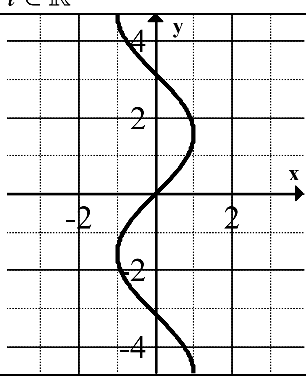
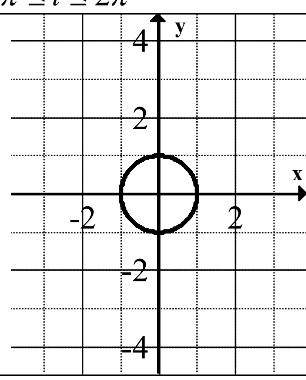
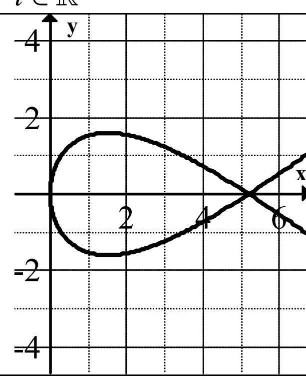
Parametric Curves:

Curves that are not functions can be thought of the path of a point/particle moving through the plane with respect to time. The point/particle can move as freely as it chooses, and the path is the collection of points where the point/particle has been located over time.
Every coordinate is specific to a moment of time, so $(x,y)\leftrightarrow(x(t),y(t))$, where $x(t)$ and $y(t)$ are functions independent of each other, but both a function of time. A point/particle can return to a previous location at different moments in time.

$C=\begin{cases}x(t)=\sin(t)\\ y(t)=t\end{cases}$ for $t\in\mathbb{R}$	$C=\begin{cases}x(t)=\cos(t)\\ y(t)=\sin(t)\end{cases}$ $-2\pi\le t\le 2\pi$	$C=\begin{cases}x(t)=t^2\\ y(t)=t-3\sin(t)\end{cases}$ for $t\in\mathbb{R}$
		

Eliminating the Parameter:

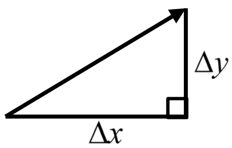
Sometimes you can use the two function $x(t)$ and $y(t)$ to eliminate the parameter *t*. That is, with some manipulation, get an expression that involves *x*'s and *y*'s, that does not involve any *t*'s.

$\begin{Bmatrix}x(t)=3t-1\\ y(t)=2t+1\end{Bmatrix}$	$x=3t-1\leftrightarrow t=\dfrac{x+1}{3}$ $y=2t+1$ $\boxed{y=2\left(\dfrac{x+1}{3}\right)+1}$ Parameter eliminated (no *t*'s)
$\begin{Bmatrix}x(t)=2t^2\\ y(t)=t^4+1\end{Bmatrix}$	$x=2t^2\leftrightarrow t^2=\dfrac{x}{2}$ $y=t^4+1$ $y=(t^2)^2+1$ $y=\left(\dfrac{x}{2}\right)^2+1$ Parameter eliminated (no *t*'s)

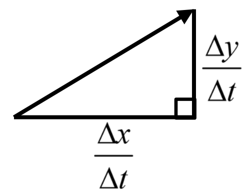
Calculus of Parametric Curves

It turns out that a parametric curve is smooth if the derivative of the curve at that moment in time exists. Just like in the past - if a function is differentiable, then the function is smooth.

So how do you find the derivative of a parametric function? Let $x = x(t)$ and $y = y(t)$



$$\frac{\Delta y}{\Delta x} \sim \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{y'(t)}{x'(t)}$$



Since you cannot have a change in y or x without a change in time, $\Delta y \sim \frac{\Delta y}{\Delta t}$ and $\Delta x \sim \frac{\Delta x}{\Delta t}$.

Vertical Tangent Lines	Horizontal Tangent lines
$y'(t) \neq 0$ and $x'(t) = 0$	$x'(t) \neq 0$ and $y'(t) = 0$

To find the ***second derivative***, you must do the following:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{\frac{d}{dt}}{\frac{dx}{dt}}\left[\frac{dy}{dx}\right] = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}}$$

That is, you must:

(1) Determine $\frac{dy}{dx}$.

(2) Differentiate $\frac{dy}{dx}$ with respect to t.

(3) Divide the result of step #2 by the expression of $\frac{dx}{dt}$.

$x(t) = \cos(t)$ $y(t) = 3\sin(t)$	
$\frac{dy}{dx} = \frac{\frac{d}{dt}\left[3\sin(t)\right]}{\frac{d}{dt}\left[\cos(t)\right]}$ $= \frac{3\cos(t)}{-\sin(t)}$ $= -3\cot(t)$	$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[-3\cot(t)\right]}{\frac{d}{dt}\left[\cos(t)\right]}$ $= \frac{3\csc^2(t)}{-\sin(t)}$ $= -3\csc^3(t)$

Area under the curve:

$\int_a^b f(x)\,dx$ is found by considering the following:

The value of t that corresponds to $x = a$ is will become the lower bound in the parametric form of the integral.

The value of t that corresponds to $x = b$ is will become the upper bound in the parametric form of the integral.

The value of $f(x)$ is represented by $y(t)$.

