$$\begin{aligned}
\sum_{k=1}^{\infty}\frac{2^{k-1}\cdot 3^{k-1}}{k^k} &= \sum_{k=1}^{\infty}\frac{3}{3}\cdot\frac{2}{2}\cdot\frac{2^{k-1}\cdot 3^{k-1}}{k^k}\\
&= \sum_{k=1}^{\infty}\frac{3}{3}\cdot\frac{2}{2}\cdot\frac{2^{k-1}\cdot 3^{k-1}}{k^k}\\
&= \sum_{k=1}^{\infty}\frac{1}{3}\cdot\frac{1}{2}\cdot\frac{2\cdot 2^{k-1}\cdot 3\cdot 3^{k-1}}{k^k}\\
&= \sum_{k=1}^{\infty}\frac{1}{6}\cdot\frac{2^k\cdot 3^k}{k^k}\\
&= \sum_{k=1}^{\infty}\frac{1}{6}\cdot\left(\frac{2\cdot 3}{k}\right)^k\\
&= \sum_{k=1}^{\infty}\frac{1}{6}\cdot\left(\frac{6}{k}\right)^k
\end{aligned}$$

$$\begin{aligned}
\lim_{k\to\infty}\left[\sqrt[k]{\frac{1}{6}\cdot\left(\frac{6}{k}\right)^k}\right] &= \lim_{k\to\infty}\left[\sqrt[k]{\frac{1}{6}}\cdot\sqrt[k]{\left(\frac{6}{k}\right)^k}\right]\\
&= \lim_{k\to\infty}\frac{6}{k}\\
&= 0
\end{aligned}$$

Therefore, by the root test $\sum_{k=1}^{\infty}\frac{2^{k-1}\cdot 3^{k-1}}{k^k}$ converges absolutely.

#16

$$\sum_{n=1}^{\infty}\frac{n^2+1}{n^3+1}$$

$$\lim_{n\to\infty}\frac{n^2+1}{n^3+1}\sim\lim_{n\to\infty}\frac{n^2}{n^3}=\lim_{n\to\infty}\frac{1}{n}$$

$$\lim_{n\to\infty}\frac{\left[\frac{1}{n}\right]}{\left(\frac{n^2+1}{n^3+1}\right)}=\lim_{n\to\infty}\frac{\left[\frac{1}{n}\right]}{\left(\frac{n^2}{n^3}\right)}=\lim_{n\to\infty}\frac{\left[\frac{1}{n}\right]}{\left(\frac{1}{n}\right)}=1$$

Since $\sum_{n=1}^{\infty}\frac{1}{n}$ is the divergent harmonic series, by the limit comparison test $\sum_{n=1}^{\infty}\frac{n^2+1}{n^3+1}$ also diverges.

#17

$$\sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2\cdot 5\cdot 8\cdot\cdots\cdot(3n-1)}$$

$$\lim_{n\to\infty}\left|\frac{\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2(n+1)-1)}{2\cdot 5\cdot 8\cdot\cdots\cdot(3(n+1)-1)}}{\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2\cdot 5\cdot 8\cdot\cdots\cdot(3n-1)}}\right|=\lim_{n\to\infty}\left|\frac{\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n+1)}{2\cdot 5\cdot 8\cdot\cdots\cdot(3n+2)}}{\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2\cdot 5\cdot 8\cdot\cdots\cdot(3n-1)}}\right|$$

$$=\lim_{n\to\infty}\left|\frac{\cancel{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}\,(2n+1)}{\cancel{2\cdot 5\cdot 8\cdot\cdots\cdot(3n-1)}\,(3n+2)}\cdot\frac{\cancel{2\cdot 5\cdot 8\cdot\cdots\cdot(3n-1)}}{\cancel{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}}\right|$$

$$=\lim_{n\to\infty}\frac{2n+1}{3n+2}$$

$$=\frac{2}{3}$$

By the ratio test, $\sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2\cdot 5\cdot 8\cdot\cdots\cdot(3n-1)}$ converges absolutely.

