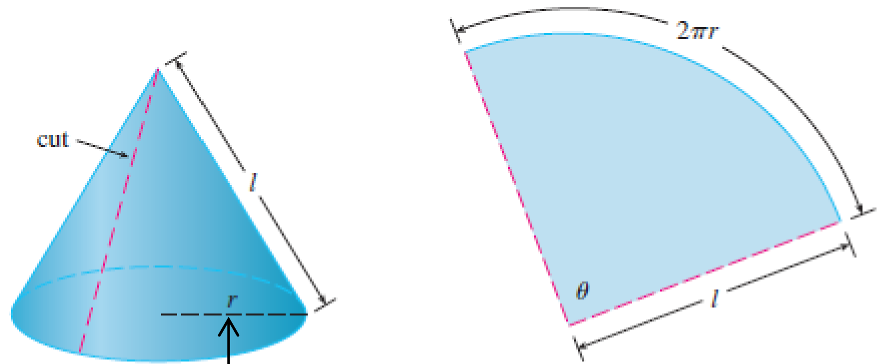
Derivation of Surface Area of the Frustum of a Cone & Surface Area of Revolution

Consider a right circular cone that is made by connecting the radii of a sector of a circle together as illustrated in the figure below:



Circumference of the Base of the Cone = Arc Length of corresponding Sector

$$2\pi r = \underbrace{2\pi l}_{\substack{\text{circumference}\\ \text{unfolded}}} \cdot \underbrace{\left[\frac{\theta}{2\pi}\right]}_{\substack{\text{fraction of}\\ \text{unfolded circle}}}$$

$$\langle 1.1\rangle \quad 2\pi r = 2\pi l \cdot \frac{\theta}{2\pi}$$

$$2\pi r = \theta l$$

$$r = \frac{\theta l}{2\pi}$$

The area of the sector is given by

$$\frac{\text{degrees}}{360}\cdot(\text{area of circle}) = \text{area of sector}$$

$$\frac{\theta}{2\pi}\left[\pi l^2\right] \Rightarrow \frac{\theta l^2}{2}$$

Since the Lateral Surface Area of the Cone = Area of the corresponding Sector we have

$$L.A. = \frac{\theta l^2}{2}$$

$$\langle 1.2\rangle \qquad = \frac{\theta l^2}{2}\cdot\frac{\pi}{\pi}$$

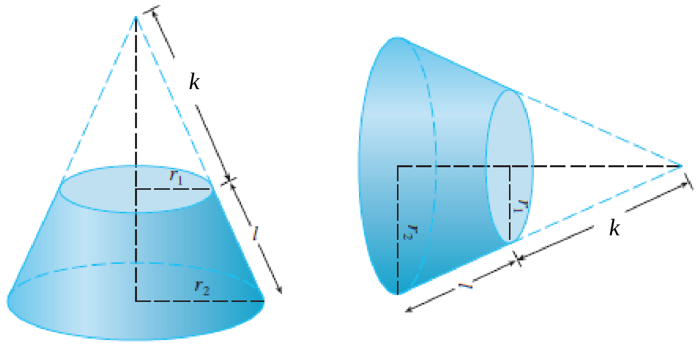
$$= \pi\left[\frac{\theta l}{2\pi}\right] l$$

$$= \pi r l \qquad \text{(by equation 1.1)}$$

So the Lateral Surface Area of a Right Circular Cone is given by:

$$L.A. = \pi(\text{base radius})(\text{slant height})$$

The Frustum of a Cone is illustrated as the shaded figure below



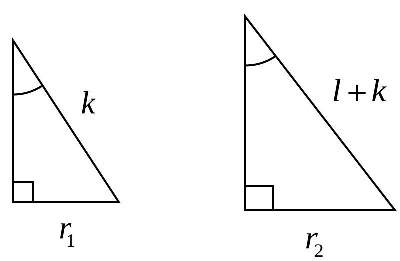
The lateral surface area of the frustum of the cone is the difference in lateral surface area of the larger cone and smaller cone.

$$\text{Lateral Area of Frustum} = \text{Lateral Area of Large Cone } - \text{Lateral Area of Small cone}$$

Since we can now calculate both the surface areas on the right hand side of the equation, we get that:

$$\begin{aligned} L.A. &= \pi r_2 (l+k) - \pi r_1 k \\ \langle 1.3 \rangle \qquad &= \pi r_2 l + \pi r_2 k - \pi r_1 k \\ &= \pi r_2 l + \pi (r_2 - r_1) k \end{aligned}$$