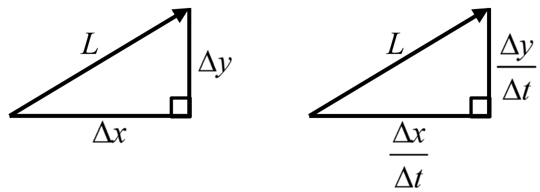
dx in the standard integral is obtained by the following: $dx = \frac{dx}{dt} \cdot dt = x'(t)\,dt$

$$\int_a^b y \cdot dx \sim \int_{t_1 \leftrightarrow a}^{t_2 \leftrightarrow b} y(t) \cdot \frac{dx}{dt} \cdot dt$$

$$\sim \int_{t_1 \leftrightarrow a}^{t_2 \leftrightarrow b} y(t) \cdot x'(t) \cdot dt$$

Note: One needs to be careful that the curve does not “loopdi-loop” on the time interval $t_1 \le t \le t_2$. If so, more analysis and breakdown of the time interval will be necessary to make sure that no area is double-counted.

Arc Length in Parametric form:



$$L^2 = \left(\Delta x\right)^2 + \left(\Delta y\right)^2$$

$$L^2 = \left[\left(\Delta x\right)^2 + \left(\Delta y\right)^2\right]\frac{\left(\Delta t\right)^2}{\left(\Delta t\right)^2}$$

$$L^2 = \left[\left(\Delta x\right)^2 + \left(\Delta y\right)^2\right]\frac{\left(\Delta t\right)^2}{\left(\Delta t\right)^2}$$

$$L^2 = \left[\frac{\left(\Delta x\right)^2}{\left(\Delta t\right)^2} + \frac{\left(\Delta y\right)^2}{\left(\Delta t\right)^2}\right]\left(\Delta t\right)^2$$

$$L = \sqrt{\left[\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2\right]\left(\Delta t\right)^2}$$

$$L = \sqrt{\left[\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2\right]}\sqrt{\left(\Delta t\right)^2}$$

$$L = \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2} \cdot \Delta t$$

$$\downarrow$$

$$L = \int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$L = \int_{t=a}^{t=b} \sqrt{\left[x'(t)\right]^2 + \left[y'(t)\right]^2}\, dt$$

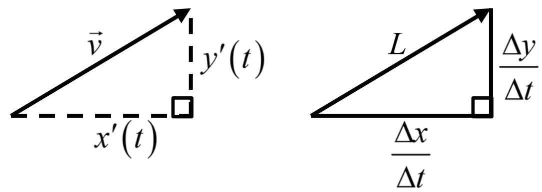
Motion and Parametric Curves:

A particle's position in parametric form is given by $(x, y) = (x(t), y(t))$.

The velocity vector of the particle, which describes the direction and speed in which the partile is moving is given by

$$\vec{v} = \langle x'(t), y'(t) \rangle$$

Note the use of vector brackets $\langle \ \rangle$.



The length of the velocity vector is the speed of the particle.

$$|\vec{v}| = \sqrt{[x'(t)]^2 + [y'(t)]^2}$$

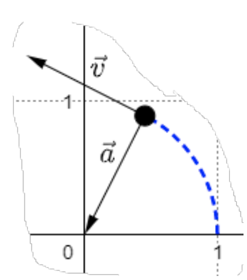
Therefore, the distance traveled by the particle is given by

$$\begin{aligned} d &= \int_{t_1}^{t_2} |\vec{v}| \, dt \\ &= \int_{t_1}^{t_2} \sqrt{[x'(t)]^2 + [y'(t)]^2} \, dt \\ &= \text{Length of the curve from } t_1 \text{ to } t_2 \end{aligned}$$

The acceleration vector of the particle is given by

$$\vec{a} = \langle x''(t), y''(t) \rangle$$

The acceleration vector can be thought of as the force acting upon the particle. It is the force that influences/changes the velocity vector.

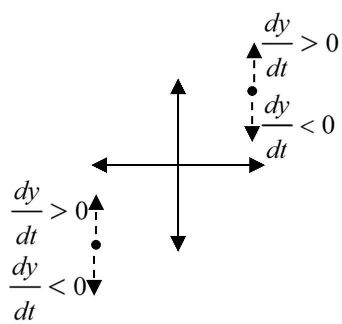
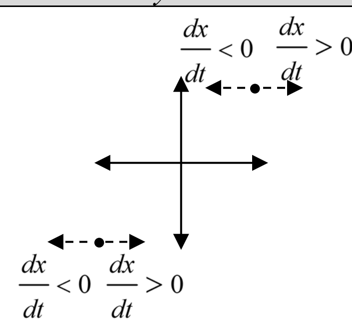


A common question asked on the free response questions is whether a particle is moving towards or away from the x-axis or the y-axis. The following table/decision tree helps you determine whether a particle is moving towards or away from an axis by identifying the following:

I. Which axis the motion is relative to.
II. The current position of the particle.
III. The component of the velocity vector perpendicular to the axis.

How to determine whether a point is moving towards or away from the *x*-axis or *y*-axis:

Choose a column of the table, then identify the conditions in the rows below to determine the result.

<table>
<tr><th colspan="4">the x-axis</th><th colspan="4">the y-axis</th></tr>
<tr><td colspan="4">
</td><td colspan="4">
</td></tr>
<tr><td colspan="2">$y > 0$</td><td colspan="2">$y < 0$</td><td colspan="2">$x < 0$</td><td colspan="2">$x > 0$</td></tr>
<tr><td>$\frac{dy}{dt} < 0$</td><td>$\frac{dy}{dt} > 0$</td><td>$\frac{dy}{dt} < 0$</td><td>$\frac{dy}{dt} > 0$</td><td>$\frac{dx}{dt} < 0$</td><td>$\frac{dx}{dt} > 0$</td><td>$\frac{dx}{dt} < 0$</td><td>$\frac{dx}{dt} > 0$</td></tr>
<tr><td>towards x-axis</td><td>away from x-axis</td><td>away from x-axis</td><td>towards x-axis</td><td>away from y-axis</td><td>towards y-axis</td><td>towards y-axis</td><td>away from y-axis</td></tr>
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