#18

$$\sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}-1}$$

$$\lim_{n\to\infty}\frac{1}{\sqrt{n}-1}=0$$

Since the series is alternating and $\lim_{n\to\infty}\frac{1}{\sqrt{n}-1}=0$, $\sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}-1}$ converges by the alternating series test.

#19

$$\sum_{n=1}^{\infty}(-1)^{n}\cdot\frac{\ln(n)}{\sqrt{n}}$$

$$\lim_{n\to\infty}\frac{\ln(n)}{\sqrt{n}}=0$$

Since the series is alternating and $\lim_{n\to\infty}\frac{\ln(n)}{\sqrt{n}}=0$, $\sum_{n=1}^{\infty}(-1)^{n}\cdot\frac{\ln(n)}{\sqrt{n}}$ converges by the alternating series test.

#20

$$\sum_{k=1}^{\infty}\frac{\sqrt[3]{k}-1}{k\left(\sqrt{k}+1\right)}$$

$$\lim_{k\to\infty}\frac{\sqrt[3]{k}-1}{k\left(\sqrt{k}+1\right)}=\lim_{k\to\infty}\frac{\sqrt[3]{k}}{k\left(\sqrt{k}\right)}=\lim_{k\to\infty}\frac{k^{\frac{1}{3}}}{k^{\frac{3}{2}}}=\lim_{k\to\infty}\frac{1}{k^{\frac{7}{6}}}$$

$$\lim_{k\to\infty}\frac{\left[\dfrac{1}{k^{\frac{7}{6}}}\right]}{\left(\dfrac{\sqrt[3]{k}-1}{k\left(\sqrt{k}+1\right)}\right)}=\lim_{k\to\infty}\frac{\left[\dfrac{1}{k^{\frac{7}{6}}}\right]}{\left(\dfrac{1}{k^{\frac{7}{6}}}\right)}=1$$

Since $\sum_{k=1}^{\infty}\frac{1}{k^{\frac{7}{6}}}$ is a convergent *p*-series, by the limit comparison test $\sum_{k=1}^{\infty}\frac{\sqrt[3]{k}-1}{k\left(\sqrt{k}+1\right)}$ also converges.

#21

$$\sum_{n=1}^{\infty}(-1)^n \cdot \cos\left(\frac{1}{n^2}\right)$$

$$\lim_{n\to\infty}\left[(-1)^n \cdot \cos\left(\frac{1}{n^2}\right)\right] = \lim_{n\to\infty}\left[(-1)^n\right] \text{ DNE} \neq 0$$

$\displaystyle\sum_{n=1}^{\infty}(-1)^n \cdot \cos\left(\frac{1}{n^2}\right)$ does not converge by the limit of the n^{th} term test.

#22

$$\sum_{k=1}^{\infty}\frac{1}{2+\sin(k)}$$

$$\lim_{k\to\infty}\frac{1}{2+\sin(k)} \neq 0$$

$\displaystyle\sum_{k=1}^{\infty}\frac{1}{2+\sin(k)}$ does not converge by the limit of the n^{th} term test.

#23

$$\sum_{n=1}^{\infty}\frac{1}{\tan\left(\frac{1}{n}\right)}$$

Skip.

#24

$$\sum_{n=1}^{\infty} n \cdot \sin\left(\frac{1}{n}\right)$$

$$\begin{aligned}
y &= \lim_{n\to\infty}\left[n \cdot \sin\left(\frac{1}{n}\right)\right] \\
&= \lim_{n\to\infty}\left[\frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}}\right] \\
&= \lim_{n\to\infty}\left[\frac{\cos\left(\frac{1}{n}\right)\cdot\left(-n^{-2}\right)}{-n^{-2}}\right] \\
&= \lim_{n\to\infty}\left[\cos\left(\frac{1}{n}\right)\right] \\
&= 1
\end{aligned}$$

$\sum_{n=1}^{\infty} n \cdot \sin\left(\frac{1}{n}\right)$ does not converge by the limit of the n^{th} term test.