Consider the similar triangles



We can conclude that

$$\begin{aligned} \frac{r_1}{r_2} &= \frac{k}{l+k} \\ r_1 (l+k) &= r_2 k \\ \langle 1.4 \rangle \qquad r_1 l + r_1 k &= r_2 k \\ r_1 l &= r_2 k - r_1 k \\ r_1 l &= (r_2 - r_1) k \end{aligned}$$

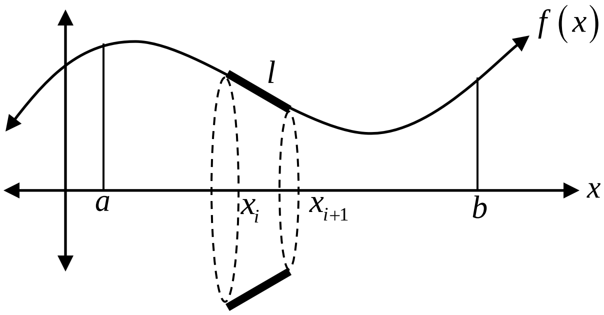
Substituting the result in equation 1.4 into equation 1.3 we get that

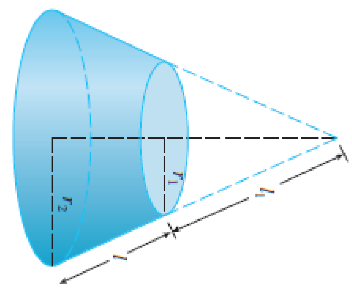
$$\begin{aligned} L.A. &= \pi r_2 l + \pi (r_2 - r_1) k \\ &= \pi r_2 l + \pi r_1 l \\ &= \pi (r_2 + r_1) l \\ &= 2\pi \left(\frac{r_2 + r_1}{2} \right) l \end{aligned}$$

Letting $\bar{r} = \frac{r_2 + r_1}{2}$, we conclude that $L.A. = 2\pi \cdot \bar{r} \cdot l$

Where $\bar{r}$ is the average of the radii of the Frustum of the cone.

To find the surface area of a volume revolution of a continuous function that is being revolved around the *x*-axis, we revolve a linear estimate of the arc length of the function around the *x*-axis and determine the area of the frustum of a cone that is generated.





The surface area of the furstum of the cone illustrated above is given by

$$\langle 1.5\rangle \quad S.A. = 2\pi\left(\frac{f(x_i)+f(x_{i+1})}{2}\right)l_{i+1}$$

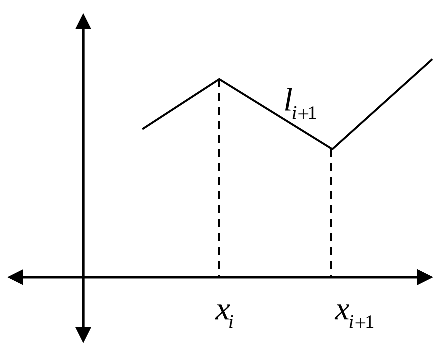
Since $f(x)$ is a continuous function on the interval $[x_i, x_{i+1}]$, by IVT there must exist a c_{i+1} where $x_i \le c_{i+1} \le x_{i+1}$ where

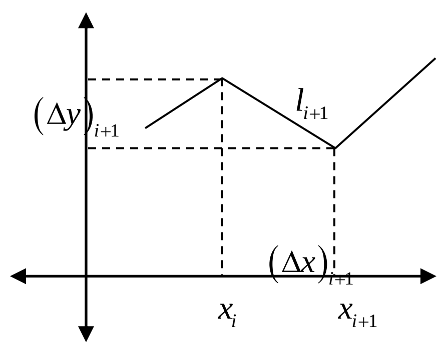
$$\langle 1.6\rangle \quad f(c_{i+1}) = \frac{f(x_i)+f(x_{i+1})}{2}$$

Using the expression in equation 1.6 with equation 1.5 we get that

$$S.A. = 2\pi \cdot f(c_{i+1}) \cdot l_{i+1}$$

Now we must discuss how to determine l





We approximate the length of the curve on the interval $[x_i, x_{i+1}]$ using a line l. We can determine the value of l by using Pythagorean Theorem.

$$\langle 1.7\rangle \quad \begin{aligned} l_{i+1}^2 &= \left([\Delta x]_{i+1}\right)^2 + \left([\Delta y]_{i+1}\right)^2 \\ l_{i+1} &= \sqrt{\left([\Delta x]_{i+1}\right)^2 + \left([\Delta y]_{i+1}\right)^2} \end{aligned}$$

To get the length of the overall function we must sum all the l_{i+1}, giving us the following sum:

$$
\begin{aligned}
\text{Length of curve} &= \sum_{i=0}^{n-1} \sqrt{\left([\Delta x]_{i+1}\right)^2 + \left([\Delta y]_{i+1}\right)^2} \\
&= \sum_{i=0}^{n-1} \sqrt{\left[\left([\Delta x]_{i+1}\right)^2 + \left([\Delta y]_{i+1}\right)^2\right] \cdot \frac{\left([\Delta x]_{i+1}\right)^2}{\left([\Delta x]_{i+1}\right)^2}} \\
&= \sum_{i=0}^{n-1} \sqrt{\left[\left([\Delta x]_{i+1}\right)^2 + \left([\Delta y]_{i+1}\right)^2\right] \cdot \frac{1}{\left([\Delta x]_{i+1}\right)^2} \cdot \left([\Delta x]_{i+1}\right)^2} \\
&= \sum_{i=0}^{n-1} \sqrt{\left[\left([\Delta x]_{i+1}\right)^2 + \left([\Delta y]_{i+1}\right)^2\right] \cdot \frac{1}{(\Delta x)_{i+1}{}^2}} \sqrt{(\Delta x)_{i+1}{}^2} \\
&= \sum_{i=0}^{n-1} \sqrt{\frac{\left([\Delta x]_{i+1}\right)^2}{\left([\Delta x]_{i+1}\right)^2} + \frac{\left([\Delta y]_{i+1}\right)^2}{\left([\Delta x]_{i+1}\right)^2}} \cdot \left|[\Delta x]_{i+1}\right| \\
&= \sum_{i=0}^{n-1} \sqrt{1 + \left(\frac{[\Delta y]_{i+1}}{[\Delta x]_{i+1}}\right)^2} \cdot \left|[\Delta x]_{i+1}\right| \\
&\downarrow \\
&\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx
\end{aligned}
$$

Therefore $l_{i+1} = \sqrt{1 + \left(\frac{[\Delta y]_{i+1}}{[\Delta x]_{i+1}}\right)^2} \cdot [\Delta x]_{i+1}$. We will use this to derive the formula for the surface area of revolution. To finish this we conclude with the arc length formula.

The formula for the length of the curve of $f(x)$ on the interval $[a,b]$ is given by

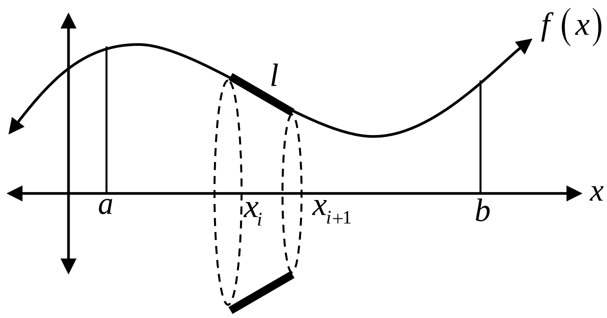
$$\int_a^b \sqrt{1 + \left[f'(x)\right]^2}\, dx$$

or

$$\int_a^b \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\, dx$$

Now to return to the surface area of a volume of revolution.

To determine the surface area of the volume of revolution, we need to determine the surface area of each frustum and sum them all together. The surface area of the frustum below is given by



$$L.A. = 2\pi\left(\frac{f(x_i)+f(x_{i+1})}{2}\right)l_{i+1}$$

Since $f(x)$ is a continuous function, by IVT there must exist a c_i , where $x_i \le c_i \le x_{i+1}$ and

$$f(c_{i+1}) = \frac{f(x_i)+f(x_{i+1})}{2}$$

Therefore, the surface area of the frustum can be rewritten as

$$\begin{aligned} L.A. &= 2\pi\left(\frac{f(x_i)+f(x_{i+1})}{2}\right)l_{i+1} \\ &= 2\pi\cdot f(c_{i+1})\cdot l_{i+1} \end{aligned}$$

Previously we determined that

$$l_{i+1} = \sqrt{1+\left(\frac{[\Delta y]_{i+1}}{[\Delta x]_{i+1}}\right)^2}\cdot[\Delta x]_{i+1}$$