#25

$$\sum_{n=1}^{\infty} \frac{n!}{e^{n^2}}$$

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{\frac{(n+1)!}{e^{(n+1)^2}}}{\frac{n!}{e^{n^2}}}\right| &= \lim_{n\to\infty}\left|\frac{(n+1)!}{e^{(n+1)^2}}\cdot\frac{e^{n^2}}{n!}\right| \\
&= \lim_{n\to\infty}\left|\frac{(n+1)!}{n!}\cdot\frac{e^{n^2}}{e^{(n+1)^2}}\right| \\
&= \lim_{n\to\infty}\left|\frac{(n+1)\cdot[n!]}{n!}\cdot\frac{e^{n^2}}{e^{n^2+2n+1}}\right| \\
&= \lim_{n\to\infty}\left|\frac{(n+1)}{1}\cdot\frac{1}{e^{2n+1}}\right| \\
&= \lim_{n\to\infty}\left|\frac{(n+1)}{e^{2n+1}}\right| \\
&= 0
\end{aligned}$$

By the ratio test, $\sum_{n=1}^{\infty} \frac{n!}{e^{n^2}}$ converges absolutely.

#26

$\sum_{n=1}^{\infty} \frac{n^2+1}{5^n}$

$$\lim_{n \to \infty} \sqrt[n]{\frac{n^2+1}{5^n}} = \lim_{n \to \infty} \frac{\sqrt[n]{n^2+1}}{\sqrt[n]{5^n}} = \frac{1}{5}$$

By the root test, $\sum_{n=1}^{\infty} \frac{n^2+1}{5^n}$ converges absolutely.

#27

$$\sum_{k=1}^{\infty}\frac{k\cdot\ln(k)}{(k+1)^3}$$

$$\frac{k\cdot\ln(k)}{(k+1)^3}<\frac{k\cdot\ln(k)}{k^3}=\frac{\ln(k)}{k^2}$$

$$\sum_{k=1}^{\infty}\frac{k\cdot\ln(k)}{(k+1)^3}<\sum_{k=1}^{\infty}\frac{\ln(k)}{k^2}$$

$$\int_1^{\infty}\frac{\ln(x)}{x^2}dx=\lim_{t\to\infty}\int_1^{\infty}x^{-2}\cdot\ln(x)\,dx$$

$$\text{let}\quad \begin{matrix} u=\ln(x) & v'=x^{-2} \\ du=\frac{1}{x} & v=-x^{-1}\end{matrix}$$

$$=\lim_{t\to\infty}\left[-x^{-1}\cdot\ln(x)-\int\frac{1}{x}\cdot\left(-x^{-1}\right)dx\right]_1^t$$

$$=\lim_{t\to\infty}\left[-x^{-1}\cdot\ln(x)+\int x^{-2}dx\right]_1^t$$

$$=\lim_{t\to\infty}\left[-x^{-1}\cdot\ln(x)-x^{-1}\right]_1^t$$

$$=\lim_{t\to\infty}\left(\left[-(t)^{-1}\cdot\ln(t)-(t)^{-1}\right]-\left[-(1)^{-1}\cdot\ln(1)-(1)^{-1}\right]\right)$$

$$=\lim_{t\to\infty}\left(\left[-\frac{\ln(t)}{t}-\frac{1}{t}\right]-\left[-(1)^{-1}\cdot\ln(1)-(1)^{-1}\right]\right)$$

$$=-\left[-(1)^{-1}\cdot\ln(1)-(1)^{-1}\right]$$

Since $\int_1^{\infty}\frac{\ln(x)}{x^2}dx$ converges, by the integral test so does $\sum_{k=1}^{\infty}\frac{\ln(k)}{k^2}$. Since $\sum_{k=1}^{\infty}\frac{\ln(k)}{k^2}$ converges, by the direct comparison test $\sum_{k=1}^{\infty}\frac{k\cdot\ln(k)}{(k+1)^3}$ converges.