We can rewrite the surface area for the frustum of a cone again as

$$L.A. = \sum 2\pi\cdot f(c_{i+1})\cdot\sqrt{1+\left(\frac{[\Delta y]_{i+1}}{[\Delta x]_{i+1}}\right)^2}\cdot[\Delta x]_{i+1}$$

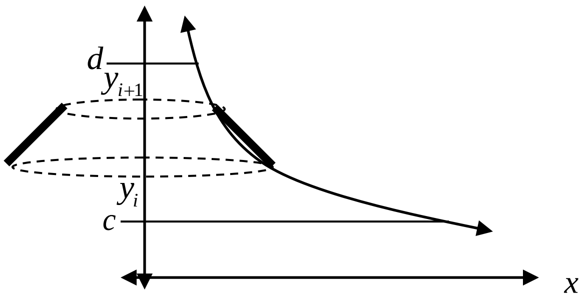
Taking the sum of all the frustums of the cones over the closed interval, we get the formula for the surface area of revolution of $f(x)$ about the *x*-axis is

$$\int_a^b 2\pi f(x)\sqrt{1+\left[f'(x)\right]^2}\,dx$$

This formula can be described as

$$\int 2\pi(\text{radius})(\text{arc length})$$

To find the surface area of revolution when the axis of revolution is vertical, the process is similar to the case where the axis of revolution is horizontal, except that we change our perspective to be from the *y*-axis. Note that x is now a function of y. That is $f(y)=x$.



We can start with the generalized formula $\int 2\pi(\text{radius})(\text{arc length})$. As with the horizontal axis of revolution case, by IVT there must exist a k_{i+1} in the interval $[y_i, y_{i+1}]$ such that $f(k_{i+1})=\dfrac{f(y_i)+f(y_{i+1})}{2}$. So that our formula turns in to $\sum 2\pi f(k_i)(\text{arc length})$

However, since *f* is a function of *y*, we must have the arc length formula written in such a way that dy is a part of this formula. From equation 1.7 we have that

$$\sqrt{\left[\left([\Delta x]_{i+1}\right)^2+\left([\Delta y]_{i+1}\right)^2\right]\left[\frac{\left([\Delta x]_{i+1}\right)^2}{\left([\Delta x]_{i+1}\right)^2}\right]}=l_{i+1}=\sqrt{\left[\left([\Delta x]_{i+1}\right)^2+\left([\Delta y]_{i+1}\right)^2\right]\left[\frac{\left([\Delta y]_{i+1}\right)^2}{\left([\Delta y]_{i+1}\right)^2}\right]}$$

$$\sqrt{\left[\left([\Delta x]_{i+1}\right)^2+\left([\Delta y]_{i+1}\right)^2\right]\left[\frac{\left([\Delta x]_{i+1}\right)^2}{\left([\Delta x]_{i+1}\right)^2}\right]}=\sqrt{\left[\left([\Delta x]_{i+1}\right)^2+\left([\Delta y]_{i+1}\right)^2\right]\left[\frac{\left([\Delta y]_{i+1}\right)^2}{\left([\Delta y]_{i+1}\right)^2}\right]}$$

$$\sqrt{\left[\left([\Delta x]_{i+1}\right)^2+\left([\Delta y]_{i+1}\right)^2\right]\left[\frac{1}{\left([\Delta x]_{i+1}\right)^2}\right]\left([\Delta x]_{i+1}\right)^2}=\sqrt{\left[\left([\Delta x]_{i+1}\right)^2+\left([\Delta y]_{i+1}\right)^2\right]\left[\frac{1}{\left([\Delta y]_{i+1}\right)^2}\right]\left([\Delta y]_{i+1}\right)^2}$$

$$\sqrt{\left[\frac{\left([\Delta x]_{i+1}\right)^2}{\left([\Delta x]_{i+1}\right)^2}+\frac{\left([\Delta y]_{i+1}\right)^2}{\left([\Delta x]_{i+1}\right)^2}\right]}\sqrt{\left([\Delta x]_{i+1}\right)^2}=\sqrt{\left[\frac{\left([\Delta x]_{i+1}\right)^2}{\left([\Delta y]_{i+1}\right)^2}+\frac{\left([\Delta y]_{i+1}\right)^2}{\left([\Delta y]_{i+1}\right)^2}\right]}\sqrt{\left([\Delta y]_{i+1}\right)^2}$$

$$\sqrt{1+\left[\frac{\left([\Delta y]_{i+1}\right)}{\left([\Delta x]_{i+1}\right)}\right]^2}\cdot[\Delta x]_{i+1}=\sqrt{1+\left[\frac{\left([\Delta x]_{i+1}\right)}{\left([\Delta y]_{i+1}\right)}\right]^2}\cdot[\Delta y]_{i+1}$$

$$\downarrow$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

Hence we have two equivalent expressions for arc length, one with a perspective from the *x* axis on the left, and one from the perspective of the *y*-axis on the right.
It should be noted that the length of the line segment should be the same from both perspectives, as the length does not change no matter how you look at it.

Now that we have a formula for arc length with respect to *y*, we can complete our surface area of revolution formula to be

$$\sum 2\pi f\left(k_{i+1}\right)\sqrt{1+\left[\frac{(\Delta x)_{i+1}}{(\Delta y)_{i+1}}\right]^2}\,(\Delta y)_{i+1}$$

$$\downarrow$$

$$\int_c^d 2\pi f\left(y\right)\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy = \int_c^d 2\pi f\left(y\right)\sqrt{1+\left[f'\left(y\right)\right]^2}\,dy$$

However, the formula can be rewritten with respect to *x* by noting the following:

(1) $f\left(y\right)=x$

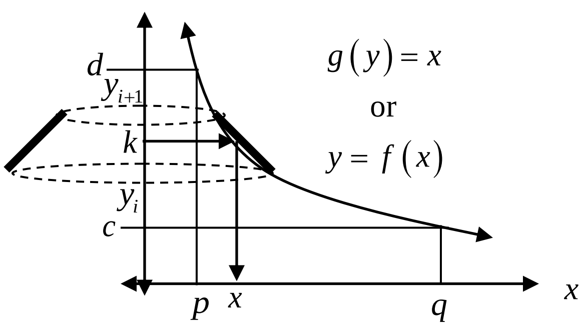
(2) $\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

Therefore, we can rewrite our integral as

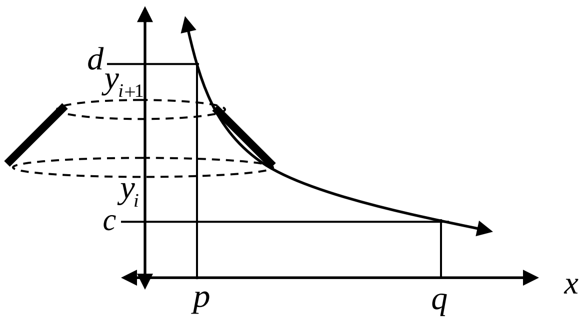
$$\int_{???}^{???} 2\pi x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

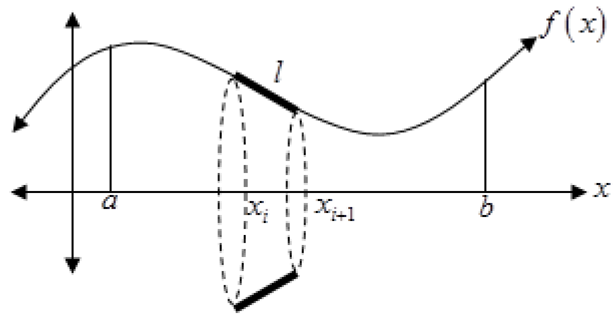
However, our bounds may not turn out to be *c* and *d* because our integral is with respect to *x* now, and *c* and *d* were with respect to *y*. Therefore, we must rewrite the bounds with respect to *x*.

If we are given the following diagram and view the curve so that $g\left(y\right)=x$ produces the same graph of $y=f\left(x\right)$, then



$$\int_c^d 2\pi g\left(y\right)\sqrt{1+\left[g'\left(y\right)\right]^2}\,dy = \int_p^q 2\pi x\sqrt{1+\left[f'\left(x\right)\right]^2}\,dx$$





$$\int 2\pi(\text{radius})(\text{arc length})$$

y-axis:

- $x = g(y)$: $$\int_c^d 2\pi g(y)\sqrt{1+\left[g'(y)\right]^2}\,dy$$
- $y = f(x)$: $$\int_p^q 2\pi x\sqrt{1+\left[f'(x)\right]^2}\,dx$$

x-axis:

$$\int_a^b 2\pi f(x)\sqrt{1+\left[f'(x)\right]^2}\,dx$$