#28

$$\sum_{n=1}^{\infty}\frac{e^{\frac{1}{n}}}{n^2}$$

$$\frac{e^{\frac{1}{n}}}{n^2}<\frac{3}{n^2}$$

$$\sum_{n=1}^{\infty}\frac{e^{\frac{1}{n}}}{n^2}<\sum_{n=1}^{\infty}\frac{3}{n^2}=3\cdot\left[\sum_{n=1}^{\infty}\frac{1}{n^2}\right]\ \frac{e^{\frac{1}{n}}}{n^2}<\frac{3}{n^2}$$

Since $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is a convergent *p*-series, by the direct comparison test $\sum_{n=1}^{\infty}\frac{e^{\frac{1}{n}}}{n^2}$ also converges.

#29
Skip, we do not cover hyperbolic trig functions.

#30

$$\sum_{j=1}^{\infty}(-1)^j\cdot\frac{\sqrt{j}}{j+5}$$

The series is alternating with $\lim_{j\to\infty}\frac{\sqrt{j}}{j+5}\sim\lim_{j\to\infty}\frac{\sqrt{j}}{j}=\lim_{j\to\infty}\frac{1}{\sqrt{j}}=0$, therefore $\sum_{j=1}^{\infty}(-1)^j\cdot\frac{\sqrt{j}}{j+5}$ converges by the alternating series test.

#31

$$\sum_{k=1}^{\infty}\frac{5^k}{3^k+4^k}$$

$$\lim_{k\to\infty}\frac{5^k}{3^k+4^k}=\lim_{k\to\infty}\frac{5^k}{3^k+4^k}\cdot\frac{\left(\frac{1}{4^k}\right)}{\left(\frac{1}{4^k}\right)}$$

$$=\lim_{k\to\infty}\frac{\frac{5^k}{4^k}}{\frac{3^k}{4^k}+\frac{4^k}{4^k}}$$

$$=\lim_{k\to\infty}\frac{\left(\frac{5}{4}\right)^k}{\left(\frac{3}{4}\right)^k+1}$$

$$\to\infty$$

Therefore $\sum_{k=1}^{\infty}\frac{5^k}{3^k+4^k}$ does not converge by the limit of the n^{th} term test.

#32

$$\sum_{n=1}^{\infty}\frac{(n!)^n}{n^{4n}}$$

$$\lim_{n\to\infty}\sqrt[n]{\frac{(n!)^n}{n^{4n}}}=\lim_{n\to\infty}\frac{\sqrt[n]{(n!)^n}}{\sqrt[n]{n^{4n}}}=\lim_{n\to\infty}\frac{n!}{n^4}\neq 0$$

Therefore $\sum_{n=1}^{\infty}\frac{(n!)^n}{n^{4n}}$ does not converge by the limit of the n^{th} term test.

#33

$$\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^{n^2}$$

$$\begin{aligned}\lim_{n\to\infty}\sqrt[n]{\left(\frac{n}{n+1}\right)^{n^2}} &= \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{n} \\ &= \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^{-n} \\ &= \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^{n}\right]^{-1} \\ &= e^{-1}\end{aligned}$$

By the root test $\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^{n^2}$ converges.

#34

$$\sum_{n=1}^{\infty}\frac{1}{n+n\cdot\cos^2(n)}$$

$$\frac{1}{2n}<\frac{1}{n+n\cdot\cos^2(n)}$$

$$\frac{1}{2}\cdot\left[\sum_{n=1}^{\infty}\frac{1}{n}\right]=\sum_{n=1}^{\infty}\frac{1}{2n}<\sum_{n=1}^{\infty}\frac{1}{n+n\cdot\cos^2(n)}$$

Since $\sum_{n=1}^{\infty}\frac{1}{n}$ is the divergent harmonic series, by the direct comparison test $\sum_{n=1}^{\infty}\frac{1}{n+n\cdot\cos^2(n)}$ also diverges.